

Fixed-domain asymptotic properties of maximum composite likelihood estimators for max-stable Brown-Resnick random fields

Nicolas CHENAVIER* and Christian Y. ROBERT†

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Abstract

Likelihood inference for max-stable random fields is in general impossible because their finite-dimensional probability density functions are unknown or cannot be computed efficiently. The weighted composite likelihood approach that utilizes lower dimensional marginal likelihoods (typically pairs or triples of sites that are not too distant) is rather favored. In this paper, we consider the family of spatial max-stable Brown-Resnick random fields associated with isotropic fractional Brownian fields. We assume that the sites are given by only one realization of a homogeneous Poisson point process restricted to $\mathbf{C} = (-1/2, 1/2]^2$ and that the random field is observed at these sites. As the intensity increases, we study the asymptotic properties of the composite likelihood estimators of the scale and Hurst parameters of the fractional Brownian fields using different weighting strategies: we exclude either pairs that are not edges of the Delaunay triangulation or triples that are not vertices of triangles.

Keywords: Brown-Resnick random fields, Composite likelihood estimators, Fixed-domain asymptotics, Gaussian random fields, Poisson random sampling, Delaunay triangulation.

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1 Introduction

Gaussian random fields are widely used to model spatial data because their finite-dimensional distributions are only characterized by the mean and covariance functions. In general it is assumed that these functions

*Université du Littoral Côte d'Opale, 50 rue F. Buisson 62228 Calais. nicolas.chenavier@univ-littoral.fr

†1. Laboratory in Finance and Insurance - LFA CREST - Center for Research in Economics and Statistics, ENSAE, Palaiseau, France; 2. Université de Lyon, Université Lyon 1, Institut de Science Financière et d'Assurances, 50 Avenue Tony Garnier, F-69007 Lyon, France. chrobert@ensae.fr

belong to some parametric models which leads to a parametric estimation problem. When extreme value phenomena are of interest and meaningful spatial patterns can be discerned, max-stable random field models are preferred to describe such phenomena. However, likelihood inference is challenging for such models because their corresponding finite-dimensional probability density functions are unknown or cannot be computed efficiently. In this paper we study composite likelihood estimators in a fixed-domain asymptotic framework for a widely used class of stationary max-stable random fields: the Brown-Resnick random fields. As a preliminary, we provide brief reviews of work on maximum likelihood estimators and on composite likelihood estimators for Gaussian random fields under fixed-domain asymptotics and present max-stable random fields with their canonical random tessellations.

1.1 Maximum likelihood estimators for Gaussian random fields under fixed-domain asymptotics

The fixed-domain asymptotic framework is sometimes called infill asymptotics (Stein (1999), Cressie (1993)) and corresponds to the case where more and more data are observed in some fixed bounded sampling domain (usually a region of \mathbf{R}^d , $d \in \mathbf{N}_*$). Within this framework, the maximum likelihood estimators (MLE) of the covariance parameters of Gaussian random fields have been deeply studied in the last three decades.

It is noteworthy that two types of covariance parameters have to be distinguished: microergodic and non-microergodic parameters. A parameter is said to be microergodic if, for two different values of it, the two corresponding Gaussian measures are orthogonal (Ibragimov and Rozanov (1978), Stein (1999)). It is non-microergodic if, even for two different values of it, the two corresponding Gaussian measures are equivalent. Non-microergodic parameters cannot be estimated consistently under fixed-domain asymptotics. No general results are available for the asymptotic properties of microergodic MLE. Most available results are specific to particular covariance models.

The initial covariance model that has been studied is the exponential model with its variance and scale parameters. When $d = 1$, only a reparameterized quantity obtained from the variance and scale parameters is microergodic (Ying (1991)). It is shown that the MLE of this microergodic parameter is consistent and asymptotically normal. When $d > 1$ and for a separable exponential covariance function, all the covariance parameters are microergodic, and the asymptotic normality of the MLE is proved in Ying (1993). Other results are also given in van der Vaart (1996) and in Abt and Welch (1998).

The Matern covariance model (Matern (1960)) is very popular in spatial statistics for its flexibility with respect to the parameterization of smoothness (in the mean square sense) of the underlying Gaussian field. This model has three parameters: the variance, the scale and the smoothness parameters. Zhang (2004) showed that when the smoothness parameter is known and fixed, not all parameters can be estimated

consistently when $d = 1, 2, 3$; only the ratio of variance and scale parameters (to the power of the smoothing parameter) is microergodic. Kaufman and Shaby (2013) proved strong consistency and provided the asymptotic distributions of the microergodic parameters when estimating jointly the scale and variance parameters (see also Du et al. (2009) and Wang and Loh (2011) for tapered MLE as well as Loh et al. (2021) for quadratic variation estimators). For $d = 5$, Anderes (2010) proved the orthogonality of two Gaussian measures with different Matern covariance functions. In this case, all the parameters are microergodic. The case $d = 4$ is still open.

More recently Bevilacqua et al. (2019) considered the generalized Wendland (GW) covariance model. They characterized conditions for equivalence of two Gaussian measures and they established strong consistency and asymptotic normality of the MLE for the microergodic parameters associated with the GW covariance model. Bevilacqua and Faouzi (2019) considered the generalized Cauchy covariance model that is able to separate the characterizations of the fractal dimension and the long range dependence of the associated Gaussian random fields. They also characterized conditions for the equivalence of two Gaussian measures, and established strong consistency and asymptotic normality of the MLE of the microergodic parameters.

1.2 Maximum composite likelihood estimators for Gaussian random fields under fixed-domain asymptotics

From a theoretical point of view, the maximum likelihood method is the best approach for estimating the covariance parameters of a Gaussian random field. Nevertheless, the evaluation of the likelihood function under the Gaussian assumption requires a computational burden of order $O(n^3)$ for n observations (because of the inversion of the $n \times n$ covariance matrix), making this method computationally impractical for large datasets. The composite likelihood (CL) methods rather use objective functions based on the likelihood of lower dimensional marginal or conditional events (Varin et al. (2011)). These methods are generally appealing when dealing with large data sets or when it is difficult to specify the full likelihood, and provide estimation methods with a good balance between computational complexity and statistical efficiency.

There is not a lot of results under fixed domain asymptotics for maximum CL estimators (MCLE). However, Bachoc et al. (2019) studied the problem of estimating the covariance parameters of a Gaussian process ($d = 1$) with exponential covariance function. They showed that the weighted pairwise maximum likelihood estimator of the microergodic parameters can be consistent, but also inconsistent, according to the objective function; e.g. the weighted pairwise conditional maximum likelihood estimator is always consistent (and also asymptotically Gaussian). Bachoc and Lagnoux (2020) considered a Gaussian process ($d = 1$) whose covariance function is parametrized by variance, scale and smoothness parameters. They focused on CL objective functions based on the conditional log likelihood of the observations given the K

(resp. L) observations corresponding to the left (resp. right) nearest neighbor observation points. They examined the case where only the variance parameter is unknown and the case where the variance and the spatial scale are jointly estimated. In the first case they proved that for small values of the smoothness parameter, the composite likelihood estimator converges at a sub-optimal rate and they showed that the asymptotic distribution is not Gaussian. For large values of the smoothness parameter, they proved that the estimator converges at the optimal rate.

1.3 Fixed-domain asymptotics for non-Gaussian random fields

To the best of our knowledge, there is a few papers that study MLE or MCLE for non-Gaussian random fields under fixed-domain asymptotics. For example, Li (2013) proposed approximate maximum-likelihood estimation for diffusion processes ($d = 1$) and provided closed-form asymptotic expansion for transition density. But diffusion processes may not be generalized for $d \geq 2$.

Other papers rather considered variogram-based or power variation-based estimators. Chan and Wood (2004) considered a random field of the form $g(X)$, where $g : \mathbf{R} \rightarrow \mathbf{R}$ is an unknown smooth function and X is a real-valued stationary Gaussian field on \mathbf{R}^d ($d = 1$ or 2) whose covariance function obeys a power law at the origin. The authors addressed the question of the asymptotic properties of variogram-based estimators when $g(X)$ is observed instead of X under a fixed-domain framework. They established that the asymptotic distribution theory for nonaffine g is somewhat richer than in the Gaussian case (i.e. when g is an affine transformation). Although the variogram-based estimators are not MLE or MCLE, this study shows that their asymptotic properties can differ significantly from the Gaussian random field case. Robert (2020) considered a particular class of max-stable processes ($d = 1$), the class of simple Brown-Resnick max-stable processes whose spectral processes are continuous exponential martingales. He developed the asymptotic theory for the realized power variations of these max-stable processes, that is, sums of powers of absolute increments. He considered a fixed-domain asymptotic setting and obtained a biased central limit theorem whose bias depends on the local times of the differences between the logarithms of the underlying spectral processes.

1.4 Max-stable random fields

Max-stable random fields appear as the only possible non-degenerate limits for normalized pointwise maxima of independent and identically distributed (i.i.d.) random fields with continuous sample paths (see e.g. de Haan and Ferreira (2006)). The one-dimensional marginal distributions of max-stable fields belong to the parametric class of Generalized Extreme Value distributions. Since we are interested in the estimation of parameters characterizing the dependence structure, we restrict our attention to max-stable random

fields $\eta = (\eta(x))_{x \in \mathcal{X}}$ on $\mathcal{X} \subset \mathbf{R}^d$ with standard unit Fréchet margins, that is, satisfying

$$\mathbb{P}[\eta(x) \leq z] = \exp(-z^{-1}), \quad \text{for all } x \in \mathcal{X} \text{ and } z > 0.$$

The max-stability property has then the simple form

$$n^{-1} \bigvee_{i=1}^n \eta_i \stackrel{d}{=} \eta$$

where $(\eta_i)_{1 \leq i \leq n}$ are i.i.d. copies of η , \bigvee is the pointwise maximum, and $\stackrel{d}{=}$ denotes the equality of finite-dimensional distributions. Max-stable random fields are characterized by their spectral representation (see e.g., de Haan (1984), Giné et al. (1990)): any stochastically continuous max-stable process η can be written as

$$\eta(x) = \bigvee_{i \geq 1} U_i Y_i(x), \quad x \in \mathcal{X}, \quad (1)$$

where $(U_i)_{i \geq 1}$ is the decreasing enumeration of the points of a Poisson point process on $(0, +\infty)$ with intensity measure $u^{-2} du$, $(Y_i)_{i \geq 1}$ are i.i.d. copies of a non-negative stochastic random field Y on \mathcal{X} such that $\mathbb{E}[Y(x)] = 1$ for all $x \in \mathcal{X}$, the sequences $(U_i)_{i \geq 1}$ and $(Y_i)_{i \geq 1}$ are independent.

The spectral representation (1) makes it possible to construct a canonical tessellation of \mathcal{X} as in Dombry and Kabluchko (2018). We define the cell associated with each index $i \geq 1$ by $C_i = \{x \in \mathcal{X} : U_i Y_i(x) = \eta(x)\}$. It is a (possibly empty) random closed subset of \mathcal{X} and each point $x \in \mathcal{X}$ belongs almost surely (a.s.) to a unique cell (the point process $\{U_i Y_i(x)\}_{i \geq 1}$ is a Poisson point process with intensity $u^{-2} du$ so that the maximum $\eta(x)$ is almost surely attained for a unique i). It is noteworthy that the terms *cell* and *tessellation* are meant in a broader sense than in Stochastic Geometry where they originated. Here, a cell is a general (not necessarily convex or connected) random closed set and a tessellation is a random covering of \mathcal{X} by closed sets with pairwise disjoint interiors.

Likelihood inference is challenging for max-stable random fields because their finite-dimensional probability density functions are unknown or cannot be computed efficiently. Padoan et al. (2010) proposed to use a composite-likelihood approach but only discussed asymptotic properties of the estimators when the data-sites are fixed and when there is a large number of i.i.d. data replications.

1.5 Contributions of the paper

In this paper, we consider the class of spatial max-stable Brown-Resnick random fields ($d = 2$) associated with isotropic fractional Brownian random fields as defined in Kabluchko et al. (2009). We assume a Poisson stochastic spatial sampling scheme and use the Poisson-Delaunay triangulation to select the pairs and triples of sites with their associated marginal distributions that will be integrated into the CL objective

functions (we exclude pairs that are not edges of the Delaunay triangulation or triples that are not vertices of triangles of this triangulation). Note that using the Delaunay triangulation is relatively natural here since we only use the distributions of pairs and triples. Moreover, the Delaunay triangulation appears to be the most “regular” triangulation in the sense that it is the one that maximises the minimum of the angles of the triangles.

We study for the first time the asymptotic properties of the MCLE of the scale and Hurst parameters of the max-stable Brown-Resnick random fields under fixed domain asymptotics (for only one realization of a Poisson point process). Pairwise and triplewise CL objective functions (considering all pairs and triples) have been proposed for inference for max-stable processes, but the properties of the MCLE have only been studied when the sites are fixed and when there is a large number of independent observations over time of the max-stable random field (see, e.g., Blanchet and Davison (2011), Davison et al. (2012) or Huser and Davison (2013)). Note that the tapered CL estimators for max-stable random field excluding pairs that are at a too large distance apart have also been studied in Sang and Genton (2014) (here again with independent observations), but this is the first time that a Delaunay triangulation is used to select the pairs and triples.

To obtain the asymptotic distributions of the MCLEs, we proceed in several steps. First we consider sums of square increments of an isotropic fractional Brownian field on the edges of the Delaunay triangles and provide asymptotic results using Malliavin calculus (see Theorem 1). Zhu and Stein (2002) also studied sums of generalized variations for this random field but assumed data-sites on a regular grid. Second we consider sums of square increments of the pointwise maximum of two independent isotropic fractional Brownian fields and show that the asymptotic behaviors of the sums now depend on the local time at the level 0 of the difference between the two fractional Brownian fields (see Theorem 2). Third we generalize these results to the max-stable Brown-Resnick random field which is built as the pointwise maximum of an infinite number of isotropic fractional Brownian fields (see Theorem 3). Using approximations of the pairwise and triplewise CL objective functions, we derive the asymptotic properties of the MCLEs (see Theorem 4).

The family of stationary Brown-Resnick random fields defined in Kabluchko et al. (2019) is presented in Section 2. We also provide the asymptotic distributions of pairs and triples as the distances between sites tend to zero. In Section 3, we introduce the randomized sampling scheme and define the CL estimators of the scale and Hurst parameters. Our main results are stated in Section 4. The proofs and some intermediate results are deferred into a Supplementary Material.

2 The max-stable Brown-Resnick random fields

2.1 Definition of the max-stable Brown-Resnick random fields

This paper concerns the class of max-stable random fields known as Brown-Resnick random fields. This class of random fields is based on Gaussian random fields with stationary increments and was introduced in Kabluchko et al. (2009). Recall that a random process $(W(x))_{x \in \mathbf{R}^d}$ is said to have stationary increments if the law of $(W(x+x_0) - W(x_0))_{x \in \mathbf{R}^d}$ does not depend on the choice of $x_0 \in \mathbf{R}^d$. A prominent example is the isotropic fractional Brownian field where $W(0) = 0$ a.s. and semi-variogram given by $\gamma(x) = \text{var}(W(x))/2 = \sigma^2 \|x\|^\alpha / 2$ for some $\alpha \in (0, 2)$ and $\sigma^2 > 0$, where $\|x\|$ is the Euclidean norm of x . The parameter σ is called the scale parameter while α is called the range parameter ($H = \alpha/2$ is also known as the Hurst parameter and relates to the Hölder continuity exponent of W). It is noteworthy that W is a self-similar random field with linear stationary increments as presented in Definition 3.3.1 of Cohen and Istas (2013) and it differs from the fractional Brownian sheet which is a self-similar random field with stationary rectangular increments (see e.g. Section 3.3.2 of the same book). Functional limit theorems for generalized variations of this fractional Brownian sheet have been studied in Pakkanen and Reveillac (2016), but these theorems cannot be extended to the isotropic fractional Brownian field whose rectangular increments are not stationary.

In this paper we consider spatial max-stable random fields ($d = 2$) and assume that the random field Y introduced in the spectral representation (1) has the following form

$$Y(x) = \exp(W(x) - \gamma(x)), \quad x \in \mathbf{R}^2.$$

With this choice, η is a stationary random field while W is not stationary but has (linear) stationary increments (see Kabluchko et al. (2009)).

2.2 Pairwise joint distributions and asymptotic score contributions

Let us consider two sites $x_1, x_2 \in \mathbf{R}^2$ and denote by $d = \|x_2 - x_1\|$ the distance between these sites. Let $z_1, z_2 \in \mathbf{R}_+$, $a = \sigma d^{\alpha/2}$, $u = \log(z_2/z_1)/a$ and $v(u) = a/2 + u$. It is well known that the joint probability distribution function of $(\eta(x_1), \eta(x_2))$ is given by (see e.g. Huser and Davison (2013))

$$F_{x_1, x_2}(z_1, z_2) = \mathbb{P}[\eta(x_1) \leq z_1, \eta(x_2) \leq z_2] = \exp(-V_{x_1, x_2}(z_1, z_2)),$$

where

$$V_{x_1, x_2}(z_1, z_2) = \frac{1}{z_1} \Phi(v(u)) + \frac{1}{z_2} \Phi(v(-u)), \quad z_1, z_2 > 0.$$

Here Φ denotes the cumulative distribution function of the standard Gaussian distribution. The term V_{x_1, x_2} is referred to as the pairwise exponent function. Let us now consider the “normalized” (linear) increments of the logarithm of the Brown-Resnick random field

$$U = d^{-\alpha/2} \sigma^{-1} \log(\eta(x_2) / \eta(x_1)).$$

The following proposition provides the conditional and marginal distributions of U and allows us to deduce that it has asymptotically a standard Gaussian distribution as the distance d tends to 0. Such a result generalizes Proposition 3 in Robert (2020).

Proposition 1 *The conditional distribution of U given $\eta(x_1) = \eta > 0$ is characterized by*

$$\mathbb{P}[U \leq u | \eta(x_1) = \eta] = \exp\left(-\frac{1}{\eta} \left[V_{x_1, x_2}(1, e^{\sigma d^{\alpha/2} u}) - 1 \right]\right) \Phi(v(u)), \quad u \in \mathbf{R},$$

and its marginal distribution by

$$\mathbb{P}[U \leq u] = \frac{\Phi(v(u))}{V_{x_1, x_2}(1, e^{\sigma d^{\alpha/2} u})}, \quad u \in \mathbf{R}.$$

It follows that

$$\lim_{d \rightarrow 0} \mathbb{P}[U \leq u] = \Phi(u), \quad u \in \mathbf{R}.$$

The fact that the asymptotic distribution of U (as d tends to 0) is a standard Gaussian distribution is not a surprise since the probability that x_1 and x_2 belong to the same cell of the canonical tessellation of the max-stable random field tends to 1. Indeed, in a common cell, the values of the max-stable random field are generated by the same isotropic fractional Brownian random field. It is natural to first study the asymptotic behaviors of the increment sums for an isotropic fractional Brownian random field before considering a Brown-Resnick random field (see Section 4.2).

The distribution of $(\eta(x_1), \eta(x_2))$ is absolutely continuous with respect to the Lebesgue measure on $(\mathbf{R}_+)^2$. Its density function satisfies

$$f_{x_1, x_2}(z_1, z_2) = \frac{\partial}{\partial z_1 \partial z_2} F_{x_1, x_2}(z_1, z_2), \quad z_1, z_2 \in \mathbf{R}_+,$$

and will be used for the contribution of the pair $(\eta(x_1), \eta(x_2))$ to the pairwise CL function. For any $\alpha \in (0, 2)$, $\sigma^2 > 0$ and $z_1, z_2 \in \mathbf{R}_+$, this joint density function is a differentiable function with respect to (α, σ) . The following proposition provides the asymptotic contributions of the pair to the pairwise score functions.

Proposition 2 *Let $u \in \mathbf{R}$ be fixed. Let $x_1, x_2 \in \mathbf{R}^2$ and $z_1, z_2 \in \mathbf{R}_+$ be such that $d^{-\alpha/2} \sigma^{-1} \log(z_2/z_1) = u$,*

where $d = \|x_2 - x_1\| > 0$. Then

$$\begin{aligned}\lim_{d \rightarrow 0} \frac{\partial}{\partial \sigma} \log f_{x_1, x_2}(z_1, z_2) &= \frac{1}{\sigma} (u^2 - 1), \\ \lim_{d \rightarrow 0} \frac{1}{\log d} \frac{\partial}{\partial \alpha} \log f_{x_1, x_2}(z_1, z_2) &= \frac{1}{2} (u^2 - 1).\end{aligned}$$

The asymptotic score contributions of a pair are therefore proportional to $(u^2 - 1)$. Further, u will be replaced by the normalized increment of $\log(\eta)$ which has asymptotically a standard Gaussian distribution as stated in Proposition 1. This fact ensures that the asymptotic score contributions are asymptotically unbiased.

2.3 Triplewise joint distributions and asymptotic score contributions

Let us now consider three sites $x_1, x_2, x_3 \in \mathbf{R}^2$ and denote by $d_{1,2} = \|x_2 - x_1\|$, $d_{1,3} = \|x_3 - x_1\|$, $d_{2,3} = \|x_3 - x_2\|$ the distances between two different sites. Let $z_1, z_2, z_3 \in \mathbf{R}_+$ and, for $i, j = 1, 2, 3$ such that $i \neq j$, let $a_{i,j} = \sigma d_{i,j}^{\alpha/2}$, $u_{i,j} = \log(z_j/z_i)/a_{i,j}$ and $v_{i,j}(u) = a_{i,j}/2 + u_{i,j}$. The joint probability distribution function of $(\eta(x_1), \eta(x_2), \eta(x_3))$ is given by (see e.g. Huser and Davison (2013))

$$F_{x_1, x_2, x_3}(z_1, z_2, z_3) = \mathbb{P}[\eta(x_1) \leq z_1, \eta(x_2) \leq z_2, \eta(x_3) \leq z_3] = \exp(-V_{x_1, x_2, x_3}(z_1, z_2, z_3)),$$

where

$$\begin{aligned}V_{x_1, x_2, x_3}(z_1, z_2, z_3) &= \frac{1}{z_1} \Phi_2 \left(\begin{pmatrix} v_{1,2}(u_{1,2}) \\ v_{1,3}(u_{1,3}) \end{pmatrix}; \begin{pmatrix} 1 & R_1 \\ R_1 & 1 \end{pmatrix} \right) + \frac{1}{z_2} \Phi_2 \left(\begin{pmatrix} v_{1,2}(-u_{1,2}) \\ v_{2,3}(u_{2,3}) \end{pmatrix}; \begin{pmatrix} 1 & R_2 \\ R_2 & 1 \end{pmatrix} \right) \\ &\quad + \frac{1}{z_3} \Phi_2 \left(\begin{pmatrix} v_{1,3}(-u_{1,3}) \\ v_{2,3}(-u_{2,3}) \end{pmatrix}; \begin{pmatrix} 1 & R_3 \\ R_3 & 1 \end{pmatrix} \right) \quad (2)\end{aligned}$$

with

$$R_1 = \frac{d_{1,2}^\alpha + d_{1,3}^\alpha - d_{2,3}^\alpha}{2(d_{1,2}d_{1,3})^{\alpha/2}}, \quad R_2 = \frac{d_{1,2}^\alpha + d_{2,3}^\alpha - d_{1,3}^\alpha}{2(d_{1,2}d_{2,3})^{\alpha/2}}, \quad R_3 = \frac{d_{1,3}^\alpha + d_{2,3}^\alpha - d_{1,2}^\alpha}{2(d_{1,3}d_{2,3})^{\alpha/2}}.$$

Here $\Phi_2(\bullet, \Sigma)$ denotes the bivariate cumulative distribution function of the centered Gaussian distribution with covariance matrix Σ . As for the pairs, let us also consider the ‘‘normalized’’ (linear) increments of the logarithm of the Brown-Resnick random field

$$U_{1,2} = d_{1,2}^{-\alpha/2} \sigma^{-1} \log(\eta(x_2)/\eta(x_1)), \quad U_{1,3} = d_{1,3}^{-\alpha/2} \sigma^{-1} \log(\eta(x_3)/\eta(x_1)).$$

The following proposition provides the conditional and marginal distributions of the vector $(U_{1,2}, U_{1,3})$ and allows us to deduce that it has asymptotically a bivariate Gaussian distribution as the distances $d_{1,2}$ and $d_{1,3}$ tend to 0 proportionally.

Proposition 3 *The conditional distribution of $(U_{1,2}, U_{1,3})$ given $\eta(x_1) = \eta > 0$ is characterized by*

$$\mathbb{P}[U_{1,2} \leq u_2, U_{1,3} \leq u_3 | \eta(x_1) = \eta] = \exp\left(-\frac{1}{\eta} \left[V_{x_1, x_2, x_3}(1, e^{\sigma d_{12}^{\alpha/2} u_2}, e^{\sigma d_{13}^{\alpha/2} u_3}) - 1 \right]\right) \\ \times \Phi_2\left(\begin{pmatrix} v_{1,2}(u_2) \\ v_{1,3}(u_3) \end{pmatrix}; \begin{pmatrix} 1 & R_1 \\ R_1 & 1 \end{pmatrix}\right), \quad u_1, u_2 \in \mathbf{R},$$

and its marginal distribution by

$$\mathbb{P}[U_{1,2} \leq u_2, U_{1,3} \leq u_3] = \frac{\Phi_2\left(\begin{pmatrix} v_{1,2}(u_2) \\ v_{1,3}(u_3) \end{pmatrix}; \begin{pmatrix} 1 & R_1 \\ R_1 & 1 \end{pmatrix}\right)}{V_{x_1, x_2, x_3}(1, e^{\sigma d_{12}^{\alpha/2} u_2}, e^{\sigma d_{13}^{\alpha/2} u_3})}, \quad u_1, u_2 \in \mathbf{R}.$$

It follows that, if $\|x_2 - x_1\| = \delta d_{1,2}$, $\|x_3 - x_1\| = \delta d_{1,3}$, $\|x_3 - x_2\| = \delta d_{2,3}$, where $d_{i,j}$, $i \neq j$, is fixed, then

$$\lim_{\delta \rightarrow 0} \mathbb{P}[U_{1,2} \leq u_2, U_{1,3} \leq u_3] = \Phi_2\left(\begin{pmatrix} u_2 \\ u_3 \end{pmatrix}; \begin{pmatrix} 1 & R_1 \\ R_1 & 1 \end{pmatrix}\right), \quad u_1, u_2 \in \mathbf{R}.$$

The comment concerning the asymptotic distribution of U also holds for $(U_{1,2}, U_{1,3})$. The probability that x_1 , x_2 and x_3 belong to the same cell of the canonical tessellation of the max-stable random field tends to 1 as δ tends to 0. Therefore the vector $(U_{1,2}, U_{1,3})$ tends to have the same distribution as the vector of normalized linear increments of an isotropic fractional Brownian random field.

The distribution of $(\eta(x_1), \eta(x_2), \eta(x_3))$ is absolutely continuous with respect to the Lebesgue measure on $(\mathbf{R}_+)^3$. Its density function satisfies

$$f_{x_1, x_2, x_3}(z_1, z_2, z_3) = \frac{\partial}{\partial z_1 \partial z_2 \partial z_3} F_{x_1, x_2, x_3}(z_1, z_2, z_3), \quad z_1, z_2, z_3 \in \mathbf{R}_+,$$

and will be used for the contribution of the triple $(\eta(x_1), \eta(x_2), \eta(x_3))$ to the triplewise CL function. For any $\alpha \in (0, 2)$, $\sigma^2 > 0$ and $z_1, z_2, z_3 \in \mathbf{R}_+$, this joint density function is a differentiable function with respect to (α, σ) . The following proposition provides the asymptotic contributions of the triple $(\eta(x_1), \eta(x_2), \eta(x_3))$ to the triplewise score functions.

Proposition 4 *Let $u_2, u_3 \in \mathbf{R}$ be fixed. Let $x_1, x_2, x_3 \in \mathbf{R}^2$ and $z_1, z_2, z_3 \in \mathbf{R}_+$ be such that*

$$\delta^{-\alpha/2} d_{1,2}^{-\alpha/2} \sigma^{-1} \log(z_2/z_1) = u_2 \text{ and } \delta^{-\alpha/2} d_{1,3}^{-\alpha/2} \sigma^{-1} \log(z_3/z_1) = u_3,$$

where $\delta d_{1,2} = \|x_2 - x_1\|$, $\delta d_{1,3} = \|x_3 - x_1\|$, $\delta d_{2,3} = \|x_3 - x_2\|$. Then

$$\begin{aligned} \lim_{\delta \rightarrow 0} \frac{\partial}{\partial \sigma} \log f_{x_1, x_2, x_3}(z_1, z_2, z_3) &= \frac{1}{\sigma} \left(\begin{pmatrix} u_2 & u_3 \end{pmatrix} \begin{pmatrix} 1 & R_1 \\ R_1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} - 2 \right), \\ \lim_{\delta \rightarrow 0} \frac{1}{\log \delta} \frac{\partial}{\partial \alpha} \log f_{x_1, x_2, x_3}(z_1, z_2, z_3) &= \frac{1}{2} \left(\begin{pmatrix} u_2 & u_3 \end{pmatrix} \begin{pmatrix} 1 & R_1 \\ R_1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} - 2 \right). \end{aligned}$$

The asymptotic contributions of a triple are therefore proportional to a quadratic function of (u_2, u_3) . Further, u_2 and u_3 will be replaced by normalized increments of $\log(\eta)$ over a triangle of the Delaunay triangulation (see Section 3.1) which have asymptotically a bivariate standard Gaussian distribution with correlation coefficient R_1 , as stated in Proposition 3. We can also conclude that the asymptotic score contributions are asymptotically unbiased.

If we let $\delta^{-\alpha/2} d_{1,2}^{-\alpha/2} \sigma^{-1} \log(z_1/z_2) = \tilde{u}_1$ and $\delta^{-\alpha/2} d_{2,3}^{-\alpha/2} \sigma^{-1} \log(z_3/z_2) = \tilde{u}_3$ with fixed $\tilde{u}_1, \tilde{u}_3 \in \mathbf{R}$, we also get

$$\lim_{\delta \rightarrow 0} \frac{\partial}{\partial \sigma} \log f_{x_1, x_2, x_3}(z_1, z_2, z_3) = \frac{1}{\sigma} \left(\begin{pmatrix} \tilde{u}_1 & \tilde{u}_3 \end{pmatrix} \begin{pmatrix} 1 & R_2 \\ R_2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_3 \end{pmatrix} - 2 \right).$$

In particular, an invariance property with respect to the choice of the order of the points x_1 , x_2 and x_3 holds for the triplewise score functions. However, it will be necessary to order these points later.

3 The weighted CL approach

3.1 The randomized sampling scheme

We assume that the data-sites are given by a realization of a homogeneous Poisson point process of intensity N in \mathbf{R}^2 , denoted by P_N , which is independent of the Brown-Resnick random field. Let us denote by $\mathbf{C} = (-1/2, 1/2]^2$ the square where we will consider the sites for the observations of the max-stable random field.

The Delaunay graph $\text{Del}(P_N)$ based on P_N is our connection scheme and is defined as the unique triangulation with vertices in P_N such that the circumball of each triangle contains no point of P_N in its interior. With a slight abuse of notation, we identify $\text{Del}(P_N)$ to its skeleton. When $x_1, x_2 \in P_N$ are Delaunay neighbors, we write $x_1 \sim x_2$ in $\text{Del}(P_N)$.

For a Borel subset \mathbf{B} in \mathbf{R}^2 , let $E_{N, \mathbf{B}}$ be the set of couples (x_1, x_2) such that the following conditions

hold:

$$x_1 \sim x_2 \text{ in } \text{Del}(P_N), \quad x_1 \in \mathbf{B}, \quad \text{and} \quad x_1 \preceq x_2,$$

where \preceq denotes the lexicographic order. When $\mathbf{B} = \mathbf{C}$, we only write $E_N = E_{N,\mathbf{C}}$.

For a Borel subset \mathbf{B} in \mathbf{R}^2 , let $DT_{N,\mathbf{B}}$ be the set of triples (x_1, x_2, x_3) satisfying the following properties

$$\Delta(x_1, x_2, x_3) \in \text{Del}(P_N), \quad x_1 \in \mathbf{B}, \quad \text{and} \quad x_1 \preceq x_2 \preceq x_3,$$

where $\Delta(x_1, x_2, x_3)$ is the convex hull of (x_1, x_2, x_3) . When $\mathbf{B} = \mathbf{C}$, we only write $DT_N = DT_{N,\mathbf{C}}$.

3.2 The weighted CL objective functions and the CL estimators

The (tapered) pairwise CL objective function is defined as

$$\ell_{2,N}(\sigma, \alpha) = \sum_{(x_1, x_2) \in E_N} \log f_{x_1, x_2}(\eta(x_1), \eta(x_2)),$$

while the (tapered) triplewise CL objective function is defined as

$$\ell_{3,N}(\sigma, \alpha) = \sum_{(x_1, x_2, x_3) \in DT_N} \log f_{x_1, x_2, x_3}(\eta(x_1), \eta(x_2), \eta(x_3)).$$

Thereby, in the CL objective functions, we exclude pairs that are not edges of the Delaunay triangulation or triples that are not vertices of triangles of this triangulation. Restricting the CL objective functions to the most informative pairs and triples for the estimation of the parameters does not modify the approach that follows, but allows us to simplify the presentation and the proofs.

From Section 4.4 of Dombry et al. (2018), we know that there exist families of positive functions $(l_{x_1, x_2})_{x_1, x_2 \in \mathbf{R}^2}$ and $(l_{x_1, x_2, x_3})_{x_1, x_2, x_3 \in \mathbf{R}^2}$ with $l_{x_1, x_2} : \mathbf{R}^2 \rightarrow \mathbf{R}$ and $l_{x_1, x_2, x_3} : \mathbf{R}^3 \rightarrow \mathbf{R}$ such that the following Lipschitz conditions hold: for any $\sigma_1, \sigma_2 > 0$ and $\alpha_1, \alpha_2 \in (0, 2)$

$$\left| \log \frac{f_{x_1, x_2}(z_1, z_2; (\sigma_2, \alpha_2))}{f_{x_1, x_2}(z_1, z_2; (\sigma_1, \alpha_1))} \right| \leq l_{x_1, x_2}(z_1, z_2) (|\sigma_2 - \sigma_1| + |\alpha_2 - \alpha_1|)$$

and

$$\left| \log \frac{f_{x_1, x_2, x_3}(z_1, z_2, z_3; (\sigma_2, \alpha_2))}{f_{x_1, x_2, x_3}(z_1, z_2, z_3; (\sigma_1, \alpha_1))} \right| \leq l_{x_1, x_2, x_3}(z_1, z_2, z_3) (|\sigma_2 - \sigma_1| + |\alpha_2 - \alpha_1|).$$

Let us denote by (σ_0, α_0) the true parameters. We assume that σ_0 belongs to a compact set S_σ of \mathbf{R}_+ and that α_0 belongs to a compact set S_α of $(0, 2)$. We can now define the MCLEs of σ and α .

When α_0 is assumed to be known, the pairwise and triplewise maximum (tapered) CL estimators of

$\sigma_0, \hat{\sigma}_{j,N}$, are respectively defined as a solution of the maximization problems

$$\max_{\sigma \in \mathcal{S}_\sigma} \ell_{j,N}(\sigma, \alpha_0), \quad j = 2, 3.$$

When σ_0 is assumed to be known, the pairwise and triplewise maximum (tapered) CL estimators of α_0 , $\hat{\alpha}_{j,N}$, are respectively defined as a solution of the maximization problems

$$\max_{\alpha \in \mathcal{S}_\alpha} \ell_{j,N}(\sigma_0, \alpha), \quad j = 2, 3.$$

Note that the solutions of these maximization problems become unique as $N \rightarrow \infty$. This can be viewed from the first-order optimality conditions and the asymptotic approximations of the score functions obtained in Propositions 2 and 4.

4 Main results

Our aim is to characterize the asymptotic distributions of the MCLEs. We provide intermediate results for different random fields in order to understand how we obtained the different families of asymptotic distributions of our estimators. We first provide some definitions and notations related to the Poisson-Delaunay triangulation. Then we consider sums of square increments of an isotropic fractional Brownian field on the edges of the Delaunay triangles and provide Central Limit Theorems using Malliavin calculus. We only consider the case $\alpha \in (0, 1)$ for which the asymptotic distributions are Gaussian. This is not a very restrictive constraint since almost all empirical studies that use the spatial Brown-Resnick random field obtain values for α in this interval (see e.g. Davison et al. (2012), Engelke et al. (2014), Einmahl et al. (2015) or de Fondeville and Davison (2018)). Third we consider sums of square increments of the pointwise maximum of two independent isotropic fractional Brownian fields and show that the asymptotic behaviors of the sums now depend on the local time at the level 0 of the difference between the two fractional Brownian fields. Fourth we generalize these results to the max-stable Brown-Resnick random field and, using approximation of the pairwise and triplewise CL objective functions, we derive the asymptotic properties of the MCLEs.

4.1 Definitions and notations

A classical object in Stochastic Geometry is the typical cell. To define it, let us consider a Delaunay triangulation $\text{Del}(P_1)$ based on a homogeneous Poisson point process of intensity 1. With each cell $C \in \text{Del}(P_1)$, we associate the circumcenter $z(C)$ of C . Now, let \mathbf{B} be a Borel subset in \mathbf{R}^2 with area $a(\mathbf{B}) \in$

$(0, \infty)$. The cell intensity β_2 of $\text{Del}(P_1)$ is defined as the mean number of cells per unit area, i.e.

$$\beta_2 = \frac{1}{a(\mathbf{B})} \mathbb{E} [\#\{C \in \text{Del}(P_1) : z(C) \in \mathbf{B}\}].$$

It is known that $\beta_2 = 2$, see e.g. Theorem 10.2.9. in Schneider and Weil (2008). Then, we define the typical cell as a random triangle \mathcal{C} with distribution given as follows: for any positive measurable and translation invariant function $g : \mathcal{K}_2 \rightarrow \mathbf{R}$, we have

$$\mathbb{E} [g(\mathcal{C})] = \frac{1}{\beta_2 a(\mathbf{B})} \mathbb{E} \left[\sum_{C \in \text{Del}(P_1) : z(C) \in \mathbf{B}} g(C) \right],$$

where \mathcal{K}_2 denotes the set of convex compact subsets in \mathbf{R}^2 , endowed with the Fell topology (see Section 12.2 in Schneider and Weil (2008) for the definition). The distribution of \mathcal{C} has the following integral representation (see e.g. Theorem 10.4.4. in Schneider and Weil (2008)):

$$\mathbb{E} [g(\mathcal{C})] = \frac{1}{6} \int_0^\infty \int_{(\mathbf{S}^1)^3} r^3 e^{-\pi r^2} a(\Delta(u_1, u_2, u_3)) g(\Delta(ru_1, ru_2, ru_3)) \sigma(du_1) \sigma(du_2) \sigma(du_3) dr, \quad (3)$$

where \mathbf{S}^1 is the unit sphere of \mathbf{R}^2 and σ is the spherical Lebesgue measure on \mathbf{S}^1 with normalization $\sigma(\mathbf{S}^1) = 2\pi$. It means that \mathcal{C} is equal in distribution to $R\Delta(U_1, U_2, U_3)$, where R and (U_1, U_2, U_3) are independent with probability density functions given respectively by $2\pi^2 r^3 e^{-\pi r^2}$ and $a(\Delta(u_1, u_2, u_3))/(12\pi^2)$.

In a similar way, we can define the notion of typical edge. The edge intensity β_1 of $\text{Del}(P_1)$ is defined as the mean number of edges per unit area and is equal to $\beta_1 = 3$ (see e.g. Theorem 10.2.9. in Schneider and Weil (2008)). The distribution of the length of the typical edge is the same as the distribution of $D = R\|U_1 - U_2\|$. Its probability density function f_D satisfies the following equality

$$\mathbb{P} [D \leq \ell] = \int_0^\ell f_D(d) dd = \frac{\pi}{3} \int_0^\infty \int_{(\mathbf{S}^1)^2} r^3 e^{-\pi r^2} a(\Delta(u_1, u_2, e_1)) \mathbb{I} [r \|u_1 - u_2\| \leq \ell] \sigma(du_1) \sigma(du_2) dr, \quad (4)$$

where $e_1 = (1, 0)$ and $\ell > 0$. Following Eq. (3), a typical couple of (distinct) Delaunay edges with a common vertex can be defined as a 3-tuple of random variables (D_1, D_2, Θ) , where $D_1, D_2 \geq 0$ and $\Theta \in [-\frac{\pi}{2}, \frac{\pi}{2})$, with distribution given by

$$\begin{aligned} \mathbb{P}[(D_1, D_2, \Theta) \in B] &= \frac{1}{6} \int_0^\infty \int_{(\mathbf{S}^1)^3} r^3 e^{-\pi r^2} a(\Delta(u_1, u_2, u_3)) \\ &\quad \times \mathbb{I}[(r\|u_3 - u_2\|, r\|u_2 - u_1\|, \arcsin(\cos(\zeta_{u_1, u_2}/2))) \in B] \sigma(du_1) \sigma(du_2) \sigma(du_3) dr, \end{aligned}$$

where ζ_{u_1, u_2} is the measure of the angle (u_1, u_2) and where B is any Borel subset in $\mathbf{R}_+^2 \times [-\frac{\pi}{2}, \frac{\pi}{2})$. The random variables D_1, D_2 (resp. Θ) can be interpreted as the lengths of the two typical edges (resp. as the angle between the edges). In particular, the length of a typical edge is equal in distribution to $D = R\|U_2 - U_1\|$ with distribution given in Eq. (4).

4.2 Asymptotic distributions of squared increment sums for an isotropic fractional Brownian field

Let $(W(x))_{x \in \mathbf{R}^2}$ be an isotropic fractional Brownian field where $W(0) = 0$ a.s. and $\text{var}(W(x)) = \sigma^2 \|x\|^\alpha$ for some $\alpha \in (0, 1)$ and $\sigma^2 > 0$. For two sites $x_1, x_2 \in \mathbf{R}^2$, let us define the normalized increment between x_1 and x_2 as

$$U_{x_1, x_2}^{(W)} = \sigma^{-1} d_{1,2}^{-\alpha/2} (W(x_2) - W(x_1))$$

with $d_{1,2} = \|x_2 - x_1\|$.

The (normalized) squared increment sum for the edges of the Delaunay triangulation is given by

$$V_{2,N}^{(W)} = \frac{1}{\sqrt{|E_N|}} \sum_{(x_1, x_2) \in E_N} \left((U_{x_1, x_2}^{(W)})^2 - 1 \right),$$

while the (normalized) squared increment sum for the pairs of edges of Delaunay triangles is defined as

$$V_{3,N}^{(W)} = \frac{1}{\sqrt{|DT_N|}} \sum_{(x_1, x_2, x_3) \in DT_N} \left(\begin{pmatrix} U_{x_1, x_2}^{(W)} & U_{x_1, x_3}^{(W)} \end{pmatrix} \begin{pmatrix} 1 & R_{x_1, x_2, x_3} \\ R_{x_1, x_2, x_3} & 1 \end{pmatrix}^{-1} \begin{pmatrix} U_{x_1, x_2}^{(W)} \\ U_{x_1, x_3}^{(W)} \end{pmatrix} - 2 \right),$$

where

$$R_{x_1, x_2, x_3} = \text{corr}(U_{x_1, x_2}^{(W)}, U_{x_1, x_3}^{(W)}) = \frac{d_{1,2}^\alpha + d_{1,3}^\alpha - d_{2,3}^\alpha}{2(d_{1,2} d_{1,3})^{\alpha/2}}, \quad (5)$$

with $d_{1,3} = \|x_3 - x_1\| > 0$ and $d_{2,3} = \|x_3 - x_2\| > 0$. Let

$$\tilde{U}_{x_1, x_2, x_3}^{(W)} = (1 - R_{x_1, x_2, x_3}^2)^{-1/2} \left(U_{x_1, x_2}^{(W)} - R_{x_1, x_2, x_3} U_{x_1, x_3}^{(W)} \right) \quad \text{and} \quad \tilde{U}_{x_1, x_3}^{(W)} = U_{x_1, x_3}^{(W)}.$$

Note that $\tilde{U}_{x_1, x_2, x_3}^{(W)}$ is a normalized increment based on the three points x_1, x_2, x_3 (see e.g. Chan and Wood (2002)) and that

$$\text{corr}(\tilde{U}_{x_1, x_2, x_3}^{(W)}, \tilde{U}_{x_1, x_3}^{(W)}) = 0.$$

The sum $V_{3,N}^{(W)}$ may be rewritten as

$$V_{3,N}^{(W)} = \frac{1}{\sqrt{|DT_N|}} \sum_{(x_1, x_2, x_3) \in DT_N} \left([(\tilde{U}_{x_1, x_2, x_3}^{(W)})^2 - 1] + [(\tilde{U}_{x_1, x_3}^{(W)})^2 - 1] \right).$$

The following theorem states that the asymptotic distributions of $V_{2,N}^{(W)}$ and $V_{3,N}^{(W)}$ are Gaussian. Their asymptotic variances are known, but quite intricate. We provide their integral representations in Section 1 in the Supplementary Material.

Theorem 1 *Let $\alpha \in (0, 1)$. Then there exist constants $\sigma_{V_2}^2 > 0$ and $\sigma_{V_3}^2 > 0$ such that, as $N \rightarrow \infty$,*

$$V_{2,N}^{(W)} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_{V_2}^2), \quad V_{3,N}^{(W)} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_{V_3}^2).$$

We note that the rates of convergences of both sums are the same as in Theorem 3.2 of Chan and Wood (2000) or in Theorem 1 of Zhu and Stein (2002) where statistics based on square increments on regular grids have been considered.

4.3 Asymptotic distributions of squared increment sums for the (pointwise) maximum of two independent fractional Brownian fields

Let $(W^{(1)}(x))_{x \in \mathbf{R}^2}$ and $(W^{(2)}(x))_{x \in \mathbf{R}^2}$ be two independent isotropic fractional Brownian fields, where $W^{(1)}(0) = W^{(2)}(0) = 0$ a.s. and $\text{var}(W^{(1)}(x)) = \text{var}(W^{(2)}(x)) = \sigma^2 \|x\|^\alpha$ for some $\alpha \in (0, 1)$ and $\sigma^2 > 0$. We denote by W_\vee the pointwise maximum of the two isotropic fractional Brownian fields, i.e.

$$W_\vee(x) = W^{(1)}(x) \vee W^{(2)}(x), \quad x \in \mathbf{R}^2.$$

For two distinct sites $x_1, x_2 \in \mathbf{R}^2$, let

$$U_{x_1, x_2}^{(W_\vee)} = \sigma^{-1} d_{1,2}^{-\alpha/2} (W_\vee(x_2) - W_\vee(x_1)).$$

Then we define

$$V_{2,N}^{(W_\vee)} = \frac{1}{\sqrt{|E_N|}} \sum_{(x_1, x_2) \in E_N} \left((U_{x_1, x_2}^{(W_\vee)})^2 - 1 \right)$$

$$V_{3,N}^{(W_\vee)} = \frac{1}{\sqrt{|DT_N|}} \sum_{(x_1, x_2, x_3) \in DT_N} \left(\begin{pmatrix} U_{x_1, x_2}^{(W_\vee)} & U_{x_1, x_3}^{(W_\vee)} \end{pmatrix} \begin{pmatrix} 1 & R_{x_1, x_2, x_3} \\ R_{x_1, x_2, x_3} & 1 \end{pmatrix}^{-1} \begin{pmatrix} U_{x_1, x_2}^{(W_\vee)} \\ U_{x_1, x_3}^{(W_\vee)} \end{pmatrix} - 2 \right),$$

where R_{x_1, x_2, x_3} is given in Eq. (5).

The main result of this section concerns the asymptotic behaviors of $V_{2,N}^{(W_\vee)}$ and $V_{3,N}^{(W_\vee)}$. To state it, let us denote the difference between both fractional Brownian fields as $W^{(2 \setminus 1)}(x) = W^{(2)}(x) - W^{(1)}(x)$ for any $x \in \mathbf{R}^2$. Similarly to Section 5.1 in Robert (2020), we observe that, for any real measurable function $f : \mathbf{R} \rightarrow \mathbf{R}$ and for any $(x_1, x_2) \in E_N$,

$$f(U_{x_1, x_2}^{(W_\vee)}) = f(U_{x_1, x_2}^{(1)}) \mathbb{I}[W^{(2 \setminus 1)}(x_1) < 0] + f(U_{x_1, x_2}^{(2)}) \mathbb{I}[W^{(2 \setminus 1)}(x_1) > 0]$$

$$+ \Psi_f \left(U_{x_1, x_2}^{(1)}, U_{x_1, x_2}^{(2)}, W^{(2 \setminus 1)}(x_1) / (\sigma d_{1,2}^{\alpha/2}) \right), \quad (6)$$

where

$$U_{x_1, x_2}^{(1)} = \frac{1}{\sigma d_{1,2}^{\alpha/2}} \left(W^{(1)}(x_2) - W^{(1)}(x_1) \right), \quad U_{x_1, x_2}^{(2)} = \frac{1}{\sigma d_{1,2}^{\alpha/2}} \left(W^{(2)}(x_2) - W^{(2)}(x_1) \right)$$

and

$$\Psi_f(x, y, w) = (f(y+w) - f(x))\mathbb{I}[x-y \leq w \leq 0] + (f(x-w) - f(y))\mathbb{I}[0 \leq w \leq x-y].$$

In particular, taking $f(u) = H_2(u) = u^2 - 1$, for all $u \in \mathbf{R}$, and $\Psi = \Psi_{H_2}$, the above decomposition implies that

$$V_{2,N}^{(W_\vee)} = V_{2,N}^{(1)} + V_{2,N}^{(2)} + V_{2,N}^{(2/1)}, \quad (7)$$

where

$$\begin{aligned} V_{2,N}^{(1)} &= \frac{1}{\sqrt{|E_N|}} \sum_{(x_1, x_2) \in E_N, W^{(2 \setminus 1)}(x_1) < 0} \left((U_{x_1, x_2}^{(1)})^2 - 1 \right) \\ V_{2,N}^{(2)} &= \frac{1}{\sqrt{|E_N|}} \sum_{(x_1, x_2) \in E_N, W^{(2 \setminus 1)}(x_1) > 0} \left((U_{x_1, x_2}^{(2)})^2 - 1 \right) \\ V_{2,N}^{(2/1)} &= \frac{1}{\sqrt{|E_N|}} \sum_{(x_1, x_2) \in E_N} \Psi(U_{x_1, x_2}^{(1)}, U_{x_1, x_2}^{(2)}, W^{(2 \setminus 1)}(x_1) / (\sigma d_{1,2}^{\alpha/2})). \end{aligned}$$

To obtain a similar decomposition for the triples, let us denote, for $-1 < R < 1$, by Ω the following function

$$\begin{aligned} \Omega(u_1, v_1, u_2, v_2, w_1, w_2; R) &= \frac{1}{1-R^2} [\Psi_{H_2}(u_1, v_1, w_1) + \Psi_{H_2}(u_2, v_2, w_2)] \\ &\quad - 2 \frac{R}{1-R^2} \Psi_I(u_1, v_1, w_1) \Psi_I(u_2, v_2, w_2) \\ &\quad - 2 \frac{R}{1-R^2} [u_1 \Psi_I(u_2, v_2, w_2) + u_2 \Psi_I(u_1, v_1, w_1)] \mathbb{I}[w_1 < 0] \\ &\quad - 2 \frac{R}{1-R^2} [v_1 \Psi_I(u_2, v_2, w_2) + v_2 \Psi_I(u_1, v_1, w_1)] \mathbb{I}[w_1 > 0] \end{aligned} \quad (8)$$

with $I(u) = u$ for all $u \in \mathbf{R}$. Then we have (see Section 3.3.2 in the Supplementary Material)

$$V_{3,N}^{(W_\vee)} = V_{3,N}^{(1)} + V_{3,N}^{(2)} + V_{3,N}^{(2/1)}, \quad (9)$$

where

$$\begin{aligned} V_{3,N}^{(1)} &= \frac{1}{\sqrt{|DT_N|}} \sum_{\substack{(x_1, x_2, x_3) \in DT_N, \\ W^{(2 \setminus 1)}(x_1) < 0}} \left(\begin{pmatrix} U_{x_1, x_2}^{(1)} & U_{x_1, x_3}^{(1)} \end{pmatrix} \begin{pmatrix} 1 & R_{x_1, x_2, x_3} \\ R_{x_1, x_2, x_3} & 1 \end{pmatrix}^{-1} \begin{pmatrix} U_{x_1, x_2}^{(1)} \\ U_{x_1, x_3}^{(1)} \end{pmatrix} - 2 \right) \\ V_{3,N}^{(2)} &= \frac{1}{\sqrt{|DT_N|}} \sum_{\substack{(x_1, x_2, x_3) \in DT_N, \\ W^{(2 \setminus 1)}(x_1) > 0}} \left(\begin{pmatrix} U_{x_1, x_2}^{(2)} & U_{x_1, x_3}^{(2)} \end{pmatrix} \begin{pmatrix} 1 & R_{x_1, x_2, x_3} \\ R_{x_1, x_2, x_3} & 1 \end{pmatrix}^{-1} \begin{pmatrix} U_{x_1, x_2}^{(2)} \\ U_{x_1, x_3}^{(2)} \end{pmatrix} - 2 \right) \\ V_{3,N}^{(2/1)} &= \frac{1}{\sqrt{|DT_N|}} \sum_{(x_1, x_2, x_3) \in DT_N} \Omega \left(U_{x_1, x_2}^{(1)}, U_{x_1, x_3}^{(1)}, U_{x_1, x_2}^{(2)}, U_{x_1, x_3}^{(2)}, \frac{W^{(2 \setminus 1)}(x_1)}{\sigma d_{1,2}^{\alpha/2}}, \frac{W^{(2 \setminus 1)}(x_1)}{\sigma d_{1,3}^{\alpha/2}}; R_{x_1, x_2, x_3} \right). \end{aligned}$$

An adaptation of the proof of Theorem 1 shows that, for $\alpha \in (0, 1)$, as $N \rightarrow \infty$,

$$V_{2,N}^{(1)} + V_{2,N}^{(2)} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_{V_2}^2) \quad (10)$$

and

$$V_{3,N}^{(1)} + V_{3,N}^{(2)} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_{V_3}^2). \quad (11)$$

To obtain the asymptotic behaviors of $V_{2,N}^{(W_\vee)}$ and $V_{3,N}^{(W_\vee)}$, the asymptotic behaviors of $V_{2,N}^{(2/1)}$ and $V_{3,N}^{(2/1)}$ are investigated. This requires to introduce the notion of local time of $W^{(2\setminus 1)}$.

The local time of $W^{(2\setminus 1)}$. Let $\nu^{(2\setminus 1)}$ be the occupation measure of $W^{(2\setminus 1)}$ over \mathbf{C} defined by

$$\nu^{(2\setminus 1)}(A) = \int_{\mathbf{C}} \mathbb{I} [W^{(2\setminus 1)}(x) \in A] dx,$$

for any Borel measurable set $A \subset \mathbf{R}$. Observe that, for any $s, t \in [0, 1]^2$,

$$\Delta(s, t) := \mathbb{E} \left[(W^{(2\setminus 1)}(s) - W^{(2\setminus 1)}(t))^2 \right] = 2\sigma^2 \|s - t\|^\alpha.$$

Because $\int_{\mathbf{C}} (\Delta(s, t))^{-1/2} ds$ is finite for all $t \in \mathbf{C}$, it follows from Section 22 in Geman and Horowitz (1980) that the occupation measure $\nu^{(2\setminus 1)}$ admits a Lebesgue density, referred to as the local time, that we denote by

$$L_{W^{(2\setminus 1)}}(\ell) := \frac{d\nu^{(2\setminus 1)}}{d\ell}(\ell).$$

An immediate consequence of the existence of the local time is the occupation time formula, which states that

$$\int_{\mathbf{C}} g(W^{(2\setminus 1)}(x)) dx = \int_{\mathbf{R}} g(\ell) L_{W^{(2\setminus 1)}}(\ell) d\ell$$

for any Borel function g on \mathbf{R} . Adapting the proof of Lemma 1.1 in Jaramillo et al. (2021), we can easily show that, for any $\ell \in \mathbf{R}$,

$$L_{W^{(2\setminus 1)}}(\ell) = \lim_{\varepsilon \rightarrow 0} \int_{\mathbf{C}} \frac{1}{\sqrt{2\pi\varepsilon}} \exp\left(-\frac{1}{2\varepsilon} (W^{(2\setminus 1)}(x) - \ell)^2\right) dx$$

or

$$L_{W^{(2\setminus 1)}}(\ell) = \frac{1}{2\pi} \lim_{M \rightarrow \infty} \int_{-[M, M]} \int_{\mathbf{R}} e^{i\xi(W^{(2\setminus 1)}(x) - \ell)} dx d\xi, \quad (12)$$

where the limits hold in L^2 .

The asymptotic behaviors of $V_{2,N}^{(W_\vee)}$ and $V_{3,N}^{(W_\vee)}$. Let F_2 be the function defined, for any $z \in \mathbf{R}$, by

$$F_2(z) = \int_{\mathbf{R}^2 \times \mathbf{R}_+} \Psi_{H_2}(x, y, z/d^{\alpha/2}) \frac{1}{2\pi} e^{-(x^2+y^2)/2} f_D(d) dx dy dd,$$

where f_D is the density function of the length of the typical edge defined in Eq. (4). Let us also define

$$F_3(z) = \int_{\mathbf{R}^4 \times (\mathbf{R}_+)^3} \Omega(x_1, y_1, x_2, y_2, z/d_1^{\alpha/2}, z/d_3^{\alpha/2}; R(d_1, d_2, d_3)) \\ \times \varphi_2(x_1, y_1; R(d_1, d_2, d_3)) \varphi_2(x_2, y_2; R(d_1, d_2, d_3)) \\ \times f_{D_1, D_2, D_3}(d_1, d_2, d_3) dx_1 dy_1 dx_2 dy_2 dd_1 dd_2 dd_3,$$

where

$$\varphi_2(x, y; R) = \frac{1}{2\pi} \frac{1}{(1-R^2)} \exp\left(-\frac{1}{2} \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & R \\ R & 1 \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \end{pmatrix}\right), \\ R(d_1, d_2, d_3) = \frac{d_1^\alpha + d_3^\alpha - d_2^\alpha}{2(d_1 d_3)^{\alpha/2}},$$

and where f_{D_1, D_2, D_3} is the density function of the edge lengths of the typical Delaunay triangle \mathcal{C} .

Moreover let

$$c_{V_2} = \int_{\mathbf{R}} F_2(z) dz \quad \text{and} \quad c_{V_3} = \int_{\mathbf{R}} F_3(z) dz.$$

The following proposition provides the asymptotic behaviors of $V_{2,N}^{(2/1)}$ and $V_{3,N}^{(2/1)}$.

Proposition 5 *Let $\alpha \in (0, 1)$. Then, as $N \rightarrow \infty$,*

$$\frac{\sqrt{3}}{3} N^{-(2-\alpha)/4} V_{2,N}^{(2/1)} \xrightarrow{\mathbb{P}} c_{V_2} L_{W^{(2 \setminus 1)}}(0) \\ \frac{\sqrt{2}}{2} N^{-(2-\alpha)/4} V_{3,N}^{(2/1)} \xrightarrow{\mathbb{P}} c_{V_3} L_{W^{(2 \setminus 1)}}(0).$$

Note that the factors $\sqrt{3}/3$ and $\sqrt{2}/2$ come from the facts that $|E_N|/N \xrightarrow{a.s.} 3$ and $|DT_N|/N \xrightarrow{a.s.} 2$ as $N \rightarrow \infty$, respectively. As a consequence of the above proposition, we obtain the following result.

Theorem 2 *Let $\alpha \in (0, 1)$. Then, as $N \rightarrow \infty$,*

$$\frac{\sqrt{3}}{3} N^{-(2-\alpha)/4} V_{2,N}^{(W_\vee)} \xrightarrow{\mathbb{P}} c_{V_2} L_{W^{(2 \setminus 1)}}(0) \\ \frac{\sqrt{2}}{2} N^{-(2-\alpha)/4} V_{3,N}^{(W_\vee)} \xrightarrow{\mathbb{P}} c_{V_3} L_{W^{(2 \setminus 1)}}(0).$$

An important observation is that the rates of convergence of $V_{2,N}^{(W_\vee)}$ and $V_{3,N}^{(W_\vee)}$ differ from those of $V_{2,N}^{(W)}$ and $V_{3,N}^{(W)}$. The sums of square increments in $V_{2,N}^{(2/1)}$ and $V_{3,N}^{(2/1)}$ are actually the dominant terms. These

increments depend on both isotropic fractional Brownian fields and they reveal the local time of $W^{(2\setminus 1)}$ at level 0 in the limits. It is also noteworthy that the convergence is now in probability.

4.4 Asymptotic distributions of squared increment sums for the max-stable Brown-Resnick random field

Let $(\eta(x))_{x \in \mathbf{R}^2}$ be a max-stable Brown-Resnick random field such that $\eta(x) = \bigvee_{i \geq 1} U_i Y_i(x)$ for any $x \in \mathbf{R}^2$, where $(U_i)_{i \geq 1}$ is a decreasing enumeration of the points of a Poisson point process on $(0, +\infty)$ with intensity measure $u^{-2} du$, and $(Y_i)_{i \geq 1}$ are i.i.d. copies of

$$Y(x) = \exp(W(x) - \gamma(x)), \quad x \in \mathbf{R}^2,$$

where $(W(x))_{x \in \mathbf{R}^2}$ is an isotropic fractional Brownian field satisfying $W(0) = 0$ a.s. and $\gamma(x) = \text{var}(W(x))/2 = \sigma^2 \|x\|^\alpha / 2$ for some $\alpha \in (0, 1)$ and $\sigma^2 > 0$.

Let us define, for $k \neq j \geq 1$,

$$Z_{k \setminus j}(x) = Z_k(x) - Z_j(x), \quad x \in \mathbf{R}^2,$$

where

$$Z_i(x) = \log U_i + \log Y_i(x), \quad x \in \mathbf{R}^2.$$

In the same spirit as Dombry and Kabluchko (2018), we build a random tessellation of \mathbf{C} , $(\mathbf{C}_{k,j})_{k \neq j \geq 1}$ where

$$\mathbf{C}_{k,j} = \left\{ x \in \mathbf{C} : Z_k(x) \wedge Z_j(x) > \bigvee_{i \neq j,k} Z_i(x) \right\}. \quad (13)$$

If $\mathbf{C}_{k,j} \neq \emptyset$, we define for any Borel subset A of \mathbf{R} the occupation measure of $Z_{k \setminus j}$ over $\mathbf{C}_{k,j}$ by

$$\nu^{(k \setminus j)}(A) = \int_{\mathbf{C}_{k,j}} \mathbb{I}[Z_{k \setminus j}(x) \in A] dx.$$

The associated local time at level 0 is given by $L_{Z_{k \setminus j}}(0) := \frac{d\nu^{(k \setminus j)}}{d\ell}(0)$. If $\mathbf{C}_{k,j} = \emptyset$, we let $L_{Z_{k \setminus j}}(0) := 0$.

Let $U_{x_1, x_2}^{(\eta)}$ be the (normalized) increment of $\log(\eta)$ defined as

$$U_{x_1, x_2}^{(\eta)} = \frac{1}{\sigma \|x_2 - x_1\|^{\alpha/2}} \log \left(\frac{\eta(x_2)}{\eta(x_1)} \right).$$

The square increment sums are given respectively by

$$V_{2,N}^{(\eta)} = \frac{1}{\sqrt{|E_N|}} \sum_{(x_1, x_2) \in E_N} \left((U_{x_1, x_2}^{(\eta)})^2 - 1 \right)$$

$$V_{3,N}^{(\eta)} = \frac{1}{\sqrt{|DT_N|}} \sum_{(x_1, x_2, x_3) \in DT_N} \left(\begin{pmatrix} U_{x_1, x_2}^{(\eta)} & U_{x_1, x_3}^{(\eta)} \end{pmatrix} \begin{pmatrix} 1 & R_{x_1, x_2, x_3} \\ R_{x_1, x_2, x_3} & 1 \end{pmatrix}^{-1} \begin{pmatrix} U_{x_1, x_2}^{(\eta)} \\ U_{x_1, x_3}^{(\eta)} \end{pmatrix} - 2 \right),$$

where R_{x_1, x_2, x_3} is given in Eq. (5).

Theorem 3 *Let $\alpha \in (0, 1)$. Then, as $N \rightarrow \infty$,*

$$\frac{\sqrt{3}}{3} N^{-(2-\alpha)/4} V_{2,N}^{(\eta)} \xrightarrow{\mathbb{P}} c_{V_2} \sum_{j \geq 1} \sum_{k > j} L_{Z_{k \setminus j}}(0)$$

$$\frac{\sqrt{2}}{2} N^{-(2-\alpha)/4} V_{3,N}^{(\eta)} \xrightarrow{\mathbb{P}} c_{V_3} \sum_{j \geq 1} \sum_{k > j} L_{Z_{k \setminus j}}(0).$$

The results in Theorem 3 are quite similar with those in Theorem 2. It can be noted that there is an a.s. finite number of local times $L_{Z_{k \setminus j}}(0)$, $j \geq 1$ and $k > j$, which are positive. This is related to the fact that there is an a.s. finite number of non-empty cells of the canonical tessellation in \mathbf{C} .

Using the Slivnyak-Mecke formula (see e.g. Theorem 3.2.5 in Schneider-Weil (2008)) and the same arguments as in the proof of Proposition 3 in Robert (2020), we can state that

$$\lim_{N \rightarrow \infty} N^{\alpha/4} \mathbb{E} \left[\frac{1}{N} \sum_{(x_1, x_2) \in E_N} \left((U_{x_1, x_2}^{(\eta)})^2 - 1 \right) \right] = 4\sigma \mathbb{E} \left[D^{\alpha/2} \right] \psi$$

with

$$\psi = \int_0^\infty u \varphi(u) \left[1/2 - \bar{\Phi}(u) - u \bar{\Phi}(u) \Phi(u) / \varphi(u) \right] du \simeq -0.094.$$

As a consequence we deduce that c_{V_2} is negative.

4.5 Asymptotic properties of the MCLEs

We are now able to present the asymptotic properties of $\hat{\sigma}_{j,N}^2$ and $\hat{\alpha}_{j,N}$ for $j = 2, 3$. Let us recall that the sums of the contributions of the observations to the composite likelihood are proportional to the square increment statistics (see Propositions 2 and 4). Moreover the asymptotic behaviors of these statistics are characterized in Theorem 3.

Theorem 4 Assume that σ_0 belongs to the interior of a compact set of \mathbf{R}_+ , and that α_0 belongs to the interior of a compact set of $(0, 1)$. Then, as $N \rightarrow \infty$,

$$\begin{aligned} \frac{\sqrt{3}}{3} \sqrt{|E_N|} N^{-(2-\alpha_0)/4} (\hat{\sigma}_{2,N}^2 - \sigma_0^2) &\xrightarrow{\mathbb{P}} c_{V_2} \sigma_0^2 \sum_{j \geq 1} \sum_{k > j} L_{Z_{k \setminus j}}(0) \\ \frac{\sqrt{3}}{6} \sqrt{|E_N|} N^{-(2-\alpha_0)/4} \log(N) (\hat{\alpha}_{2,N} - \alpha_0) &\xrightarrow{\mathbb{P}} -c_{V_2} \sum_{j \geq 1} \sum_{k > j} L_{Z_{k \setminus j}}(0) \end{aligned}$$

and

$$\begin{aligned} \frac{\sqrt{2}}{2} \sqrt{|E_N|} N^{-(2-\alpha_0)/4} (\hat{\sigma}_{3,N}^2 - \sigma_0^2) &\xrightarrow{\mathbb{P}} c_{V_3} \sigma_0^2 \sum_{j \geq 1} \sum_{k > j} L_{Z_{k \setminus j}}(0) \\ \frac{\sqrt{2}}{4} \sqrt{|E_N|} N^{-(2-\alpha_0)/4} \log(N) (\hat{\alpha}_{3,N} - \alpha_0) &\xrightarrow{\mathbb{P}} -c_{V_3} \sum_{j \geq 1} \sum_{k > j} L_{Z_{k \setminus j}}(0). \end{aligned}$$

Several important points have to be highlighted. First the MCLEs of σ_0^2 and α_0 (when the other parameter is known) are consistent in our infill asymptotic setup. They have rates of convergence proportional to $N^{\alpha_0/4}$ for $\hat{\sigma}_{2,N}^2$ and $\log(N) N^{\alpha_0/4}$ for $\hat{\alpha}_{2,N}$ that differ from the expected rates of convergence $N^{1/2}$ and $\log(N) N^{1/2}$ as in Zhu and Stein (2002) for the isotropic fractional Brownian field. Second the type of convergence is in probability. The random variables appearing in the limits in Theorem 4 are proportional to a sum of local times. However these local times have unknown distributions and they cannot be estimated from the data since the underlying random fields $(Y_i)_{i \geq 1}$ and the point process $(U_i)_{i \geq 1}$ are not observed. In particular, if the spatial data are only observed for a single date, the Gaussian approximation for the MCLEs given in Padoan et al. (2010) (when several independent replications over time of the spatial data are available) should not be used.

The problem of joint parameter estimation of (σ_0^2, α_0) is left for future work, but it is expected that the respective rates of convergence will be modified into $N^{\alpha_0/4}/\log(N)$ and $N^{\alpha_0/4}$ as suggested by Brouste and Fukasawa (2018) in the case of a fractional Gaussian process ($d = 1$) observed on a regular grid.

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