IDLA aggregates with infinitely many sources and the directed IDLA forest

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Abstract

We investigate three types of Internal Diffusion Limited Aggregation (IDLA) models. These models are based on simple random walks on $\mathbb{Z}^2$ with infinitely many sources that are the points of the vertical axis $I(\infty) = \{0\} \times \mathbb{Z}$. Various properties are provided, such as stationarity, mixing, stabilization and shape theorems. Our results allow us to define a new directed (w.r.t. the horizontal direction) random forest spanning $\mathbb{Z}^2$, based on an IDLA protocol, which is invariant in distribution w.r.t. vertical translations.

Keywords : Internal Diffusion Limited Aggregation; Cluster growth; Random trees and forests; Shape theorems; Random walks.

AMS 2020 classification : Primary: 60K35; 05C80; 82C24; Secondary: 82B41; 60G50.

1 Introduction

The Internal Diffusion Limited Aggregation (IDLA) is a random growth model first introduced for chemical applications in 1986 by Meakin and Deutch [30] and then, in a mathematical framework, by Diaconis and Fulton in [14]. In this model, the aggregate is recursively defined by adding to the aggregate the first site out of the current aggregate visited by a random walk starting from some source point. The classical IDLA model is constructed in $\mathbb{Z}^d$ as follows. We start with $A_0 = \emptyset$. At step n, a simple symmetric random walk starts from the origin 0 until it exits the current aggregate A_{n-1}, say at some vertex z, which is added to A_{n-1} to get $A_n = A_{n-1} \cup \{z\}$. In the classical IDLA model (and also in this paper), the word *particle* is used to refer to the random walk which is stopped when it exits the current aggregate A_{n-1}, and settled on the new vertex z.

A first shape theorem was established by Lawler, Bramson and Griffeath in [25] for the classical IDLA model. It asserts that the aggregate A_n (when it is suitably normalized) converges a.s. to an Euclidean ball as n goes to infinity, with fluctuations (w.r.t. the limit shape) which are at most linear. Since then, several papers (by Lawler [24], Asselah and Gaudillière [1, 2, 3] and Jerison, Levine and Sheffield [20, 21, 22]) have improved the bounds for fluctuations which are known to be logarithmic in $2D$ and sublogarithmic in higher dimensions.

Recently, many variants of this problem have been considered. In particular, IDLA on discrete groups with polynomial or exponential growth have been studied in [5, 10], on non-amenable graphs in [18], with multiple sources in [27], on supercritical percolation clusters in [15, 33], on comb lattices in [4, 19], on cylinder graphs in [23, 28, 34], constructed with drifted random walks in [29] or with uniform starting points in [7].

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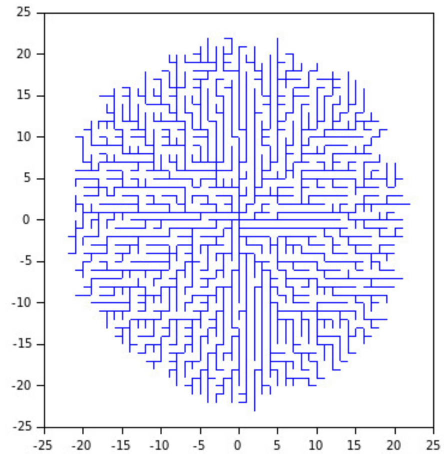

Figure 1: A realization of $\mathscr{T}_{1500}$.

A random infinite tree $\mathscr{T}_\infty$ can be associated with the sequence of IDLA aggregates $(A_n)_{n\geq 0}$ defined above in a very natural way. To our knowledge, this object has not been introduced in the literature. The tree $\mathscr{T}_1$ only consists of the root 0. By induction, $\mathscr{T}_n$ is obtained by adding to $\mathscr{T}_{n-1}$ the new vertex z such that $A_n = A_{n-1} \cup \{z\}$ and the edge used by the n-th particle to reach z from A_{n-1} (see Figure 1). Hence, we can define a.s. a random graph

$$\mathscr{T}_\infty = \bigcup_{n\geq 1} \uparrow \mathscr{T}_n,$$

which actually is a tree (since each vertex of $\mathbb{Z}^2$ may only be added once) rooted at the origin. The lower bound for the shape theorem specifies that its edge set spans the whole set $\mathbb{Z}^2$.

The first question about the random tree $\mathscr{T}_\infty$ concerns the existence of (many) infinite branches with asymptotic directions (of course, by compacity, $\mathscr{T}_\infty$ contains at least one infinite branch). In [17], Howard and Newman have developed an efficient strategy leading to such results. The key point would be to prove that the fluctuations of the branch in $\mathscr{T}_\infty$ joining the origin to a given vertex $z \in \mathbb{Z}^2$ w.r.t. the segment $[0,z]$ (in $\mathbb{R}^2$) are negligible w.r.t. $|z|_2$, where $|\cdot|_2$ denotes the Euclidean norm in $\mathbb{R}^2$. In this case, the tree $\mathscr{T}_\infty$ is said *straight*. Although one might strongly conjecture such a result (especially because fluctuations in the shape theorem are logarithmic), it is difficult to prove it for several reasons. First, any branch γ of the IDLA tree $\mathscr{T}_\infty$ is not produced by a single particle but by many particles, each of them adding exactly one edge depending on the shape of the current aggregate. Secondly, the radial character of $\mathscr{T}_\infty$ (its branches are directed to the origin) prevents its distribution to satisfy any useful invariance properties.

A way to overcome this (second) obstacle would be to consider a directed forest w.r.t. to some vector $u \in \mathbb{R}^2$ (i) whose distribution admits some invariant translation (w.r.t. any orthogonal vector to u) properties making its study easier than the one of $\mathscr{T}_\infty$ and (ii) which approximates locally and far from the origin, i.e. on the ball $B(-nu,R)$ with $n \gg 1$ and R constant, the distribution of the random tree $\mathscr{T}_\infty$ so that we could transfer results about this directed forest to $\mathscr{T}_\infty$. This strategy relies on the following remark: the radial character of the branches of $\mathscr{T}_\infty$ restricted to $B(-nu,R)$ should fade away as $n \to \infty$ (with R constant) so that the branches should be directed w.r.t. the vector u and not toward the origin. In particular, this strategy has been used successfully by Baccelli and Bordenave in [5] to approximate the *Radial Spanning Tree* by the *Directed spanning Forest* and to show that it is straight. See also [11] in which the author exploits this link between trees and the associated forests to quantify the density of infinite branches of these trees. Furthermore, the associated forest generally is much more than a tool: in [12], for example, the authors prove that the Directed spanning Forest mentioned above converges in distribution to the Brownian web.

This strategy is the original motivation for this work.

One of our main results is the construction of a new random forest $\mathscr{F}_\infty$, called the *directed infinite-volume IDLA forest*, and directed w.r.t. the horizontal vectors $u = (\pm 1, 0)$ (see Section 7.3). This con-

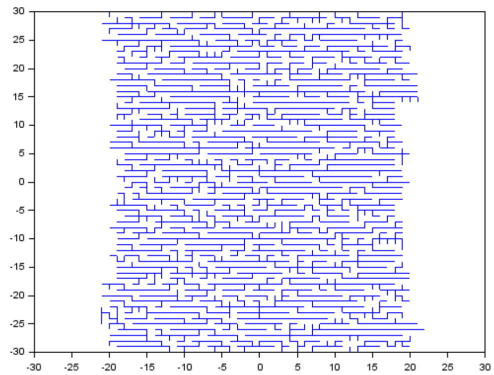

Figure 2: A realization of $\mathscr{F}\left(A^{\dagger}_{40}[200]\right)$ observed in $\mathbb{Z}_{30}$.

struction relies on the use of three IDLA processes (and their properties) whose limits are denoted by $A_n[\infty]$, $A^*_n[\infty]$ and $A^{\dagger}_n[\infty]$, and based on infinitely many sources given by the sites of the vertical axis $I(\infty) = \{0\} \times \mathbb{Z}$. Thus we show that the distribution of $\mathscr{F}_\infty$ is invariant (and even mixing) w.r.t. vertical translations (Theorem 7.4). Finally, we think that the directed IDLA forest $\mathscr{F}_\infty$ is an interesting mathematical object with several conjectures (finiteness of its trees, straightness of its branches, scaling limit). In particular, we conjecture that $\mathscr{F}_\infty$ approximates in distribution, locally and far from the origin, the IDLA tree $\mathscr{T}_\infty$ (see Conjecture 1 in Section 7.4).

Let us note that Berger, Kagan and Procaccia proposed in [9] a random forest model based on an IDLA protocol. Nevertheless, their construction is based on oriented random walks which certainly simplifies the construction but also prohibits U-turns that are possible in $\mathscr{T}_\infty$ and also in our directed IDLA forest $\mathscr{F}_\infty$ (see e.g. the edge with vertex $(8,-8)$ in Figure 2). Hence, their model does not seem to be a good candidate to mimic the infinite IDLA tree $\mathscr{T}_\infty$ and to capture its properties.

Let us describe the construction of the three infinite aggregates, namely $A_n[\infty]$, $A^*_n[\infty]$ and $A^{\dagger}_n[\infty]$. We say that a particle is *emitted* or *sent from level* i if the underlying random walk starts from the source $(0,i)$. First, we consider finite aggregates $A_n[M]$, with $M \geq 0$, in which n particles are sent from each site of $I(M) = \{0\} \times [\![-M, M]\!]$, where $[\![a,b]\!] = [a,b] \cap \mathbb{Z}$ for any $a \leq b$. The first n ones are sent from level 0, then the following n particles from level 1, next from level -1 and so on up to last particles sent from levels M thus $-M$. In the rest of the paper, this specific order for sending particles is referred to as the *usual order*. The sequence $(A_n[M])_{M \geq 0}$ is increasing and allows us to define a limiting infinite aggregate $A_n[\infty]$. Note that this model is not covered by [27].

The second infinite IDLA aggregate, denoted by $A^*_n[\infty]$, is defined in the same spirit as $A_n[\infty]$ but this time by launching a random number N_i of particles from each level i w.r.t. the usual order. The N_i's are i.i.d. Poisson random variables with parameter n and are independent of the underlying random walks.

Although $A_n[\infty]$ and $A^*_n[\infty]$ are not identically distributed (see Proposition 3.6), they have in common the usual order in which particles are sent and for this reason they are not conducive to define a translation invariant IDLA forest. However, this usual order will be particularly useful to derive the stabilization properties for $A_n[\infty]$ and $A^*_n[\infty]$. A natural way to get back an invariance property in distribution w.r.t. vertical translations consists in sending particles *uniformly* from the vertical axis $I(\infty)$. This idea motivates the definition of the third infinite IDLA aggregate $A^{\dagger}_n[\infty]$. So let us consider a family $(\mathscr{N}_i)_{i\in\mathbb{Z}}$ of i.i.d. Poisson point processes (PPP's) in $\mathbb{R}^+$ with intensity 1 (and also independent of the underlying random walks). Each PPP $\mathscr{N}_i$ is attached to the level i in the sense that particles are sent from the source $(0,i)$ according to the clocks given by $\mathscr{N}_i$. Let us specify that the trajectories of particles are instantaneously realized. Hence, for any M, the aggregate $A^{\dagger}_n[M]$ is built by sending particles from the source set $I(M)$

according to the clocks given by the corresponding PPP's up to time n. Remark that this construction ensures that, at each time, the next particle (if it exists) is sent from a source chosen uniformly on $I(M)$. Thus, $A_n^\dagger[\infty]$ is defined as the increasing union of the $A_n^\dagger[M]$'s.

Defining an IDLA forest $\mathscr{F}(A_n^\dagger[M])$ from the (finite) aggregate $A_n^\dagger[M]$ is easy to do (see Section 7.1 and Figure 2). But taking the limit $M \to \infty$ in the sequence $(\mathscr{F}(A_n^\dagger[M]))_{M\geq 0}$ needs to take some precautions. Indeed, given $M' > M$, any particle starting from a level $M' \geq |i| > M$ may generate a set of discrepancies between both forests $\mathscr{F}(A_n^\dagger[M])$ and $\mathscr{F}(A_n^\dagger[M'])$ through a tricky phenomenon that we have called a *chain of changes* and described in Section 7.2. The existence of arbitrarily long chains of changes in the sequence $(\mathscr{F}(A_n^\dagger[M]))_{M\geq 0}$ is the main obstacle to define the directed IDLA forest $\mathscr{F}_\infty$.

To overcome this obstacle, we proceed as follows. We first establish two stabilization results for both infinite aggregates $A_n[\infty]$ and $A_n^*[\infty]$. The first one (Theorem 3.1) asserts that $A_n[\infty]$ and $A_n^*[\infty]$, restricted to a neighborhood of the origin, are not sensitive to particles coming from far levels. On the opposite, the second one (Theorem 4.1) claims that $A_n[\infty]$ and $A_n^*[\infty]$, restricted to high levels, are not sensitive to particles sent from a neighborhood of the origin. These fruitful tools imply that the infinite aggregates $A_n[\infty]$ and $A_n^*[\infty]$ are mixing w.r.t. vertical translations. Combining with Proposition 3.6, we obtain that $A_n^*[\infty]$ a.s. avoids an infinite number of horizontal lines which means that $A_n^*[\infty]$ is made up with infinitely many finite and disjoint connected components. The same holds for $A_n^\dagger[\infty]$ since $A_n^\dagger[\infty]$ and $A_n^*[\infty]$ are equally distributed thanks to the Abelian property (see [14], p. 3). This is this latter statement which prevents the existence of arbitrarily long chains of changes in the sequence $(\mathscr{F}(A_n^\dagger[M]))_{M\geq 0}$ and allows us to take first the (vertical) limit $M \to \infty$ in the sequence $(\mathscr{F}(A_n^\dagger[M]))_{M\geq 0}$ and thus the (horizontal) limit $n \to \infty$ to finally define the directed infinite-volume IDLA forest $\mathscr{F}_\infty$.

Finally, we prove a shape theorem (Theorem 6.1) for the three aggregates $A_n[\infty]$, $A_n^*[\infty]$ and $A_n^\dagger[\infty]$ restricted to the strip $\mathbb{Z}_{n^\alpha} = \mathbb{Z} \times [\![-n^\alpha, n^\alpha]\!]$ (for any $\alpha > 0$) as n tends to infinity. Adapting to our context the strategy developed by Asselah and Gaudillière [1, 2], we prove that, with probability 1 and for any n large enough, the fluctuations (w.r.t. the Hausdorff distance) between $A_n[\infty] \cap \mathbb{Z}_{n^\alpha}$ and the rectangle $[\![-\frac{n}{2}, \frac{n}{2}]\!] \times [\![-n^\alpha, n^\alpha]\!]$ are at most logarithmic. The same result holds for $A_n^*[\infty]$ and $A_n^\dagger[\infty]$. As a consequence of this shape theorem, the vertex set of the directed IDLA forest $\mathscr{F}_\infty$ fulfills the whole set $\mathbb{Z}^2$.

The rest of this paper is organized as follows. In Section 2, we define the infinite aggregates $A_n[\infty]$, $A_n^*[\infty]$ and $A_n^\dagger[\infty]$ and we state their first properties that mostly rely on the particular order in which the particles are sent in $A_n[\infty]$ and $A_n^*[\infty]$. Then, in Section 3, we prove that a.s. particles emitted above some random level contribute to the aggregate before visiting the strip $\mathbb{Z}_M$ and that $A_n[\infty]$ and $A_n^*[\infty]$ are not equally distributed. In Section 4, we show that the aggregates above some random levels a.s. do not depend on particles which are sent arround the origin. The mixing properties of the aggregates are discussed in Section 5 and used to deduce that $A_n^*[\infty]$ and $A_n^\dagger[\infty]$ a.s. avoid infinitely many lines $\mathbb{Z} \times \{i\}$, $i \in \mathbb{Z}$. Section 6 is devoted to the shape theorems. Finally, Section 7 contains the construction of the infinite volume IDLA forest that spans $\mathbb{Z}^2$ and whose distribution is invariant w.r.t. vertical translations. Our paper ends with three conjectures on the random forest $\mathscr{F}_\infty$ and on the random tree $\mathscr{T}_\infty$.

2 Construction and properties of $A_n[\infty]$, $A_n^*[\infty]$ and $A_n^\dagger[\infty]$

Let $n \geq 1$ be fixed. In this section, we introduce three infinite random aggregates with sources on $I(\infty) = \{0\} \times \mathbb{Z}$. Then we give some properties of these aggregates.

2.1 Construction of the aggregates

2.1.1 Construction of $A_n[\infty]$

The aggregate $A_n[\infty]$ is a natural extension of the standard IDLA cluster (see *e.g.* [25]) with sources on the $I(\infty)$ instead of the origin. To construct it, we first introduce a family of finite random aggregates $A_n[M]$, $M \geq 0$. When $M = 0$, the random set $A_n[0]$ is the standard IDLA cluster with volume n. It is obtained

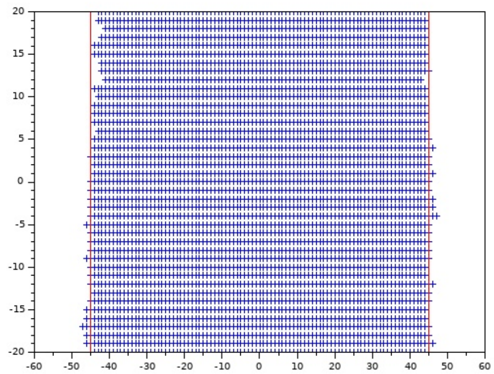

Figure 3: A realization of the aggregate $A_{90}[200] \cap \mathbb{Z}_{20}$ based on 90 particles per site $(0,i)$, with $|i| \leq 200$, and intersected by the strip $\mathbb{Z}_{20}$.

inductively as follows. One by one, particles perform independent simple symmetric 2-dimensional random walks. Each particle starts from the origin and moves until it reaches a site that has not been visited previously, at which point it stops. Then $A_n[0]$ is the cluster of occupied sites after the n-th particle stops. In a similar way, given a realization of $A_n[M-1]$, we send n particles from the site $(0,M)$ then n particles from the site $(-M,0)$. The set $A_n[M]$ denotes the aggregate which is produced by these $2n$ particles and by the aggregate $A_n[M-1]$. As an illustration, Figure 3 gives a realization of $A_{90}[200]$ when it is observed in the strip $\mathbb{Z}_{20} = \{0\} \times [\![-20,20]\!]$.

By construction, the sequence of random aggregates $(A_n[M])_{M\geq 0}$ is increasing and the number of sites of $A_n[M]$ is

$$\#A_n[M] = (2M+1)n.$$

The infinite random aggregate $A_n[\infty]$ is then defined as:

$$A_n[\infty] = \bigcup_{M\geq 0} A_n[M].$$

2.1.2 Construction of $A_n^*[\infty]$

The aggregate $A_n^*[\infty]$ is constructed in the same spirit as above but this time the number of particles which are sent from each site of $I(\infty)$ is no longer equal to n but random. To define it, let $(N_i)_{i\in\mathbb{Z}}$ be a family of independent Poisson random variables with parameter n. The random variable N_i is the number of particles starting from $(0,i)$. Then, for each $j \in \mathbb{Z}$, we consider a family of simple random walks, starting from $(0,j)$, which are independent of $(N_i)_{i\in\mathbb{Z}}$. Given a realization of $(N_i)_{i\in\mathbb{Z}}$, we denote by $A_n^*[M]$, $M \geq 0$, the IDLA aggregate obtained by sending particles from levels $|i| \leq M$ in the usual order, *i.e.* first by sending the N_0 particles from $(0,0)$, thus the N_1 particles from $(0,1)$, thus the N_{-1} particles from level $(0,-1)$ and so on. By construction, the cardinality of the set $A_n^*[M]$ is random and can be equal to zero with positive probability, so that $A_n^*[M]$ does not have the same distribution as $A_n[M]$. However, notice that the mean size of $A_n^*[M]$ is

$$\mathbb{E}\left[\#A_n^*[M]\right] = (2M+1)n.$$

Similarly to the above, the sequence of random aggregates $(A_n^*[M])_{M\geq 0}$ is increasing and the infinite random aggregate $A_n^*[\infty]$ is defined as:

$$A_n^*[\infty] = \bigcup_{M\geq 0} A_n^*[M].$$

This new aggregate will be used to derive some properties of a third infinite random aggregate that we introduce below.

2.1.3 Construction of $A_n^\dagger[\infty]$

As in the previous constructions, we first define an increasing family of finite random aggregates. The number of particles from each site is still Poisson but this time the order for which they are sent is modified: the particles are not sent w.r.t. the usual order but w.r.t. a family of random clocks. To do it, let $(\mathcal{N}_i)_{i\in\mathbb{Z}}$ be a family of independent and identically distributed Poisson point processes (PPP's) in $\mathbb{R}^+$, with intensity 1. Each PPP $\mathcal{N}_i$ provides an increasing sequence $\left(\tau_{i,j}\right)_{j\geq 1}$ of random clocks. Then, we attach to the collection $\left\{\tau_{i,j} : i\in\mathbb{Z}, j\geq 1\right\}$ a family of independent and identically distributed symmetric random walks $\left\{S_{i,j} : i\in\mathbb{Z}, j\geq 1\right\}$ which are also independent of the PPP's. In other words, at time $\tau_{i,j}$, the j-th particle from level i starts and its trajectory, associated with $S_{i,j}$, is instantaneously realized and adds a new site to the current aggregate.

Now, let $M\geq 0$. We denote by $A_n^\dagger[M]$ the IDLA aggregate obtained, with the same protocol as above, by sending particles from levels $|i|\leq M$ according to the PPP's $(\mathcal{N}_i)_{i\in\mathbb{Z}}$ until time n. As in Section 2.1.2, the random number $N_i = \#\mathcal{N}_i([0,n])$ of particles starting from level i in $A_n^\dagger[M]$ is Poisson with parameter n. Notice that, conditional on the r.v.'s $N_i = \#\mathcal{N}_i([0,n])$, with $|i|\leq M$, only the order of the particles is changed between the aggregates $A_n^\dagger[M]$ and $A_n^*[M]$. A remarkable property of (finite) IDLA aggregates is the so-called *Abelian property*. This property claims that the distribution of the aggregate does not depend on the order in which particles are sent (see [14], p. 3). In particular, for each $M\geq 0$,

$$A_n^\dagger[M] \overset{\text{law}}{=} A_n^*[M]. \tag{1}$$

The above equality will be used to derive some properties of $A_n^\dagger[M]$ from those which deal with $A_n^*[M]$. However, the sequences $(A_n^\dagger[M])_{M\geq 0}$ and $(A_n^*[M])_{M\geq 0}$ are not equally distributed.

One of our motivations to introduce $A_n^*[M]$, and then $A_n^*[\infty]$, is to define a random forest which is stationary w.r.t. to vertical translations (see Section 7). To get it, it is crucial to base our model on a family of PPP's. Indeed, a random forest which would be based only on a family of Poisson random variables and on the same protocol as $A_n^*[M]$ (*i.e.* by sending first particles from level 0, then from level 1, then from level $-1,\dots$) is not stationary.

The aggregates $A_n^\dagger[M]$, with $n,M\geq 0$, are based on the same family of PPP's and on the same random walks. The following lemma shows that the sequences $(A_n^\dagger[M])_{n\geq 0}$ (with M fixed) and $(A_n^\dagger[M])_{M\geq 0}$ (with n fixed) are increasing.

Lemma 2.1 *For any integers $n,M\geq 0$, the following inclusions hold a.s.*

(i) $A_n^\dagger[M]\subset A_n^\dagger[M+1]$;

(ii) $A_n^\dagger[M]\subset A_{n+1}^\dagger[M]$.

Proof of Lemma 2.1. First, we prove (i). Let

$$\kappa = \sum_{|i|\leq M+1} \#\mathcal{N}_i([0,n])$$

be the number of particles which are sent from levels $|i|\leq M+1$ until time n. We index them by $j=1,\dots,\kappa$ according to their starting times $0<\tau_1<\ldots<\tau_\kappa<n$ (notice that they are a.s. all different). Recall that these particles are based on the same random walks for $A_n^\dagger[M]$ and for $A_n^\dagger[M+1]$. Some of these particles come from levels $\pm(M+1)$ and only concern the aggregate $A_n^\dagger[M+1]$. For $j=1,\dots,\kappa$, we denote by $A[M,j]$ (resp. $A[M+1,j]$) the aggregate obtained until (or at) time τ_j with particles from levels $|i|\leq M$ (resp. from levels $|i|\leq M+1$). We set $A[M,0]=A[M+1,0]=\emptyset$. Let $1\leq j\leq\kappa$ and assume that a.s. $A[M,j-1]\subset A[M+1,j-1]$. If particle j (which is sent at time τ_j) comes from level $\pm(M+1)$ then $A[M,j] = A[M,j-1]\subset A[M+1,j-1]\subset A[M+1,j]$. Otherwise, let y be the site added to $A[M,j-1]$ by particle j. If $y\notin A[M+1,j-1]$ then

$$A[M+1,j] = A[M+1,j-1]\cup\{y\}\supset A[M,j-1]\cup\{y\} = A[M,j]\ .$$

Otherwise, the random walk associated with particle j continues its trajectory till exiting $A[M+1, j-1]$ on a site y'. In this case,

$$A[M+1, j] = A[M+1, j-1] \cup \{y'\} \supset A[M, j-1] \cup \{y\} = A[M, j] \ .$$

By induction over $j = 0, \ldots, \kappa$, we a.s. get $A_n^\dagger[M+1] = A[M+1, \kappa] \supset A[M, \kappa] = A_n^\dagger[M]$.

Assertion (ii) is easy to check by letting increase $A_n^\dagger[M]$ with particles from levels $|i| \le M$ on the time interval $(n, n+1]$. □

Similarly to the previous subsections, we let

$$A_n^\dagger[\infty] = \bigcup_{M \ge 0} A_n^\dagger[M].$$

Because the sequences $(A_n^\dagger[M])_{M\ge 0}$ and $(A_n^*[M])_{M\ge 0}$ are increasing, it follows from (1) that

$$A_n^\dagger[\infty] \overset{\text{law}}{=} A_n^*[\infty].$$

2.2 First properties of the aggregates

2.2.1 Invariance w.r.t. symmetries and translations

Let $k \in \mathbb{Z}$. In what follows, we denote by τ_k (resp. $S_{k/2}$) the translation operator w.r.t. vector $(0, k)$ (resp. w.r.t. the horizontal axis $\mathbb{R} \times \{k/2\}$). The following proposition claims that the three infinite random aggregates are invariant w.r.t. translations and symmetries.

Proposition 2.2 *Let $n \ge 0$. The following properties hold:*

(i) the distributions of $A_n[\infty]$, $A_n^*[\infty]$ and $A_n^\dagger[\infty]$ are invariant w.r.t. τ_k, $k \in \mathbb{Z}$;

(ii) the distributions of $A_n[\infty]$, $A_n^*[\infty]$ and $A_n^\dagger[\infty]$ are invariant w.r.t. $S_{k/2}$, $k \in \mathbb{Z}$.

The above result is intuitively clear for $A_n^\dagger[\infty]$ since the Poisson clocks are independent. However, this is not so intuitive for $A_n[\infty]$ and $A_n^*[\infty]$ because, in their constructions, the particles are sent in the specific "usual" order.

Notice that $\tau_k A_n[\infty]$ (resp. $S_{k/2} A_n[\infty]$) is the increasing union of the aggregates $\tau_k A_n[M]$ (resp. $S_{k/2} A_n[M]$), $M \ge 0$. In distribution, they are obtained by sending n particles per level in the order $k, k+1, k-1, k+2, \ldots$ (resp. in the order $k, k-1, k+1, k-2, \ldots$). The construction is similar for $\tau_k A_n^*[\infty]$ and $S_{k/2} A_n^*[\infty]$.

Proof of Proposition 2.2. Since $\tau_k = S_{k/2} \circ S_0$, we only need to show (ii). We only give the proof for $A_n[\infty]$. The proof is similar for $A_n^*[\infty]$ and implies the result for $A_n^\dagger[\infty]$.

To do so, it suffices to check that the random sets $A_n[\infty]$ and $S_{k/2} A_n[\infty]$ have the same probability to intersect any given compact set (see [32, Theorem 2.1.3]). Let $C \subset \mathbb{R}^2$ be a compact and $\varepsilon > 0$. Let M_0 be such that for all $M \ge M_0$:

$$|\mathbb{P}(A_n[\infty] \cap C \neq \emptyset) - \mathbb{P}(A_n[M] \cap C \neq \emptyset)| \le \varepsilon$$

and

$$|\mathbb{P}(S_{k/2} A_n[\infty] \cap C \neq \emptyset) - \mathbb{P}(S_{k/2} A_n[M] \cap C \neq \emptyset)| \le \varepsilon.$$

Now, let us grow the aggregate $A_n[M]$ by sending n particles per level from $M+1$ to $M+k$. We denote by A_1 the resulting aggregate. Similarly, let us grow the aggregate $S_{k/2} A_n[M]$ by sending n particles per level from $k-M-1$ to $-M$. We denote by A_2 the resulting aggregate. In both A_1 and A_2, n particles are sent per level from $-M$ to $M+k$, but not in the same order. Nevertheless, by the *Abelian property for finite aggregates*, they are equally distributed. So,

$$\mathbb{P}(A_1 \cap C \neq \emptyset) = \mathbb{P}(A_2 \cap C \neq \emptyset).$$

The aggregates can be coupled in such a way that

$$A_n[M] \subset A_1 \subset A_n[M+k]$$

and

$$S_{k/2}A_n[M] \subset A_2 \subset S_{k/2}A_n[M+k],$$

with probability 1, respectively. This implies that

$$|\mathbb{P}(A_n[\infty] \cap C \neq \emptyset) - \mathbb{P}(A_1 \cap C \neq \emptyset)| \leq \varepsilon \tag{2}$$

and

$$|\mathbb{P}(S_{k/2}A_n[\infty] \cap C \neq \emptyset) - \mathbb{P}(A_2 \cap C \neq \emptyset)| \leq \varepsilon. \tag{3}$$

Indeed, for (2), we have

$$\begin{aligned}\mathbb{P}(A_n[\infty] \cap C \neq \emptyset) - \varepsilon &\leq \mathbb{P}(A_n[M] \cap C \neq \emptyset) \leq \mathbb{P}(A_1 \cap C \neq \emptyset)\\ &\leq \mathbb{P}(A_n[M+k] \cap C \neq \emptyset) \leq \mathbb{P}(A_n[\infty] \cap C \neq \emptyset) + \varepsilon.\end{aligned}$$

The bound (3) is obtained by the same way.

Finally, we get by collecting bounds that:

$$\begin{aligned}&|\mathbb{P}(A_n[\infty] \cap C \neq \emptyset) - \mathbb{P}((S_{k/2}A_n[\infty]) \cap C \neq \emptyset)|\\ &\leq |\mathbb{P}(A_n[\infty] \cap C \neq \emptyset) - \mathbb{P}(A_1 \cap C \neq \emptyset)|\\ &\quad + |\mathbb{P}(A_1 \cap C \neq \emptyset) - \mathbb{P}(A_2 \cap C \neq \emptyset)|\\ &\quad + |\mathbb{P}(A_2 \cap C \neq \emptyset) - \mathbb{P}((S_{k/2}A_n[\infty]) \cap C \neq \emptyset)|\\ &\leq 2\varepsilon.\end{aligned}$$

□

2.2.2 Mean size of the aggregates per level

As a consequence of Proposition 2.2, the following result shows that the expected width of $A_n[\infty]$ (resp. $A_n^*[\infty]$ and $A_n^\dagger[\infty]$) equals n when it is restricted to a horizontal line.

Proposition 2.3 *For any $i \in \mathbb{Z}$, we have:*

(i) $\mathbb{E}[\#A_n[\infty] \cap (\mathbb{Z} \times \{i\})] = n$;

(ii) $\mathbb{E}\left[\#A_n^*[\infty] \cap (\mathbb{Z} \times \{i\})\right] = n$;

(iii) $\mathbb{E}\left[\#A_n^\dagger[\infty] \cap (\mathbb{Z} \times \{i\})\right] = n$.

Roughly, the above result means that it is as if all the sites of $A_n[\infty] \cap (\mathbb{Z} \times \{i\})$ are created by the particles from level i.

Proof of Proposition 2.3. We begin with (i). To do it, for any $i \in \mathbb{Z}$, we denote by $Q(i)$ the mean number of sites in $A_n[\infty]$ with ordinate i, *i.e.*

$$Q(i) = \mathbb{E}[\#A_n[\infty] \cap (\mathbb{Z} \times \{i\})].$$

For $j \in \mathbb{Z}$, we also denote by $Q(i,j)$ the mean number of sites in $A_n[\infty] \cap (\mathbb{Z} \times \{i\})$ which are created by particles from level j. Thus $Q(i) = \sum_{j \in \mathbb{Z}} Q(i,j)$ a.s.. According to Proposition 2.2, the random variables $Q(i,j)$ and $Q(j,i)$ have the same distribution. Since $\sum_{i \in \mathbb{Z}} Q(i,j) = n$, we obtain

$$\mathbb{E}[Q(i)] = \sum_{j \in \mathbb{Z}} \mathbb{E}\left[Q(i,j)\right] = \sum_{j \in \mathbb{Z}} \mathbb{E}\left[Q(j,i)\right] = n\,.$$

In a similar way, we obtain (ii), (iii) by noticing that $\mathbb{E}[N_i] = \mathbb{E}[\#\mathcal{N}_i] = n$. □

To derive Proposition 2.3, we used the fact that $\mathbb{E}\left[Q(i,j)\right] = \mathbb{E}\left[Q(j,i)\right]$. Such an equality can be understood as a mass transport principle (see *e.g.* [8]).

2.2.3 Stabilization

The following proposition claims that, given a strip $\mathbb{Z}_M$, all the aggregates $A_n[M']$ (resp. $A_n^*[M']$) coincide with $A_n[\infty]$ (resp. $A_n^*[\infty]$) on $\mathbb{Z}_M$ when M' is large enough.

Theorem 2.4 *Let $n \geq 1$.*

(i) A.s., for all integer M, there exists M_0 such that for all $M' \geq M_0$:

$$A_n[M'] \cap \mathbb{Z}_M = A_n[\infty] \cap \mathbb{Z}_M \,.$$

(ii) A.s., for all integer M, there exists M_0 such that for all $M' \geq M_0$,

$$A_n^*[M'] \cap \mathbb{Z}_M = A_n^*[\infty] \cap \mathbb{Z}_M \,.$$

Proof of Theorem 2.4. First, notice that the random variable $X_{n,M} = \#A_n[\infty] \cap \mathbb{Z}_M$ is a.s. finite since, according to Proposition 2.3, we have $\mathbb{E}\left[X_{n,M}\right] = (2M+1)n$. Now, let $z_1, \ldots, z_{X_{n,M}}$ be an enumeration of the sites in $A_n[\infty] \cap \mathbb{Z}_M$. Denote by t_i the level from which the particle which creates z_i is emitted. The conclusion readily follows by setting $M_0 = \max\{|t_i| : i = 1, \ldots, X_{n,M}\} + 1$. Note that M_0 is a.s. finite since $X_{n,M}$ is a.s. finite. This proves (i). The same arguments can be used to get (ii). □

In the proof of Theorem 2.4, we strongly used the fact that $A_n[\infty]$ and $A_n^*[\infty]$ are constructed w.r.t. the usual order. Indeed, all the sites of $A_n[\infty] \cap \mathbb{Z}_M$ (resp. $A_n^*[\infty] \cap \mathbb{Z}_M$)) are produced by particles which are necessarily sent until a *finite* level $\pm M_0$ since all of the particles which are beyond this level are sent *after* those with level $|i| \leq M_0$. This argument does not hold for $A_n^\dagger[\infty]$ because $A_n^\dagger[\infty]$ is based on a family of independent PPP's. In particular, for any level i, an *infinite* number of particles, beyond i, are sent *before* those with level i. However, as we will see in Section 7, Theorem 2.4 remains true for $A_n^\dagger[\infty]$. The proof will require a specific treatment by introducing a notion of chain of changes.

Note also that the conclusions of Theorem 2.4 hold if we apply one of the transformations τ_k or $S_{k/2}$, $k \in \mathbb{Z}$, to the aggregates and strips.

3 Far particles do not touch central strips

The main result of this section (Theorem 3.1) claims that upon adding a new site to the aggregate, each particle from levels $(0, \pm M')$, with M' large enough, does not visit the strip $\mathbb{Z}_M$. In particular, it provides a (more sophisticated) alternative proof of Theorem 2.4 since it ensures that, a.s. for M large enough, all the aggregates $A_n[M']$ with $M' > M^\alpha$ coincide on $\mathbb{Z}_M$.

Not only Theorem 3.1 is more precise than Theorem 2.4, it also has its own interest and appears as a fruitful tool throughout this paper. First, it is one of the main ingredients to derive Propositions 3.6 and 5.1, which themselves are fundamental because they imply that the aggregate $A_n^\dagger[\infty]$ only consists of finite connected components (see Corollary 5.2). This fact allows us to define the infinite IDLA forest without controlling tricky chains of changes (see Section 7). Such a control would be a challenging question and contains technical difficulties. Another result which is based on Theorem 3.1 is a shape theorem (Theorem 2.4). Proving this result requires fine estimates which cannot be deduced directly from Theorem 2.4. The arguments in the proof of Theorem 3.1 are also extensively re-used in Section 4.

Theorem 3.1 *Let $n \geq 1$.*

(i) *Let* $\alpha > 1$. *The following property holds with probability* 1. *There exists a random integer* $M_0 = M_0(n) \geq 1$ *such that, for any* $M \geq M_0$, *the trajectory of any particle contributing to* $A_n[\infty]$ *and starting from* $(0,i)$, *with* $|i| > M^\alpha$, *does not visit the horizontal strip* $\mathbb{Z}_M$.

(ii) *The same holds for* $A_n^*[\infty]$.

3.1 Proof of Theorem 3.1, (i)

Let $\alpha > 1$. For any integers $M \geq 1$, $k \geq 0$, let

$$J_{M,k} = \{0\} \times [\![\lfloor M^\alpha \rfloor + kM + 1, \lfloor M^\alpha \rfloor + (k+1)M]\!] . \tag{4}$$

Given a realization of $A_n[M^\alpha + kM]$, we send nM particles from the interval $J_{M,k}$ first from level $\lfloor M^\alpha \rfloor + nM + 1$, then from level $\lfloor M^\alpha \rfloor + kM + 2$, and so on. In what follows, the *current aggregate* associated with a particle P denotes the aggregate which is produced just before sending P. Now, let $E_{M,k}$ be the following event:

$$E_{M,k} = \left\{ \begin{array}{c} \text{At least one of the } nM \text{ particles starting from } J_{M,k} \\ \text{visits the strip } \mathbb{Z}_M \text{ before exiting the current aggregate.} \end{array} \right\} . \tag{5}$$

The event $E_{M,k}$ describes the unpleasant situation where particles started far away from the origin, more precisely from $J_{M,k}$, may modify the aggregate close to the origin. The following lemma shows that $E_{M,k}$ occurs with small probability.

Lemma 3.2 *There exist constants* $0 < \eta < 1$, $c_1, c_2 > 0$ *such that, for any* $M, n \geq 1$ *and for any* $k \geq 0$

$$\mathbb{P}\left(E_{M,k}\right) \leq nM(1-\eta)^{\frac{M^\alpha + (k-2)M - c_1 n}{c_2 n^2 M}} .$$

Since $\alpha > 1$, it follows from Lemma 3.2 that

$$\sum_{M \geq 1} \mathbb{P}\left(\bigcup_{k \geq 0} E_{M,k} \right) \leq \sum_{M \geq 1} \sum_{k \geq 0} \mathbb{P}\left(E_{M,k}\right) < \infty.$$

This together with the Borel-Cantelli lemma implies Theorem 3.1 (i).

For $x \in \mathbb{R}^2$ and $r > 0$, we denote by $B(x,r)$ the intersection of $\mathbb{Z}^2$ and of the Euclidean ball, centered at x with radius r. We set $B(r) = B(0,r)$.

Proof of Lemma 3.2 . First, we introduce some notation. For any $\ell \geq 0$, let $B_\ell = J_{M,k} \oplus B(n + \ell r)$ and $S_{\ell+1} = B_{\ell+1} \setminus B_\ell$, where $r > 0$ will be specified later. Notice that if a particle starting from $J_{M,k}$ meets the strip $\mathbb{Z}_M$ then necessarily it crosses the annulus S_ℓ, for any $\ell \leq \ell_{\max}$, with

$$\ell_{\max} = \left\lfloor r^{-1} \left(\lfloor M^\alpha \rfloor + (k-1)M + 1 - n \right) \right\rfloor . \tag{6}$$

One of the key ingredients to derive Lemma 3.2 is Lemma 3.4. To apply it, the width r is chosen in such a way that $nr \leq \eta_0 r^2$, where η_0 is as in Lemma 3.4. More precisely, we let

$$r = \frac{4n}{\eta_0}.$$

Now, to be in the framework of Lemma 3.2, we introduce the notion of good (resp. bad) annuli as follows. Given a realization of $A_n[M^\alpha + kM]$, we say that S_ℓ is *good* if

$$\#\left(A_n[M^\alpha + kM] \cap S_\ell\right) \leq \eta_0 r^2.$$

Otherwise, we say that the annulus is *bad*. Notice that the number N^{bad} of bad annuli is deterministically bounded since a.s.

$$N^{\text{bad}} \times \eta_0 r^2 \leq \# A_n[M^\alpha + kM] \leq n\left(2(M^\alpha + kM) + 1\right).$$

Denoting by $N^{\text{good}} = \ell_{\max} - N^{\text{bad}}$ the number of good annuli, it follows from the above inequality and from the definitions of r, $\ell_{\max}$ that

$$N^{\text{good}} \geq r^{-1}\left(\frac{1}{2}M^{\alpha} + \frac{1}{2}kM - M\right) - \frac{\eta_0}{4} - \frac{\eta_0}{16n} - 1.$$

Thus

$$N^{\text{good}} \geq \frac{M^{\alpha} + (k-2)M}{2r} - c_0, \tag{7}$$

where $c_0 = 1 + \frac{5\eta_0}{16}$.

Now, for $i = 1, \dots, nM$, let us call particle i the i-th particle which is sent from the interval $J_{M,k}$. The main idea to prove Lemma 3.2 is to use the fact that, if particle i hits the strip $\mathbb{Z}_M$ then, according to Equation (7), it necessarily has to cross a large number of good annuli. But, as we will see in Lemma 3.4, for each new good annulus that particle i meets, it has probability at least η to be stuck inside. Roughly, this will imply that the event $E_{M,k}$ cannot occur with high probability.

More precisely, given a realization of $A_n[M^{\alpha} + kM]$, let X_1 be the number of good annuli which are crossed by particle 1. Given $X_1, \dots, X_i$, with $i \geq 1$, let X_{i+1} be the number of good annuli which are crossed by particle $i+1$, *but not* already crossed by particles $1, \dots, i$. Notice that after the launches of the first i particles, i new sites are added to the aggregate. With nonnull probability, some of these particles will settle on the first good annuli. Hence, some of these good annuli will contain more than $\eta_0 r^2$ sites of the current aggregate. To deal with this problem, we choose to only focus on *new good annuli* for the random variable X_{i+1}. Notice that the random variables X_i, $i \geq 1$, are not identically distributed and that the sequence $(\mathbb{P}(X_i = 0))_{i\geq 1}$ is non-decreasing. The following lemma shows that the random variable X_i is stochastically dominated by a geometric distribution with parameter independent of i.

Lemma 3.3 *There exists $\eta \in (0,1)$ such that, for any $i \geq 1$ and for any $s, t \geq 0$, the following properties hold:*

(i) $\mathbb{P}\big(X_1 > t \,|\, A_n[M^{\alpha} + kM]\big) \leq (1-\eta)^t$;

(ii) $\mathbb{P}\big(X_{i+1} > t \,|\, A_n[M^{\alpha} + kM], \sum_{j\leq i} X_j \leq s\big) \leq (1-\eta)^t$.

If one of the nM particles starting from $J_{M,k}$ hits the strip $\mathbb{Z}_M$ then the sum of X_i's is necessarily larger than N^{good}. Thus, according to (7), we have

$$E_{M,k} \subset \left\{ \sum_{i=1}^{nM} X_i \geq \frac{M^{\alpha} + (k-2)M}{2r} - c_0 \right\}.$$

This implies that

$$\begin{aligned}\mathbb{P}\left(E_{M,k}\right) \leq \mathbb{P}&\left(\sum_{i=1}^{nM} X_i \geq \frac{M^{\alpha} + (k-2)M}{2r} - c_0 \;\middle|\; \sum_{i=1}^{nM-1} X_i \leq \frac{nM-1}{nM}\left(\frac{M^{\alpha} + (k-2)M}{2r} - c_0\right)\right) \\ &+\mathbb{P}\left(\sum_{i=1}^{nM-1} X_i \geq \frac{nM-1}{nM}\left(\frac{M^{\alpha} + (k-2)M}{2r} - c_0\right)\right).\end{aligned} \tag{8}$$

Thanks to Assertion (ii) in Lemma 3.3, we can bound the conditional probability in (8) by $(1-\eta)^{\frac{M^{\alpha}+(k-2)M-2c_0 r}{2rnM}}$. By induction and Lemma 3.3, we get

$$\mathbb{P}(E_{M,k}) \leq nM(1-\eta)^{\frac{M^{\alpha}+(k-2)M-2c_0 r}{2rnM}}.$$

This concludes the proof of Lemma 3.2 by taking $c_1 = \frac{8c_0}{\eta_0}$ and $c_2 = \frac{8}{\eta_0}$. □

Proof of Lemma 3.3. One of the key ingredients to derive Lemma 3.3 is the following result, referred to as the *crossing lemma*.

Lemma 3.4 (Crossing Lemma) *There exist $\eta_0, \eta > 0$ such that for any $S, V \subset \mathbb{Z}^2$ and $r > 0$ satisfying $S \subset V \oplus B(r)$ and $\#S \setminus V \leq \eta_0 r^2$, and for any particle ξ^x starting from $x \in V$ and stopped upon exiting $V \oplus B(r)$, the following inequality holds:*

$$\mathbb{P}\big(\xi^x \cap \big((V \oplus B(r)) \setminus (S \cup V)\big) \neq \emptyset\big) \geq \eta \,. \tag{9}$$

The above result is an adaptation of [15, Lemma 3.2] written in the context of $\mathbb{Z}^2$. It expresses the difficulty for a particle to cross an annulus when the aggregate occupies only a small portion of it.

First, we prove (i). To do it, we first show that, for any $t \geq 0$,

$$\mathbb{P}\big(X_1 > t \,|\, A_n[M^\alpha + kM],\, X_1 > t-1\big) \leq 1 - \eta \,. \tag{10}$$

Given a realization of $A_n[M^\alpha + kM]$, we denote by T the index of the $\lfloor t \rfloor + 1$-th good annulus which is reached by particle 1. Now, let

$$S = A_n[M^\alpha + kM] \cap B_T \quad \text{and} \quad V \equiv B_{T-1}.$$

Notice that $\#S \setminus V \equiv \#A_n[M^\alpha + kM] \cap S_T \leq \eta_0 r^2$ since S_T is a good annulus. Conditional on the event $\{X_1 > t-1\}$, if $X_1 > t$, then necessarily particle 1 crosses the annulus S_T. According to Lemma 3.4, the event $\{X_1 > t\}$ occurs with probability smaller than $1-\eta$, which proves (10). By induction and because

$$\begin{aligned}&\mathbb{P}\left(X_1 > t \,|\, A_n[M^\alpha + kM]\right) \\ &\qquad \leq \mathbb{P}\left(X_1 > t \,|\, A_n[M^\alpha + kM],\, X_1 > t-1\right) \mathbb{P}\left(X_1 > t-1 \,|\, A_n[M^\alpha + kM]\right),\end{aligned}$$

we get Assertion (i).

Now, we prove (ii). We proceed in the same spirit as above. To do it, it is sufficient to show that for any $t \geq 0$,

$$\mathbb{P}\left(X_{i+1} > t \,\middle|\, A_n[M^\alpha + kM],\, \sum_{j \leq i} X_j \leq s,\; X_{i+1} > t-1\right) \leq 1 - \eta.$$

Given $X_1, \ldots, X_i$ and $A_n[M^\alpha + kM]$, we denote by T the index of the $\lfloor t \rfloor + 1$-th good annulus which is counted from the $\lfloor s \rfloor + 1$-th good one. Conditional on the event $\{X_{i+1} > t-1\}$, if $X_{i+1} > t$, then particle $i+1$ has to cross the good annulus S_T through the current aggregate $\tilde{A}$, which has been augmented from $A_n[M^\alpha + kM]$ by exactly i sites corresponding to the first i particles sent from $J_{M,k}$. But, by definition of X_{i+1}, the annulus S_T has not been visited by the first i particles. In particular, $\tilde{A} \cap S_T$ equals $A_n[M^\alpha + kM] \cap S_T$. Since $\#A_n[M^\alpha + kM] \cap S_T$ is smaller than $\eta_0 r^2$, we can apply Lemma 3.4. The end of the proof follows from the same lines as in (i). □

Proof of Lemma 3.4. It relies on an adaptation of [15, Lemma 3.2]. For sake of completeness, we recall the main arguments of [15]. Let Y be the outer boundary of $V \oplus B(r/2)$ in $\mathbb{Z}^2$, *i.e.*

$$Y = \left\{y \in \mathbb{Z}^2 :\; y \notin V \oplus B(r/2) \text{ and } \exists y' \in V \oplus B(r/2) \text{ s.t. } |y - y'|_1 = 1\right\},$$

where $|\cdot|_1$ denotes the 1-norm on $\mathbb{Z}^2$, *i.e.* $|z| = |z(1)| + |z(2)|$ for any $z = (z(1), z(2)) \in \mathbb{Z}^2$. First, notice that every path from $x \in V$ to the complement of $V \oplus B(r)$ must hit Y. Thus, by Markov's property, it suffices to prove (9) for random walks starting from Y.

Let $y \in Y$, $B = B(y, r/3)$ and $Q = B \setminus S$. According to [15, Lemma 3.1], for any $t > 0$, we have

$$\begin{aligned}\mathbb{P}\left(\xi^y \cap \big((V \oplus B(r)) \setminus (S \cup V)\big) \neq \emptyset\right) &\geq \mathbb{P}\left(\xi^y \cap Q \neq \emptyset\right) \\ &\geq \mathbb{P}\left(B \subset A_t(y \mapsto r/3)\right)\right) \times \frac{\#Q}{t},\end{aligned} \tag{11}$$

where $A_t(y \mapsto r/3)$ denotes the aggregate obtained by letting t particles starting from y and stopped upon exiting $B(y, r/3)$.

To deal with (11), recall that from [15, Section 2], there exists $\alpha > 0$ such that

$$\mathbb{P}\big(B \subset A_t(y \mapsto r/3)\big) \geq \alpha$$

for $t = \#B(y, r/(3\alpha))$. The previous inequality is referred to as the *weaker lower bound* in [15]. Now, let $c > 0$ be such that $\#B \geq c(r/3)^2$. Since $B \subset V^c$ and $\#S \setminus V \leq \eta_0 r^2$, we have

$$\#Q = \#B - \#B \cap S \geq c(r/3)^2 - \eta_0 r^2 = \eta_0 r^2,$$

with $\eta_0 = c/18$. Taking $C > 0$ in such a way that $t \leq C(r/(3\alpha))^2$, we have

$$\mathbb{P}\left(B \subset A_t(y \mapsto r/3)\right)\right) \times \frac{\#Q}{t} \geq \alpha \times \frac{\eta_0 r^2}{C(r/(3\alpha))^2} =: \eta.$$

□

3.2 Proof of Theorem 3.1, (ii)

This proof will be sketched because it relies on a simple adaptation of the proof of Theorem 3.1, (i). The main difference is that we have to provide estimates for the number of particles which are sent per site.

Let $\alpha > 1$, $M, n \geq 1$ and $k \geq 0$. We have to show that the series $\sum_{M\geq 1} \mathbb{P}\left(\bigcup_{k\geq 0} E^*_{M,k}\right)$ is convergent, where the event $E^*_{M,k}$ is defined in the same spirit as (5) by replacing the aggregate $A^*_n[M^\alpha + kM]$ by $A_n[M^\alpha + kM]$. To control the size of the aggregate $A^*_n[M^\alpha + kM]$, let

$$F_{M,k} = \left\{ \sum_{|i|\leq M^\alpha + kM} N_i \leq 2n\big(2(M^\alpha + kM) + 1\big) \right\}.$$

On the event $F_{M,k}$, we can adapt the main arguments of Section 3.1 by considering good and bad annuli S_ℓ, $\ell \geq 1$, with width r as follows. First, recall that each particle starting from $J_{M,k}$ has to cross $\ell_{\max}$ annuli to hit the strip $\mathbb{Z}_M$, where $J_{M,k}$ and $\ell_{\max}$ are defined in (4) and (6), respectively. Then, conditional on $A^*_n[M^\alpha + kM]$, we say that the annulus S_ℓ is *good* if

$$\#\left(A^*_n[M^\alpha + kM] \cap S_\ell\right) \leq \eta_0 r^2$$

and we say that it is *bad* otherwise. Since the aggregate $A^*_n[M^\alpha + kM]$ potentially contains twice more particles than $A_n[M^\alpha + kM]$, the width r has to be chosen as twice the width appearing in the proof of (i). Thus we let $r = 8n/\eta_0$. On the event $F_{M,k}$, the number N^{bad} of bad annuli is still deterministically bounded since, a.s.,

$$N^{\text{bad}} \times \eta_0 r^2 \leq \#A^*_n[M^\alpha + kM] \leq 2n\big(2(M^\alpha + kM) + 1\big)$$

and thus N^{good} can be bounded in the same spirit as (7).

Now, to control the number of particles starting from $J_{M,k}$, let

$$G_{M,k} = \left\{ \sum_{(0,i)\in J_{M,k}} N_i \leq 2nM \right\}.$$

Proceeding exactly in the same spirit as in the proof of Lemma 3.2, we obtain that, for some constants $c'_1, c'_2 > 0$:

$$\mathbb{P}\left(E^*_{M,k} \,\middle|\, F_{M,k} \cap G_{M,k} \right) \leq 2nM(1-\eta)^{\frac{M^\alpha + (k-2)M - c'_1 n}{c'_2 n^2 M}}.$$

Moreover, we can easily prove that the series $\sum_{M\geq 1}\sum_{k\geq 0} \mathbb{P}\left(F^c_{M,k}\right)$ and $\sum_{M\geq 1}\sum_{k\geq 0} \mathbb{P}\left(G^c_{M,k}\right)$ are finite. Therefore $\sum_{M\geq 1} \mathbb{P}\left(\bigcup_{k\geq 0} E^*_{M,k}\right)$ is finite, which concludes the proof of Theorem 3.1, (ii).

Remark 3.5 *To prove Theorem 3.1, (ii), we introduced the event $F_{M,k}$, which roughly means that the aggregate $A^*_n[M^\alpha]$ is thin. In particular, we used the fact that the thinner the aggregate $A^*_n[M^\alpha]$ is (which appears when the sum of Poisson random variables N_i, $|i| \leq M^\alpha + kM$, is not big), the bigger is the probability that the particles, which start from levels $|i| > M^\alpha$, do not get the strip $\mathbb{Z}_M$. This remark will be used in the proof of Proposition 3.6.*

3.3 $A^*_n[\infty]$ avoids $\mathbb{Z}\times\{0\}$ with positive probability

The next result ensures that the infinite aggregates $A_n[\infty]$ and $A^*_n[\infty]$ are not identically distributed. The first one contains, by construction, the vertical axis $\{0\}\times\mathbb{Z}$ with probability 1, whereas the second one does not intersect the axis $\mathbb{Z}\times\{0\}$ with positive probability.

Proposition 3.6 *Let $n \geq 1$. With positive probability, the aggregate $A^*_n[\infty]$ does not intersect the axis $\mathbb{Z}\times\{0\}$.*

Proof of Proposition 3.6. To prove that $\mathbb{P}\left(A^*_n[\infty]\cap(\mathbb{Z}\times\{0\})=\emptyset\right)$ is positive, it is sufficient to show that, for M large enough,

$$\mathbb{P}\left(\bigcap_{k\geq 0}\left(E^*_{M,k}\right)^c \cap \left\{\sum_{|i|\leq M^\alpha} N_i = 0\right\}\right) > 0, \tag{12}$$

where $E^*_{M,k}$ is as in the proof of Theorem 3.1, (ii). To do it, we recall that the sums $\sum_{M\geq 1}\sum_{k\geq 0}\mathbb{P}\left(E^*_{M,k}\right)$ and $\sum_{M\geq 1}\sum_{k\geq 0}\mathbb{P}\left(F^c_{M,k}\right)$ are finite (see the proof of Theorem 3.1, (ii)). Thus, for M large enough, we have

$$\mathbb{P}\left(\bigcap_{k\geq 0}\left(E^*_{M,k}\right)^c \cap \left\{\sum_{|i|\leq M^\alpha} N_i \leq 2n(2M^\alpha+1)\right\}\right) > 0. \tag{13}$$

Now, recall that the aggregate $A^*_n[\infty]$ is constructed w.r.t. two independent random variables: the first one, say ω_1 (resp. the second one denoted by ω_2) concerns random walks starting from $I(M^\alpha)=\{0\}\times[\![-M^\alpha,M^\alpha]\!]$ (resp. starting from $(\{0\}\times\mathbb{Z})\setminus I(M^\alpha)$) and Poisson random variables N_i indexed by the set of $|i|\leq M^\alpha$ (resp. by the set of $|i|>M^\alpha$). In what follows, for any set $A\subset\mathbb{Z}^2$, we write

$$\mathscr{E}(A) = \left\{\begin{array}{c}\omega_2\text{: particles associated with } \omega_2 \text{ (starting from } |i|>M^\alpha\text{) do not visit}\\ \text{the strip } \mathbb{Z}_M \text{ when they are used to grow the initial aggregate } A\end{array}\right\}.$$

Notice that

$$\bigcap_{k\geq 0}\left(E^*_{M,k}\right)^c = \left\{(\omega_1,\omega_2):\omega_2\in\mathscr{E}\left(A^*_n[M^\alpha](\omega_1)\right)\right\}.$$

Remark 3.5 expresses that $\mathbb{P}(\mathscr{E}(A))$ is decreasing w.r.t. A, *i.e.* for any $A, A'\subset\mathbb{Z}^2$,

$$A\subset A' \Longrightarrow \mathbb{P}(\mathscr{E}(A)) \geq \mathbb{P}\left(\mathscr{E}(A')\right). \tag{14}$$

We are now prepared to prove (12). Indeed, since $\sum_{|i|\leq M^\alpha} N_i = 0$ if and only if $A^*_n[M^\alpha]=\emptyset$, we have

$$\begin{aligned}\mathbb{P}\left(\bigcap_{k\geq 0}\left(E^*_{M,k}\right)^c \cap \left\{\sum_{|i|\leq M^\alpha} N_i = 0\right\}\right) &= \int \mathbf{1}_{\omega_2\in\mathscr{E}\left(A^*_n[M^\alpha](\omega_1)\right)}\mathbf{1}_{\sum_{|i|\leq M^\alpha} N_i(\omega_1)=0}\,\mathrm{d}\omega_1\mathrm{d}\omega_2\\ &= \int \mathbf{1}_{\omega_2\in\mathscr{E}(\emptyset)}\,\mathrm{d}\omega_2\times\mathbb{P}\left(\sum_{|i|\leq M^\alpha} N_i = 0\right).\end{aligned}$$

The second term of the last equality is positive. For the first one, according to (14), we know that

$$\begin{aligned}\int \mathbf{1}_{\omega_2\in\mathscr{E}(\emptyset)}\,\mathrm{d}\omega_2 &\geq \int \mathbf{1}_{\omega_2\in\mathscr{E}\left(A^*_n[M^\alpha](\omega_1)\right)}\mathbf{1}_{\sum_{|i|\leq M^\alpha} N_i(\omega_1)\leq 2n(2M^\alpha+1)}\,\mathrm{d}\omega_1\mathrm{d}\omega_2\\ &= \mathbb{P}\left(\bigcap_{k\geq 0}\left(E^*_{M,k}\right)^c \cap \left\{\sum_{|i|\leq M^\alpha} N_i \leq 2n(2M^\alpha+1)\right\}\right),\end{aligned}$$

which is positive according to (13). This concludes the proof of Proposition 3.6.

□

4 Central particles do not touch far levels

In this section, we show that the aggregates $A_n[\infty]$ and $A_n^*[\infty]$ above some random levels a.s. do not depend on particles which are sent arround the origin. Such a property is one of the key ingredients to prove that the aggregates $A_n[\infty]$, $A_n^*[\infty]$ and $A_n^\dagger[\infty]$ satisfy a mixing property (Proposition 5.1).

Let $n, M \geq 0$. We define two random aggregates which are coupled to $A_n[\infty]$ as follows. First, we write $A_{n,M}(M) = A_n[M]$ and $B_{n,M}(M) = \emptyset$. Then we launch particles from levels $M+1$, $M+2$ and so on by using the same random walks as $A_n[\infty]$. In particular, we couple the growth (only from the top) of the aggregates based on $A_n^*[M]$ and $\emptyset$. For any $t > M$, we denote by $A^*_{n,M}(t)$ (resp. $B^*_{n,M}(t)$) these two aggregates after sending particles from levels $M+1$, thus $M+2$ etc. till level t included. Using the same arguments than those used in Theorem 3.1, we prove that the increasing sequences $(A_{n,M}(t))_{t\geq M}$ and $(B_{n,M}(t))_{t\geq M}$ a.s. converge to infinite aggregates denoted respectively by $A_{n,M}(\infty)$ and $B_{n,M}(\infty)$. With these notations, we emphasize the dependence of those aggregates on parameters n, M and the level t is referred to as the time.

Now, let $(N_i)_{i\in\mathbb{Z}}$ be a family of independent Poisson random variables with parameter n. In a similar way, we define two aggregates based on $A^*_{n,M}(M) = A^*_n[M]$ and $B^*_{n,M}(M) = \emptyset$, by launching (the same) particles from levels $M+1$, $M+2$ and so on, with N_i particles per level i. We denote the aggregates at time $t > M$ (resp. the limits of the aggregates) by $A^*_{n,M}(t)$ and $B^*_{n,M}(t)$ (resp. $A^*_{n,M}(\infty)$ and $B^*_{n,M}(\infty)$).

For any $K \geq 0$, we let $\mathscr{H}_K = \{(x,y) \in \mathbb{Z}^2 : y \geq K\}$. We are now prepared to state the main result of this section.

Theorem 4.1 *Let $n \geq 1$.*

(i) A.s. there exists a (random) integer $M_0 = M_0(n) \geq 0$ such that, for any $M \geq M_0$, there exists $t_0 = t_0(n,M) \geq M$ such that, for any $t \geq t_0$,

$$A_n[\infty] \cap \mathscr{H}_t = B_{n,M}(\infty) \cap \mathscr{H}_t.$$

(ii) The same property holds if we replace $A_n[\infty]$ and $B_{n,M}(\infty)$ by $A^*_n[\infty]$ and $B^*_{n,M}(\infty)$ respectively.

Theorem 4.1 asserts that the infinite aggregates $A_n[\infty]$ and $A^*_n[\infty]$ restricted to the half-plane $\mathscr{H}_t$ do not depend on particles starting from levels $i < M$. We call this property *Stabilization w.r.t. the origin.* It is a consequence of the following proposition.

Proposition 4.2 *Let $n \geq 1$.*

(i) For any $M \geq 1$, a.s. there exists t_0 such that, for any $t \geq t_0$, we have

$$A_{n,M}(\infty) \cap \mathscr{H}_t = A_{n,M}(\infty) \cap \mathscr{H}_t.$$

(ii) The same property holds if we replace $A_{n,M}(\infty)$ and $A_{n,M}(\infty)$ by $A^*_{n,M}(\infty)$ and $A^*_{n,M}(\infty)$ respectively.

In what follows, we detail the proof of the above results for the Poisson case because the fact that $A^*_n[\infty]$ does not intersect horizontal lines with positive probability (Proposition 3.6) leads to a more natural strategy. However, we explain how we adapt the proof for the deterministic case when modifications are required.

Obviously, by symmetry, a similar result to Theorem 4.1 but for negative levels also holds, *i.e.* aggregates $A_n[\infty]$ and $A^*_n[\infty]$ restricted to the half-plane $\{(x,y) \in \mathbb{Z}^2 : y \leq -t\}$ do not depend on particles starting from levels $i > -M$, for M, t large enough.

Proof of Theorem 4.1 . Let $n \geq 1$. Recall that the aggregate $A^*_n[\infty]$ (resp. $A^*_{n,M}(\infty)$) is based on $A^*_n[M]$ and on particles which are sent from levels $|i| > M$ (resp. from levels $i > M$), *i.e.* from the top and from the bottom w.r.t. the usual order (resp. from the top). The aggregates are coupled in the sense that they are based on the same particles. Theorem 3.1 states that a.s. there exists M_0 such that for any $M \geq M_0$, particles from levels $i < -M^2$ (taking $\alpha = 2$ for instance) do not visit the horizontal strip $\mathbb{Z}_M$. This means that both aggregates $A_n^*[\infty]$ and $A_{n,M^2}^*(\infty)$ coincide on the hyperplane $\mathscr{H}_M$. This together with Proposition 4.2 implies Theorem 4.1. $\square$

The end of this section is devoted to the proof of Proposition 4.2. Thanks to our coupling, the inclusion $B_{n,M}^*(t) \subset A_{n,M}^*(t)$ holds a.s. for any $t \geq M$, and we denote by $\Delta(t)$ the random (symmetric) difference between these aggregates, *i.e.*

$$\Delta(t) = A_{n,M}^*(t) \setminus B_{n,M}^*(t) .$$

Let us explain how the difference set $\Delta(t)$ evolves over time. Let $x \in \Delta(t)$. Let $t' \geq t$ be the first level (if it exists) for which a particle, say P, starting from level t' reaches the site x and thus exits the current aggregate through some site y. Here, "current aggregate" denotes the aggregate which is produced just before sending P. Just after P is sent, the site x is added to $B_{n,M}^*(t')$ and is no longer a difference between $A_{n,M}^*(\cdot)$ and $B_{n,M}^*(\cdot)$, but y becomes a new difference between both.

If there is no particle starting from level $t' \geq t$ which visits x, then x is a difference forever, *i.e.* $x \in \bigcap_{t' \geq t} \Delta(t')$. Although the symmetric difference evolves, its cardinality remains constant over time and equals

$$\#\Delta(0) = \#\left(A_{n,M}^*(0) \setminus B_{n,M}^*(0)\right) = \#A_n^*[M].$$

The above expression is random and Poisson distributed with parameter $(2M+1)n$.

Notice that $\left(\cup_{t' \geq M} \Delta(t')\right) \cap \mathscr{H}_t = \emptyset$ if and only if the aggregates $A_{n,M}^*(t')$ and $B_{n,M}^*(t')$ coincide on the hyperplane $\mathscr{H}_t$ for any time t'. The same holds for their limits. Henceforth, to prove Proposition 4.2, (ii) it suffices to prove that for any $M \geq 1$,

$$\text{a.s.}\,,\text{ for } t \text{ large enough, } \left(\bigcup_{t' \geq M} \Delta(t')\right) \cap \mathscr{H}_t = \emptyset. \tag{15}$$

To do it, our strategy consists in bounding the growth of the aggregate $(A_{n,M}^*(t))_{t \geq M}$ from the top at some specific (and random) times t. In the same spirit as [28], we introduce the (excess) *height* of the aggregate $A_{n,M}^*(t)$ as

$$h\left(A_{n,M}^*(t)\right) = \max\left\{y \in \mathbb{Z} : (x,y) \in A_{n,M}^*(t)\right\} - t .$$

The above quantity is the difference, w.r.t. the y axis, between the highest vertex of $A_{n,M}^*(t)$ and the level t at which the last particles are sent. Roughly speaking, $h(A_{n,M}^*(t))$ expresses how much the aggregate $A_{n,M}^*(t)$ drools beyond level t. Given $\zeta > 0$, let us set $\tau_1(\zeta)$ as the first time $t > M$ at which the height of $A_{n,M}^*(t)$ becomes smaller than ζ:

$$\tau_1(\zeta) = \inf\{t > M : h\left(A_{n,M}^*(t)\right) \leq \zeta\} .$$

Thus, by induction, we define a sequence of random times $(\tau_m(\zeta))_{m \geq 1}$ as follows:

$$\tau_{m+1}(\zeta) = \inf\{t > \tau_m(\zeta) : h\left(A_{n,M}^*(t)\right) \leq \zeta\},$$

with the convention $\tau_m(\zeta) = \infty$ implies $\tau_{m+1}(\zeta) = \infty$. The next result claims that a.s. infinitely often, the height of the aggregate $A_{n,M}^*(\cdot)$ is smaller than ζ.

Proposition 4.3 *Let $n, M \geq 1$. Then there exists a positive real number $\zeta = \zeta(n)$ such that, a.s., the random times $(\tau_m(\zeta))_{m \geq 1}$ are all finite.*

As we will see in Section 4.1, the choice of ζ depends on n but not on M. Proposition 4.3 is one of the main ingredients to prove (15).

4.1 Proposition 4.3 implies (15)

For any $t \geq M$, we denote by $\mathscr{G}_t$ the σ-algebra generated by the Poisson r.v.'s N_i, $i \leq t$, and by the random walks starting from level $i \leq t$. In particular, $A^*_{n,M}(t)$ is $\mathscr{G}_t$-measurable.

According to Proposition 4.3, we know that the random time τ_m is finite for each $m \geq 1$. Let us now extract a subsequence $(\sigma_m)_{m\geq 1}$ from the sequence $(\tau_m)_{m\geq 1}$. First, we let $\sigma_1 = \tau_1$. As a by product of the proof of Theorem 3.1, we know that the aggregate which is built by using only particles from levels R, thus $R+1$ and so on, does not intersect the horizontal line $\mathbb{Z} \times \{0\}$, with probability $c(n,R) > 0$ for R large enough. This event only concerns randomness above level R. Now, given $R > 0$ so that $c = c(n,R) > 0$, let Gap_1 be the following event:

- $N_t = 0$ for any $\sigma_1 < t < \sigma_1 + \zeta + R$;
- the aggregate built using only particles from levels $\sigma_1 + \zeta + R$, thus $\sigma_1 + \zeta + R + 1$ and so on, does not intersect the horizontal line $\mathbb{Z} \times \{\sigma_1 + \zeta\}$.

Conditional on $\mathscr{G}_{\sigma_1}$, the probability of Gap_1 is larger than:

$$c' = c'(n,R,\zeta) = \mathbb{P}\left(N_t = 0 \text{ for any } \sigma_1 < t < \sigma_1 + \zeta + R\right) \times c > 0.$$

Although the event Gap_1 depends on σ_1, this is not the case for the lower bound c'.

On the event Gap_1, the aggregate $A^*_{n,M}(\sigma_1)$ remains below the line $\mathbb{Z} \times \{\sigma_1 + \zeta\}$ and then cannot help particles coming from levels $t \geq \sigma_1 + \zeta + R$ to merge both aggregates, namely $A^*_{n,M}(\sigma_1)$ and the one which is built with particles from levels $t \geq \sigma_1 + \zeta + R$. Hence, the aggregates $A^*_{n,M}(t)$, for any $t \geq \sigma_1$, does not intersect the horizontal line $\mathbb{Z} \times \{\sigma_1 + \zeta\}$. This implies that the set $\Delta(t)$ is stuck below $\mathbb{Z} \times \{\sigma_1 + \zeta\}$ for any $t \geq M$. To sum up, conditional on the event Gap_1, we have

$$\left(\bigcup_{t\geq M} \Delta(t)\right) \cap \mathscr{H}_{\sigma_1+\zeta+1} = \emptyset,$$

which implies (15) since σ_1 is a.s. finite.

In order to get back independence when Gap_1 does not occur, we have to proceed step by step. Let $t > \sigma_1$. Given a realization of $A^*_{n,M}(t-1)$, we say that the level t *fails* if one of the following properties holds:

- $N_t \neq 0$ provided that $t < \sigma_1 + \zeta + R$;
- a particle starting from level t touches $\mathbb{Z} \times \{\sigma_1 + \zeta\}$ before exiting the current aggregate provided that $t \geq \sigma_1 + \zeta + R$.

Thus $\mathrm{Gap}_1^c = \{\text{there exists } t > \sigma_1 \text{ such that level } t \text{ fails}\}$.

Now, we set

$$\bar{\sigma}_1 = \inf\{t > \sigma_1 : \text{level } t \text{ fails for } \mathrm{Gap}_1\} .$$

Notice that $\mathrm{Gap}_1^c = \{\bar{\sigma}_1 < \infty\}$ is $\mathscr{G}_{\bar{\sigma}_1}$ measurable. In particular, Gap_1^c is $\mathscr{G}_{\sigma_2}$ measurable, where σ_2 is defined as the first random time τ_m which is larger than $\bar{\sigma}_1$ (such a quantity exists according to Proposition 4.3, provided that $\bar{\sigma}_1$ is finite). Thus we define the event Gap_2 in the same spirit as we did for Gap_1 but this time by replacing σ_1 by σ_2. Conditional on $\mathscr{G}_{\sigma_2}$, the event Gap_2 occurs with probability larger than $c' > 0$. Therefore,

$$\begin{aligned}
\mathbb{P}\left(\mathrm{Gap}_2^c \cap \mathrm{Gap}_1^c \mid \mathscr{G}_{\sigma_1}\right) &= \mathbb{P}\left(\{\bar{\sigma}_2 < \infty\} \cap \{\bar{\sigma}_1 < \infty\} \mid \mathscr{G}_{\sigma_1}\right) \\
&= \mathbb{E}\left[\mathbb{P}\left(\bar{\sigma}_2 < \infty \mid \mathscr{G}_{\sigma_2}\right) \mathbf{1}_{\bar{\sigma}_1 < \infty} \mid \mathscr{G}_{\sigma_1}\right] \\
&\leq (1-c')\mathbb{P}\left(\bar{\sigma}_1 < \infty \mid \mathscr{G}_{\sigma_1}\right) \\
&\leq (1-c')^2 .
\end{aligned}$$

Thus we proceed as previously by introducing

$$\bar{\sigma}_2 = \inf\{t > \sigma_2 : \text{level } t \text{ fails for Gap}_2\} .$$

If $\bar{\sigma}_2 = \infty$ then Gap_2 occurs and $\bigcup_{t\geq M}\Delta(t)$ does not overlap $\mathscr{H}_{\sigma_2+\zeta}$, which implies (15). If $\bar{\sigma}_2 < \infty$ then we restart the procedure.

By induction, we deduce that $\bigcap_{m\geq 1}\mathrm{Gap}_m^c$ has null probability. So, with probability 1, there exists some (random) number m_0 such that Gap_{m_0} occurs. This means that $\bigcup_{t\geq M}\Delta(t)$ does not overlap $\mathscr{H}_{\sigma_{m_0}+\zeta}$, where the random time σ_{m_0} is a.s. finite. This concludes the proof of (15) follows.

Adaptation of the proof of (15) **for** $A_n[\infty]$**.** The track is to introduce a family of events in the context of $A_n[\infty]$ as we did for $A_n^*[\infty]$ by introducing the events Gap_j, $j \geq 1$. To do it, let R be sufficiently large so that the particles which allow us to construct $A_n[\infty]$ and which are sent from levels $i \geq R$ do not visit the horizontal axis $\mathbb{Z}\times\{1\}$ with probability $\tilde{c} = \tilde{c}(n,R) > 0$ (such a quantity R exists according to Theorem 3.1).

Now, conditional on $\mathscr{G}_{\sigma_1}$, let Thine_1 be the following event:

- any particle from level $\sigma_1 < t < \sigma_1+\zeta+R$ settles on the segment $\{0\}\times[\![\sigma_1+1, \sigma_1+\zeta+R-1]\!]$ or goes directly to the site $(0,\sigma_1+\zeta+1)$ and then goes to the right of $(0,\sigma_1+\zeta+1)$ (on the axis $\mathbb{Z}\times\{\sigma_1+\zeta+1\}$) until exiting the aggregate;

- the aggregate built from the initial set $\{0\}\times[\![\sigma_1+\zeta+1, \sigma_1+\zeta+R-1]\!]$ by sending n particles from level $\sigma_1+\zeta+R$, thus n particles from level $\sigma_1+\zeta+R+1$ and so on, does not intersect the horizontal line $\mathbb{Z}\times\{\sigma_1+\zeta+1\}$.

Conditional on $\mathscr{G}_{\sigma_1}$, the two above properties are independent. The first one is realized with positive probability (depending on n,R,ζ). Indeed, the number of steps imposed to the trajectories in the first property only depends on n,R and ζ (note that no point of the aggregate lies in $\mathbb{Z}\times\{\sigma_1+\zeta+1\}$ before that the particles of level σ_1+1 are sent). The second one is realized with probability at least $\tilde{c}$ due to the choice of R. Thus, the event Thine_1 occurs with probability at least $\tilde{c}'(n,R,\zeta) > 0$. In a similar way, we can introduce events Thine_j, $j \geq 2$, as we did for Gap_j. Observe that, on Thine_j, a difference in $\Delta(\sigma_j)$ can be relayed by particles sent from levels σ_j+1 to $\sigma_j+\zeta+R-1$ at level $\sigma_j+\zeta+1$ but not above. Since, on this event, any particle emitted above or at level $\sigma_j+\zeta+R$ settles before it reaches the line $\mathbb{Z}\times\{\sigma_1+\zeta+1\}$, we deduce that on Thine_j:

$$\left(\bigcup_{t\geq M}\Delta(t)\right)\cap\mathscr{H}_{\sigma_j+\zeta+2} = \emptyset.$$

The rest of the proof works as before.

4.2 Proof of Proposition 4.3

Let $n, M \geq 1$. Our aim is to determine a threshold ζ in such a way that $\tau_m(\zeta)$ is finite for any $m \geq 1$. It is sufficient to deal with the case $m = 1$ since the general case can be dealt in a similar way.

For brievety, we write $h_t = h(A_{n,M}^*(t))$. First remark that the process $(h_t)_{t\geq M}$ is not Markov because h_{t+1} depends on the whole aggregate $A_{n,M}^*(t)$ and not only on its height. To prove Proposition 4.3, we first have to state three lemmas. The first one contains the main idea and claims that a *negative drift* holds for $(h_t)_{t\geq M}$ far away from 0.

Lemma 4.4 *There exists $\zeta_0(n)$ such that, for any $\zeta \geq \zeta_0(n)$, on the event $\{h_t > \zeta\}$, we have a.s.*

$$\mathbb{E}\left[h_{t+1} - h_t \mid \mathscr{G}_t\right] \leq -\frac{1}{2} .$$

To state the second lemma, let $t \geq M$ be fixed and assume that $h_t > \zeta$. We denote by $H_{t,\zeta}$ the following event:

$$H_{t,\zeta} = \left\{ \begin{array}{c} \text{At least one of the } N_{t+1} \text{ particles starting from level } t+1 \\ \text{hits the line } \mathbb{Z}\times\{t+\zeta\} \text{ before exiting the current aggregate} \end{array} \right\},$$

with the convention $H_{t,\zeta} = \emptyset$ if $N_{t+1} = 0$. The following lemma claims that, conditional on $\mathscr{G}_t$ and *uniformly on t*, the event $H_{t,\zeta}$ does not occur with high probability.

Lemma 4.5 *The following limit holds a.s.:*

$$\lim_{\zeta\to\infty}\sup_{t\geq M}\mathbb{P}\left(H_{t,\zeta}\mid\mathcal{G}_t\right)=0\,.$$

The above result is also one of the main ingredients to derive Lemma 4.4. The next one comes from [12] and provides finiteness (and also tail decay but we omit this part here) for the hitting time to 0 for a discrete-time, non-negative valued process $\{Y_t : t\geq 0\}$ which is not necessarily Markov. Only supermartingale structure and moment conditions for increments are assumed. To state it, we denote by ν^Y the first hitting time to 0, *i.e.*

$$\nu^Y=\inf\{t\geq 1 : Y_t=0\}\,.$$

Lemma 4.6 [12, THEOREM 5.2]. *Let $\{Y_t : t\geq 0\}$ be a $\{\mathcal{G}_t : t\geq 0\}$ discrete-time adapted stochastic process taking values in $\mathbb{R}_+$. Suppose that there exist constants $C_0, C_1>0$ such that, for any $t\geq 0$ and a.s. on the event $\{Y_t>0\}$, we have:*

(i) $\mathbb{E}\left[(Y_{t+1}-Y_t)\mid\mathcal{G}_t\right]\leq 0$;

(ii) $\mathbb{E}\left[(Y_{t+1}-Y_t)^2\mid\mathcal{G}_t\right]\geq C_0$;

(iii) $\mathbb{E}\left[|Y_{t+1}-Y_t|^3\mid\mathcal{G}_t\right]\leq C_1$.

Then $\nu^Y<\infty$ a.s..

Proof of Proposition 4.3. Let $\zeta\geq\zeta_0(n)$, where $\zeta_0(n)$ is as in Lemma 4.4. To prove that $\tau_1(\zeta)$ is finite a.s., it is sufficient to apply Lemma 4.6 to the process $\{Y_t : t\geq 0\}$, where

$$Y_t=h_t\mathbf{1}_{h_t>\zeta}\ ,$$

for any $t\geq 0$. We check below the three assumptions of Lemma 4.6. First, on the event $\{Y_t>0\}$, we notice that $h_t>\zeta$ and $Y_t=h_t$. Hence $Y_{t+1}-Y_t\leq h_{t+1}-h_t$ and Assumption (i) immediately follows from Lemma 4.4.

To prove (ii), we consider two cases. First assume that $h_t>\zeta+1$. In this case, we have $h_{t+1}>\zeta$ and $Y_{t+1}=h_{t+1}$. Since $h_{t+1}-h_t=-1$ on $\{h_t>\zeta\}\cap H^c_{t,\zeta}$, we get

$$\mathbb{E}\left[(Y_{t+1}-Y_t)^2\mid\mathcal{G}_t\right]\geq\mathbb{E}\left[(h_{t+1}-h_t)^2\mathbf{1}_{H^c_{t,\zeta}}\mid\mathcal{G}_t\right]=\mathbb{P}\left(H^c_{t,\zeta}\mid\mathcal{G}_t\right),$$

which is larger than $C_0=1/2$ for ζ large enough thanks to Lemma 4.5. Now, if $h_t=\zeta+1$, we have $h_{t+1}=\zeta$ on $\{h_t>\zeta\}\cap H^c_{t,\zeta}$ and we conclude similarly. This proves (ii).

Assumption (iii) is easy to check since $-1\leq h_{t+1}-h_t\leq N_{t+1}$. □

Proof of Lemma 4.4. Let us work conditional on $\mathcal{G}_t$ and assume that $h_t>\zeta$. If the event $H^c_{t,\zeta}$ occurs, then the highest ordinate which is reached by the aggregate $A^*_{n,M}(\cdot)$ does not change between times t and $t+1$ since $h_t>\zeta$, and thus $h_{t+1}-h_t=-1$. If not, we bound $h_{t+1}-h_t$ by the Poisson random variable N_{t+1}. So, using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\mathbb{E}\left[h_{t+1}-h_t\mid\mathcal{G}_t\right]&\leq-\mathbb{P}\left(H^c_{t,\zeta}\mid\mathcal{G}_t\right)+\mathbb{E}\left[N_{t+1}\mathbf{1}_{H_{t,\zeta}}\mid\mathcal{G}_t\right]\\&\leq-1+\mathbb{P}\left(H_{t,\zeta}\mid\mathcal{G}_t\right)+C\,\mathbb{P}\left(H_{t,\zeta}\mid\mathcal{G}_t\right)^{1/2}\end{aligned}$$

where $C=\mathbb{E}\left[N^2_{t+1}\mid\mathcal{G}_t\right]^{1/2}=\mathbb{E}\left[N^2_{t+1}\right]^{1/2}=\mathbb{E}\left[N^2_0\right]^{1/2}$ is finite and independent of t. According to Lemma 4.4, we can choose ζ large enough so that, for any $t\geq M$, the expectation $\mathbb{E}[h_{t+1}-h_t\mid\mathcal{G}_t]$ is smaller than $-1/2$. □

Proof of Lemma 4.5. We do not give all the details because it relies on an adaptation of the proof of Theorem 3.1. In particular, we will introduce a notion of good and bad annuli.

Let $\zeta > 0$. Because N_{t+1} is Poisson distributed with parameter n for any $t \geq M$, we have

$$\sup_{t\geq M} \mathbb{P}\left(N_{t+1} > \zeta^{\delta} \right) \leq \sup_{t\geq M} \zeta^{-\delta} \mathbb{E}\left[N_{t+1} \right] = \zeta^{-\delta} n \,.$$

So it is sufficient to prove that, a.s.,

$$\lim_{\zeta\to\infty} \sup_{t\geq M} \mathbb{P}\left(H_{t,\zeta} \cap \left\{ N_{t+1} \leq \zeta^{\delta} \right\} \mid \mathcal{G}_t \right) = 0 \,.$$

On the event $H_{t,\zeta} \cap \{N_{t+1} \leq \zeta^{\delta}\}$, there exists $1 \leq i \leq \zeta^{\delta}$ such that the i-th particle starting from level $t+1$ reaches ordinate $t+\zeta$ before exiting the current aggregate. To deal with this event, we have to control the sizes of the current aggregates. Because they are a.s. included in $A^*_{n,M}(\infty)$, we introduce the random variable

$$X(t,\zeta,n) = \#\left(A^*_{n,M}(\infty) \cap \left\{ (x,y) \in \mathbb{Z}^2 : t+1 \leq y \leq t+\zeta \right\} \right) \,.$$

Since $A^*_{n,M}(\infty)$ is included in $A^*_n[\infty]$, we have

$$\mathbb{E}\left[X(t,\zeta,n) \right] \leq \mathbb{E}\left[\#\left(A^*_n[\infty] \cap \{(x,y) \in \mathbb{Z}^2 : t+1 \leq y \leq t+\zeta\} \right) \right] = n\zeta,$$

where the last equality comes from Propositions 2.2 (i) and 2.3. Taking $0 < \delta' < 1/2$, we have

$$\sup_{t\geq M} \mathbb{P}\left(X(t,\zeta,n) > \zeta^{1+\delta'} \right) \leq \frac{n\zeta}{\zeta^{1+\delta'}} = n\zeta^{-\delta'} \,.$$

Hence, it suffices to show that the following conditional probability tends to 0 as ζ goes to ∞, uniformly on t:

$$\mathbb{P}\left(\bigcup_{i\leq\zeta^{\delta}} \left\{ \begin{array}{c} \text{The } i\text{-th particle starting from level } t+1 \\ \text{reaches ordinate } t+\zeta \\ \text{before exiting the current aggregate} \end{array} \right\} \cap \left\{ X(t,\zeta,n) \leq \zeta^{1+\delta'} \right\} \middle| \mathcal{G}_t \right) \,. \tag{16}$$

Let $B_0 = A^*_{n,M}(t) \cap \{(x,y) \in \mathbb{Z}^2 : y \leq t+1\}$ and let us set for any integer ℓ

$$B_\ell = B_0 \oplus B(0,\ell r),$$

where $r = \zeta^{1/2}$. Thus $S_{\ell+1} = B_{\ell+1} \backslash B_\ell$ is an annulus with width r. Hence, a particle starting from the source $(0,t+1)$ has to cross at least $\ell_{\max} = \lfloor \zeta/r \rfloor$ such annuli to reach the horizontal line with ordinate $t+\zeta$.

An annulus S_ℓ is said *good* if

$$\#\left(A^*_{n,M}(t) \cap S_\ell \right) \leq \eta_0 r^2 \,.$$

Otherwise, we say that it is *bad*. Since the aggregate $A^*_{n,M}(t)$ is $\mathcal{G}_t$-measurable, we know which annuli are good or not conditional on $\mathcal{G}_t$. Let N^{good} and N^{bad} be the numbers of good and bad annuli. We know that

$$N^{\text{good}} + N^{\text{bad}} = \ell_{\max} = \lfloor \zeta/r \rfloor.$$

On the event $\left\{ X(t,\zeta,n) \leq \zeta^{1+\delta'} \right\}$, we have

$$N^{\text{bad}} \times \eta_0 r^2 \leq \#\left(A^*_{n,M}(\infty) \cap \left\{ (x,y) \in \mathbb{Z}^2 : t+1 \leq y \leq \zeta \right\} \right) \leq \zeta^{1+\delta'} \,.$$

We deduce from these inequalities that

$$N^{\text{good}} \geq \left\lfloor \frac{\zeta}{r} \right\rfloor - \frac{\zeta^{1+\delta'}}{\eta_0 r^2} \,.$$

The choice of the parameter $r = \zeta^{1/2}$ (and $\delta' < 1/2$) implies that N^{good} is larger than $c\zeta^{1/2}$, for some universal constant $c > 0$.

The sequel is very close to the proof of Theorem 3.1. Conditional on $\mathscr{G}_t$, we denote by X_1 the number of good annuli crossed by particle 1. Thus, given $X_1,\dots,X_i$ with $1\le i<\zeta^{\delta}$, we denote by X_{i+1} the number of good annuli crossed by particle $i+1$, *but not* already crossed by particles $1,\dots,i$. On the one hand, the Crossing Lemma (Lemma 3.4) claims that, conditional on $\mathscr{G}_t$ and $X_1,\dots,X_i$, the r.v. X_{i+1} is stochastically dominated by a geometric distribution with parameter $1-\eta$, where η is as in Lemma 3.4. On the other hand, the fact that at least one particle starting from level $t+1$ reaches the ordinate $t+\zeta$ means that $\sum_{i\le\zeta^{\delta}} X_i$ is larger than N^{good}. In particular, there exists $i\le\zeta^{\delta}$ such that $X_i\ge c\zeta^{1/2-\delta}$. Therefore, the probability appearing in (16) is bounded by $\zeta^{\delta}(1-\eta)^{c\zeta^{1/2-\delta}}$, which tends to 0 uniformly on t. This concludes the proof of Lemma 4.5. □

5 Mixing property for $A_n[\infty]$, $A_n^*[\infty]$ and $A_n^{\dagger}[\infty]$

Combining the stabilization results obtained in the previous sections (Theorems 3.1 and 4.1), we get a mixing property for aggregates $A_n[\infty]$, $A_n^*[\infty]$ and $A_n^{\dagger}[\infty]$. An alternative (longer) proof of Proposition 5.1 could be provided, without using the latter theorems, by introducing once again good and bad annuli as described in Section 3.

Proposition 5.1 *Let $n\ge 1$.*

(i) The distribution of $A_n[\infty]$ is mixing w.r.t. vertical translations, i.e.

$$\lim_{|k|\to\infty} \mathbb{P}\left(A_n[\infty]\in\mathscr{A},\, A_n[\infty]\in\tau_k\mathscr{B}\right)=\mathbb{P}\left(A_n[\infty]\in\mathscr{A}\right)\mathbb{P}\left(A_n[\infty]\in\mathscr{B}\right)$$

for any events $\mathscr{A},\mathscr{B}$.

(ii) The same holds for $A_n^*[\infty]$ and $A_n^{\dagger}[\infty]$.

The next result is a direct consequence of the above mixing property and Proposition 3.6. It claims that the infinite aggregates $A_n^*[\infty]$ and $A_n^{\dagger}[\infty]$ have a.s. infinitely many finite connected components. Here, we say that $C\subset\mathbb{Z}^2$ is *connected* if for any $z,z'\in C$ there exist $z_0=z,z_1,\dots,z_n=z'\in C$ with $|z_j-z_{j-1}|_1=1$ for every $j\in[\![1,n]\!]$. Corollary 5.2 will be used to define the IDLA forest in Section 7.

Corollary 5.2 *(i) Let $n\ge 1$. With probability 1, for any integer M there are (infinitely many) levels $i\ge M$ and $j\le -M$ such that the aggregate $A_n^*[\infty]$ does not intersect the axes $\mathbb{Z}\times\{i\}$ and $\mathbb{Z}\times\{j\}$. In particular, a.s. $A_n^*[\infty]$ has only finite connected components included in disjoint strips.*

(ii) The same holds for $A_n^{\dagger}[\infty]$.

Proof of Corollary 5.2. Notice that from Propositions 2.2 and 3.6, we have, for any integer $m\in\mathbb{Z}$,

$$\mathbb{P}\big(A_n^*[\infty]\cap(\mathbb{Z}\times\{m\})=\emptyset\big)=\mathbb{P}\big(A_n^*[\infty]\cap(\mathbb{Z}\times\{0\})=\emptyset\big)>0\,.$$

Because the infinite aggregate $A_n^*[\infty]$ is mixing w.r.t. vertical translations (Proposition 5.1) and thus ergodic, we then get that, with probability 1, for any M, there exist levels $i>M$ and $j<-M$ such that $A_n^*[\infty]$ avoids the axes $\mathbb{Z}\times\{i\}$ and $\mathbb{Z}\times\{j\}$. The same holds for $A_n^{\dagger}[\infty]$ because $A_n^*[\infty]$ and $A_n^{\dagger}[\infty]$ are equally distributed. □

Proof of Proposition 5.1. We only prove (i). The proof is similar for $A_n^*[\infty]$ and so for $A_n^{\dagger}[\infty]$.

It is enough to prove that

$$\lim_{k\to\infty}\mathbb{P}\left(A_n[\infty]\cap(C_1\cup\tau_kC_2)=\emptyset\right)=\mathbb{P}\left(A_n[\infty]\cap C_1=\emptyset\right)\mathbb{P}\left(A_n[\infty]\cap C_2=\emptyset\right)$$

for all compact sets $C_1, C_2 \subset \mathbb{Z}^2$ (see *e.g.* [32, Theorem 9.3.2]). The case where $k \to -\infty$ works as well.

Let $\varepsilon > 0$. By Theorem 3.1 (with $\alpha = 2$), we can choose M large enough so that $C_1 \subset \mathbb{Z}_M$ and

$$\mathbb{P}\big(A_n[\infty] \cap \mathbb{Z}_M = A_n[M^2] \cap \mathbb{Z}_M\big) \geq 1 - \varepsilon/2 .$$

Furthermore, Theorem 4.1 asserts that

$$\lim_{M\to\infty} \lim_{t\to\infty} \mathbb{P}\left(A_n[\infty] \cap \mathscr{H}_t = B_{n,M}(\infty) \cap \mathscr{H}_t \right) = 1,$$

where $B_{n,M}(\infty)$ denotes the aggregate, coupled with $A_n[\infty]$ and included in $A_n[\infty]$, which is built by sending particles from levels $M+1$, $M+2$ and so on (see Section 4). Taking M sufficiently large, we can choose $t \geq M^2$ large enough so that

$$\mathbb{P}\left(A_n[\infty] \cap \mathscr{H}_t = B_{n,M^2}(\infty) \cap \mathscr{H}_t \right) \geq 1 - \varepsilon/2 .$$

Now, let $F_{M,t}$ be the following event:

$$F_{M,t} = \left\{A_n[\infty] \cap \mathbb{Z}_M = A_n[M^2] \cap \mathbb{Z}_M \text{ and } A_n[\infty] \cap \mathscr{H}_t = B_{n,M^2}(\infty) \cap \mathscr{H}_t\right\}.$$

Let k be such that $\tau_k C_2$ is included in $\mathscr{H}_t$. Since $\mathbb{P}\left(F_{M,t}\right) \geq 1-\varepsilon$, we have

$$\begin{aligned}
&\mathbb{P}\left(A_n[\infty] \cap C_1 = \emptyset\,,\, A_n[\infty] \cap \tau_k C_2 = \emptyset\right) \\
&\qquad = \mathbb{P}\left(\left\{A_n[M^2] \cap C_1 = \emptyset\,,\, B_{n,M^2}(\infty) \cap \tau_k C_2 = \emptyset\right\} \cap F_{M,t}\right) \pm \varepsilon \\
&\qquad = \mathbb{P}\left(A_n[M^2] \cap C_1 = \emptyset, B_{n,M^2}(\infty) \cap \tau_k C_2 = \emptyset\right) \pm 2\varepsilon \\
&\qquad = \mathbb{P}\left(A_n[M^2] \cap C_1 = \emptyset\right) \mathbb{P}\left(B_{n,M^2}(\infty) \cap \tau_k C_2 = \emptyset\right) \pm 2\varepsilon,
\end{aligned}$$

where the last line comes from the fact that $A_n[M^2]$ and $B_{n,M^2}(\infty)$ are based on independent particles. In the above equation, the notation $p = p' \pm \varepsilon$ means that $|p - p'| \leq \varepsilon$ for any $p, p' \in \mathbb{R}$. Using once again that $\mathbb{P}\left(F_{M,t}\right) \geq 1-\varepsilon$, we have

$$\begin{aligned}
&\mathbb{P}\left(A_n[M^2] \cap C_1 = \emptyset\right) \mathbb{P}\left(B_{n,M^2}(\infty) \cap \tau_k C_2 = \emptyset\right) \\
&\qquad = \mathbb{P}\left(\left\{A_n[M^2] \cap C_1 = \emptyset\right\} \cap F_{M,t}\right) \mathbb{P}\left(\left\{B_{n,M^2}(\infty) \cap \tau_k C_2 = \emptyset\right\} \cap F_{M,t}\right) \pm 2\varepsilon \\
&\qquad = \mathbb{P}\left(A_n[\infty] \cap C_1 = \emptyset\right) \mathbb{P}\left(A_n[\infty] \cap \tau_k C_2 = \emptyset\right) \pm 4\varepsilon \\
&\qquad = \mathbb{P}\left(A_n[\infty] \cap C_1 = \emptyset\right) \mathbb{P}\left(A_n[\infty] \cap C_2 = \emptyset\right) \pm 4\varepsilon
\end{aligned}$$

since the distribution of $A_n[\infty]$ is invariant w.r.t. vertical translations. This concludes the proof of Proposition 5.1. □

6 Shape theorems

Proposition 2.3 claims that, in expectation, the infinite aggregates $A_n[\infty]$, $A_n^*[\infty]$ and $A_n^\dagger[\infty]$ can be approximated with the rectangle $R_{n/2}$, where

$$R_r = [\![-r, r]\!] \times \mathbb{Z}.$$

for any $r > 0$. The following result ensures that the fluctuations of these aggregates, when they are restricted to a strip, are at most logarithmic.

Theorem 6.1 *(i) There exists $A > 0$ such that, for any $\alpha > 0$, a.s. there exists $N \geq 1$ such that for any $n \geq N$,*

$$R_{n/2 - A\log(n)} \cap \mathbb{Z}_{n^\alpha} \subset A_n[\infty] \cap \mathbb{Z}_{n^\alpha} \subset R_{n/2 + A\log(n)} \cap \mathbb{Z}_{n^\alpha}.$$

(ii) The same property holds for $A_n^*[\infty]$ and $A_n^\dagger[\infty]$.

As an illustration of Theorem 6.1, Figure 3 shows that $A_n[\infty]$, when it is restricted to a strip, is very close to a rectangle. Notice that the intersection with the horizontal strip $\mathbb{Z}_M$ is unavoidable. Indeed, for any integer n, there exists a random level i such that (n^2, i) belongs to the aggregate with probability 1. Note also that a weak version of Theorem 6.1 for $A_n^*[\infty]$ and $A_n^\dagger[\infty]$ – in the vein of [25] – could be simply derived from Theorem 6.1 for $A_n[\infty]$ and from standard deviations on Poisson distribution.

Theorem 6.1 is classical in the sense that many results dealing with the shape of aggregates were established for various IDLA models, see *e.g.* [1, 2, 7, 28], but these results cannot be applied in our context. The proof of Theorem 6.1 relies on an adaptation of [1, 2]. This adaptation is not direct at some places because our random aggregates are based on an infinite number of particles and are *not isotropic*. The counterpart is that our models are invariant w.r.t vertical translations and satisfy a mass transport principle (see Proposition 2.3). These properties will be extensively used in our proof (the main differences w.r.t. [1, 2] will be stressed).

Notice that Theorem 6.1 provides upper bounds for the fluctuations of our random aggregates in terms of Hausdorff distance. In the same spirit as [2], we could show that the fluctuations are *at least* logarithmic.

With a slight abuse of notation, lower and upper integer parts are omitted along this section.

6.1 The lower bound for $A_n[\infty]$

In this section, we prove the following result (referred to as the lower bound for the shape theorem): there exists $A > 0$ such that, for any $\alpha > 0$, a.s. there exists $N \geq 1$ such that for any $n \geq N$,

$$R_{n/2-A\log(n)} \cap \mathbb{Z}_{n^\alpha} \subset A_n[\infty] \cap \mathbb{Z}_{n^\alpha}. \tag{17}$$

To do it, we mainly follows the strategy developed in [1, 2] and, then we will make several references to these articles. In order to make clearer these references, we adopt – only for Section 6 – their notation $A(\eta)$ to denote the aggregate generated by an initial configuration η. The notation $\eta = n\mathbf{1}_C$, where $n \geq 1$ and $C \subset \mathbb{Z}^2$, means that we send n particles from each site of C. In particular, we have $A\big(n\mathbf{1}_{I(M)}\big) = A_n[M]$ and $A\big(n\mathbf{1}_{I(\infty)}\big) = A_n[\infty]$.

For each integer k, we let

$$S_k = \big(R_{(k+1)\log(n)} \setminus R_{k\log(n)}\big) \cap \mathbb{Z}_{n^\alpha}.$$

The set S_k is referred to as a *shell*. Now, for each $z \in \partial R_{k\log(n)}$, with

$$\partial R_{k\log(n)} = \{-\lfloor k\log(n)\rfloor, \lfloor k\log(n)\rfloor\} \times \mathbb{Z},$$

we define the so-called *tile* and *cell* centered at z as

$$\tau(z) = B(z, \log(n)/2) \cap \partial R_{k\log(n)} \qquad \text{and} \qquad C(z) = B(z, \log(n)) \cap R^c_{k\log(n)}.$$

Notice that

$$S_k \subset \bigcup_{z\in\partial_{k,n}} C(z),$$

where $\partial_{k,n} = \partial R_{k\log(n)} \cap \mathbb{Z}_{n^\alpha}$. Given η and $B \subset R_{k\log(n)}$, the number of particles (resp. random walks), with initial configuration η, hitting B before or when they exit $R_{k\log(n)}$, are denoted by $W_{k\log(n)}(\eta, B)$ (resp. $M_{k\log(n)}(\eta, B)$). With a slight abuse of notation, when $\eta = \mathbf{1}_C$, we simply write $W_{k\log(n)}(C, B)$ (resp. $M_{k\log(n)}(C, B)$) instead of $W_{k\log(n)}(\mathbf{1}_C, B)$ (resp. $M_{k\log(n)}(\mathbf{1}_C, B)$). Recall that we use the word *particle* for a random walk which is stopped when exiting the aggregate and adding site; thus trajectories of particles depend on the aggregate while random walks do not.

We say that a set B is *not covered* if $B \not\subset A\big(n\mathbf{1}_{I(\infty)}\big)$. According to the Borel-Cantelli lemma, it is sufficient to prove that there exists A such that, for any $L > 0$, $n \geq 1$ and $k \leq \frac{n}{2\log(n)} - A$, we have:

$$\mathbb{P}(S_k \text{ is not covered}) \leq cn^{-L}.$$

As in [2, 24], it is useful to stop the particles when they reach $\partial R_{k\log(n)}$. The strategy can be divided into two steps. Roughly, it consists in, first, showing that each tile τ of $\partial R_{k\log(n)}$ is likely to capture many particles and then arguing that, if many particles exit $\partial R_{k\log(n)}$ from τ, then they are likely to cover the corresponding cell C. To do it, let A be fixed and let $k \le \frac{n}{2\log(n)} - A$. We write

$$\mathbb{P}\left(S_k \text{ is not covered}\right) \le p(n,k) + q(n,k), \tag{18}$$

where

$$p(n,k) = \mathbb{P}\left(\exists \tau \in \mathcal{T}_{k\log(n)}, W_{k\log(n)}(n\mathbf{1}_{I(\infty)}, \tau) \le \frac{1}{3}\mu(\tau)\right)$$

and

$$q(n,k) = \mathbb{P}\left(\forall \tau \in \mathcal{T}_{k\log(n)}, W_{k\log(n)}(n\mathbf{1}_{I(\infty)}, \tau) \ge \frac{1}{3}\mu(\tau) \text{ and } S_k \text{ is not covered}\right).$$

The term $\mu(\tau)$ appearing in the above equations will be defined later in (20), and the set $\mathcal{T}_{k\log(n)}$ denotes the family of tiles, *i.e.*

$$\mathcal{T}_{k\log(n)} = \{\tau(z) : z \in \partial_{k,n}\}.$$

We prove below that $p(n,k)$ and $q(n,k)$ are smaller than any power of n^{-1} when A is large enough.

6.1.1 Upper bound for $p(n,k)$

Since $\#\mathcal{T}_{k\log(n)} \le 4n^{\alpha} + 2$, it is sufficient to prove that

$$\mathbb{P}\left(W_{k\log(n)}(n\mathbf{1}_{I(\infty)}, \tau) \le \frac{1}{3}\mu(\tau)\right)$$

is lower than any power of n^{-1}, for any tile $\tau \in \mathcal{T}_{k\log(n)}$. To do it, we will apply an analog of Lemma 2.4 of [2] that deals with series of Bernoulli random variables instead of sums of Bernoulli variables. It is obtained by straightforward modifications of the proof of [2, Lemma 2.4] and is stated as follows.

Lemma 6.2 *Suppose that a sequence of random variables $\{W_n, M_n, L_n, \widetilde{M}_n; n \ge 0\}$ and a sequence of real numbers $(c_n)_{n\ge 0}$ satisfy for any $n \ge 0$:*

$$W_n + L_n + c_n \ge \widetilde{M}_n \qquad \text{and} \qquad \widetilde{M}_n \overset{law}{=} M_n.$$

Assume that W_n and L_n are independent and that L_n and M_n are both series of independent Bernoulli random variables with finite first moment. Assume also that

(H1) *the Bernoulli variables $\{Y_1^{(n)}, Y_2^{(n)}, \ldots\}$ whose series is L_n satisfy for some $\kappa > 1$:*

$$\sup_n \sup_i \mathbb{E}\left[Y_i^{(n)}\right] < \frac{\kappa - 1}{\kappa};$$

(H2) $\mu_n = \mathbb{E}[M_n - L_n] \ge 0$.

Then, for any $n \ge 0$ and $\xi_n \in \mathbb{R}$, we have for any $\lambda \ge 0$,

$$\mathbb{P}\left(W_n < \xi_n\right) \le \exp\left(-\lambda\left(\mu_n - \xi_n - c_n\right) + \frac{\lambda^2}{2}\left(\mu_n + \kappa \sum_{i=1}^{\infty} \mathbb{E}\left[Y_i^{(n)}\right]^2\right)\right).$$

The desired upper bound for 6.1.1 will be obtained thanks to Lemma 6.2. Hence, we must check that hypotheses of Lemma 6.2 are satisfied. First of all, note that, following the strategy initiated in [25], similar arguments to [2] show that

$$W_{k\log(n)}(n\mathbf{1}_{I(\infty)}, \tau) + M_{k\log(n)}(R_{k\log(n)}, \tau) \ge \widetilde{M}_{k\log(n)}(n\mathbf{1}_{I(\infty)}, \tau), \tag{19}$$

with

$$\widetilde{M}_{k\log(n)}(n\mathbf{1}_{I(\infty)}, \tau) = W_{k\log(n)}(n\mathbf{1}_{I(\infty)}, \tau) + M_{k\log(n)}(A_{k\log(n)}(n\mathbf{1}_{I(\infty)}), \tau),$$

where $A_r(\eta)$ denotes the aggregate produced by particles with initial configuration η stopped when they leave R_r.

For $z \in \partial R_{k\log(n)}$, let $\mathscr{Z} = \mathscr{Z}(z, b, n)$ be the set

$$\mathscr{Z} = \{z' \in R_{k\log(n)} : d(z', \tau(z)) \leq b\log(n)\}.$$

The set $\mathscr{Z}$ will be useful to check that **(H1)** holds, for some $b > 0$. Set $c_n = |\mathscr{Z}| \leq c(b\log(n))^2$, so that (19) leads to:

$$W_{k\log(n)}(n\mathbf{1}_{I(\infty)}, \tau) + M_{k\log(n)}(R_{k\log(n)} \setminus \mathscr{Z}, \tau) + c_n \geq \widetilde{M}_{k\log(n)}(n\mathbf{1}_{I(\infty)}, \tau).$$

We will apply Lemma 6.2 with $W_n = W_{k\log(n)}(n\mathbf{1}_{I(\infty)}, \tau)$, $M_n = M_{k\log(n)}(n\mathbf{1}_{I(\infty)}, \tau)$, $L_n = M_{k\log(n)}(R_{k\log(n)} \setminus \mathscr{Z}, \tau)$ and $\widetilde{M}_n = \widetilde{M}_{k\log(n)}(n\mathbf{1}_{I(\infty)}, \tau)$. So, $\mu(\tau)$ in (18) must be defined as:

$$\mu(\tau) = \mathbb{E}\left[M_{k\log(n)}(n\mathbf{1}_{I(\infty)}, \tau) \right] - \mathbb{E}\left[M_{k\log(n)}(R_{k\log(n)} \setminus \mathscr{Z}, \tau) \right]. \tag{20}$$

Verification of (H1) Assumption **(H1)** is ensured for some $b > 0$ by the following lemma which is the counterpart in our context of [1, Lemma 5.1]. Its proof is very similar to the one of Lemma 5.1 in [1] and is omitted here.

Lemma 6.3 *There exists $\kappa > 0$ such that, for any r, any $y \in R_r$ and any $x \in \partial R_r \setminus \{y\}$, we have*

$$\mathbb{P}_y(S(H_r) = x) \leq \frac{\kappa}{|x - y|},$$

where H_r denotes the hitting time of ∂R_r for the simple random walk $(S(t))_{t \geq 0}$.

Lower bound for $\mu(\tau)$ and verification of (H2) We show that, if z is at a distance at least $A\log(n)$ from $\partial R_{\frac{n}{2}}$, then

$$\mu(\tau) \geq cA\log(n)^2 \tag{21}$$

for some $c > 0$. Note that this ensures **(H2)**. In what follows, given a multiset η and $r \leq r'$, we denote by $M_{r'}(\eta, \tau)$ the number of random walks starting from η that exit $R_{r'}$ in a site of τ. Inequality (21) will be derived from the following lemma. Its proof relies on the invariance w.r.t vertical translations of our model and cannot be adapted from [1, 2].

Lemma 6.4 *Let $r \leq r'$ and let $\tau \subset \partial R_{r'}$ be finite. Then*

$$\mathbb{E}[M_{r'}(R_r, \tau)] = \frac{2r+1}{2}\#\tau.$$

In particular, as a consequence of Lemma 6.4 applied to $R_0 = I(\infty)$, we have for any $r' \geq 1$,

$$\mathbb{E}[M_{r'}(I(\infty), \tau)] = \frac{\#\tau}{2}. \tag{22}$$

Proof of Lemma 6.4. With the notation of Lemma 6.3, we have:

$$\mathbb{E}[M_{r'}(R_r, \tau)] = \sum_{y \in \tau} \sum_{z \in R_r} \mathbb{P}_z\left(S(H_{r'}) = y\right) = \#\tau \sum_{z \in R_r} \mathbb{P}_z\left(S(H_{r'}) = (r', 0)\right),$$

where we used invariance w.r.t. vertical translations and the symmetry w.r.t. $I(\infty)$ in the second equality. Hence,

$$\begin{aligned}
\mathbb{E}[M_{r'}(R_r, \tau)] &= \#\tau \sum_{i=-r}^{r} \sum_{j \in \mathbb{Z}} \mathbb{P}_{(i,j)}\left(S(H_{r'}) = (r', 0)\right) \\
&= \#\tau \sum_{i=-r}^{r} \sum_{j \in \mathbb{Z}} \mathbb{P}_{(i,0)}\left(S(H_{r'}) = (r', j)\right) \\
&= \#\tau \sum_{i=-r}^{r} \mathbb{P}_{(i,0)}\left(S(H_{r'}) \in \{r'\} \times \mathbb{Z}\right)
\end{aligned}$$

The result then follows since, by symmetry w.r.t. $I(\infty)$,

$$\mathbb{P}_{(i,0)}\left(S(H_{r'}) \in \{r'\} \times \mathbb{Z}\right) + \mathbb{P}_{(-i,0)}\left(S(H_{r'}) \in \{r'\} \times \mathbb{Z}\right) = 1.$$

□

Now, to get (21), we write

$$\begin{aligned}\mu(\tau) &\geq \mathbb{E}\left[M_{k\log(n)}(n\mathbf{1}_{I(\infty)}, \tau) - M_{k\log(n)}(R_{k\log(n)}, \tau) \right] \\ &= n\mathbb{E}\left[M_{k\log(n)}(I(\infty), \tau) \right] - \mathbb{E}\left[M_{k\log(n)}(R_{k\log(n)}, \tau) \right]\end{aligned}$$

According to (22) and Lemma 6.4, we have

$$\begin{aligned}n\mathbb{E}\left[M_{k\log(n)}(I(\infty), \tau) \right] - \mathbb{E}\left[M_{k\log(n)}(R_{k\log(n)}, \tau) \right] &= \frac{1}{2}\left(n - 2k\log(n) - 1\right)\#\tau \\ &\geq cA\log(n)^2.\end{aligned}$$

This gives (21).

Second-order estimate In order to exploit the upper bound given by Lemma 6.2, we have to control $\sum_i \mathbb{E}\left[Y_i^n \right]^2$. Due to the definition of $\mathcal{Z} = \mathcal{Z}(z, b, n)$ and $\tau = \tau(z)$, and according to Lemma 6.3, we have for all $y \in R_{k\log(n)} \setminus \mathcal{Z}$ (corresponding to index i in the sum defining L_n):

$$\mathbb{E}\left[Y_i^n \right] = \mathbb{P}_y(S(H_{k\log(n)}) \in \tau) \leq \#\tau \max_{x\in\tau} \mathbb{P}_y(S(H_{k\log(n)}) = x) \leq \#\tau \max_{x\in\tau} \frac{\kappa}{|x-y|} \leq c\frac{\#\tau}{|z-y|}.$$

Summing over y, it follows that

$$\begin{aligned}\sum_{y\in R_{k\log(n)}\setminus\mathcal{Z}} \mathbb{P}_y(S(H_{k\log(n)}) \in \tau)^2 &\leq c\#\tau^2 \sum_{y\in R_{k\log(n)}\setminus\mathcal{Z}} \frac{1}{|z-y|^2} \\ &\leq \#\tau^2 \left(c + 2\sum_{j=1}^{2k\log(n)} \int_1^\infty \frac{1}{j^2 + x^2}\,\mathrm{d}x \right) \leq c\log(n)^3.\end{aligned}$$

Application of Lemma 6.2 Lemma 6.2 and similar computations as in [2, Section 3.1.2] imply that

$$\mathbb{P}\left(W_{k\log(n)}(n\mathbf{1}_{I(\infty)}, \tau) \leq \frac{1}{3}\mu(\tau) \right) \leq \exp\left(-cA^2\log(n)\right). \tag{23}$$

The last term converges to 0 faster than any power of n^{-1} for A large enough.

6.1.2 Upper bound for $q(n,k)$

We prove below that $q(n,k)$ is bounded by any power of n^{-1}.

Recall that [2, Lemma 1.3] roughly says that if many particles (w.r.t its radius) initially lie in the middle of a ball, then the aggregate they produce is very likely to cover the ball. By using (21) and [2, Lemma 1.3], one has:

$$\begin{aligned}&\mathbb{P}\left(\forall \tau \in \mathcal{T}_{k\log(n)}, W_{k\log(n)}(n\mathbf{1}_{I(\infty)}, \tau) \geq \frac{1}{3}\mu(\tau) \text{ and } S_k \text{ is not covered} \right) \\ &\quad\leq \mathbb{P}\left(\exists z \in \partial_{k,n}, C(z) \text{ is not covered } \middle| \forall \tau \in \mathcal{T}_{k\log(n)}, W_{k\log(n)}(n\mathbf{1}_{I(\infty)}, \tau) \geq \frac{1}{3}\mu(\tau) \right) \\ &\quad\leq cn^\alpha \exp\left(-c\frac{A\log(n)^2}{\log(\log(n))} \right).\end{aligned}$$

The last term converges to 0 faster than any given power n^{-1}. This together with (18) and (23) concludes the proof of (17).

6.2 The upper bound for $A_n[\infty]$

In this section, we prove the following result (referred to as the upper bound for the shape theorem): there exists $A > 0$ such that for any $\alpha > 0$, there a.s. exists $N \geq 1$ such that for any $n \geq N$,

$$A\big(n\mathbf{1}_{I(\infty)}\big) \cap \mathbb{Z}_{n^\alpha} \subset R_{n/2+A\log(n)} \cap \mathbb{Z}_{n^\alpha}. \tag{24}$$

Let $\alpha > 0$ be fixed and let $A > 0$ (which will be chosen sufficiently large later). One can extract from the proof of Theorem 3.1 that, for any $L > 0$, for any $\gamma > \alpha$, and for n large enough,

$$\mathbb{P}\left(A\big(n\mathbf{1}_{I(\infty)}\big) \cap \mathbb{Z}_{n^\alpha} \neq A(n\mathbf{1}_{\{0\}\times[\![-n^\gamma,n^\gamma]\!]}) \cap \mathbb{Z}_{n^\alpha} \right) \leq n^{-L}. \tag{25}$$

The above inequality is crucial to derive the upper-bound because it reduces our problem to an aggregate which is generated by a *finite* number of particles. According to the Borel-Cantelli lemma, it is sufficient to prove that, for n large enough,

$$\mathbb{P}\left(A(n\mathbf{1}_{\{0\}\times[\![-n^\gamma,n^\gamma]\!]}) \cap \mathbb{Z}_{n^\alpha} \not\subset R_{n/2+A\log(n)} \cap \mathbb{Z}_{n^\alpha} \right) \tag{26}$$

is smaller than any power of n^{-1}. To do it, we bound (26) by

$$\mathbb{P}\left(t_n \geq \frac{n}{2} + A\log(n) \right),$$

where $t_n = \max\{|z(1)| : z \in A(n\mathbf{1}_{\{0\}\times[\![-n^\gamma,n^\gamma]\!]})\}$. Remark that t_n is a.s. smaller than $n(2n^\gamma+1)$. Taking the supremum over the point $z \in \mathbb{Z}_{n^\alpha} \cap \{z : \frac{n}{2} + A\log(n) \leq |z(1)| \leq n(2n^\gamma+1)\}$, it is enough to prove that

$$\sup_z \mathbb{P}\left(z \in A(n\mathbf{1}_{\{0\}\times[\![-n^\gamma,n^\gamma]\!]}), |z(1)| = t_n \right)$$

is lower than any power of n^{-1}. We do so in the same spirit as [2] and we recall several arguments included in [2] in order to make the paper self-contained. Let $z \in \mathbb{Z}_{n^\alpha}$, with $\frac{n}{2} + A\log(n) \leq |z(1)| \leq n(2n^\gamma+1)$. In what follows, we set

$$h(n) = |z(1)| - \frac{n}{2}.$$

First, we write

$$\begin{aligned}
&\mathbb{P}\left(z \in A(n\mathbf{1}_{\{0\}\times[\![-n^\gamma,n^\gamma]\!]}), |z(1)| = t_n \right) \\
&\quad\leq \mathbb{P}\left(\#(A(n\mathbf{1}_{\{0\}\times[\![-n^\gamma,n^\gamma]\!]}) \cap B(z,h(n))) > \beta h^2(n), |z(1)| = t_n \right) &(27)\\
&\qquad + \mathbb{P}\left(z \in A(n\mathbf{1}_{\{0\}\times[\![-n^\gamma,n^\gamma]\!]}), \#(A(n\mathbf{1}_{\{0\}\times[\![-n^\gamma,n^\gamma]\!]}) \cap B(z,h(n))) \leq \beta h^2(n) \right), &(28)
\end{aligned}$$

where β is chosen as in Lemma 6.5 below in order to ensure that (28) vanishes quickly. The later lemma is an adaptation of [20, Lemma A]. It allows us to avoid the use of an analog of the *flashing process* introduced in [1, 2] to derive the shape theorem. However, such a process remains useful to prove that the fluctuations in the shape theorem are of correct order and could be used in our context.

Lemma 6.5 *Let $\gamma > 0$ and let $A(\eta)$ be the aggregate with initial configuration η, such that η has support in $\{0\} \times [\![-n^\gamma, n^\gamma]\!]$ and $\#\eta < \infty$. Then there exist positive universal constants β, C_0 and c such that for any real number $m > 0$ and all $z \in \mathbb{Z}^2$ with $|z(1)| > m$,*

$$\mathbb{P}\left(z \in A(\eta), \#(A(\eta) \cap B(z,m)) \leq \beta m^2 \right) \leq C_0 e^{-cm^2/\log m}.$$

According to Lemma 6.5, Equation (28) is bounded by any power of n^{-1}. To deal with (27), we need to introduce some notation. In what follows, for any set $\Gamma \subset \mathbb{Z}^2$, we denote by $M^*_{n/2+h(n)}\big(\eta, \Gamma\big)$ the number of *random walks*, with initial configuration η, satisfying the following two properties:

- the particle associated with the random walk hits $\partial R_{n/2}$ before exiting the aggregate;
- the random walk intersects Γ before exiting $R_{n/2+h(n)}$.

In particular, on the event $\{|z(1)| = t_n\}$ and by definition of $h(n)$, we have

$$\#\left(A(n\mathbf{1}_{\{0\}\times[\![-n^\gamma,n^\gamma]\!]})\cap B(z,h(n))\right) \le M^*_{n/2+h(n)}\left(n\mathbf{1}_{\{0\}\times[\![-n^\gamma,n^\gamma]\!]},B(z,h(n))\right)$$

since no particle can escape $R_{n/2+h(n)}$. Therefore

$$\mathbb{P}\left(\#\left(A(n\mathbf{1}_{\{0\}\times[\![-n^\gamma,n^\gamma]\!]})\cap B(z,h(n))\right) > \beta h^2(n), |z(1)| = t_n\right) \\ \le \mathbb{P}\left(M^*_{n/2+h(n)}\left(n\mathbf{1}_{\{0\}\times[\![-n^\gamma,n^\gamma]\!]},B(z,h(n))\right) > \beta h^2(n)\right).$$

It is clear that $M^*_{n/2+h(n)}\left(n\mathbf{1}_{\{0\}\times[\![-n^\gamma,n^\gamma]\!]},B(z,h(n))\right)$ is a.s. finite. Conditional on the fact that this random variable equals some integer k, the trajectories of the k random walks *are not independent* since they have to satisfy the first property mentioned above. However, one of the key ingredients is to remark that, conditional on the later event, these random walks *are independent after they reach $\partial R_{n/2}$*, and so they evolve independently after reaching $\partial B(z,h(n))$. Now, recall that a random walk starting from some point $x \in \partial B(z,h(n))$ has a probability at least $\rho > 0$, wich does not depend on n, to hit $B(z,2h(n))\cap(R_{n/2+h(n)})^c$ when it exits $B(z,2h(n))$. In particular, it has a probability at least ρ to hit the tile

$$\tau(z) = B(z,2h(n))\cap\partial R_{n/2+h(n)}.$$

Therefore, conditional on the fact that $M^*_{n/2+h(n)}\left(n\mathbf{1}_{\{0\}\times[\![-n^\gamma,n^\gamma]\!]},B(z,h(n))\right)$ is larger than $\beta h^2(n)$, the random variable $M^*_{n/2+h(n)}(n\mathbf{1}_{\{0\}\times[\![-n^\gamma,n^\gamma]\!]},\tau(z))$ stochastically dominates a binomial distribution with parameters $(\beta h^2(n),\rho)$. Thus, there exists $I>0$ such that

$$\mathbb{P}\left(M^*_{\frac{n}{2}+h(n)}(n\mathbf{1}_{\{0\}\times[\![-n^\gamma,n^\gamma]\!]},\tau(z)) \le \frac{\beta h^2(n)}{2}\,\middle|\, M^*_{\frac{n}{2}+h(n)}(n\mathbf{1}_{\{0\}\times[\![-n^\gamma,n^\gamma]\!]},B(z,h(n))) > \beta h^2(n)\right) \\ \le \exp(-Ih^2(n)).$$

The right-hand side is negligeable compared to any power of n^{-1}. Thus, it is sufficient to prove that for n large enough, for any $L>0$,

$$\mathbb{P}\left(M^*_{n/2+h(n)}(n\mathbf{1}_{\{0\}\times[\![-n^\gamma,n^\gamma]\!]},\tau(z)) > \frac{\beta}{2}h^2(n)\right) \le n^{-L}. \tag{29}$$

Since

$$M^*_{n/2+h(n)}(n\mathbf{1}_{\{0\}\times[\![-n^\gamma,n^\gamma]\!]},\tau(z)) \le M^*_{n/2+h(n)}(n\mathbf{1}_{I(\infty)},\tau(z)),$$

we have to prove that $\mathbb{P}\left(M^*_{n/2+h(n)}(n\mathbf{1}_{I(\infty)},\tau(z)) > \frac{\beta}{2}h^2(n)\right)$ is lower than any power of n^{-1}.

We will use the following adaptation of [2, Lemma 2.5] stated in the context of series of Bernoulli random variables.

Lemma 6.6 *Let $W_n, L_n, \widetilde{M}_n$ be independent random variables and let $\mathscr{A}_n$ be an event independent of L_n. Assume that L_n and $M_n \overset{law}{=} \widetilde{M}_n$ are series of independent Bernoulli random variables with finite expectations such that $\mu_n = \mathbb{E}[M_n] - \mathbb{E}[L_n] \ge 0$ and write $L_n = \sum_{i\ge0} Y_i^{(n)}$. If the following holds:*

$$(W_n + L_n)\mathbf{1}_{\mathscr{A}_n} \underset{sto}{\le} \widetilde{M}_n,$$

then, for all $n \ge 0$, $\xi_n \in \mathbb{R}$ and $\lambda \in [0,\log 2]$,

$$\mathbb{P}(W_n \ge \xi_n, \mathscr{A}_n) \le \exp\left(-\lambda(\xi_n-\mu_n) + \lambda^2\left(\mu_n + 4\sum_{i\ge0}\mathbb{E}\left[Y_i^{(n)}\right]^2\right)\right).$$

Similarly to (2.4) in [2], it follows from the definition of $M^*_{n/2+h(n)}(n\mathbf{1}_{I(\infty)},\tau(z))$ that

$$M^*_{n/2+h(n)}(n\mathbf{1}_{I(\infty)},\tau(z)) + M_{n/2+h(n)}(A_{n/2}(n\mathbf{1}_{I(\infty)}),\tau(z)) \overset{\text{law}}{=} \widetilde{M}_{n/2+h(n)}(n\mathbf{1}_{I(\infty)},\tau(z)),$$

where $A_{n/2}(n\mathbf{1}_{I(\infty)})$ stands for the positions of settled particles before the associated random walks leave $R_{n/2}$ and where $\widetilde{M}_{n/2+h(n)}(n\mathbf{1}_{I(\infty)},\tau(z))$ is an independent copy of $M_{n/2+h(n)}(n\mathbf{1}_{I(\infty)},\tau(z))$. Let us denote by $\delta_I(n)$ the internal error of $A_{n/2}(n\mathbf{1}_{I(\infty)})$ on the strip $\mathbb{Z}_{n^\alpha}$, *i.e.*

$$\delta_I(n) = \max\left\{\frac{n}{2} - |z(1)| : z \in \left(R_{n/2} \setminus A_{n/2}\left(n\mathbf{1}_{I(\infty)}\right)\right) \cap \mathbb{Z}_{n^\alpha}\right\},$$

with the convention $\max\emptyset = 0$. Notice that in the proof of the lower bound (see Section 6.1), no particle exits $R_{n/2}$; hence we actually have $\delta_I(n) \le C\log(n)$ w.h.p. provided that C is large enough. Here, we say that a sequence of events $(E_n)_{n\ge 0}$ occurs w.h.p. if $\mathbb{P}\left(E_n^c\right)$ is lower than any power of n^{-1} for n large enough.

Since $R_{n/2-\delta_I(n)} \cap \mathbb{Z}_{n^\alpha} \subset A\left(n\mathbf{1}_{I(\infty)}\right)$, it follows that

$$\left(M^*_{n/2+h(n)}(n\mathbf{1}_{I(\infty)},\tau(z)) + M_{n/2+h(n)}\left(R_{n/2-\alpha_2\frac{h(n)}{2A}},\tau(z)\right)\right)\mathbf{1}_{\delta_I(n)\le\alpha_2 h(n)/2A} \underset{\text{sto}}{\le} \widetilde{M}_{n/2+h(n)}(n\mathbf{1}_{I(\infty)},\tau(z)),$$

where α_2 is some constant which will be chosen later. We are now ready to apply Lemma 6.6 with

$$W_n = M^*_{n/2+h(n)}\left(n\mathbf{1}_{I(\infty)},\tau(z)\right), \qquad L_n = M_{n/2+h(n)}\left(R_{n/2-\alpha_2\frac{h(n)}{2A}},\tau(z)\right)$$

$$\widetilde{M}_n = \widetilde{M}_{n/2+h(n)}(n\mathbf{1}_{I(\infty)},\tau(z)) \qquad \text{and} \qquad \mathscr{A}_n = \left\{\delta_I(n) \le \alpha_2\frac{h(n)}{2A}\right\}.$$

Note that L_n is independent of $\mathscr{A}_n$ by definition. It remains to control μ_n and $\sum_i \mathbb{E}[Y_i^{(n)}]^2$. According to (22) and Lemma 6.4, we know that

$$\begin{aligned}\mathbb{E}\left[\widetilde{M}_{n/2+h(n)}(n\mathbf{1}_{I(\infty)},\tau(z))\right] - \mathbb{E}\left[M_{n/2+h(n)}\left(R_{n/2-\alpha_2\frac{h(n)}{2A}},\tau(z)\right)\right] \\ = n\mathbb{E}\left[M_{n/2+h(n)}(I(\infty),\tau(z))\right] - \mathbb{E}\left[M_{n/2+h(n)}\left(R_{n/2-\alpha_2\frac{h(n)}{2A}},\tau(z)\right)\right] \\ = \frac{\alpha_2}{A}h^2(n) + O(h(n)).\end{aligned}$$

Moreover, it follows from the same computations as in Section 6.1 that

$$\sum_{y\in R_{n/2}} \mathbb{P}_y(S(H(\Sigma_1)) \in \tau(z))^2 \le ch(n)^3.$$

Taking successively A and α_2 large enough, and proceeding exactly in the same spirit as in [1, Section 4.3], it follows from Lemma 6.6 that $\mathbb{P}\left(M^*_{n/2+h(n)}(n\mathbf{1}_{I(\infty)},\tau(z)) > \frac{\beta}{2}h^2(n)\right)$ is negligeable compared to any power of n^{-1}. This concludes the proof of (24).

6.3 The lower bound for $A_n^*[\infty]$ and $A_n^\dagger[\infty]$

Since $A_n^*[\infty]$ and $A_n^\dagger[\infty]$ are equally distributed, we only deal with $A_n^*[\infty]$. Recall that the aggregate $A_n^*[\infty]$ is generated by random walks starting from $\{0\}\times\mathbb{Z}$ and by a family of independent Poisson random variables $(N_i, i\in\mathbb{Z})$ with parameter n. In what follows, we denote by η the initial configuration of $A_n^*[\infty]$, *i.e.*

$$\eta = \sum_{i\in\mathbb{Z}} N_i\mathbf{1}_{\{(0,i)\}},$$

and we simply write $A(\eta)$ instead of $A_n^*[\infty]$ to make explicit the dependence in η. Given a realization of η, we recall that $M_r(\eta,B)$ (resp. $W_r(\eta,B)$) denotes the number of random walks (resp. particles) with initial configuration η, that hit B before or when they exit R_r. The proof of Theorem 6.1, (ii) will be sketched because it relies on a simple adaptation of the proof of Theorem 6.1.

We proceed in the same spirit as in Section 6.1. The only difference is that, for a tile $\tau\in\mathscr{T}_{k\log(n)}$, we have to consider the random variables $M_{k\log(n)}(\eta,\tau)$ and $W_{k\log(n)}(\eta,\tau)$ instead of $M_{k\log(n)}(n\mathbf{1}_{I(\infty)},\tau)$ and $W_{k\log(n)}(n\mathbf{1}_{I(\infty)},\tau)$, respectively. Notice that, for almost all realization of η, the random variable

$M_{k\log(n)}(\eta,\tau)$ has a finite expectation. Indeed, denoting by $\mathbb{E}[\cdot|\eta]$ the conditional expectation w.r.t. η, we have

$$\begin{aligned}\mathbb{E}\left[M_{k\log(n)}(\eta,\tau)\right] &= \mathbb{E}\left[\mathbb{E}[M_{k\log(n)}(\eta,\tau)|\eta]\right] \\ &= \mathbb{E}\left[\sum_{i\in\mathbb{Z}} N_i \mathbb{P}_{(0,i)}(S(H_{k\log(n)})\in\tau)\right] \\ &= n\sum_{i\in\mathbb{Z}} \mathbb{P}_{(0,i)}(S(H_{k\log(n)})\in\tau) \\ &= n\mathbb{E}\left[M_{k\log(n)}(I(\infty),\tau)\right] \\ &= \#\tau\frac{n}{2},\end{aligned}$$

where the last line comes from (22).

Now, to apply Lemma 6.2, we need an estimate which is similar to (21). The number $\mu(\tau)$, as defined in (20), has to be replaced with the random variable

$$\mu(\tau|\eta) = \mathbb{E}\left[M_{k\log(n)}(\eta,\tau)\middle|\eta\right] - \mathbb{E}\left[M_{k\log(n)}(R_{k\log(n)}\setminus\mathcal{Z},\tau)\right].$$

Notice that our above computations show that $\mathbb{E}\left[\mu(\tau|\eta)\right] = \mu(\tau)$. The following lemma provides an estimate for $\mu(\tau|\eta)$.

Lemma 6.7 *Let $\varepsilon > 0$. Then w.h.p., for any $k \le \frac{n}{2\log(n)} - A$ and for any tile $\tau \in \mathcal{T}_{k\log(n)}$, we have*

$$\mu(\tau|\eta) \ge (1-\varepsilon)\mu(\tau).$$

The lower-bound in Theorem 6.1, (ii) follows from Lemma 6.7, Inequality (21) and from computations which are similar to the one appearing in Section 6.1.

Proof of Lemma 6.7. Lemma 6.7 is mainly based on the following result.

Lemma 6.8 *Let $\varepsilon \in (0,1)$, $\gamma > 0$, and let $(N_i, i \in \mathbb{Z})$ be a sequence of independent Poisson distributed random variables with parameter n. Let*

$$\mathcal{E}_n = \bigcap_{|i|\le n^\gamma} \{(1-\varepsilon)n \le N_i \le (1+\varepsilon)n)\}. \tag{30}$$

Then $(\mathcal{E}_n)_{n\ge 0}$ occurs w.h.p..

The proof of the above lemma is omitted and mainly follows from Stirling's approximation. To prove Lemma 6.7, it is sufficient to show that, for any $\varepsilon > 0$, w.h.p.

$$\mathbb{E}[M_{k\log(n)}(\eta,\tau)|\eta] \ge (1-2\varepsilon)\mathbb{E}\left[M_{k\log(n)}(\eta,\tau)\right].$$

To do it, we write

$$\mathbb{E}[M_{k\log(n)}(\eta,\tau)|\eta] = \sum_{i\in\mathbb{Z}} N_i \mathbb{P}_{(0,i)}(S(H_{k\log(n)})\in\tau).$$

According to (30), we have

$$\begin{aligned}\mathbb{E}[M_{k\log(n)}(\eta,\tau)|\eta] &\ge \sum_{|i|\le n^\gamma}(1-\varepsilon)n\mathbb{P}_{(0,i)}(S(H_{k\log(n)})\in\tau)\mathbf{1}_{\mathcal{E}_n} \\ &= (1-\varepsilon)\left(\mathbb{E}\left[M_{k\log(n)}(\eta,\tau)\right] - n\sum_{|i|>n^\gamma}\mathbb{P}_{(0,i)}(S(H_{k\log(n)})\in\tau)\right)\mathbf{1}_{\mathcal{E}_n}.\end{aligned}$$

Moreover, taking $\gamma > \alpha$ and applying [26, Proposition 8.1.5], we can easily show that for each $k \le \frac{n}{2\log(n)} - A$,

$$\sum_{|i|>n^\gamma}\mathbb{P}_{(0,i)}(S(H_{k\log(n)})\in\tau) \le cn^{-\gamma} = \frac{2c}{\#\tau}n^{-\gamma-1}\mathbb{E}\left[M_{k\log(n)}(\eta,\tau)\right].$$

Thus, for n large enough, we have

$$\mathbb{E}[M_{k\log(n)}(\eta,\tau)|\eta] \geq (1-2\varepsilon)\mathbb{E}\left[M_{k\log(n)}(\eta,\tau)\right]\mathbf{1}_{\mathscr{E}_n}.$$

This together with Lemma 6.8 concludes the proof of Lemma 6.7. □

6.4 The upper bound for $A_n^*[\infty]$ and $A_n^\dagger[\infty]$

Let $\alpha > 0$ and $\gamma > \alpha$. In what follows, we assume that we are on the event $\mathscr{E}_n$, as defined in (30). According to Lemma 6.8, we know that $\mathscr{E}_n$ occurs w.h.p.. Now, similarly to (25), we have

$$\mathbb{P}\left(A\left(\sum_{i\in\mathbb{Z}} N_i \mathbf{1}_{\{(0,i)\}}\right)\cap\mathbb{Z}_{n^\alpha} \neq A\left(\sum_{|i|\leq n^\gamma} N_i \mathbf{1}_{\{(0,i)\}}\right)\cap\mathbb{Z}_{n^\alpha}\right) \leq n^{-L}$$

for any $L > 0$ and for n large enough. To derive the upper-bound in Theorem 6.1, (ii), we follow the same lines as in Section 6.2 by considering aggregates, random walks, particles with initial configuration $\sum_{|i|\leq n^\gamma} N_i \mathbf{1}_{\{(0,i)\}}$ instead of $n\mathbf{1}_{\{0\}\times[\![-n^\gamma,n^\gamma]\!]}$. Similarly to (29), we have to prove that

$$\mathbb{P}\left(M^*_{n/2+h(n)}\left(\sum_{|i|\leq n^\gamma} N_i \mathbf{1}_{\{(0,i)\}},\tau(z)\right) > \frac{\beta}{2}h^2(n)\right) \leq n^{-L},$$

for any tile $\tau(z)$, with $z\in\mathbb{Z}_{n^\alpha}$ and $\frac{n}{2}+A\log(n)\leq|z(1)|\leq n(2n^\gamma+1)$. To do it, recall that we work on the event $\mathscr{E}_n$. Thus

$$M^*_{n/2+h(n)}\left(\sum_{|i|\leq n^\gamma} N_i \mathbf{1}_{\{(0,i)\}},\tau(z)\right) \leq M^*_{n/2+h(n)}\left(n(1+\varepsilon)\mathbf{1}_{I(\infty)},\tau(z)\right).$$

Now, proceeding exactly along the same lines as in Section 6.2, and taking ε small enough, we can easily show that

$$\mathbb{P}\left(M^*_{n/2+h(n)}\left(n(1+\varepsilon)\mathbf{1}_{I(\infty)},\tau(z)\right) > \frac{\beta}{2}h^2(n)\right) \leq n^{-L}.$$

This together with Lemma 6.7 concludes the proof of Theorem 6.1, (ii).

7 The directed IDLA forest

In this section, we introduce a new random forest $\mathscr{F}_\infty$ spanning all $\mathbb{Z}^2$ and based on the IDLA protocol with sources on $I(\infty)=\{0\}\times\mathbb{Z}$. For any n,M, we first build in Section 7.1 a random forest $\mathscr{F}(A_n^\dagger[M])$ w.r.t. the aggregate $A_n^\dagger[M]$. By letting $M\to\infty$ (vertical limit), we define a random forest $\mathscr{F}_n$ as the limit of the sequence $(\mathscr{F}(A_n^\dagger[M]))_{M\geq 0}$. The existence of $\mathscr{F}_n$ is based on a non-trivial stabilization result (Proposition 7.2) which is stated in the same spirit as Theorem 2.4. The key argument to define the limiting forest $\mathscr{F}_n$ lies on the fact that the infinite aggregate $A_n^\dagger[\infty]$ is made up with finite connected components. This property prevents the existence of *chain of changes* (see Section 7.2) which could come from far levels and perturbate the evolution of $(\mathscr{F}(A_n^\dagger[M]))_{M\geq 0}$ in the neighborhood of the origin. It will be shown that the sequence $(\mathscr{F}_n)_{n\geq 1}$ is consistent (Lemma 7.3). By letting $n\to\infty$ (horizontal limit), this fact allows us to define easily our random directed IDLA forest $\mathscr{F}_\infty$. This section ends with some properties of $\mathscr{F}_\infty$ (Theorem 7.4).

7.1 Each aggregate $A_n^\dagger[M]$ generates a forest

Let $n,M\geq 1$ be fixed and let $\kappa=\sum_{|i|\leq M}\#\mathscr{N}_i([0,n])$. As in the proof of Lemma 2.1, we index all the particles starting from level $|i|\leq M$ by some integer $j=1,\ldots,\kappa$ according to their starting times $0<\tau_1<\ldots<\tau_\kappa<n$. For $j=1,\ldots,\kappa$, we denote by $A[j]$ the aggregate obtained until (or at) time τ_j. In particular, we have $A[0]=\emptyset$ and $A[\kappa]=A_n^\dagger[M]$.

We define a (finite) random forest with vertices in $A_n^\dagger[M]$ inductively as follows. First, we let $\mathscr{F}_n[M,0] = (\emptyset,\emptyset)$. Then, for some $1 \le j \le \kappa$, assume that a graph $\mathscr{F}_n[M, j-1]$ is built, with set of vertices and edges denoted by $V[j-1]$ and $E[j-1]$, respectively. Let z be the site which is added by particle j, *i.e.* $A[j] = A[j-1] \cup \{z\}$.

- If $z = (0,i)$ and if particle j is (the first one) which is sent from level i then z is the root of a new tree in the graph. In this case, we set $\mathscr{F}_n[M, j] = (V[j], E[j])$, where

$$V[j] = V[j-1] \cup \{z\} \text{ and } E[j] = E[j-1] \ .$$

- Otherwise, the site of $\{0\} \times \mathbb{Z}$ from which particle j starts already belongs to $A[j-1]$ and we let z' as the last site of $A[j-1]$ which is visited by particle j before reaching z. Then, we set $\mathscr{F}_n[M, j] = (V[j], E[j])$, where

$$V[j] = V[j-1] \cup \{z\} \text{ and } E[j] = E[j-1] \cup \{(z', z)\} \ .$$

In other words, from $\mathscr{F}_n[M, j-1]$ to $\mathscr{F}_n[M, j]$ we merely add the new site created by particle j and the (directed) edge from which this new site is reached. In what follows, we set

$$\mathscr{F}(A_n^\dagger[M]) = \mathscr{F}_n[M,\kappa] \ .$$

Figure 4 gives a realization of $\mathscr{F}(A_n^\dagger[M])$. The next lemma claims that $\mathscr{F}(A_n^\dagger[M])$ is a (finite) random forest with vertices in $A_n^\dagger[M]$.

Lemma 7.1 *Let $n, M \ge 1$. The following properties hold a.s.*

(i) The set of vertices of $\mathscr{F}(A_n^\dagger[M])$ is $A_n^\dagger[M]$;

(ii) The edges of $\mathscr{F}(A_n^\dagger[M])$ are edges of the square lattice $\mathbb{Z}^2$ plus a direction;

(iii) The random graph $\mathscr{F}(A_n^\dagger[M])$ is a (finite) union of directed trees with roots in $I(\infty)$.

Proof of Lemma 7.1. Properties (i)-(ii) are satisfied by construction. Property comes from the fact that the random graph $\mathscr{F}(A_n^\dagger[M])$ contains no loop since a site cannot be added twice to the aggregate. □

7.2 Absence of infinite chain of changes and stabilization

Let $n, M \ge 0$. For both constructions of aggregates $A_n[M]$ and $A_n^*[M]$, recall that the particles are sent w.r.t. the usual order, *i.e.* from levels 0, thus ± 1, ± 2 and so on by moving away from the origin step by step. Hence, these aggregates are first built around the origin and thus grow mainly from their upper and lower parts. In Theorem 2.4, we proved that the particles which come from far levels cannot visit a neighborhood of the origin in this setting.

The situation is quite different for aggregates $A_n^\dagger[M]$, $M \ge 0$. Indeed, particles are sent according to the clocks given by $(\mathcal{N}_i)_{i\in\mathbb{Z}}$ and some pathological situations may occur. Basically, for $M' > M \ge 0$, any particle sent from a level $M < i \le M'$ works for (the growth of) $A_n^\dagger[M']$ but not for $A_n^\dagger[M]$. Hence, this particle may create several discrepancies between the forests $\mathscr{F}(A_n^\dagger[M])$ and $\mathscr{F}(A_n^\dagger[M'])$ through a mechanism called a *chain of changes* that we describe now.

Assume that a particle, referred to as particle 1, starts at time $t_1 \in (0,n)$ (from a level $M < |i_1| \le M'$) and adds a site z_1 to $A_{t_1-}^\dagger[M']$. The aggregate at time t_1 becomes

$$A_{t_1}^\dagger[M'] = A_{t_1-}^\dagger[M'] \cup \{z_1\}$$

while $A_{t_1}^\dagger[M]$ remains unchanged. In the above equation, the set $A_{t_1-}^\dagger[M']$ denotes the (current) aggregate produced just before sending particle 1. The site z_1 is a *discrepancy* at time t_1 between aggregates $A_{t_1}^\dagger[M]$

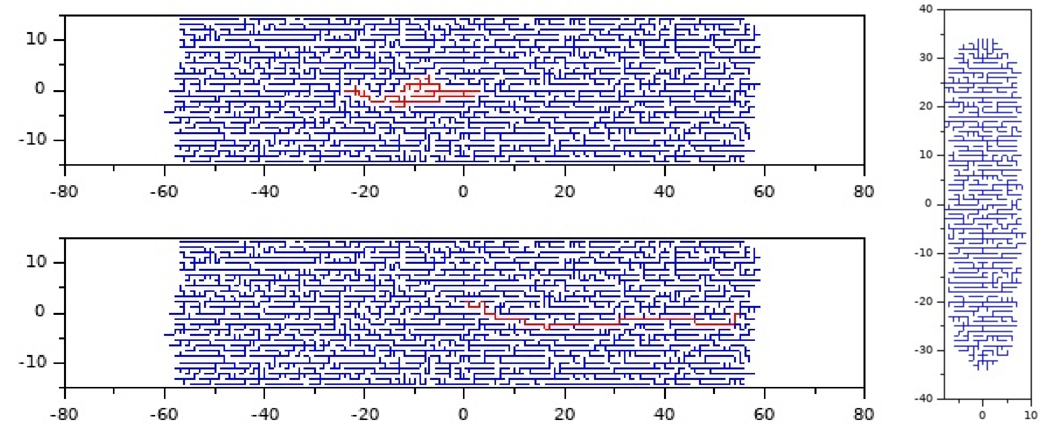

Figure 4: The top left corner depicts a realization of the random forest $\mathscr{F}(A^{\dagger}_{120}[120])$ with particles starting from levels $|i| \le 120$ and during the time interval $[0,120]$, viewed through the strip $\mathbb{Z}_{15}$. The tree of $\mathscr{F}(A^{\dagger}_{120}[120])$ containing the origin is in red. A second realization of $\mathscr{F}(A^{\dagger}_{120}[120])$ is given on the bottom left corner. The branch passing through $(55,0)$ (red) remains close to the x-axis and comes from the source $(0,2)$. A realization of $\mathscr{F}(A^{\dagger}_{15}[30])$ is depicted on the right. One can imagine that the vertical edges at the top of $\mathscr{F}(A^{\dagger}_{15}[30])$ are due to border effects and will not be present in the limiting forest $\mathscr{F}_{15}$.

and $A^{\dagger}_{t_1}[M']$. If there is no other particles starting from a level $|i| \le M$, at time $t \in (t_1, n)$ and going through z_1 then, at the final time n, the site z_1 constitutes a discrepancy (created by particle 1) between the aggregates $A^{\dagger}_n[M]$ and $A^{\dagger}_n[M']$. It also defines a discrepancy between the forests $\mathscr{F}(A^{\dagger}_n[M])$ and $\mathscr{F}(A^{\dagger}_n[M'])$. Otherwise, we set

$$t_2 = \min\left\{t \in (t_1, n) : \text{a particle, starting from a level } |i| \le M \text{ at time } t\text{, goes through } z_1\right\}.$$

The particle starting from time t_2 is referred to as particle 2. This particle works for both aggregates. By definition, it adds the site z_1 to $A^{\dagger}_{t_2-}[M]$, so that the aggregate at time t_2 becomes $A^{\dagger}_{t_2}[M] = A^{\dagger}_{t_2-}[M] \cup \{z_1\}$. Thus, it continues its trajectory until adding a site z_2 (but only) to $A^{\dagger}_{t_2-}[M']$ which then becomes $A^{\dagger}_{t_2}[M'] = A^{\dagger}_{t_2-}[M'] \cup \{z_2\}$. At this time:

- the site z_1 now belongs to both aggregates but it could be reached via two different edges respectively in $A^{\dagger}_{t_2}[M]$ and $A^{\dagger}_{t_2}[M']$ so that the forests $\mathscr{F}(A^{\dagger}_n[M])$ and $\mathscr{F}(A^{\dagger}_n[M'])$ may differ at the edge leading to z_1;
- the site z_2 is become a discrepancy between both aggregates at time t_2. This discrepancy is generated via a relay between particle 1 and particle 2.

Thus, we iterate this step while the current discrepancy is visited by a new particle starting from a level $|i| \le M$. After a random number ℓ of steps (a.s. finite), we finally get the set of possible discrepancies between the forests $\mathscr{F}(A^{\dagger}_n[M])$ and $\mathscr{F}(A^{\dagger}_n[M'])$, generated by particle 1. This set consists of edges leading to $z_1, \ldots, z_\ell$ and the final vertex z_ℓ itself. The mechanism producing this set of discrepancies is called a chain of changes, initiated by particle 1, between the forests $\mathscr{F}(A^{\dagger}_n[M])$ and $\mathscr{F}(A^{\dagger}_n[M'])$. Notice that the aggregates $A^{\dagger}_n[M]$ and $A^{\dagger}_n[M']$ may have other chains of changes initiated by other particles starting from levels $M < |i| \le M'$.

Roughly speaking, the existence of an infinite chain of changes involving an infinite number of relaying particles and initiated by a "Big Bang particle", *i.e.* a particle coming from a level arbitrarily far from the origin and born arbitrarily early, could modify infinitely often (in M) the forests $\mathscr{F}(A^{\dagger}_n[M])$, for $M \ge 0$, in the neighborhood of the origin. Proving that such infinite chain of changes does not exist with probability 1 leads to the next stabilization result and to the existence of the random forest $\mathscr{F}_n$.

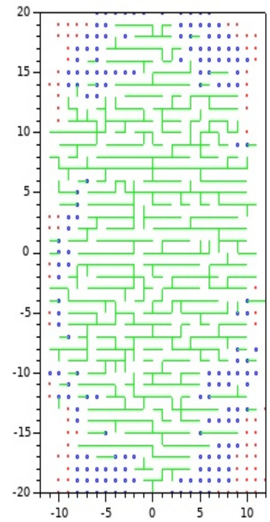

Figure 5: Realizations of the forests $\mathscr{F}(A^{\dagger}_{20}[20])$ and $\mathscr{F}(A^{\dagger}_{20}[50])$, defined on the same time interval $[0,20]$, with different sets of sources, and restricted to the strip $\mathbb{Z}_{20}$, are depicted. The associated aggregates are coupled in the sense that they are based on the same clocks and random walks with level $|i| \le 20$. In particular, $A^{\dagger}_{20}[20]$ is included in $A^{\dagger}_{20}[50]$. The edges created in both forests by the same particles are depicted in green. The red points are vertices of $A^{\dagger}_{20}[50] \setminus A^{\dagger}_{20}[20]$. The blue circles represent vertices in $A^{\dagger}_{20}[20]$ (and then also in $A^{\dagger}_{20}[50]$) which are reached by different particles in both aggregates and whose corresponding edges may differ in both forests $\mathscr{F}(A^{\dagger}_{20}[20])$ and $\mathscr{F}(A^{\dagger}_{20}[50])$. These blue vertices are possible discrepancies generated by chains of changes between forests $\mathscr{F}(A^{\dagger}_{20}[20])$ and $\mathscr{F}(A^{\dagger}_{20}[50])$.

Proposition 7.2 *Let $K \ge 1$. Then, a.s. there exists some (random) integer $M_0(K)$ such that, for any $M' > M \ge M_0(K)$, we have*

$$\mathscr{F}(A^{\dagger}_n[M]) \cap \mathbb{Z}_K = \mathscr{F}(A^{\dagger}_n[M']) \cap \mathbb{Z}_K \,.$$

Proposition 7.2 allows us to define a.s. the random forest $\mathscr{F}_n$ as the increasing union

$$\mathscr{F}_n = \bigcup_{K \ge 0} \uparrow \mathscr{F}(A^{\dagger}_n[M_0(K)]) \cap \mathbb{Z}_K,$$

with vertex set $V(\mathscr{F}_n) = A^{\dagger}_n[\infty]$.

Proof of Proposition 7.2. Let $K \ge 1$. Corollary 5.2 provides the almost sure existence of an infinite subset $\mathscr{L} \subset \mathbb{Z}$ such that, for any $i \in \mathscr{L}$,

$$A^{\dagger}_n[\infty] \cap (\mathbb{Z} \times \{i\}) = \emptyset \,.$$

Now, let us define $M_0(K)$ as the following (random) integer:

$$M_0(K) = \max\Big\{ \min\{i \in \mathscr{L} : i \ge K\}, -\max\{i \in \mathscr{L} : i \le -K\} \Big\} \,.$$

For any $M' > M \ge M_0(K)$, there is no chain of changes initiated by a particle starting from some level $M < |i| \le M'$ between the forests $\mathscr{F}(A^{\dagger}_n[M])$ and $\mathscr{F}(A^{\dagger}_n[M'])$ which visits the strip $\mathbb{Z}_K$. This implies that, for any $M' > M \ge M_0(K)$, the forests $\mathscr{F}(A^{\dagger}_n[M])$ and $\mathscr{F}(A^{\dagger}_n[M'])$ coincide on the strip $\mathbb{Z}_K$.

□

The following lemma is a direct consequence of our construction.

Lemma 7.3 *The sequence of forests* $(\mathscr{F}_n)_{n\geq 1}$ *is consistent in the sense that*

$$a.s.\ \forall n \geq 1, \quad V(\mathscr{F}_n) \subset V(\mathscr{F}_{n+1}) \quad \text{and} \quad E(\mathscr{F}_n) \subset E(\mathscr{F}_{n+1}) \, . \tag{31}$$

Moreover, for any $n \geq 1$, $\mathscr{F}_n$ *is a.s. made up with infinitely many directed trees rooted at* $I(\infty)$.

Proof of Lemma 7.3. Inclusion (31) is an immediate consequence of our construction (the same argument was used in the proof of Lemma 2.1 (ii)). The fact that $\mathscr{F}_n$ is made up with infinitely many trees is due to Corollary 5.2 which asserts that $A_n^\dagger[\infty] = V(\mathscr{F}_n)$ is itself made up with an infinite number of disjoint connected components. □

7.3 The directed infinite-volume IDLA forest

Equation (31) allows us to define a.s. the *directed infinite-volume IDLA forest*

$$\mathscr{F}_\infty = \bigcup_{n\geq 1} \uparrow \mathscr{F}_n.$$

The following theorem states the main properties of $\mathscr{F}_\infty$.

Theorem 7.4 *The directed infinite-volume IDLA forest* $\mathscr{F}_\infty$ *satisfies the following properties:*

(i) A.s. the random forest $\mathscr{F}_\infty$ *spans the whole set* $\mathbb{Z}^2$, *i.e.* $V(\mathscr{F}_\infty) = \mathbb{Z}^2$, *and its edge set* $E(\mathscr{F}_\infty)$ *is made up with edges of the square lattice* $\mathbb{Z}^2$ *plus a direction;*

(ii) A.s. $\mathscr{F}_\infty$ *is a countable (infinite) union of directed trees rooted at* $I(\infty)$*;*

(iii) the distributions of $\mathscr{F}_\infty$ *and* $\mathscr{F}_n$, $n \geq 1$ *are invariant w.r.t. vertical translations;*

(iv) the distributions of $\mathscr{F}_\infty$ *and* $\mathscr{F}_n$, $n \geq 1$ *are mixing w.r.t. vertical translations;*

(v) the distributions of $\mathscr{F}_\infty$ *and* $\mathscr{F}_n$, $n \geq 1$ *are symmetric invariant w.r.t. the* y*-axis and w.r.t.* $S_{k/2}$, $k \in \mathbb{Z}$.

Item (iii) means that the forests $\mathscr{F}_\infty$ and $\mathscr{F}_n$, $n \geq 1$, are no longer sensitive to the border effects whereas the $\mathscr{F}(A_n^\dagger[M])$'s are (see the right-hand side of Figure 4). This result is one of the original motivation of this paper. Let us also remark that Item (iii) is not an immediate consequence of the translation invariance of the aggregates $A_n^\dagger[\infty]$, $n \geq 1$, (see Proposition 2.2) since $\mathscr{F}_\infty$ actually is a richer object including edges which depend on the trajectories of particles.

Proof of Theorem 7.4. The fact that $E(\mathscr{F}_\infty)$ is made up with edges of the square lattice $\mathbb{Z}^2$ plus a direction and Item (ii) are direct consequences of Proposition 7.2 and Lemma 7.3. To prove that $V(\mathscr{F}_\infty) = \mathbb{Z}^2$, we use the lower bound appearing in our Shape Theorem. Indeed, Theorem 6.1 implies that, for any finite subset S of $\mathbb{Z}^2$,

$$\mathbb{P}\left(S \subset A_n^\dagger[\infty]\right) \underset{n\to\infty}{\longrightarrow} 1 \, . \tag{32}$$

Since $\mathbb{P}(S \subset V(\mathscr{F}_\infty)) \geq \mathbb{P}(S \subset V(\mathscr{F}_n)) = \mathbb{P}\left(S \subset A_n^\dagger[\infty]\right)$ for any n, we get $\mathbb{P}(S \subset V(\mathscr{F}_\infty)) = 1$. This concludes the proof of (i).

Let us prove (iii). Let $K \subset \mathbb{R}^2$ be a compact set and let $k \geq 0$. Recall that $\tau_k\mathscr{F}_\infty$ (resp. $\tau_k\mathscr{F}_n$) denotes the directed IDLA forest $\mathscr{F}_\infty$ (resp. $\mathscr{F}_n$) translated w.r.t. the vector $(0, k)$. According to [32, Theorem 2.1.3], it is is sufficient to prove that both forests $\mathscr{F}_\infty$ and $\tau_k\mathscr{F}_\infty$ (resp. $\mathscr{F}_n$ and $\tau_k\mathscr{F}_n$) have the same probability to intersect K, where all these graphs are seen as subsets of $\mathbb{R}^2$. First, assume that this holds for the $\mathscr{F}_n$'s, *i.e.*

$$\mathbb{P}(\mathscr{F}_n \cap K \neq \emptyset) = \mathbb{P}(\tau_k\mathscr{F}_n \cap K \neq \emptyset) \, . \tag{33}$$

Notice that the random set $\mathscr{F}_n \cap C$ is $A_n^{\dagger}[\infty] \cap (C \oplus B(0,1))$ measurable for any compact set C. Now, let $\varepsilon > 0$. Thanks to (32) applied to the set $(C \oplus B(0,1)) \cap \mathbb{Z}^2$, with $C = K$ and $C = \tau_{-k}K$, there exists an integer n_0 such that

$$\left|\mathbb{P}\left(\mathscr{F}_\infty \cap K \neq \emptyset\right) - \mathbb{P}\left(\mathscr{F}_{n_0} \cap K \neq \emptyset\right)\right| \text{ and } \left|\mathbb{P}\left(\tau_k\mathscr{F}_\infty \cap K \neq \emptyset\right) - \mathbb{P}\left(\tau_k\mathscr{F}_{n_0} \cap K \neq \emptyset\right)\right|$$

are both smaller than ε. By (33), we deduce the translation invariance in distribution for the limiting forest $\mathscr{F}_\infty$.

It then remains to prove (33) for any $n \geq 1$. Let $n \geq 1$ and M large enough so that $K \cap \mathbb{Z}^2 \subset \mathbb{Z}_M$. By Proposition 7.2, there exists a (deterministic) integer $M_0 = M_0(n,\varepsilon)$ such that, with probability at least $1-\varepsilon$, we have for any $M' \geq M_0$,

$$\mathscr{F}_n \cap \mathbb{Z}_M = \mathscr{F}(A_n^{\dagger}[M']) \cap \mathbb{Z}_M \text{ and } \tau_k\mathscr{F}_n \cap \mathbb{Z}_M = \tau_k\mathscr{F}(A_n^{\dagger}[M']) \cap \mathbb{Z}_M \,. \tag{34}$$

Then,

$$\begin{aligned}&\left|\mathbb{P}\left(\mathscr{F}_n \cap K \neq \emptyset\right) - \mathbb{P}\left(\tau_k\mathscr{F}_n \cap K \neq \emptyset\right)\right| \\ &\qquad\leq \left|\mathbb{P}\left(\mathscr{F}(A_n^{\dagger}[M_0]) \cap K \neq \emptyset\right) - \mathbb{P}\left(\tau_k\mathscr{F}(A_n^{\dagger}[M_0]) \cap K \neq \emptyset\right)\right| + 2\varepsilon \,.\end{aligned}$$

Now let us increase the forests $\mathscr{F}(A_n^{\dagger}[M_0])$ and $\tau_k\mathscr{F}(A_n^{\dagger}[M_0])$ as follows. Let $\mathfrak{F}_1$ be the random forest obtained by sending the particles used to build $\mathscr{F}(A_n^{\dagger}[M_0])$ plus those from levels $M_0+1,\ldots,M_0+k$ (according to their own PPP's). Besides, let $\mathfrak{F}_2$ be the translation w.r.t. the vector $(0,k)$ of the forest induced by the particles used to build $\mathscr{F}(A_n^{\dagger}[M_0])$ plus those from levels $-M_0-k,\ldots,-M_0-1$. On the one hand, using (34), we know that the forests $\mathfrak{F}_1$ and $\mathfrak{F}_2$ coincide respectively with $\mathscr{F}(A_n^{\dagger}[M_0])$ and $\tau_k\mathscr{F}(A_n^{\dagger}[M_0])$ on the strip $\mathbb{Z}_M$, with probability at least $1-\varepsilon$. This implies

$$\left|\mathbb{P}\left(\mathscr{F}_n \cap K \neq \emptyset\right) - \mathbb{P}\left(\tau_k\mathscr{F}_n \cap K \neq \emptyset\right)\right| \leq \left|\mathbb{P}\left(\mathfrak{F}_1 \cap K \neq \emptyset\right) - \mathbb{P}\left(\mathfrak{F}_2 \cap K \neq \emptyset\right)\right| + 4\varepsilon \,.$$

On the other hand, the forests $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are produced by the same IDLA protocol from the same levels $i = -M_0,\ldots,M_0+k$ and during the time interval $[0,n]$. Of course, the corresponding clocks and random walks used for $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are different but the Abelian property states they have the same distribution. This concludes the proof of (iii).

Let us prove (iv). By [32, Theorem 9.3.2], it is enough to check that

$$\lim_{k\to\infty} \mathbb{P}\left(\mathscr{F}_\infty \cap (C_1 \cup \tau_k C_2) = \emptyset\right) = \mathbb{P}\left(\mathscr{F}_\infty \cap C_1 = \emptyset\right)\mathbb{P}\left(\mathscr{F}_\infty \cap C_2 = \emptyset\right)$$

holds for any compact sets C_1 and C_2 in $\mathbb{R}^2$. Let $\varepsilon > 0$ and C_1, C_2 be two compact sets in $\mathbb{R}^2$. Let $r > 0$ be such that $C_1 \cup C_2$ is included in the ball $B(0,r-1)$. By (32) we can choose n so that

$$\mathbb{P}\left(B(0,r) \cap \mathbb{Z}^2 \subset A_n^{\dagger}[\infty]\right) \geq 1-\varepsilon \,.$$

Replacing $r-1$ with r in the above probability allows us to take into account all the edges incident to the vertices of $(C_1 \cup C_2) \cap \mathbb{Z}^2$. On the above event, $\mathscr{F}_\infty \cap C_i$ and $\mathscr{F}_n \cap C_i$ are equal, for $i \in \{1,2\}$. The translation invariance in distribution of $A_n^{\dagger}[\infty]$ (Proposition 2.2) implies that the probability of the event $\{\tau_k(C_1 \cup C_2) \cap \mathbb{Z}^2 \subset A_n^{\dagger}[\infty]\}$ is also larger than $1-\varepsilon$, for any integer k (which does not depend on ε). Hence, with probability larger than $1-2\varepsilon$, $\mathscr{F}_\infty \cap (C_1 \cup \tau_k C_2)$ and $\mathscr{F}_n \cap (C_1 \cup \tau_k C_2)$ are equal. Henceforth,

$$\begin{aligned}&\left|\mathbb{P}\left(\mathscr{F}_\infty \cap (C_1 \cup \tau_k C_2) = \emptyset\right) - \mathbb{P}\left(\mathscr{F}_\infty \cap C_1 = \emptyset\right)\mathbb{P}\left(\mathscr{F}_\infty \cap C_2 = \emptyset\right)\right| \\ &\qquad\leq \left|\mathbb{P}\left(\mathscr{F}_n \cap (C_1 \cup \tau_k C_2) = \emptyset\right) - \mathbb{P}\left(\mathscr{F}_n \cap C_1 = \emptyset\right)\mathbb{P}\left(\mathscr{F}_n \cap C_2 = \emptyset\right)\right| + 4\varepsilon \,.\end{aligned}$$

Let us consider the event $D_{n,k}$ defined by "there is no connected component of $A_n^{\dagger}[\infty]$ overlapping simultaneously C_1 and $\tau_k C_2$". By Corollary 5.2, given n, the probability of $D_{n,k}$ is larger than $1-\varepsilon$ for any k

large enough. For any such integer k, the events $\{\mathscr{F}_n \cap C_1 = \emptyset\}$ and $\{\mathscr{F}_n \cap \tau_k C_2 = \emptyset\}$ are independent on $D_{n,k}$. Thus

$$\mathbb{P}\left(\{\mathscr{F}_n \cap (C_1 \cup \tau_k C_2) = \emptyset\} \cap D_{n,k}\right) = \mathbb{P}\left(\{\mathscr{F}_n \cap C_1 = \emptyset\} \cap D_{n,k}\right) \mathbb{P}\left(\{\mathscr{F}_n \cap \tau_k C_2 = \emptyset\} \cap D_{n,k}\right) .$$

Since $\mathscr{F}_n$ is invariant in distribution w.r.t. vertical translations, we have

$$|\mathbb{P}(\mathscr{F}_n \cap (C_1 \cup \tau_k C_2) = \emptyset) - \mathbb{P}(\mathscr{F}_n \cap C_1 = \emptyset)\mathbb{P}(\mathscr{F}_n \cap C_2 = \emptyset)| \leq 3\varepsilon.$$

Therefore

$$|\mathbb{P}(\mathscr{F}_\infty \cap (C_1 \cup \tau_k C_2) = \emptyset) - \mathbb{P}(\mathscr{F}_\infty \cap C_1 = \emptyset)\mathbb{P}(\mathscr{F}_\infty \cap C_2 = \emptyset)| \leq 7\varepsilon .$$

(v) By construction, the forests $\mathscr{F}_n$ and $\mathscr{F}_\infty$ are invariant under symmetries w.r.t. x-axis and y-axis. Since $S_{k/2} = \tau_k \circ S_0$ for any $k \in \mathbb{Z}$, it follows from (iii) that $\mathscr{F}_n$ and $\mathscr{F}_\infty$ are invariant under any horizontal symmetries. □

7.4 Conjectures

We end this section with three conjectures about the directed IDLA forest $\mathscr{F}_\infty$.

7.4.1 Conjecture 1

Consider the infinite IDLA tree $\mathscr{T}_\infty$ rooted at the origin (as described in the Introduction) and focus on its branches only through the ball $B((n,0),R)$, where R is fixed and where n is intended to go to infinity. The idea is that the radial character of its branches restricted to $B((n,0),R)$ should fade away as $n \to \infty$ since R is constant. Hence, the infinite IDLA tree $\mathscr{T}_\infty$ restricted to the ball $B((n,0),R)$ should look like to a directed forest with direction the vector $(-1,0)$; as if, roughly speaking, the root of the tree is sent to infinity according to the vector $(-1,0)$. Approximating the distribution of a tree (locally and far away from the root) by the distribution of a directed forest is classical in the literature, see *e.g.* [5, 11]. We conjecture that the directed IDLA forest $\mathscr{F}_\infty$ is the natural candidate to approximate the distribution of the tree $\mathscr{T}_\infty$. More precisely, we conjecture that the infinite IDLA tree $\mathscr{T}_\infty$ and the directed IDLA forest $\mathscr{F}_\infty$, both restricted to $B((n,0),R)$, are asymptotically equally distributed.

7.4.2 Conjecture 2

Directed forests in $\mathbb{R}^d$ may coalesce or not according to the dimension d, see *e.g.* [13, 16, 31]. But whatever the dimension, in the backward sense (*i.e.* in the opposite direction to the one in which the branches coalesce), branches are finite (for the models previously cited) so that the directed forests do not contain bi-infinite branches. In regards to the directed IDLA forest $\mathscr{F}_\infty$, branches coalesce when they get closer to the source axis $I(\infty)$ so that the backward sense is moving away from $I(\infty)$. Then, we conjecture that the same holds for the directed IDLA forest $\mathscr{F}_\infty$: a.s. all the (infinitely many) trees making up the directed IDLA forest and rooted on the axis $I(\infty)$, are finite. See, for instance, the tree associated with the origin in Figure 4. The main difficulties to get such a result are the lack of free energy property and the fact that $\mathscr{F}_\infty$ is not invariant in distribution w.r.t. *horizontal* translations.

Notice that, in [9], the authors proved in that all the trees are a.s. finite in the forest they defined. The counterpart of this result for our IDLA forest is an open and challenging question.

7.4.3 Conjecture 3

A challenging question about the infinite IDLA tree $\mathscr{T}_\infty$ is the existence of (many) infinite branches with asymptotic directions. Following the strategy initiated by Howard and Newman [17], the key point would be to control the fluctuations w.r.t. the segment $[0,z]$ (in $\mathbb{R}^2$) of the branch in $\mathscr{T}_\infty$ joining the root 0 to any given vertex $z \in \mathbb{Z}^2$, with $|z|_2 \gg 1$. This question is difficult for various reasons. First, any branch γ of the IDLA tree $\mathscr{T}_\infty$ is not produced by a single particle but by many particles, each of them adding exactly one edge depending on the shape of the current aggregate. Moreover, this random subgraph of the lattice $\mathbb{Z}^2$ is radial since its branches are directed to the origin and then it does not satisfy any useful invariance properties in distribution.

An intermediate step would be to control the fluctuations of branches in the directed IDLA forest $\mathscr{F}_\infty$ which presents the advantage to be invariant in distribution (and even mixing) w.r.t. vertical translations. Let $n \geq 1$ and let $(z_i)_{0\leq i\leq\kappa_n}$ be the branch joining a source z_0 on $I(\infty)$ to $z_{\kappa_n} = (n,0)$. As an illustration, Figure 4 depicts the branch associated with $(55,0)$. Denote by Δ_n the maximal fluctuation (or maximal deviation) of the branch $(z_i)_{0\leq i\leq\kappa_n}$ w.r.t. the x-axis, *i.e.*

$$\Delta_n = \max_{0\leq i\leq\kappa_n} z_i(2).$$

We conjecture that with probability tending to 1 with n, the maximal deviation Δ_n is negligible w.r.t. n. Thus, from a macroscopic point of view, we expect that the branch $(z_i)_{0\leq i\leq\kappa_n}$ asymptotically merges with the horizontal axis.

Acknowledgements

This work was partially supported by the French ANR grant ASPAG (ANR-17-CE40-0017), by the French GDR GeoSto (GDR-3477), and by the French PEPS-JCJC 2019.

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