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Valeurs extrêmes de mosaïques aléatoires

Résumé

Une mosaïque aléatoire est une partition aléatoire de l'espace euclidien en des polytopes appelés *cellules*. Ce type de structure apparaît dans divers domaines tels que la biologie cellulaire, les télécommunications et la segmentation d'images. Beaucoup de travail a déjà été effectué sur la *cellule typique* c'est-à-dire sur une cellule "choisie uniformément". Cependant, ces travaux ne tiennent pas compte de l'irrégularité de la mosaïque et d'éventuelles cellules pathologiques (par exemple, celles qui sont anormalement allongées ou anormalement grandes).

Dans cette thèse, on étudie les mosaïques aléatoires par une approche inédite : celle des valeurs extrêmes. En pratique, on observe la mosaïque aléatoire dans une fenêtre et on considère une certaine caractéristique géométrique (comme le volume, le nombre de sommets ou le diamètre des cellules). Le problème de base est d'étudier le comportement du maximum et du minimum, voire des statistiques d'ordre, de cette caractéristique pour toutes les cellules de la fenêtre lorsque la taille de celle-ci tend vers l'infini. Une telle approche permet non seulement de mieux comprendre la régularité de la mosaïque mais aussi d'étudier la qualité d'une approximation discrète d'un ensemble par des cellules d'une mosaïque aléatoire. Cette approche pourrait également fournir une piste inédite pour discriminer les processus ponctuels.

Les résultats de cette thèse portent principalement sur des théorèmes limites des extrêmes et des statistiques d'ordre pour diverses caractéristiques géométriques et diverses mosaïques aléatoires. En particulier, on obtient des vitesses de convergence en établissant de fines estimations géométriques. On déduit de l'étude du maximum des diamètres une majoration de la distance de Hausdorff entre un ensemble et son approximation dite de Poisson-Voronoï. On traite, notamment, de plusieurs aspects géométriques comme les problèmes de bord et la forme des cellules optimisantes. Enfin, dans le but de savoir comment se répartissent les cellules excédentes (celles dont la caractéristique est grande), on s'intéresse à la convergence de processus ponctuels associés et à la taille moyenne d'un cluster d'excédents. Les outils utilisés sont issus à la fois de la géométrie aléatoire (mesure de Palm, probabilités de recouvrement, formule de Slivnyak) et de la théorie des valeurs extrêmes (graphes de dépendance, méthode de Chen-Stein, indice extrême).

Extreme values of random tessellations

Abstract

A random tessellation is a partition of the Euclidean space into polytopes that are called *cells*. Such a structure appears in many domains such as cellular biology, telecommunications and image segmentation. Many results were established on the *typical cell* i.e. a cell which is "chosen uniformly" in the tessellation. Nevertheless, these works do not reflect the regularity of the tessellation and the pathology of several cells (e.g. elongated or big cells).

In this PhD thesis, we investigate the random tessellations by a new approach which is Extreme Value Theory. In practice, we observe the random tessellation in a window and we consider a geometrical characteristic (e.g. the volume, the number of vertices or the diameter of the cells). Our problem is to investigate the behaviour of the maximum and minimum (and more generally the order statistics) of this characteristic for the cells of the window when the size of the window tends to infinity. Such an approach leads to a better description of the regularity of the tessellation. It provides also some tools to investigate the quality of a discrete approximation between a set and the cells of a random tessellation. Another potential application field is the statistics of point processes.

Our results concern mainly limit theorems on the extremes and order statistics of various geometrical characteristics and random tessellations. In particular, we provide the rates of convergence with some delicate geometric estimates. We derive an upper bound of the Hausdorff distance between a set and its so-called Poisson-Voronoi approximation from the investigation of the maximum of diameters. Besides, we deal with geometrical aspects such as boundary effects and shape of the optimizing cells. Finally, in order to study the repartition of the exceedance cells (i.e. cells with a large characteristic), we are interested by the convergence of underlying point processes and by the mean size of a cluster of exceedances. Our tools come from stochastic geometry (Palm measure, Slivnyak's formula, covering probabilities) and Extreme Value Theory (dependency graphs, Chen-Stein method, extremal index).

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1. Les personnes qui n'ont pas été citées peuvent être rassurées : je pense bien à tout le monde !

A ma femme, Ania.

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Introduction (version française)

Géométrie aléatoire et mosaïques. La géométrie stochastique, ou géométrie aléatoire, est une branche des probabilités qui traite de modèles spatiaux aléatoires. Ce domaine s'est considérablement développé dans le dernier tiers du 20$^{\text{ème}}$ siècle notamment grâce aux travaux de Matheron qui développe une théorie d'ensembles aléatoires [89] en 1975 et de Miles qui traite de certains objets géométriques comme les processus ponctuels [95] et les mosaïques aléatoires [96] (voir également les travaux de Fallert [38] sur les modèles Booléens). La géométrie aléatoire offre un large champ d'applications dans diverses sciences telles que les statistiques spatiales [44], [101], la stéréologie [8], l'analyse d'image [142], l'astronomie [88], les télécommunications [6], les sciences des matériaux [141] et, de façon plus éloignée, la mécanique statistique [45]. Pour un large panorama de résultats et d'applications, on renvoie aux ouvrages de Schneider et Weil [130], Stoyan *et al.* [140] et Santaló [126].

On s'intéresse dans cette thèse à une structure spatiale particulière appelée mosaïque (convexe) aléatoire de l'espace euclidien $\mathbf{R}^d$. Un tel objet est une collection dénombrable de polytopes convexes d'intérieurs deux à deux disjoints appelés *cellules*, partitionnant l'espace, et telle que le nombre de cellules intersectant un sous-ensemble borné quelconque soit fini. Nous considérons essentiellement deux types de mosaïques aléatoires.

Le premier type est la mosaïque de Poisson-Voronoï. Considérons, pour cela, un ensemble fermé localement fini Φ de $\mathbf{R}^d$. A chaque point x de Φ, on se donne le lieu des points de $\mathbf{R}^d$ qui sont plus proches de x que de tout autre point de Φ i.e.

$$C_\Phi(x) = \{y \in \mathbf{R}^d, |y - x| \leq |y - x'|, x' \in \Phi\}$$

où $|\cdot|$ désigne la norme euclidienne de $\mathbf{R}^d$. Un tel ensemble s'appelle la *cellule de Voronoï* de germe x et est un polyèdre puisque c'est l'intersection finie des demi-espaces bordés par les hyperplans médiateurs. Les cellules de Voronoï sont introduites par Descartes en 1644 lorsqu'il étudie la répartition de la matière dans le système solaire. Ce concept est formalisé en 1850 par Dirichlet [34] dans un cadre déterministe et bi-dimensionnel lors de ses travaux sur les formes quadratiques définies positives puis étendu en dimension quelconque par Voronoï [143] en 1908. Une cellule de Voronoï délimite la zone d'influence d'un point. Les tâches des girafes, par exemple, sont des cellules de Voronoï car les cellules productrices de mélanine s'étendent jusqu'à rencontrer la matière issue d'autres tâches.

En 1953, pour modéliser la formation des cristaux, Meijering [93] utilise les cellules de Voronoï en considérant un ensemble de germes aléatoires et plus précisément un processus ponctuel de Poisson (qu'on suppose homogène). Ce modèle est appelé, par la suite, la mosaïque de Poisson-Voronoï et est utilisé dans un grand nombre de domaines dans lesquels interviennent des territoires de prédominance. En géographie, Boots [13], [14] s'en sert pour évaluer les aires de service des transports publics de l'Angleterre et des pays de Galles en modélisant les stations de bus par un processus ponctuel de Poisson. En biologie moléculaire, Gerstein *et al.* [46] approchent les protéines par des cellules de Voronoï en considérant les centres des atomes comme un ensemble aléatoire. Plus généralement, un grand nombre de domaines qui relèvent de territoires de prédominance utilisent les mosaïques de Poisson-Voronoï. Citons, par exemple, les applications en astronomie [120], télécommunications [6], [41], segmentation d'image [21] et en modèles de percolation [68], [69].

Le second type de modèle auquel on s'intéresse est la mosaïque de Poisson-Delaunay qui s'obtient par dualité de la mosaïque de Poisson-Voronoï. Considérons, d'abord, un ensemble déterministe localement fini Φ de $\mathbf{R}^d$. On relie deux points x et x' de Φ par une arête dès lors que les cellules de Voronoï sont voisines c'est-à-dire lorsque $C_\Phi(x) \cap C_\Phi(x') \neq \varnothing$. L'ensemble de ces arêtes crée une partition de l'espace en des polyèdres appelés *cellules de Delaunay.* Ce type de partition a fourni le premier algorithme pour calculer l'arbre couvrant de poids minimal (MST) [62] et est utile dans divers domaines comme la segmentation d'images [137] et la méthode des éléments finis [70] pour construire un maillage adéquat.

Lorsque les germes forment un processus ponctuel de Poisson (homogène), on parle de mosaïque de Poisson-Delaunay. Les cellules obtenues sont alors des simplexes. En théorie quantique des champs, cette mosaïque aléatoire fournit un bon réseau aléatoire, invariant par isométrie, ce que ne permettent pas les réseaux réguliers. Christ *et al.* [26], par exemple, proposent de modéliser les noyaux par un processus ponctuel de Poisson et les liens entre les particules par les arêtes de la mosaïque de Poisson-Delaunay. En considérant un modèle d'Ising et un modèle de Potts sur cette mosaïque, et grâce à la méthode de Monte Carlo, Espriu *et al.* [37], Janke *et al.* [65] et Janke et Villanova [66] montrent que la mosaïque de Poisson-Delaunay fournit un support qui se comporte comme celui d'un réseau régulier et montrent l'efficacité de ce modèle. En mécanique statistique, David et Drouffe [30] l'utilisent également pour modéliser des membranes. En microstructures, Ostoja-Starzewski [109] propose la mosaïque de Poisson-Delaunay pour les matériaux fibreux. Pour plus d'informations sur l'utilisation des mosaïques de Poisson-Voronoï et Poisson-Delaunay, on renvoie le lecteur au paragraphe 5.3 du livre d'Okabe *et al* [108].

Une notion fondamentale pour étudier les mosaïques aléatoires est celle de *cellule typique.* Cette dernière est un polytope aléatoire qui décrit le "comportement moyen" de la mosaïque c'est-à-dire une cellule "prise au hasard". On peut distinguer trois types de résultats : ces derniers portent respectivement sur des calculs de moments ou de lois de la cellule typique, sur la forme de cette cellule et sur le comportement global de la mosaïque. Pour les calculs de moments et de lois, l'un des premiers travaux sur la cellule typique de Poisson-Voronoï remonte à Gilbert [47] qui, à ce jour, donne le meilleur encadrement non asymptotique de la queue de distribution du volume. Par la suite, les travaux de Miles [94] et Møller [98], [99], Hayen et Quine [52] et Muche et Ballani [103] ont permis notamment d'obtenir des calculs de moyennes de diverses caractéristiques de la cellule typique. En 1970, Miles [95] donne également une représentation intégrale de la cellule typique de Poisson-Delaunay et détermine les moments à tout ordre de son volume. Plusieurs résultats distributionnels ont été découverts depuis : Rathie [121] et Muche [102] donnent respectivement les lois du volume et de l'aire d'une face de la cellule typique de Poisson-Delaunay, Zuyev [148] calcule la loi du volume de la fleur de Voronoï. Plus récemment, d'autres résultats sur les lois de diverses caractéristiques sont dus à Calka qui détermine, en particulier, les lois du rayon circonscrit [15] et du nombre de côtés [16] dans le cas d'une mosaïque de Poisson-Voronoï planaire. Un deuxième type de travail porte sur la forme de la cellule typique. Une conjecture, due à Kendall, et initialement réservée à une certaine mosaïque aléatoire appelée la mosaïque Poissonienne de droites, affirme, informellement, que la cellule typique de cette mosaïque aléatoire ressemble à une sphère lorsqu'elle est grande. Kovalenko [76] obtient un résultat similaire dans le cas d'une mosaïque de Poisson-Voronoï planaire tandis qu'une version beaucoup plus générale et en dimension quelconque est traitée par Hug *et al.* [61]. Dans les articles [62], [63] Hug et Schneider étudient des variantes de cette célèbre conjecture et montrent notamment que la cellule typique de Poisson-Delaunay tend à être un simplexe régulier lorsqu'elle est grande. Enfin, un troisième type de travail porte sur le comportement global de ces mosaïques aléatoires et est étudié dans plusieurs directions. Avram et Bertsimas [5] puis Heinrich et Muche [54] donnent des théorèmes centraux limites pour diverses caractéristiques géométriques de la mosaïque de Poisson-Voronoï. Dans le cadre de la stéréologie, Heinrich [53] puis Muche et Nieminen [104] étudient l'intersection de cette mosaïque avec des sous-espaces affines. En proposant un modèle de routage pour les réseaux mobiles, Baccelli *et al.* [7] étudient des chaînes de Markov sur des graphes de Poisson-Delaunay planaires. Mentionnons enfin des travaux effectués sur l'approximation dite de Poisson-Voronoï. Cette dernière consiste à discrétiser un corps convexe par des cellules de Poisson-Voronoï. Une telle approximation a été introduite par Khmaladze et Torondjaze [75]. Elle a, depuis, été approfondie par Heveling et Reitzner [57] puis par Schulte [131] qui établissent respectivement des résultats de grandes déviations et un théorème central limite.

Cependant, le travail sur ces mosaïques est principalement consacré à tout ce qui relève de la moyenne (cellule typique, théorèmes centraux limites . . .). Dans cette thèse, nous étudions ces modèles par une approche qui, à notre connaissance, est inédite : celle des valeurs extrêmes.

Valeurs extrêmes. La théorie des valeurs extrêmes est un domaine à l'interface des probabilités et des statistiques dont le problème de base est d'étudier des événements de type extrême comme, par exemple, la température maximale lors d'une vague de chaleur, la force des vents, les pics de pollution ou encore la résistance des matériaux. En hydrologie [74], en particulier, l'étude des niveaux élevés de la mer permet de savoir à quelle hauteur construire les digues. Les valeurs extrêmes jouent également un rôle fondamental en finances [36] notamment pour les assurances et les risques dans les marchés. Pour d'autres applications de la théorie des valeurs extrêmes, on renvoie au livre de Beirlant *et al.* [11].

Cette théorie a d'abord été développée dans un contexte univarié pour des observations réelles unidimensionnelles et pour des échantillons indépendants c'est-à-dire pour une suite de variables aléatoires réelles iid $(X_i)_{i\geq 1}$. Elle démarre en 1927 avec les travaux de Fréchet [42], Fisher et Tipett [40] puis Gnedenko [48] et Gumbel [51] qui montrent, notamment, que les seules lois apparaissant naturellement sont de trois types. Ces lois, dites max-stables, sont respectivement les lois de Fréchet, Gumbel et Weibull. Plusieurs travaux ont été effectués pour étendre l'étude à des suites dépendantes vérifiant une condition d'indépendance asymptotique. Loynes [86] et Welsch [145], par exemple, considèrent des suites avec une propriété de mélange fort. Leadbetter [80] introduit, en 1974, une des hypothèses les plus faibles en supposant deux conditions sur la suite $(X_i)_{i\geq 1}$ (l'une globale et l'autre locale) et établit un résultat sur les statistiques d'ordre de la suite c'est-à-dire sur les r plus grandes valeurs. Lorsque l'hypothèse locale de Leadbetter n'est pas satisfaite, les variables aléatoires dépassant un certain seuil se font par clusters. La notion sous-jacente est celle *d'indice extrême*, terme introduit par Leadbetter *et al.* [82] en 1983. Celle-ci conduit à de nombreuses recherches en probabilités [39], [83] et en statistiques [79], [136]. Parallèlement au travail sur l'indice extrême et en prenant des conditions un peu plus fortes que les deux conditions de Leadbetter, Smith [135] obtient des vitesses de convergence grâce à la méthode de Chen-Stein sur l'approximation Poissonienne. Plus récemment, l'étude s'est portée sur des champs discrets [25], [84] ou continus comme les processus Gaussiens [32], [72]. Nous ne pouvons présenter de façon exhaustive la théorie des valeurs extrêmes tant ce domaine a pris une extension considérable de nos jours. Cependant, nous citons deux types de travaux fondamentaux pour montrer la richesse de cette théorie bien que cela ne nous serve pas dans cette thèse. Le premier type de travail concerne la modélisation des excès (la méthode POT), essentiellement dû à Pickands [116] ainsi que les valeurs extrêmes dans un contexte multivarié [117]. Un autre type de travail porte sur les processus max-stables avec les travaux de Schlather [128], et plus récemment de Kabluchko [71]. Pour un large panorama de résultats sur les valeurs extrêmes, on renvoie aux livres de de Haan et Ferreira [31], Leadbetter *et al.* [82], et Resnick [124].

Sujet de la thèse. Dans cette thèse, on propose une approche inédite qui consiste à étudier les mosaïques aléatoires par les valeurs extrêmes. En pratique, on observe la mosaïque aléatoire dans une fenêtre vouée à décrire l'espace c'est-à-dire dans un ensemble $\mathbf{W}_\rho = \rho^{1/d}W$ où W est un Borélien borné de $\mathbf{R}^d$ et de volume non nul. Les quantités considérées sont des caractéristiques géométriques des cellules comme le diamètre, le nombre de sommets ou le volume. On s'intéresse au comportement asymptotique du maximum, voire plus généralement des statistiques d'ordre, de ces quantités prises sur toutes les cellules de la fenêtre $\mathbf{W}_\rho$ lorsque ρ tend vers l'infini. Concrètement, il s'agit de déterminer, pour différentes caractéristiques et différentes mosaïques aléatoires, des paramètres de normalisation c'est-à-dire des fonctions dépendant de ρ de sorte que le maximum (ou les statistiques d'ordre) convenablement renormalisé par ces fonctions converge vers une loi non dégénérée.

L'étude des mosaïques aléatoires par les valeurs extrêmes peut avoir un grand nombre d'applications. Elle permet, en particulier, de décrire la régularité de la mosaïque aléatoire. Une telle notion est utile notamment en méthode des éléments finis parce que la qualité de l'approximation dépend de la régularité du maillage. La question de l'existence de pavages pseudo-réguliers d'espaces métriques divers semble par ailleurs d'intérêt pour les géomètres. Une deuxième application porte sur l'approximation de Poisson-Voronoï. Si plusieurs résultats ont été établis sur la loi du volume et du volume de la différence symétrique, la distance de Hausdorff entre un corps convexe et son approximation n'a pas été étudiée. Celle-ci, cependant, est majorée par le maximum des diamètres des cellules de la mosaïque. En particulier, l'étude de cette caractéristique fournit un résultat sur cette approximation. Enfin, un autre domaine, potentiel, porte sur les statistiques de processus ponctuels. L'idée serait d'identifier le processus ponctuel à partir des extrêmes de diverses mosaïques qu'il induit, comme les mosaïques de Voronoï et de Delaunay. Plusieurs travaux ont été faits sur des méthodes d'inférence [100] ou des comparaisons de processus ponctuels (voir, par exemple, la comparaison entre un processus ponctuel déterminental et un processus ponctuel de Poisson dans [85]). Une discrimination des processus ponctuels fondée sur les extrêmes de mosaïques aléatoires serait inédite et mériterait d'être comparée aux méthodes existantes.

La théorie classique des valeurs extrêmes n'est pas suffisante pour résoudre le problème car les variables aléatoires que nous considérons ne sont définies ni comme une suite, ni comme un champ aléatoire. De plus, les lois de probabilité sous-jacentes ne sont en général pas connues et les cellules dépendent, en général, les unes des autres. Cependant, ces points peuvent être contrecarrés car les mosaïques aléatoires que nous considérons présentent une propriété de mélange. Nos résultats portent principalement sur des théorèmes limites des extrêmes et des statistiques d'ordre et sur des aspects géométriques comme les problèmes de bord et la forme des cellules optimisantes.

Cette thèse s'articule autour de quatre chapitres. Le premier est introductif, les deux suivants sont présentés sous la forme d'articles en anglais, précédés d'introductions partielles, et le dernier porte sur des travaux en cours et des perspectives.

Dans le premier chapitre, on présente des outils de géométrie aléatoire et de théorie des valeurs extrêmes. En particulier, on trouvera des rappels sur les processus ponctuels, les mosaïques aléatoires, la mesure de Palm et la notion de cellule typique ainsi que quelques résultats connus (section 1.1). Dans les outils de valeurs extrêmes, on présente les conditions faibles de Leadbetter, un résultat d'approximation Poissonienne ainsi que la notion d'indice extrême qui reflète la taille des clusters d'excédents pour une suite de variables aléatoires réelles (section 1.2). On trouvera, à la fin du chapitre, une description formelle du problème que nous considérons (section 1.3).

Dans le deuxième chapitre, on étudie les caractéristiques radiales de la mosaïque de Poisson-Voronoï c'est-à-dire le rayon inscrit qui est le rayon de la plus grande boule incluse dans la cellule et centrée en le germe et le rayon circonscrit qui est le rayon de la plus petite boule contenant la cellule et également centrée en le germe. En particulier, on détermine les comportements asymptotiques des maxima et minima des rayons circonscrits (respectivement inscrits) et on montre que la participation des cellules frontières n'affecte pas le comportement des extrêmes. Cela signifie qu'on peut indifféremment considérer divers types de cellules : celles dont le germe est dans la fenêtre, celles qui intersectent la fenêtre ou encore les cellules incluses dans la fenêtre (plus faciles à manipuler pour des données concrètes). On obtient également une majoration de la distance de Hausdorff entre un corps convexe et son approximation de Poisson-Voronoï et on montre que la cellule qui minimise le rayon circonscrit est un simplexe. Les notions utilisées sont essentiellement géométriques et reposent principalement sur des problèmes de recouvrement. Ce travail a fait l'objet d'un article co-écrit avec P. Calka et accepté dans *Extremes*.

Dans le troisième chapitre, on établit une méthode plus large permettant de traiter n'importe quelle caractéristique géométrique et un certain type de mosaïques aléatoires satisfaisant une condition de mélange. On établit un théorème principal nécessitant deux propriétés de la mosaïque aléatoire, l'une globale et l'autre locale, qui ramène l'étude des extrêmes à celle de la cellule typique. En appliquant ce théorème à diverses caractéristiques géométriques, on obtient des résultats sur la mosaïque de Poisson-Delaunay, la mosaïque de Poisson-Voronoï et la mosaïque de Voronoï induite par un processus ponctuel non Poissonien. Des exemples d'application comprennent les maxima et minima des aires d'une mosaïque de Poisson-Delaunay planaire, le minimum des volumes des fleurs de Voronoï ou encore le minimum des distances aux germes les plus éloignés. Pour décrire les lois jointes des statistiques d'ordre, on s'intéresse également à la répartition des cellules dont la caractéristique géométrique est au-dessus d'un certain seuil. On montre alors que le processus ponctuel des germes de ces cellules converge vers un processus ponctuel de Poisson non homogène. Enfin, lorsque l'hypothèse locale du théorème principal n'est pas satisfaite, on montre un résultat analogue à celui de Leadbetter [81] sur l'indice extrême. Ce travail a fait l'objet d'un second article (soumis).

Dans le dernier chapitre, on présente quelques travaux en cours et quelques perspectives. En particulier, on s'intéresse, en dimension 2, au minimum des aires et au maximum du nombre de sommets d'une mosaïque de Poisson-Voronoï. On étudie également le minimum des angles des triangles d'une mosaïque de Poisson-Delaunay planaire. Cette partie contient notamment des simulations et fait l'objet d'un travail en cours avec R. Hemsley.

Introduction (english version)

Stochastic Geometry and tessellations. Stochastic geometry, or random geometry, is a branch of probability theory which deals with random geometrical structures and spatial data. This domain was considerably developed in the last third of the 20^{th} century. In 1975, Matheron created a theory of random sets [89] while Miles investigated several geometrical objects as point processes [95] and random tessellations [96] (see also Boolean models by Fallert [38]). Stochastic geometry provides many applications in different sciences such as spatial statistics [44], [101], stereology [8], image analysis [142], astronomy [88], telecommunications [6], materials science [141] and statistical mechanics [45]. For a panorama of results and applications, we refer to the books by Santaló [126], Schneider and Weil [130] and Stoyan *et al.* [140].

In the present PhD thesis, we investigate random (convex) tessellations in the Euclidean space $\mathbf{R}^d$. Such a model is a subdivision of the space into a countable collection of convex polytopes called *cells* with disjoint interiors such that the number of cells intersecting any bounded subset of $\mathbf{R}^d$ is finite. We consider generally two random tessellations.

Our first model is the Poisson-Voronoi tessellation. To construct this tessellation, we consider a closed subset Φ in $\mathbf{R}^d$ which is locally finite. To each point x of Φ, we consider that set of points $C_\Phi(x)$ in $\mathbf{R}^d$ which are closer to x than the other points of Φ i.e.

$$C_\Phi(x) = \{y \in \mathbf{R}^d, |y - x| \leq |y - x'|, x' \in \Phi\}$$

where $|\cdot|$ is the Euclidean norm of $\mathbf{R}^d$. Such a subset is called the *Voronoi cell* of nucleus x. It is a polyhedra since it is the intersection of half-spaces delimited by the bisecting hyperplanes. In 1644, Descartes introduced Voronoi cells to investigate the placement of the matter in the universe. In 1850, Dirichlet [34] specified this concept in a deterministic context in $2D$ in order to study positive definite quadratic forms. In 1908, Voronoi [143] extended this notion to any dimension. A Voronoi cell is the influence space of a point. The patches of a giraffe, for example, are Voronoi cells. It comes from the fact that the cells producing melanin grow until they meet the melanin of other patches.

In 1953, Meijering [93] used Voronoi tessellation to model crystal aggregates and considered the set of nuclei as the points of a (homogeneous) Poisson point process. Later, this model was called a Poisson-Voronoi tessellation. It is extensively used in various sciences. In geography, Boots [13], [14] uses this concept to evaluate areas associated with public bus services in England and Wales. In molecular biology, Gerstein, Tsai and Levitt [46] approximate proteins by Voronoi cells considering the centers of nuclei as a random set. More generally, Poisson-Voronoi tessellation is used in modelling of territories of predominance. See e.g. applications in astronomy [120], telecommunications [6],[41], image segmentation [21] and in percolation models [68], [69].

Another model that we investigate is the Poisson-Delaunay tessellation. It corresponds to the dual graph of the Poisson-Voronoi tessellation in the following sense: there exists an edge between two points $x, x' \in \chi$ in the Delaunay graph if and only if they are Voronoi neighbors i.e. $C_\chi(x) \cap C_\chi(x') \neq \varnothing$. The set of these edges induces a partition of the space into polyhedra that are called *Delaunay cells*. Such a partition gives the first algorithm in order to calculate the minimum spanning tree (MST) [62] and is used in a large number of domains such as image segmentation [137] and finite element method [70] to build meshes.

When the set of nuclei is a (homogeneous) Poisson point process, it is the so-called Poisson-Delaunay tessellation. The cells of such a tessellation are simplices. In quantum field theory, this random tessellation is very interesting since it provides a random network which is motion invariant, quite the opposite to regular networks. Considering the set of nuclei as a Poisson point process, Christ *et al.* [26] model the links between particles as an edge of a Poisson-Delaunay tessellation. Espriu *et al.* [37], Janke *et al.* [65] and Janke and Villanova [66] defined the model of Ising and the model of Potts on this random tessellation and showed that the Poisson-Delaunay tessellation provides a support which has the same behaviour as a regular network so that this model is an efficient one. In statistical mechanics, David and Drouffe [30] use this tessellation as well to model membranes. In microstructure, Ostoja-Starzewski [109] proposes to use this model for fibrous materials. For a wider panorama of use of Poisson-Voronoi and of Poisson-Delaunay tessellations, see section 5.3. in [108].

In order to investigate random tessellations, a notion of *typical cell* is fundamental. The *typical cell* is a random polytope which describes the "mean behaviour" of a random tessellation. Roughly speaking, it has the same distribution as a randomly chosen cell selected in such a way that every cell has the same chance of being sampled. The results can be divided into three types : moments and distributional computations of the typical cell, shape of the typical cell and global behaviour of the random tessellation. For moments and distributional results, one of the first works on the typical cell of the Poisson-Voronoi tessellation is due to Gilbert [47] who established the best lower and upper bounds on the tail of the distribution function of the volume. Later, Miles [94], Møller [98], [99], Hayen and Quine [52] and Muche and Ballani [103] calculated several means of several geometrical characteristics of the typical cell. In 1970, Miles [95] gave an integral representation of the Poisson-Delaunay typical cell and obtained the moments of its volume. Rathie [121] and Muche [102] derived respectively the distribution functions of the volume and the area of a face of the Poisson-Delaunay typical cell and Zuyev [148] calculated the distribution function of the Voronoi flower. More recently, Calka calculated several distributions of many characteristics such as the circumradius [15] and the number of vertices [16] of the Poisson-Voronoi typical cell in $2D$. The second type of results concerns the shape of the typical cell. Due to a conjecture of Kendall, if the typical cell of the Poisson line tessellation (which is a particular case of random tessellation) is large it has the shape of a sphere. Kovalenko [76] obtained a similar result in the case of a planar Poisson-Voronoi tessellation. The general case was proved and largely extended by Hug *et al.* in [61]. Besides, Hug and Schneider [62], [63] investigated variants of this famous conjecture and showed that the Poisson-Delaunay typical cell tends to be a regular simplex when it is large. The third type of results concerns the global behaviour of these random tessellations and can be divided into many directions. Avram and Bersimas [5] and Heinrich and Muche [54] established central limit theorems for several geometrical characteristics of the Poisson-Voronoi tessellation. In stereology, Heinrich [53] and Muche and Nieminen [104] investigated the intersection of this tessellation with affine subspaces. To model mobile networks, Baccelli *et al.* [7] studied Markov chains on planar Poisson-Delaunay graphs. Finally, we mention several works devoted to the so-called Poisson-Voronoi approximation. Such an approximation was introduced by Khmaladze and Torondjaze [75]. It has been extended by Heveling and Reitzner [57] and Schulte [131] who established large deviations and a central limit theorem respectively.

Nevertheless, the work on Poisson-Voronoi and Poisson-Delaunay tessellations was mainly devoted to describe their means (typical cell, central limit theorems . . .). In this PhD thesis, we investigate these models by a new approach (to the best of our knowledge) which is Extreme Value Theory.

Extreme Values. Extreme Value Theory is a domain of probability theory and statistics where the main problem is to investigate extreme events such as maximal temperature, strong winds, pollution peaks and strength of materials. One typical example of science that largely uses this theory is hydrology [74]. Indeed, we need to know the extreme values of the level of the sea to build dikes. Extreme values are also useful in finance [36] especially in insurance and in market risk estimation as well as in climatology [146]. For a wide panorama of applications of Extreme Value Theory, see Beirlant *et al.* [11].

Historically this theory has been developed for a sequence of iid real random variables $(X_i)_{i\geq 1}$. It starts in 1927 by the works of Fréchet [42], Fisher and Tipett [40], Gnedenko [48] and Gumbel [51] who showed that the "natural distributions" that appear in extremes can be divided into three types. These so-called *max-stable* distributions belong to the Fréchet's, Gumbel's and Weibull's distribution families. There are many studies devoted to extend this notion up to dependent sequences of random variables with a weak condition of independence. For example, Loynes [86] and Welsch [145] consider sequences which satisfy a strong mixing property. In 1974, Leadbetter [80] introduced one of the weakest assumptions with two conditions on the sequence $(X_i)_{i\geq 1}$: a global condition and a local one. He provided with these conditions the asymptotic behaviour of the order statistics i.e. of the r largest values of the sequence. When the local condition is not satisfied, the random variables larger than a given threshold are aggregated into clusters. The underlying notion here is *extremal index*. It was introduced by Leadbetter *et al.* [82] in 1983. This notion leads to many problems in probability theory [39], [83] and in statistics [79], [136]. Smith [135], who considered restrictive conditions than Leadbetter, obtained several rates of convergence thanks to a Poisson approximation and to the Chen-Stein method. More recently, the investigation of extreme values has been devoted to discrete random fields [25], [84] and to continuous random fields such as Gaussian processes [32], [72]. This domain is quite large and there were a lot of works published recently on this topic, so we cannot mention all its branches. Still, we quote two fundamental concepts, without being exhaustive, even if we do not use them in our work. One of them is the Point Over Threshold (POT) method, which due to Pickands [116], deals with the exceedances over a threshold. Another Pickands' work concerns extreme values in the multivariate case [117]. A large branch of this domain concerns max-stable processes in the works of Schlather [128] and, more recently, Kabluchko [71]. For a wider panorama of results in Extreme Value Theory, see de Haan and Ferreira [31], Leadbetter *et al.* [82] and Resnick [124].

Thesis topic. In our PhD thesis, our objective is to investigate random tessellations by a new approach which is Extreme Value Theory. More precisely, we are interested in the following problem: only a part of the tessellation is observed in the window $W_\rho = \rho^{1/d}W$ where W is a bounded Borel subset of $\mathbf{R}^d$ with non-zero volume. The quantities considered are geometrical characteristics of the cells e.g. the diameter, the number of vertices and the volume. We investigate the asymptotic behaviour of the maximum and, more generally, the order statistics of these quantities for the cells of the window W_ρ when ρ tends to infinity. Concretely, for different characteristics and different random tessellations we are looking for suitable parameters which are functions, depending on ρ, so that the maximum (or the order statistics) normalized by these functions converges to a non degenerate random variable.

Here are some applications of the extreme value approach in random tessellations. Firstly, the study of extremes gives an idea of the regularity of a tessellation. For instance, in finite element method, the quality of the approximation depends on consistency measurements over the partition. Moreover, the existence of pseudo-regular tessellations in metric space can provide useful tools in geometry. The second application concerns the Poisson-Voronoi approximation. We have said that the majority of results deals with the distributions of the volume and the volume of the symmetric difference. But the Hausdorff distance between a convex body and its approximation has never been studied. Nevertheless, it is connected to the maximum of the diameters of cells, which intersect the boundary of the convex body. Hence, the investigation of the diameter maximum provides an upper bound of the Hausdorff distance of this approximation. Finally, another potential application field is the statistics of point processes. The key idea is to identify a point process from the extremes of its underlying Voronoi tessellation. A lot of inference methods have been developed for spatial point processes [100]. A comparison based on Voronoi extremes may or may not provide stronger results. At least, the regularity seems to discriminate to some extent some point processes (see for instance a comparison between a determinantal point process and a Poisson point process in [85]).

Classical Extreme Value Theory cannot be applied to examine our objective for several reasons: unknown distribution of the characteristic for one fixed cell and inter-dependence between cells. Besides, the set of considered random variables is not a sequence neither a discrete random field. Still, we can elude this problem because we consider that the random tessellations have a mixing property. Our results concern limit theorems of extremes and order statistics and geometrical problems such as boundary effects and the shape of the optimal cells.

Our thesis is divided into four chapters. The first one is an introduction, the second and the third ones are based on articles which highlight geometric and probabilistic aspects. The last chapter is devoted to our works in progress and to perspectives.

In the first chapter, we present basic tools of stochastic geometry and Extreme Value Theory. In particular, we recall several notions about point processes, random tessellations, Palm measure and *typical cell* and finally we present already known results (section 1.1). Regarding Extreme Value Theory, we present weak conditions of Leadbetter and Poisson approximation results. We also mention the works about extremal index which describes the size of the clusters of exceedances for a sequence of real random variables (section 1.2). At the end of the chapter, we describe precisely the topic of our thesis (section 1.3).

In the second chapter, we investigate the characteristic radii of the Poisson-Voronoi tessellation: the inradius, i.e. the radius of the largest ball centered at the nucleus and included into the cell, and the circumscribed radius, i.e. the radius of the smallest ball centered at the nucleus and containing the cell. In particular, we describe the asymptotic behaviours of circumradii (and respectively inradii) of their maxima and minima and we show that boundary cells are negligible. This fact shows that any kind of cells could be considered i.e. cells with the nucleus in the window as well as cells intersecting the window and cells included in the window (which are more convenient to deal with for practical purpose). We obtain also an upper bound of the Hausdorff distance between a convex body and its Poisson-Voronoi approximation and then we show that the cell minimizing the circumradii is a simplex. The notions that we use are mainly geometrical. For instance, we use coverings of the sphere. This work led to a paper (accepted in *Extremes*) in collaboration with P. Calka.

In the third chapter, we develop a more general method. We consider any geometrical characteristic and we restrict our investigation to a certain kind of random tessellation satisfying a strong mixing property. In our main theorem, we assume two conditions which are global and local respectively. Our theorem guarantees that knowing the characteristic of the typical cell is enough to investigate the extremes. When applying this theorem to several characteristics, we derive a large number of results on a Poisson-Delaunay tessellation, a Poisson-Voronoi tessellation and a Voronoi tessellation induced by a non-Poisson point process. In particular, we investigate the maximum and minimum of the areas of the Poisson-Delaunay cells. For a Poisson-Voronoi tessellation, we study the minimum of distances of the farthest nucleus and the minimum of the volume of flowers. To describe the joint distributions of order statistics of a certain geometrical characteristic, we investigate the placement of the cells so that the characteristic is larger than a given threshold. We show that the point process of nuclei of the cells converges to a non homogeneous Poisson point process. Finally, when the local condition of the main theorem is not satisfied, we get the result (close to the result of Leadbetter [81]) about the extremal index. This work led to another (submitted) paper.

In the last chapter, we present our works in progress. In $2D$, we investigate the minimum of the areas and the maximum of the number of vertices of a Poisson-Voronoi tessellation. We also investigate the minimum of the angles of a Poisson-Delaunay tessellation. We present in this part several simulations. This is a work in collaboration with R. Hemsley. The end of the chapter is devoted to perspectives.

Chapitre 1

Notions de géométrie aléatoire et de valeurs extrêmes

Sommaire

Dans ce chapitre, on présente des notions de géométrie aléatoire et de valeurs extrêmes qui nous serviront dans les chapitres 2, 3 et 4. La fin du présent chapitre est consacrée à une présentation formelle de notre sujet de thèse.

1.1 Outils de Géométrie aléatoire

1.1.1 Processus ponctuels et mosaïques aléatoires

Dans cette section, on présente quelques notions et outils issus de la géométrie aléatoire. L'objet de base de cette branche des probabilités est ce qu'on appelle un *ensemble aléatoire*. On présente, en particulier, deux exemples d'ensembles aléatoires que sont les processus ponctuels et les mosaïques aléatoires. Pour donner un sens à ce qu'on appelle *ensemble aléatoire*, il convient de construire une tribu adéquate. Cette dernière est présentée ci-dessous dans un cadre général.

Topologie et tribu de Fell Désignons par E un espace topologique localement compact et respectivement par $\mathcal{F}(E)$, $\mathcal{C}(E)$ et $\mathcal{O}(E)$ l'ensemble de ses fermés, de ses compacts et de ses ouverts. Pour tout $C \in \mathcal{C}(E)$ et pour tout $O \in \mathcal{O}(E)$ de E, on désigne par $\mathcal{F}^C(E)$ et $\mathcal{F}_O(E)$ les ensembles

$$\mathcal{F}^C(E) := \{F \in \mathcal{F}(E), F \cap C = \varnothing\} \text{ et } \mathcal{F}_O(E) := \{F \in \mathcal{F}(E), F \cap O \neq \varnothing\}.$$

La famille décrite par ces ensembles génère sur $\mathcal{F}(E)$ une topologie *compacte* dite de *Fell*. On munit $\mathcal{F}(E)$ de sa tribu Borélienne appelée *tribu de Fell* et notée $\mathfrak{F}(E)$. De façon équivalente, on peut construire cette tribu comme la tribu engendrée par les $\mathcal{F}^C$, $C \in \mathcal{C}(E)$. Par *fermé aléatoire*, on entend une variable aléatoire définie sur un espace probabilisé $(\Omega, \mathfrak{A}, \mathbb{P})$ hypothétique et à valeurs dans $\mathcal{F}(E)$.

Lorsque $E = \mathbf{R}^d$, on pose $\mathcal{F}_d = \mathcal{F}(\mathbf{R}^d)$, $\mathcal{C}_d = \mathcal{C}(\mathbf{R}^d)$, $\mathcal{O}_d = \mathcal{O}(\mathbf{R}^d)$ et l'on désigne par $\mathcal{K}_d$ l'ensemble des corps convexes de $\mathbf{R}^d$ c'est-à-dire l'ensemble de ses convexes compacts. La topologie de Fell coïncide alors avec la topologie induite par la distance de Hausdorff sur l'ensemble des corps convexes non vides noté $\mathcal{K}'_d = \mathcal{K}_d - \{\varnothing\}$. On désigne plus simplement par $\mathfrak{F}_d = \mathfrak{F}(\mathbf{R}^d)$ la tribu de Fell associée.

Processus ponctuels La notion de fermé aléatoire est très générale. Pour avoir des modèles adéquats et plus concrets, on restreint notre attention au cas particulier des processus ponctuels. Un *processus ponctuel* Φ de E est un fermé aléatoire prenant presque sûrement ses valeurs dans l'ensemble $\mathcal{F}_{lf}(E)$ des fermés localement finis de E i.e.

$$\mathcal{F}_{lf}(E) = \{F \in \mathcal{F}(E), \#F \cap C < \infty \text{ pour tout } C \in \mathcal{C}(E)\}$$

où $\#A$ désigne le cardinal d'un ensemble $A \subset E$. A un processus ponctuel Φ, on peut associer ce qu'on appelle une mesure d'intensité. Cette dernière, notée Θ_Φ, est définie par

$$\Theta_\Phi(B) = \mathbb{E}\left[\#\Phi \cap B\right]$$

où B est un Borélien de E. En particulier, Θ_Φ est une mesure sur E. On suppose, en pratique, que Θ_Φ soit finie sur les compacts. Lorsque $E = \mathbf{R}^d$ et que Φ est stationnaire, c'est-à-dire invariante par translation en loi, la mesure d'intensité est stationnaire et localement finie, donc proportionnelle à la mesure de Lebesgue. Le coefficient de proportionnalité s'appelle *l'intensité* de Φ et est notée $\gamma_\Phi \in [0, \infty)$. Autrement dit

$$\Theta_\Phi = \gamma_\Phi \lambda_d$$

où λ_d désigne la mesure de Lebesgue d-dimensionnelle.

Dans ce qui suit, on présente la notion fondamentale de convergence de processus ponctuels puis on donne deux exemples de processus ponctuels.

Convergence de processus ponctuels Replaçons-nous dans le cas d'un espace topologique localement compact quelconque E. On dit d'une suite de processus ponctuels $(\Phi_n)_{n \geq 1}$ qu'elle *converge* vers un processus ponctuel Φ si, pour toute fonction continue bornée f définie sur $\mathcal{F}_{lf}(E)$, on a

$$\mathbb{E}\left[f(\Phi_n)\right] \underset{n\to\infty}{\longrightarrow} \mathbb{E}\left[f(\Phi)\right].$$

Ce type de convergence est équivalent à la convergence des lois fini-dimensionnelles (voir Théorème 4.2 de [73]) i.e. pour tout K-uplet de Boréliens bornés $B_1, \ldots, B_K$ de E avec $\Theta_\Phi(\partial B_i) = 0$, $1 \leq i \leq K$, on a la convergence en loi

$$(\#\Phi_n \cap B_1, \ldots, \#\Phi_n \cap B_K) \underset{n\to\infty}{\overset{\mathcal{D}}{\longrightarrow}} (\#\Phi \cap B_1, \ldots, \#\Phi \cap B_K)$$

où l'ensemble ∂B_i désigne le bord de B_i et où $\underset{n\to\infty}{\overset{\mathcal{D}}{\longrightarrow}}$ désigne la convergence en loi dans E^K. En pratique, la convergence d'un processus ponctuel telle qu'elle a été définie est difficile à manipuler.

En 1975, Kallenberg [73] donne un critère plus explicite. Nous énonçons ci-dessous son théorème tel qu'il a été réécrit par Resnick (voir la Proposition 3.22 de [124]) car c'est sous cette forme que nous l'utiliserons dans le troisième chapitre.

Théorème 1.1.1. *(Kallenberg) Supposons que Φ soit un processus ponctuel de E et $\mathcal{I}$ une base d'ouverts relativement compacts telle que $\mathcal{I}$ soit stable par réunions et intersections finies et telle que*

$$\mathbb{P}\left(\#\Phi \cap I = 0\right) = 1$$

pour tout $I \in \mathcal{I}$ vérifiant $\Theta_\Phi(I) = 0$. Si $(\Phi_n)_{n\geq 1}$ est une suite de processus ponctuels sur E et si pour tout $I \in \mathcal{I}$, on a

$$\lim_{n\to\infty} \mathbb{P}\left(\#\Phi_n \cap I = 0\right) = \mathbb{P}\left(\#\Phi \cap I = 0\right)$$

et

$$\lim_{n\to\infty} \mathbb{E}\left[\#\Phi_n \cap I\right] = \mathbb{E}\left[\#\Phi \cap I\right]$$

alors Φ_n converge en loi vers Φ.

On présente ci-dessous deux types de processus ponctuels que sont le processus ponctuel de Poisson et les processus à particules.

Processus ponctuel de Poisson Le *processus ponctuel de Poisson* est un exemple fondamental de processus ponctuel. Dans ce qui suit, on désigne ce processus par $\Phi = \mathbf{X}$. Celui-ci satisfait, par définition, les deux propriétés suivantes :

1. Pour tout Borélien B borné de $\mathbf{R}^d$, le nombre de points de $\mathbf{X}$ tombant dans A suit une loi de Poisson de paramètre $\Theta_{\mathbf{X}}(B)$. Autrement dit, pour tout $k \geq 0$

$$\mathbb{P}\left(\#\mathbf{X} \cap B = k\right) = e^{-\Theta_{\mathbf{X}}(B)} \frac{(\Theta_{\mathbf{X}}(B))^k}{k!}.$$

2. Pour tousBoréliens deux à deux disjoints $B_1, \ldots, B_K$ de $\mathbf{R}^d$, les variables aléatoires $\#\mathbf{X} \cap B_1$, ..., $\#\mathbf{X} \cap B_K$ sont indépendantes.

Pour toute mesure Θ sur E sans atome, il existe un processus ponctuel de Poisson $\mathbf{X}$ de mesure d'intensité $\Theta_{\mathbf{X}} = \Theta$. De plus, ce processus est unique en loi (voir, par exemple, le Théorème 3.2.1 de [130]). Un processus ponctuel ne satisfaisant que la première propriété et dont la mesure d'intensité est sans atome est nécessairement un processus ponctuel de Poisson. Une telle observation est due à Rényi [123]. Réciproquement, si un processus ponctuel Φ de mesure d'intensité sans atome vérifie la seconde condition (en fait, l'indépendance 2 à 2 des cardinaux est également suffisante), il s'agit là-aussi d'un processus ponctuel de Poisson (voir, par exemple, Daley et Vere-Jones [29], Lemme 2.VI).

Dans le cas où $E = \mathbf{R}^d$ et pour tout réel positif $\gamma \in [0, \infty)$, il existe un unique processus ponctuel de Poisson stationnaire d'intensité γ. De plus, ce processus est isotrope. On parle alors de processus ponctuel de Poisson *homogène* et on utilisera essentiellement ce type de processus dans les chapitres 2, 3 et 4. Une propriété du processus ponctuel de Poisson est que ses points sont en position générale (au sens de [147]) i.e. chaque sous-ensemble à n points avec $n < d+1$ est affinement indépendant et qu'il n'existe pas de sphère contenant au moins $d+2$ points de $\mathbf{X}$.

Le résultat suivant donne une caractérisation du processus ponctuel de Poisson à partir de la transformée de Laplace.

Théorème 1.1.2. *(Transformée de Laplace) Soit Φ un processus ponctuel de E de mesure d'intensité Θ_Φ sans atome. Alors Φ est un processus ponctuel de Poisson si et seulement si*

$$\mathbb{E}\left[\prod_{x\in\Phi} f(x)\right] = \exp\left(\int_E (f-1)d\Theta_\Phi\right)$$

pour toute fonction mesurable $f : E \to [0,1]$.

Processus à particules Un second type de processus ponctuels sur lesquels nous nous appuierons sont les processus à particules. Par *processus à particules* dans $\mathbf{R}^d$, on entend un processus ponctuel dans $\mathcal{F}'_d = \mathcal{F}_d - \{\varnothing\}$ concentré sur le sous-ensemble $\mathcal{C}'_d = \mathcal{C}_d - \{\varnothing\}$ des sous-ensembles compacts non vides c'est-à-dire de mesure d'intensité Θ_Φ satisfaisant $\Theta_\Phi(\mathcal{F}'_d - \mathcal{C}'_d) = 0$. Dans ce qui suit, on restreint notre attention à un cas particulier des processus à particules que sont les mosaïques aléatoires.

Mosaïques aléatoires On appelle *mosaïque aléatoire* (convexe) de $\mathbf{R}^d$ un processus à particules $\mathfrak{m}$ dans $\mathbf{R}^d$ satisfaisant presque sûrement les quatre propriétés suivantes :

1. Les ensembles $C \in \mathfrak{m}$ sont compacts, convexes et d'intérieur non vide.
2. $\mathfrak{m}$ est un recouvrement de $\mathbf{R}^d$ i.e.
$$\bigcup_{K\in\mathfrak{m}} K = \mathbf{R}^d.$$
3. Si $C_1, C_2 \in \mathfrak{m}$ et $C_1 \neq C_2$, alors C_1 et C_2 sont d'intérieurs disjoints.

Il est d'usage d'appeler *cellules* les élements de $\mathfrak{m}$. On peut en fait montrer (voir le Lemme 10.1.1 de [130]) que les cellules sont nécessairement des polytopes convexes si bien qu'une mosaïque aléatoire est une partition aléatoire en des polytopes. On dit d'une mosaïque aléatoire qu'elle est *face-à-face* si pour toute paire de cellules $C_1, C_2 \in \mathfrak{m}$, l'intersection $C_1 \cap C_2$ est une k-face commune de C_1 et de C_2, $0 \leq k \leq d$. On dit qu'une telle mosaïque est de plus *normale* si toute k-face de $\mathfrak{m}$ est contenue dans la frontière d'exactement $d-k+1$ cellules.

On désigne, dans ce qui suit, par $z(\cdot)$ une fonction définie sur l'ensemble des corps convexes $\mathcal{K}_d$ de $\mathbf{R}^d$ invariante par translation. Pour toute cellule $C \in \mathfrak{m}$, le point $z(C)$ s'appelle le *germe* de la cellule. L'ensemble des germes $\Phi_{\mathfrak{m}} = \{z(C), C \in \mathfrak{m}\}$ est un processus ponctuel dans $\mathbf{R}^d$. Lorsque $\mathfrak{m}$ est stationnaire (c'est-à-dire invariante par translation en loi), on appelle *intensité* de $\mathfrak{m}$ l'intensité de ce processus ponctuel. Autrement dit, l'intensité de la mosaïque est le nombre γ défini par

$$\gamma = \frac{1}{\lambda_d(B)}\mathbb{E}\left[\sum_{C\in\mathfrak{m}} \mathbb{1}_{z(C)\in B}\right]$$

où B est un Borélien de $\mathbf{R}^d$ de volume $\lambda_d(B) \in (0,\infty)$. On reviendra sur la dépendance éventuelle de γ en le choix de la fonction $z(\cdot)$. On présente ci-dessous deux types de mosaïques aléatoires que sont les mosaïques de Voronoï et de Delaunay.

Mosaïques de Voronoï Soit Φ un processus ponctuel dans $\mathbf{R}^d$ tel que l'enveloppe convexe de Φ, noté conv(Φ), est égal à $\mathbf{R}^d$. Pour tout point $x \in \Phi$, on note

$$C_\Phi(x) = \{y \in \mathbf{R}^d, |y-x| \leq |y-x'|, x' \in \Phi\}.$$

Le sous-ensemble défini ci-dessus s'appelle la *cellule de Voronoï* de germe x et la collection $\{C_\Phi(x), x \in \Phi\}$ de ces cellules s'appelle la *mosaïque de Voronoï* associée au processus ponctuel Φ. Si l'on suppose de plus que cette collection est un processus à particules *localement fini* (ce qui ne résulte pas nécessairement du fait que Φ soit localement fini), il s'agit, en particulier, d'une mosaïque aléatoire face-à-face. Pour toute cellule $C_\Phi(x)$, $x \in \Phi$, de cette mosaïque, on pose $z(C_\Phi(x)) = x$. En particulier, si Φ est stationnaire d'intensité γ_Φ alors la mosaïque aléatoire est, elle aussi, stationnaire et son intensité est égale à γ_Φ.

Un cas particulier est lorsque le processus ponctuel sous-jacent Φ est un processus ponctuel de Poisson $\mathbf{X}$ homogène. Dans ce cas, on parle de *mosaïque de Poisson-Voronoï* ou de *tessellation de Poisson-Voronoï*, et l'on désigne par $\mathfrak{m}_{PVT}$ cette mosaïque. Parce que, presque sûrement, les points de $\mathbf{X}$ sont en position générale et qu'aucun sous-ensemble à $d+2$ points de $\mathbf{X}$ n'est contenu dans une sphère, la mosaïque aléatoire qui en résulte est normale. Cette mosaïque est de plus stationnaire par stationnarité de $\mathbf{X}$ et a même intensité que $\mathbf{X}$. Pour une réalisation de la mosaïque de Poisson-Voronoï, voir la Figure 1.1 (a).

Mosaïques de Delaunay Soit Φ un processus ponctuel dans $\mathbf{R}^d$ tel que $\mathrm{conv}(\Phi) = \mathbf{R}^d$ et soit $\Sigma(\Phi)$ l'ensemble des sommets de la mosaïque de Voronoï associée. Pour tout sommet $s \in \Sigma(\Phi)$, on désigne par

$$D_\Phi(s) = \mathrm{conv}\{x \in \Phi, s \in C_\Phi(x)\}.$$

La collection de ces sous-ensembles s'appelle la *mosaïque de Delaunay* associée à Φ. Il s'agit, en particulier, d'une mosaïque aléatoire face-à-face.

Lorsque Φ est un processus ponctuel de Poisson $\mathbf{X}$, on parle de *mosaïque de Poisson-Delaunay* et on la note $\mathfrak{m}_{PDT}$. Dans ce cas, la mosaïque est une partition en des simplexes de l'espace du fait que la mosaïque de Poisson-Voronoï $\mathfrak{m}_{PVT}$ est normale. On peut construire la mosaïque de Poisson-Delaunay de plusieurs façons. En effet, deux germes x, x' de $\mathbf{X}$ forment une arête de Poisson-Delaunay si et seulement si leurs cellules de Voronoï sont voisines i.e.

$$C_\mathbf{X}(x) \cap C_\mathbf{X}(x') \neq \varnothing.$$

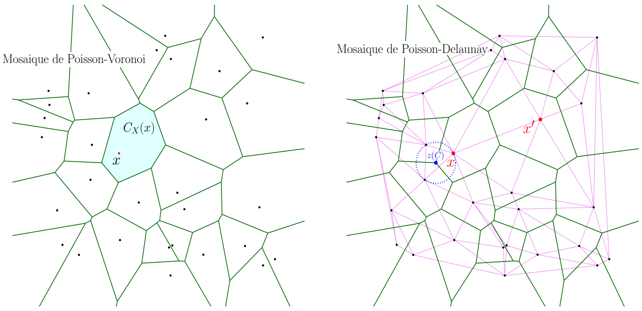
Une autre façon de construire la mosaïque de Poisson-Delaunay est de procéder comme suit : une famille de $d+1$ points de $\mathbf{X}$ définit un simplexe de Delaunay si et seulement si la boule circonscrite à ces points ne contient aucun point dans son intérieur. Notons qu'il y a un sens à parler de la boule circonscrite à $d+1$ points puisque ces derniers sont en position générale du fait que $\mathbf{X}$ est un processus ponctuel de Poisson. Pour toute cellule de Poisson-Delaunay $C \in \mathfrak{m}_{PDT}$, on choisit $z(C)$ comme le centre circonscrit du simplexe C. La mosaïque de Poisson-Delaunay est stationnaire et son intensité est égale à

$$\gamma = \beta_d^{-1} \cdot \gamma_\mathbf{X}$$

où

$$\beta_d = \frac{(d^3+d^2)\Gamma\left(\frac{d^2}{2}\right)\Gamma^d\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d^2+1}{2}\right)\Gamma^d\left(\frac{d+2}{2}\right)2^{d+1}\pi^{\frac{d-1}{2}}}. \tag{1.1.1}$$

et où $\gamma_\mathbf{X}$ est l'intensité de $\mathbf{X}$. Les mosaïques de Poisson-Voronoï et Poisson-Delaunay sont duales au sens où l'on peut construire l'une à partir de l'autre. Dans la Figure 1.1 (b), on donne une réalisation de la mosaïque de Poisson-Delaunay.



(a) *(b)*

FIGURE 1.1 – (a) Mosaïque de Poisson-Voronoï d'intensité 30, observée dans le carré unité. (b) La même mosaïque et sa mosaïque duale de Delaunay en violet.

1.1.2 Mesure de Palm et cellule typique

Pour décrire le comportement moyen d'une mosaïque aléatoire, on introduit la notion de cellule typique. Pour cela, nous avons besoin de présenter une mesure naturelle associée à un processus ponctuel appelée, communément, la mesure de Palm.

Mesure de Palm d'un processus ponctuel stationnaire dans $\mathbf{R}^d$ Dans ce qui suit, on considère un processus ponctuel Φ de $\mathbf{R}^d$ que l'on suppose stationnaire et d'intensité $\gamma_\Phi \in (0, \infty)$. Il est naturel de comprendre à quoi ressemble le processus ponctuel vu d'un point quelconque c'est-à-dire d'un point x choisi "uniformément au hasard" dans Φ, autrement dit la loi de Φ sachant que Φ contient x. Cette notion nécessite quelques précautions car le nombre de points de Φ pouvant être infini, l'événement "$\Phi \ni x$" peut être de mesure nulle. Pour rendre rigoureuse cette notion, Palm a introduit une mesure qui porte son nom dans [110] dans le cadre réel et qui, plus tard, a été approfondie par Khinchin et Kaplan en 1955 et Slivnyak en 1962 et 1966. On appelle *mesure de Palm* de Φ la mesure de probabilité $\mathbb{P}^0$ définie sur l'ensemble des fermés localement finis de $\mathbf{R}^d$ i.e. $\mathcal{F}_{lf} = \mathcal{F}_{lf}(\mathbf{R}^d)$ et donnée par

$$\mathbb{P}^0(A) = \frac{1}{\gamma \lambda_d(B)} \mathbb{E}\left[\sum_{x \in \Phi \cap B} \mathbb{1}_{\Phi - x} \in A\right]$$

pour tout $A \in \mathfrak{F}_d$ et tout Borélien $B \in \mathcal{B}(\mathbf{R}^d)$. Il s'agit de la loi du processus vu d'un "point typique", en l'occurrence qu'on a supposé être 0 par stationarité de Φ. Un résultat équivalent à la définition de la mesure de Palm est le théorème de Campbell.

Théorème 1.1.3. *(Théorème de Campbell) Soit Φ un processus ponctuel stationnaire dans $\mathbf{R}^d$ d'intensité $\gamma \in (0, \infty)$ et soit $f : \mathbf{R}^d \times \mathcal{F}_{lf} \to \mathbf{R}$ une fonction mesurable positive. Alors*

$$\mathbb{E}\left[\sum_{x \in \Phi} f(x, \Phi)\right] = \gamma \int_{\mathbf{R}^d} \int_{\mathcal{F}_{lf}} f(x, \eta + x) d\mathbb{P}^0(\eta) dx.$$

Dans le cas où $\Phi = \mathbf{X}$ est un processus ponctuel de Poisson (homogène), le théorème précédent peut se réécrire en un résultat plus précis. Celui-ci porte le nom de *formule de Slivnyak.*

Théorème 1.1.4. *(Formule de Slivnyak) Soit $\mathbf{X}$ un processus ponctuel de Poisson dans $\mathbf{R}^d$,*

$K \in \mathbb{N}$ *et* $f : \mathcal{F}_{lf} \times (\mathbf{R}^d)^K \to \mathbf{R}$ *une fonction mesurable et positive. Alors*

$$\mathbb{E}\left[\sum_{(x_1,\ldots,x_K)_{\neq}\in\mathbf{X}^K} f(\mathbf{X}, x_1, \ldots, x_K)\right] = \gamma^K \int_{\mathbf{R}^d} \cdots \int_{\mathbf{R}^d} \mathbb{E}\left[f\left(\mathbf{X}\cup\{x_1,\ldots,x_K\}, x_1,\ldots,x_K\right)\right] dx_1 \ldots dx_K.$$

Dans ce qui précède, on entend par $(x_1, \ldots, x_K)_{\neq}$ un K-uplet de points distincts. Nous utiliserons essentiellement cette formule dans les chapitres 2 et 3. Une conséquence immédiate de cette formule est qu'elle donne une représentation explicite de la mesure de Palm de $\mathbf{X}$: pour tout $A \in \mathfrak{F}_d$, on a

$$\mathbb{P}^0(A) = \mathbb{P}\left(\mathbf{X}\cup\{0\} \in A\right). \tag{1.1.2}$$

Autrement dit, vu d'un point typique de $\mathbf{X}$, le processus ponctuel observé de ce point reste un processus ponctuel de Poisson. Cette propriété est intrinsèque au processus ponctuel de Poisson.

Maintenant que nous avons présenté la mesure de Palm, nous pouvons introduire la notion de cellule typique d'une mosaïque aléatoire.

Cellule typique d'une mosaïque aléatoire Dans ce qui suit, on désigne par $\mathfrak{m}$ une mosaïque aléatoire stationnaire de $\mathbf{R}^d$. Rappelons que $\Phi_{\mathfrak{m}} = \{z(C), C \in \mathfrak{m}\}$ est l'ensemble des germes de la mosaïque. La cellule typique de $\mathfrak{m}$ est la cellule dont le germe est un point typique de $\Phi_{\mathfrak{m}}$ placé en 0. Formellement, il s'agit d'un polytope aléatoire $\mathscr{C}$, unique en loi et dont la loi est définie par

$$\mathbb{E}\left[f(\mathscr{C})\right] = \frac{1}{\gamma\lambda_d(B)}\mathbb{E}\left[\sum_{\substack{C\in\mathfrak{m},\\ z(C)\in B}} f(C - z(C))\right]$$

où $f : \mathcal{K}_d \to \mathbf{R}$ est une fonction positive mesurable et bornée et $B \in \mathcal{B}(\mathbf{R}^d)$ un Borélien de volume $\lambda_d(B) \in (0, \infty)$. L'expression précédente décrit bien ce qu'est la cellule typique : il s'agit d'une moyenne portant sur n'importe quelle caractéristique et prise sur toutes les cellules de la mosaïque lorsqu'on observe cette mosaïque dans une fenêtre quelconque. En particulier, puisque l'intensité γ est le nombre moyen de cellules par unité de volume, le volume moyen de la cellule typique est égal à

$$\mathbb{E}\left[\lambda_d(\mathscr{C})\right] = \gamma^{-1}.$$

A priori, la cellule typique dépend du choix de $z(\cdot)$. En fait, lorsqu'on suppose que la mosaïque aléatoire est ergodique, c'est-à-dire que le système dynamique canonique associé aux translations est ergodique, la définition de la cellule typique est indépendante du choix des germes. Une telle propriété est due a Cowan qui, dans les articles [27] et [28], montrent que la cellule typique peut être définie comme une moyenne ergodique i.e.

$$\mathbb{E}\left[f(\mathscr{C})\right] = \frac{1}{\mathbb{E}\left[1/\lambda_d(C(0))\right]}\mathbb{E}\left[\frac{f(C(0))}{\lambda_d(C(0))}\right]$$

où la cellule $C(0)$, dite de *Crofton*, est la cellule de $\mathfrak{m}$ qui contient l'origine. Notons que la cellule de Crofton est en général plus grande que la cellule typique au sens où son volume moyen est plus important que celui de la cellule typique. En dimension 1, un tel fait est plus connu sous le nom de *paradoxe de l'autobus*.

Une troisième façon de définir la cellule typique, toujours dans le cas d'une mosaïque ergodique, est de la construire comme une limite *presque sûre* : en désignant par $\mathbf{B}_\rho$ la dilatation de B i.e. $\mathbf{B}_\rho = \rho^{1/d} B$ où $B \in \mathcal{B}(\mathbf{R}^d)$ contient l'origine dans son intérieur, on a

$$\mathbb{E}\left[f(\mathscr{C})\right] = \lim_{\rho \to \infty} \frac{1}{\gamma \lambda_d(\mathbf{B}_\rho)} \mathbb{E}\left[\sum_{\substack{C \in \mathbf{m}, \\ C \cap \mathbf{B}_\rho \neq \varnothing}} f(C - z(C))\right].$$

En général, il est difficile de donner une représentation géométrique simple de la cellule typique. Cependant, on peut expliciter cette dernière dans le cas des mosaïques de Poisson-Voronoï et Poisson-Delaunay.

Cellule typique de la mosaïque de Poisson-Voronoï En vertu de l'expression (1.1.2) ou, de façon équivalente, de la formule de Slivnyak (voir Théorème 1.1.4), la cellule typique de la mosaïque de Poisson-Voronoï est égale en loi à

$$\mathscr{C} = C_{\mathbf{X} \cup \{0\}}(0). \tag{1.1.3}$$

Autrement dit, prendre "une cellule au hasard" revient à se placer en un point d'observation fixe ajouté à l'ensemble des germes et à "regarder" la cellule de Voronoï dans laquelle on se trouve. Une telle caractérisation permet d'avoir un grand nombre de résultats sur la cellule typique comme on le verra dans le prochain paragraphe. Elle sera, par ailleurs, fondamentale pour obtenir le comportement de diverses valeurs extrêmes dans le chapitre 3.

Cellule typique de la mosaïque de Poisson-Delaunay En 1970, Miles [95] donne une représentation explicite de la cellule typique d'une mosaïque de Poisson-Delaunay qui, aujourd'hui, se présente comme un cas particulier des résultats en loi de Baumstark et Last [10]. Au préalable, on donne quelques notations. On désigne par $d\sigma(u)$ la loi uniforme sur la sphère unité $\mathbf{S}^{d-1}$ de $\mathbf{R}^d$ et $d\sigma(\mathbf{u}_{1:d+1}) = d\sigma(u_1) \cdots d\sigma(u_{d+1})$ avec $\mathbf{u}_{1:d+1} = (u_1, \ldots, u_{d+1}) \in (\mathbf{S}^{d-1})^{d+1}$.

Etant donnée une fonction $f : \mathcal{K}_d \to \mathbf{R}$, mesurable bornée et invariante par translation, on a

$$\mathbb{E}\left[f(\mathscr{C})\right] = \delta'_d \cdot \gamma_{\mathbf{X}}^d \int_0^\infty \int_{(\mathbf{S}^{d-1})^{d+1}} r^{d^2-1} e^{-\gamma_{\mathbf{X}} \kappa_d r^d} \lambda_d(\Delta(\mathbf{u}_{1:d+1})) f(\Delta(r\mathbf{u}_{1:d+1})) d\sigma(\mathbf{u}_{1:d+1}) dr \tag{1.1.4}$$

où $\Delta(\mathbf{u}_{1:d+1}) = \mathrm{conv}(u_1, \ldots, u_{d+1})$ et $\delta'_d = (d+1) \cdot \beta_d$. L'expression précédente donne une construction explicite de la cellule typique de Poisson-Delaunay : on commence d'abord par tirer le rayon circonscrit (à la puissance d) selon une loi exponentielle. Puis, conditionnellement à r, on tire un simplexe inscrit dans la sphère $S(0, r)$ de densité proportionnelle au volume du simplexe. Un tel résultat découle directement de la formule de Slivnyak et du changement de variables :

$$\int_{(\mathbf{R}^d)^{d+1}} f(x_1, \ldots, x_{d+1}) dx_1 \ldots dx_{d+1}$$
$$= d! \int_{\mathbf{R}^d} \int_0^\infty \int_{\mathbf{S}^{d-1}} \cdots \int_{\mathbf{S}^{d-1}} f(z + ru_0, \ldots, z + ru_d) r^{d^2-1} \lambda_d(\Delta(\mathbf{u}_{1:d+1})) dz dr d\sigma(\mathbf{u}_{1:d+1}).$$

On dit d'un tel changement de variables qu'il est de *type Blaschke-Petkantschin*. Ce résultat de géométrie intégrale sera utile dans le chapitre 3.

1.1.3 Quelques résultats connus sur les mosaïques de Poisson-Voronoï et Poisson-Delaunay

Dans ce paragraphe, on donne quelques résultats connus sur les mosaïques de Poisson-Voronoï et Poisson-Delaunay homogènes en se limitant à ceux qui nous seront utiles dans la thèse. En particulier, on énonce des résultats distributionnels sur diverses caractéristiques géométriques de leurs cellules typiques.

Mosaïque de Poisson-Voronoï Une propriété fondamentale de la mosaïque de Poisson-Voronoï engendrée par un processus ponctuel de Poisson homogène est qu'elle est stationnaire et *mélangeante*. Plus précisément, on se place sur l'espace canonique associé à la mosaïque. Autrement dit, on pose $\Omega = \mathbb{M}$, $\mathfrak{A} = \mathfrak{F}_{|\mathbb{M}}$ et $\mathbb{P} = \mathbb{P}_{\mathfrak{m}_X}$ où $\mathfrak{F}_{|\mathbb{M}}$ est la tribu de Fell induite sur l'espace des mosaïques $\mathbb{M}$. Pour tout $t \in \mathbf{R}^d$, on pose $T_t : \Omega \to \Omega, \mathfrak{m} \mapsto \mathfrak{m} + t$ la translation de vecteur t. L'espace $(\Omega, \mathfrak{A}, \mathbb{P}, T_t)$ définit un système dynamique.

Propriété 1. *Les transformations T_t, $t \geq 1$ sont mélangeantes i.e. pour tout $A, B \in \mathfrak{F}_{|\mathbb{M}}$, on a*

$$\mathbb{P}\left(\mathfrak{m} \in A, \mathfrak{m} + t \in B\right) \underset{|t| \to \infty}{\longrightarrow} \mathbb{P}\left(\mathfrak{m} \in A\right) \cdot \mathbb{P}\left(\mathfrak{m} \in B\right).$$

Une telle propriété est démontrée dans la proposition 6.4.1 de [129]. En particulier, cela montre que la cellule typique de Poisson-Voronoï, vue comme une moyenne ergodique, est indépendante du choix des germes.

Dans les paragraphes suivants, on donne quelques résultats exacts et asymptotiques sur les distributions de diverses caractéristiques géométriques de la cellule typique.

Loi du rayon inscrit et circonscrit D'après l'égalité en loi (1.1.3), la cellule typique de Poisson-Voronoï est la cellule de germe 0 quand on rajoute 0 au processus ponctuel de Poisson. Le résultat le plus immédiat porte sur le rayon inscrit de cette cellule, c'est-à-dire sur le rayon de la plus grande boule centrée en l'origine et incluse dans la cellule. On désigne cette variable aléatoire par $r(\mathscr{C})$. Puisque $r(\mathscr{C}) = r(C_{\mathbf{X}\cup\{0\}})$ est inférieure à r si et seulement si $\mathbf{X} \cap B(0, 2r)$ est vide, où r est un réel positif, et puisque $\mathbf{X}$ est un processus ponctuel de Poisson, on a

$$\mathbb{P}(r(\mathscr{C}) \leq r) = e^{-2^d \kappa_d r^d \gamma}$$

où κ_d désigne le volume de la boule unité de $\mathbf{R}^d$.

Une autre caractéristique intéressante est le rayon circonscrit qui est le rayon de la plus petite boule centrée en l'origine et contenant la cellule. On désigne cette grandeur par $R(\mathscr{C}) = R(C_{\mathbf{X}\cup\{0\}}(0))$. En interprétant l'événement "$R(\mathscr{C}) \leq r$" comme un recouvrement de la sphère $S(0, r)$ par des calottes sphériques et indépendantes, Calka [15] donne une représentation explicite de la loi du rayon circonscrit. Plus précisément, on a

$$\mathbb{P}(R(\mathscr{C}) \leq r) = e^{-2^d \kappa_d r^d \gamma} \sum_{k=0}^{\infty} \frac{(2^d \kappa_d r^d \gamma)^k}{k!} p(k)$$

où $p(k)$ est la probabilité pour que k calottes sphériques indépendantes, invariantes en loi par rotation et dont le diamètre angulaire suit la loi ν recouvrent la sphère et où ν a pour densité $f_\nu(\theta) = d\pi \sin(\pi\theta) \cos^{d-1}(\pi\theta) \mathbb{1}_{[0,1/2]}(\theta)$. La proposition suivante donne des estimations de la queue du rayon circonscrit (voir Théorèmes 3 et 5 de [15]) :

Proposition 1.1.5. *(Calka) Soit $\mathfrak{m}$ une mosaïque de Poisson-Voronoï d'intensité γ dans le plan. Alors*

1. *Il existe une constante* $r_0 \simeq 0.337$ *telle que pour tout* $r > r_0$*, on a*

$$2\pi r^2 \gamma e^{-\pi r^2 \gamma} \leq \mathbb{P}(R(\mathscr{C}) \geq r) \leq 4\pi r^2 \gamma e^{-\pi r^2 \gamma}.$$

2. *Il existe une constante* $c > 0$ *telle que pour tout* $\delta \in \left[-1, \frac{d-1}{d+1}\right]$*, on a*

$$\mathbb{P}(R(\mathscr{C}) \geq r + r^{-\delta} \,|r(\mathscr{C}) = r) = O\left(e^{-cr^{\frac{1}{2}((d-1)+\delta(d+1))}}\right)$$

quand r *tend vers l'infini.*

Le premier point n'est valable qu'en dimension 2 et résulte d'une extension [134] de la formule de Stevens [139] qui rend explicite la probabilité $p(k)$. Le second montre que, si le rayon inscrit est grand, alors le rayon circonscrit est "proche" de ce dernier. Un tel fait peut être relié à la conjecture de Kendall. Cette dernière, qui affirme que les grandes cellules ont tendance à avoir une "forme sphérique", a été depuis démontrée par Hug *et al.* [61].

Loi du volume Il n'existe pas d'expression simple de la queue de la fonction de répartition du volume. En revanche, la proposition suivante, due à Gilbert [47], donne un encadrement de sa fonction de répartition.

Proposition 1.1.6. *(Gilbert) Pour tout* $t > 0$*, on a*

$$e^{-2^d t\gamma} \leq \mathbb{P}(\lambda_d(\mathscr{C}) \geq t) \leq \frac{\gamma t - 1}{e^{\gamma t - 1} - 1}.$$

Ce résultat est, à ce jour, le meilleur encadrement connu. Depuis, il a été prouvé par Hug et Schneider [62] que la borne inférieure fournit un bon équivalent logarithmique i.e.

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}(\lambda_d(\mathscr{C}) \geq t) = -2^d\gamma.$$

En d'autres mots, la queue du volume et du rayon inscrit se comportent asymptotiquement de la même façon. Un tel fait est relié, lui aussi, à la conjecture de Kendall.

Loi du volume de la fleur de Voronoï On appelle *fleur de Voronoï* ou *domaine fondamental* de la mosaïque de Poisson-Voronoï l'ensemble

$$\mathcal{F}(\mathscr{C}) = \bigcup_{x\in\mathscr{C}} B(x, |x|).$$

En désignant par $N(\mathscr{C})$ le nombre de faces de $\mathscr{C}$, Zuyev [148] donne la loi conditionnelle du volume de la fleur sachant $N(\mathscr{C})$:

Proposition 1.1.7. *(Zuyev) Pour tout* $k \geq 0$*, conditionnellement au fait que* $N(\mathscr{C}) = k$*, le volume de* $\mathcal{F}(\mathscr{C})$ *suit une loi Gamma de paramètres* $(k, 1)$*.*

Pour des résultats sur la loi du nombre d'arêtes de la cellule typique en dimension 2, on peut consulter [16]. Cette loi se concentre sur 6 qui est aussi le nombre moyen d'arêtes de la cellule typique.

On termine ce paragraphe par une application célèbre de la mosaïque de Poisson-Voronoï.

Approximation de Poisson-Voronoï Désignons par $W \in \mathcal{K}_d$ un corps convexe de $\mathbf{R}^d$ d'intérieur non vide. Une façon d'approcher le volume de W est de discrétiser le corps convexe par des cellules de Voronoï. Plus précisément, on pose

$$\mathscr{V}_{\mathbf{X}}(W) = \bigcup_{x \in \mathbf{X} \cap W} C_{\mathbf{X}}(x)$$

où $\mathbf{X}$ est un processus ponctuel de Poisson d'intensité γ. Une telle approximation (voir Figure 1.1.3) s'appelle *l'approximation de Poisson-Voronoï* et a été introduite par Khmaladze et Torondjaze dans [75] en dimension 1.

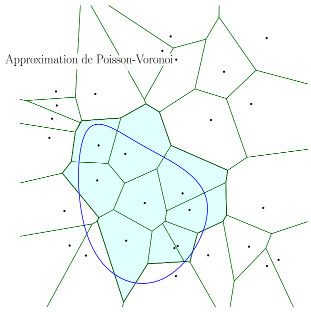


FIGURE 1.2 – Approximation d'un corps convexe (en bleu) par des cellules de Voronoï (en vert)

Elle fournit, en particulier, un estimateur sans biais du volume grâce à la formule de Slivnyak et peut avoir des applications dans divers domaines tels que les statistiques non paramétriques (voir la section 3 de [35]), l'analyse d'image [75] et des problèmes de quantification (voir le chapitre 9 du livre de Graf et Luschgy [50]). En 2009, Heveling et Reitzner [57] établissent une estimation de la variance et un principe de grandes déviations pour le volume de K ainsi que sur le volume de la différence symétrique $\lambda_d(K \Delta \mathcal{V}_{\mathbf{X}}(K))$ lorsque l'intensité tend vers l'infini. L'approximation est d'autant meilleure que l'intensité est grande. En utilisant la décomposition en chaos de Wiener-Itô et la méthode de Stein, Schulte [131] obtient un théorème central limite avec encadrement de la variance que nous énonçons ci-dessous :

Proposition 1.1.8. *(Schulte) Soit $W \in \mathcal{K}_d$ un corps convexe de $\mathbf{R}^d$. Alors, on a la convergence en loi*

$$\frac{\lambda_d(\mathscr{V}_{\mathbf{X}}(W)) - \lambda_d(W)}{\sqrt{Var(\lambda_d(\mathcal{V}_{\mathbf{X}}(W)))}} \xrightarrow[\gamma \to \infty]{\mathcal{D}} \mathcal{N}(0,1)$$

où $\mathcal{N}(0,1)$ est la loi normale centrée réduite.

Un tel résultat montre un des intérêts que la mosaïque de Poisson-Voronoï peut avoir puisqu'il donne un critère pratique pour approcher le volume d'un corps convexe. Dans le chapitre 2, nous donnerons une majoration de la distance de Hausdorff entre le corps convexe et son approximation.

Mosaïque de Poisson-Delaunay De même que pour la mosaïque de Poisson-Voronoï, la mosaïque de Poisson-Delaunay est stationnaire et vérifie une propriété de mélange, si bien que la cellule typique ne dépend pas des germes considérés. L'expression intégrale (1.1.4) donne des lois explicites pour diverses caractéristiques comme les angles ou le rayon circonscrit. Dans [95], Miles donne les moments d'ordre quelconque du volume de la cellule typique. En utilisant les transformées de Fourier et le théorème des résidus, Rathie [121] en déduit la loi du volume sous forme intégrale et explicite sa densité en dimension 1 et 2. Nous réécrivons son résultat en dimension 2 et l'utiliserons dans le troisième chapitre.

Proposition 1.1.9. *(Rathie) Soit* $\mathbf{X}$ *un processus ponctuel de Poisson d'intensité* $\gamma_{\mathbf{X}}$ *dans le plan et* $\mathscr{C}$ *la cellule typique de la mosaïque de Delaunay associée. Pour tout* $v \geq 0$*, on a*

$$\mathbb{P}(\lambda_2(\mathscr{C}) \leq v) = \frac{8\pi}{9} \int_0^{\gamma_{\mathbf{X}} v} x K_{1/6}^2(2\pi x / 3\sqrt{3}) dx$$

où $K_{1/6}(\cdot)$ *désigne la fonction de Bessel modifié d'ordre* $1/6$.

Il existe un analogue à la conjecture de Kendall pour les mosaïques de Poisson-Delaunay. Cette dernière affirme que lorsque la cellule typique de Poisson-Delaunay est grande, alors cette dernière tend à être régulière. Un tel fait a été formalisée par Hug et Schneider et prouvé dans [62].

On termine par quelques références, non exhaustives, où l'on trouvera des résultats sur les mosaïques. Pour des résultats portant sur des mosaïques plus générales que celles présentées ci-dessus, on peut consulter Mecke pour les dimensions 2 et 3 [91], [92] et Møller [98], [99] où l'on trouvera des calculs portant sur des moyennes et variances ainsi que des relations entre diverses caractéristiques géométriques en dimension quelconque. Pour des résultats distributionnels très généraux sur les mosaïques de Poisson-Voronoï et Poisson-Delaunay, on peut voir Baumstark et Last [10]. Enfin, on trouvera chez Heinrich et Muche [54] et chez Penrose et Yukich [119] des théorèmes centraux limites pour de telles mosaïques voire des modèles plus généraux. Pour un large panorama sur les mosaïques aléatoires, on peut consulter les livres [108], [130] et [140].

1.2 Notions sur les valeurs extrêmes

Soit $(X_i)_{i\geq 1}$ une suite de variables aléatoire réelles iid de fonction de répartition F_X. La théorie classique des valeurs extrêmes a pour but d'étudier le maximum

$$M_n = \max_{i \leq n} X_i.$$

Lorsque n tend vers l'infini, le maximum converge vers $x^* = \sup\{x \in \mathbf{R}, F_X(x) < 1\}$ et par conséquent vers une limite dégénérée. Il convient de renormaliser M_n par un seuil adéquat, c'est-à-dire de déterminer un seuil u_n s'écrivant sous la forme $u_n = u_n(t) = a_n t + b_n$ où $a_n >$, $b_n \in \mathbf{R}$ et où $t \in \mathbf{R}$ est un paramètre, de sorte que $\mathbb{P}\left(M_n \leq u_n(t)\right) = F_X^n(u_n(t))$ converge vers une limite non dégénérée c'est-à-dire une fonction de répartition qui n'est pas une fonction saut. En d'autres termes, il s'agit d'obtenir une convergence en loi du type

$$a_n^{-1}(M_n - b_n) \xrightarrow[n\to\infty]{\mathcal{D}} Y$$

où Y est une variable aléatoire dont la fonction de répartition est non dégénérée. Les seules limites possibles ne peuvent être que de trois types d'après le théorème de classification suivant :

Théorème 1.2.1. *(Fisher-Tippett-Gnedenko) Supposons qu'il existe* $a_n > 0$, $b_n \in \mathbf{R}$, $n \geq 1$ *tels que* $\mathbb{P}\left(a_n^{-1}(M_n - b_n) \leq t\right) = F_X^n(a_n t + b_n)$ *converge vers une limite non dégénérée* $\tau(t)$*. Alors* $\tau(\cdot)$ *appartient à l'un des trois types suivants :*

1. *Type Fréchet d'indice* $\alpha > 0$ *:*

$$\Phi_\alpha(t) = \begin{cases} 0, & t \leq 0 \\ e^{-t^{-\alpha}}, & t > 0 \end{cases}.$$

2. *Type Gumbel :*

$$\Lambda(t) = e^{-e^{-t}}, \quad t \in \mathbf{R}.$$

3. *Type Weibull d'indice* $\alpha > 0$ *:*

$$\Psi_\alpha(t) = \begin{cases} e^{-(-t)^\alpha}, & t < 0 \\ 1, & t \geq 0 \end{cases}.$$

En d'autres termes, il existe deux termes $a > 0$ et $b \in \mathbf{R}$ tels que $\tau(t)$ soit la composée de la fonction $t \mapsto at + b$ et de l'une des trois fonctions citées ci-dessus. Le théorème précédent a d'abord été démontré par Fisher et Tippett en 1927 puis approfondi par Gnedenko [48] en 1943. Pour une preuve plus récente, on peut consulter le chapitre 1, [31] de de Haan et Ferreira.

Plusieurs travaux ont été effectués pour généraliser l'étude du cas iid à des suites à dépendance faible. On cite, par exemple, les travaux de Watson [144] sur la k-dépendance, Loynes [86] et Welsch [145] sur des suites satisfaisant une condition de mélange fort. L'une des conditions les plus faibles sur la dépendance de la suite est la condition $D(u_n)$ de Leadbetter, introduite en 1974 dans [80]. On présente ci-dessous la condition $D(u_n)$ et une seconde condition, notée $D'(u_n)$, dont il est également l'auteur et qui concerne une propriété locale de la suite.

Conditions $D(u_n)$ et $D'(u_n)$ Soit $(X_i)_{i\geq 1}$, une suite de variables aléatoires stationnaires réelles et $(u_i)_{i\geq 1}$ une suite déterministe réelle. Désignons par $F_{i_1,\ldots,i_n}(x_1,\ldots,x_n)$ la probabilité $\mathbb{P}\left(X_{i_1} \leq x_{i_1},\ldots,X_{i_n} \leq x_{i_n}\right)$ et plus brièvement $F_{i_1,\ldots,i_n}(u) = F_{i_1,\ldots,i_n}(u,\ldots,u)$ pour tout n, $i_1,\ldots,i_n$ et u.

On dit que la suite $(X_i)_{i\geq 1}$ satisfait la *condition* $D(u_n)$ si pour tout n, l et pour tout entier $i_1,\ldots,i_p,j_1,\ldots,j_p$ tels que $1 \leq i_1 < i_2 < \cdots < i_p < j_1 < \cdots < j_{p'} \leq n$ avec $j_1 - i_p \geq l$, on a

$$\left|F_{i_1,\ldots,i_p,j_1,\ldots,j_{p'}}(u_n) - F_{i_1,\ldots,i_p}(u_n)F_{j_1,\ldots,j'_p}(u_n)\right| \leq \alpha_{n,l}$$

où $\alpha_{n,l_n} \to 0$ quand $n \to \infty$ pour une certaine suite $(l_n)_{n\geq 1}$ avec $l_n = o(n)$. Cette condition de dépendance faible est peu restrictive car elle ne porte que sur des événements A du type

$$A = \bigcap_{r=1}^{p} \{X_{i_r} \leq u_n\}$$

et pas nécessairement sur les lois jointes. Sous cette condition et le fait que $\mathbb{P}\left(M_n \leq u_n(t)\right)$ converge vers une limite non dégénérée, avec $u_n = u_n(t) = a_n t + b_n$, le résultat de Fisher-Tippett-Gnedenko reste vrai.

Un autre résultat évident, dans le cas d'une suite iid, est que la convergence

$$\mathbb{P}\left(M_n \leq u_n\right) \underset{n\to\infty}{\longrightarrow} e^{-\tau}$$

est équivalente à

$$n\mathbb{P}\left(X_i > u_n\right) \underset{n\to\infty}{\longrightarrow} \tau \tag{1.2.1}$$

où $\tau \geq 0$ est une constante fixée. Lorsque la suite ne satisfait que la condition $D(u_n)$, cette équivalence n'est pas toujours vraie. Pour remédier à cela, Leadbetter introduit une seconde condition.

On dit que la suite $(X_i)_{i\geq 1}$ satisfait la *condition* $D'(u_n)$ si

$$\limsup_{n\to\infty} n \sum_{j=2}^{\lfloor n/k \rfloor} \mathbb{P}\left(X_1 > u_n, X_j > u_n\right) \longrightarrow 0 \text{ lorsque } k \to \infty$$

avec $u_n = u_n(t) = a_n t + b_n$. Sous ces deux conditions, Leadbetter obtient le théorème suivant :

Théorème 1.2.2. *(Leadbetter) Soit $(X_i)_{i\geq 1}$, une suite de variables aléatoires réelles satisfaisant les conditions $D(u_n)$ et $D'(u_n)$ et $0 \geq \tau < \infty$. Alors*

$$\mathbb{P}\left(M_n \leq u_n\right) \underset{n\to\infty}{\longrightarrow} e^{-\tau}$$

si et seulement si

$$n\mathbb{P}\left(X_i > u_n\right) \underset{n\to\infty}{\longrightarrow} \tau.$$

Le théorème précédent peut être étendu aux statistiques d'ordre c'est-à-dire aux variables aléatoires $M_n^{(r)}$ où $M_n^{(r)}$ désigne la r-ième plus grande valeur, $r \geq 1$. En effet, grâce aux conditions $D(u_n)$ et $D'(u_n)$, le nombre d'excédents, c'est-à-dire le nombre U_n de variables aléatoires se situant au-dessus du seuil u_n et donné par

$$U_n = \sum_{i=1}^{n} \mathbb{1}_{X_i > u_n}$$

peut être approché par une variable aléatoire de Poisson U_n' de moyenne $\mathbb{E}[U_n'] = \mathbb{E}[U_n] = n\mathbb{P}\left(X_i > u_n\right)$. Comme le nombre moyen d'excédents $n\mathbb{P}\left(X_i > u_n\right)$ converge vers τ, on peut approcher U_n par une variable aléatoire de Poisson de paramètre τ. Parce que les événements $\{M_n^{(r)} \leq u_n\}$ et $\{U_n \leq r-1\}$ sont égaux, le Théorème 1.2.2 peut être étendu en le résultat plus précis (voir Théorème 5.2 de [80]) :

$$\mathbb{P}\left(M_n^{(r)} \leq u_n\right) \underset{n\to\infty}{\longrightarrow} \sum_{s=0}^{r-1} \frac{e^{-\tau}\tau^s}{s!}. \tag{1.2.2}$$

Processus ponctuel des excédents Dans le cas d'une suite $(X_i)_{i\geq 1}$ iid, on peut caractériser la répartition des excédents. Plus précisément, on désigne par Φ_n le processus ponctuel dans $[0,1] \times \mathbf{R}$ défini par

$$\Phi_n = \left\{ \left(\frac{j}{n}, a_n^{-1}(X_j - b_n) \right), 1 \leq j \leq n \right\}$$

où $u_n = u_n(t) = a_n t + b_n$ satisfait (1.2.1) avec $\tau = \tau(t)$. Désignons respectivement par x^* et ${}_*x$ les extrémités de l'intervalle sur lequel $\tau(\cdot)$ est définie c'est-à-dire

$${}_*x = \inf\{t \in \mathbf{R}, \tau(t) < \infty\} \text{ et } x^* = \sup\{t \in \mathbf{R}, \tau(t) > 0\}$$

et considérons un processus ponctuel de Poisson $\Phi \subset \mathbf{R}_+ \times ({}_*x, x^*]$, de mesure d'intensité ν donnée par

$$\nu((a,b] \times (s,t]) = \mathbb{E}\left[\#\Phi \cap ((a,b] \times (s,t])\right] = (b-a) \cdot (\tau(s) - \tau(t))$$

pour tout $0 \leq a \leq b \leq 1$ et pour tout segment $(s, t] \subset ({}_*x, x^*]$. Le processus ponctuel des excédents Φ_n converge vers le processus ponctuel de Poisson Φ (voir Théorème 2.1.2 de [31]) i.e.

$$\Phi_n \xrightarrow[n\to\infty]{\mathcal{D}} \Phi \tag{1.2.3}$$

où la convergence en loi $\xrightarrow[n\to\infty]{\mathcal{D}}$ de processus ponctuels a été introduite à la page 24. Ce résultat montre que les statistiques d'ordre peuvent être vues comme des points d'un processus ponctuel de Poisson (non homogène). Il est encore valable pour des suites non iid satisfaisant plus généralement la condition $D'(u_n)$ et une condition, dite $D_r(\mathbf{u}_n)$ (voir page 107 de [82]), un peu plus forte que la condition $D(u_n)$.

La démarche de Leadbetter, cependant, ne permet pas d'avoir une vitesse de convergence de (1.2.2) lorsque la suite satisfait seulement les conditions $D(u_n)$ et $D'(u_n)$ notamment parce que celles-ci sont trop faibles. Une façon de l'obtenir est d'utiliser la méthode de Chen-Stein qui fournit une majoration de l'erreur entre une variable aléatoire, ici le nombre d'excédents U_n, et une loi de Poisson. En choisissant de bonnes hypothèses, plus contraignantes que celles de Leadbetter, Smith [135] détermine une vitesse de convergence de $\mathbb{P}(M_n \leq u_n) - e^{-\tau}$ pour une suite de variables aléatoires pas nécessairement stationnaires. Nous présentons, ci-dessous, un résultat fondamental issu de cette méthode et nous utiliserons ce résultat dans le chapitre 3.

Un résultat d'approximation Poissonienne Dans ce paragraphe, on présente la méthode, dite de Chen-Stein, sur laquelle Smith [135] s'est appuyé pour donner une vitesse de convergence du maximum d'une suite de variables aléatoires. La méthode de Chen-Stein est un des outils les plus puissants pour approcher une variable aléatoire par une loi de Poisson. Elle a été introduite par Chen [22] en 1975. On accolle le nom de Stein car l'idée sous-jacente à cette méthode a d'abord été introduite par Stein [138] en 1972 pour obtenir une vitesse de convergence d'une approximation par une loi normale. Pour des travaux récents sur l'amélioration de la méthode de Chen-Stein et l'approximation poissonienne, on peut consulter Peccati [112], Nourdin et Peccati [106] et Lachièze-Rey et Peccati [77]. Nous ne présentons pas formellement cette méthode et nous nous contentons d'énoncer un résultat dû à Arratia *et al.* dont elle est l'outil.

Désignons par $(X_i)_{i\in\mathcal{I}}$ un champ aléatoire discret, c'est-à-dire un ensemble $\mathcal{I}$ dénombrable et une variable aléatoire X_i pour tout $i \in \mathcal{I}$. On suppose que les X_i soient des variables aléatoires de Bernoulli de paramètre $p_i > 0$. Notons

$$U = \sum_{i\in\mathcal{I}} X_i \text{ et } \tau = \mathbb{E}[U] = \sum_{i\in\mathcal{I}} p_i.$$

Pour tout $i \in \mathcal{I}$, on suppose qu'on ait choisi un voisinage $V(i) \subset \mathcal{I}$ contenant i. On peut voir $V(i)$ comme le lieu des indices $j \in \mathcal{I}$ tels que les variables aléatoires X_i et X_j soient dépendantes. Désignons par b_1, b_2 et b_3 les trois termes suivants :

$$b_1 = \sum_{i\in V}\sum_{j\in V(i)} p_i p_j, \quad b_2 = \sum_{i\in V}\sum_{i\neq j\in V(i)} p_{ij} \text{ et } b_3 = \sum_{i\in V} \mathbb{E}\left[\left|\mathbb{E}\left[X_i - p_i | \sigma(X_j : j \notin V(i))\right]\right|\right]$$

où $p_{ij} = \mathbb{E}[X_i X_j]$. Le terme b_1 mesure la taille du voisinage, b_2 le nombre moyen de voisins pour une occurrence donnée tandis que b_3 mesure la dépendance entre un événement et le nombre d'occurrences en dehors de son voisinage. Avec les notations précédentes et en appliquant la méthode de Chen-Stein, Arratia *et al.* [4] obtiennent le théorème suivant :

Théorème 1.2.3. *(Arratia et al.) Soit* $U = \sum_{i \in \mathcal{I}} X_i$ *le nombre d'occurrences et* Z *une loi de Poisson de paramètre* $\tau = \mathbb{E}[U] < \infty$. *Alors*

$$d_{TV}(U, Z) \leq 2 \cdot (b_1 + b_2 + b_3).$$

Dans l'expression précédente, le terme $d_{TV}(U, Z)$ désigne la distance en variation totale entre les variables aléatoires U et Z i.e.

$$d_{TV}(U, Z) = \sup_{A \subset \mathbb{N}} |\mathbb{P}(Z \in A) - \mathbb{P}(U \in A)|.$$

Nous utiliserons ce résultat dans les problèmes que nous considérerons dans le chapitre 3.

Notion d'indice extrême Dans ce paragraphe, nous revenons à l'un des travaux de Leadbetter. Nous avons vu que, sous les conditions $D(u_n)$ et $D'(u_n)$, le nombre d'excédents U_n peut être approché par une loi de Poisson et que, sous des conditions plus fortes, on peut obtenir une vitesse de convergence pour étudier les statistiques d'ordre. Dans ce paragraphe, on ne suppose que la condition $D(u_n)$. Dans ce cas, on ne peut rien dire sur les statistiques d'ordre mais on a le résultat suivant (voir le Théorème 2.2 de [81]) :

Théorème 1.2.4. *(Leadbetter) Soit* $(X_i)_{i \geq 1}$, *une suite de variables aléatoires réelles et stationnaires et* $(u_i)_{i \geq 1}$ *une suite déterministe telle que* $n\mathbb{P}(X_1 > u_n(\tau)) \underset{n \to \infty}{\longrightarrow} \tau$ *pour tout* $\tau > 0$ *et satisfaisant la condition* $D(u_n(\tau_0))$ *pour un certain* $\tau_0 > 0$. *Alors, il existe des constantes* $0 \leq \theta \leq \theta' \leq 1$ *telles que*

$$\limsup_{n \to \infty} \mathbb{P}(M_n \leq u_n(\tau)) = e^{-\theta \tau} \textit{ et } \liminf_{n \to \infty} \mathbb{P}(M_n \leq u_n(\tau)) = e^{-\theta' \tau}$$

pour tout $0 < \tau \leq \tau_0$. *En particulier, si* $\mathbb{P}(M_n \leq u_n(\tau))$ *converge alors* $\theta = \theta'$ *et*

$$\mathbb{P}(M_n \leq u_n(\tau)) \underset{n \to \infty}{\longrightarrow} e^{-\theta \tau}$$

pour tout τ.

Si $n\mathbb{P}(X_1 > u_n(\tau)) \underset{n \to \infty}{\longrightarrow} \tau$ et $\mathbb{P}(M_n \leq u_n(\tau)) \underset{n \to \infty}{\longrightarrow} e^{-\theta \tau}$ pour tout $\tau > 0$, on dit que la suite (u_i) est *d'indice extrême* θ. Cet indice n'existe pas nécessairement et peut être égal à 0. L'indice extrême a une interprétation géométrique et est dû au fait qu'on n'ait pas supposé la condition $D'(u_n)$. En effet, tandis que la condition $D(u_n)$ est une condition globale portant sur "la dépendance faible de variables aléatoires éloignées" et ne peut être éradiquée, la condition $D'(u_n)$, locale, signifie que "deux variables aléatoires proches ne sont pas toutes les deux des excédents" avec grande probabilité. Lorsque celle-ci est supposée, les excédents se font de façon isolée et cela explique que le maximum se comporte comme un maximum pris sur exactement n variables aléatoires indépendantes. En revanche, lorsque la condition $D'(u_n)$ n'est pas garantie, les excédents peuvent se regrouper en des clusters et l'indice extrême représente alors l'inverse de la taille moyenne d'un cluster d'excédents. Un résultat dû à Leadbetter et Nandagopalan [83] et approfondi par Chernick *et al.* (voir le Corollaire 1.3 de [24]) montre que, sous de bonnes hypothèses, l'indice extrême θ existe si et seulement si

$$\frac{\mathbb{P}(X_2 \leq u_n(\tau) < X_1)}{\mathbb{P}(X_1 > u_n(\tau))} \underset{n \to \infty}{\longrightarrow} \theta$$

pour tout $\tau > 0$. La relation précédente reflète bien ce qu'est l'indice extrême : il s'agit du nombre moyen de sauts divisé par le nombre moyen d'excédents. Plusieurs travaux ont été effectués pour estimer cet indice, voir par exemple Smith et Weissman [136] et Hsing *et al.* [60].

Les résultats ci-dessus peuvent être généralisés à des champs aléatoires, notamment pour ce qui concerne les conditions $D(u_n)$ et $D'(u_n)$ [25], [84] et la notion d'indice extrême [114]. Pour une présentation générale de la théorie des valeurs extrêmes, on peut consulter les ouvrages classiques de de Haan et Ferreira [31], Leadbetter*et al.* [82] et Resnick [124].

1.3 Présentation du problème

Nous avons vu, dans la première section du présent chapitre, que beaucoup de travail a été effectué sur le comportement moyen d'une mosaïque aléatoire et notamment sur la cellule typique. Dans cette thèse, nous étudions la mosaïque aléatoire par les valeurs extrêmes, une approche qui, à notre connaissance, est jusqu'ici inédite. Dans cette section, nous nous proposons de fixer formellement le cadre de notre travail.

Désignons par $\mathfrak{m}$ une mosaïque aléatoire stationnaire de $\mathbf{R}^d$ d'intensité γ fixée et par W une fenêtre d'observation qui est un Borélien borné de $\mathbf{R}^d$ de volume non nul. Pour décrire l'ensemble de la mosaïque, on fait tendre la fenêtre vers l'infini, autrement dit, on considère l'ensemble

$$\mathbf{W}_\rho = \rho^{1/d} W$$

où $\rho \to \infty$ est un réel positif. Rappelons que pour tout $C \in \mathfrak{m}$, le point $z(C)$ désigne le germe de la cellule et considérons une fonction $f : \mathcal{K}_d \to \mathbf{R}$ mesurable définie sur l'ensemble des corps convexes de $\mathbf{R}^d$. On peut voir $f(\cdot)$ comme une caractéristique géométrique (par exemple le volume ou le diamètre) et on applique $f(\cdot)$ à chaque cellule de $\mathfrak{m}$. L'objet de cette thèse est d'étudier le comportement des extrêmes, voire plus généralement des statistiques d'ordre, de la fonction $f(\cdot)$ pris sur toutes les cellules dont le germe est dans $\mathbf{W}_\rho$ i.e. $z(C) \in \mathbf{W}_\rho$ lorsque ρ tend vers l'infini. En particulier, on s'intéresse au maximum :

$$M_{f,\mathbf{W}_\rho} = \max_{\substack{C \in \mathfrak{m}, \\ z(C) \in \mathbf{W}_\rho}} f(C).$$

Concrètement, ainsi que nous l'avons vu dans la section précédente, il s'agit de déterminer deux fonctions $a_\rho > 0$ et $b_\rho \in \mathbf{R}$ de sorte qu'on ait une convergence en loi du type

$$a_\rho^{-1} \left(M_{f,\mathbf{W}_\rho} - b_\rho \right) \xrightarrow[\rho \to \infty]{\mathcal{D}} Y \tag{1.3.1}$$

où Y est une variable aléatoire non dégénérée.

La théorie classique des valeurs extrêmes n'est pas suffisante pour résoudre le problème car les variables aléatoires que nous considérons ne sont définies ni comme une suite, ni comme un champ aléatoire. De plus, les lois de probabilité sous-jacentes ne sont en général pas connues et les cellules dépendent les unes des autres. Cependant, ces points peuvent être contrecarrés lorsqu'on suppose une propriété de mélange sur la mosaïque, ce qui est le cas pour les mosaïques de Poisson-Voronoï ou de Poisson-Delaunay. En particulier, on peut penser que le maximum pris sur les cellules dont le germe est dans la fenêtre se comporte comme celui pris sur des cellules *typiques* et *indépendantes*. Nos résultats portent principalement sur des théorèmes limites des extrêmes et des statistiques d'ordre et sur des aspects géométriques comme les problèmes de bord et la forme des cellules optimisantes.

Chapitre 2

Maxima et minima des caractéristiques radiales pour une mosaïque de Poisson-Voronoi

Sommaire

Désignons par $\mathbf{X}_\gamma$ un processus ponctuel de Poisson (homogène) d'intensité γ dans $\mathbf{R}^d$ et par $\mathfrak{m}_{PVT}$ la mosaïque de Poisson-Voronoï associée. On rappelle que pour tout $x \in \mathbf{X}_\gamma$, la cellule de Voronoï de germe x est l'ensemble

$$C_{\mathbf{X}_\gamma}(x) = \{y \in \mathbf{R}^d, |y - x| \leq |y - x'|, x' \in \mathbf{X}_\gamma\}.$$

Dans ce chapitre, on s'intéresse au cas où la fonction $f(\cdot)$, introduite dans la section 1.3, est une caractéristique radiale. Plus précisément, pour toute cellule $C_{\mathbf{X}_\gamma}(x)$, on désigne par $r(C_{\mathbf{X}_\gamma}(x))$ et $R(C_{\mathbf{X}_\gamma}(x))$ le rayon inscrit, respectivement circonscrit, de la cellule i.e.

$$r(C_{\mathbf{X}_\gamma}(x)) = \max\{r \geq 0, B(x, r) \subset C_{\mathbf{X}_\gamma}(x)\} \text{ et } R(C_{\mathbf{X}_\gamma}(x)) = \min\{R \geq 0, B(x, R) \supset C_{\mathbf{X}_\gamma}(x)\}.$$

On observe la mosaïque $\mathfrak{m}_{PVT}$ dans une fenêtre W qui, dans ce chapitre, est un *corps convexe* de $\mathbf{R}^d$ de volume 1 et l'on étudie le maximum et le minimum de ces caractéristiques prises sur toutes les cellules de la fenêtre. Autrement dit, on s'intéresse aux variables aléatoires

$$r_{\max}(\gamma) = \max_{x \in \mathbf{X}_\gamma \cap W} r(C_{\mathbf{X}_\gamma}(x)), \qquad r_{\min}(\gamma) = \min_{x \in \mathbf{X}_\gamma \cap W} r(C_{\mathbf{X}_\gamma}(x))$$
$$R_{\max}(\gamma) = \max_{x \in \mathbf{X}_\gamma \cap W} R(C_{\mathbf{X}_\gamma}(x)), \qquad R_{\min}(\gamma) = \min_{x \in \mathbf{X}_\gamma \cap W} R(C_{\mathbf{X}_\gamma}(x)).$$

En vue de nous placer dans le cadre précis de l'approximation de Poisson-Voronoï (voir page 33), on ne choisit pas de prendre une intensité fixée et une fenêtre qui tend vers l'infini. A la place, on étudie le comportement limite des variables aléatoires ci-dessus dans une *fenêtre fixée*, en l'occurrence W, pour une *intensité* γ qui tend vers l'*infini*. Le problème revient au même par changement d'échelle.

Résultats nouveaux Dans ce travail, on obtient quatre types de résultats nouveaux :

1. Pour chacune de ces variables aléatoires, on détermine les paramètres de normalisation a_γ et b_γ de sorte qu'on ait une convergence de type (1.3.1) (Théorème 2.1.1). En particulier, on montre que les maxima des rayons inscrits et circonscrits sont asymptotiquement proches, de l'ordre de $(\log \gamma/\gamma)^{1/d}$, ce qui peut être relié à la conjecture de Kendall (voir page 32). Les lois limites sont alors des variables aléatoires de Gumbel. Les minima, en revanche, ne sont pas du même ordre puisque on montre que $r_{\min}(\gamma)$ est de l'ordre de $\gamma^{-2/d}$ tandis que l'ordre de $R_{\min}(\gamma)$ est $\gamma^{-(d+2)/(d(d+1))}$. Les lois limites apparaissant sont alors des lois de Weibull. Pour obtenir ces résultats, on réinterprète ces variables aléatoires en terme de recouvrement. On utilise abondamment la formule de Slivnyak et on adapte et étend, dans notre cadre, un lemme de Henze qui ramène l'étude des extrêmes à des lois fini-dimensionnelles.
2. On déduit du comportement asymptotique du maximum des rayons circonscrits, une majoration de la distance de Hausdorff entre un corps convexe et son approximation de Poisson-Voronoï (Corollaire 2.4.2). Cette majoration est *a priori* le bon ordre de la distance car elle est de l'ordre $\log \gamma/\gamma$.
3. Dans le cas du rayon circonscrit, on donne la nature de la cellule qui le minimise. Plus précisément, on montre que cette cellule est un simplexe (Corollaire 2.3.4).
4. Un dernier résultat porte sur les cellules frontières, c'est-à-dire les cellules qui intersectent le bord de W. On montre que la participation de ces dernières est négligeable et n'affecte pas le comportement asymptotique des extrêmes (Proposition 2.2.1).

Les résultats précédents font l'objet d'un premier article [19] accepté dans *Extremes* et en collaboration avec Pierre Calka.

Extreme values for characteristic radii of a Poisson-Voronoi Tessellation

P. Calka and N. Chenavier

2.1 Introduction

Let χ be a locally finite subset of $\mathbf{R}^d$ endowed with its natural norm $|\cdot|$. The Voronoi cell of nucleus $x \in \chi$ is the set

$$C_\chi(x) = \{y \in \mathbf{R}^d, |y - x| \leq |y - x'|, x \neq x' \in \chi\}.$$

When $\chi = \mathbf{X}_\gamma$ is a homogeneous Poisson point process of intensity γ, the family $\{C_{\mathbf{X}_\gamma}(x), x \in \mathbf{X}_\gamma\}$ is the so-called Poisson-Voronoi tessellation. Such model is extensively used in many domains such as cellular biology [118], astrophysics [120], telecommunications [6] and ecology [125]. For a complete account, we refer to the books [108], [130], [99] and the survey [18].

To describe the mean behaviour of the tessellation, the notion of typical cell is introduced. The distribution of this random polytope can be defined as

$$\mathbb{E}[f(\mathscr{C}_\gamma)] = \frac{1}{\gamma \lambda_d(B)} \mathbb{E}\left[\sum_{x \in \mathbf{X}_\gamma \cap B} f(C_{\mathbf{X}_\gamma}(x) - x)\right]$$

where $f : \mathcal{K}_d \to \mathbf{R}$ is any bounded measurable function on the set of convex bodies $\mathcal{K}_d$ (endowed with the Hausdorff topology), λ_d is the d-dimensional Lebesgue measure and B is a Borel subset of $\mathbf{R}^d$ with finite volume $\lambda_d(B) \in (0, \infty)$. Equivalently, $\mathscr{C}_\gamma$ is the Voronoi cell $C_{\mathbf{X}_\gamma \cup \{0\}}(0)$ when we add the origin to the Poisson point process: this fact is a consequence of Slivnyak's Theorem, see e.g. Theorem 3.3.5 in [130]. The study of the typical cell in the literature includes mean values calculations [98], second order properties [54] and distributional estimates [16], [10], [102]. A long standing conjecture due to D.G. Kendall about the asymptotic shape of large typical cell is proved in [61].

To the best of our knowledge, extremes of geometric characteristics of the cells, as opposed to their means, have not been studied in the literature up to now. In this paper, we are interested in the following problem: only a part of the tessellation is observed in a convex body W (i.e. a convex compact set with non-empty interior) of volume $\lambda_d(W) = 1$ where λ_d denotes the Lebesgue measure in $\mathbf{R}^d$. Let $f : \mathcal{K}_d \to \mathbf{R}$ be a measurable function, e.g. the volume or the diameter of the cells. What is the limit behaviour of

$$M_f(\gamma) = \max_{x \in \mathbf{X}_\gamma \cap W} f(C_{\mathbf{X}_\gamma}(x))$$

when γ goes to infinity ? By scaling invariance of $\mathbf{X}_\gamma$, it is the same as considering a tessellation with fixed intensity and observed in a window $W_\rho := \mathbf{W}_{\rho^d} = \rho W$ with $\rho \to \infty$. We give below some applications of such approach.

First, the study of extremes describes the regularity of the tessellation. For instance, in finite element method, the quality of the approximation depends on some consistency measurements over the partition, see e.g. [70].

Another potential application field is statistics of point processes. The key idea would be to identify a point process from the extremes of its underlying Voronoi tessellation. A lot of inference methods have been developed for spatial point processes [100]. A comparison based on Voronoi extremes may or may not provide stronger results. At least, the regularity seems to discriminate to some extent some point processes (see for instance a comparison between a determinantal point process and a Poisson point process in [85]).

A third application is the so-called Poisson-Voronoi approximation i.e. a discretization of a convex body W by the following union of Voronoi cells

$$\mathscr{V}_{\mathbf{X}_\gamma}(W) = \bigcup_{x \in \mathbf{X}_\gamma \cap W} C_{\mathbf{X}_\gamma}(x).$$

The first breakthrough is due to Heveling and Reitzner [57] and includes variance estimates of the volume of symmetric difference. However, the Hausdorff distance between the convex body and its approximation has not been studied yet. It is strongly connected to the maximum of the diameter of the cells which intersect the boundary of ∂W. We discuss this in section 2.4 and prove a rate of convergence of the approximation to the convex body with a suitable assumption on W.

Concretely, we are looking for two parameters $a_f(\gamma)$ and $b_f(\gamma)$ such that

$$a_f(\gamma) M_f(\gamma) + b_f(\gamma) \underset{\gamma \to \infty}{\overset{\mathscr{D}}{\longrightarrow}} Y$$

where Y is a non degenerate random variable and $\overset{\mathscr{D}}{\longrightarrow}$ denotes the convergence in distribution. Up to a normalization, the extreme distributions of real random variables which are iid or with a mixing property are of three types: Fréchet, Gumbel or Weibull (see e.g. [86] and [80]). More about extreme value theory can be found in the reference books by De Haan and Ferreira [31] and by Resnick [124]. Some extremes have been studied in stochastic geometry, for instance the maximum and minimum of inter-point distances of some point processes [64], [90], [113], extremes of particular random fields [78] or in the field of stereology [58], [111] but, to the best of our knowledge, nothing has been done for random tessellations. In our framework, the general theory cannot directly be applied for several reasons: unknown distribution of the characteristic for one fixed cell, dependency between cells and boundary effects. Moreover, the exceedances can be realized in clusters. For example, when the distance between the boundary of the cell and its nucleus is small, this is the same for one of its neighbors. Such clusters lead to the notion of extremal index, which was introduced by Leadbetter in [81], and that we will study in a future work.

In this paper, we are interested in the characteristic radii i.e. inscribed and circumscribed radii of the Voronoi cell $C_{\mathbf{X}_\gamma}(x)$ defined as

$$r(C_{\mathbf{X}_\gamma}(x)) = \max\{r \geq 0, B(x,r) \subset C_{\mathbf{X}_\gamma}(x)\} \text{ and } R(C_{\mathbf{X}_\gamma}(x)) = \min\{R \geq 0, B(x,R) \supset C_{\mathbf{X}_\gamma}(x)\}$$

where $B(x,r)$ is the ball of radius r centered at x. Two reasons led us to the study of these quantities. First, the distribution tails of the inradius and circumscribed radius of the typical cell are easier to deal with [15] compared to other characteristics such as the volume or the number of hyperfaces. Secondly, knowing these two radii provides a better understanding of the cell shape since the boundary of $C_{\mathbf{X}_\gamma}(x)$ is included in the annulus $B(x, R(C_{\mathbf{X}_\gamma}(x))) - B(x, r(C_{\mathbf{X}_\gamma}(x)))$. We consider the extremes

$$\begin{aligned} r_{\max}(\gamma) &= \max_{x \in \mathbf{X}_\gamma \cap W} r(C_{\mathbf{X}_\gamma}(x)), \qquad r_{\min}(\gamma) = \min_{x \in \mathbf{X}_\gamma \cap W} r(C_{\mathbf{X}_\gamma}(x)) \\ R_{\max}(\gamma) &= \max_{x \in \mathbf{X}_\gamma \cap W} R(C_{\mathbf{X}_\gamma}(x)), \qquad R_{\min}(\gamma) = \min_{x \in \mathbf{X}_\gamma \cap W} R(C_{\mathbf{X}_\gamma}(x)). \end{aligned} \tag{2.1.1}$$

In the following theorem, we derive the convergence in distribution of these quantities over cells with nucleus in W.

Theorem 2.1.1. *Let $\mathbf{X}_\gamma$ be a Poisson point process of intensity γ and W a convex body of volume 1 in $\mathbf{R}^d$. Then*

$$\mathbb{P}\left(2^d\kappa_d\gamma r_{\max}(\gamma)^d - \log(\gamma) \leq t\right) \underset{\gamma\to\infty}{\longrightarrow} e^{-e^{-t}}, \quad t \in \mathbf{R}, \tag{2.1.2a}$$

$$\mathbb{P}\left(2^{d-1}\kappa_d\gamma^2 r_{\min}(\gamma)^d \geq t\right) \underset{\gamma\to\infty}{\longrightarrow} e^{-t}, \quad t \geq 0, \tag{2.1.2b}$$

$$\mathbb{P}\left(\kappa_d\gamma R_{\max}(\gamma)^d - \log\left(\alpha_1\gamma(\log\gamma)^{d-1}\right) \leq t\right) \underset{\gamma\to\infty}{\longrightarrow} e^{-e^{-t}}, \quad t \in \mathbf{R}, \tag{2.1.2c}$$

$$\mathbb{P}(\alpha_2\kappa_d\gamma^{(d+2)/(d+1)}R_{\min}(\gamma)^d \geq t) \underset{\gamma\to\infty}{\longrightarrow} e^{-t^{d+1}}, \quad t \geq 0, \tag{2.1.2d}$$

where α_1 and α_2 are given in (2.4.2) *and* (2.3.7) *and $\kappa_d = \lambda_d(B(0,1))$.*

The limit distributions are of type II and III and do not depend on the shape of W. One can note that the ratios $r_{\max}(\gamma)/r_{\min}(\gamma)$ and $R_{\max}(\gamma)/R_{\min}(\gamma)$ are of respective orders $(\gamma\log\gamma)^{1/d}$ and $(\gamma^{1/(d+1)}\log\gamma)^{1/d}$. This quantifies to some extent the irregularity of the Poisson-Voronoi tessellation. Moreover, the ratio $r_{\max}(\gamma)/R_{\max}(\gamma)$ is bounded. It suggests that large cells tend to be spherical around the nucleus. This fact seems to confirm the D.G. Kendall's conjecture.

As it is written, Theorem 2.1.1 is not applicable for concrete data. Indeed, in practice, the only cells which can be measured are included in the window. The following proposition addresses this problem.

Proposition 2.1.2. *The extremes of characteristic radii over all cells included in W or over all cells intersecting ∂W have the same limit distributions as $r_{\max}(\gamma)$, $r_{\min}(\gamma)$, $R_{\max}(\gamma)$ and $R_{\min}(\gamma)$.*

The convergences are illustrated in Figure 2.1 for the cells which are included in $W = [0,1]^2$. For sake of simplicity, the Poisson point process has been realized only in W. Because of Proposition 2.1.2 and related arguments, this does not affect the distribution over cells included in W. Simulations suggest that the rates of convergence are not the same for all these quantities. Indeed, in a future work, we will show that the rate is of the order of γ^{-1}, $\gamma^{-1/4}$ and $\gamma^{-1/6}$ for $r_{\min}(\gamma)$, $r_{\max}(\gamma)$ and $R_{\min}(\gamma)$ respectively.

All results of Theorem 2.1.1 use geometric interpretations. For the circumscribed radii $R_{\max}(\gamma)$ and $R_{\min}(\gamma)$, we write the distributions as covering probabilities of spheres. The inscribed radii can be interpreted as interpoint distances. A study of the extremes of these distances has been done in several works such as [64] and [56]. For sake of completeness, we have rewritten these results in our setting in particular because the boundary effects are highly non trivial. Convergences (2.1.2a) and (2.1.2d) could be obtained by considering underlying random fields and using methods inherited for [4] and [135]. However, this approach does not provide (2.1.2b) and (2.1.2c). We will develop this idea in a future work and deduce some rates of convergence.

The paper is organized as follows. In section 2.2, we provide some preliminary result which shows that the boundary cells are negligible and implies Proposition 2.1.2. In sections 2.3, 2.4 and 2.5, proofs of (2.1.2d), (2.1.2a), (2.1.2c) and (2.1.2b) are respectively given. Section 2.3 requires a technical lemma about deterministic covering of the sphere by caps which is proved in appendix. Section 2.4 contains an application of (2.1.2c) to the Hausdorff distance between W and its Poisson-Voronoi approximation. In section 2.5, we get a specific treatment of boundary effects which is more precise than in section 2.2.

In the rest of the paper, c denotes a generic constant which does not depend on γ but may depend on other quantities. The term u_γ denotes a generic function of t, depending on γ, which is specified at the beginning of sections 2.3, 2.4 and 2.5.

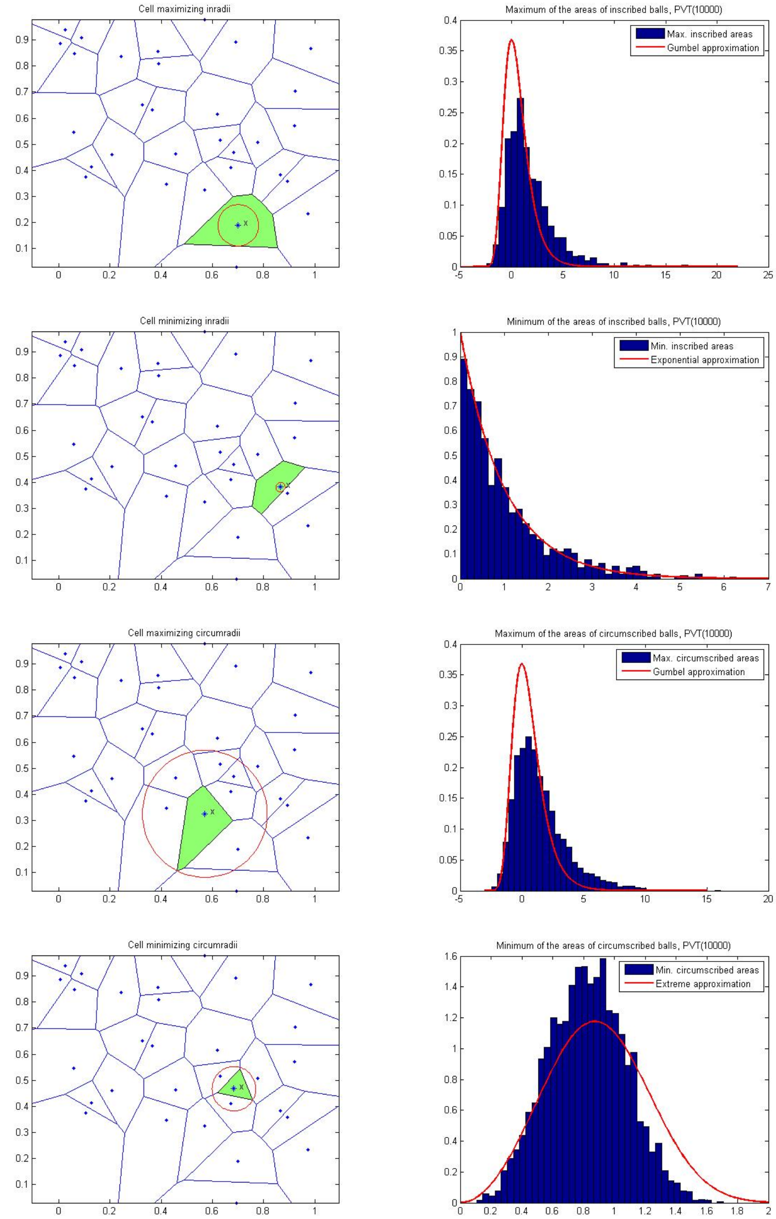


FIGURE 2.1 – Empirical densities of the extremes based on 3500 simulations of PVT in 2D with $\gamma = 10000$, for the cells included in $W = [0,1]^2$, on Matlab©. *(Line 1)* Cell maximizing the inradius. *(Line 2)* Cell minimizing the inradius. *(Line 3)* Cell maximizing the circumradius. *(Line 4)* Cell minimizing the circumradius.

2.2 Preliminaries on boundary effects

In this section, we show that the asymptotic behaviour of an extreme is in general not affected by boundary cells. We apply that result directly to the extremes of characteristic radii in order to show that Theorem 2.1.1 implies Proposition 2.1.2.

Let $f : \mathcal{K}_d \to \mathbf{R}$ be a k-homogeneous measurable function, $0 \leq k \leq d$ (i.e. $f(\lambda C) = \lambda^k f(C)$ for all $\lambda \in \mathbf{R}_+$ and $C \in \mathcal{K}_d$). We consider for any $l \in \mathbf{R}$

$$M_f^b(\gamma, l) = \max_{x \in \mathbf{X}_\gamma, C_{\mathbf{X}_\gamma}(x) \cap W_{1+l} \neq \varnothing} f(C_{\mathbf{X}_\gamma}(x)),$$

$$M_f(\gamma, l) = \max_{x \in \mathbf{X}_\gamma \cap W_{1+l}} f(C_{\mathbf{X}_\gamma}(x)),$$

$$M_f^i(\gamma, l) = \max_{x \in \mathbf{X}_\gamma, C_{\mathbf{X}_\gamma}(x) \subset W_{1+l}} f(C_{\mathbf{X}_\gamma}(x)),$$

where $W_{1+l} = (1+l)W$. When $l = 0$, these maxima are simply denoted by $M_f^b(\gamma)$, $M_f(\gamma)$ and $M_f^i(\gamma)$. We define, for all $\epsilon > 0$, a function l_γ as

$$l_\gamma = \gamma^{-(1-\epsilon)/d}. \tag{2.2.2}$$

Under suitable conditions, the following proposition shows that $M_f^b(\gamma)$, $M_f(\gamma)$ and $M_f^i(\gamma)$ satisfy the same convergence in distribution.

Proposition 2.2.1. *Let Y be a random variable and a_γ, b_γ two functions such that*

$$\frac{a_\gamma}{a_{\gamma_\pm}} \underset{\gamma\to\infty}{\longrightarrow} 1, \quad l_\gamma b_\gamma \underset{\gamma\to\infty}{\longrightarrow} 0 \text{ and } \frac{b_\gamma a_{\gamma_\pm} - a_\gamma b_{\gamma_\pm}}{a_\gamma} \underset{\gamma\to\infty}{\longrightarrow} 0 \tag{2.2.3}$$

with $\gamma_+ = (1+l_\gamma)^k \gamma$ and $\gamma_- = (1-l_\gamma)^k\gamma$ for a certain ϵ. Then

$$a_\gamma M_f^b(\gamma) + b_\gamma \underset{\gamma\to\infty}{\overset{\mathscr{D}}{\longrightarrow}} Y \iff a_\gamma M_f(\gamma) + b_\gamma \underset{\gamma\to\infty}{\overset{\mathscr{D}}{\longrightarrow}} Y \iff a_\gamma M_f^i(\gamma) + b_\gamma \underset{\gamma\to\infty}{\overset{\mathscr{D}}{\longrightarrow}} Y.$$

Before proving Proposition 2.2.1, we need an intermediary result due to Heinrich, Schmidt and Schmidt (Lemma 4.1 of [55]) which shows that, with high probability, the cells which intersect ∂W have nucleus close to ∂W. Actually, they showed it for any stationary tessellation of intensity 1 which is observed in a window ρW with $\rho \to \infty$. For sake of completeness, we rewrite their result in a more explicit version for a Poisson-Voronoi tessellation.

Lemma 2.2.2. *(Heinrich, Schmidt and Schmidt) Let us denote by A_γ and B_γ the events*

$$A_\gamma = \left\{ \bigcap_{x \in \mathbf{X}_\gamma} \{C_{\mathbf{X}_\gamma}(x) \cap W = \varnothing\} \cup \{x \in W_{1+l_\gamma}\} \right\}$$

and

$$B_\gamma = \left\{ \bigcap_{x \in \mathbf{X}_\gamma} \left\{C_{\mathbf{X}_\gamma}(x) \subset W\right\} \cup \{x \notin W_{1-l_\gamma}\} \right\}$$

where l_γ is given in (2.2.2). *Then $\mathbb{P}(A_\gamma)$ and $\mathbb{P}(B_\gamma)$ converge to 1 as γ goes to infinity.*

Proof of Lemma 2.2.2. In [55], it is shown that the probability of the event

$$\left\{ \bigcap_{x \in X_1} \{C_{X_1}(x) \cap W_\rho = \varnothing\} \cup \{x \in W_{\rho+q(\rho)}\} \right\} \cap \left\{ \bigcap_{x \in X_1} \{C_{X_1}(x) \subset W_\rho\} \cup \{x \notin W_{\rho - q(\rho)}\} \right\}$$

(2.2.4)

converges to 1 as ρ goes to infinity where $q(\rho)$ is the solution of the functional equation

$$\rho^d = H(q^d(\rho)).$$

The function $H : \mathbf{R}_+ \to \mathbf{R}_+$ is convex, strictly increasing on its support (x_0, ∞) (for some $x_0 \geq 0$), such that $H(x)/x$ is non-decreasing for $x > 0$, $\lim_{x\to\infty} H(x)/x = \infty$ and $\mathbb{E}[H(D^d(\mathscr{C}_1))] < \infty$ where $D(\mathscr{C}_1)$ is the diameter of the typical cell.

In the case of a Poisson-Voronoi tessellation, $q(\rho)$ can be made explicit. Indeed, we can show that all moments of $D(\mathscr{C}_1)$ exist since $D(\mathscr{C}_1) \leq 2R(\mathscr{C}_1)$ and $R(\mathscr{C}_1)$ is shown to have an exponentially decreasing tail in any dimension by an argument similar to Lemma 1 of [148]. Consequently, for a fixed $\epsilon \in (0,1)$, the functions H and q can be chosen as $H(x) = x^{1/\epsilon}$ and $q(\rho) = \rho^\epsilon$. Using the scaling property of Poisson point process,

$$X_1 \cap W_\rho \overset{\mathscr{D}}{=} \gamma^{1/d}(\mathbf{X}_\gamma \cap W) \text{ and } X_1 \cap W_{\rho \pm q(\rho)} \overset{\mathscr{D}}{=} \gamma^{1/d}(\mathbf{X}_\gamma \cap W_{1\pm l_\gamma})$$

where $\gamma = \rho^d$ and $l_\gamma = \gamma^{-(1-\epsilon)/d}$. We deduce Lemma 2.2.2 from the fact that the event given in (2.2.4) converges to 1 as ρ goes ton infinity. □

Proof of Proposition 2.2.1. *First equivalence:* Let us assume that $a_\gamma M_f^b(\gamma) + b_\gamma$ converges in distribution to Y. On the event A_γ, $\forall x \in \mathbf{X}_\gamma$, $C_{\mathbf{X}_\gamma}(x) \cap W \neq \varnothing \Longrightarrow x \in W_{1+l_\gamma}$. Hence

$$M_f^b(\gamma) \leq M_f(\gamma, l_\gamma) \leq M_f^b(\gamma, l_\gamma). \tag{2.2.5}$$

Because of Lemma 2.2.2, it is enough to show the convergence in distribution of the random variables conditionally on A_γ. Thanks to the scaling property of Poisson point process and the k-homogeneity of f

$$M_f^b(\gamma, l_\gamma) \overset{\mathscr{D}}{=} (1 + l_\gamma)^k M_f^b(\gamma_+) \tag{2.2.6}$$

with $\gamma_+ = (1+l_\gamma)^k\gamma$. According to (2.2.5) and (2.2.6), it remains to show that $a_\gamma(1+l_\gamma)^k M_f^b(\gamma_+) + b_\gamma$ converges in distribution to Y. To do so, it is enough by (2.2.3) to write the equality

$$a_\gamma(1 + l_\gamma)^k M_f^b(\gamma_+) + b_\gamma = \frac{a_\gamma}{a_{\gamma_+}}(1 + l_\gamma)^k \left(a_{\gamma_+} M_f^b(\gamma_+) + b_{\gamma_+}\right) + \frac{b_\gamma a_{\gamma_+} - a_\gamma b_{\gamma_+}}{a_{\gamma_+}} + \frac{a_\gamma}{a_{\gamma_+}} \cdot \frac{1 - (1 + l_\gamma)^k}{l_\gamma} l_\gamma b_{\gamma_+}.$$

In conclusion, we get

$$a_\gamma M_f(\gamma) + b_\gamma \xrightarrow[\gamma\to\infty]{\mathscr{D}} Y. \tag{2.2.7}$$

Conversely, if (2.2.7) holds then, using the fact that

$$M_f(\gamma) \leq M_f^b(\gamma) \leq M_f(\gamma, l_\gamma) \overset{\mathscr{D}}{=} (1 + l_\gamma)^k M_f(\gamma_+)$$

and proceeding along the same lines, we get $a_\gamma M_f^b(\gamma) + b_\gamma \xrightarrow[\gamma\to\infty]{\mathscr{D}} Y$.

Second equivalence: On the event B_γ, $\forall x \in \mathbf{X}_\gamma$, $x \in W_{1-l_\gamma} \Longrightarrow C_{\mathbf{X}_\gamma}(x) \subset W$. We prove the second equivalence as previously noting that, conditionally on B_γ

$$M_f^i(\gamma, -l_\gamma) \leq M_f(\gamma, -l_\gamma) \leq M_f^i(\gamma) \leq M_f(\gamma). \tag{2.2.8}$$

□

2.3 Proof of (2.1.2d) and (2.1.2a)

Proof of (2.1.2d)**.** Let $t \geq 0$ be fixed. We denote by u_γ the following function:

$$u_\gamma = u_\gamma(t) = \left(\alpha_2^{-1}\kappa_d^{-1}\gamma^{-(d+2)/(d+1)}t\right)^{1/d} \tag{2.3.1}$$

where α_2 is given by (2.3.7). Our aim is to prove that $\mathbb{P}(R_{\min}(\gamma) \geq u_\gamma)$ converges to $e^{-t^{d+1}}$ where $R_{\min}(\gamma)$ has been defined in (2.1.1). The main idea is to deduce the asymptotic behaviour of $R_{\min}(\gamma)$ from the study of finite dimensional distributions

$$\left(R(C_{\mathbf{X}_\gamma \cup \{\mathbf{x}_K\}}(x_1)), \ldots, R(C_{\mathbf{X}_\gamma \cup \{\mathbf{x}_K\}}(x_K))\right)$$

for all $\{\mathbf{x}_K\} = \{x_1, \ldots, x_K\}$ and $K \geq 1$. To do this, we write a new adapted version of a lemma due to Henze (see Lemma p. 345 in [56]) in a context of Poisson point process.

Lemma 2.3.1. *Let $f : \mathcal{K}_d \to \mathbf{R}$, $F : \mathcal{K}_d \to \mathbf{R}$ be two measurable functions and A a Borel subset of $\mathbf{R}$. Let us assume that for any $K \geq 1$,*

$$\gamma^K \int_{W^K} \mathbb{P}\left(\forall i \leq K, f(C_{\mathbf{X}_\gamma \cup \{\mathbf{x}_K\}}(x_i)) < u_\gamma, F(C_{\mathbf{X}_\gamma \cup \{\mathbf{x}_K\}}(x_i)) \in A\right) d\mathbf{x}_K \underset{\gamma\to\infty}{\longrightarrow} \lambda^K \tag{2.3.2}$$

where $d\mathbf{x}_K = dx_1 \cdots dx_K$. Then

$$\mathbb{P}\left(\min_{x \in \mathbf{X}_\gamma \cap W, F(C_{\mathbf{X}_\gamma}(x)) \in A} f(C_{\mathbf{X}_\gamma}(x)) \geq u_\gamma\right) \underset{\gamma\to\infty}{\longrightarrow} e^{-\lambda}.$$

Proof of Lemma 2.3.1. Let K be a fixed integer. The proof is close to the proof of Henze's Lemma and uses Bonferroni inequalities: one can show that if $A_{x,\mathbf{X}_\gamma}$ is an $\mathbf{X}_\gamma$-measurable event for all $x \in \mathbf{X}_\gamma \cap W$, then

$$\begin{aligned}\sum_{k=0}^{2K} \frac{(-1)^{k+1}}{k!} \mathbb{E}\left[\sum_{(x_1,\ldots,x_k)_{\neq} \in \mathbf{X}_\gamma \cap W} \mathbb{1}_{A_{x_1,\mathbf{X}_\gamma}} \cdots \mathbb{1}_{A_{x_k,\mathbf{X}_\gamma}}\right] &\leq \mathbb{P}\left(\bigcup_{x \in \mathbf{X}_\gamma \cap W} A_{x,\mathbf{X}_\gamma}\right) \\ &\leq \sum_{k=0}^{2K+1} \frac{(-1)^{k+1}}{k!} \mathbb{E}\left[\sum_{(x_1,\ldots,x_k)_{\neq} \in \mathbf{X}_\gamma \cap W} \mathbb{1}_{A_{x_1,\mathbf{X}_\gamma}} \cdots \mathbb{1}_{A_{x_k,\mathbf{X}_\gamma}}\right].\end{aligned} \tag{2.3.3}$$

where $(x_1, \ldots, x_k)_{\neq}$ means that $(x_1, \ldots, x_k)$ is a k-tuple of distinct points. Applying (2.3.3) to

$$A_{x,\mathbf{X}_\gamma} = \{f(C_{\mathbf{X}_\gamma}(x)) < u_\gamma\} \cap \{F(C_{\mathbf{X}_\gamma}(x)) \in A\},$$

from Slivnyak's formula (see Corollary 3.2.3 in [130]), we obtain

$$\begin{aligned}&\sum_{k=0}^{2K+1} \frac{(-1)^k}{k!}\gamma^k \int_{W^k} \mathbb{P}\left(\forall i \leq K, f(C_{\mathbf{X}_\gamma \cup \{\mathbf{x}_K\}}(x_i)) < u_\gamma, F(C_{\mathbf{X}_\gamma \cup \{\mathbf{x}_K\}}(x_i)) \in A\right) d\mathbf{x}_k \\ &\leq \mathbb{P}\left(\min_{x \in \mathbf{X}_\gamma \cap W, F(C_{\mathbf{X}_\gamma}(x)) \in A} f(C_{\mathbf{X}_\gamma}(x)) \geq u_\gamma\right) \\ &\quad \leq \sum_{k=0}^{2K} \frac{(-1)^k}{k!}\gamma^k \int_{W^k} \mathbb{P}\left(\forall i \leq K, f(C_{\mathbf{X}_\gamma \cup \{\mathbf{x}_K\}}(x_i)) < u_\gamma, F(C_{\mathbf{X}_\gamma \cup \{\mathbf{x}_K\}}(x_i)) \in A\right) d\mathbf{x}_k.\end{aligned}$$

From (2.3.2), we obtain

$$\sum_{k=0}^{2K+1} \frac{(-1)^k}{k!}\lambda^k \leq \liminf_{\gamma\to\infty} \mathbb{P}\left(\min_{x\in\mathbf{X}_\gamma\cap W, F(C_{\mathbf{X}_\gamma}(x))\in A} f(C_{\mathbf{X}_\gamma}(x)) \geq u_\gamma\right)$$
$$\leq \limsup_{\gamma\to\infty} \mathbb{P}\left(\min_{x\in\mathbf{X}_\gamma\cap W, F(C_{\mathbf{X}_\gamma}(x))\in A} f(C_{\mathbf{X}_\gamma}(x)) \geq u_\gamma\right) \leq \sum_{k=0}^{2K} \frac{(-1)^k}{k!}\lambda^k.$$

We conclude the proof by taking $K \to \infty$. □

We apply Lemma 2.3.1 to $f(C_{\mathbf{X}_\gamma}(x)) = R(C_{\mathbf{X}_\gamma}(x))$. The function $F(C_{\mathbf{X}_\gamma}(x)) = F_{d-1}(C_{\mathbf{X}_\gamma}(x))$ denotes the number of hyperfaces of the cell $C_{\mathbf{X}_\gamma}(x)$. In all the proof, the event considered is $A = \mathbf{R}$. We notice that the choice of the function F is of no importance here but will be essential in the proof of Propositions 2.3.3 and 2.3.5. From Lemma 2.3.1, it is sufficient to study the limit behaviour of

$$\gamma^K \int_{W^K} \mathbb{P}\left(\forall i \leq K, R(C_{X\cup\{\mathbf{x}_K\}}(x_i)) < u_\gamma\right) d\mathbf{x}_K \tag{2.3.4}$$

for all integer K. We divide the proof into two parts.

Step 1 When $K = 1$, using the stationarity of $\mathbf{X}_\gamma$ and the fact that $\lambda_d(W) = 1$, we show that the integral (2.3.4) is $\gamma\mathbb{P}(R(C_{\mathbf{X}_\gamma\cup\{0\}}(0)) < u_\gamma)$. As in [18] section 5.2.3, we can reinterpret the distribution function of $R(C_{\mathbf{X}_\gamma\cup\{0\}}(0))$ as a covering probability to get

$$\gamma\mathbb{P}(R(C_{\mathbf{X}_\gamma\cup\{0\}}(0)) < u_\gamma) = \gamma\sum_{k=0}^{\infty} e^{-2^d\kappa_d\gamma u_\gamma^d}\frac{(2^d\kappa_d\gamma u_\gamma^d)^k}{k!}p_k \tag{2.3.5}$$

where p_k is the probability to cover the unit sphere with k independent spherical caps such that their normalized radii are distributed as $d\nu(\theta) = d\pi\sin(\pi\theta)\cos^{d-1}(\pi\theta)\mathbb{1}_{[0,1/2]}(\theta)d\theta$. The equality comes from the fact that

$$\begin{aligned} R(C_{\mathbf{X}_\gamma\cup\{0\}}(0)) < u_\gamma &\iff \text{the family } \{\mathscr{A}_y(0), y \in \mathbf{X}_\gamma\} \text{ covers } S(0,u_\gamma) \\ &\iff \text{the family } \{\mathscr{A}_y(0), y \in \mathbf{X}_\gamma \cap B(0,2u_\gamma)\} \text{ covers } S(0,u_\gamma) \end{aligned}$$

where $S(0,u_\gamma)$ denotes the sphere of radius u_γ and centered in 0,

$$\mathscr{A}_y(x) = S(x,u_\gamma) \cap H_y^+(x) \tag{2.3.6}$$

and $H_y^+(x)$ is the half-space which contains y and delimited by the bisecting hyperplane of $[x,y]$.

We denote by

$$\alpha_2 := \left(\frac{2^{d(d+1)}}{(d+1)!}p_{d+1}\right)^{1/(d+1)} > 0. \tag{2.3.7}$$

For example, when $d = 2$, $\alpha_2 = \left(\frac{5}{12} - \frac{4}{\pi^2}\right)^{1/3}$.

Since $p_k = 0$ for all $k \leq d$, (2.3.5) gives

$$\gamma\mathbb{P}(R(C_{\mathbf{X}_\gamma\cup\{0\}}(0)) < u_\gamma) = \gamma\frac{(2^d\kappa_d\gamma u_\gamma^d)^{d+1}}{(d+1)!}e^{-2^d\kappa_d\gamma u_\gamma^d}p_{d+1} + \gamma\sum_{k=d+2}^{\infty} e^{-2^d\kappa_d\gamma u_\gamma^d}\frac{(2^d\kappa_d\gamma u_\gamma^d)^k}{k!}p_k.$$

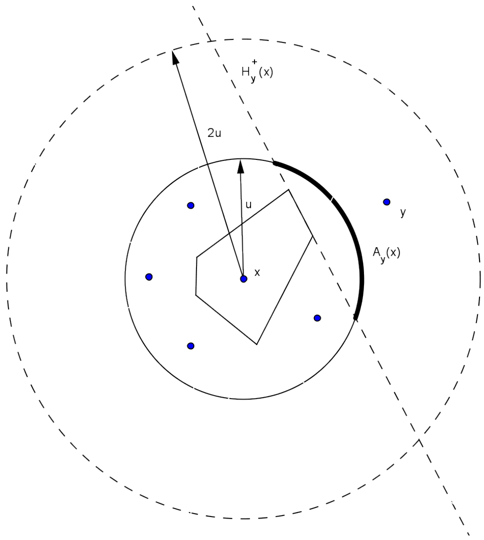


Figure 2.2 – Interpretation of the circumscribed radius as a covering of sphere.

The first term converges to t^{d+1} from (2.3.1) and (2.3.7). The second term is negligible since $\gamma(\gamma u_\gamma^d)^{d+2} = c \cdot \gamma^{-1/(d+1)}$ converges to 0 as γ tends to infinity. This shows that

$$\gamma \int_W \mathbb{P}(R(C_{\mathbf{X}_\gamma \cup \{x\}}(x)) < u_\gamma)dx \underset{\gamma\to\infty}{\longrightarrow} t^{d+1}. \tag{2.3.8}$$

Step 2 When $K \geq 2$, we use the same interpretation as in step 1: for all $\mathbf{x}_K = (x_1, \dots, x_K) \in W^K$ and $i \leq K$, we have

$$\begin{aligned} &R(C_{\mathbf{X}_\gamma \cup \{\mathbf{x}_K\}}(x_i)) < u_\gamma \\ &\qquad \iff \text{the family } \{\mathscr{A}_y(x_i), y \in \mathbf{X}_\gamma \cup \{\mathbf{x}_K\} - \{x_i\}\} \text{ covers } S(x_i, u_\gamma) \\ &\qquad \iff \text{the family } \{\mathscr{A}_y(x_i), y \in (\mathbf{X}_\gamma \cup \{\mathbf{x}_K\} - \{x_i\}) \cap B(x_i, 2u_\gamma)\} \text{ covers } S(x_i, u_\gamma). \end{aligned}$$

Hence, writing the previous event as "$S(x_i, u_\gamma)$ covered", we have

$$\mathbb{P}\left(\forall i \leq K, R(C_{\mathbf{X}_\gamma \cup \{\mathbf{x}_K\}}(x_i)) < u_\gamma\right) = \mathbb{P}\left(\bigcap_{i\leq K} \{S(x_i, u_\gamma) \text{ covered}\}\right). \tag{2.3.9}$$

We have now to consider the spherical caps induced by both the points $x_j, j \neq i$ and the points from $\mathbf{X}_\gamma$. For all $\mathbf{x}_K = (x_1, \dots, x_K) \in W^K$, we denote by $n_l(\mathbf{x}_K)$ the number of connected components of $\bigcup_{i=1}^K B(x_i, 2u_\gamma)$ with exactly l balls. Given $n_1, \dots, n_K$ such that $\sum_{l=1}^K ln_l = K$, we define

$$W_K(n_1, \dots, n_K) = \{\mathbf{x}_K \in W^K, n_l(\mathbf{x}_K) = n_l \text{ for all } l \leq K\}. \tag{2.3.10}$$

Let us note that the subsets $W_K(n_1, \dots, n_K)$, with $\sum_{l=1}^K ln_l = K$, partition W^K. We then deal with two cases.

1. If $B(x_i, 2u_\gamma) \cap B(x_j, 2u_\gamma) = \varnothing$ for all $i \neq j \leq K$ i.e. $\mathbf{x}_K \in W_K(K, \cdots, 0)$, the events considered in the right-hand side of (2.3.9) are independent.
2. If not, we are going to show that the contribution of such $\mathbf{x}_K$ in (2.3.4) is negligible.

More precisely, we write the integral (2.3.4) in the following way

$$\begin{aligned}&\gamma^K \int_{W^K} \mathbb{P}\left(\forall i \leq K, R(C_{\mathbf{X}_\gamma \cup \{\mathbf{x}_K\}}(x_i)) < u_\gamma\right) d\mathbf{x}_K \\ &= \gamma^K \int_{W_K(K,0,\ldots,0)} \mathbb{P}\left(\forall i \leq K, R(C_{\mathbf{X}_\gamma \cup \{\mathbf{x}_K\}}(x_i)) < u_\gamma\right) d\mathbf{x}_K \\ &\qquad + \gamma^K \int_{W^K - W_K(K,0,\ldots,0)} \mathbb{P}\left(\forall i \leq K, R(C_{\mathbf{X}_\gamma \cup \{\mathbf{x}_K\}}(x_i)) < u_\gamma\right) d\mathbf{x}_K. \quad (2.3.11)\end{aligned}$$

Step 2.1 (Case of disjoint balls) For all $\mathbf{x}_K = (x_1, \ldots, x_K) \in W_K(K, 0, \ldots, 0)$, we obtain from (2.3.9) and (2.3.8)

$$\gamma^K \mathbb{P}\left(\forall i \leq K, R(C_{\mathbf{X}_\gamma \cup \{\mathbf{x}_K\}}(x_i)) < u_\gamma\right) = \prod_{i=1}^{K} \gamma \mathbb{P}(R(C_{\mathbf{X}_\gamma \cup \{x_i\}}(x_i)) < u_\gamma) \underset{\gamma \to \infty}{\longrightarrow} (t^{d+1})^K. \quad (2.3.12)$$

Moreover, $\lambda_{dK}(W_K(K, 0, \ldots, 0)) \underset{\gamma \to \infty}{\longrightarrow} 1$. This shows that

$$\gamma^K \int_{W_K(K,0,\ldots,0)} \mathbb{P}\left(\forall i \leq K, R(C_{\mathbf{X}_\gamma \cup \{\mathbf{x}_K\}}(x_i)) < u_\gamma\right) d\mathbf{x}_K \underset{\gamma \to \infty}{\longrightarrow} (t^{d+1})^K. \quad (2.3.13)$$

Step 2.2 (Case of non disjoint balls) In this step, we show that the second integral in the right-hand side of (2.3.11) converges to 0. In particular, we study the limit behaviour of the integrand of (2.3.4) for all $\mathbf{x}_K = (x_1, \ldots, x_K) \in W_K(n_1, \ldots, n_K)$ with $(n_1, \ldots, n_K) \neq (K, 0, \ldots, 0)$. The number of points of $\mathbf{X}_\gamma \cap \bigcup_{i=1}^K B(x_i, 2u_\gamma)$ is Poisson distributed of mean $\gamma \lambda_d \left(\bigcup_{i=1}^K B(x_i, 2u_\gamma)\right)$. From (2.3.9), we deduce that

$$\begin{aligned}&\gamma^K \mathbb{P}\left(\forall i \leq K, R(C_{\mathbf{X}_\gamma \cup \{\mathbf{x}_K\}}(x_i)) < u_\gamma\right) \\ &= \gamma^K \sum_{k=0}^{\infty} \frac{\left(\gamma \lambda_d \left(\bigcup_{i=1}^k B(x_i, 2u_\gamma)\right)\right)^k}{k!} e^{-\gamma \lambda_d \left(\bigcup_{i=1}^k B(x_i, 2u_\gamma)\right)} \times p_k(x_1, \ldots, x_K). \quad (2.3.14)\end{aligned}$$

The term $p_k(x_1, \ldots, x_K)$ denotes the probability to cover the spheres $S(x_i, u_\gamma)$, $i = 1 \ldots K$, with the spherical caps $\{\mathscr{A}_{x_j}(x_i), i \neq j \leq K\}$ and $\{\mathscr{A}_{y_m}(x_i), m \leq k\}$, defined in (2.3.6), where $y_1, \ldots, y_k$ are k independent points which are uniformly distributed in $\bigcup_{i=1}^K B(x_i, 2u_\gamma)$. This probability satisfies the following property:

Lemma 2.3.2. *Let $\mathbf{x}_K = (x_1, \ldots, x_K) \in W_K(n_1, \ldots, n_K)$ and*

$$N = \sum_{l=1}^{K} (d+1) n_l. \quad (2.3.15)$$

Then, for all $k < N$

$$p_k(x_1, \ldots, x_K) = 0. \quad (2.3.16)$$

The proof of Lemma 2.3.2 is postponed to the appendix of this chapter. From (2.3.14), (2.3.16) and the trivial inequalities $0 \leq p_k(x_1, \ldots, x_K) \leq 1$ and $\lambda_d\left(\bigcup_{i=1}^k B(x_i, 2u_\gamma)\right) \leq k2^d\kappa_d u_\gamma^d$, we deduce that there exists a constant c, depending on K, such that

$$\gamma^K \mathbb{P}\left(\forall i \leq K, R(C_{\mathbf{X}_\gamma \cup \{\mathbf{x}_K\}}(x_i)) < u_\gamma\right) \leq \gamma^K \sum_{k=N}^{\infty} \frac{(k2^d\kappa_d\gamma u_\gamma^d)^k}{k!} \underset{\gamma\to\infty}{\sim} c \cdot \gamma^K(\gamma u_\gamma^d)^N$$

where $\phi(\gamma) \underset{\gamma\to\infty}{\sim} \psi(\gamma)$ means $\frac{\phi(\gamma)}{\psi(\gamma)} \underset{\gamma\to\infty}{\longrightarrow} 1$. Using (2.3.1), (2.3.15) and the fact that $K = \sum_{l=1}^K ln_l$, we obtain for γ large enough

$$\gamma^K \mathbb{P}\left(\forall i \leq K, R(C_{\mathbf{X}_\gamma \cup \{\mathbf{x}_K\}}(x_i)) < u_\gamma\right) \leq c \cdot \prod_{l=2}^{K} \gamma^{(l-1)n_l}. \tag{2.3.17}$$

Moreover, according to (2.3.10), we have

$$\lambda_{dK}(W_K(n_1, \ldots, n_K)) \leq c \cdot \prod_{l=2}^{K} (u_\gamma^d)^{(l-1)n_l} = c \cdot \prod_{l=2}^{K} \gamma^{-\frac{(d+2)(l-1)}{d+1}n_l}.$$

This together with (2.3.17) shows that

$$\begin{aligned}&\gamma^K \int_{W^K - W_K(K,0,\ldots,0)} \mathbb{P}\left(\forall i \leq K, R(C_{\mathbf{X}_\gamma \cup \{\mathbf{x}_K\}}(x_i)) < u_\gamma\right) d\mathbf{x}_K \\ &\quad = \sum \gamma^K \int_{W_K(n_1,\ldots,n_K)} \mathbb{P}\left(\forall i \leq K, R(C_{\mathbf{X}_\gamma \cup \{\mathbf{x}_K\}}(x_i)) < u_\gamma\right) d\mathbf{x}_K \leq c \cdot \sum \prod_{l=2}^{K} \gamma^{-\frac{l-1}{d+1}n_l}.\end{aligned} \tag{2.3.18}$$

The sum above runs over all the K-tuples $(n_1, \ldots, n_K)$ such that $\sum_{l=1}^K ln_l = K$ and $n_1 \neq K$. Since $(n_1, \ldots, n_K) \neq (K, 0, \ldots, 0)$, there exists $l \geq 2$ such that $n_l \neq 0$. Consequently, we get from (2.3.18)

$$\gamma^K \int_{W^K - W_K(K,0,\ldots,0)} \mathbb{P}\left(\forall i \leq K, R(C_{\mathbf{X}_\gamma \cup \{\mathbf{x}_K\}}(x_i)) < u_\gamma\right) d\mathbf{x}_K = O\left(\gamma^{-1/(d+1)}\right) \tag{2.3.19}$$

where $\phi(\gamma) = O(\psi(\gamma))$ means that $\frac{\phi(\gamma)}{\psi(\gamma)}$ is bounded.

Conclusion From (2.3.13) and (2.3.19), we deduce that for all $K \geq 1$

$$\gamma^K \int_{W^K} \mathbb{P}\left(\forall i \leq K, R(C_{\mathbf{X}_\gamma \cup \{\mathbf{x}_K\}}(x_i)) < u_\gamma\right) d\mathbf{x}_K \underset{\gamma\to\infty}{\longrightarrow} (t^{d+1})^K.$$

We then apply Lemma 2.3.1, with $A = \mathbf{R}$, to conclude that

$$\mathbb{P}\left(R_{\min}(\gamma) \geq u_\gamma\right) \underset{\gamma\to\infty}{\longrightarrow} e^{-t^{d+1}}.$$

□

The cell which minimizes the circumscribed radius is asymptotically a simplex. To show it, we denote by

$$R'_{\min}(\gamma) = \min_{x\in \mathbf{X}_\gamma \cap W, F_{d-1}(C_{\mathbf{X}_\gamma}(x))\geq d+2} R(C_{\mathbf{X}_\gamma}(x))$$

where $F_{d-1}(C_{\mathbf{X}_\gamma}(x))$ is the number of hyperfaces of $C_{\mathbf{X}_\gamma}(x)$. The order of convergence of $R'_{\min}(\gamma)$ is greater than u_γ according to the following proposition.

Proposition 2.3.3. *Let $\mathbf{X}_\gamma$ be a Poisson point process of intensity γ and W a convex body of volume 1. Then, for all $t \geq 0$,*

$$\mathbb{P}\left(\alpha_2 \kappa_d \gamma^{(d+2)/(d+1)} R'^{d}_{\min}(\gamma) \geq t\right) \underset{\gamma\to\infty}{\longrightarrow} 1.$$

Proof of Proposition 2.3.3. We apply Lemma 2.3.1 to $f(C_{\mathbf{X}_\gamma}(x)) = R(C_{\mathbf{X}_\gamma}(x))$ and $A = [d+2,\infty)$. We then study the finite dimensional distributions i.e.

$$\gamma^K \int_{W^K} \mathbb{P}\left(\forall i \leq K, R(C_{\mathbf{X}_\gamma \cup \{\mathbf{x}_K\}}(x_i)) < u_\gamma, F_{d-1}(C_{\mathbf{X}_\gamma \cup \{\mathbf{x}_K\}}(x_i)) \geq d+2\right) d\mathbf{x}_K \tag{2.3.20}$$

for all $K \geq 1$. When $K = 1$, the integrand of (2.3.20) is

$$\begin{aligned}
&\gamma \mathbb{P}\left(R(C_{\mathbf{X}_\gamma \cup \{0\}}(0)) < u_\gamma, F_{d-1}(C_{\mathbf{X}_\gamma \cup \{0\}}(0)) \geq d+2\right)\\
&\leq \gamma \mathbb{P}\left(R(C_{\mathbf{X}_\gamma \cup \{0\}}(0)) < u_\gamma, \#(\mathbf{X}_\gamma \cap B(0, 2u_\gamma)) \geq d+2\right)\\
&\qquad = \gamma \sum_{k=d+2}^{\infty} \frac{(2^d \kappa_d \gamma u_\gamma^d)^k}{k!} e^{-2^d\kappa_d \gamma u_\gamma^d} p_k \underset{\gamma\to\infty}{\sim} c\cdot \gamma^{-1/(d+1)}.
\end{aligned}$$

We deduce that $\gamma \int_W \mathbb{P}\left(R(C_{\mathbf{X}_\gamma \cup \{x\}}(x)) < u_\gamma, F_{d-1}(C_{\mathbf{X}_\gamma \cup \{x\}}(x)) \geq d+2\right) dx$ converges to 0. More generally, for all $K \geq 1$, we get

$$\gamma^K \int_{W_K(K,0,\ldots,0)} \mathbb{P}\left(\forall i \leq K, R(C_{\mathbf{X}_\gamma \cup \{\mathbf{x}_K\}}(x_i)) < u_\gamma, F_{d-1}(C_{\mathbf{X}_\gamma \cup \{\mathbf{x}_K\}}(x_i)) \geq d+2\right) d\mathbf{x}_K \underset{\gamma\to\infty}{\longrightarrow} 0. \tag{2.3.21}$$

Moreover, from (2.3.19)

$$\gamma^K \int_{W^K - W_K(K,0,\ldots,0)} \mathbb{P}\left(\forall i \leq K, R(C_{\mathbf{X}_\gamma \cup \{\mathbf{x}_K\}}(x_i)) < u_\gamma, F_{d-1}(C_{\mathbf{X}_\gamma \cup \{\mathbf{x}_K\}}(x_i)) \geq d+2\right) d\mathbf{x}_K \underset{\gamma\to\infty}{\longrightarrow} 0. \tag{2.3.22}$$

From (2.3.21), (2.3.22) and Lemma 2.3.1 applied to $A = [d+2, \infty)$, we get

$$\mathbb{P}\left(R'_{\min}(\gamma) \geq u_\gamma\right) \underset{\gamma\to\infty}{\longrightarrow} 1.$$

□

Corollary 2.3.4. Let $\mathbf{X}_\gamma$ be a Poisson point process of intensity γ and W a convex body of volume 1. Then

$$\mathbb{P}\left(\forall x \in \mathbf{X}_\gamma, R(C_{\mathbf{X}_\gamma}(x)) = R_{\min}(\gamma) \Longrightarrow F_{d-1}(C_{\mathbf{X}_\gamma}(x)) = d+1\right) \underset{\gamma\to\infty}{\longrightarrow} 1.$$

Proposition 2.3.3 implies Corollary 2.3.4 but does not provide the exact order of $R'_{\min}(\gamma)$. Nevertheless, when $d=2$, it can be made explicit. The key idea is contained in Lemma 2.3.6 and cannot unfortunately be extended to higher dimensions.

Proposition 2.3.5. *Let $\mathbf{X}_\gamma$ be a Poisson point process of intensity γ and W a convex body of volume 1 in $\mathbf{R}^2$. Then, for all $t \geq 0$,*

$$\mathbb{P}\left(\alpha_2' \pi \gamma^{5/4} R'^2_{\min}(\gamma) \geq t\right) \underset{\gamma\to\infty}{\longrightarrow} e^{-t^4}$$

where α_2' is defined in (2.3.25).

Proof of Proposition 2.3.5. Let $t \geq 0$ be fixed and let us denote by

$$u'_\gamma = u'_\gamma(t) = \left(\alpha_2'^{-1}\pi^{-1}\gamma^{-5/4}t\right)^{1/2} \tag{2.3.23}$$

where α_2' is specified in (2.3.25). As in the proof of (2.1.2d), we interpret the distribution function of $R'_{\min}(\gamma)$ as a covering probability of the circle. Let μ_k be the probability that $S(0,u'_\gamma)$ is covered with the circular caps $\{\mathscr{A}_{y_m}(0), m \leq k\}$ where $y_1,\ldots,y_k$ are k independent points which are uniformly distributed in $B(0,2u'_\gamma)$ and such that $F_1(C_{\{0\}\cup\{\mathbf{y}_k\}}(0)) \geq 4$ i.e.

$$\mathbb{P}\left(R(C_{\mathbf{X}_\gamma\cup\{0\}}(0)) < u'_\gamma, F_1(C_{\mathbf{X}_\gamma\cup\{0\}}(0)) \geq 4\right) = \sum_{k=4}^{\infty}\frac{1}{k!}(4\pi\gamma u'^2_\gamma)^k e^{-4\pi\gamma u'^2_\gamma}\mu_k. \tag{2.3.24}$$

The constant α_2' is defined as

$$\alpha_2' = \left(\frac{32}{3}\mu_4\right)^{1/4} > 0. \tag{2.3.25}$$

We are going to apply Lemma 2.3.1 to the event $A=[4,\infty)$ replacing u_γ by u'_γ. To do it, we need to get the limit behaviour of

$$\gamma^K \int_{W^K} \mathbb{P}\left(\forall i \leq K, R(C_{\mathbf{X}_\gamma\cup\{\mathbf{x}_K\}}(x_i)) < u'_\gamma, F_1(C_{\mathbf{X}_\gamma\cup\{\mathbf{x}_K\}}(x_i)) \geq 4\right) d\mathbf{x}_K \tag{2.3.26}$$

for all $K \geq 1$.

When $K=1$, from (2.3.24) and (2.3.23), we deduce that

$$\gamma\int_W \mathbb{P}\left(R(C_{\mathbf{X}_\gamma\cup\{x\}}(x)) < u'_\gamma, F_{d-1}(C_{\mathbf{X}_\gamma\cup\{x\}}(x)) \geq 4\right) dx \underset{\gamma\to\infty}{\longrightarrow} t^4.$$

More generally, for all $K \geq 1$,

$$\gamma^K \int_{W_K(K,0,\ldots,0)} \mathbb{P}\left(\forall i \leq K, R(C_{\mathbf{X}_\gamma\cup\{\mathbf{x}_K\}}(x_i)) < u'_\gamma, F_1(C_{\mathbf{X}_\gamma\cup\{\mathbf{x}_K\}}(x_i)) \geq 4\right) d\mathbf{x}_K \underset{\gamma\to\infty}{\longrightarrow} t^{4K}. \tag{2.3.27}$$

Otherwise, for all $\mathbf{x}_K \in W_K(n_1,\ldots,n_K)$ with $(n_1,\ldots,n_K) \neq (K,0,\ldots,0)$, the integrand of (2.3.26) is

$$\begin{aligned}&\gamma^K\mathbb{P}\left(\forall i \leq K, R(C_{\mathbf{X}_\gamma\cup\{\mathbf{x}_K\}}(x_i)) < u'_\gamma, F_1(C_{\mathbf{X}_\gamma\cup\{\mathbf{x}_K\}}(x_i)) \geq 4\right)\\ &\quad = \gamma^K\sum_{k=0}^{\infty}\frac{\left(\gamma\lambda_d\left(\bigcup_{i=1}^k B(x_i,2u'_\gamma)\right)\right)^k}{k!}e^{-\gamma\lambda_d\left(\bigcup_{i=1}^k B(x_i,2u'_\gamma)\right)} \times \mu_k(x_1,\ldots,x_K).\end{aligned} \tag{2.3.28}$$

The term $\mu_k(x_1,\ldots,x_K)$ denotes the probability that $S(x_i,u'_\gamma)$ is covered with the spherical caps $\{\mathscr{A}_{x_j}(x_i), i\neq j\leq K\}$ and $\{\mathscr{A}_{y_m}(x_i), m\leq k\}$ where $y_1,\ldots,y_k$ are k independent points which are uniformly distributed in $\bigcup_{i=1}^K B(x_i,2u'_\gamma)$ and such that $F_1(C_{\mathbf{X}_\gamma\cup\{\mathbf{x}_K\}}(x_i))\geq 4$ for all $i\leq K$. This probability satisfies the following property:

Lemma 2.3.6. *Let* $\mathbf{x}_K=(x_1,\ldots,x_K)\in W_K(n_1,\ldots,n_K)\subset\mathbf{R}^2$ *and*

$$N'=4n_1+4n_2+\sum_{l=3}^K 3n_l. \tag{2.3.29}$$

Then, for all $k<N'$

$$\mu_k(x_1,\ldots,x_K)=0. \tag{2.3.30}$$

The proof of Lemma 2.3.6 is postponed to the appendix. From (2.3.28), (2.3.30) and (2.3.29), we deduce for γ large enough that

$$\gamma^K\mathbb{P}\left(\forall i\leq K, R(C_{\mathbf{X}_\gamma\cup\{\mathbf{x}_K\}}(x_i))<u'_\gamma, F_1(C_{\mathbf{X}_\gamma\cup\{\mathbf{x}_K\}}(x_i))\geq 4\right)\leq c\cdot\gamma^K(\gamma u'^2_\gamma)^{N'}=c\cdot\gamma^{n_2}\prod_{l=3}^K\gamma^{\frac{4l-3}{4}n_l}.$$

Moreover, $\lambda_{2K}(W_K(n_1,\ldots,n_K))\leq c\cdot\prod_{l=2}^K(u'^2_\gamma)^{(l-1)n_l}=c\cdot\gamma^{-\frac{5}{4}n_2}\prod_{l=3}^K\gamma^{\frac{-5l+5}{4}n_l}$. This shows that

$$\gamma^K\int_{W^K-W_K(K,0,\ldots,0)}\mathbb{P}\left(\forall i\leq K, R(C_{\mathbf{X}_\gamma\cup\{\mathbf{x}_K\}}(x_i))<u'_\gamma, F_1(C_{\mathbf{X}_\gamma\cup\{\mathbf{x}_K\}}(x_i))\geq 4\right)d\mathbf{x}_K=O\left(\gamma^{-1/4}\right). \tag{2.3.31}$$

From (2.3.27), (2.3.31) and Lemma 2.3.1, we get

$$\mathbb{P}\left(R'_{\min}(\gamma)\geq u'_\gamma\right)\underset{\gamma\to\infty}{\longrightarrow}e^{-t^4}.$$

□

We conclude the section with a quick sketch of proof for (2.1.2a).

Proof of (2.1.2a). We notice that

$$r_{\max}(\gamma)=\max_{x\in\mathbf{X}_\gamma\cap W}r(C_{\mathbf{X}_\gamma}(x))=\frac{1}{2}\max_{x\in\mathbf{X}_\gamma\cap W}\min_{y\neq x\in\mathbf{X}_\gamma}d(x,y).$$

The behaviour of the maximum of nearest neighbor distances was studied by Henze in Theorem 1 of [56] when the input is a binomial process. His result did not include the contribution of boundary effects and is consequently limited to the set of points in $W\ominus B(0,u_\gamma)$. With Lemma 2.3.1 and proceeding along the same lines as in the proof of (2.1.2d), we are able to show the convergence in distribution of the maximal inradius of Voronoi tessellation when the input is a Poisson point process in W. □

2.4 Proof of (2.1.2c), consequence on Poisson-Voronoi approximation

Proof of (2.1.2c). First, we notice that

$$R_{\max}(\gamma) = \max_{x \in \mathbf{X}_\gamma \cap W} R(C_{\mathbf{X}_\gamma}(x)) = \max_{x \in \mathbf{X}_\gamma \cap W} \max_{y \in C_{\mathbf{X}_\gamma}(x)} d(x, y).$$

In order to avoid boundary effects, we start by studying an intermediary radius $R'_{\max}(\gamma)$ defined as

$$R'_{\max}(\gamma) = \max_{x \in \mathbf{X}_\gamma, C_{\mathbf{X}_\gamma}(x) \cap W \neq \varnothing} \max_{y \in C_{\mathbf{X}_\gamma}(x) \cap W} d(x, y).$$

In a first step, we provide the asymptotic behaviour of $R'_{\max}(\gamma)$. Secondly, we study the effects of Voronoi cells astride W and W^c.

Step 1 The distribution function of $R'_{\max}(\gamma)$ can be interpreted as a covering probability. Indeed, if we denote by

$$u_\gamma = u_\gamma(t) = \left(\frac{1}{\kappa_d \gamma} t + \frac{1}{\kappa_d \gamma} \log\left(\alpha_1 \gamma (\log \gamma)^{d-1} \right) \right)^{1/d} \tag{2.4.1}$$

where

$$\alpha_1 := \frac{1}{d!} \left(\frac{\pi^{1/2} \Gamma\left(\frac{d}{2} + 1\right)}{\Gamma\left(\frac{d+1}{2}\right)} \right)^{d-1} \tag{2.4.2}$$

and t is a fixed parameter, we have

$$\begin{aligned} R'_{\max}(\gamma) \leq u_\gamma &\iff \forall x \in \mathbf{X}_\gamma, \text{ s.t. } C_{\mathbf{X}_\gamma}(x) \cap W \neq \varnothing, \forall y \in C_{\mathbf{X}_\gamma}(x) \cap W, d(x, y) \leq u_\gamma \\ &\iff \forall y \in W, \exists x \in \mathbf{X}_\gamma, d(x, y) \leq u_\gamma \\ &\iff \{B(x, u_\gamma), x \in \mathbf{X}_\gamma\} \text{ covers } W. \end{aligned}$$

We have to deal with the probability to cover a region with a large number of balls having a small radius when $\gamma \to \infty$. Asymptotics of such covering probabilities have been studied by Janson. We apply Lemma 7.3 of [67] which is rewritten in our particular framework. Actually, Lemma 7.3 of [67] investigates covering with copies of a general convex body and requires conditions which are clearly satisfied in the case of the ball (see Lemmas 5.2, 5.4 and (9.24) therein).

Lemma 2.4.1. *(Janson) Let W be a bounded subset of $\mathbf{R}^d$ such that $\lambda_d(\partial W) = 0$ and $\mathbf{X}_\gamma$ a Poisson point process of intensity γ. Let R be a random variable such that $\mathbb{E}[R] > 0$ and $\mathbb{E}[R^{d+\epsilon}] < \infty$ for some $\epsilon > 0$. We denote by $\alpha(B(0, R)) = \alpha_1 \mathbb{E}[R^{d-1}]^d \mathbb{E}[R^d]^{-(d-1)}$. If $a = a(\gamma)$ is a function such that $a(\gamma) \underset{\gamma\to\infty}{\longrightarrow} 0$ and*

$$\mathbb{E}\left[\lambda_d(aB(0,R))\right]\gamma - \log \frac{\lambda_d(W)}{\mathbb{E}\left[\lambda_d(aB(0,R))\right]} - d \log\log \frac{\lambda_d(W)}{\mathbb{E}\left[\lambda_d(aB(0,R))\right]} - \log \alpha(B(0,R)) \underset{\gamma\to\infty}{\longrightarrow} t \tag{2.4.3}$$

where $t \in \mathbf{R}$, then

$$\mathbb{P}\left(\{B(x, R), x \in \mathbf{X}_\gamma\} \text{ covers } W\right) \underset{\gamma\to\infty}{\longrightarrow} e^{-e^{-t}}.$$

Taking $a = u_\gamma$, $R = 1$, $\lambda_d(W) = 1$ and noting that $\mathbb{E}\left[\lambda_d(aB(0,R))\right] = \kappa_d u_\gamma^d$ and $\alpha(B(0,R)) = \alpha_1$, we check easily (2.4.3). From Lemma 2.4.1, we deduce that $\mathbb{P}\left(\{B(x,u_\gamma), x \in \mathbf{X}_\gamma\} \text{ covers } W\right)$ converges to $e^{-e^{-t}}$. Hence, for all $t \in \mathbf{R}$,

$$\lim_{\gamma\to\infty} \mathbb{P}\left(R'_{\max}(\gamma) \leq u_\gamma\right) = e^{-e^{-t}}. \tag{2.4.4}$$

Step 2 Taking $f(C_{\mathbf{X}_\gamma}(x)) = \kappa_d(\max_{y\in C_{\mathbf{X}_\gamma}(x)\cap W} d(x,y))^d$, $a_\gamma = \gamma$, $b_\gamma = \log\left(\alpha_1\gamma(\log\gamma)^{d-1}\right)$ and Y a Gumbel distribution (i.e. $\mathbb{P}(Y \leq t) = e^{-e^{-t}}, t \in \mathbf{R}$), one can check condition (2.2.3) with $k = d$. From (2.4.4) and Proposition 2.2.1, we deduce that

$$\mathbb{P}\left(\max_{x\in\mathbf{X}_\gamma\cap W} \max_{y\in C_{\mathbf{X}_\gamma}(x)\cap W} d(x,y) \leq u_\gamma\right) \underset{\gamma\to\infty}{\longrightarrow} e^{-e^{-t}}$$

for all $t \in \mathbf{R}$. Using the fact that, on the event A_γ (given in Lemma 2.2.2),

$$\max_{x\in\mathbf{X}_\gamma\cap W} \max_{y\in C_{\mathbf{X}_\gamma}(x)\cap W} d(x,y) \leq \max_{x\in\mathbf{X}_\gamma\cap W} \max_{y\in C_{\mathbf{X}_\gamma}(x)} d(x,y) \leq \max_{x\in\mathbf{X}_\gamma\cap W_{1+l_\gamma}} \max_{y\in C_{\mathbf{X}_\gamma}(x)\cap W_{1+l_\gamma}} d(x,y)$$

and proceeding along the same lines as in the proof of Proposition 2.2.1, we get

$$\mathbb{P}\left(R_{\max}(\gamma) \leq u_\gamma\right) \underset{\gamma\to\infty}{\longrightarrow} e^{-e^{-t}}. \tag{2.4.5}$$

□

We can note that the asymptotic behaviour of $R_{\max}(\gamma)$ gives an interpretation of Lemma 7.3 in [67]. Indeed, (2.4.5) shows that the Gumbel distribution which appears as a limit probability of a covering is actually the limit distribution of a maximum.

We now apply this convergence result to the so-called Poisson-Voronoi approximation defined as

$$\mathscr{V}_{\mathbf{X}_\gamma}(W) = \bigcup_{x\in\mathbf{X}_\gamma\cap W} C_{\mathbf{X}_\gamma}(x).$$

It consists in discretizing a given convex window W with a finite union of convex polyhedra. This approximation has various applications such as image analysis (reconstructing an image from its intersection with a Poisson point process, see [75]) or quantization (see chapter 9 of [50]). Estimates of the first two moments of the symmetric difference between the convex body and its approximation are given in [57] and extended to higher moments in [122]. To the best of our knowledge, the convergence of $\mathscr{V}_{\mathbf{X}_\gamma}(W)$ to W in the sense of Hausdorff distance, denoted by $d_H(\cdot,\cdot)$, has not been investigated. Corollary 2.4.2 addresses that question with an assumption on the regularity of W which is in the spirit of the n-regularity (see Definition 3 in [20]).

Corollary 2.4.2. Let us assume that there exists $\alpha > 0$ such that, for v small enough and for all $y \in W$,

$$\lambda_d(B(y,v)\cap W) \geq \alpha\lambda_d(B(y,v)). \tag{2.4.6}$$

Then

$$\mathbb{P}\left(d_H(W, \mathscr{V}_{\mathbf{X}_\gamma}(W)) \leq \left(c(\alpha)\gamma^{-1}\log\left(\alpha_1\gamma(\log\gamma)^{d-1}\right)\right)^{1/d}\right) \underset{\gamma\to\infty}{\longrightarrow} 1 \tag{2.4.7}$$

where $c(\alpha) = \kappa_d^{-1} + 2^d\kappa_d^{-1}\alpha^{-1}$.

Proof of Corollary 2.4.2. Let us denote by

$$v_\gamma = \left(c(\alpha)\gamma^{-1}\log\left(\alpha_1\gamma(\log\gamma)^{d-1}\right)\right)^{1/d}. \tag{2.4.8}$$

First, we show that $\max_{y\in\mathscr{V}_{\mathbf{X}_\gamma}(W)} d(y,W)\leq v_\gamma$ with high probability. For all $t\in\mathbf{R}$, using the fact that $u_\gamma\leq v_\gamma$ for γ large enough, where $u_\gamma=u_\gamma(t)$ is given in (2.4.1), we get

$$\mathbb{P}\left(\max_{y\in\mathscr{V}_{\mathbf{X}_\gamma}(W)} d(y,W)\leq v_\gamma\right)\geq \mathbb{P}\left(R_{\max}(\gamma)\leq v_\gamma\right)\geq \mathbb{P}\left(R_{\max}(\gamma)\leq u_\gamma\right).$$

From (2.4.5) and Proposition 2.2.1, the last term converges to $e^{-e^{-t}}$ as γ goes to infinity. Taking $t\to\infty$, we get

$$\lim_{\gamma\to\infty}\mathbb{P}\left(\max_{y\in\mathscr{V}_{\mathbf{X}_\gamma}(W)} d(y,W)\leq v_\gamma\right)\geq \lim_{t\to\infty} e^{-e^{-t}}=1. \tag{2.4.9}$$

In a second step, we are going to show that $\max_{y\in W} d(y,\mathbf{X}_\gamma\cap W)\leq v_\gamma$ with high probability via the use of a covering of W by balls as in the proof of (2.1.2c). Now, the convex body W is covered by $\mathscr{N}=O\left(v_\gamma^{-d}\right)$ deterministic balls $B_1,\dots,B_{\mathscr{N}}$ with center in W and radius equal to $v_\gamma/2$. From (2.4.6), (2.4.8) and the fact that $\#(B_i\cap(\mathbf{X}_\gamma\cap W))$ is Poisson distributed with mean $\gamma\lambda_d(B_i\cap W)$, we get for γ large enough

$$\begin{aligned}\mathbb{P}\left(\max_{y\in W} d(y,\mathbf{X}_\gamma\cap W)>v_\gamma\right)&\leq \mathbb{P}\left(\bigcup_{i=1}^{\mathscr{N}}\{\#(B_i\cap(\mathbf{X}_\gamma\cap W))=0\}\right)\leq \mathscr{N}e^{-\gamma\alpha\kappa_d(v_\gamma/2)^d}\\&\leq \alpha_1^{-1}\gamma^{-1}(\log\gamma)^{-(d-1)}\mathscr{N}.\end{aligned}$$

Using the fact that $\mathscr{N}=O(v_\gamma^{-d})$ i.e. $\mathscr{N}=O(\gamma(\log\gamma)^{-1})$ according to (2.4.8), the right-hand side is $O\left((\log\gamma)^{-d}\right)$. Hence

$$\mathbb{P}\left(\max_{y\in W} d(y,\mathbf{X}_\gamma\cap W)\leq v_\gamma\right)\underset{\gamma\to\infty}{\longrightarrow}1. \tag{2.4.10}$$

Since $d_H(W,\mathscr{V}_{\mathbf{X}_\gamma}(W))\leq\max\left\{\max_{y\in\mathscr{V}_{\mathbf{X}_\gamma}(W)} d(y,W),\max_{y\in W} d(y,\mathbf{X}_\gamma\cap W)\right\}$, we deduce from (2.4.9) and (2.4.10) that

$$\mathbb{P}\left(d_H(W,\mathscr{V}_{\mathbf{X}_\gamma}(W))\leq v_\gamma\right)\underset{\gamma\to\infty}{\longrightarrow}1.$$

□

In [57], Heveling and Reitzner obtain that the volume of the symmetric difference between W and $\mathscr{V}_{\mathbf{X}_\gamma}(W)$ is of the order of $\gamma^{-1/d}$. The result above makes sense and could provide the right order of the Hausdorff distance. Obviously, the constant $c(\alpha)=\kappa_d^{-1}+2^d\kappa_d^{-1}\alpha^{-1}$ is not optimal. From Lemma 2.2.2, it would have been possible to get an upper-bound of the order of $\gamma^{-(1-\epsilon)/d}$ but it is less precise than Corollary 2.4.2.

2.5 Proof of (2.1.2b)

Proof of (2.1.2b)**.** Let $t \geq 0$ be fixed. We denote by u_γ the following function:

$$u_\gamma = u_\gamma(t) = \left(2^{-(d-1)}\kappa_d^{-1}\gamma^{-2}t\right)^{1/d}. \tag{2.5.1}$$

We start by finding a different expression of $r_{\min}(\gamma)$ which does not rely on the Voronoi structure. Indeed, for all $x \in \mathbf{X}_\gamma \cap W$ we have

$$r(C_{\mathbf{X}_\gamma}(x)) = \max\{r \geq 0, B(x,r) \subset C_{\mathbf{X}_\gamma}(x)\} = \frac{1}{2}\min_{y \neq x \in \mathbf{X}_\gamma} d(x,y).$$

Hence, $r_{\min}(\gamma)$ can be rewritten as

$$r_{\min}(\gamma) = \frac{1}{2}\min_{(x,y)_{\neq} \in (\mathbf{X}_\gamma \cap W) \times \mathbf{X}_\gamma} d(x,y). \tag{2.5.2}$$

The equality (2.5.2) implies that the problem is reduced to a study of inter-point distance. Such study is well known for a binomial process $X^{(n)}$ of intensity n in W. In particular, Jammalamadaka and Janson (see [64], §4) have shown that for all $t \geq 0$,

$$\mathbb{P}\left(r'_{\min,n} \geq u_n\right) \underset{n\to\infty}{\longrightarrow} e^{-t} \tag{2.5.3}$$

where $r'_{\min,n}$ is defined as

$$r'_{\min,n} = \frac{1}{2}\min_{(x,y)_{\neq} \in X^{(n)} \times X^{(n)}} d(x,y)$$

and u_n given in (2.5.1). In a first elementary step, we extend the limit to a Poisson point process. Our main contribution is then to compare the obtained limit with $r_{\min}(\gamma)$ by dealing with boundary effects. In particular, our study provides a far more accurate estimate of the contribution of boundary cells (see (2.5.15)) than what we could have deduced from Proposition 2.2.1.

Step 1 We extend (2.5.3) to a Poisson point process. We define

$$r'_{\min}(\gamma) = \frac{1}{2}\min_{(x,y)_{\neq} \in (\mathbf{X}_\gamma \cap W)^2} d(x,y). \tag{2.5.4}$$

Let us note that for all $0 < \alpha < \beta < 1$, and for all $n \in \{0,1,2,\ldots\}$, $|n-\gamma| \leq \gamma^\alpha \Longrightarrow |n-\gamma| \leq n^\beta$ for γ large enough. Consequently, since u_γ is non-increasing in γ, we have for γ large enough

$$\begin{aligned}\left|\mathbb{P}\left(r'_{\min}(\gamma) \geq u_\gamma\right) - e^{-t}\right| &\leq \sum_{n=0}^{\infty}\left|\mathbb{P}(r'_{\min,n} \geq u_\gamma) - e^{-t}\right|\mathbb{P}(\#(\mathbf{X}_\gamma \cap W) = n)\\ \leq \sum_{|n-\gamma|\leq\gamma^\alpha} &\max\left\{\left|\mathbb{P}(r'_{\min,n} \geq u_{n-n^\beta}) - e^{-t}\right|, \left|\mathbb{P}(r'_{\min,n} \geq u_{n+n^\beta}) - e^{-t}\right|\right\}\mathbb{P}(\#(\mathbf{X}_\gamma \cap W) = n)\\ &+ \mathbb{P}(|\#(\mathbf{X}_\gamma \cap W) - \gamma| > \gamma^\alpha).\end{aligned} \tag{2.5.5}$$

The second term of (2.5.5) converges to 0 thanks to a concentration inequality for Poisson variables (see e.g. Lemma 1.4 in [113]). The first term is lower than

$$\max_{n\geq\gamma-\gamma^\alpha} \max\left\{\left|\mathbb{P}(r'_{\min,n}\geq u_{n-n^\beta}(t))-e^{-t}\right|,\left|\mathbb{P}(r'_{\min,n}\geq u_{n+n^\beta}(t))-e^{-t}\right|\right\}$$

which tends to 0 according to (2.5.3). This shows that, for all $t\geq 0$,

$$\lim_{\gamma\to\infty}\mathbb{P}\left(r'_{\min}(\gamma)\geq u_\gamma\right)=e^{-t}. \tag{2.5.6}$$

Step 2 We show that $r_{\min}(\gamma)=r'_{\min}(\gamma)$ with probability of order of $O(\gamma^{-\epsilon})$ with $\epsilon\in(0,\frac{2}{d})$. Indeed, the random variables $r_{\min}(\gamma)$ and $r'_{\min}(\gamma)$, defined in (2.5.2) and (2.5.4), are equal if and only if no point of $\mathbf{X}_\gamma\cap W^c$ falls into the union of the balls $B(x,2r'_{\min}(\gamma))$ for $x\in\mathbf{X}_\gamma\cap W$ such that $d(x,\partial W)<2r'_{\min}(\gamma)$ i.e.

$$\begin{aligned}\mathbb{P}(r_{\min}(\gamma)\neq r'_{\min}(\gamma)) &= \mathbb{P}\left(\#\left(\mathbf{X}_\gamma\cap W^c\cap\bigcup_{\substack{x\in\mathbf{X}_\gamma\cap W,\\ d(x,\partial W)<2r'_{\min}(\gamma)}}B(x,2r'_{\min}(\gamma))\right)\neq 0\right)\\ &\leq \mathbb{E}\left[\sum_{\substack{x\in\mathbf{X}_\gamma\cap W,\\ d(x,\partial W)<2r'_{\min}(\gamma)}}\#\left(\mathbf{X}_\gamma\cap W^c\cap B(x,2r'_{\min}(\gamma))\right)\right].\end{aligned} \tag{2.5.7}$$

From Slivnyak-Mecke formula (see e.g. Corollary 3.2.3 of [130]), we get

$$\begin{aligned}&\mathbb{E}\left[\sum_{\substack{x\in\mathbf{X}_\gamma\cap W,\\ d(x,\partial W)<2r'_{\min}(\gamma)}}\#\left(\mathbf{X}_\gamma\cap W^c\cap B(x,2r'_{\min}(\gamma))\right)\right]\\ &\qquad=\int_W\gamma\mathbb{E}\left[\#\left(\mathbf{X}_\gamma\cap W^c\cap B(x,2r'^{(x)}_{\min}(\gamma))\right)\mathbb{1}_{d(x,\partial W)<2r'^{(x)}_{\min}(\gamma)}\right]dx\end{aligned}$$

where $r'^{(x)}_{\min}(\gamma)=\frac{1}{2}\min_{(x',y)_{\neq}\in(\mathbf{X}_\gamma\cup\{x\}\cap W)^2}d(x',y)$ for all $x\in\mathbf{X}_\gamma\cap W$. Noting that $r'^{(x)}_{\min}(\gamma)\leq r'_{\min}(\gamma)$, we then obtain

$$\begin{aligned}&\mathbb{E}\left[\sum_{\substack{x\in\mathbf{X}_\gamma\cap W,\\ d(x,\partial W)<2r'_{\min}(\gamma)}}\#\left(\mathbf{X}_\gamma\cap W^c\cap B(x,2r'_{\min}(\gamma))\right)\right]\\ &\leq\int_W\gamma\mathbb{E}\left[\#\left(\mathbf{X}_\gamma\cap W^c\cap B(x,2r'_{\min}(\gamma))\right)\mathbb{1}_{d(x,\partial W)<2r'_{\min}(\gamma)}\right]dx\\ &\quad=\int_W\gamma\mathbb{E}\left[\mathbb{E}\left[\#\left(\mathbf{X}_\gamma\cap W^c\cap B(x,2r'_{\min}(\gamma))\right)|\mathbf{X}_\gamma\cap W\right]\mathbb{1}_{d(x,\partial W)<2r'_{\min}(\gamma)}\right]dx.\end{aligned} \tag{2.5.8}$$

Since $\#\left(\mathbf{X}_\gamma \cap W^c \cap B(x, 2r'_{\min}(\gamma))\right)$ is Poisson distributed, we get

$$\begin{aligned}&\gamma\mathbb{E}\left[\mathbb{E}\left[\#\left(\mathbf{X}_\gamma \cap W^c \cap B(x, 2r'_{\min}(\gamma))\right) | \mathbf{X}_\gamma \cap W\right] \mathbb{1}_{d(x,\partial W)<2r'_{\min}(\gamma)}\right] \\ &= \gamma^2\mathbb{E}\left[\lambda_d(W^c \cap B(x, 2r'_{\min}(\gamma)))\mathbb{1}_{d(x,\partial W)<2r'_{\min}(\gamma)}\right] \\ &\qquad\qquad \leq 2^d\kappa_d \cdot \gamma^2\mathbb{E}\left[r'^d_{\min}(\gamma)\mathbb{1}_{d(x,\partial W)<2r'_{\min}(\gamma)}\right]. \end{aligned} \tag{2.5.9}$$

Using (2.5.7), (2.5.8), (2.5.9) and Fubini's theorem, we obtain

$$\mathbb{P}(r_{\min}(\gamma) \neq r'_{\min}(\gamma)) \leq 2^d\kappa_d \cdot \gamma^2\mathbb{E}\left[r'^d_{\min}(\gamma) \int_W \mathbb{1}_{d(x,\partial W)<2r'_{\min}(\gamma)}dx\right] \leq c \cdot \gamma^2\mathbb{E}\left[r'^{d+1}_{\min}(\gamma)\right]. \tag{2.5.10}$$

The last inequality comes from Steiner formula (see (14.5) in [130]) and c denotes a constant depending on W. Hence, to show that

$$\mathbb{P}\left(r_{\min}(\gamma) \neq r'_{\min}(\gamma)\right) \underset{\gamma\to\infty}{\longrightarrow} 0 \tag{2.5.11}$$

we have to find some upper-bound of $\gamma^2\mathbb{E}[r'^{d+1}_{\min}(\gamma)]$. We know, from (2.5.6) and (2.5.1) , that $\gamma^2 r'^d_{\min}(\gamma)$ tends to 0 in distribution but it does not imply (2.5.11). Lemma 2.5.1 below provides an estimate of the deviation probabilities of $\gamma^2 r'^d_{\min}(\gamma)$ when the window W is a cube.

Lemma 2.5.1. *Let C be a cube of side M and $\mathbf{X}_\gamma$ a Poisson point process of intensity γ. Let us denote by*

$$r'_{\min|C}(\gamma) = \frac{1}{2}\min_{(x,y)_{\neq}\in(\mathbf{X}_\gamma\cap C)^2} d(x,y).$$

Then, for all $u \leq \min\{\frac{1}{4}Md^{1/2}, \frac{1}{2}d^{1/2}\gamma^{-1/d}\}$, there exists a constant $c(M)$ such that

$$\mathbb{P}(r'_{\min|C}(\gamma) \geq u) \leq e^{-c(M)\gamma^2u^d}.$$

Proof of Lemma 2.5.1. Let $u \leq \min\{\frac{1}{4}Md^{1/2}, \frac{1}{2}d^{1/2}\gamma^{-1/d}\}$ be fixed.

We subdivide the cube $C = [0, M]^d$ into a set of $\mathcal{N}$ subcubes $C_1, \ldots, C_{\mathcal{N}}$ of equal size c with $c = 2d^{-1/2}u$ and $\mathcal{N} = \lfloor Mc^{-1}\rfloor^d$. Since $\mathrm{diam}(C_i) = 2u$ for each $i \leq \mathcal{N}$, we obtain

$$\mathbb{P}(r'_{\min|C}(\gamma) \geq u) \leq \mathbb{P}\left(\bigcap_{i=1}^{\mathcal{N}}\{\#(C_i \cap \mathbf{X}_\gamma) \leq 1\}\right) = \left(e^{-\gamma c^d}(1+\gamma c^d)\right)^{\mathcal{N}}.$$

Replacing c^d by $2^d d^{-d/2}u^d$ and $\mathcal{N}$ by $\lfloor 2^{-1}Md^{1/2}u^{-1}\rfloor^d$ we obtain the following inequality:

$$\mathbb{P}(r'_{\min|C}(\gamma) \geq u) \leq e^{\lfloor 2^{-1}Md^{1/2}u^{-1}\rfloor^d\left(\log(1+\gamma 2^d d^{-d/2}u^d) - \gamma 2^d d^{-d/2}u^d\right)}.$$

Since $\gamma 2^d d^{-d/2}u^d \leq 1$ and $2M^{-1}d^{-1/2}u \leq \frac{1}{2}$, we have $\log(1+\gamma 2^d d^{-d/2}u^d) - \gamma 2^d d^{-d/2}u^d \leq -\frac{1}{4}2^{2d}d^{-d}\gamma^2u^{2d}$ and $\lfloor 2^{-1}Md^{1/2}u^{-1}\rfloor^d \geq (2^{-1}Md^{1/2}u^{-1}-1)^d \geq 2^{-2d}M^d d^{d/2}u^{-d}$. Hence

$$\mathbb{P}(r'_{\min|C}(\gamma) \geq u) \leq e^{-\frac{1}{4}d^{-d/2}M^d\gamma^2u^d} = e^{-c(M)\gamma^2u^d}$$

where $c(M) = \frac{1}{4}d^{-d/2}M^d$. □

Now, we can derive an upper-bound of $\gamma^2\mathbb{E}[r'^{d+1}_{\min}(\gamma)]$. Indeed, since W has non-empty interior, there exists a cube C of side M included in W. Using the fact that $\#(\mathbf{X}_\gamma \cap C) \geq 2 \Longrightarrow r'_{\min}(\gamma) \leq r'_{\min|C}(\gamma)$, we get

$$\gamma^2\mathbb{E}[r'^{d+1}_{\min}(\gamma)] = \gamma^2 \int_0^{\mathrm{diam}(W)} \mathbb{P}(r'^{d+1}_{\min}(\gamma) \geq s)ds$$
$$\leq \mathrm{diam}(W)\gamma^2\mathbb{P}(\#(\mathbf{X}_\gamma \cap C) \leq 1) + \gamma^2 \int_0^{Md^{1/2}} \mathbb{P}(r'^{d+1}_{\min|C}(\gamma) \geq s)ds. \quad (2.5.12)$$

The first term of the right-hand side of (2.5.12) is decreasing exponentially fast to 0 since $\#(\mathbf{X}_\gamma \cap C)$ is Poisson distributed of mean γM^d. For the second term, let us consider a fixed ϵ in $(0, \frac{2}{d})$. Then

$$\gamma^2 \int_0^{Md^{1/2}} \mathbb{P}(r'^{d+1}_{\min|C}(\gamma) \geq s)ds$$
$$= \int_0^{\gamma^{-(2+\epsilon)}} \gamma^2\mathbb{P}\left(r'_{\min|C}(\gamma) \geq s^{1/(d+1)}\right)ds + \int_{\gamma^{-(2+\epsilon)}}^{Md^{1/2}} \gamma^2\mathbb{P}\left(r'_{\min|C}(\gamma) \geq s^{1/(d+1)}\right)ds$$
$$\leq \gamma^{-\epsilon} + Md^{1/2}\gamma^2\mathbb{P}\left(r'_{\min|C}(\gamma) \geq \gamma^{-(2+\epsilon)/(d+1)}\right). \quad (2.5.13)$$

Since $\epsilon > 0$, we have $\gamma^{-(2+\epsilon)/(d+1)} \leq \min\{\frac{1}{4}Md^{1/2}, \frac{1}{2}d^{1/2}\gamma^{-1/d}\}$ for γ large enough. Hence, from Lemma 2.5.1 applied to $u := \gamma^{-(2+\epsilon)/(d+1)}$, we deduce that for γ large enough,

$$\gamma^2 \int_0^{Md^{1/2}} \mathbb{P}(r'^{d+1}_{\min|C}(\gamma) \geq s)ds \leq \gamma^{-\epsilon} + Md^{1/2}\gamma^2 e^{-c(M)\gamma^{(2-\epsilon d)/(d+1)}}. \quad (2.5.14)$$

The last term of the right-hand side of (2.5.14) converges exponentially fast to 0 as γ goes to infinity since $\epsilon < \frac{2}{d}$. Combining that argument with (2.5.10), (2.5.12) and (2.5.14), we deduce that

$$\mathbb{P}\left(r_{\min}(\gamma) \neq r'_{\min}(\gamma)\right) = O\left(\gamma^{-\epsilon}\right). \quad (2.5.15)$$

We then deduce from (2.5.6) and (2.5.15) that

$$\left|\mathbb{P}\left(r'_{\min}(\gamma) \geq u_\gamma\right) - e^{-t}\right| \leq \left|\mathbb{P}\left(r'_{\min}(\gamma) \geq u_\gamma\right) - e^{-t}\right| + 2\mathbb{P}\left(r_{\min}(\gamma) \neq r'_{\min}(\gamma)\right) \underset{\gamma\to\infty}{\longrightarrow} 0.$$

□

Remark 2.5.2. The rate for the convergence in distribution of $r_{\min}(\gamma)$ to the Weibull distribution can be estimated. For instance, we can show that Theorem 2.1 in [132] implies the rate of convergence of $r'_{\min}(\gamma)$. Another way to get it is to use Theorem 1 in [4]. We then obtain that there exist positive constants $c(W)$ and $\Gamma(W)$ such that, for all $\epsilon < \frac{2}{d}$, $t \geq 0$ and $\gamma \geq \Gamma(W)$,

$$\left|\mathbb{P}\left(2^{d-1}\kappa_d\gamma^2 r_{\min}(\gamma)^d \geq t\right) - e^{-t}\right| \leq c(W)\gamma^{-\min\{\frac{1}{2},\epsilon\}}.$$

The study of more extremes for general tessellations and their rates of convergence will be developed in a future paper.

Appendix

Proof of Lemma 2.3.2. Actually, we show the following deterministic result: let $K \geq 2$, $k < N$, $(x_1, \ldots, x_K) \in W_K(n_1, \ldots, n_K)$ with $(n_1, \ldots, n_K) \neq (K, 0, \ldots, 0)$ and let

$$(y_1, \ldots, y_k) \in \bigcup_{i=1}^{K} B(x_i, 2u_\gamma)$$

such that $\{\mathbf{x}_K\} \cup \{\mathbf{y}_k\}$ are in general position i.e. each subset of size $n < d+1$ is affinely independent (see [147]). Then there exists $i \leq K$ such that sphere $S(x_i, u_\gamma)$ is not covered by the induced spherical caps $\{\mathscr{A}_{x_j}(x_i), i \neq j \leq K\} \cup \{\mathscr{A}_{y_m}(x_i), m \leq k\}$.

Indeed, from (2.3.15) there exists a connected component of $\bigcup_{i=1}^{K} B(x_i, 2u_\gamma)$ of size $1 \leq l \leq K$, say $\bigcup_{i=1}^{l} B(x_i, 2u_\gamma)$ without loss of generality, such that $N_l < d+1$ with

$$N_l = \#\left(\{\mathbf{y}_k\} \cap \bigcup_{i=1}^{l} B(x_i, 2u_\gamma)\right) \tag{2.5.16}$$

Since $\{\mathbf{x}_l\} \cup \{\mathbf{y}_{N_l}\}$ are in general position, the family $\{\mathbf{x}_l\}$ is not included in the convex hull of $\{\mathbf{y}_{N_l}\}$. In particular, there exists $i \leq l$ such that x_i is not in the convex hull of $\{\mathbf{x}_l\} \cup \{\mathbf{y}_{N_l}\} - \{x_i\}$. Since a Voronoi cell induced by a finite number of points is not bounded if and only if its nucleus is an extremal point of the polytope induced by the points, it implies that the circumscribed radius of $C_{\{\mathbf{x_l}\} \cup \{\mathbf{y}_{N_l}\}}(x_i)$ is not finite i.e. $S(x_i, u_\gamma)$ is not covered. □

Proof of Lemma 2.3.6. We show the following deterministic result: let $K \geq 2$, $k < N'$, $(x_1, \ldots, x_K) \in W_K(n_1, \ldots, n_K)$ with $(n_1, \ldots, n_K) \neq (K, 0, \ldots, 0)$ and let

$$(y_1, \ldots, y_k) \in \bigcup_{i=1}^{K} B(x_i, 2u_\gamma)$$

such that $\{\mathbf{x}_K\} \cup \{\mathbf{y}_k\}$ are in general position. Then there exists $i \leq K$ such that either the sphere $S(x_i, u_\gamma)$ is not covered by the induced spherical caps $\{\mathscr{A}_{x_j}(x_i), i \neq j \leq K\} \cup \{\mathscr{A}_{y_m}(x_i), m \leq k\}$ or $F_1(C_{\{\mathbf{x}_K\} \cup \{\mathbf{y}_k\}}(x_i)) \leq 3$.

Indeed, from (2.3.29) there exists a connected component of $\bigcup_{i=1}^{K} B(x_i, 2u_\gamma)$ of size $1 \leq l \leq K$, say $\bigcup_{i=1}^{l} B(x_i, 2u_\gamma)$ without loss of generality, such that $N_l < 4$ if $l = 1, 2$ and $N_l < 3$ if $l \geq 3$ where N_l is given in (2.5.16).

- If $l = 1$, either $S(x_1, u_\gamma)$ is covered or $F_1(C_{\{\mathbf{x}_K\} \cup \{\mathbf{y}_k\}}(x_1)) = F_1(C_{\{x_1\} \cup \{\mathbf{y}_{N_1}\}}(x_1)) \leq 3$ since $N_1 \leq 3$.
- If $l \geq 3$, from Lemma 2.3.2 there exists $i \leq l$ such that $S(x_i, u_\gamma)$ is not covered.
- If $l = 2$, we can assume that $N_2 = 3$. We have to prove that if $\mathbf{y}_3 = \{y_1, y_2, y_3\}$ is a set of three points in $B(x_1, 2u_\gamma) \cup B(x_2, 2u_\gamma)$, then the following properties 1 and 2 below cannot hold simultaneously.
  1. The circles $S(x_1, u_\gamma)$ and $S(x_2, u_\gamma)$ are covered by the induced circular caps

     $$\{\mathscr{A}_{x_1}(x_2), \mathscr{A}_{x_2}(x_1)\} \cup \{\mathscr{A}_{y_m}(x_i), m \leq 3\}.$$

  2. The number of edges of the Voronoi cells satisfy the following inequalities

     $$F_1(C_{\{x_1, x_2, \mathbf{y}_3\}}(x_1)) \geq 4 \text{ and } F_1(C_{\{x_1, x_2, \mathbf{y}_3\}}(x_2)) \geq 4.$$

Let us assume that Properties 1 and 2 hold simultaneously. Let us denote by G the Delaunay graph associated to $\{x_1, x_2, y_1, y_2, y_3\}$. Then G is a connected planar graph with $v = 5$ vertices and e edges. From Euler's formula on planar graphs, $e \leq 3v - 6$ i.e.

$$e \leq 9. \tag{2.5.17}$$

From Property 1 and according to the proof of Lemma 2.3.2, x_1, x_2 are in the convex hull of $\{y_1, y_2, y_3\}$ i.e. $\{y_1, y_2\}$, $\{y_1, y_3\}$ and $\{y_2, y_3\}$ are edges of the associated Delaunay triangulation. From Property 2, x_1, x_2 are connected to every point i.e. $\{x_1, x_2\}$, $\{x_1, y_1\}$, $\{x_1, y_2\}$, $\{x_1, y_3\}$, $\{x_2, y_1\}$, $\{x_2, y_2\}$ and $\{x_2, y_3\}$ are also edges of the Delaunay triangulation. The total number of these edges is $e = 10$. This contradicts (2.5.17).

□

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Chapitre 3

Une étude générale des statistiques d'ordre pour des mosaïques aléatoires stationnaires

Sommaire

Dans le chapitre précédent, on s'est intéressé aux maxima et minima d'une certaine mosaïque aléatoire, observée dans un corps convexe, et pour un certain type de caractéristiques géométriques. Dans ce chapitre, on donne une méthode plus générale.

On considère une mosaïque aléatoire $\mathfrak{m}$ stationnaire. Pour tout $C \in \mathfrak{m}$, on rappelle que $z(C)$ désigne le germe de la cellule où $z(\cdot) : \mathcal{K}_d \to \mathbf{R}^d$ est une fonction telle que $z(C+x) = z(C)+x$. On observe la mosaïque aléatoire dans un Borélien W qu'on suppose seulement borné et de volume non nul et on considère une caractéristique géométrique $f : \mathcal{K}_d \to \mathbf{R}$ quelconque. On étend l'étude aux statistiques d'ordre c'est-à-dire aux variables aléatoires $M^{(r)}_{f,\mathbf{W}_\rho}$ où $M^{(r)}_{f,\mathbf{W}_\rho}$ désigne le r-ième maximum de $f(\cdot)$ pris sur toutes les cellules $C \in \mathfrak{m}$ dont le germe $z(C)$ appartient à $\mathbf{W}_\rho$ et où $\mathbf{W}_\rho = \rho^{1/d} W$ est la dilatation de W.

Dans la théorie classique des valeurs extrêmes, nous avons vu que les statistiques d'ordre d'une suite de variables aléatoires satisfaisant des conditions, dites $D(u_n)$ et $D'(u_n)$, sont déterminées par le nombre moyen d'excédents (voir (1.2.2)) et par conséquent par la queue de distribution des variables aléatoires. Par analogie, il est naturel de se demander si l'étude des statistiques d'ordre $M_{f,\mathbf{W}_\rho}^{(r)}$ d'une mosaïque aléatoire peut être ramenée à l'étude de la queue "pour une seule cellule" : la cellule typique. Pour répondre à cette question, nous avons besoin de supposer deux conditions :

- CONDITION 1 : elle porte sur une propriété de mélange de la mosaïque et est l'analogue de la condition $D(u_n)$.
- CONDITION 2 : elle garantit que deux cellules proches ne sont pas toutes les deux des excédents. Il s'agit de l'analogue de la condition $D'(u_n)$.

Ainsi que nous l'avons vu dans la section 1.2, des conditions de ce type sont nécessaires car elles garantissent : *primo* que les cellules se comportent "comme si elles étaient indépendantes" et *secundo* que les excédents ne se font pas par cluster.

Résultats nouveaux

1. Sous les deux conditions mentionnées ci-dessus, on montre un théorème général (Théorème 3.1.1) garantissant que la seule connaissance de la queue de $f(\mathscr{C})$, où $\mathscr{C}$ désigne la cellule typique, suffit à déterminer le comportement des statistiques d'ordre et on donne les vitesses de convergence. Un tel résultat permet, en particulier, de ramener l'étude des extrêmes de plusieurs cellules qui ne sont ni *typiques* ni *indépendantes* à celle de la cellule typique. Cela permet d'avoir une multitude d'applications puisque plusieurs lois de diverses caractéristiques géométriques de la cellule typique sont connues (voir section 1.1).
2. Toujours sous les CONDITIONS 1 et 2, on obtient une convergence du type (1.2.3) sur le processus ponctuel des excédents. Ceci permet, en particulier, de voir comment se répartissent les excédents de la mosaïque et d'avoir une approximation des lois jointes des statistiques d'ordre (Théorème 3.1.2).
3. On applique le théorème général à plusieurs mosaïques aléatoires et à plusieurs caractéristiques géométriques pour déterminer le comportement asymptotique de plusieurs extrêmes.
   (a) Pour une mosaïque de Poisson-Delaunay, on détermine le minimum des rayons circonscrits, en dimension quelconque, ainsi que le maximum et le minimum des aires des cellules dans le cas d'une mosaïque planaire (Propositions 3.3.3, 3.3.4 et 3.3.7).
   (b) Pour une mosaïque de Poisson-Voronoï, on détermine le minimum des volumes des fleurs ainsi que le minimum des distances du germe d'une cellule au germe voisin le plus éloigné (Propositions 3.4.1).
   (c) Pour une mosaïque de Voronoï engendrée par un processus ponctuel de Gauss-Poisson, on étudie le maximum des rayons inscrits (Proposition 3.5.1). En particulier, on étend le premier résultat du Théorème 2.1.1.
4. Enfin, lorsqu'on on ne suppose plus la seconde condition, on montre un théorème analogue à celui de Leadbetter (voir Théorème 1.2.4, page 38) sur l'indice extrême. En particulier, on donne l'indice extrême du minimum des rayons inscrits pour une mosaïque de Poisson-Voronoi et du maximum des rayons circonscrits pour une mosaïque de Poisson-Delaunay (Exemples 1 et 2 pages 106, 107).

Le théorème général, dans lequel on suppose les CONDITIONS 1 et 2, est l'analogue du Théorème 1.2.2 de Leadbetter mais la preuve est fondamentalement différente, d'abord parce que les variables aléatoires considérées ne sont ni une suite, ni un champ aléatoire mais aussi parce que les outils utilisés ne sont pas les mêmes. On s'appuie notamment sur le Théorème d'Arratia *et al.* 1.2.3 ce qui permet, en outre, d'avoir des vitesses de convergence. Pour ramener notre étude au Théorème 1.2.3, on utilise une notion de graphe de dépendance.

Les résultats précédents font l'objet d'un second article [23] (soumis).

A general study of extremes of stationary tessellations with applications

N. Chenavier

3.1 Introduction

A tessellation of $\mathbf{R}^d$, endowed with its natural norm $|\cdot|$, is a countable collection of compact subsets, called *cells*, with disjoint interiors which subdivides the space and such that the number of cells intersecting any bounded subset of $\mathbf{R}^d$ is finite. By a random tessellation $\mathfrak{m}$, we mean a random variable defined on a hypothetical probability space $(\Omega, \mathfrak{A}, \mathbb{P})$ with values in the set of tessellations of $\mathbf{R}^d$ endowed with a specific σ-algebra induced by the Fell topology. It is said to be stationary if its distribution is invariant under translation of the cells. For a complete account on random tessellations, we refer to the books [130], [140] and the survey [18].

Given a fixed realization of $\mathfrak{m}$, we associate to each cell $C \in \mathfrak{m}$, in a deterministic way, a point $z(C)$ which is called the *nucleus* of the cell, such that $z(C+x) = z(C) + x$ for all $x \in \mathbf{R}^d$. To describe the mean behaviour of the tessellation, the notions of intensity and typical cell are introduced as follows. Let B be a Borel subset of $\mathbf{R}^d$ such that $\lambda_d(B) \in (0, \infty)$ where λ_d is the d-dimensional Lebesgue measure. The *intensity* γ of the tessellation is defined as

$$\gamma = \frac{1}{\lambda_d(B)} \cdot \mathbb{E}\left[\#\{C \in \mathfrak{m}, z(C) \in B\}\right]$$

and we assume that $\gamma \in (0, \infty)$. Since $\mathfrak{m}$ is stationary, γ is independent of B and we suppose, without loss of generality, that $\gamma = 1$. The *typical cell* $\mathscr{C}$ is a random polytope such that the distribution is given by

$$\mathbb{E}[f(\mathscr{C})] = \frac{1}{\lambda_d(B)} \cdot \mathbb{E}\left[\sum_{\substack{C \in \mathfrak{m}, \\ z(C) \in B}} f(C - z(C))\right] \tag{3.1.1}$$

where $f : \mathcal{K}_d \to \mathbf{R}$ is any bounded measurable function on the set of convex bodies $\mathcal{K}_d$ (endowed with the Hausdorff topology).

We are interested in the following problem: only a part of the tessellation is observed in the window $\mathbf{W}_\rho = \rho^{1/d} W$ where W is a bounded Borel subset of $\mathbf{R}^d$, i.e. included in a cube $\mathbf{C}^{(W)}$, and such that $\lambda_d(W) \neq 0$. Let $f : \mathcal{K}_d \to \mathbf{R}$ be a translationally invariant measurable function, i.e. $f(C + x) = f(C)$ for all $C \in \mathcal{K}_d$ and $x \in \mathbf{R}^d$. We denote by $M^{(r)}_{f,\mathbf{W}_\rho}$ the r-th order statistic of f over the cells $C \in \mathfrak{m}$ such that $z(C) \in \mathbf{W}_\rho$. When $r = 1$, the 1-st order statistic is denoted by $M_{f,\mathbf{W}_\rho}$ i.e.

$$M_{f,\mathbf{W}_\rho} = M^{(1)}_{f,\mathbf{W}_\rho} = \max_{\substack{C \in \mathfrak{m}, \\ z(C) \in \mathbf{W}_\rho}} f(C).$$

In this paper, we investigate the limit behaviour of $M^{(r)}_{f,\mathbf{W}_\rho}$ when ρ goes to infinity.

The study of extremes describes the regularity of the tessellation. For instance, in finite element method, the quality of the approximation depends on some consistency measurements over the partition, see e.g. [70]. Another potential application field is statistics of point processes. The key idea would be to identify a point process from the extremes of a tessellation induced by the point process.

To the best of our knowledge, one of the first works on extreme values in stochastic geometry is due to Penrose. In chapters 6,7 and 8 in [113], he investigates the maximum and minimum degrees of random geometric graphs. More recently, Schulte and Thäle [132] establish a theorem to derive the smallest values of a functional $f_k(x_1, \ldots, x_k)$ of k points on a homogeneous Poisson point process. Nevertheless, their approach cannot be applied to our problem for several reasons: first, they consider a Poisson point process. Moreover, studying extremes of the tessellation requires to use functionals which depend on the all point process of nuclei and not only on a fixed number of points. In this paper, we consider any function $f(\cdot)$ and we restrict our investigation to a certain kind of random tessellation satisfying a strong mixing property. We give a general theorem, with the rates of convergence, which is followed by numerous examples in the particular setting of Poisson-Voronoi and Poisson-Delaunay tessellations. This improves in particular some extremes that are investigated in [19]. Before stating our main theorems, we need some preliminaries which contain notations and conditions on the random tessellation.

Preliminaries. Let $\mathbf{C}^{(W)}$ be a cube in $\mathbf{R}^d$ containing W. We partition $\mathbf{C}_\rho^{(W)} = \rho^{1/d}\mathbf{C}^{(W)}$ by a set V_ρ of N_ρ sub-cubes of equal size with $N_\rho \underset{\rho\to\infty}{\longrightarrow} \infty$. These sub-cubes are denoted by indices $\mathbf{i} = (i_1, \ldots, i_d) \in V_\rho$. Let us define a distance between sub-cubes $\mathbf{i}$ and $\mathbf{j}$ as

$$d(\mathbf{i}, \mathbf{j}) = \max_{1\leq r\leq d}\{|i_r - j_r|\}.$$

Moreover, if A, B are two sets of sub-cubes, we let $d(A, B) = \min_{\mathbf{i}\in A,\mathbf{j}\in B} d(\mathbf{i}, \mathbf{j})$. For each $\mathbf{i} \in V_\rho$, we denote by

$$M_{f,\mathbf{i}} = \max_{\substack{C\in\mathfrak{m},\\ z(C)\in\mathbf{i}\cap\mathbf{W}_\rho}} f(C).$$

When $\{C \in \mathfrak{m}, z(C) \in \mathbf{i} \cap \mathbf{W}_\rho\}$ is empty, we take $M_{f,\mathbf{i}} = -\infty$.

Let us consider a threshold v_ρ that is a function depending on ρ. Studying the order statistics amounts to investigate the number of exceedance cells $U_\rho(v_\rho)$ defined as

$$U_\rho(v_\rho) = \sum_{\substack{C\in\mathfrak{m},\\ z(C)\in\mathbf{W}_\rho}} \mathbb{1}_{f(C)>v_\rho}. \tag{3.1.2}$$

Thanks to (3.1.1), the mean of this random variable is

$$\mathbb{E}\left[U_\rho(v_\rho)\right] = \lambda_d(\mathbf{W}_\rho)\cdot\mathbb{P}\left(f(\mathscr{C}) > v_\rho\right). \tag{3.1.3}$$

We assume the following condition which is referred as the typical cell property (TCP):

Condition (TCP): *The mean number of exceedance cells converges to a limit denoted by* $\tau \geq 0$ *i.e.*

$$\lambda_d(\mathbf{W}_\rho)\cdot\mathbb{P}\left(f(\mathscr{C}) > v_\rho\right) \underset{\rho\to\infty}{\longrightarrow} \tau.$$

Moreover, we denote by $G_1(\rho)$ the rate of convergence i.e.

$$G_1(\rho) = \left|\lambda_d(\mathbf{W}_\rho)\cdot\mathbb{P}(f(\mathscr{C}) > v_\rho) - \tau\right|. \tag{3.1.4}$$

We assume two conditions on $\mathfrak{m}$ and f that are respectively global and local. The first is a property of R-dependence on the maxima.

CONDITION 1: *There exist an integer R and an event A_ρ with $\mathbb{P}(A_\rho) \underset{\rho\to\infty}{\longrightarrow} 1$ such that, conditional on A_ρ , the σ-algebras $\sigma\{M_{f,\mathbf{i}}, \mathbf{i} \in A\}$ and $\sigma\{M_{f,\mathbf{i}}, \mathbf{i} \in B\}$ are independent when $d(A, B) > R$.*

To introduce the second condition, we consider a second function defined as

$$G_2(\rho) = N_\rho \mathbb{E}\left[\sum_{\substack{(C_1,C_2)_{\neq}\in\mathfrak{m}^2,\\ z(C_1),z(C_2)\in\mathfrak{C}_\rho}} \mathbb{1}_{f(C_1)>v_\rho, f(C_2)>v_\rho}\right] \tag{3.1.5}$$

where

$$\mathfrak{C}_\rho = \left[0, (2R+1)\cdot\lambda_d(\mathbf{W}_\rho)^{1/d} N_\rho^{-1/d}\right]^d. \tag{3.1.6}$$

The following local condition means that with high probability two neighbor cells are not simultaneously exceedances.

CONDITION 2: *The function $G_2(\rho)$ converges to 0 as ρ goes to infinity.*

Order statistics We are now prepared to present our first theorem.

Theorem 3.1.1. *Let $\mathfrak{m}$ be a stationary random tessellation of intensity 1 such that* CONDITIONS (TCP) *and* CONDITIONS 1 *and* 2 *hold. Then*

$$\left|\mathbb{P}(M_{f,\mathbf{W}_\rho}^{(r)} \le v_\rho) - e^{-\tau}\sum_{k=0}^{r-1}\frac{\tau^k}{k!}\right| = O\left(N_\rho^{-1} + \mathbb{P}(A_\rho^c) + G_1(\rho) + G_2(\rho)\right).$$

where $\phi(\rho) = O(\psi(\rho))$ means that $\phi(\rho)/\psi(\rho)$ is bounded. In particular, $\mathbb{P}(M_{f,\mathbf{W}_\rho}^{(r)} \le v_\rho)$ converges to $e^{-\tau}\sum_{k=0}^{r-1}\frac{\tau^k}{k!}$ as ρ goes to infinity.

Theorem 3.1.1 could be extended to more general models such as Boolean models and marked point processes. When the random tessellation is ergodic with respect to the group of tessellations of $\mathbf{R}^d$, the order statistics are asymptotically independent of the choice of nuclei $z(\cdot)$. This will be the case for the examples that we deal with. Indeed, they only depend on the asymptotic behaviour of $G_1(\rho)$ given by (3.1.4) and the typical cell $\mathscr{C}$ itself does not depend on the set of nuclei thanks to Wiener ergodic's theorem. Moreover, we notice that the order statistics do not depend on the shape of the window W. Actually, a method similar to Proposition 3 of [19] shows that the contribution of boundary cells is negligible.

CONDITION 1 is a global property of the tessellation whereas CONDITION 2 is a local property. In fact, there exists an analogy between CONDITIONS 1 and 2 and Conditions $D(u_n)$ and $D'(u_n)$ of Leadbetter [80] respectively. The general theory of extreme values deals with sequences [59] or random fields [84], [25], see also the reference books [31] and [124]. Unfortunately, we are unable to apply it in our setting. Indeed, the set of random variables that we consider is not a discrete random field in a classical meaning. More precisely, the process $\{M_{f,\mathbf{i}}\}_{\mathbf{i}\in V_\rho}$ is a triangular array indexed by $\mathbb{N}^d$ and the process $\{f(C_x)\}_{x\in\mathbf{R}^d}$ is not a Gaussian continuous random field, where C_x is the cell of the tessellation containing x.

Point process of exceedances In practice, the threshold is often of the form $v_\rho = v_\rho(t) = a_\rho t + b_\rho$, $t \in \mathbf{R}$ with $a_\rho > 0$. In that case, we can be more specific about the joint distributions of the order statistics. Before stating our second theorem, we need some preliminaries. We denote by $\tau(t) \in [0, +\infty]$, $t \in \mathbf{R}$, the limit of $\lambda_d(\mathbf{W}_\rho) \cdot \mathbb{P}(f(\mathscr{C}) > v_\rho(t))$ and by ${}_*x = \inf\{t \in \mathbf{R}, \tau(t) < \infty\}$ and $x^* = \sup\{t \in \mathbf{R}, \tau(t) > 0\}$ the lower and upper endpoints of $\tau(\cdot)$. Since a_ρ is positive, the function $\tau(\cdot)$ is not increasing so that $\tau(\cdot)$ is finite on $({}_*x, x^*]$.

Under Conditions 1 and 2, we consider the random collection

$$\Phi_\rho = \left\{ \left(\rho^{-1/d} z(C), a_\rho^{-1}(f(C) - b_\rho) \right), C \in \mathfrak{m} \text{ and } z(C) \in \mathbf{W}_\rho \right\} \subset W \times \mathbf{R}$$

which is referred as the point process of exceedances. Moreover, we consider a Poisson point process $\Phi \subset W \times ({}_*x, x^*]$, with intensity measure ν given by

$$\nu(B \times (s,t]) = \mathbb{E}\left[\#\Phi \cap (B \times (s,t])\right] = \frac{\lambda_d(B)}{\lambda_d(W)} \cdot (\tau(s) - \tau(t))$$

for all Borel subset $B \subset W$ and all segment $(s,t] \subset ({}_*x, x^*]$. We then obtain the following limit theorem.

Theorem 3.1.2. *Let $\mathfrak{m}$ be a stationary random tessellation of intensity 1 such that* Conditions *(TCP) and* Conditions *1 and 2 hold. Then the family of point processes Φ_ρ, $\rho > 0$ converges in distribution to the Poisson point process Φ i.e. for any Borel subset $\mathscr{B}_1, \ldots, \mathscr{B}_k \subset W \times ({}_*x, x^*]$ with $\nu(\partial \mathscr{B}_i) = 0$ for all $i = 1, 2, \ldots, k$*

$$(\#\Phi_\rho \cap \mathscr{B}_1, \ldots, \#\Phi_\rho \cap \mathscr{B}_k) \xrightarrow{\mathcal{D}} (\#\Phi \cap \mathscr{B}_1, \ldots, \#\Phi \cap \mathscr{B}_k).$$

This result suggests that the largest order statistics can be seen as points of a (non homogeneous) Poisson point process. Theorem 3.1.2 gives their joint distributions so that Theorem 3.1.1 is a particular case of the latter when $k = 1$ and $B = W \times (t, \infty)$. For a wide panorama on results of the point process of exceedances associated to the extremes of a sequence of non independent random variables, we refer to chapter 5 in [82]. When $W = \mathbf{C}^{(W)} = [0,1]^d$ and when $\tau(\cdot)$ is not constant, the function $\tau(\cdot)$ belongs to either the Fréchet, the Gumbel or the Weibull family. This fact is a rewriting of the proof of Theorem 4.1 in [84].

Extremal index When Condition 2 does not hold, the exceedance locations can be divided into clusters and the order statistics cannot be investigated when $r \geq 2$. Yet, the behaviour of $M_{f,\mathbf{W}_\rho}$ can be deduced up to a constant according to the following proposition. For sake of simplicity, we assume in Proposition 3.1.3 that $W = [0,1]^d$.

Proposition 3.1.3. *Let $\mathfrak{m}$ be a stationary random tessellation of intensity 1 such that* Condition *(TCP) and* Condition *1 hold and let $W = [0,1]^d$. Let us assume that for all $\tau \geq 0$, there exists a deterministic function $v_\rho(\tau)$ depending on ρ such that $\rho \cdot \mathbb{P}(f(\mathscr{C}) > v_\rho(\tau))$ converges to τ as ρ goes to infinity. Then there exist constants $\theta, \theta', 0 \leq \theta \leq \theta' \leq 1$ such that, for all $\tau \geq 0$,*

$$\limsup_{\rho\to\infty} \mathbb{P}(M_{f,\mathbf{W}_\rho} \leq v_\rho(\tau)) = e^{-\theta\tau} \text{ and } \liminf_{\rho\to\infty} \mathbb{P}(M_{f,\mathbf{W}_\rho} \leq v_\rho(\tau)) = e^{-\theta'\tau}.$$

In particular, if $\mathbb{P}\left(M_{f,\mathbf{W}_\rho} \leq v_\rho(\tau)\right)$ converges, then $\theta = \theta'$ and

$$\mathbb{P}\left(M_{f,\mathbf{W}_\rho} \leq v_\rho(\tau)\right) \underset{\rho\to\infty}{\longrightarrow} e^{-\theta\tau}.$$

Proposition 3.1.3 is similar to the result due to Leadbetter for stationary sequences of real random variables (see Theorem 2.2 of [81]). Its proof relies notably on the adaptation to our setting of several arguments included in [81]. According to Leadbetter, we say that the random tessellation $\mathfrak{m}$ has *extremal index* θ if, for each $\tau \geq 0$, $\rho \cdot \mathbb{P}\left(f(\mathscr{C}) > v_\rho(\tau)\right) \underset{\rho\to\infty}{\longrightarrow} \tau$ and $\mathbb{P}\left(M_{f,\mathbf{W}_\rho} \leq v_\rho(\tau)\right) \underset{\rho\to\infty}{\longrightarrow} e^{-\theta\tau}$. In a future work, we hope to develop a general method to estimate the extremal index.

The paper is organized as follows. In section 3.2, we show how to reduce our problem to the study of extreme values on a dependency graph. We use a result of [4] to derive an estimation of exceedances by a Poisson distribution. We then deduce Theorems 3.1.1 and 3.1.2 from a discretization of $\mathbf{W}_\rho$ into sub-cubes. Sections 3.3, 3.4 and 3.5 are devoted to numerous applications on Delaunay and Voronoi random tessellations. We investigate the asymptotic behaviours of:

- the minimum of circumradii of a Poisson-Delaunay tessellation in any dimension and the maximum and minimum of the areas in the planar case (section 3.3),
- the minimum of distances of the farthest nucleus and the minimum of the volume of flowers for a Poisson-Voronoi tessellation (section 3.4),
- the maximum of inradii for a Voronoi tessellation induced by a Gauss-Poisson process (section 3.5).

For each tessellation and each characteristic, we need to find a suitable threshold v_ρ and to check CONDITION 2 which requires some delicate geometric estimates. In section 3.6, we prove Proposition 3.1.3 and we give two examples where the extremal index differs from 1.

In the rest of the paper, c or c' denotes a generic constant which does not depend on ρ but may depend on other quantities.

3.2 Proof of Theorems 3.1.1 and 3.1.2

3.2.1 Extreme values on a dependency graph and proof of Theorem 3.1.1

We first outline the methodology of the proof of Theorem 3.1.1 with some additional notations. A classical method in extreme value theory is to investigate the exceedances. We consider two random variables that are the number of exceedance cells $U_\rho(v_\rho)$, introduced in (3.1.2), and the number of exceedance cubes $U'_{V_\rho}(v_\rho)$ defined as

$$U_\rho(v_\rho) = \sum_{\substack{C\in\mathfrak{m},\\ z(C)\in\mathbf{W}_\rho}} \mathbb{1}_{f(C)>v_\rho} \text{ and } U'_{V_\rho}(v_\rho) = \sum_{\mathbf{i}\in V_\rho} \mathbb{1}_{M_{f,\mathbf{i}}>v_\rho} \tag{3.2.1}$$

where V_ρ and $M_{f,\mathbf{i}}$ are introduced in the preliminaries. We denote by μ_ρ the mean of $U'_{V_\rho}(v_\rho)$ i.e .

$$\mu_\rho = \mathbb{E}\left[U'_{V_\rho}(v_\rho)\right] = \sum_{\mathbf{i}\in V_\rho} \mathbb{P}\left(M_{f,\mathbf{i}} > v_\rho\right). \tag{3.2.2}$$

The proof of Theorem 3.1.1 can be displayed as the three following results.

Lemma 3.2.1. *With the same assumptions as in Theorem 3.1.1, we get for all $r \in \mathbb{N}^*$*

$$\left|\mathbb{P}\left(U_\rho(v_\rho) \leq r-1\right) - \mathbb{P}\left(U'_{V_\rho}(v_\rho) \leq r-1\right)\right| \leq 2 \cdot G_2(\rho) \tag{3.2.3}$$

The above lemma is a consequence of CONDITION 2.

Lemma 3.2.2. *Let μ_ρ be as in* (3.2.2). *With the same assumptions as in Theorem 3.1.1, we get for all $r \in \mathbb{N}^*$*

$$\left|\mathbb{P}(U'_{V_\rho}(v_\rho) \leq r-1) - e^{-\mu_\rho}\sum_{k=0}^{r-1}\frac{\mu_\rho^k}{k!}\right| = O\left(N_\rho^{-1} + \mathbb{P}(A_\rho^c) + G_2(\rho)\right) \tag{3.2.4}$$

The derivation of the latter constitutes the major part of the proof of Theorem 3.1.1. It means that the number of exceedance cubes is approximately a Poisson random variable. The fundamental concept to prove this lemma is that of a dependency graph. We first establish a Poisson approximation on the number of exceedances on such graph and we show how we can reduce our problem to this graph. Finally, the following result gives the convergence of μ_ρ.

Lemma 3.2.3. *Let μ_ρ as in* (3.2.2). *With the same assumptions as in Theorem 3.1.1, we get*

$$|\mu_\rho - \tau| \leq G_1(\rho) + G_2(\rho). \tag{3.2.5}$$

Proof of Theorem 3.1.1. Since $M^{(r)}_{f,\mathbf{W}_\rho}$ is lower than v_ρ if and only if $U_\rho(v_\rho) \leq r-1$, we deduce Theorem 3.1.1 from the three lemmas above and the fact that the function $x \mapsto e^{-x}\sum_{k=0}^{r-1}\frac{x^k}{k!}$ is Lipschitz. □

In the rest of the subsection, we establish the Poisson approximation on a dependency graph (Proposition 3.2.4) and we deduce from it Lemma 3.2.2. Then we prove Lemmas 3.2.1 and 3.2.3.

Extreme values on a dependency graph By a dependency graph, we mean a graph $G = (V, E)$ and a collection of real random variables $X_{\mathbf{i}}, \mathbf{i} \in V$ (not necessarily stationary) which satisfy the following property: for any pair of disjoint sets $A_1, A_2 \subset V$ such that no edge in E has one endpoint in A_1 and the other in A_2, the σ-field $\sigma(X_{\mathbf{i}}, \mathbf{i} \in A_1)$ and $\sigma(X_{\mathbf{i}}, \mathbf{i} \in A_2)$ are mutually independent. Introduced by Petrovskaya and Leontovitch in [115], this concept was applied by Baldi and Rinott (e.g. [9]) to obtain central limit theorems and normal approximations. Furthermore, Arratia *et al.* give a Poisson approximation of a sum of (non independent) Bernoulli random variables for a random field (see Theorem 1 in [4]). We write their result in our context to approximate the number of exceedances on a dependency graph by a Poisson random variable.

First, we give some notations. We denote by $|V|$ the number of vertices of $G = (V, E)$, D the maximal degree and $J \subset \mathbf{R}$ a finite union of disjoint intervals. Let $\mathbf{U}'_V(J)$ be the number of exceedances i.e.

$$\mathbf{U}'_V(J) = \sum_{\mathbf{i}\in V}\mathbb{1}_{X_{\mathbf{i}}\in J}$$

and $p_{\mathbf{i}} = \mathbb{P}(X_{\mathbf{i}} \in J)$, $p_{\mathbf{ij}} = \mathbb{P}(X_{\mathbf{i}} \in J, X_{\mathbf{j}} \in J)$ for all $\mathbf{i} \in V$ and $\mathbf{j} \in V(\mathbf{i}) - \{\mathbf{i}\}$ where $V(\mathbf{i})$ is the set of neighbors of $\mathbf{i}$ i.e.

$$V(\mathbf{i}) = \{\mathbf{j} \in V, (\mathbf{i}, \mathbf{j}) \in E\} \cup \{\mathbf{i}\}. \tag{3.2.6}$$

Let us consider a Poisson random variable Z of mean

$$\boldsymbol{\mu}_J = \mathbb{E}[Z] = \mathbb{E}[\mathbf{U}'_V(J)] = \sum_{\mathbf{i}\in V}\mathbb{P}\left(X_{\mathbf{i}} \in J\right).$$

Chen-Stein method can be applied to approximate the number of occurrences of dependent events by a Poisson random variable (e.g. [4]). In particular, this is a powerful tool to derive some results in extreme value theory for a sequence of real random variables (e.g. [135]). We write below a slightly modified version of Theorem 1 of [4] to derive an upper bound of the total variation distance between the number of exceedances $\mathbf{U}'_V(J)$ and its Poisson approximation Z for a dependency graph.

Proposition 3.2.4. *(Arratia et al. 1989) Let* $p(V) = \sup_{i\in V} p_{\mathbf{i}}$ *and* $q(V)^2 = \sup_{(\mathbf{i},\mathbf{j})\in E} p_{\mathbf{ij}}$. *Then*

$$\sup_{A\subset\mathbb{N}} |\mathbb{P}(\mathbf{U}'_V(J)\in A) - \mathbb{P}(Z\in A)| \leq 2D\cdot|V|\cdot\left(p(V)^2+q(V)^2\right). \tag{3.2.7}$$

In particular, for all $r\in\mathbb{N}^*$, *we get*

$$\left|\mathbb{P}(\mathbf{U}'_V(J)\leq r-1) - e^{-\boldsymbol{\mu}_J}\sum_{k=0}^{r-1}\frac{\boldsymbol{\mu}_J^k}{k!}\right| \leq 2D\cdot|V|\cdot\left(p(V)^2+q(V)^2\right). \tag{3.2.8}$$

Proof of Proposition 3.2.4. The upper bound (3.2.8) is a direct consequence of (3.2.7). From Theorem 1 of [4], we get

$$\sup_{A\subset\mathbb{N}} |\mathbb{P}(\mathbf{U}'_V(J)\in A) - \mathbb{P}(Z\in A)| \leq 2(b_1+b_2+b_3) \tag{3.2.9}$$

where

$$b_1 = \sum_{\mathbf{i}\in V}\sum_{\mathbf{j}\in V(\mathbf{i})} p_{\mathbf{i}}p_{\mathbf{j}}, \quad b_2 = \sum_{\mathbf{i}\in V}\sum_{\mathbf{i}\neq\mathbf{j}\in V(\mathbf{i})} p_{\mathbf{ij}} \text{ and } b_3 = \sum_{i\in V}\mathbb{E}\left[\left|\mathbb{E}\left[X_{\mathbf{i}}-p_{\mathbf{i}}|\sigma(X_{\mathbf{j}}:\mathbf{j}\notin V(\mathbf{i}))\right]\right|\right].$$

Since $|V(\mathbf{i})|\leq D+1$, we obtain $b_1\leq|V|\cdot D\cdot p(V)^2$ and $b_2\leq|V|\cdot D\cdot q(V)^2$. Moreover, using the fact that if $\mathbf{j}\notin V(\mathbf{i})$, the random variable $X_{\mathbf{j}}$ is independent of $X_{\mathbf{i}}$, we get $b_3=0$. We then deduce (3.2.7) from (3.2.9). □

Central limit theorems in geometric probability have been deduced from normal approximation on a dependency graph by a discretization technique (see e.g. [5]). In the same spirit, we derive Lemma 3.2.2 from Proposition 3.2.4. We need first to explain how we construct the dependency graph from our random tessellation.

Construction of the dependency graph We define a graph $G_\rho=(V_\rho,E_\rho)$ as follows. The set V_ρ consists of the sub-cubes $\mathbf{i}$ ($|V_\rho|=N_\rho$) which cover $\mathbf{W}_\rho$ whereas an edge $(\mathbf{i},\mathbf{j})\in E_\rho$ if $d(\mathbf{i},\mathbf{j})\leq R$ where R is introduced in Condition 1. The maximal degree D_ρ of this graph satisfies

$$D_\rho\leq(2R+1)^d. \tag{3.2.10}$$

For all $\mathbf{i}\in V_\rho$, we define the random variable $X_{\mathbf{i}}$ as

$$X_{\mathbf{i}} = M_{f,\mathbf{i}}. \tag{3.2.11}$$

From Condition 1, conditional on A_ρ, the graph G_ρ and the collection $(M_{f,\mathbf{i}})_{\mathbf{i}\in V_\rho}$ define a dependency graph.

Proof of Lemma 3.2.2. We apply Proposition 3.2.4 to $X_{\mathbf{i}}=M_{f,\mathbf{i}}$ and $J=(v_\rho,\infty)$. It is enough to give upper bounds of $\mathbb{P}(M_{f,\mathbf{i}}>v_\rho|A_\rho)$ and $\mathbb{P}(M_{f,\mathbf{i}}>v_\rho, M_{f,\mathbf{j}}>v_\rho|A_\rho)$. According to (3.2.11), we get

$$\mathbb{P}(M_{f,\mathbf{i}} > v_\rho) = \mathbb{P}\left(\bigcup_{\substack{C\in\mathfrak{m},\\ z(C)\in\mathbf{i}\cap\mathbf{W}_\rho}} \{f(C) > v_\rho\}\right) \leq \mathbb{E}\left[\sum_{\substack{C\in\mathfrak{m},\\ z(C)\in\mathbf{i}\cap\mathbf{W}_\rho}} \mathbb{1}_{f(C)>v_\rho}\right].$$

Since f is translation invariant and $\lambda_d(\mathbf{i}) = \lambda_d(W)\cdot\rho/N_\rho$, we deduce from (3.1.1) that

$$\mathbb{P}(M_{f,\mathbf{i}} > v_\rho) \leq \frac{1}{N_\rho}\cdot\lambda_d(W)\cdot\rho\cdot\mathbb{P}(f(\mathscr{C}) > v_\rho). \tag{3.2.12}$$

Using the trivial inequalities $\mathbb{P}(M_{f,\mathbf{i}} > v_\rho|A_\rho) \leq \mathbb{P}(M_{f,\mathbf{i}} > v_\rho)/\mathbb{P}(A_\rho)$ and $\lambda_d(W)\cdot\rho\cdot\mathbb{P}(f(\mathscr{C}) > v_\rho) \leq G_1(\rho)+\tau$ where $G_1(\rho)$ is defined in (3.1.4), we obtain

$$p_{\mathbf{i}} := \mathbb{P}(M_{f,\mathbf{i}} > v_\rho|A_\rho) \leq \frac{G_1(\rho)+\tau}{\mathbb{P}(A_\rho)N_\rho}. \tag{3.2.13}$$

Moreover, for any $\mathbf{i}\in V_\rho$ and $\mathbf{j}\in V_\rho(\mathbf{i})-\{\mathbf{i}\}$, we get

$$\begin{aligned}\mathbb{P}(M_{f,\mathbf{i}} > v_\rho, M_{f,\mathbf{j}} > v_\rho) &= \mathbb{P}\left(\bigcup_{\substack{C_1\in\mathfrak{m},\\ z(C_1)\in\mathbf{i}\cap\mathbf{W}_\rho}}\ \bigcup_{\substack{C_2\in\mathfrak{m},\\ z(C_2)\in\mathbf{j}\cap\mathbf{W}_\rho}} \{f(C_1) > v_\rho, f(C_2) > v_\rho\}\right)\\ &\leq \mathbb{E}\left[\sum_{\substack{(C_1,C_2)_{\neq}\in\mathfrak{m}^2\\ z(C_1),z(C_2)\in V_\rho(\mathbf{i})}} \mathbb{1}_{f(C_1)>v_\rho, f(C_2)>v_\rho}\right]\end{aligned} \tag{3.2.14}$$

where $(C_1,C_2)_{\neq}\in\mathfrak{m}^2$ means that (C_1,C_2) is a couple of distinct cells. With the slight abuse of notation, we will write in the rest of the paper $V_\rho(\mathbf{i})$ for the union of the sub-cubes $\bigcup_{\mathbf{j}\in V_\rho(\mathbf{i})}\mathbf{j}$.

Besides, the set of neighbors $V_\rho(\mathbf{i})$ can be rewritten as $V_\rho(\mathbf{i}) = \{\mathbf{j}\in V_\rho, d(\mathbf{i},\mathbf{j})\leq R\}$. Hence $V_\rho(\mathbf{i})$ is a convex union of disjoints sub-cubes of volume $\lambda_d(W)\cdot\rho/N_\rho$, which are at most $(2R+1)^d$, and can be included in the cube $\mathfrak{C}_\rho$ defined in (3.1.6) up to a translation. Since f is translation invariant, we obtain

$$\mathbb{E}\left[\sum_{\substack{(C_1,C_2)_{\neq}\in\mathfrak{m}^2\\ z(C_1),z(C_2)\in V_\rho(\mathbf{i})}} \mathbb{1}_{f(C_1)>v_\rho, f(C_2)>v_\rho}\right] \leq \frac{G_2(\rho)}{N_\rho}. \tag{3.2.15}$$

Using the fact that $\mathbb{P}(M_{f,\mathbf{i}} > v_\rho, M_{f,\mathbf{j}} > v_\rho|A_\rho) \leq \mathbb{P}(M_{f,\mathbf{i}} > v_\rho, M_{f,\mathbf{j}} > v_\rho)/\mathbb{P}(A_\rho)$ we deduce from (3.2.14) that

$$p_{\mathbf{ij}} = \mathbb{P}(M_{f,\mathbf{i}} > v_\rho, M_{f,\mathbf{j}} > v_\rho|A_\rho) \leq \frac{G_2(\rho)}{\mathbb{P}(A_\rho)N_\rho}. \tag{3.2.16}$$

From (3.2.8) written for the conditional probability $\cdot|A_\rho$, (3.2.10), (3.2.13), (3.2.16) and the fact that $|V_\rho| = N_\rho$, we get

$$\left|\mathbb{P}(U'_{V_\rho}(v_\rho) \leq r-1|A_\rho) - e^{-\mu_\rho}\sum_{k=0}^{r-1}\frac{\mu_\rho^k}{k!}\right| \leq \frac{2(2R+1)^d}{\mathbb{P}(A_\rho)^2}\cdot\left(\frac{(G_1(\rho)+\tau)^2}{N_\rho} + \mathbb{P}(A_\rho)G_2(\rho)\right).$$

The rate of convergence (3.2.4) results directly from the previous upper bound and the fact that $\mathbb{P}(A_\rho)$ and $G_1(\rho)$ converge respectively to 1 and 0. □

We prove below Lemmas 3.2.1 and 3.2.3. Both need only the local CONDITION 2.

Proof of Lemma 3.2.1. Let us notice that Lemma 3.2.1 is trivial when $r = 1$. More generally, for all $r \in \mathbb{N}^*$, we have

$$\left|\mathbb{P}\left(U_\rho(v_\rho) \leq r-1\right) - \mathbb{P}\left(U'_{V_\rho}(v_\rho) \leq r-1\right)\right| \leq 2\mathbb{P}\left(U_\rho(v_\rho) \neq U'_{V_\rho}(v_\rho)\right). \tag{3.2.17}$$

According to (3.2.1), the above random variables differ if and only if there are at least two exceedances in the same sub-cube $\mathbf{i}$ i.e.

$$\begin{aligned}\mathbb{P}\left(U_\rho(v_\rho) \neq U'_{V_\rho}(v_\rho)\right) &= \mathbb{P}\left(\bigcup_{\mathbf{i}\in V_\rho} \bigcup_{\substack{(C_1,C_2)_{\neq}\in\mathfrak{m}^2,\\ z(C_1),z(C_2)\in\mathbf{i}\cap\mathbf{W}_\rho}} \{f(C_1) > v_\rho, f(C_2) > v_\rho\}\right)\\ &\leq \sum_{\mathbf{i}\in V_\rho} \mathbb{E}\left[\sum_{\substack{(C_1,C_2)_{\neq}\in\mathfrak{m}^2\\ z(C_1),z(C_2)\in\mathbf{i}\cap\mathbf{W}_\rho}} \mathbb{1}_{f(C_1)>v_\rho, f(C_2)>v_\rho}\right].\end{aligned}$$

Since $|V_\rho| = N_\rho$, the right-hand side is bounded by $G_2(\rho)$ thanks to (3.2.15). This shows that $\mathbb{P}\left(U_\rho(v_\rho) \neq U'_{V_\rho}(v_\rho)\right) \leq G_2(\rho)$ and consequently we deduce (3.2.3) from (3.2.17). □

Proof of Lemma 3.2.3. From (3.2.2) and the triangle inequality, we get

$$|\mu_\rho - \tau| \leq |\mathbb{E}\left[U_\rho(v_\rho)\right] - \tau| + \mathbb{E}\left[U_\rho(v_\rho) - U'_{V_\rho}(v_\rho)\right] \tag{3.2.18}$$

where $U_\rho(v_\rho) \geq U'_{V_\rho}(v_\rho)$ a.s. According to (3.1.3) and (3.1.4), we obtain that

$$|\mathbb{E}[U_\rho(v_\rho)] - \tau| = G_1(\rho). \tag{3.2.19}$$

To give an upper bound of the second term of the right-hand side of (3.2.18), we use the fact that the family V_ρ covers $\mathbf{W}_\rho$. Intuitively, the number of exceedance sub-cubes U_{V_ρ} can be approximated by the number of exceedance cells $U_\rho(v_\rho)$ since $G_2(\rho)$ is negligible. We justify this fact below. From (3.2.1), we obtain a.s. that

$$\begin{aligned}U_\rho(v_\rho) - U'_{V_\rho}(v_\rho) &= \sum_{\mathbf{i}\in V_\rho} \sum_{\substack{C\in\mathfrak{m},\\ z(C)\in\mathbf{i}\cap\mathbf{W}_\rho}} \mathbb{1}_{f(C)>v_\rho} - \mathbb{1}_{M_{f,\mathbf{i}}>v_\rho}\\ &= \sum_{\mathbf{i}\in V_\rho}\left(\sum_{\substack{C\in\mathfrak{m},\\ z(C)\in\mathbf{i}\cap\mathbf{W}_\rho}} \mathbb{1}_{f(C)>v_\rho} - 1\right)\mathbb{1}_{M_{f,\mathbf{i}}>v_\rho} \leq \sum_{\mathbf{i}\in V_\rho} \sum_{\substack{(C_1,C_2)_{\neq}\in\mathfrak{m}^2\\ z(C_1),z(C_2)\in\mathbf{i}\cap\mathbf{W}_\rho}} \mathbb{1}_{f(C_1)>v_\rho, f(C_2)>v_\rho}.\end{aligned} \tag{3.2.20}$$

The last inequality comes from the fact that if there is 0 or 1 exceedance cell, the sums inside the expectations are null. Otherwise, if the number of exceedances is $k \geq 2$, we use that fact that $k-1 \leq \frac{k(k-1)}{2} =: \binom{k}{2}$ which is the number of exceedance couples.

Taking the means in (3.2.20) and using the fact that the mean of the right-hand side of (3.2.20) is bounded by $G_2(\rho)$ as in the proof of Lemma 3.2.1, we get

$$\mathbb{E}\left[U_\rho(v_\rho) - U'_{V_\rho}(v_\rho)\right] \leq G_2(\rho) \tag{3.2.21}$$

From (3.2.18), (3.2.19) and (3.2.21) we obtain that $|\mu_\rho - \tau|$ is lower than $G_1(\rho) + G_2(\rho)$. □

3.2.2 Proof of Theorem 3.1.2

By Kallenberg's theorem (see Proposition 3.22, p. 156 in [124], see also the proof of Theorem 2.1.2 in [31]) it is enough to check that:

- For all Borel subset $B \subset W$ and ${}_*x < s \leq t \leq x^*$

$$\mathbb{E}\left[\#\Phi_\rho \cap (B \times (s,t])\right] \underset{\rho\to\infty}{\longrightarrow} \mathbb{E}\left[\#\Phi \cap (B \times (s,t])\right] \tag{3.2.22}$$

- For all $\mathscr{P} = \bigcup_{l=1}^{L} B^{(l)} \times (s_l, t_l]$ where $B^{(l)}$ is the intersection of W and a rectangular solid in $\mathbf{C}^{(W)}$ and ${}_*x < s_l \leq t_l \leq x^*$

$$\mathbb{P}\left(\#\Phi_\rho \cap \mathscr{P} = 0\right) \underset{\rho\to\infty}{\longrightarrow} \mathbb{P}\left(\#\Phi \cap \mathscr{P} = 0\right) \tag{3.2.23}$$

Proof of (3.2.22)**.** From (3.1.1), we have

$$\begin{aligned}\mathbb{E}\left[\#\Phi_\rho \cap (B \times (s,t])\right] &= \mathbb{E}\left[\sum_{\substack{C\in\mathfrak{m},\\ z(C)\in\mathbf{B}_\rho}} \mathbb{1}_{a_\rho s + b_\rho < f(C) \leq a_\rho t + b_\rho}\right] \\ &= \lambda_d(\mathbf{B}_\rho)\cdot\left(\mathbb{P}\left(f(\mathscr{C}) > a_\rho s + b_\rho\right) - \mathbb{P}\left(f(\mathscr{C}) > a_\rho t + b_\rho\right)\right)\end{aligned}$$

where we recall that $\mathbf{B}_\rho = \rho^{1/d}B$. According to the trivial equality $\lambda_d(\mathbf{B}_\rho) = \frac{\lambda_d(B)}{\lambda_d(W)}\cdot\lambda_d(W)\cdot\rho$ and the fact that $\lambda_d(\mathbf{W}_\rho)\cdot\mathbb{P}\left(f(\mathscr{C}) > v_\rho(t)\right)$ converges to $\tau(t)$ for all $t \in \mathbf{R}$, we get

$$\mathbb{E}\left[\#\Phi_\rho \cap (B \times (s,t])\right] \underset{\rho\to\infty}{\longrightarrow} \frac{\lambda_d(B)}{\lambda_d(W)}\cdot(\tau(s) - \tau(t)) = \mathbb{E}\left[\#\Phi \cap (B \times (s,t])\right]$$

and consequently (3.2.22). □

Proof of (3.2.23)**.** We can write $\mathscr{P}$ as a disjoint union of strips i.e.

$$\mathscr{P} = \bigsqcup_{l=1}^{L} B^{(l)} \times J^{(l)} \tag{3.2.24}$$

such that the Borel subsets $B^{(l)} \subset W$ are disjoint and such that $J^{(l)}$ is a finite union of half-open intervals for all $l = 1, \ldots, L$. The following lemma shows that it is enough to investigate the case where $\mathscr{P}$ is a strip.

Lemma 3.2.5. *Let $\mathscr{P}$ be as in* (3.2.24). *With the same hypothesis as in Theorem 3.1.1, we have*

$$\mathbb{P}\left(\#\Phi_\rho \cap \mathscr{P} = 0\right) - \prod_{l=1}^{L} \mathbb{P}\left(\#\Phi_\rho \cap (B^{(l)} \times J^{(l)}) = 0\right) \underset{\rho\to\infty}{\longrightarrow} 0. \tag{3.2.25}$$

The proof of Lemma 3.2.5 is postponed at the end of the present subsection. Thanks to Lemma 3.2.5, we can assume that $\mathscr{P}$, defined in (3.2.24), is a strip i.e. $\mathscr{P} = B \times J$ where J is a finite union of half-open intervals. Without loss of generality, we can assume that these intervals are disjoint i.e.

$$J = \bigsqcup_{j=1}^{k} (s_j, t_j]$$

with ${}_*x < s_j \leq t_j \leq x^*$ and $s_j \leq t_{j+1}$, $j = 1, \ldots, k$. As in the proof of Theorem 3.1.1, we introduce two random variables that are

$$\mathscr{U}_\rho(B\times J) = \#\Phi_\rho \cap (B\times J) = \sum_{\substack{C\in\mathfrak{m},\\ z(C)\in\mathbf{B}_\rho}} \mathbb{1}_{a_\rho^{-1}(f(C)-b_\rho)\in J} \text{ and } \mathscr{U}'_{V_\rho}(B\times J) = \sum_{\mathbf{i}\in V_\rho} \mathbb{1}_{a_\rho^{-1}(M_{f,\mathbf{i}}(B)-b_\rho)\in J}$$

where

$$M_{f,\mathbf{i}}(B) = \max_{\substack{C\in\mathfrak{m},\\ z(C)\in\mathbf{i}\cap\mathbf{B}_\rho}} f(C).$$

In particular, $\mathscr{U}_\rho(W \times (s,\infty)) = U_\rho(v_\rho(s))$ and $\mathscr{U}'_{V_\rho}(W \times (s,\infty)) = U'_{V_\rho}(v_\rho(s))$ where $U_\rho(v_\rho(s))$ and $U'_{V_\rho}(v_\rho(s))$ have been defined in (3.2.1). We denote by $\boldsymbol{\mu}_\rho(B \times J)$ the mean of $\mathscr{U}'_{V_\rho}(B \times J)$ i.e.

$$\boldsymbol{\mu}_\rho(B \times J) = \mathbb{E}\left[\mathscr{U}'_{V_\rho}(B \times J)\right] = \sum_{\mathbf{i}\in V_\rho} \mathbb{P}\left(a_\rho^{-1}(M_{f,\mathbf{i}}(B) - b_\rho) \in J\right).$$

As in the proof of Theorem 3.1.1, we subdivide the proof into three steps. More precisely, we show that

$$\mathbb{P}\left(\mathscr{U}_\rho(B \times J) = 0\right) - \mathbb{P}\left(\mathscr{U}'_{V_\rho}(B \times J) = 0\right) \underset{\rho\to\infty}{\longrightarrow} 0 \tag{3.2.26a}$$

$$\mathbb{P}\left(\mathscr{U}'_{V_\rho}(B \times J) = 0\right) - e^{-\boldsymbol{\mu}_\rho(B\times J)} \underset{\rho\to\infty}{\longrightarrow} 0 \tag{3.2.26b}$$

$$\boldsymbol{\mu}_\rho(B \times J) \underset{\rho\to\infty}{\longrightarrow} \nu(B \times J). \tag{3.2.26c}$$

Let us notice that the convergences (3.2.26a),(3.2.26b) and (3.2.26c) are generalisations of Lemmas 3.2.1, 3.2.2 and 3.2.3 respectively. For the proof of (3.2.26a), it is enough to show that $\mathbb{P}\left(\mathscr{U}_\rho(B \times J) \neq \mathscr{U}'_{V_\rho}(B \times J)\right)$ converges to 0 as ρ goes to infinity. Since $\mathscr{U}_\rho(B\times J) \geq \mathscr{U}'_{V_\rho}(B\times J)$ for all Borel subsets, we have

$$\begin{aligned}\mathbb{P}\left(\mathscr{U}_\rho(B \times J) \neq \mathscr{U}'_{V_\rho}(B \times J)\right) &\leq \sum_{j=1}^{k} \mathbb{P}\left(\mathscr{U}_\rho(B \times (s_j,t_j]) \neq \mathscr{U}'_{V_\rho}(B \times (s_j,t_j])\right)\\ &\leq \sum_{j=1}^{k} \mathbb{P}\left(\mathscr{U}_\rho(W \times (s_j,\infty)) \neq \mathscr{U}'_{V_\rho}(W \times (s_j,\infty))\right) = \sum_{j=1}^{k} \mathbb{P}\left(\mathscr{U}_\rho(v_\rho(s_j)) \neq \mathscr{U}'_{V_\rho(v_\rho(s_j))}\right).\end{aligned} \tag{3.2.27}$$

The last term converges to 0 according to the proof of Lemma 3.2.1.

Secondly, we prove (3.2.26b). In the same spirit as in the proof of Lemma 3.2.2, we apply Proposition 3.2.4 conditional on A_ρ to $X_{\mathbf{i}} = a_\rho^{-1}(M_{f,\mathbf{i}}(B) - b_\rho)$ and $J = \bigsqcup_{j=1}^k (s_j, t_j]$. Let $\mathbf{i} \in V_\rho$ and $\mathbf{j} \in V_\rho(\mathbf{i}) - \{\mathbf{i}\}$. Using the fact that $M_{f,\mathbf{i}}(B) \leq M_{f,\mathbf{i}}$, we get

$$p_{\mathbf{i}} = \mathbb{P}\left(a_\rho^{-1}(M_{f,\mathbf{i}}(B) - b_\rho) \in J | A_\rho\right) \leq \mathbb{P}\left(M_{f,\mathbf{i}}(B) > v_\rho(s_1) | A_\rho\right) = O(N_\rho^{-1})$$

according to (3.2.12). Moreover

$$\begin{aligned} p_{\mathbf{ij}} &= \mathbb{P}\left(a_\rho^{-1}(M_{f,\mathbf{i}}(B) - b_\rho) \in J, a_\rho^{-1}(M_{f,\mathbf{j}}(B) - b_\rho) \in J | A_\rho\right) \\ &\leq \mathbb{P}\left(M_{f,\mathbf{i}}(B) > v_\rho(s_1), M_{f,\mathbf{i}}(B) > v_\rho(s_1) | A_\rho\right) = O\left(G_2(\rho) \cdot N_\rho^{-1}\right) \end{aligned}$$

according to (3.2.16). We deduce (3.2.26b) from the previous inequalities and Proposition 3.2.4.

At last we prove (3.2.26c). Proceeding along the same lines as in the proof of (3.2.22) and using the fact that J is a union of disjoint intervals, we show that

$$\mathbb{E}\left[\mathscr{U}_\rho(B \times J)\right] \underset{\rho\to\infty}{\longrightarrow} \frac{\lambda_d(B)}{\lambda_d(W)} \sum_{j=1}^{k} (\tau(s_j) - \tau(t_j)) = \nu(B \times J). \tag{3.2.28}$$

Moreover

$$\mathbb{E}\left[\mathscr{U}_\rho(B \times J)\right] - \mu_\rho(B \times J) = \mathbb{E}\left[\mathscr{U}_\rho(B \times J) - U_{V_\rho}(B \times J)\right] \leq \mathbb{E}\left[U_\rho(v_\rho(s_1)) - U'_{V_\rho}(v_\rho(s_1))\right] \tag{3.2.29}$$

converges to 0 according to (3.2.21). We deduce (3.2.26c) from (3.2.28) and (3.2.29).

Conclusion of the proof of (3.2.23). According to (3.2.26a), (3.2.26b) and (3.2.26c) and the fact that $\mathscr{U}_\rho(B \times J) = \#\Phi_\rho \cap (B \times J)$, we deduce that

$$\mathbb{P}\left(\#\Phi_\rho \cap (B \times J) = 0\right) \underset{\rho\to\infty}{\longrightarrow} e^{-\nu(N \times J)} = \mathbb{P}\left(\#\Phi \cap (B \times J) = 0\right)$$

and consequently (3.2.23). □

The end of the subsection is devoted to the proof of Lemma 3.2.5.

Proof of Lemma 3.2.5. Let $\mathscr{P} = \bigsqcup_{l=1}^L B^{(l)} \times J^{(l)}$ and $B^{(l)} = B_l \cap W$ where the rectangular solids $B_l \subset \mathbf{C}^{(W)}$ are disjoint. First, we introduce some notations. We denote respectively by $V_\rho(B^{(l)})$, $S_\rho(B^{(l)})$ and $V_\rho^\circ(B^{(l)})$ the sets

$$\begin{cases} V_\rho(B^{(l)}) = \{\mathbf{i} \in V_\rho, \mathbf{i} \cap B_l \neq \varnothing\} \\ S_\rho(B^{(l)}) = \{\mathbf{i} \in V_\rho, \mathbf{i} \cap \partial B_l \neq \varnothing\} \\ V_\rho^\circ(B^{(l)}) = \{\mathbf{i} \in V_\rho(B^{(l)}), d(\mathbf{i}, S_\rho(B^{(l)})) > R\} \end{cases}.$$

Finally, we denote by $\mathscr{U}'_{V_\rho^\circ}(B^{(l)} \times J^{(l)}) \leq \mathscr{U}_\rho(B^{(l)} \times J^{(l)})$ the number of exceedances in $V_\rho^\circ(B^{(l)})$ i.e.

$$\mathscr{U}'_{V_\rho^\circ}(B^{(l)} \times J^{(l)}) = \sum_{\mathbf{i} \in V_\rho^\circ(B^{(l)})} \mathbb{1}_{a_\rho^{-1}(M_{f,\mathbf{i}}(B^{(l)}) - b_\rho) \in J^{(l)}}.$$

Let $l \in \{1, \ldots, L\}$ be fixed. Since B_l is a rectangular solid in $\mathbf{C}^{(W)}$ which is covered with at most N_ρ sub-cubes $\mathbf{i}$, we have $\#S_\rho(B^{(l)}) \leq c \cdot N_\rho^{(d-1)/d}$. This shows that

$$\mathbb{P}\left(\mathscr{U}'_{V_\rho^\circ}(B^{(l)} \times J^{(l)}) \neq \mathscr{U}'_{V_\rho}(B^{(l)} \times J^{(l)})\right) \leq \#S_\rho(B^{(l)}) \cdot \mathbb{P}\left(M_{f,\mathbf{i}} > v_\rho\right) = O\left(N_\rho^{-1/d}\right)$$

according to (3.2.12) and Condition (TCP). Thanks to (3.2.26a), we deduce that

$$\mathbb{P}\left(\mathscr{U}_\rho(B^{(l)} \times J^{(l)}) = 0\right) - \mathbb{P}\left(\mathscr{U}'_{V_\rho^\circ}(B^{(l)} \times J^{(l)}) = 0\right) \underset{\rho\to\infty}{\longrightarrow} 0. \tag{3.2.30}$$

Moreover, conditional on A_ρ, the random variables $\mathscr{U}'_{V_\rho^\circ}(B^{(l)} \times J^{(l)})$, $l = 1, \ldots, L$ are independent since the rectangular solids B_l, $l = 1, \ldots, L$ are at distance higher than R. In particular, we get

$$\mathbb{P}\left(\bigcap_{l=1}^{L} \mathscr{U}'_{V_\rho^\circ}(B^{(l)} \times J^{(l)}) = 0 \,|A_\rho\right) = \prod_{l=1}^{L} \mathbb{P}\left(\mathscr{U}'_{V_\rho^\circ}(B^{(l)} \times J^{(l)}) = 0 \,|A_\rho\right).$$

Lemma 3.2.5 is a consequence of the previous equality, the convergence (3.2.30) and the fact that

$$\mathbb{P}\left(\#\Phi_\rho \cap \mathscr{P} = 0\right) = \mathbb{P}\left(\bigcap_{l=1}^{L} \{\mathscr{U}_\rho(B^{(l)} \times J^{(l)}) = 0\}\right).$$

□

Remark 3.2.6. When Condition 2 does not hold, Lemma 3.2.5 remains true when $\mathscr{P} = \bigsqcup_{l=1}^{L} B^{(l)} \times (s_l, \infty)$. This comes from the fact that the left-hand side of (3.2.26a) equals 0 when $J = (s, \infty)$. In the same spirit, we can show that if $B^{(1)}, \ldots, B^{(l)}$, $1 \leq l \leq L$ is a set of $L \geq 1$ disjoint Borel subsets included in W, we have:

$$\mathbb{P}\left(M_{f,\mathbf{W}_\rho} \leq v_\rho\right) - \prod_{l=1}^{L} \mathbb{P}\left(M_{f,\mathbf{B}_\rho^{(l)}} \leq v_\rho\right) \underset{\rho\to\infty}{\longrightarrow} 0 \tag{3.2.31}$$

where $\mathbf{B}_\rho^{(l)} = \rho^{1/d}\mathbf{B}^{(l)}$, $1 \leq l \leq L$. Let us note that the previous convergence holds for a threshold v_ρ which is *not necessarily* of the form $v_\rho = v_\rho(t) = a_\rho t + b_\rho$. We will use this remark in section 3.6.

Remark 3.2.7. The inequalities appearing in (3.1.4), (3.1.5) and Theorem 3.1.1 have to be reversed when we deal with the r smallest values. This fact will be extensively used in the rest of the paper.

In the three following sections, we apply Theorem 3.1.1 to derive the order statistics for different geometrical characteristics and random tessellations. For aesthetic reasons, we only investigate maxima and minima for the particular case $W = \mathbf{C}^{(W)} = [0,1]^d$ keeping in mind that these results can be generalized to order statistics and to any bounded set with $\lambda_d(W) \neq 0$. Up to a normalization, all the thresholds v_ρ can be written as $v_\rho = v_\rho(t) = a_\rho t + b_\rho$ (excepted in section 3.5) so that Theorem 3.1.2 is also available.

3.3 Extreme Values of a Poisson-Delaunay tessellation

Before applying Theorem 3.1.1 to different geometrical characteristics of a Poisson-Delaunay tessellation, we introduce some notations and preliminaries.

Notations

- Let z be a point in $\mathbf{R}^d$ and r be a positive real number. We denote by $B(z,r)$ and $S(z,r)$ the ball and the sphere of radius r centered in z. When $z=0$ and $r=1$, we denote by $\mathbf{S}^{d-1}=S(0,1)$ the unit sphere. Moreover, we denote by κ_d the volume of the unit ball i.e.

$$\kappa_d = \lambda_d(B(0,1)).$$

- Let C be a simplex in $\mathbf{R}^d$. We denote respectively by $B(C)$, $S(C)$, $z(C)$ and $R(C)$ the circumball, the circumsphere, the circumcenter and the circumradius of C.
- Let k be an integer and $x_1,\ldots,x_k$ be k points in $\mathbf{R}^d$ and let $f:\mathcal{K}_d\to\mathbf{R}$ be a measurable function.
  - We denote by $\mathbf{x}_{1:k}$ the k-tuple $(x_1,\ldots,x_k)$ and by $\{\mathbf{x}_{1:k}\}$ the set of points $\{x_1,\ldots,x_k\}$.
  - If r is a positive number, we define $r\mathbf{x}_{1:k}=(rx_1,\ldots,rx_k)$ and $r\{\mathbf{x}_{1:k}\}=\{rx_1,\ldots,rx_k\}$.
  - When $k=d+1$ and when the $d+1$ points $x_1,\ldots,x_{d+1}$ lie on a sphere, we denote by $\Delta(\mathbf{x}_{1:d+1})$ the convex hull of $x_1,\ldots,x_{d+1}$. Moreover, we define $f(\mathbf{x}_{1:d+1})$ as

$$f(\mathbf{x}_{1:d+1}) = f\left(\Delta(\mathbf{x}_{1:d+1})\right).$$

  In particular, $B(\mathbf{x}_{1:d+1})$, $S(\mathbf{x}_{1:d+1})$, $z(\mathbf{x}_{1:d+1})$, $R(\mathbf{x}_{1:d+1})$ and $\lambda_d(\mathbf{x}_{1:d+1})$ are respectively the circumball, the circumsphere, the circumcenter, the circumradius and the volume of the simplex $\Delta(\mathbf{x}_{1:d+1})$.
  - If $k\leq d+1$ and if $\{\mathbf{y}_{k+1:d+1}\}=\{y_{k+1},\ldots,y_{d+1}\}$ is a set of $d+1-k$ points in $\mathbf{R}^d$ such that $x_1,\ldots,x_k$ and $y_{k+1},\ldots,y_{d+1}$ lie on a sphere, we denote in the same spirit by $\Delta(\mathbf{x}_{1:k},\mathbf{y}_{k+1:d+1})$ the convex hull of $x_1,\ldots,x_k,y_{k+1},\ldots,y_{d+1}$. Moreover, we define $f(\mathbf{x}_{1:k},\mathbf{y}_{k+1:d+1})$ as

$$f(\mathbf{x}_{1:k},\mathbf{y}_{k+1:d+1}) = f\left(\Delta\left(\mathbf{x}_{1:k},\mathbf{y}_{k+1:d+1}\right)\right).$$

  - Finally, we denote by $d\sigma(u)$ the uniform distribution over the unit sphere $\mathbf{S}^{d-1}$ and $d\sigma(\mathbf{u}_{1:d+1})=d\sigma(u_1)\cdots d\sigma(u_{d+1})$.

Preliminaries Let χ be a locally finite subset of $\mathbf{R}^d$ such that each subset of size $n<d+1$ are affinely independent and no $d+2$ points lie on a sphere. If $d+1$ points $x_1,\ldots,x_{d+1}$ of χ lie on a sphere that contains no point of χ in its interior, then the convex hull of $x_1,\ldots,x_{d+1}$ is called a cell. The set of such cells defines a partition of $\mathbf{R}^d$ into simplices and such partition is called the Delaunay tessellation. Such model is the key ingredient of the first algorithm for computing the minimum spanning tree [133]. It is extensively used in medical image segmentation [137], in finite element method to build meshes [70] and is a powerful tool for reconstructing a $3D$ set from a discrete point set [127].

When $\chi=\mathbf{X}$ is a Poisson point process, we speak about Poisson-Delaunay tessellation and we denote this random tessellation by $\mathfrak{m}_{PDT}$. For each cell $C\in\mathfrak{m}_{PDT}$ which is a.s. a simplex, we define $z(C)$ as the circumcenter of C. The relation between the intensity γ of $\mathfrak{m}_{PDT}$ and the intensity $\gamma_{\mathbf{X}}$ of the underlying Poisson point process is given by (see section 7 in [98])

$$\gamma = \beta_d^{-1}\cdot\gamma_{\mathbf{X}}$$

where

$$\beta_d = \frac{(d^3+d^2)\Gamma\left(\frac{d^2}{2}\right)\Gamma^d\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d^2+1}{2}\right)\Gamma^d\left(\frac{d+2}{2}\right)2^{d+1}\pi^{\frac{d-1}{2}}}. \tag{3.3.1}$$

Without loss of generality, we assume that $\gamma = 1$ i.e.

$$\gamma_{\mathbf{X}} = \beta_d.$$

The window $\mathbf{W}_\rho = \rho^{1/d}[0,1]^d$ is partitioned into N_ρ sub-cubes $\mathbf{i} \in V_\rho$ where

$$N_\rho = \left\lfloor \frac{\rho}{2\log\rho} \right\rfloor.$$

Moreover, we define the event A_ρ as

$$A_\rho = \bigcap_{i \in V_\rho} \{\mathbf{X} \cap \mathbf{i} \neq \varnothing\}. \tag{3.3.2}$$

Lemma 3.3.1. *The event A_ρ defined in* (3.3.2) *satisfies* Condition 1.

Proof of Lemma 3.3.1.

We use the same arguments as in the proof of Proposition 3 in [5]. Let $\mathbf{i} \in V_\rho$ be a sub-cube in $\mathbf{W}_\rho$ and let $C \in \mathfrak{m}_{PDT}$ such that $z(C) \in \mathbf{i}$. Since a $d+1$-tuple of points of $\mathbf{X}$ is a Delaunay cell if and only if its circumball contains no point in its interior, we have $R(C) = \min_{x \in \mathbf{X}}\{|z(C) - x|\}$. Moreover, conditional on A_ρ, there exists a point x_0 in $\mathbf{X} \cap \mathbf{i}$. In particular, we have $|z(C) - x_0| \leq \sqrt{d} \cdot c_\rho$ where c_ρ is the length of the sides of each sub-cube. Consequently, we obtain

$$R(C) \leq \sqrt{d} \cdot c_\rho. \tag{3.3.3}$$

This shows that the circumsphere $S(C)$ of C is included in $V_\rho(\mathbf{i}, D)$ where $D = \lfloor \sqrt{d} \rfloor + 1$ and

$$V_\rho(\mathbf{i}, D) = \{\mathbf{j} \in V_\rho, d(\mathbf{i}, \mathbf{j}) \leq D\}.$$

Indeed if not, there exists a point $y \in S(C)$ such that y is in a sub-cube $\mathbf{j}$ with $d(\mathbf{i}, \mathbf{j}) \geq D + 1$. This shows that $|y - z(C)| > (\lfloor \sqrt{d} \rfloor + 1) \cdot c_\rho$ and contradicts (3.3.3) since $R(C) = |y - z(C)|$.

Since $S(C)$ is included in $V_\rho(\mathbf{i}, D)$ for any cell $C \in \mathfrak{m}_{PDT}$ such that $z(C) \in \mathbf{i}$, this shows that $M_{f,\mathbf{i}}$ is $\mathbf{X} \cap V_\rho(\mathbf{i}, D)$ measurable. Because $d(A, B) > 2D$ implies that $\{\mathbf{i}, d(\mathbf{i}, A) < D\}$ and $\{\mathbf{i}, d(\mathbf{i}, B) < D\}$ are disjoint and because $\mathbf{X} \cap \{\mathbf{i}, d(\mathbf{i}, A) < D\}$ and $\mathbf{X} \cap \{\mathbf{i}, d(\mathbf{i}, B) < D\}$ are independent, the σ-algebras $\sigma(M_{f,\mathbf{i}}, \mathbf{i} \in A)$ and $\sigma(M_{f,\mathbf{i}}, \mathbf{i} \in B)$ are independent, yielding $R = 2D$.

Moreover the probability of the event A_ρ converges to 1. Indeed, since $\mathbf{X}$ is a Poisson point process, we get

$$\mathbb{P}(A_\rho^c) = \mathbb{P}\left(\bigcup_{\mathbf{i} \in V_\rho} \{\mathbf{X} \cap \mathbf{i} = \varnothing\}\right) \leq N_\rho e^{-\rho/N_\rho} = O\left((\log\rho)^{-1} \times \rho^{-1}\right). \tag{3.3.4}$$

□

The distribution function of the typical cell can be made explicit. Let $f : \mathcal{K}_d \to \mathbf{R}$ be a translation invariant function on the set of convex bodies. An integral representation of $f(\mathscr{C})$, due to Miles [95] (the proof can also be found in Theorem 10.4.4. of [130]), is given by

$$\mathbb{E}\left[f(\mathscr{C})\right] = \delta_d' \cdot \int_0^\infty \int_{(\mathbf{S}^{d-1})^{d+1}} r^{d^2-1} e^{-\delta_d r^d} \lambda_d(\mathbf{u}_{1:d+1}) f(r\mathbf{u}_{1:d+1}) d\sigma(\mathbf{u}_{1:d+1}) dr \tag{3.3.5}$$

where

$$\delta'_d = (d+1) \cdot \beta_d \text{ and } \delta_d = \kappa_d \cdot \beta_d. \tag{3.3.6}$$

We recall that a $(d+1)$-tuple of points of $\mathbf{X}$ is a Delaunay cell if its circumball contains no point of $\mathbf{X}$ in its interior. This justifies the exponential term since it is the probability that $\mathbf{X} \cap B(0,r)$ is empty. Thanks to (3.3.5), the typical cell can be built explicitly: it is a random simplex inscribed in the ball $B(0,r)$ such that the vector $\mathbf{u}_{1:d+1}$ is independent of r and has a density proportional to the volume of the simplex $\Delta(\mathbf{u}_{1:d+1})$.

For practical reasons, we write below a generic lemma which gives an integral representation of the function $G_2(\cdot)$. To do it, we introduce some notations. As defined in (3.1.5), $G_2(\cdot)$ brings up two cells Δ_1, Δ_2 that are two different simplices such that $f(\Delta_i) > v_\rho$ and $z(\Delta_i) \in \mathfrak{C}_\rho$, $i = 1, 2$. The intersection of these cells is a k-dimensional simplex with $0 \leq k \leq d-1$. Translating the circumcenter of the cell which has the largest circumradius say Δ_1 at the origin, the cells can we written as $\Delta_1 = \Delta(r\mathbf{u}_{1:d+1})$ and $\Delta_2 = \Delta(r\mathbf{u}_{1:k}, \mathbf{y}_{k+1:d+1})$ with $r \geq 0$, $u_1, \ldots, u_{d+1} \in \mathbf{S}^{d-1}$ and $y_{k+1}, \ldots, y_{d+1} \in \mathbf{R}^d$. We consider two properties $\mathscr{P}_1, \mathscr{P}_2$ that are

$$\mathscr{P}_1 : f(r\mathbf{u}_{1:k}, \mathbf{y}_{k+1:d+1}) > v_\rho, R(r\mathbf{u}_{1:k}, \mathbf{y}_{k+1:d+1}) \leq r \text{ and } z(r\mathbf{u}_{1:k}, \mathbf{y}_{k+1:d+1}) \in \mathfrak{C}_\rho. \tag{3.3.7a}$$

$$\mathscr{P}_2 : y_j \notin B(r\mathbf{u}_{1:d+1}) \text{ and } ru_j \notin B(r\mathbf{u}_{1:k}, \mathbf{y}_{k+1:d+1}) \text{ for all } j = k+1, \ldots, d+1. \tag{3.3.7b}$$

The first property concerns the cell Δ_2 which has the smallest circumradius whereas the second property means that the two simplices are Delaunay cells. Moreover, introduce the set

$$E_{k,r,\mathbf{u}_{1:d+1}} = \{\mathbf{y}_{k+1:d+1} \in (\mathbf{R}^d)^{d+1-k} \text{ satisfying } \mathscr{P}_1 \text{ and } \mathscr{P}_2\}. \tag{3.3.8}$$

At last, in the same spirit as (3.3.5), we consider the volume of the union of the two circumballs i.e.

$$\lambda_d^{(\cup)}(r, \mathbf{u}_{1:k}, \mathbf{y}_{k+1:d+1}) = \lambda_d\left(B(0,r) \cup B(r\mathbf{u}_{1:k}, \mathbf{y}_{k+1:d+1})\right).$$

We are now prepared to state the generic lemma.

Lemma 3.3.2. *Let $\mathfrak{m}_{PDT}$ be a Poisson-Delaunay tessellation of intensity $\gamma = 1$. Then*

$$G_2(\rho) = 2 \cdot \sum_{k=0}^{d} G_{2,k}(\rho) \tag{3.3.9}$$

where

$$G_{2,k}(\rho) = \rho \int_0^\infty \int_{(\mathbf{S}^{d-1})^{d+1}} \int_{(\mathbf{R}^d)^{d+1-k}} g_{2,k}(\rho, r, \mathbf{u}_{1:d+1}, \mathbf{y}_{k+1:d+1}) dr d\sigma(\mathbf{u}_{1:d+1}) d\mathbf{y}_{k+1:d+1} \tag{3.3.10}$$

and

$$\begin{aligned} & g_{2,k}(\rho, r, \mathbf{u}_{1:d+1}, \mathbf{y}_{k+1:d+1}) \\ & \quad = r^{d^2-1} e^{-\beta_d \lambda_d^{(\cup)}(r, \mathbf{u}_{1:k}, \mathbf{y}_{k+1:d+1})} \lambda_d(\mathbf{u}_{1:d+1}) \mathbb{1}_{f(r\mathbf{u}_{1:d+1}) > v_\rho} \mathbb{1}_{E_{k,r,\mathbf{u}_{1:d+1}}}(\mathbf{y}_{k+1:d+1}). \end{aligned} \tag{3.3.11}$$

Proof of Lemma 3.3.2. This will be sketched since it in the same spirit as in the proof of (3.3.5). Considering that the intersection of the two Delaunay cells Δ_1, Δ_2 which appear in

(3.1.5) is a k-dimensional simplex with $0 \leq k \leq d$ and assuming that $R(\Delta_1) \geq R(\Delta_2)$, we have

$$\begin{aligned}&\mathbb{P}\left(f(\mathscr{C}) > v_\rho\right)\\&= 2\sum_{k=0}^{d}\mathbb{E}\Bigg[\sum_{\substack{(x_1,\ldots,x_{d+1})_{\neq}\in\mathbf{X}^{d+1}\\(y_1,\ldots,y_k)_{\neq}\in\mathbf{X}^k}}\mathbb{1}_{f(\mathbf{x}_{d+1})>v_\rho}\mathbb{1}_{f(\mathbf{x}_{1:k},\mathbf{y}_{k+1:d+1})>v_\rho}\mathbb{1}_{R(\mathbf{x}_{1:d+1})\geq R(\mathbf{x}_{1:k},\mathbf{y}_{k+1:d+1})}\\&\qquad\qquad\times\mathbb{1}_{\mathbf{X}\cap B^{(\cup)}(\mathbf{x}_{1:d+1},\mathbf{y}_{k+1:d+1})-\{\mathbf{x}_{1:d+1}\}\cup\{\mathbf{y}_{k+1:d+1}\}}=\varnothing\Bigg].\end{aligned}$$

where $B^{(\cup)}(\mathbf{x}_{1:d+1},\mathbf{y}_{k+1:d+1}) = B(\mathbf{x}_{1:d+1}) \cup B(\mathbf{x}_{1:k},\mathbf{y}_{k+1:d+1})$. It results of Slivnyak's formula (see e.g. Theorem 3.3.5 in [130]) that

$$\begin{aligned}&\mathbb{P}\left(f(\mathscr{C}) > v_\rho\right)\\&= 2\sum_{k=0}^{d}\int_{(\mathbf{R}^d)^{d+1-k}}\int_{(\mathbf{R}^d)^{d+1}}\mathbb{1}_{f(\mathbf{x}_{d+1})>v_\rho}\mathbb{1}_{f(\mathbf{x}_{1:k},\mathbf{y}_{k+1:d+1})>v_\rho}\mathbb{1}_{R(\mathbf{x}_{1:d+1})\geq R(\mathbf{x}_{1:k},\mathbf{y}_{k+1:d+1})}\\&\qquad\times\mathbb{P}\left(\#\mathbf{X}\cap\left(B(\mathbf{x}_{1:d+1})\cup B(\mathbf{x}_{1:k},\mathbf{y}_{k+1:d+1})\right)=0\right)d\mathbf{x}_{1:d+1}d\mathbf{y}_{k+1:d+1}.\end{aligned}$$

We conclude the proof noting that $\#\mathbf{X} \cap B^{(\cup)}(\mathbf{x}_{1:d+1},\mathbf{y}_{k+1:d+1})$ is Poisson distributed of mean $\beta_d\lambda_d\left(B^{(\cup)}(\mathbf{x}_{1:d+1},\mathbf{y}_{k+1:d+1})\right)$ and using for all $y_{k+1},\ldots,y_{d+1}$ the (Blaschke-Petkantschin type) change of variables

$$\begin{aligned}\phi_1 :&\mathbf{R}_+ \times \mathbf{R}^d \times (\mathbf{S}^{d-1})^{d+1} \longrightarrow (\mathbf{R}^d)^{d+1}\\&(r, z, \mathbf{u}_{1:d+1}) \longmapsto \mathbf{x}_{1:d+1} \text{ with } x_i = z + ru_i\end{aligned}\tag{3.3.12}$$

where the Jacobian matrix is given by $|D\phi_1(r,z,\mathbf{u}_{1:d+1})| = r^{d^2-1}\lambda_d(\mathbf{u}_{1:d+1})$.

□

In Lemma 3.3.2, we have assumed that $R\left(r\mathbf{u}_{1:k},\mathbf{y}_{k+1:d+1}\right)$ is less than $R(r\mathbf{u}_{1:d+1})$. It overcomes the difficulty to consider elongated cells. This property will be needed in sections 3.3.2 and 3.3.3 but not in section 3.3.1 since we consider small circumradii.

3.3.1 Minimum of the circumradii

Let us recall that $R(C)$ denotes the circumradius of the cell $C \in \mathfrak{m}_{PDT}$. In this paragraph, we investigate the minimum

$$R_{\min,PDT}(\rho) = \min_{\substack{C\in\mathfrak{m}_{PDT},\\ z(C)\in\mathbf{W}_\rho}} R(C).$$

The asymptotic behaviour of $R_{\min,PDT}(\rho)$ is given in the following proposition.

Proposition 3.3.3. *Let $\mathfrak{m}_{PDT}$ be a Poisson-Delaunay tessellation of intensity $\gamma = 1$ in $\mathbf{R}^d$, $d \geq 2$. Then for all $t \geq 0$*

$$\left|\mathbb{P}\left(\alpha_{d,1}^{1/d}\rho^{1/d}R_{\min,PDT}(\rho)^d \geq t\right) - e^{-t^d}\right| = O\left(\rho^{-1/d}\right)\tag{3.3.13}$$

where

$$\alpha_{d,1} = \frac{\delta_d^d}{d!} = \frac{(\kappa_d \beta_d)^d}{d!} = \frac{1}{d!} \cdot \left(\frac{(d^3+d^2)\Gamma\left(\frac{d^2}{2}\right)\Gamma^d\left(\frac{d+1}{2}\right)\pi^{1/2}}{2^{d+1}\Gamma\left(\frac{d^2+1}{2}\right)\Gamma^{d+1}\left(\frac{d+2}{2}\right)} \right)^d .$$

The asymptotic behaviour of the maximum of circumradii has been investigated in [19] and will be recalled in section 3.6.

Proof of Proposition 3.3.3. First, we give the asymptotic behaviour of the distribution function of $R(\mathscr{C})$. According to (3.3.5), the random variable $R(\mathscr{C})^d$ is Gamma distributed of parameters $\left(d^2, \delta_d^{-1}\right)$. Thanks to consecutive integration by parts, this provides that

$$\mathbb{P}(R(\mathscr{C}) < v) = \sum_{i=d}^{\infty} \frac{1}{i!} (\delta_d v^d)^i e^{-\delta_d v^d} \tag{3.3.14}$$

for all $v \geq 0$. A Taylor approximation of the right-hand side when v is small shows that $|\mathbb{P}(R(\mathscr{C}) < v) - \alpha_{d,1} \cdot v^{d^2}|$ is of order v^{d^2+d}. Hence, taking for all $t \geq 0$

$$v_\rho = v_\rho(t) = \left(\alpha_{d,1}^{-1}\rho^{-1}\right)^{1/d^2} t^{1/d} \tag{3.3.15}$$

we obtain

$$G_1(\rho) = |\rho\mathbb{P}(R(\mathscr{C}) < v_\rho) - t^d| = O\left(\rho^{-1/d}\right). \tag{3.3.16}$$

To calculate the order of $G_2(\rho)$, it is enough to give a suitable upper bound of $G_{2,k}(\rho)$ for all $k = 0, \ldots, d$ according to Lemma 3.3.2. Bounding the exponential in (3.3.11) by 1 (a suitable estimate when considering small cells) and $\lambda_d(\mathbf{u}_{1:d+1})$ by a constant, we deduce for all $r \in \mathbf{R}_+$, $\mathbf{u}_{1:d+1} \in (\mathbf{S}^{d-1})^{d+1}$ and $\mathbf{y}_{k+1:d+1} \in (\mathbf{R}^d)^{d+1-k}$ that

$$g_{2,k}(\rho, r, \mathbf{u}_{1:d+1}, \mathbf{y}_{k+1:d+1}) \leq c \cdot r^{d^2-1} \mathbb{1}_{r < v_\rho} \mathbb{1}_{E_{k,r,\mathbf{u}_{1:d+1}}}(\mathbf{y}_{k+1:d+1}). \tag{3.3.17}$$

When $k = 0$, we bound $\mathbb{1}_{E_{0,r,\mathbf{u}_{1:d+1}}}(\mathbf{y}_{1:d+1})$ by $\mathbb{1}_{R(\mathbf{y}_{1:d+1}) < v_\rho} \cdot \mathbb{1}_{z(\mathbf{y}_{1:d+1}) \in \mathfrak{C}_\rho}$. We can omit the last condition in (3.3.7a) and the two conditions in (3.3.7b) since having a small circumradius almost guarantees that they are satisfied. Integrating the right-hand side of (3.3.17) and taking the same change of variables as in (3.3.12) i.e. $y_i = z' + r'u_i'$, $i = 1, \ldots, d+1$, we deduce from (3.3.10) and (3.3.15) that

$$G_{2,0}(\rho) \leq c \cdot \rho \int_0^{v_\rho} r^{d^2-1} dr \times \lambda_d(\mathfrak{C}_\rho) \int_0^{v_\rho} r'^{d^2-1} dr' = O\left(\log \rho \cdot \rho^{-1}\right). \tag{3.3.18}$$

When $k = 1, \ldots, d$, we use the fact that $R(r\mathbf{u}_{1:k}, \mathbf{y}_{k+1:d+1}) < v_\rho \Longrightarrow y_i \in B(ru_1, 2v_\rho)$ for all $i = k+1, \ldots, d+1$. Bounding $\mathbb{1}_{E_{k,r,\mathbf{u}_{1:d+1}}}(\mathbf{y}_{k+1:d+1})$ by $\mathbb{1}_{y_{k+1},\ldots,y_{d+1} \in B(ru_1, 2v_\rho)}$ and integrating (3.3.17), we deduce from (3.3.10) that

$$\begin{aligned} G_{2,k}(\rho) &\leq c \cdot \rho \int_0^{v_\rho} \int_{\mathbf{S}^{d-1}} \int_{(\mathbf{R}^d)^{d+1}} r^{d^2-1} \mathbb{1}_{y_{k+1},\ldots,y_{d+1} \in B(ru_1, 2v_\rho)} dr d\sigma(u_1) d\mathbf{y}_{k+1:d+1} \\ &\leq c \cdot \rho \int_0^{v_\rho} r^{d^2-1} dr \times v_\rho^{d(d+1-k)} = O\left(\rho^{-(d+1-k)/d}\right). \end{aligned} \tag{3.3.19}$$

Since $k = 0, \ldots, d$, the right-hand side of (3.3.19) is less than $\rho^{-1/d}$ for ρ large enough. Indeed, $G_{2,k}(\rho)$ is maximal when $k = d$ i.e. when the two distinct Delaunay cells have d common vertices. From (3.3.18),(3.3.19) and (3.3.9) we deduce that

$$G_2(\rho) = O\left(\rho^{-1/d}\right). \tag{3.3.20}$$

The rate of convergence (3.3.13) is now a direct consequence of (3.3.16), (3.3.20) and Theorem 3.1.1. □

When $d = 1$, the rate of convergence in (3.3.13) is $\log\rho \cdot \rho^{-1}$ since this is the order of $\mathbb{P}(A_\rho)$ and N_ρ^{-1} which appear in Theorem 3.1.1.

Let us remark that a slightly weaker version of Proposition 3.3.3 in $\mathbf{R}^d$ could have been deduced from a theorem due Schulte and Thäle (see Theorem 1.1 in [132]). It comes from the fact that $R_{\min,PDT}(\rho)$ can be written as a minimum of a U-statistic. More precisely

$$R_{\min,PDT}(\rho) = \min_{\substack{\mathbf{x}_{1:d+1}\in\mathbf{X}^{d+1},\\ z(\mathbf{x}_{1:d+1})\in\mathbf{W}_\rho}} R(\mathbf{x}_{1:d+1}).$$

Indeed, if a simplex induced by a set of $(d+1)$ distinct points $\mathbf{x}_{1:d+1}$ of $\mathbf{X}$ minimizes the circumradius, it is necessarily a Delaunay cell: otherwise, the circumball $B(\mathbf{x}_{1:d+1})$ contains a point of $\mathbf{X}$ in its interior which contradicts the minimality of $R(\mathbf{x}_{1:d+1})$. Nevertheless, the rate of convergence $O\left(\rho^{-1/d}\right)$ of Proposition 3.3.3 is more accurate than the rate deduced from Theorem 1.1. in [132] since the latter is of order $O\left(\rho^{-1/2d}\right)$. To the best of our knowledge, the convergence of the point process provided by Theorem 3.1.2 applied to the circumscribed radius of Delaunay cells is new.

3.3.2 Maximum of the areas, $d = 2$

Here and in the subsequent subsection, we investigate the extremes of the areas of a planar Poisson-Delaunay tessellation of intensity 1. The extension to higher dimension would be intricate since the integral formula for the distribution function of the volume of the typical cell becomes intractable. The intensity of the underlying Poisson point process is

$$\gamma_{\mathbf{X}} = \beta_2 = \frac{1}{2}. \tag{3.3.21}$$

In this subsection, we investigate the maximum of the areas i.e.

$$A_{\max,PDT}(\rho) = \max_{\substack{C\in\mathfrak{m}_{PDT},\\ z(C)\in\mathbf{W}_\rho}} \lambda_2(C).$$

The following proposition shows that $A_{\max,PDT}(\rho)$ is of order $\log\rho$.

Proposition 3.3.4. *Let $\mathfrak{m}_{PDT}$ be a Poisson-Delaunay tessellation of intensity $\gamma = 1$ in $\mathbf{R}^2$. Then for all $t \in \mathbf{R}$*

$$\left|\mathbb{P}\left(\alpha_2 A_{\max,PDT}(\rho) - \log\left(\frac{3}{2}\rho\right) \leq t\right) - e^{-e^{-t}}\right| = O\left(1/\log\rho\right) \tag{3.3.22}$$

where

$$\alpha_2 = \frac{2\pi}{3\sqrt{3}}. \tag{3.3.23}$$

Proof of Proposition 3.3.4. Thanks to (3.3.5), the distribution function of $\lambda_2(\mathscr{C})$ can be made explicit. Indeed, an integral representation of $\mathbb{P}\left(\lambda_2(\mathscr{C})>v\right)$ due to Rathie (see (3.2) in [121]) is

$$\mathbb{P}(\lambda_2(\mathscr{C})>v)=\frac{6}{\pi}\int_{\alpha_2\beta_2 v}^{\infty} xK_{1/6}^2(x)dx \tag{3.3.24}$$

where $K_{1/6}(\cdot)$ denotes the modified Bessel function of order 1/6. When x goes to infinity, a Taylor approximation of $K_{1/6}(x)$ is given by (see Formula 9.7.2, page 378 in [1])

$$K_{1/6}(x)=\sqrt{\frac{\pi}{2x}}e^{-x}\left(1+O\left(\frac{1}{x}\right)\right). \tag{3.3.25}$$

We deduce from (3.3.21), (3.3.24) and (3.3.25) that for v large enough

$$\left|\mathbb{P}\left(\lambda_2(\mathscr{C})>v\right)-\frac{3}{2}e^{-\alpha_2 v}\right|\leq c\cdot\int_{\frac{1}{2}\alpha_2 v}^{\infty}\frac{e^{-2x}}{x}dx\leq c\cdot\frac{e^{-\alpha_2 v}}{v}. \tag{3.3.26}$$

Taking for all $t\in\mathbf{R}$

$$v_\rho=v_\rho(t)=\frac{1}{\alpha_2}\left(\log\left(\frac{3}{2}\rho\right)+t\right). \tag{3.3.27}$$

we obtain from (3.3.26) that

$$G_1(\rho)=\left|\rho\mathbb{P}\left(\lambda_2(\mathscr{C})>v_\rho\right)-e^{-t}\right|=O\left(1/\log\rho\right). \tag{3.3.28}$$

In the rest of the proof, we give a suitable upper bound of $G_2(\rho)$. Taking $f(\cdot)=\lambda_2(\cdot)$ in (3.3.11) and using the facts that $\lambda_2(r\mathbf{u}_{1:3})=r^2\lambda_2(\mathbf{u}_{1:3})$ and $\lambda_2(\mathbf{u}_{1:3})\leq c$, we have

$$g_{2,k}(\rho,r,\mathbf{u}_{1:3},\mathbf{y}_{k+1:3})\leq c\cdot r^3 e^{-\frac{1}{2}\lambda_d^{(\cup)}(r,\mathbf{u}_{1:k},\mathbf{y}_{k+1:3})}\mathbb{1}_{r^2\lambda_2(\mathbf{u}_{1:3})>v_\rho}\mathbb{1}_{E_{k,r,\mathbf{u}_{1:3}}}(\mathbf{y}_{k+1:3}). \tag{3.3.29}$$

for all $k=0,1,2$. To bound $g_{2,k}(\cdot)$, the key idea is to give a suitable lower bound of the area of the union of two disks (see Figure 3.1 (a)). This is provided in the following fundamental lemma.

Lemma 3.3.5. *Let* $\{\mathbf{x}_{1:3}\}=\{x_1,x_2,x_3\}$ *and* $\{\mathbf{x}'_{1:3}\}=\{x'_1,x'_2,x'_3\}$ *be two 3-tuples of points in* $\mathbf{R}^2$ *such that* $x_i\notin B(\mathbf{x}'_{1:3})$ *and* $x'_j\notin B(\mathbf{x}_{1:3})$ *for all* $i,j=1,2,3$. *Let us assume that* $R:=R(\mathbf{x}_{1:3})\geq R(\mathbf{x}'_{1:3})$. *Then*

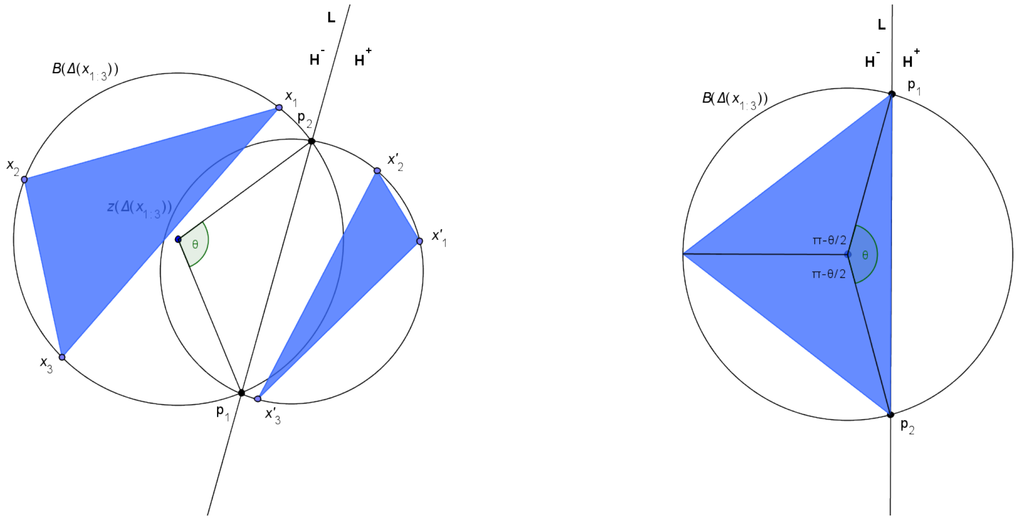
$$\lambda_2\left(B(\mathbf{x}_{1:3})\cup B(\mathbf{x}'_{1:3})\right)\geq\left(\frac{\pi}{2}-1\right)R^2+\lambda_2(\mathbf{x}_{1:3})+\lambda_2(\mathbf{x}'_{1:3}). \tag{3.3.30}$$

Proof of Lemma 3.3.5. Let $\{\mathbf{x}_{1:3}\}$ and $\{\mathbf{x}'_{1:3}\}$ be two 3-tuples in $\mathbf{R}^2$. If $\lambda_2(B(\mathbf{x}_{1:3}))\cap B(\mathbf{x}'_{1:3})$ equals 0, we have

$$\lambda_2(B(\mathbf{x}_{1:3})\cup B(\mathbf{x}'_{1:3}))=\lambda_2(B(\mathbf{x}_{1:3}))+\lambda_2(B(\mathbf{x}'_{1:3}))\geq\pi R^2+\lambda_2(\mathbf{x}'_{1:3}). \tag{3.3.31}$$

Moreover, the maximal area of a triangle inscribed in a ball of radius R is $\frac{3\sqrt{3}}{4}R^2$ which is the area of an equilateral triangle. In particular, we have $\lambda_2(\mathbf{x}_{1:3})\leq\frac{3\sqrt{3}}{4}R^2$. This together with (3.3.31) implies that

$$\lambda_2(B(\mathbf{x}_{1:3})\cup B(\mathbf{x}'_{1:3}))\geq\left(\pi-\frac{3\sqrt{3}}{4}\right)R^2+\lambda_2(\mathbf{x}_{1:3})+\lambda_2(\mathbf{x}'_{1:3})\geq\left(\frac{\pi}{2}-1\right)R^2+\lambda_2(\mathbf{x}_{1:3})+\lambda_2(\mathbf{x}'_{1:3})$$



(a) (b)

FIGURE 3.1 – (a). A union of two disks. (b). The triangle which maximizes the area.

If $B(\mathbf{x}_{1:3}) \cap B(\mathbf{x}'_{1:3})$ has non empty interior, the intersection of the circumspheres induced by the points $\mathbf{x}_{1:3}$ and $\mathbf{x}'_{1:3}$ is reduced to two points, say $p_1, p_2 \in \mathbf{R}^2$. Let us denote by $\mathbf{L}$ the affine line (p_1, p_2) and $\mathbf{H}^-$ (respectively $\mathbf{H}^+$) the half plane delimited by $\mathbf{L}$ and containing (respectively not containing) the circumcenter $z(\mathbf{x}_{1:3})$. Since $x_i \notin B(\mathbf{x}'_{1:3})$ and $x'_j \notin B(\mathbf{x}_{1:3})$, $i, j = 1, 2, 3$, the triangle $\Delta(\mathbf{x}'_{1:3})$ is included in $\mathbf{H}^+$. Hence

$$\begin{aligned} \lambda_2\left(B(\mathbf{x}_{1:3}) \cup B(\mathbf{x}'_{1:3})\right) &= \lambda_2\left(\left(B(\mathbf{x}_{1:3}) \cup B(\mathbf{x}'_{1:3})\right) \cap \mathbf{H}^-\right) + \lambda_2\left(\left(B(\mathbf{x}_{1:3}) \cup B(\mathbf{x}'_{1:3})\right) \cap \mathbf{H}^+\right) \\ &\geq \lambda_2(B(\mathbf{x}_{1:3}) \cap \mathbf{H}^-) + \lambda_2(\mathbf{x}'_{1:3}). \end{aligned} \tag{3.3.32}$$

In the rest of the proof, we provide a suitable lower bound of $\lambda_2(B(\mathbf{x}_{1:3}) \cap \mathbf{H}^-)$. To do it, we denote by $\theta \in [0, 2\pi]$ the angle $\angle p_1 z(\mathbf{x}_{1:3}) p_2$. Actually $\theta \in [0, \pi]$: this comes from the fact that $\lambda_2(B(\mathbf{x}_{1:3}) \cap \mathbf{H}^-) \geq \frac{\pi}{2} R^2$ since $R := R(\mathbf{x}_{1:3}) \geq R(\mathbf{x}'_{1:3})$. The area of the cap $B(\mathbf{x}_{1:3}) \cap \mathbf{H}^-$ is given by

$$\lambda_2(B(\mathbf{x}_{1:3}) \cap \mathbf{H}^-) = \left(\pi - \frac{1}{2}(\theta - \sin\theta)\right) R^2. \tag{3.3.33}$$

We discuss below two cases depending on θ.

If $\theta \in [0, 2\pi/3]$, we deduce from (3.3.33) that

$$\lambda_2(B(\mathbf{x}_{1:3}) \cap \mathbf{H}^-) \geq \left(\frac{2\pi}{3} + \frac{\sqrt{3}}{4}\right) R^2. \tag{3.3.34}$$

Since $\lambda_2(\mathbf{x}_{1:3})$ is less than $\frac{3\sqrt{3}}{4} R^2$, we deduce from (3.3.34) that

$$\lambda_2(B(\mathbf{x}_{1:3}) \cap \mathbf{H}^-) \geq \lambda_2(\mathbf{x}_{1:3}) + \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) R^2 \geq \lambda_2(\mathbf{x}_{1:3}) + \left(\frac{\pi}{2} - 1\right) R^2. \tag{3.3.35}$$

In that case, the inequality (3.3.30) results from (3.3.32) and (3.3.35).

If $\theta \in [2\pi/3, \pi]$, with a standard method of geometry, we can show that the maximal area of a triangle inscribed in $B(\mathbf{x}_{1:3}) \cap \mathbf{H}^-$, denoted by $M(\theta)$, is

$$M(\theta) = \left(\sin\frac{\theta}{2} + \frac{1}{2}\sin\theta\right) R^2. \tag{3.3.36}$$

Actually, the triangle which maximizes the area is isoscele with central angles $\pi - \theta/2, \pi - \theta/2$ and θ (see Figure 3.1 (b)). In particular, we have

$$\lambda_2(\mathbf{x}_{1:3}) \leq M(\theta). \tag{3.3.37}$$

We obtain from (3.3.33) and (3.3.36) that

$$\lambda_2(B(\mathbf{x}_{1:3}) \cap \mathbf{H}^-) \geq M(\theta) + \left(\frac{\pi}{2} - 1\right) R^2 + \left(\frac{\pi}{2} + 1 - \left(\frac{1}{2}\theta + \sin\frac{\theta}{2}\right)\right) R^2. \tag{3.3.38}$$

The last term of the right-hand side is a decreasing function on $[0, \pi]$. Its minimum equals 0 at $\theta = \pi$ i.e.

$$\frac{\pi}{2} + 1 - \left(\frac{1}{2}\theta + \sin\frac{\theta}{2}\right) \geq 0$$

for all $\theta \in [0, \pi]$. This shows that

$$\lambda_2(B(\mathbf{x}_{1:3}) \cap \mathbf{H}^-) \geq M(\theta) + \left(\frac{\pi}{2} - 1\right) R^2. \tag{3.3.39}$$

The inequality (3.3.30) is a direct consequence of (3.3.32), (3.3.39) and (3.3.37).

□

We can now derive an upper bound of $g_{2,k}(\cdot)$ for all $k = 0, 1, 2$. Indeed, if $\mathbf{y}_{k+1:3} \in E_{k,r,\mathbf{u}_{1:3}}$, where $E_{k,r,\mathbf{u}_{1:3}}$ has been defined in (3.3.8), the set of points $\{\mathbf{x}_{1:3}\} = \{r\mathbf{u}_{1:3}\}$ and $\{\mathbf{x}'_{1:3}\} = \{r\mathbf{u}_{1:k}, \mathbf{y}_{k+1:3}\}$ satisfies the assumptions of Lemma 3.3.5 since $R(r\mathbf{u}_{1:3}) = r$ and $R(r\mathbf{u}_{1:3}) \geq R(r\mathbf{u}_{1:k}, \mathbf{y}_{k+1:3})$. Using the fact that $B(r\mathbf{u}_{1:3}) = B(0, r)$, $\lambda_2(r\mathbf{u}_{1:3}) > v_\rho$ and $\lambda_2(r\mathbf{u}_{1:k}, \mathbf{y}_{k+1:3}) > v_\rho$, we deduce from (3.3.29) and (3.3.30) that

$$g_{2,k}(\rho, r, \mathbf{u}_{1:3}, \mathbf{y}_{k+1:3}) \leq c \cdot r^3 e^{-\frac{1}{2}\left(\left(\frac{\pi}{2}-1\right)r^2 + 2v_\rho\right)} \mathbb{1}_{r^2\lambda_2(\mathbf{u}_{1:3}) > v_\rho} \mathbb{1}_{E_{k,r,\mathbf{u}_{1:3}}}(\mathbf{y}_{k+1:3}). \tag{3.3.40}$$

Since $\frac{3\sqrt{3}}{4}r^2 \geq r^2\lambda_2(\mathbf{u}_{1:3})$, we deduce from (3.3.23) and (3.3.27) that

$$r^2\lambda_2(\mathbf{u}_{1:3}) > v_\rho \Longrightarrow r^2 > 4v_\rho/3\sqrt{3} \Longrightarrow r > \left(2\left(\log\rho + c\right)/\pi\right)^{1/2} \tag{3.3.41}$$

where $c = \log(3/2) + t$. Integrating the right-hand side on $\mathbf{y}_{k+1:3}$, we obtain

$$G_{2,k}(\rho) \leq c \cdot \rho \int_{(2(\log\rho + c)/\pi)^{1/2}}^{\infty} \int_{(\mathbf{S}^1)^3} r^3 e^{-\frac{1}{2}\left(\left(\frac{\pi}{2}-1\right)r^2 + 2v_\rho\right)} \times \lambda_{2(3-k)}(E_{k,r,\mathbf{u}_{1:3}}) dr d\sigma(\mathbf{u}_{1:3}). \tag{3.3.42}$$

The following lemma gives a uniform upper bound of $\lambda_{2(3-k)}(E_{k,r,\mathbf{u}_{1:3}})$.

Lemma 3.3.6. *Let $\mathbf{u}_{1:3} \in (\mathbf{S}^1)^k$ and $r > \left(2\left(\log\rho + c\right)/\pi\right)^{1/2}$. Then for ρ large enough*

$$\lambda_{2(3-k)}(E_{k,r,\mathbf{u}_{1:3}}) \leq c \cdot r^{2(3-k)}. \tag{3.3.43}$$

Proof of Lemma 3.3.6. We discuss three cases that depend on k.

If $k = 2$, we show that $E_{2,r,\mathbf{u}_{1:3}}$ is included in a ball of radius r up to a multiplicative constant and centered at 0. Let y_3 be in $E_{2,r,\mathbf{u}_{1:3}}$. From the triangle inequality, we have

$$|y_3| \leq |y_3 - z(r\mathbf{u}_{1:2}, y_3)| + |z(r\mathbf{u}_{1:2}, y_3)| \leq r + \mathrm{diam}(\mathfrak{C}_\rho). \tag{3.3.44}$$

The last inequality comes from the fact that $|y_3 - z(r\mathbf{u}_{1:2}, y_3)|$ is the circumradius of $\Delta(r\mathbf{u}_{1:2}, y_3)$, which is less than r, and the fact that $z(r\mathbf{u}_{1:2}) \in \mathfrak{C}_\rho$. Moreover

$$\mathrm{diam}(\mathfrak{C}_\rho) \leq c \cdot (\log \rho)^{1/2} \leq c \cdot r \tag{3.3.45}$$

where the last inequality holds for ρ large enough since $r > (2(\log \rho + c)/\pi)^{1/2}$ converges to ∞ as ρ goes to infinity. We deduce from (3.3.44) and (3.3.45) that

$$|y_3| \leq c \cdot r \tag{3.3.46}$$

The upper bound (3.3.46) shows that $E_{2,r,\mathbf{u}_{1:3}} \subset B(0, c \cdot r)$. In particular,

$$\lambda_2(E_{2,r,\mathbf{u}_{1:3}}) \leq c \cdot r^2.$$

If $k = 0$ or $k = 1$, proceeding along the same lines as in the case $k = 2$, we show that $E_{k,r,\mathbf{u}_{1:3}} \subset B(0, c \cdot r)^{3-k}$ and consequently we get $\lambda_{2(3-k)}(E_{k,r,\mathbf{u}_{1:3}}) \leq c \cdot r^{2(3-k)}$.

□

We can now derive an upper bound of $G_{2,k}(\rho)$. Indeed, integrating $\mathbf{u}_{1:3}$ on $(\mathbf{S}^1)^3$, we deduce from (3.3.42) and (3.3.43) that

$$G_{2,k}(\rho) \leq c \cdot \rho \int_{(2(\log \rho + c)/\pi)^{1/2}}^{\infty} r^{9-k} e^{-\frac{1}{2}\left(\left(\frac{\pi}{2}-1\right)r^2 + 2v_\rho\right)} dr.$$

Integrating the right-hand side, we obtain from (3.3.27) that

$$G_{2,k}(\rho) \leq c \cdot (\log \rho)^{8-2k} \rho^{(\pi+2-3\sqrt{3})/2\pi} = O\left((\log \rho)^8 \rho^{-\epsilon}\right) \tag{3.3.47}$$

with $\epsilon = -\pi - 2 + 3\sqrt{3} > 0$. Proposition 2 results of (3.3.47), Lemma 3.3.2 and Theorem 3.1.1.
□

Lemma 3.3.5 provides the main tool of the proof. Note that the inequality (3.3.30) is obvious when we replace $\frac{\pi}{2} - 1$ by a constant $\alpha \leq \pi - \frac{3\sqrt{3}}{2}$. Indeed, if $\Delta(\mathbf{x}_{1:3})$ and $\Delta(\mathbf{x}'_{1:3})$ are two triangles with $R := R(\mathbf{x}_{1:3}) \geq R(\mathbf{x}'_{1:3})$, a trivial inequality is

$$\lambda_2(B(\mathbf{x}_{1:3}) \cup B(\mathbf{x}'_{1:3})) \geq \pi R^2.$$

Consequently

$$\lambda_2(B(\mathbf{x}_{1:3}) \cup B(\mathbf{x}'_{1:3})) \geq \left(\pi - \frac{3\sqrt{3}}{2}\right) R^2 + \lambda_2(\mathbf{x}_{1:3}) + \lambda_2(\mathbf{x}'_{1:3})$$

since $\lambda_2(\mathbf{x}_{1:3})$ and $\lambda_2(\mathbf{x}'_{1:3})$ are less than $\frac{3\sqrt{3}}{4} R^2$. Nevertheless, the previous lower bound is not enough to guarantee that $G_{2,k}(\rho)$ converges to 0. The important fact in Lemma 3.3.5 is that we consider the more precise constant $\frac{\pi}{2} - 1 > \pi - \frac{3\sqrt{3}}{2}$.

Another remark deals with the shape of the cell maximizing the area. As we will see in Example 2 of section 3.6, the maximum of circumradii of a planar Poisson-Delaunay tessellation, denoted by $R_{\max,PDT}(\rho)$, is of order $(\delta_2^{-1}\log\rho)^{1/2} = (2\pi^{-1}\log\rho)^{1/2}$. Thanks to (3.3.4), this shows that $A_{\max,PDT}(\rho)$ equals asymptotically $\frac{3\sqrt{3}}{4}R^2_{\max,PDT}(\rho)$ which is the area of an equilateral triangle of circumradius $R_{\max,PDT}(\rho)$. It seems that the shape of the cell maximizing the area tends to that of an equilateral triangle. This fact can be connected to the D.G. Kendall's conjecture and the work of Hug and Schneider in [62].

3.3.3 Minimum of the areas, $d = 2$

In our third example, we calculate the asymptotic behaviour of the minimum of the areas of the cells of a Poisson-Delaunay tessellation (of intensity 1) in $\mathbf{R}^2$ i.e.

$$A_{\min,PDT}(\rho) = \min_{\substack{C\in\mathfrak{m}_{PDT},\\ z(C)\in\mathbf{W}_\rho}} \lambda_2(C).$$

The asymptotic behaviour is given in the following proposition.

Proposition 3.3.7. *Let $\mathfrak{m}_{PDT}$ be a Poisson-Delaunay tessellation of intensity $\gamma = 1$ in $\mathbf{R}^2$. Then for all $t \geq 0$*

$$\mathbb{P}\left(\alpha_3^{3/5}\rho^{3/5}A_{\min,PDT}(\rho) \geq t\right) \underset{\rho\to\infty}{\longrightarrow} e^{-t^{5/3}} \tag{3.3.48}$$

where

$$\alpha_3 = 2^{-2/3}\cdot 3^{-1/2}\cdot 5^{-1}\cdot \pi^{2/3}\cdot \Gamma(1/6)^2.$$

In [132], Schulte and Thäle investigate the behaviour of the smallest area S_ρ of all triangles that can be formed by three points of the Poisson point process i.e.

$$S_\rho = \min_{\substack{\mathbf{x}_{1:3}\in\mathbf{X}^3,\\ z(\mathbf{x}_{1:3})\in\mathbf{W}_\rho}} \lambda_2(\mathbf{x}_{1:3}).$$

The asymptotic behaviour of S_ρ is given by (see Theorem 2.5. in [132])

$$\mathbb{P}\left(\rho S_\rho \geq t\right) \underset{\rho\to\infty}{\longrightarrow} e^{-\beta t}$$

where β is a constant which can be made explicit. The previous limit compared to (3.3.48) shows that the smallest area of the Delaunay cells is much larger than the smallest area of all triangles.

Proof of Proposition 3.3.7. First, we calculate the asymptotic behaviour of the distribution function of $\lambda_2(\mathscr{C})$. Such function is given in (3.3.24). A Taylor expansion of the modified Bessel function of order 1/6 is given by (see Formula 9.6.9., page 375 in [1])

$$K_{1/6}(x) = 2^{-5/6}\Gamma(1/6)\,x^{-1/6} + o(x^{-1/6}). \tag{3.3.49}$$

This together with (3.3.24) and (3.3.21) shows that

$$\mathbb{P}\left(\lambda_2(\mathscr{C}) < v\right) = \frac{6}{\pi}\cdot 2^{-5/3}\Gamma(1/6)^2\int_0^{\alpha_2\beta_2 v}\left(x^{2/3} + o\left(x^{2/3}\right)\right)dx = \alpha_3\cdot v^{5/3} + o\left(v^{5/3}\right). \tag{3.3.50}$$

Taking for all $t \geq 0$

$$v_\rho = v_\rho(t) = (\alpha_3^{-1}\rho^{-1})^{3/5}t \tag{3.3.51}$$

we obtain

$$G_1(\rho) = |\rho\mathbb{P}\left(\lambda_2(\mathscr{C}) < v_\rho\right) - t^{5/3}| \underset{\rho\to\infty}{\longrightarrow} 0 \tag{3.3.52}$$

We investigate below the rate of convergence of $G_2(\rho)$. Taking $f(r\mathbf{u}_{1:3}) = r^2\lambda_2(\mathbf{u}_{1:3})$ and using the fact that $\lambda_2(B(0,r)\cup B(r\mathbf{u}_{1:k},\mathbf{y}_{k+1:3}))$ is greater than πr^2, for all $k = 0,1,2$, we have

$$g_{2,k}(\rho, r, \mathbf{u}_{1:3}, \mathbf{y}_{k+1:3}) \leq r^3 e^{-\frac{1}{2}\pi r^2}\lambda_2(\mathbf{u}_{1:3})\mathbb{1}_{r^2\lambda_2(\mathbf{u}_{1:3})<v_\rho}\mathbb{1}_{E_{k,r,\mathbf{u}_{1:3}}}$$

according to (3.3.11). Integrating on $\mathbf{y}_{1:3}$, this gives

$$G_{2,k}(\rho) \leq \rho\int_0^\infty\int_{(\mathbf{S}^1)^3} r^3 e^{-\frac{1}{2}\pi r^2}\lambda_2(\mathbf{u}_{1:3})\lambda_{2(3-k)}(E_{k,r,\mathbf{u}_{1:3}})\mathbb{1}_{r^2\lambda_2(\mathbf{u}_{1:3})<v_\rho}drd\sigma(\mathbf{u}_{1:3}). \tag{3.3.53}$$

As in the proof of Proposition 3.3.4, we derive a suitable upper bound of the volume of $E_{k,r,\mathbf{u}_{1:3}}$.

Lemma 3.3.8. *Let* $\mathbf{u}_{1:3} \in (\mathbf{S}^1)^3$ *and* $r \geq 0$. *Then*

$$\lambda_2(E_{2,r,\mathbf{u}_{1:3}}) \leq c\cdot v_\rho|u_1-u_2|^{-1} \tag{3.3.54a}$$

$$\lambda_4(E_{1,r,\mathbf{u}_{1:3}}) \leq c\cdot r^2 v_\rho \tag{3.3.54b}$$

$$\lambda_6(E_{0,r,\mathbf{u}_{1:3}}) \leq c\cdot\log\rho\cdot r^2 v_\rho. \tag{3.3.54c}$$

Proof of Lemma 3.3.8. Let y_3 be in $E_{2,r,\mathbf{u}_{1:3}}$. Since $R(r\mathbf{u}_{1:2}, y_3)$ is less than r, we have $|y_3 - ru_1| \leq 2R(r\mathbf{u}_{1:2}, y_3) \leq 2r$. In particular, we obtain

$$|y_3| \leq 3r \tag{3.3.55}$$

Moreover, the area of the triangle $\Delta(r\mathbf{u}_{1:2}, y_3)$ is given by

$$\lambda_2(r\mathbf{u}_{1:2}, y_3) = \frac{1}{2}r|u_1-u_2|\cdot\delta(y_3, \mathbf{L}(ru_1, ru_2)) \tag{3.3.56}$$

where $\mathbf{L}(ru_1, ru_2)$ is the affine line induced by the points $p_1 = ru_1$, $p_2 = ru_2$ and $\delta(y_3, \mathbf{L}(ru_1, ru_2))$ denotes the distance between this line and the point y_3. Since $\lambda_2(r\mathbf{u}_{1:2}, y_3) < v_\rho$, it results from (3.3.56) that

$$\delta(y_3, \mathbf{L}(ru_1, ru_2)) \leq \frac{2v_\rho}{r|u_1-u_2|}. \tag{3.3.57}$$

The inequalities (3.3.55) and (3.3.57) show that $E_{2,r,\mathbf{u}_{1:3}}$ is included in the intersection of a ball of radius $3r$ and a strip of width $\frac{4v_\rho}{r|u_1-u_2|}$ i.e.

$$\lambda_2(E_{2,r,\mathbf{u}_{1:3}}) \leq 6r\times\frac{4v_\rho}{r|u_1-u_2|} = c\cdot v_\rho|u_1-u_2|^{-1}.$$

Secondly, we bound $\lambda_4(E_{1,r,\mathbf{u}_{1:3}})$. Taking the change of variables $\phi_2 : \mathbf{R}_+\times\mathbf{S}^1 \longrightarrow \mathbf{R}^2$, $(s', u_2') \longmapsto y_2 = ru_1 + s'u_2'$ with Jacobian matrix $|D\phi_2(r', u_2')| = s'$, we obtain

$$\lambda_4(E_{1,r,\mathbf{u}_{1:3}}) \leq \int_0^{2r}\int_{\mathbf{S}^1}\int_{\mathbf{R}^2} s'\mathbb{1}_{\lambda_2(ru_1,ru_1+s'u_2',y_3)<v_\rho}\mathbb{1}_{R(ru_1,ru_1+s'u_2',y_3)\leq r}ds'd\sigma(u_2')dy_3. \tag{3.3.58}$$

The positive number s' is integrated on $[0, 2r]$. Indeed, the inequality $R(ru_1, ru_1 + s'u_2', y_3) \leq r$ implies that $s' = |(ru_1 + s'u_2') - ru_1| \leq 2r$. Proceeding along the same lines as in the proof of (3.3.54a), we show that y_3 belongs to the ball $B(0, 3r)$ and a strip of width $\frac{4v_\rho}{s'}$. Integrating (3.3.58) with respect to y_3, we deduce that

$$\lambda_4(E_{1,r,\mathbf{u}_{1:3}}) \leq 24 \int_0^{2r} \int_{\mathbf{S}^1} v_\rho r ds' d\sigma(u_2') = c \cdot r^2 v_\rho.$$

Finally, we bound $\lambda_6(E_{0,r,\mathbf{u}_{1:3}})$. Taking the same change of variables as in (3.3.12), we have

$$\begin{aligned}\lambda_6(E_{0,r,\mathbf{u}_{1:3}}) &\leq \int_{(\mathbf{R}^2)^3} \mathbb{1}_{z(\mathbf{y}_{1:3}) \in \mathfrak{C}_\rho} \mathbb{1}_{R(\mathbf{y}_{1:3}) < r} \mathbb{1}_{\lambda_2(\mathbf{y}_{1:3}) < v_\rho} d\mathbf{y}_{1:3} \\ &= \int_{\mathfrak{C}_\rho} \int_0^r \int_{(\mathbf{S}^1)^3} r'^3 \lambda_2(\mathbf{u}'_{1:3}) \mathbb{1}_{r'^2 \lambda_2(u_1', u_2', u_3') < v_\rho} dz' dr' d\sigma(\mathbf{u}'_{1:3}).\end{aligned}$$

Bounding $r'^3 \lambda_2(u_1', u_2', u_3')$ by $r' v_\rho$ and integrating with respect to $z' \in \mathfrak{C}_\rho$, $r' \in [0, r]$ and $\mathbf{u}'_{1:3} \in (\mathbf{S}^1)^3$, we show that $\lambda_6(E_{0,r,\mathbf{u}_{1:3}})$ is less than $c \cdot \lambda_2(\mathfrak{C}_\rho) r^2 v_\rho$ with $\lambda_2(\mathfrak{C}_\rho) \leq c \cdot \log \rho$. □

We can now derive a suitable upper bound of $G_{2,k}(\rho)$. Indeed, if $k = 0$, we deduce from (3.3.53) and (3.3.54c) that

$$\begin{aligned}G_{2,0}(\rho) &\leq c \cdot \log \rho \cdot \rho v_\rho \int_0^\infty \int_{(\mathbf{S}^1)^3} r^5 e^{-\frac{1}{2}\pi r^2} \lambda_2(\mathbf{u}_{1:3}) \mathbb{1}_{r^2 \lambda_2(\mathbf{u}_{1:3}) < v_\rho} dr d\sigma(\mathbf{u}_{1:3}) \\ &\leq c \cdot \log \rho \cdot \rho v_\rho^2 \int_0^\infty \int_{(\mathbf{S}^1)^3} r^3 e^{-\frac{1}{2}\pi r^2} dr d\sigma(\mathbf{u}_{1:3}).\end{aligned}$$

The integral of the right-hand side is bounded. Replacing v_ρ by $c \cdot \rho^{-3/5}$ according to (3.3.51), we show that $G_{2,0}(\rho)$ is less than $c \cdot \log \rho \cdot \rho^{-1/5}$. Proceeding along the same lines, when $k = 1$, we obtain that $G_{2,1}(\rho) \leq c \cdot \rho^{-1/5}$ according to (3.3.53) and (3.3.54b). Hence

$$G_{2,0}(\rho) = O\left(\log \rho \cdot \rho^{-1/5}\right) \text{ and } G_{2,1}(\rho) = O\left(\rho^{-1/5}\right). \tag{3.3.59}$$

Finally, if $k = 2$, we deduce from (3.3.53) and (3.3.54a) that

$$G_{2,2}(\rho) \leq c \cdot \rho v_\rho \int_0^\infty \int_{(\mathbf{S}^1)^3} r^3 e^{-\frac{1}{2}\pi r^2} \lambda_2(\mathbf{u}_{1:3}) |u_1 - u_2|^{-1} \mathbb{1}_{r^2 \lambda_2(\mathbf{u}_{1:3}) < v_\rho} dr d\sigma(\mathbf{u}_{1:3}).$$

Let ϕ_3 be the change of variables

$$\begin{aligned}\phi_3 : &[0, 2\pi)^3 \longrightarrow (\mathbf{S}^1)^3 \\ &\boldsymbol{\theta}_{1:3} \longmapsto \mathbf{u}_{1:3} \text{ with } u_1 = u(-\theta_1 + \theta_3), u_2 = u(\theta_1 + \theta_3) \text{ and } u_3 = u(\theta_2 + \theta_3)\end{aligned}$$

where $u(\theta) = (\cos \theta, \sin \theta)$. For all $\boldsymbol{\theta}_{1:3} \in [0, 2\pi)^3$, let us denote by $A(\boldsymbol{\theta}_{1:3}) = \lambda_2(\mathbf{u}_{1:3})$ with $\mathbf{u}_{1:3} = \phi_3(\boldsymbol{\theta}_{1:3})$. Since $|u_1 - u_2| = 2|\sin \theta_1|$, we have

$$G_{2,2}(\rho) \leq c \cdot \rho v_\rho \int_0^\infty \int_{[0,\pi/2) \times [0,2\pi)^2} r^3 e^{-\frac{1}{2}\pi r^2} A(\boldsymbol{\theta}_{1:3}) |\sin \theta_1|^{-1} \mathbb{1}_{r^2 A(\boldsymbol{\theta}_{1:3}) < v_\rho} dr d\boldsymbol{\theta}_{1:3}.$$

Without loss of generality, we have assumed that θ_1 belongs to $[0, \pi/2]$. To bound $G_{2,2}(\rho)$, we consider two cases that depend on the order of θ_1. Let $\epsilon > \frac{3}{5}$ be fixed. The previous inequality can be written as

$$G_{2,2}(\rho) \leq c \cdot \rho v_\rho \int_0^\infty \int_{[0,\rho^{-\epsilon}[\times[0,2\pi)^2} r^3 e^{-\frac{1}{2}\pi r^2} A(\boldsymbol{\theta}_{1:3}) |\sin\theta_1|^{-1} \mathbb{1}_{r^2 A(\boldsymbol{\theta}_{1:3})<v_\rho} dr d\boldsymbol{\theta}_{1:3}$$
$$+ c \cdot \rho v_\rho \int_0^\infty \int_{[\rho^{-\epsilon},\pi/2)\times[0,2\pi)^2} r^3 e^{-\frac{1}{2}\pi r^2} A(\boldsymbol{\theta}_{1:3}) |\sin\theta_1|^{-1} \mathbb{1}_{r^2 A(\boldsymbol{\theta}_{1:3})<v_\rho} dr d\boldsymbol{\theta}_{1:3}$$
$$= G_{2,2}^{(1)}(\rho) + G_{2,2}^{(2)}(\rho) \quad (3.3.60)$$

where $G_{2,2}^{(1)}(\rho)$ and $G_{2,2}^{(2)}(\rho)$ denote respectively the first and the second integrals of the right-hand side. Let us note that $A(\boldsymbol{\theta}_{1:3})|\sin\theta_1|^{-1}$ is bounded since, according to (3.3.56), we have $A(\boldsymbol{\theta}_{1:3}) = \frac{1}{2} \cdot 2|\sin\theta_1| \cdot d(u_3, \mathbf{L}(\mathbf{u}_{1:2}))$ where $\mathbf{u}_{1:3} = \phi_3(\boldsymbol{\theta}_{1:3})$ and $d(u_3, \mathbf{L}(\mathbf{u}_{1:2})) \leq 2$. Hence, the first integral of the right-hand side of (3.3.60) is less than

$$G_{2,2}^{(1)}(\rho) \leq c \cdot \rho v_\rho \int_0^\infty \int_{[0,\rho^{-\epsilon})\times[0,2\pi)^2} r^3 e^{-\frac{1}{2}\pi r^2} dr d\boldsymbol{\theta}_{1:3} \leq c \cdot \rho^{1-\epsilon} v_\rho = O(\rho^{-1/5}) \quad (3.3.61)$$

since $v_\rho = c \cdot \rho^{-3/5}$ and $\epsilon > \frac{3}{5}$. Moreover, bounding $A(\boldsymbol{\theta}_{1:3})$ by $r^{-2} v_\rho$ in the second integral of (3.3.60), we have

$$G_{2,2}^{(2)}(\rho) \leq c \cdot \rho v_\rho^2 \int_0^\infty \int_{[\rho^{-\epsilon},\pi/2)\times[0,2\pi)^2} r e^{-\frac{1}{2}\pi r^2} |\sin\theta_1|^{-1} dr d\boldsymbol{\theta}_{1:3} \leq c \cdot \log\rho \cdot \rho v_\rho^2 = O\left(\log\rho \cdot \rho^{-1/5}\right) \quad (3.3.62)$$

since $\int_{\rho^{-\epsilon}}^{\pi/2} \frac{1}{|\sin\theta_1|} d\theta_1$ is of order $\log\rho$.

From (3.3.61), (3.3.62), (3.3.60) and (3.3.59), we deduce that $G_2(\rho) = O\left(\log\rho \cdot \rho^{-1/5}\right)$. Proposition 3.3.7 is now a direct consequence of (3.3.52) and Theorem 3.1.1. □

The main tool to derive the asymptotic behaviour of $A_{PDT,\min}(\rho)$ is the Taylor expansion of $K_{1/6}(\cdot)$ used in (3.3.50). To the best of our knowledge, there is not more accurate result on this Taylor expansion which could provide the rate of convergence $\mathbb{P}(\lambda_2(\mathscr{C}) < v)$. Actually, the rate of convergence can be investigated with a more complicated method. Indeed, in [121], using Mellin transform, Rathie shows that the density of $\lambda_2(\mathscr{C})$ is given by

$$f(x) = 3\pi^{-1/2}(2\pi i x)^{-1} \int_L \frac{\Gamma(z+5/6)\Gamma(z+1)\Gamma(z+7/6)}{\Gamma(z+3/2)} (4\pi x^2/27)^{-z} dz$$

where L encloses all the (complex) poles of the integrand. These poles, of order 1, are $-5/6-k$, $-1-k$ and $-7/6-k$, $k = 0, 1, 2, \cdots$. Evaluating the contour integral as the sum of the residues at the poles, he shows that

$$f(x) = \sum_{k=0}^\infty c_{k,1} x^{2/3+2k} + \sum_{k=0}^\infty c_{k,2} x^{1+2k} + \sum_{k=0}^\infty c_{k,3} x^{4/3+2k}.$$

It results of a Taylor expansion of the sums that $f(x) = c_{0,1} x^{2/3} + O(x)$. Integrating $f(\cdot)$ on $[0, v]$, we obtain that

$$\mathbb{P}(\lambda_2(\mathscr{C}) < v) = c_{0,1} \cdot v^{5/3} + O(v^2).$$

Taking $v = v_\rho$ as in (3.3.51), the function $G_1(\rho) = |\rho\mathbb{P}(\lambda_2(\mathscr{C}) < v_\rho) - t^{5/3}|$ is of order $\rho v_\rho^2 = c \cdot \rho^{-1/5}$. Since $G_2(\rho) = O\left(\log\rho \cdot \rho^{-1/5}\right)$, we obtain the more precise result

$$\left|\mathbb{P}\left(\alpha_3^{3/5}\rho^{3/5}A_{PDT,\min}(\rho) \geq t\right) - e^{-t^{5/3}}\right| = O\left(\log\rho \cdot \rho^{-1/5}\right).$$

Nevertheless, we have used the Taylor expansion of the modified Bessel function to prove Proposition 3.3.7 since the method is quicker than the use of series.

When $d \geq 3$, the density of $\lambda_3(\mathscr{C})$ can also be written as an integral (see (2.5) in [121]):

$$f(x) = c_1(2\pi i x)^{-1}\int_L \triangledown_d(z)\cdot(c_2x^2)^{-z}dz$$

where c_1, c_2 are two constants depending on d which can be made explicit and

$$\triangledown_d(z) = \frac{\prod_{j=2}^d \Gamma(j/2+z)\prod_{j=0}^d \Gamma\left(\frac{d^2+1+2j}{2(d+1)}+z\right)}{\prod_{j=1}^{d-1}\Gamma(d/2+j/d+z)\Gamma^{d-1}((d+1)/2+z)}.$$

The poles of $\triangledown_d(\cdot)$ are real numbers and the largest of them is -1 which is a simple pole. Proceeding along the same lines as in the case $d = 2$, we show that $f(x) = c\cdot x + o(x)$ when x goes to 0 i.e.

$$G_1(\rho) = \left|\rho\mathbb{P}\left(\lambda_d(\mathscr{C}) < c\cdot\rho^{-1/2}t\right) - t^2\right| \underset{\rho\to\infty}{\longrightarrow} 0$$

for $d \geq 3$. Unfortunately, the same method as in the proof of Proposition 3.3.7 is not enough to show that $G_2(\rho)$ converges to 0. Nevertheless, we would be able to show that the minimum of the volumes of the cells of a Poisson-Delaunay tessellation is of order $\rho^{-1/2}$ provided that the extremal index exists and differs from 0 (see section 3.6 for more details about extremal index).

3.4 Extreme Values of a Poisson-Voronoi tessellation

Let χ be a locally finite subset of $\mathbf{R}^d$. For all $x \in \chi$, we denote by $C_\chi(x)$ the Voronoi cell of nucleus x defined as

$$C_\chi(x) = \{y \in \mathbf{R}^d, |x-y| \leq |x'-y|, x' \neq x \in \chi\}.$$

For all $x \in \chi$, we denote by $\mathcal{N}_\chi(x)$ the set of neighbors of x and $N_\chi(x)$ its cardinality i.e.

$$\mathcal{N}_\chi(x) = \{x' \in \mathbf{X}, C_\chi(x') \cap C_\chi(x) \neq \varnothing\} \text{ and } N_\chi(x) = \#\mathcal{N}_\chi(x). \tag{3.4.1}$$

Voronoi tessellation corresponds to the dual graph of Delaunay tessellation in the following sense: there exists an edge between two points $x, x' \in \chi$ in the Delaunay graph if and only if they are Voronoi neighbors i.e. $C_\chi(x) \cap C_\chi(x') \neq \varnothing$.

When $\chi = \mathbf{X}$ is a Poisson point process (of intensity 1), the family $\mathfrak{m}_{PVT} = \{C_\mathbf{X}(x), x \in \mathbf{X}\}$ is called the Poisson-Voronoi tessellation. Such model is extensively used in many domains such as cellular biology [118], astrophysics [120], telecommunications [6] and ecology [125]. For a complete account, we refer to the books [99], [108], [130] and the survey [18].

As in section 3.4, the window $\mathbf{W}_\rho = \rho^{1/d}[0,1]^d$ is partitioned into $N_\rho = \left\lfloor \frac{\rho}{2\log\rho}\right\rfloor$ sub-cubes $\mathbf{i} \in V_\rho$. The event A_ρ is the same as in (3.3.2) and we can show that it satisfies Condition 1 for the Poisson-Voronoi tessellation with arguments very similar to the proof of Lemma 3.3.1.

For each cell $C \in \mathfrak{m}_{PVT}$ i.e. $C = C_{\mathbf{X}}(x)$, we take $z(C_{\mathbf{X}}(x)) = x$. A consequence of Slivnyak's Theorem (see e.g. Theorem 3.3.5 in [130]) shows that the typical cell satisfies the equality in distribution

$$\mathscr{C} \stackrel{\mathcal{D}}{=} C_{\mathbf{X}\cup\{0\}}(0) \tag{3.4.2}$$

where $C_{\mathbf{X}\cup\{0\}}(0)$ is the Voronoi cell of nucleus 0 when we add the origin to the Poisson point process.

The function $G_2(\cdot)$ defined in (3.1.5) has an integral representation. Indeed, from Slivnyak's Formula, it can be written as

$$G_2(\rho) = \rho \int_{\mathfrak{C}_\rho} \mathbb{P}\left(f(C_{\mathbf{X}\cup\{0,y\}}(0)) > v_\rho, f(C_{\mathbf{X}\cup\{0,y\}}(y)) > v_\rho\right) dy. \tag{3.4.3}$$

Extremes of characteristic radii of Poisson-Voronoi tessellation are studied in [19]. In this paper, we give the asymptotic behaviours of two new geometrical characteristics.

The first one is the distance to the farthest neighbor. Let us consider

$$D(C_{\mathbf{X}}(x)) = \max_{x' \in \mathcal{N}_{\mathbf{X}}(x)} |x - x'|, x \in \mathbf{X} \text{ and } D_{\min,PVT}(\rho) = \min_{x \in \mathbf{X}\cap\mathbf{W}_\rho} D(C_{\mathbf{X}}(x)). \tag{3.4.4}$$

The second characteristic is the volume of the so-called Voronoi flower. We denote respectively for each point x, the Voronoi flower of nucleus $x \in \mathbf{X}$ and the minimum of their volumes as

$$\mathcal{F}(C_{\mathbf{X}}(x)) = \bigcup_{y \in C_{\mathbf{X}}(x)} B(y, |y - x|) \text{ and } F_{\min,PVT}(\rho) = \min_{x \in \mathbf{X}\cap\mathbf{W}_\rho} \lambda_d(\mathcal{F}(C_{\mathbf{X}}(x))). \tag{3.4.5}$$

Obviously, $2^{-d}\kappa_d d^d_{PVT}(\rho) \leq \kappa_d \min_{x\in\mathbf{X}\cap\mathbf{W}_\rho} R(C_{\mathbf{X}}(x))^d \leq F_{\min,PVT}(\rho)$ where $R(C_{\mathbf{X}}(x))$ denotes the circumradius of $C_{\mathbf{X}}(x)$. Actually, the following proposition shows that the two random variables $D_{\min,PVT}(\rho)$ and $F_{\min,PVT}(\rho)$ are of same order when ρ goes to infinity.

Proposition 3.4.1. *Let $\mathfrak{m}_{PVT}$ be a Poisson-Voronoi tessellation of intensity $\gamma = 1$. For all $t \geq 0$, we have*

$$\left|\mathbb{P}\left(\alpha_{d,4}^{1/(d+1)} \rho^{1/(d+1)} D^d_{\min,PVT}(\rho) \geq t\right) - e^{-t^{d+1}}\right| = O\left(\rho^{-1/(d+1)}\right) \tag{3.4.6a}$$

$$\left|\mathbb{P}\left(\alpha_{d,5}^{1/(d+1)} \rho^{1/(d+1)} F_{\min,PVT}(\rho) \geq t\right) - e^{-t^{d+1}}\right| = O\left(\rho^{-1/(d+1)}\right) \tag{3.4.6b}$$

where $\alpha_{d,4}$ and $\alpha_{d,5}$ are respectively given in (3.4.11) *and* (3.4.22).

Before proving Proposition 3.4.1, we need a practical lemma which is a new version of Lemma 3 in [19] adapted to our framework.

Lemma 3.4.2. *Let $v \geq 0$, $y \neq 0 \in \mathbf{R}^d$ and $\chi \subset \mathbf{R}^d$ locally finite such that $\chi \cup \{0, y\}$ is in general position i.e. each subset of size n¡d + 1 is affinely independent (see [147]). Let us assume that each Voronoi cell associated to the set $\chi \cup \{0, y\}$ is bounded and that*

$$\mathcal{N}_{\chi\cup\{0,y\}}(0) \subset B(0, v) \text{ and } \mathcal{N}_{\chi\cup\{0,y\}}(y) \subset B(y, v). \tag{3.4.7}$$

Then

$$\#\left(\chi \cap (B(0, v) \cup B(y, v))\right) \geq d + 1.$$

Proof of Lemma 3.4.2. Let us define $\chi_{0,y}$ as the (finite) subset:

$$\chi_{0,y} = \chi \cap (B(0,v) \cup B(y,v)).$$

Thanks to (3.4.7), we have $C_{\chi\cup\{0,y\}}(0) = C_{\chi_{0,y}\cup\{0,y\}}(0)$ and $C_{\chi\cup\{0,y\}}(y) = C_{\chi_{0,y}\cup\{0,y\}}(y)$. In particular, this shows that the cells $C_{\chi_{0,y}\cup\{0,y\}}(0)$ and $C_{\chi_{0,y}\cup\{0,y\}}(y)$ are bounded. Hence 0 and y are respectively in the convex hulls of $\chi_{0,y} \cup \{y\}$ and $\chi_{0,y} \cup \{0\}$ (see Property **V**2, page 58 in [108]). This implies that

$$\{0, y\} \subset \mathrm{conv}(\chi_{0,y}).$$

Since $\chi \cup \{0, y\}$ is in general position, this shows that $\mathrm{conv}(\chi_{0,y})$ has a non-empty interior and consequently this proves Lemma 3.4.2. □

We can now prove Proposition 3.4.1.

Proof of Proposition 3.4.1.

Proof of (3.4.6a). To find a function $v_\rho(t)$ such that $G_1(\rho) = |\rho\mathbb{P}(D(\mathscr{C}) > v_\rho) - t|$ converges to 0, we have to approximate the tail of $D(\mathscr{C})$. Let $v \geq 0$ be fixed. Since $\mathscr{C} = C_{\mathbf{X}\cup\{0\}}(0)$, we have

$$D(\mathscr{C}) < v \Longleftrightarrow \mathcal{N}_{\mathbf{X}\cup\{0\}}(0) \subset B(0,v). \tag{3.4.8}$$

In particular, we get

$$\mathbb{P}(D(\mathscr{C}) < v) = \sum_{k=d+1}^{\infty} \mathbb{P}\left(\mathcal{N}_{\mathbf{X}\cup\{0\}}(0) \subset B(0,v), N_{\mathbf{X}\cup\{0\}}(0) = k\right). \tag{3.4.9}$$

An integral representation of the right-hand side is given by (see Proposition 1 in [17])

$$\mathbb{P}\left(\mathcal{N}_{\mathbf{X}\cup\{0\}}(0) \subset B(0,v), N_{\mathbf{X}\cup\{0\}}(0) = k\right) = \frac{1}{k!}\int_{B(0,v)^k} e^{-\lambda_d(\mathcal{F}(C_{\{\mathbf{x}_{1:k}\}\cup\{0\}}(0)))}\mathbb{1}_{A_k}(\mathbf{x}_{1:k})d\mathbf{x}_{1:k}$$

where

$$A_k = \left\{\mathbf{x}_{1:k} = (x_1, \ldots, x_k) \in (\mathbf{R}^d)^k, C_{\{\mathbf{x}_{1:k}\}\cup\{0\}}(0) \text{ is a convex polytope with } k \text{ faces }\right\}.$$

We recall that $\{\mathbf{x}_{1:k}\} \cup \{0\} = \{x_1, x_2, \ldots, x_k, 0\}$. Taking the change of variables $x_i = vx_i'$, we obtain for all $k \geq d+1$

$$\mathbb{P}\left(\mathcal{N}_{\mathbf{X}\cup\{0\}}(0) \subset B(0,v), N_{\mathbf{X}\cup\{0\}}(0) = k\right) = v^{dk}\cdot\frac{1}{k!}\int_{B(0,1)^k} e^{-v^d\lambda_d(\mathcal{F}(C_{\{\mathbf{x}'_{1:k}\}\cup\{0\}}(0)))}\mathbb{1}_{A_k}(\mathbf{x}'_{1:k})d\mathbf{x}'_{1:k}. \tag{3.4.10}$$

If $k = d+1$, the previous probability converges to $\alpha_{d,4} \cdot v^{d(d+1)}$ when v goes to 0 where

$$\alpha_{d,4} = \frac{1}{(d+1)!}\int_{B(0,1)^{d+1}} \mathbb{1}_{A_{d+1}}(\mathbf{x}'_{1:d+1})d\mathbf{x}'_{1:d+1}. \tag{3.4.11}$$

If $k \geq d+2$, the right-hand side of (3.4.10) is less than $\frac{\kappa_d^k}{k!}v^{dk}$ thanks to the trivial inequalities $\mathbb{1}_{A_k} \leq 1$ and $e^{-\lambda_d(\mathcal{F}(C_{\{\mathbf{x}'_{1:k}\}\cup\{0\}}(0)))} \leq 1$. It follows from (3.4.9) that

$$\left|\mathbb{P}(D(\mathscr{C}) < v) - \alpha_{d,4}\cdot v^{d(d+1)}\right| \leq \sum_{k=d+2}^{\infty}\frac{\kappa_d^k}{k!}v^{dk} = O(v^{d(d+2)}). \tag{3.4.12}$$

Now, we can choose a suitable function v_ρ. Indeed, let $t \geq 0$ be fixed and let us denote by

$$v_\rho = v_\rho(t) = \left(\alpha_{d,4}^{-1}\rho^{-1}\right)^{1/d(d+1)} t^{1/d}. \tag{3.4.13}$$

According to (3.4.12), we have

$$G_1(\rho) = |\rho\mathbb{P}\left(D(\mathscr{C}) < v_\rho\right) - t^{d+1}| = O\left(\rho^{-1/(d+1)}\right). \tag{3.4.14}$$

Let us give now an upper bound of the function $G_2(\rho)$ defined in (3.1.5). According to (3.4.3) and with the same spirit as in (3.4.8), we obtain that

$$\begin{aligned} G_2(\rho) &= \rho \int_{\mathfrak{C}_\rho} \mathbb{P}\left(D(C_{\mathbf{X}\cup\{0,y\}}(0)) < v_\rho, D(C_{\mathbf{X}\cup\{0,y\}}(y)) < v_\rho\right) dy \\ &= \rho \int_{\mathfrak{C}_\rho} \mathbb{P}\left(\mathcal{N}_{\mathbf{X}\cup\{0,y\}}(0) \subset B(0,v_\rho), \mathcal{N}_{\mathbf{X}\cup\{0,y\}}(y) \subset B(y,v_\rho)\right) dy. \end{aligned} \tag{3.4.15}$$

To guarantee the independence of the events considered in (3.4.15) for each cells which are distant enough, we write

$$\begin{aligned} G_2(\rho) &= \rho \int_{\mathfrak{C}_\rho \cap B(0,2v_\rho)^c} \mathbb{P}\left(\mathcal{N}_{\mathbf{X}\cup\{0,y\}}(0) \subset B(0,v_\rho), \mathcal{N}_{\mathbf{X}\cup\{0,y\}}(y) \subset B(y,v_\rho)\right) dy \\ &\quad + \rho \int_{\mathfrak{C}_\rho \cap B(0,2v_\rho)} \mathbb{P}\left(\mathcal{N}_{\mathbf{X}\cup\{0,y\}}(0) \subset B(0,v_\rho), \mathcal{N}_{\mathbf{X}\cup\{0,y\}}(y) \subset B(y,v_\rho)\right) dy. \end{aligned} \tag{3.4.16}$$

For the first integral, when $y \in \mathfrak{C}_\rho \cap B(0,2v_\rho)^c$, the balls $B(0,v_\rho)$ and $B(y,v_\rho)$ are disjoint. Because $\mathbf{X}$ is a Poisson point process and because $y \notin B(0,2v_\rho)$, the first integrand of (3.4.16) can be written as the product $\mathbb{P}\left(\mathcal{N}_{\mathbf{X}\cup\{0\}}(0) \subset B(0,v_\rho)\right) \times \mathbb{P}\left(\mathcal{N}_{\mathbf{X}\cup\{y\}}(y) \subset B(y,v_\rho)\right)$. Hence, according to (3.4.8) and (3.4.14) we obtain that

$$\mathbb{P}\left(\mathcal{N}_{\mathbf{X}\cup\{0,y\}}(0) \subset B(0,v_\rho), \mathcal{N}_{\mathbf{X}\cup\{0,y\}}(y) \subset B(y,v_\rho)\right) = \mathbb{P}\left(D(\mathscr{C}) < v_\rho\right)^2 \leq c\cdot\rho^{-2}, y \in B(0,2v_\rho)^c \tag{3.4.17}$$

where c is a constant which *does not* depend on y.

For the second integral of (3.4.16), we apply Lemma 3.4.2 to $\chi = \mathbf{X}$. This gives

$$\begin{aligned} &\mathbb{P}\left(\mathcal{N}_{\mathbf{X}\cup\{0,y\}}(0) \subset B(0,v_\rho), \mathcal{N}_{\mathbf{X}\cup\{0,y\}}(y) \subset B(y,v_\rho)\right) \\ &\qquad \leq \mathbb{P}\left(\#(\mathbf{X} \cap (B(0,v_\rho) \cup B(y,v_\rho)) \geq d+1)\right), y \in B(0,2v_\rho). \end{aligned} \tag{3.4.18}$$

Since $\#(\mathbf{X} \cap B)$ is Poisson distributed of mean $\lambda_d(B)$ for each Borel subset $B \subset \mathbf{R}^d$, we obtain for ρ large enough that

$$\begin{aligned} &\mathbb{P}\left(\#(\mathbf{X} \cap (B(0,v_\rho) \cup B(y,v_\rho)) \geq d+1)\right) \\ &= \sum_{k=d+1}^{\infty} \frac{1}{k!}\left(\lambda_d(B(0,v_\rho) \cup B(y,v_\rho))\right)^k e^{-\lambda_d(B(0,v_\rho)\cup B(y,v_\rho))} \\ &\qquad \leq c \cdot v_\rho^{d(d+1)} = c' \cdot \rho^{-1}, y \in B(0,2v_\rho) \end{aligned} \tag{3.4.19}$$

according to (3.4.13) and to the trivial inequalities $e^{-\lambda_d(B(0,v_\rho)\cup B(y,v_\rho))} \leq 1$ and $\lambda_d(B(0,v_\rho)\cup B(y,v_\rho)) \leq 2\cdot\kappa_d v_\rho^d$. This together with (3.4.16), (3.4.17) and (3.4.18) shows that

$$G_2(\rho) \leq c\cdot\rho^{-1}\lambda_d(\mathfrak{C}_\rho \cap B(0,2v_\rho)^c) + c\cdot\lambda_d(\mathfrak{C}_\rho\cap B(0,2v_\rho)).$$

Since $\lambda_d(\mathfrak{C}_\rho\cap B(0,2v_\rho)^c) \leq \lambda_d(\mathfrak{C}_\rho) \leq c\cdot\log\rho$ and $\lambda_d(\mathfrak{C}_\rho\cap B(0,2v_\rho)) \leq \lambda_d(B(0,2v_\rho)) = c\cdot\rho^{-1/(d+1)}$, we deduce from the previous inequality that

$$G_2(\rho) \leq c\cdot\log\rho\times\rho^{-1} + c\cdot\rho^{-1/(d+1)} = O\left(\rho^{-1/(d+1)}\right). \tag{3.4.20}$$

We now derive directly (3.4.6a) from (3.4.14), (3.4.20), (3.3.4) and Theorem 3.1.1.

Proof of (3.4.6b). This will be sketched only since it is analogous to the proof of (3.4.6a). First, we investigate the tail of $\lambda_d(\mathcal{F}(\mathscr{C}))$. In [148], Zuyev shows that, conditional on $N_{\mathbf{X}\cup\{0\}} = k$, the volume of $\mathcal{F}(\mathscr{C})$ is Gamma distributed of parameters $(k,1)$ i.e.

$$\mathbb{P}\left(\lambda_d(\mathcal{F}(\mathscr{C})) < v\right) = \sum_{k=d+1}^{\infty}\frac{1}{(k-1)!}\int_0^v x^{k-1}e^{-x}dx\cdot p(k) \tag{3.4.21}$$

where $p(k) = \mathbb{P}\left(N_{\mathbf{X}\cup\{0\}}(0) = k\right)$. When $k = d+1$, the Taylor expansion $e^{-x} = 1 + O(x)$ shows that the term of the series in (3.4.21) equals $\alpha_{d,5}v^{d+1} + O(v^{d+2})$ where

$$\alpha_{d,5} = \frac{p(d+1)}{(d+1)!}. \tag{3.4.22}$$

If $k \geq d+2$, the term of the series in (3.4.21) is less than $\frac{1}{d!}\cdot v^{d+2}\cdot p(k)$ thanks to the trivial inequality $e^{-x}\leq 1$. According to (3.4.21), we get

$$|\mathbb{P}\left(\lambda_d(\mathcal{F}(\mathscr{C})) < v\right) - \alpha_{d,5}\cdot v^{d+1}| = O(v^{d+2}).$$

Hence, for all fixed $t \geq 0$, taking

$$v_\rho = v_\rho(t) = \left(\alpha_{d,5}^{-1}\rho^{-1}\right)^{1/(d+1)} t \tag{3.4.23}$$

we obtain

$$G_1(\rho) = |\rho\mathbb{P}\left(\lambda_d(\mathcal{F}(\mathscr{C})) < v_\rho\right) - t^{d+1}| = O(\rho^{-1/(d+1)}). \tag{3.4.24}$$

To get an upper bound of $G_2(\rho)$, we note that for each $\chi\subset\mathbf{R}^d$ locally finite and $x\in\chi$, we have

$$\frac{\kappa_d}{2^d}\cdot\left(D(C_\chi(x))\right)^d \leq \lambda_d\left(\mathcal{F}(C_\chi(x))\right)$$

where $D(C_\chi(x))$ and $\mathcal{F}(C_\chi(x))$ are defined as in (3.4.4) and (3.4.5). Applying the previous inequality to $\chi = \mathbf{X}\cup\{0,y\}$ and $x = 0, y$, we deduce from (3.4.3) that

$$\begin{aligned} G_2(\rho) &= \rho\int_{\mathfrak{C}_\rho}\mathbb{P}\left(\lambda_d(\mathcal{F}(C_{\mathbf{X}\cup\{0,y\}}(0))) < v_\rho, \lambda_d(\mathcal{F}(C_{\mathbf{X}\cup\{0,y\}}(y))) < v_\rho\right)dy \\ &\leq \rho\int_{\mathfrak{C}_\rho}\mathbb{P}\left(D(C_{\mathbf{X}\cup\{0,y\}}(0)) < v'_\rho, D(C_{\mathbf{X}\cup\{0,y\}}(y)) < v'_\rho\right)dy \end{aligned} \tag{3.4.25}$$

with

$$v'_\rho = 2\kappa_d^{1/d} \cdot v_\rho^{1/d} = (2^{d(d+1)}\kappa_d^{d+1}\alpha_{d,5}^{-1}\rho^{-1})^{1/d(d+1)}t^{1/d}$$

according to (3.4.23). The function v'_ρ equals v_ρ, defined in (3.4.13), up to a multiplicative constant. Bounding the right-hand side of (3.4.25) as in (3.4.15) and proceeding along the same lines as in the proof of (3.4.6a) (see the previous page), we show that $G_2(\rho)$ is of order $\rho^{-1/(d+1)}$. This together with (3.4.24) shows (3.4.6b).

□

The random variables $F_{\min,PVT}(\rho)$ and $D_{\min,PVT}(\rho)$ are related to the minimum of the circumradii $R_{\min,PVT}(\rho)$ which is defined in [19] since both investigate a minimax. In the same spirit as before, we could re-find the asymptotic behaviour of $R_{\min,PVT}(\rho)$ included in [19] and prove that the rate of convergence is of order $\rho^{-1/(d+1)}$.

3.5 The maximum of inradii of a Gauss-Poisson Voronoi tessellation

As an example of non-Poisson point process, a Gauss-Poisson process is analyzed. Introduced by Newman and investigated by Milne and Westcott, such process has a potential application in statistical mechanics (see [105], p. 350) and could be used as a model for molecular motion (see [97] p. 169). In the sense of [140] p. 161, a stationary planar Gauss-Poisson process $\mathbf{X}$ is a (simple) point process which can be defined as follows: let $\mathbf{X}_a$ be a Poisson point process of intensity γ_a in $\mathbf{R}^2$. Every point $x_a \in \mathbf{X}_a$ is replaced by a cluster of points $\Xi(x_a) = x_a + \Xi_0(x_a)$ where the set of points $\Xi_0(x_a), x_a \in \mathbf{X}_a$ are chosen independently and with identical distribution i.e.

$$\mathbf{X} = \bigcup_{x_a \in \mathbf{X}_a} \Xi(x_a).$$

For all $x_a \in \mathbf{X}_a$, the cluster $\Xi_0(x_a)$ equals in distribution Ξ_0 defined as follows: Ξ_0 has an isotropic distribution and is composed of zero, one or two points with probability $p_0 \neq 1$, p_1 and $p_2 = 1 - (p_0 + p_1)$. If Ξ_0 contains only one point then that point is the origin 0. If Ξ_0 is composed of two points then these are separated by a unit distance and have midpoint 0. The intensity of $\mathbf{X}$ is given by

$$\gamma_{\mathbf{X}} = (p_1 + 2p_2) \cdot \gamma_a.$$

In this subsection, we investigate the maximum of inradii of a Gauss-Poisson Voronoi tessellation $\mathfrak{m}_{GPVT}$ i.e.

$$r_{\max,GPVT}(\rho) = \max_{x \in \mathbf{X} \cap \mathbf{W}_\rho} r(C_{\mathbf{X}}(x)) \quad \text{where} \quad r(C_{\mathbf{X}}(x)) = \max\{r \geq 0, B(x,r) \subset C_{\mathbf{X}}(x)\}.$$

To apply Theorem 3.1.1, we subdivide $\mathbf{W}_\rho$ into N_ρ sub-cubes of equal size where

$$N_\rho = \left\lfloor \frac{\gamma_a(p_1 + p_2)\rho}{2\log\rho} \right\rfloor.$$

With the same method as for a Poisson-Voronoi tessellation, we can show that there exists an integer $R \geq 1$ such that Condition 1 holds when the Voronoi tessellation is induced by a Gauss-Poisson process. The asymptotic distribution of $r_{\max}(\rho)$ is given in the following proposition.

Proposition 3.5.1. *Let* $\mathbf{X}$ *be a Gauss-Poisson process of intensity 1 i.e.* $(p_1 + 2p_2)\gamma_a = 1$ *with* $p_0 \neq 1$ *and* $p_1 \neq 0$. *For all* $t \in \mathbf{R}$, *we have*

$$\left|\mathbb{P}\left(r_{\max,GPVT}(\rho) \leq v_\rho\right) - e^{-e^{-t}}\right| = O\left((\log \rho)^{-1/2}\right)$$

where v_ρ *is defined in* (3.5.4).

Proof of Proposition 3.5.1. We notice that for all $x \in \mathbf{X}$ and $v \geq 0$, the inscribed radius $r(C_{\mathbf{X}}(x))$ is greater than v if and only if $\#B(x, 2v) \cap \mathbf{X} = 1$. Consequently

$$\mathbb{P}\left(r(\mathscr{C}) > v\right) = \mathbb{P}^0(\#B(0, 2v) \cap \mathbf{X}^0 = 1)$$

where $\mathscr{C}$ is the typical cell of the Voronoi tessellation induced by $\mathbf{X}$. In the above equality, $\mathbb{P}^0$ is the Palm measure of $\mathbf{X}$ in the sense of (3.6) of [130] and $\mathbf{X}^0$ is $\mathbb{P}^0$ distributed. The planar Gauss-Poisson process is one of the rare non-Poisson processes for which the right-hand side can be made fully explicit. This one is given for each $v \geq 0$ by the following formula (see p. 161 in [140]):

$$\mathbb{P}^0(\#B(0, 2v) \cap \mathbf{X}^0 = 1) = \frac{1}{p_1 + 2p_2} e^{-\gamma_a(4p_1\pi v^2 + p_2(8\pi v^2 - a(2v)))} \cdot \begin{cases} p_1 + 2p_2 & 0 \leq 2v < 1 \\ p_1 & 2v \geq 1 \end{cases}. \tag{3.5.1}$$

and

$$a(2v) = 8v^2 \arccos \frac{1}{4v} - \frac{1}{2}\sqrt{16v^2 - 1} \text{ for } 4v \geq 1 \tag{3.5.2}$$

and equals zero otherwise. The function $a(2v)$ is the area of intersection of two disks of radius $2v$ and centers separated by unit distance. A Taylor expansion of the right-hand side of (3.5.1) shows that

$$\mathbb{P}^0(\#B(0, 2v) \cap \mathbf{X}^0 = 1) = e^{-(P(v) + R(v))}$$

where

$$P(v) = 4\gamma_a\pi(p_1 + p_2)v^2 - 4\gamma_a \cdot p_2 \cdot v - \log\left(\frac{p_1}{p_1 + p_2}\right) \text{ and } R(v) = \frac{5\gamma_a \cdot p_2}{48} \cdot \frac{1}{v} + o\left(\frac{1}{v}\right) \tag{3.5.3}$$

as v goes to infinity. In the previous line, $\phi(v) = o(\psi(v))$ means that $\phi(v)/\psi(v) \underset{v\to\infty}{\longrightarrow} 0$.

For all $t \in \mathbf{R}$, we define $v_\rho = v_\rho(t)$ so that $P(v_\rho) = \log \rho + t$ i.e.

$$v_\rho = v_\rho(t) = \frac{2\gamma_a \cdot p_2 + \left(4\gamma_a^2 \cdot p_2^2 + 4\gamma_a\pi(p_1 + p_2)\left(\log\left(\frac{p_1}{p_1 + 2p_2}\right) + \log \rho + t\right)\right)^{1/2}}{4\gamma_a\pi(p_1 + p_2)}. \tag{3.5.4}$$

Using the fact that $\rho\mathbb{P}^0(\#B(0, 2v_\rho) \cap \mathbf{X}^0 = 1) = e^{-t - R(v_\rho)}$ where $R(\cdot)$ is defined in (3.5.3), we deduce that

$$G_1(\rho) = |\rho\mathbb{P}^0(\#B(0, 2v) \cap \mathbf{X}^0 = 1) - e^{-t}| \leq e^{-t}R(v_\rho) = O\left((\log \rho)^{-1/2}\right). \tag{3.5.5}$$

Moreover, from Campbell theorem (see Theorem 3.3.3. in [130]), we have

$$G_2(\rho) := N_\rho \mathbb{E}\left[\sum_{(x,y)_{\neq}\in(\mathbf{X}\cap\mathfrak{C}_\rho)^2} \mathbb{1}_{\#B(x,2v_\rho)\cap\mathbf{X}=1}\mathbb{1}_{\#B(y,2v_\rho)\cap\mathbf{X}=1}\right]$$
$$= N_\rho \int_{\mathfrak{C}_\rho}\int_{\mathcal{F}_{lf}} \sum_{y\in\eta\cap\mathfrak{C}_\rho} \mathbb{1}_{\#(\eta+x)\cap B(x,2v_\rho)=1}\mathbb{1}_{\#(\eta+x)\cap B(y,2v_\rho)=1}d\mathbb{P}^0(\eta)dx.$$

Here $\mathcal{F}_{lf}$ denotes the space of locally finite subsets of $\mathbf{R}^2$ and $\mathbb{P}^0$ is the Palm measure of a Gauss-Poisson process. Because the integrand of the right-hand side is translation invariant (in distribution) and because $N_\rho\lambda_2(\mathfrak{C}_\rho) \underset{\rho\to\infty}{\sim} c\cdot\rho$, we obtain

$$G_2(\rho) \leq c\cdot\rho\int_{\mathcal{F}_{lf}}\sum_{y\in\eta\cap\mathfrak{C}_\rho}\mathbb{1}_{\#\eta\cap B(0,2v_\rho)=1}\mathbb{1}_{\#\eta\cap B(y,2v_\rho)=1}d\mathbb{P}^0(\eta).$$

According to Formula (5.3.2) in [140], we have $\mathbb{P}^0 = \mathbb{P}_{\mathbf{X}} * \mathbb{c}^0$ where $\mathbb{P}_{\mathbf{X}}$ is the distribution of $\mathbf{X}$ and $\mathbb{c}^0$ is the Palm measure of the cluster distribution Ξ_0 that is concentrated on the space $\mathcal{F}_{lf,2}$ of subsets of 0, 1 or 2 points in $\mathbf{R}^2$. Hence

$$G_2(\rho) = c\cdot\rho\int_{\mathcal{F}_{lf}}\int_{\mathcal{F}_{lf,2}}\sum_{y\in(\phi\cup\xi)\cap\mathfrak{C}_\rho}\mathbb{1}_{\#(\phi\cup\xi)\cap(B(0,2v_\rho)\cup B(y,2v_\rho))=2}\mathbb{1}_{|y|>2v_\rho}d\mathbb{P}_{\mathbf{X}}(\phi)d\mathbb{c}^0(\xi).$$

When $|y| > 2v_\rho$, we have $y \notin \xi$ for ρ large enough since $\mathbb{c}_0$ a.s., ξ is bounded. Moreover, $\mathbb{P}_{\mathbf{X}}$ a.s. $\phi\cap\xi\cap(B(0,2v_\rho)\cup B(y,2v_\rho))$ is empty. Consequently, calculating the integral with respect to $\mathbb{c}_0$, we get

$$G_2(\rho) = c\cdot\rho\int_{\mathcal{F}_{lf}}\sum_{y\in\phi\cap\mathfrak{C}_\rho}\mathbb{1}_{\#\phi\cap(B(0,2v_\rho)\cup B(y,2v_\rho))=1}\mathbb{1}_{|y|>2v_\rho}d\mathbb{P}_{\mathbf{X}}(\phi).$$

Proceeding as previously, we deduce from Campbell theorem and from the relation $\mathbb{P}^0 = \mathbb{P}_{\mathbf{X}} * \mathbb{c}^0$, that

$$G_2(\rho) = c\cdot\rho\int_{\mathfrak{C}_\rho}\int_{\mathcal{F}_{lf}}\int_{\mathcal{F}_{lf,2}}\mathbb{1}_{\#((\xi\cup\phi)+y)\cap(B(0,2v_\rho)\cup B(y,2v_\rho))=1}\mathbb{1}_{|y|>2v_\rho}dyd\mathbb{P}_{\mathbf{X}}(\phi)d\mathbb{c}^0(\xi).$$

Since $\mathbb{P}_{\mathbf{X}}$ a.s. $\phi\cap\Xi_0\cap(B(0,2v_\rho)\cup B(y,2v_\rho))$ is empty, we deduce after integration over $\mathcal{F}_{lf}\times\mathcal{F}_{lf,2}$ with respect to $\mathbb{P}^0\otimes\mathbb{c}^0$ that

$$G_2(\rho) \leq c\cdot\rho\int_{\mathfrak{C}_\rho}\mathbb{P}\left(\mathbf{X}\cap(B(0,2v_\rho)\cup B(y,2v_\rho))=\varnothing\right)\mathbb{1}_{|y|>2v_\rho}dy. \tag{3.5.6}$$

Let $|y| > 2v_\rho$ be fixed. To get a suitable upper bound of the integrand, we use the fact that $\mathbf{X}\cap(B(0,2v_\rho)\cup B(y,2v_\rho))=\varnothing \iff (x+\Xi_0(x))\cap(B(0,2v_\rho)\cup B(y,2v_\rho))=\varnothing$ for all $x\in\mathbf{X}_a$. From Theorem 3.2.4. of [130], Fubini's theorem and the fact that Ξ_0 is symmetric, we get

$$\begin{aligned}\mathbb{P}\left(\mathbf{X}\cap(B(0,2v_\rho)\cup B(y,2v_\rho))=\varnothing\right) &= e^{-\gamma_a\int_{\mathbf{R}^2}\mathbb{P}((x+\Xi_0(x))\cap(B(0,2v_\rho)\cup B(y,2v_\rho)\neq\varnothing)dx}\\ &= e^{-\gamma_a\mathbb{E}[\lambda_2(\Xi_0\oplus(B(0,2v_\rho)\cup B(y,2v_\rho))]}.\end{aligned} \tag{3.5.7}$$

We give below a suitable lower bound of the term appearing in the exponential. Since $|y| > 2v_\rho$, we have

$$\mathbb{E}\left[\lambda_2(\Xi_0\oplus(B(0,2v_\rho)\cup B(y,2v_\rho))|\#\Xi_0=1\right] = \lambda_2\left(B(0,2v_\rho)\cup B(y,2v_\rho)\right) \geq \frac{3}{2}\cdot 4\pi v_\rho^2$$

and

$$\mathbb{E}\left[\lambda_2(\Xi_0 \oplus (B(0,2v_\rho) \cup B(y,2v_\rho))|\#\Xi_0 = 2\right] \geq \mathbb{E}\left[\lambda_2(\Xi_0 \oplus B(0,2v_\rho))\right] \geq 8\pi v_\rho^2 - a(2v_\rho)$$

where $a(\cdot)$ is defined in (3.5.2). Since Ξ_0 is reduced to 0, 1 or 2 points with probability p_0, p_1 and p_2, we deduce from (3.5.7) that

$$\begin{aligned}\mathbb{P}\left(\mathbf{X} \cap (B(0,2v_\rho) \cup B(y,2v_\rho)) = \varnothing\right) &\leq e^{-\gamma_a\left(\frac{3}{2}p_1 \cdot 4\pi v_\rho^2 + p_2(8\pi v_\rho^2 - a(2v_\rho))\right)} \\ &= \frac{p_1 + 2p_2}{p_1}\mathbb{P}^0(\#B(0,2v_\rho) \cap \mathbf{X}^0 = 1) \cdot e^{-2\gamma_a p_1 \pi v_\rho^2} \qquad (3.5.8)\end{aligned}$$

for ρ large enough according to (3.5.1). Integrating over $\mathfrak{C}_\rho$, we deduce from (3.5.6), (3.5.8), (3.5.5) and from the inequality $\lambda_2(\mathfrak{C}_\rho) \leq c \cdot \log\rho$, that

$$G_2(\rho) \leq c \cdot \log\rho \cdot e^{-2\gamma_a p_1 \pi v_\rho^2} = O\left(\log\rho \cdot \rho^{-\alpha}\right) \qquad (3.5.9)$$

where

$$\alpha = \frac{p_1}{2(p_1 + p_2)}. \qquad (3.5.10)$$

Since $p_1 \neq 0$, we have $\alpha > 0$ so that $G_2(\rho)$ converges to 0. Proposition 3.5.1 is now a direct consequence of (3.5.5), (3.5.9) and Theorem 3.1.1.

□

According to Proposition 3.5.1 and (3.5.4), the order of $r_{\max,GPVT}(\rho)$ is

$$(4\gamma_a\pi(p_1 + p_2))^{-1/2} \cdot (\log\rho)^{1/2} = \left(\frac{p_1 + 2p_2}{4\pi(p_1 + p_2)}\right)^{1/2} \cdot (\log\rho)^{1/2}$$

since we have assumed that $(p_1 + 2p_2)\gamma_a = 1$. Let us remark that the larger p_2 is, the larger the order is. This can be explained by the following fact: the nucleus $x \in \mathbf{X}$ of the Voronoi cell which maximizes the inradius belongs to a cluster of size 1 i.e. $x \in \Xi(x_a)$, where $\#\Xi(x_a) = 1$ for some $x_a \in \mathbf{X}_a$. Moreover, $r_{\max,GPVT}(\rho)$ equals the distance between the point x and another cluster. If p_2 is large, the mean number of clusters per unit volume is smaller and smaller so that the inradii associated to the clusters of size 1 are large. In particular, this implies that the maximum of inradii increases when p_2 is large.

When $p_1 = 0$, we obtain a degenerate case since $r_{\max,GPVT}(\rho) = \frac{1}{2}$ is constant. When $p_0 = p_2 = 0$ and $p_1 = 1$, the random variable $r_{\max,GPVT}(\rho)$ is the maximum of inradii of a Poisson-Voronoi tessellation $r_{\max,PVT}(\rho)$. In that case, the order is

$$v_\rho = v_\rho(t) = (4\pi)^{-1/2} \cdot (\log\rho + t)^{1/2}.$$

The order of $r_{\max,PVT}(\rho)$ has already been investigated in [19]. Nevertheless, Proposition 3.5.1 is more precise since it provides the rate of convergence. Actually, this rate could be improved. Indeed, since $p_0 = p_2 = 0$ and $p_1 = 1$ we have $G_1(\rho) = 0$ according to (3.5.1), (3.5.4) and (3.5.5). Moreover, the term α as defined in (3.5.10) equals 1/2. Hence, according to (3.5.9), we have

$$\mathbb{P}\left(r_{\max,PVT}(\rho) \leq (4\pi)^{-1/2} \cdot (\log\rho + t)^{1/2}\right) = O\left(\log\rho \cdot \rho^{-1/2}\right).$$

Finally, let us mention that a Gauss-Poisson process belongs to the class of the so-called *Neyman-Scott* processes. We do not investigate general Neyman-Scott processes since the left-hand side of (3.5.1) cannot be made explicit excepted for some particular cases as Gauss-Poisson processes.

3.6 Proof of Proposition 3.1.3 and some extremal indices

In this section, we prove Proposition 3.1.3 and we give two examples where the extremal index differs from 1.

Proof of Proposition 3.1.3. The proof is an adaptive version to our setting of two results due to Leadbetter (see Theorem 2.2 and Lemma 2.1. in [81]). The difference is that we investigate a maximum on a random graph instead of a sequence of real numbers.

First, we investigate the superior limit. For each $\tau \geq 0$, we denote by

$$\psi(\tau) = \limsup_{\rho\to\infty} \mathbb{P}\left(M_{f,\mathbf{W}_\rho} \leq v_\rho(\tau)\right). \tag{3.6.1}$$

Let k be a fixed integer. The key idea is to show that $\psi(\tau/k^d) = \psi^{1/k^d}(\tau)$. To do it, we subdivide the proof into two steps. The first is intrinsic to the sequence $v_\rho(\tau)$ while the second step needs the mixing property of the tessellation.

Step 1. We show that

$$\limsup_{\rho\to\infty} \mathbb{P}\left(M_{f,\mathbf{W}_{\rho/k^d}} \leq v_\rho(\tau)\right) = \psi(\tau/k^d). \tag{3.6.2}$$

Indeed, if $v_\rho(\tau) \geq v_{\rho/k^d}(\tau/k^d)$, it follows that

$$\begin{aligned}
&\left|\mathbb{P}\left(M_{f,\mathbf{W}_{\rho/k^d}} \leq v_\rho(\tau)\right) - \mathbb{P}\left(M_{f,\mathbf{W}_{\rho/k^d}} \leq v_{\rho/k^d}(\tau/k^d)\right)\right| \\
&\leq \mathbb{P}\left(\bigcup_{\substack{C\in\mathfrak{m}\\ z(C)\in\mathbf{W}_{\rho/k^d}}} \{v_{\rho/k^d}(\tau/k^d) \leq f(C) \leq v_\rho(\tau)\}\right) \\
&\leq \mathbb{E}\left[\sum_{\substack{C\in\mathfrak{m}\\ z(C)\in\mathbf{W}_{\rho/k^d}}} \mathbb{1}_{v_{\rho/k^d}(\tau/k^d)\leq f(C)\leq v_\rho(\tau)}\right].
\end{aligned}$$

This together with the corresponding inequality when $v_\rho(\tau) \leq v_{\rho/k^d}(\tau/k^d)$ shows that

$$\begin{aligned}
&\left|\mathbb{P}\left(M_{f,\mathbf{W}_{\rho/k^d}} \leq v_\rho(\tau)\right) - \mathbb{P}\left(M_{f,\mathbf{W}_{\rho/k^d}} \leq v_{\rho/k^d}(\tau/k^d)\right)\right| \\
&\leq \frac{\rho}{k^d}\left|\mathbb{P}\left(f(\mathscr{C}) > v_{\rho/k^d}(\tau/k^d)\right) - \mathbb{P}\left(f(\mathscr{C}) > v_\rho(\tau)\right)\right| \\
&= \frac{\rho}{k^d}\left|\frac{\tau/k^d}{\rho/k^d} - \frac{\tau}{\rho} + o\left(\frac{1}{\rho}\right)\right| \underset{\rho\to\infty}{\longrightarrow} 0
\end{aligned} \tag{3.6.3}$$

according to (3.1.1). Moreover, from (3.6.1) we have

$$\limsup_{\rho\to\infty} \mathbb{P}\left(M_{f,\mathbf{W}_{\rho/k^d}} \leq v_{\rho/k^d}(\tau/k^d)\right) = \psi(\tau/k^d).$$

The limit (3.6.2) results of the previous equality and (3.6.3).

Step 2. Secondly, we show that

$$\limsup_{\rho\to\infty} \mathbb{P}\left(M_{f,\mathbf{W}_{\rho/k^d}} \leq v_\rho(\tau)\right) = \psi(\tau)^{1/k^d}. \tag{3.6.4}$$

Indeed, we partition $W = [0,1]^d$ into a set of k^d sub-cubes of equal volume $1/k^d$, say $B^{(1)}, \ldots, B^{(k^d)}$. According to (3.2.31) applied to $L = k^d$, we have

$$\mathbb{P}\left(M_{f,\mathbf{W}_\rho} \leq v_\rho(\tau)\right) - \prod_{l=1}^{k^d} \mathbb{P}\left(M_{f,\mathbf{B}_\rho^{(l)}} \leq v_\rho(\tau)\right) \underset{\rho\to\infty}{\longrightarrow} 0$$

where $\mathbf{B}_\rho^{(l)} = \rho^{1/d} B^{(l)}$ for all $1 \leq l \leq k^d$. Since $\mathbf{B}_\rho^{(l)}$ is a cube of volume ρ/k^d and since $\mathfrak{m}$ is stationary, the previous convergence can be rewritten as

$$\mathbb{P}\left(M_{f,\mathbf{W}_\rho} \leq v_\rho(\tau)\right) - \mathbb{P}\left(M_{f,\mathbf{W}_{\rho/k^d}} \leq v_\rho(\tau)\right)^{k^d} \underset{\rho\to\infty}{\longrightarrow} 0. \tag{3.6.5}$$

We deduce (3.6.4) thanks to (3.6.1).

Conclusion. We deduce from (3.6.2) and (3.6.4), that

$$\psi(\tau/k^d) = \psi(\tau)^{1/k^d} \text{ where } \tau \geq 0 \text{ and } k \in \mathbb{N}^* \tag{3.6.6}$$

Moreover, in the same spirit as in the proof of (3.2.12), we can show that

$$\mathbb{P}\left(M_{f,\mathbf{W}_{\rho/k^d}} \leq v_\rho(\tau)\right) \geq 1 - \frac{\rho}{k^d}\mathbb{P}\left(f(\mathscr{C}) > v_\rho(\tau)\right) \underset{\rho\to\infty}{\longrightarrow} 1 - \tau/k^d.$$

Hence, taking the kth powers and using (3.6.5), we deduce that $\mathbb{P}\left(M_{f,\mathbf{W}_\rho} \leq v_\rho(\tau)\right) \geq \left(1 - \frac{\tau}{k^d}\right)^{k^d}$ and so, letting $k \to \infty$, that

$$\liminf_{\rho\to\infty} \mathbb{P}\left(M_{f,\mathbf{W}_\rho} \leq v_\rho(\tau)\right) \geq e^{-\tau}. \tag{3.6.7}$$

This shows that $\psi(\tau) > 0$. Since $\psi(\cdot)$ is also non-increasing and since the only solution of the functional equation (3.6.6) which is strictly positive and non-increasing is an exponential function, we have $\psi(\tau) = e^{-\theta\tau}$ for some $\theta \geq 0$. Hence

$$\limsup_{\rho\to\infty} \mathbb{P}\left(M_{f,\mathbf{W}_\rho} \leq v_\rho(\tau)\right) = e^{-\theta\tau}.$$

With a similar method, we obtain that $\liminf_{\rho\to\infty} \mathbb{P}\left(M_{f,\mathbf{W}_\rho} \leq v_\rho(\tau)\right) = e^{-\theta'\tau}$ for some $\theta' \leq 1$ (according to (3.6.7)) and such that $\theta \leq \theta'$. □

As an illustration, we give below two examples where the extremal index differs from 1. The first one is the minimum of inradii of a Poisson-Voronoi tessellation.

Example 1. Let $\mathfrak{m}_{PVT}$ be a Poisson-Voronoi tessellation of intensity 1 and $\mathbf{X}$ the underlying Poisson point process. For each cell $C = C_\mathbf{X}(x)$, we consider the inradius $r(C_\mathbf{X}(x))$ in the sense of section 3.5. The distribution function $r(\mathscr{C})^d$ of the inradius of the typical cell is exponentially distributed with rate $2^d\kappa_d$. Indeed, $r(\mathscr{C})$ is lower than v, $v \geq 0$, if and only if $\mathbf{X} \cap B(0, 2v)$ is not empty. Hence

$$\rho \cdot \mathbb{P}\left(r(\mathscr{C})^d \leq \frac{1}{2^d\kappa_d\rho}t\right) \underset{\rho\to\infty}{\longrightarrow} t.$$

Moreover, according to the convergence (2b) in [19], we know that

$$\mathbb{P}\left(\min_{x\in\mathbf{X}\cap\mathbf{W}_\rho} r(C_\mathbf{X}(x))^d \geq \frac{1}{2^d\kappa_d\rho}t\right) \underset{\rho\to\infty}{\longrightarrow} e^{-t/2}.$$

Let us notice that the convergence was written in [19] for a fixed window and for a Poisson point process such that the intensity goes to infinity. By scaling property of the Poisson point process, the result of [19] can be re-written as above for a fixed intensity and for a window $\mathbf{W}_\rho$ where $\rho \to \infty$.

Therefore, the extremal index of the minimum of inradii is

$$\theta = \frac{1}{2}.$$

It can be aslo explained by a trivial heuristic argument. Indeed, if a cell minimizes the inradius, one of its neighbors has to do the same. Hence, the mean cluster size of exceedances is 2. This justifies the fact that $\theta = 1/2$.

In our second example, we give the extremal index of the maximum of circumradii of a Poisson-Delaunay tessellation.

Example 2. Let $\mathfrak{m}_{PDT}$ be a Poisson-Delaunay tessellation of intensity 1 and let $\mathbf{X}$ be the underlying Poisson point process (of intensity $\gamma_{\mathbf{X}} = \beta_d^{-1}$ where β_d^{-1} is given in (3.3.21)). Denoting by $\mathscr{C}$ the typical cell of $\mathfrak{m}_{PDT}$, we deduce from a Taylor expansion of (3.3.14) that

$$\rho \cdot \mathbb{P}\left(R(\mathscr{C})^d \geq \frac{\log\left([(d-1)!]^{-1}\rho\log(\beta_d\rho)^{d-1}\right)+t}{\delta_d}\right) \underset{\rho\to\infty}{\longrightarrow} e^{-t}$$

for all $t \in \mathbf{R}$. Moreover, considering the dual Voronoi tessellation of $\mathfrak{m}_{PDT}$, we have

$$\max_{x\in\mathbf{X}\cap\mathbf{W}_\rho} R(C_{\mathbf{X}}(x)) = \max_{\substack{C\in\mathfrak{m}_{PDT}\\ V(C)\cap\mathbf{W}_\rho\neq\varnothing}} R(C) \tag{3.6.8}$$

where $V(C)$ is the set of vertices of the Delaunay cell $C \in \mathfrak{m}_{PDT}$. The asymptotic behaviour of the maximum of circumradii of a Poisson-Voronoi tessellation is already known (see (2c) in [19]). This is given by

$$\mathbb{P}\left(\max_{x\in\mathbf{X}\cap\mathbf{W}_\rho} R(C_{\mathbf{X}}(x))^d \leq \frac{\log\left(\alpha_{d,6}\beta_d\rho\log(\beta_d\rho)^{d-1}\right)+t}{\delta_d}\right) \underset{\rho\to\infty}{\longrightarrow} e^{-e^{-t}} \tag{3.6.9}$$

where

$$\alpha_{d,6} := \frac{1}{d!}\left(\frac{\pi^{1/2}\Gamma\left(\frac{d}{2}+1\right)}{\Gamma\left(\frac{d+1}{2}\right)}\right)^{d-1}.$$

With a similar method as in Lemma 4.1. in [55], we can show that the boundary cells of the Poisson-Delaunay tessellation (i.e. the cells which intersect the boundary of $\mathbf{W}_\rho$) do not affect the behaviour of the maximum. Hence, the rate of $\max_{\substack{C\in\mathfrak{m}_{PDT}\\ V(C)\cap\mathbf{W}_\rho\neq\varnothing}} R(C)$ is the same as $\max_{\substack{C\in\mathfrak{m}_{PDT}\\ z(C)\in\mathbf{W}_\rho}} R(C)$. We then deduce from (3.6.8) and (3.6.9) that

$$\mathbb{P}\left(\max_{\substack{C\in\mathfrak{m}_{PDT}\\ z(C)\in\mathbf{W}_\rho}} R(C)^d \leq \frac{\log\left([(d-1)!]^{-1}\rho\log(\beta_d\rho)^{d-1}\right)+t}{\delta_d}\right) \underset{\rho\to\infty}{\longrightarrow} e^{-e^{-t}\times\theta}$$

where

$$\theta = \alpha_{d,6}\beta_d(d-1)! = \frac{(d^3+d^2)\Gamma\left(\frac{d^2}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}{2^{d+1}d\Gamma\left(\frac{d^2+1}{2}\right)\Gamma\left(\frac{d+2}{2}\right)}.$$

In particular, when $d = 1, 2, 3$, the extremal indices are respectively $\theta = 1$, $\theta = 1/2$ and $\theta = 35/128$. The fact that $\theta = 1$ when $d = 1$ follows from Theorem 3.1.1 which is available since the associated function $G_2(\cdot)$ converges to 0. This is not the case in higher dimension.

We hope to be able to develop a systematic method to estimate the extremal index in a future work.

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Chapitre 4

Travaux en cours et perspectives

Sommaire

In this chapter, we present our works in progress. This work is in collaboration with R. Hemsley, a PhD student in computer science at the INRIA of Sophia Antipolis. We establish below several partial results to investigate further extreme values of Poisson-Voronoi and Poisson-Delaunay tessellations in $\mathbf{R}^2$. In particular, we present many new conjectures using simulations and heuristic arguments.

Let χ be a locally finite closed subset in $\mathbf{R}^2$. We recall that for each $x \in \chi$, the Voronoi cell of nucleus x is the set

$$C_\chi(x) = \{y \in \mathbf{R}^2, |y - x| \leq |y - x'|, x' \in \chi\}.$$

When $\chi = \mathbf{X}$ is a Poisson point process in $\mathbf{R}^2$, we denote by $\mathfrak{m}_{PVT} = \{C_{\mathbf{X}}(x), x \in \mathbf{X}\}$ the Poisson-Voronoi tessellation. Moreover, let us recall that there exists an edge between two points $x, x' \in \mathbf{X}$ in the Delaunay graph if and only if they are Voronoi neighbors i.e. $C_\chi(x) \cap C_\chi(x') \neq \varnothing$. The set of these edges induces a random partition of the space into simplices which is the so-called Delaunay tessellation that we denote by $\mathfrak{m}_{PDT}$. In particular, three points of $\mathbf{X}$ define a Delaunay triangle if and only if their circumballs contains no point of $\mathbf{X}$ in its interior. The random tessellations above are observed in the square

$$\mathbf{W}_\rho = \rho^{1/2}[0,1]^2$$

where ρ goes to infinity.

In this chapter, we investigate the three following extremes :

- The minimum of the areas of the cells of a Poisson-Voronoi tessellation,
- The minimum of the angles of the triangles of a Poisson-Delaunay tessellation and the extremal index,

- The maximum of the number of vertices of the cells of a Poisson-Voronoi tessellation i.e. the maximum of the degrees of the associated Poisson-Delaunay tessellation.

The end of the chapter is devoted to several perspectives. Each section can be read independently.

4.1 Minimum of the areas of a planar Poisson-Voronoi tessellation

Let $\mathbf{X}$ be a Poisson point process of intensity 1 in $\mathbf{R}^2$. In this section, we are interested in the minimum of the areas of the Voronoi cells induced by $\mathbf{X}$ i.e. the random variable $A_{\min,PVT}(\rho)$ defined as

$$A_{\min,PVT}(\rho) = \min_{x\in\mathbf{X}\cap\mathbf{W}_\rho} \lambda_2(C_{\mathbf{X}}(x))$$

where ρ goes to infinity. We present several new results and several conjectures on the asymptotic behaviour of $A_{\min,PVT}(\rho)$ and on the shape of the cell which minimizes the area.

According to section 3.1, the knowledge of the distribution of $\lambda_2(\mathscr{C})$ is enough to investigate the minimum of the areas where $\mathscr{C}$ is the typical cell of the Poisson-Voronoi tessellation. Let us recall that

$$\mathscr{C} \overset{\mathcal{D}}{=} C_{\mathbf{X}\cup\{0\}}(0).$$

To investigate the distribution of $\lambda_2(\mathscr{C})$ in the neighborhood of 0, we proceed in the same spirit as for the minimum of volumes of the Voronoi flowers and the minimum of distances to the farthest neighbor of the nucleus (section 3.4). More precisely, denoting by $N(\mathscr{C})$ the number of vertices of the typical cell, we write that

$$\mathbb{P}\left(\lambda_2(\mathscr{C}) < v\right) = \sum_{k=3}^{\infty} \mathbb{P}\left(\lambda_2(\mathscr{C}) < v, N(\mathscr{C}) = k\right) \tag{4.1.1}$$

for all $v \geq 0$.

The terms of the series can be made explicit. To do it, we recall some notations. For all $k \geq 1$, we consider the set

$$A_k = \{\mathbf{x}_{1:k} = (x_1, \ldots, x_k) \in (\mathbf{R}^2)^k, C_{\{\mathbf{x_{1:k}}\}\cup\{0\}}(0) \text{ is a convex polytope with } k \text{ vertices }\}.$$

Moreover, let us recall that for each k-tuple of points $\mathbf{x}_{1:k} = (x_1, \ldots, x_k)$ in $\mathbf{R}^2$ we denote by $\mathcal{F}(C_{\{\mathbf{x_{1:k}}\}\cup\{0\}}(0))$ the Voronoi flower of nucleus 0 i.e.

$$\mathcal{F}(C_{\{\mathbf{x_{1:k}}\}\cup\{0\}}(0)) = \bigcup_{y\in C_{\{\mathbf{x_{1:k}}\}\cup\{0\}}(0)} B(y, |y|).$$

Now, we can provide an integral representation of the terms of the series. Indeed, according to Proposition 1 in [17], we obtain for all $k \geq 3$

$$\begin{aligned}
&\mathbb{P}\left(\lambda_2(\mathscr{C}) < v, N(\mathscr{C}) = k\right) \\
&= \frac{1}{k!}\int_{(\mathbf{R}^2)^k} \mathbb{1}_{\lambda_2(C_{\{\mathbf{x_{1:k}}\}\cup\{0\}}(0))<v} \mathbb{1}_{A_k}(\mathbf{x}_{1:k}) e^{-\lambda_2\left(\mathcal{F}(C_{\{\mathbf{x_{1:k}}\}\cup\{0\}}(0))\right)} d\mathbf{x}_{1:k} \\
&\qquad = \frac{v^k}{k!}\int_{(\mathbf{R}^2)^k} \mathbb{1}_{\lambda_2(C_{\{\mathbf{x_{1:k}}\}\cup\{0\}}(0))<1} \mathbb{1}_{A_k}(\mathbf{x}_{1:k}) e^{-v\lambda_2\left(\mathcal{F}(C_{\{\mathbf{x_{1:k}}\}\cup\{0\}}(0))\right)} d\mathbf{x}_{1:k}.
\end{aligned} \tag{4.1.2}$$

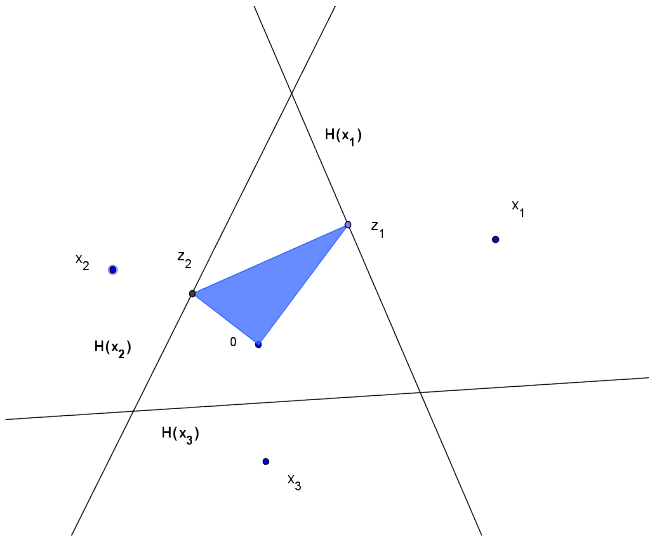

FIGURE 4.1 – Voronoi cell with three vertices.

For each $k \geq 3$, let us consider the set

$$B_k = A_k \cap \left\{ \mathbf{x}_{1:k} = (x_1, \ldots, x_k), \lambda_2(C_{\{\mathbf{x_{1:k}}\}\cup\{0\}}(0)) < 1 \right\}.$$

Let us denote by I_k the volume of B_k i.e.

$$I_k = \lambda_{2k}(B_k) = \int_{(\mathbf{R}^2)^k} \mathbb{1}_{B_k}(\mathbf{x}_{1:k}) d\mathbf{x}_{1:k}.$$

The following proposition provides a first idea to investigate the asymptotic behaviour of the minimum of the areas.

Proposition 4.1.1. *The integral $I_3 = \lambda_6(B_3)$ is finite. Moreover, we have*

$$\mathbb{P}\left(\lambda_2(\mathscr{C}) < v, N(\mathscr{C}) = 3\right) \underset{v \to 0}{\sim} \frac{I_3}{6} \cdot v^3. \tag{4.1.3}$$

Proof of Proposition 4.1.1. To show that I_3 is finite, we need the following lemma :

Lemma 4.1.2. *For all $\mathbf{x}_{1:3} = (x_1, x_2, x_3) \in B_3$, we have :*

$$|x_2| \leq 8|x_1|^{-1} \text{ and } |x_3| \leq 8|x_1|^{-1}.$$

Proof of Lemma 4.1.2. It is enough to show that $|x_2|$ is lower than $8|x_1|^{-1}$.

First, we present some notations. For each $i = 1, 2, 3$, we denote by $\mathcal{H}(x_i)$ the bisecting line of $[0, x_i]$. Let z_1, z_2 be two points in $\mathbf{R}^2$ such that $z_i \in C_{\{\mathbf{x_{1:3}}\}\cup\{0\}}(0) \cap \mathcal{H}(x_i)$, $i = 1, 2$ and such that the lines $(0z_1)$ and $(0z_2)$ are orthogonal.

Since a Voronoi cell is convex, the triangle $0z_1z_2$ is included in $C_{\{\mathbf{x_{1:3}}\}\cup\{0\}}(0)$. In particular, since $(x_1, x_2, x_3) \in B_3$, we have

$$\frac{1}{2} \cdot |z_1| \cdot |z_2| = \lambda_2(0, z_1, z_2) \leq 1.$$

Because $\frac{|x_1|}{2} \leq |z_1|$ and because $\frac{|x_2|}{2} \leq |z_2|$, we deduce from the previous inequality that

$$|x_2| \leq 8|x_1|^{-1}.$$

□

Now, we can prove that I_3 is finite. Indeed, let us write I_3 as

$$I_3 = 3\int_{B(0,1)}\int_{(\mathbf{R}^2)^2} \mathbb{1}_{B_3}(x_1,x_2,x_3)\mathbb{1}_{|x_2|\leq|x_1|,|x_3|\leq|x_1|}dx_1dx_2dx_3 \\ + 3\int_{B(0,1)^c}\int_{(\mathbf{R}^2)^2} \mathbb{1}_{B_3}(x_1,x_2,x_3)\mathbb{1}_{|x_2|\leq|x_1|,|x_3|\leq|x_1|}dx_1dx_2dx_3 = I_{3,1} + I_{3,2}.$$

The integral $I_{3,1}$ is finite since it is the volume of a bounded set. For the second integral, we deduce from Lemma 4.1.2 that

$$I_{3,2} \leq c\cdot\int_{B(0,1)^c} |x_1|^{-4}dx_1 = c\cdot\int_1^\infty r^{-3}dr.$$

The previous inequality shows that $I_{3,2}$ and I_3 are finite.

Besides, the asymptotic expansion (4.1.3) is a consequence of (4.1.2), the fact that I_3 is finite and the dominated convergence theorem. □

To obtain a heuristic estimate of I_3, we have done several simulations with the method of Monte-Carlo. More precisely, we have considered 2.10^{11} iid points uniformly distributed in the product of disks $B(0,\alpha)^3$ for many values of α which are chosen between 1 and 1000. Then we have counted the number of triplets of points $\mathbf{x}_{1:3} = (x_1,x_2,x_3)$ which fall in $B_3 \cap B(0,\alpha)^3$. The values of α which seems to be likely are in the interval $[2,10]$. Taking the mean ratio of points which fall in $B_3 \cap B(0,\alpha)^3$ for many values of $\alpha \in [2,10]$, we obtain that $I_3 \simeq 10.57$. This estimate will be legitimated at the end of the section by several simulations on Poisson-Voronoi tessellations.

Besides, to investigate the behaviour of $\mathbb{P}(\lambda_2(\mathscr{C}) < u)$ in the neighborhood of 0, we have to show that the integrals I_k converge for all $k \geq 4$. It seems to be true when we use several arguments that are given in the proof of Proposition 4.1.1. Nevertheless, the proof is technical. Our questions are the following :

Question 1. 1. Is it true that the integrals I_k, $k \geq 4$ are finite ?

2. If so, does the radius of convergence of the power series $\sum_{k=3}^\infty I_k v^k$ differ from 0 ?

If so, according to (4.1.1), (4.1.2) and (4.1.3), we should obtain that $\mathbb{P}(\lambda_2(\mathscr{C}) < v)$ is of same order as $\frac{I_3}{6}\cdot v^3$ when v tends to 0. First, we could deduce that $\mathbb{P}(N(\mathscr{C}) = 3|\lambda_2(\mathscr{C}) < v)$ converges to 1. In particular, this could imply that the typical cell is asymptotically a triangle when its area is small. Moreover, we could derive that $\rho\mathbb{P}\left(6^{-1/3}I_3^{1/3}\rho^{1/3}\lambda_2(\mathscr{C}) \leq t\right)$ converges to t^3 for all $t \geq 0$.

Besides, we have seen in section 3.1 that the so-called Condition 2 guarantees that the knowledge of the geometrical characteristic of the typical cell is enough to investigate the extremes. This leads to a second question :

Question 2. Is CONDITION 2 satisfied i.e. is it true that

$$\frac{\rho}{\log \rho}\mathbb{E}\left[\sum_{(x,y)_{\neq}\in(\mathbf{X}\cap\mathfrak{C}_\rho)^2} \mathbb{1}_{\lambda_2(C_{\mathbf{X}}(x))<v_\rho}\mathbb{1}_{\lambda_2(C_{\mathbf{X}}(y))<v_\rho}\right] \underset{\rho\to\infty}{\longrightarrow} 0$$

where

$$v_\rho = v_\rho(t) = (6I_3^{-1})^{1/3}\rho^{1/3}t, t \geq 0$$

and $\mathfrak{C}_\rho = c \cdot [0, \log \rho^{1/2}]^2$?

It should be true. Indeed, if a Voronoi cell has a small area, it should not imply the same properties for the cells of its neighborhood.

Conjecture 1. With the same notations as above, we have :

1. $\mathbb{P}\left(6^{-1/3}I_3^{1/3}\rho^{1/3}A_{\min,PVT}(\rho) \geq t\right) \underset{t\to\infty}{\longrightarrow} e^{-t^3}, t \geq 0.$
2. $\mathbb{P}\left(\forall x \in \mathbf{X}, \lambda_2(C_{\mathbf{X}}(x)) = A_{\min,PVT}(\rho) \Longrightarrow N(C_{\mathbf{X}}(x)) = 3\right) \underset{\rho\to\infty}{\longrightarrow} 1.$

To prove Conjecture 1, it is enough to answer positively to Questions 1 and 2. Indeed, the first assertion should be a consequence of Theorem 3.1.1 whereas the second one is a consequence of the fact that

$$\mathbb{P}\left(N(\mathscr{C}) = 3|\lambda_2(\mathscr{C}) < u\right) \underset{u\to 0}{\longrightarrow} 1.$$

Let us recall that we have already shown a similar result in section 2.3 i.e. the cell minimizing the circumradius is a triangle. An illustration of the first assertion of the previous conjecture is given in Figure 4.2. In this figure, we present the (normalized) empirical density of $A_{\min,PVT}(\rho_0)$ with $\rho_0 = 10^6$ for a planar Poisson-Voronoi tessellation of intensity 1 and observed in the square $\rho_0^{1/2}[0,1]^2$.

Concretely, we have done 75000 independent simulations indépendants as follows : for each simulation, we consider a Poisson point process of intensity $0.8^{-2}\cdot 10^6 \simeq 1562500$ in the window $[0,1]^2$ and we keep only the cells such that the nucleus is in $[0.1, 0.9]^2$. We choose to omit the cells in a strip of width 0.1 to avoid boundary effects. Let us remark that theses strips are quite large since the maximum of circumradii is of order $\log(1562500)/1562500 \simeq 9.10^{-6}$ according to Theorem 2.1.1. Then we normalize the results, taking $I_3 = 10.57$ (page 112), to be in the framework of a Poisson-Voronoi tessellation of intensity 1 and observed in the window $\rho_0^{1/2}[0,1]^2$, $\rho_0 = 10^6$.

To check that the estimate $I_3 \simeq 10.57$ is good, we have done a different approximation as follows : assuming the the convergence in distribution given in Conjecture 1 is also a convergence of the means, we have

$$I_3 = 6 \cdot \lim_{\rho\to\infty} \rho^{-1}\left(\frac{\int_0^\infty e^{-t^3}dt}{\mathbb{E}\left[A_{\min,PVT}(\rho)\right]}\right)^3.$$

When $\rho = \rho_0 = 10^6$, the expectation of the empirical density based on 75000 simulations of the minimum of areas (Figure 4.2) is

$$\widetilde{\mathbb{E}}[A_{\min,PVT}(\rho_0)] \simeq 0.007359.$$

Hence, an other estimate of I_3 is 10.72. This is close to the value 10.57.

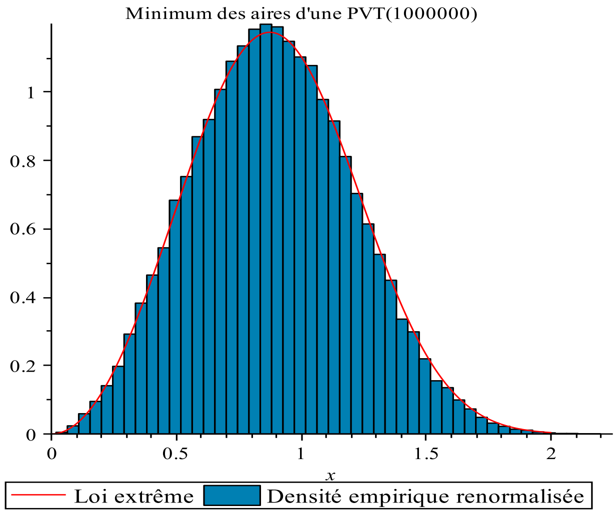

FIGURE 4.2 – (Red) Limit distribution of density $t \mapsto 3t^2 e^{-t^3}$. (Blue) Normalized empirical density of the minimum of areas of a PVT of intensity 1, for the cells included in $W_{\rho_0} = 10^3 \cdot [0,1]^2$, and based on 75000 simulations.

4.2 Minimum of the angles of a Poisson-Delaunay tessellation and clusters of exceedances

Minimum of the angles of $\mathfrak{m}_{PDT}$. Let $\mathbf{X}$ be a Poisson point process of intensity $\beta_2 = \frac{1}{2}$ in $\mathbf{R}^2$. In this section, we are interested in the minimum of the angles of the triangles of the Poisson-Delaunay tessellation $\mathfrak{m}_{PDT}$ (of intensity 1) induced by $\mathbf{X}$. We denote this random variable by :

$$\alpha_{\min,PDT}(\rho) = \min_{\substack{C \in \mathfrak{m}_{PDT} \\ z(C) \in \mathbf{W}_\rho}} \alpha_{\min}(C).$$

Here, $z(C)$ and $\alpha_{\min}(C)$ are the circumradius and the minimal angle of C.

In 1991, Bern *et al.* estimate the expectation of the above random variable. More precisely, (see Theorem 3 in [12]), they show that :

$$\mathbb{E}\left[\alpha_{\min,PDT}(\rho)\right] = \Theta(\rho^{-1/2}) \tag{4.2.1}$$

where $\phi(\rho) = \Theta(\psi(\rho))$ means $c_1 \leq \frac{\phi(\rho)}{\psi(\rho)} \leq c_2$ for all functions $\phi, \psi$with $c_1, c_2 > 0$. Our aim is to explore the two following quantities :

1. the convergence in distribution of $\alpha_{\min,PDT}(\rho)$ (when it is normalized by $\rho^{1/2}$ and the limit distribution.
2. the extremal index θ.

As usual, to investigate the limit distribution of $\alpha_{\min,PDT}(\rho)$, we first study the minimal angle of the Poisson-Delaunay typical cell $\mathscr{C}$. We denote this random variable by $\alpha_{\min}(\mathscr{C})$. In 1977, Mardia *et al.* [87] give an integral representation of this random variable. More precisely they show that (see e.g. (5.11.13) in [108])

$$\mathbb{P}\left(\alpha_{\min}(\mathscr{C}) < v\right) = \frac{2}{\pi} \int_0^{v \wedge \frac{\pi}{3}} \left((\pi - 3x)\sin 2x + \cos 2x - \cos 4x\right) dx$$

for all $v \geq 0$. In particular, $\mathbb{P}\left(\alpha_{\min}(\mathscr{C}) < v\right)$ has the same order as $2v^2$ in the neighborhood of 0. This implies that, for all $t \geq 0$,

$$\rho\mathbb{P}\left(\alpha_{\min}(\mathscr{C}) < (2\rho)^{-1/2}t\right) \underset{\rho\to\infty}{\longrightarrow} t^2.$$

Let us remark that we obtain the same order of $\alpha_{\min}(\mathscr{C})$ as Bern *et al.* according to (4.2.1).

From Proposition 3.1.3 (see section 3.1), there exists two positive numbers $0 \leq \theta \leq \theta' \leq 1$ such that, for all $t \geq 0$

$$\limsup_{\rho\to\infty} \mathbb{P}\left(\alpha_{\min,PDT}(\rho) \geq (2\rho)^{-1/2}t\right) = e^{-\theta t^2} \text{ et } \liminf_{\rho\to\infty} \mathbb{P}\left(\alpha_{\min,PDT}(\rho) \geq (2\rho)^{-1/2}t\right) = e^{-\theta' t^2}.$$

If $\theta = \theta'$, we recall that θ is the so-called extremal index of $\alpha_{\min,PDT}(\rho)$ (section3.1, page 3.1). Moreover, let us recall that, for a sequence of real random variables, the extremal index describes the mean size of a cluster of exceedances (section 1.2, page 38). A natural question is : does the extremal index of $\alpha_{\min,PDT}(\rho)$ exist ? If so, is it different from 0 ? Can we estimate it ? Is it also the mean size of a cluster of exceedances ?

We do not answer to these questions formally. Nevertheless, we give some partial ideas. We think that the extremal index exists and differs from 0 since, if not, the order of $\alpha_{\min,PDT}(\rho)$ should be different from the order of $\alpha_{\min}(\mathscr{C})$. That is why we conjecture the following fact :

Conjecture 2. The extremal index $\theta \in [0, 1]$ of the minimum of the angles exists and differs from 0 i.e. for all $t \geq 0$, we have :

$$\mathbb{P}\left((2\rho)^{1/2}\alpha_{\min,PDT}(\rho) \geq t\right) \underset{\rho\to\infty}{\longrightarrow} e^{-\theta t^2}.$$

An interesting topic is to estimate the extremal index. Many estimates about the extremal index were established for sequences of real random variables and processes, see e.g. [2], [43] et [49]. To the best of our knowledge, these works have not been applied neither for discrete random variables nor for tessellations. A heuristic method (which is derived in work in progress) to estimate the extremal index of the minimum of the angles is to proceed as follows : *assuming* that the convergence in distribution which appears in the conjecture 2 is also a convergence of the expectations, we have :

$$(2\rho)^{1/2}\mathbb{E}\left[\alpha_{\min,PDT}(\rho)\right] \underset{\rho\to\infty}{\longrightarrow} \int_0^\infty e^{-\theta t^2} dt.$$

This implies that

$$\theta = \lim_{\rho\to\infty} \frac{\pi}{8} \cdot \left(\rho\mathbb{E}\left[\alpha_{\min,PDT}(\rho)\right]^2\right)^{-1}. \tag{4.2.2}$$

When $\rho = \rho_0 = 2 \cdot 10^6$ (i.e. when the mean number of points of $\mathbf{X}$ in $W_{\rho_0} = \rho_0^{1/2}[0,1]^2$ is one million), the empirical mean of $\alpha_{\min,PDT}(\rho_0)$ based on 75000 independent simulations is :

$$\widetilde{\mathbb{E}}\left[\alpha_{\min,PDT}(\rho_0)\right] \simeq 5.12 \cdot 10^{-4}.$$

According to (4.2.2), we obtain that

$$\theta \simeq \hat{\theta} = 0.75. \tag{4.2.3}$$

In Figure 4.3, we present the empirical density of $(2\rho_0)^{1/2}\alpha_{\min,PDT}(\rho_0)$ and the graph of the function $y(t) = 2\hat{\theta} \cdot te^{-\hat{\theta}t^2}, t \geq 0$. Let us remark that Figure 4.3 corrobores Conjecture 2.

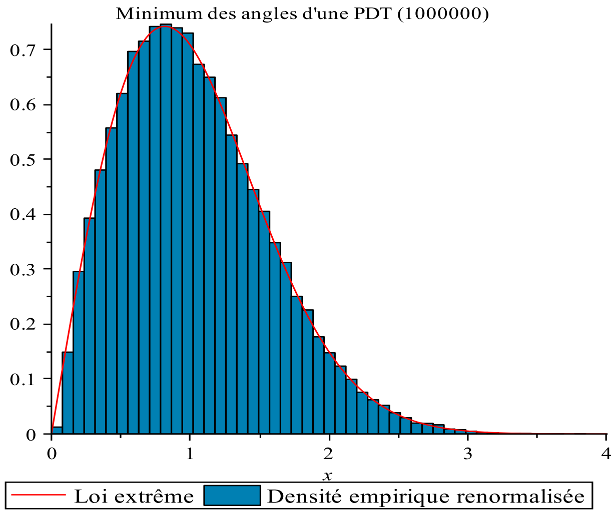

FIGURE 4.3 – (Red) Limit distribution of density $t \mapsto 2\hat{\theta} \cdot te^{-\hat{\theta}t^2}$ where $\hat{\theta} = 0.75$. (Blue) Empirical density of the minimum of the angles of a PDT of intensity $\frac{1}{2}$, for the cells included in $W_{\rho_0} = (2 \cdot 10^6)^{1/2}[0,1]^2$ and based on 75000 simulations.

Clusters of exceedances of a random tessellation In this subsection, we present a geometrical interpretation of the extremal index of a random tessellation. First, we recall the extremal index of a sequence of real random variables equals the proportion of the number of upcrossings divided by the number of exceedances. Besides, the extremal index is the inverse of the mean size of a cluster of exceedances (see section 1.2 for more details).

For a general random tessellation $\mathfrak{m}$ in $\mathbf{R}^d$, we have to explain what we mean by a cluster. Let v be a threshold, $f : \mathcal{K}_d \to \mathbf{R}$ a geometrical characteristic and

$$M_f(\rho) = \max_{\substack{C \in \mathfrak{m}, \\ z(C) \in \mathbf{W}_\rho}} f(C).$$

Let us recall that an exceedance cell (associated to v) is a cell $C \in \mathfrak{m}$ such that $f(C) > v$ (if we investigate a minimum, the inequality has to be reversed and we say that C is an exceedance when $f(C) < v$). A natural concept is to say that a "cluster of exceedances" is a family of cells of $\mathfrak{m}$ such that the union is connex and such that each cell is an exceedance.

Figure 4.4 presents an illustration of what we mean by "cluster of exceedances". This figures is a simulation of a planar Poisson-Delaunay tessellation, observed in the square $(300)^{1/2}[0,1]^2$. Yellow cells are the cells such that the minimal angle is lower than $v_0 \simeq 0.205$. Here, the number of clusters of exceedances equals 16 whereas the number of exceedances is 20. In particular, the inverse of the mean size of a cluster of exceedances is approximately 0.8. A natural question is :

Question 3. Let us assume that θ is the extremal index associated to a random tessellation $\mathfrak{m}$ and to a geometrical characteristic $f : \mathcal{K}_d \to \mathbf{R}$ i.e. $\rho\mathbb{P}\left(f(\mathscr{C}) > v_\rho\right) \underset{\rho\to\infty}{\longrightarrow} \tau$ and $\mathbb{P}\left(M_f(\rho) \leq v_\rho\right) \underset{\rho\to\infty}{\longrightarrow} e^{-\theta\tau}$. Is it true that

$$\theta = \lim_{\rho\to\infty} \frac{C_\rho}{N_\rho}$$

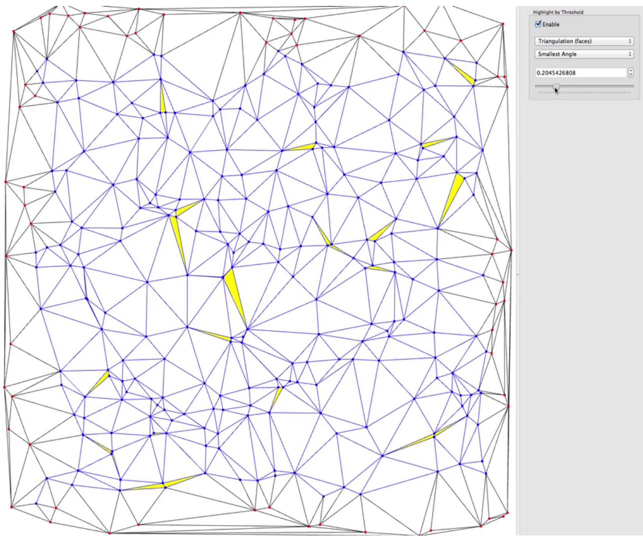

FIGURE 4.4 – Clusters of exceedances of a PDT(300) for the threshold $v \simeq 0.205$.

where N_ρ and C_ρ denote the number of exceedances and the number of clusters of exceedances respectively ?

The type of the above convergence has to be made explicit (e.g. it could be a convergence in distribution).

Let us notice that the proportion 0.8 of the number of clusters of exceedances divided by the number of exceedances (Figure 4.4) is close to the estimation of the extremal index given in (4.2.3) i.e. $\hat{\theta} = 0.75$.

4.3 Maximum of the number of vertices of a Poisson-Voronoi tessellation or maximal degree of a Poisson-Delaunay tessellation

Let $\mathbf{X}$ be a Poisson point process of intensity 1 and let us denote by $N(C_{\mathbf{X}}(x))$ the number of vertices of the Voronoi cell $C_{\mathbf{X}}(x)$ foe each $x \in \mathbf{X}$. In this section, we are interested in the maximum of this number i.e.

$$N_{\max,PVT}(\rho) = \max_{x \in \mathbf{X} \cap \mathbf{W}_\rho} N(C_{\mathbf{X}}(x))$$

when ρ goes to infinity. Let us notice that $N_{\max,PVT}(\rho)$ is also the maximal degree of the associated Delaunay graph.

It seems that Theorem 3.1.1 (see section 3.1) cannot be applied to investigate the behaviour of $N_{\max,PVT}(\rho)$. Indeed, according to several simulations given in Table 1 in [16], we think that the tail of the *discrete* number of vertices of the Poisson-Voronoi typical cell is exponentially decreasing and has the same behaviour of the tail of a Poisson random variable. Besides, a Poisson random variable does not belong to the domain of attraction of a classical extreme distribution

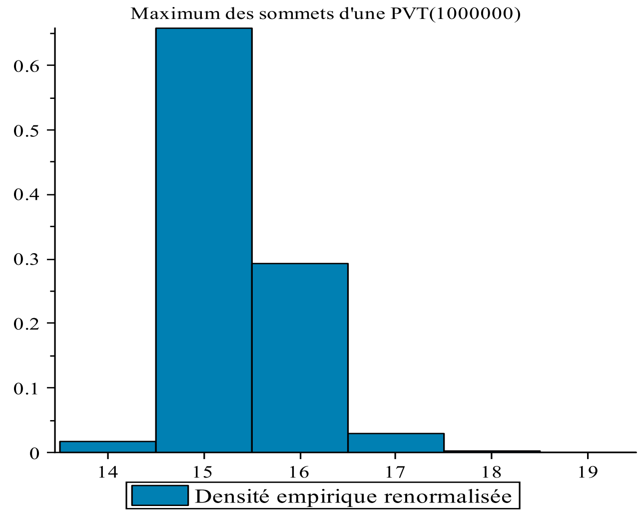

FIGURE 4.5 – Distribution of the maximum of the number of vertices of a Poisson-Voronoi tessellation, associated to a Poisson point process of intensity 1, observed in the window $W_{\rho_0} = 10^3[0,1]^2$ and based on 75000 simulations.

(i.e. Fréchet, Gumbel or Weibull distributions). More precisely, if $(X_i)_{i\geq 1}$ is a sequence of iid Poisson random variables, we cannot find a function $u_n = u_n(t) = a_n t + b_n$ such that

$$\mathbb{P}\left(M_n \leq u_n\right) \underset{n\to\infty}{\longrightarrow} e^{-\tau(t)}$$

where $M_n = \max_{i\leq n} X_i$ and where $\tau(t)$ is not degenerate. An other property of the Poisson distribution is that the maximum M_n is (asymptotically) reduced to two consecutive terms. More precisely, Anderson [3] shows that for a particular class of discrete random variables (including the Poisson distribution), there exists a sequence $(I_i)_{i\geq 1}$ such that :

$$\mathbb{P}\left(M_n = I_n \text{ ou } I_{n+1}\right) \underset{n\to\infty}{\longrightarrow} 1.$$

Such property has been used by Penrose (see Theorem 6.6 in [113]) to show that the maximal degree of a particular case of random geometric graph is reduced on two consecutive values. For the heuristic arguments given above, we think that this property hold for the maximal degree of a Poisson-Delaunay tessellation (or the maximum of the number of vertices of a Poisson-Voronoi tessellation).

In Figure 4.5, we note that the maximum of the number of vertices of a Poisson-Voronoi tessellation, induced by a Poisson point process of intensity 1 and observed in the window $W_{\rho_0} = 10^3[0,1]^2$, takes its values in $\{15, 16\}$. In particular, this corroborates our conjecture.

Moreover, Bern *et al.* estimate the mean of the maximal degree of a Poisson-Delaunay tessellation . More precisely (see Theorems 5 and 6 in [12]), they show that

$$\mathbb{E}\left[N_{\max,PVT}(\rho)\right] = \Theta\left(\frac{\log\rho}{\log\log\rho}\right).$$

According to the above facts, we conjecture the following result :

Conjecture 3. There exists a function $I_\rho \underset{\rho\to\infty}{\sim} \frac{\log \rho}{\log\log \rho}$ such that

$$\mathbb{P}\left(N_{\max,PVT}(\rho) = I_\rho \text{ ou } I_\rho + 1\right) \underset{\rho\to\infty}{\longrightarrow} 1.$$

4.4 Several perspectives

In the previous sections, we have established three conjectures that are interesting to prove. We present below six other perspectives.

Joint distributions Our first perspective concerns joint distributions. Let us consider, for example, a Poisson-Voronoi tessellation $\mathfrak{m}_{PVT}$ and let us denote for each cell $C \in \mathfrak{m}_{PVT}$ by $r(C)$ and $R(C)$ the inradius and the circumradius of C respectively and by $C^{(r)}_{\max}(\rho)$ the cell which maximizes the inradii. Of particular interest is the couple $\left(r(C^{(r)}_{\max}(\rho)), R(C^{(r)}_{\max}(\rho))\right)$ since it could provide more precise results on the shape of $C^{(r)}_{\max}(\rho)$. Another interesting couple is $\left(l(C^{(R)}_{\min}(\rho)), L(C^{(R)}_{\min}(\rho))\right)$ where $C^{(R)}_{\min}(\rho) \in \mathfrak{m}_{PVT}$ is the cell of a Poisson-Voronoi tessellation which minimizes the circumradii and where $l(C)$ and $L(C)$ denote the lengths of the largest and the smallest edges of a cell $C \in \mathfrak{m}_{PVT}$ respectively. Indeed, we have seen (Corollary 2.3.4) that $C^{(R)}_{\min}(\rho)$ is a simplex when ρ goes to infinity. The investigation of the couple $\left(l(C^{(R)}_{\min}(\rho)), L(C^{(R)}_{\min}(\rho))\right)$ can be useful to know if this simplex is regular. A third example concerns the couple $(r_{\max}(\rho), R_{\min}(\rho))$ which can be interesting to have a better understanding of the regularity of a Poisson-Voronoi tessellation. We think that an efficient method to investigate joint distributions consists in generalizing Theorem 3.1.1.

Hausdorff distance Another question concerns the Hausdorff distance between a convex body and its Poisson-Voronoi approximation. In Corollary 2.4.2, we have shown that this distance can be bounded by $\log \gamma/\gamma$ where the intensity γ converges to infinity. We think that this upper bound provides a good estimate since the Hausdorff distance should be of the same order as the maximum of circumradii. Besides, it should be interesting to find a lower bound of this distance or to establish a limit theorem.

Extremal index As announced in section 4.2, another perspective concerns the extremal index of a random tessellation. Such a notion provides also a better understanding of a random tessellation since it describes the mean size of a cluster of exceedances. The notion of extremal index can lead to many questions. In probability theory, it could be interesting to investigate its existence and to describe the distribution of a "typical cluster" i.e. a cluster which is chosen randomly and uniformly. In statistics, the main problem is to provide a good estimator of this index. A natural method to estimate it is to consider a suitable threshold and to study the mean size of the clusters. Nevertheless, this leads to the following questions : how do we choose the threshold ? Is the estimate consistent ? What is the rate of convergence of the estimator ?

Non-Poisson point processes Another perspective is to investigate more general models. First, we could consider Voronoi and Delaunay tessellations that are induced by non-Poisson point processes. In particular, it could be useful to compare the regularity of these tessellations when they are generated by "attractive" processes (e.g. Matérn cluster or determinantal point processes) or repulsive processes (e.g. Matérn hardcore or permanental point processes). Such a question could be useful since considering extremes on Voronoi or Delaunay tessellations could discriminate the underlying point processes. Nevertheless, this question seems to be hard since the Palm measure of a non-Poisson point process is not generally convenient.

Poisson hyperplane tessellations It could be interesting to investigate another type of tessellation such as the hyperplane tessellations. Such model can be constructed as follows : let Φ be a point process in $\mathbf{R}^d$. For each $x \in \Phi$, let $H_\Phi(x)$ be the hyperplane which contains x and which is orthogonal to the line $(0x)$. The intersection of the hyperplanes $H_\Phi(x)$, $x \in \Phi$ defines a partition of the Euclidean space into polyhedra. Such a partition is called "the hyperplane tessellation induced by Φ". When $\Phi = \mathbf{X}$ is a Poisson point process, we speak about Poisson hyperplane tessellation. Such a model has been introduced in 1945 in $\mathbf{R}^2$ by Goudsmit to investigate the trajectories of particles in bubble chambers. When the intensity measure of $\mathbf{X}$ is of the form $d\Theta_{\mathbf{X}} = \lambda dr d\sigma$ where $\lambda > 0$ and where $d\sigma$ is the uniform distribution on the unit sphere, the Poisson hyperplane tessellation is stationary and has a mixing property. Theorem 3.1.1 is not enough to investigate extremes on this tessellation since the k-dependence condition (see Condition 1, page 71) is not satisfied. Indeed, each hyperplane intersects an infinity number of cells so that the cells which are far away from each other can be dependent. Nevertheless, we think that Condition 1 can be weakened. Indeed, the proof could be an adaptation to our framework of several arguments of Leadbetter and Rootzen [84] who investigate random fields with a weak condition of dependence. In particular, we could establish a generalization of our theorem and apply it to derive extremes of Poisson hyperplane tessellations or more general tessellations.

Germ-grain models Another extension of our work concerns germ-grain models. These model can be constructed as follows : let Φ be a point process in $\mathbf{R}^d$ and let $\mathbb{Q}$ be a distribution on the set of convex bodies in $\mathbf{R}^d$. The germ-model associated to Φ and $\mathbb{Q}$ is the set $\bigcap_{x\in\Phi}(x + \Xi(x))$ where $\Xi(x)$, $x \in \Phi$ is a family of iid random sets that are distributed with respect to $\mathbb{Q}$. The distribution $\mathbb{Q}$ is the so-called typical distribution of a grain. When $\Phi = \mathbf{X}$ is a Poisson point process, we speak about *Boolean models*. This random set is extensively used to model sparse systems such as the deposit of potassium in sedimentation [107] or the distribution of heather in a forest [33]. For a wider panorama of applications, we refer to [140], page 62. Several results on the convergence of extremal quantities have already been established by Penrose (chapter 6 [113]). In the same spirit as in the proof of Theorem 3.1.1 (section 3.2), we can prove that Theorem 3.1.1 can be extended to Boolean models satisfying the following condition :

$$\text{there exists } M > 0, \text{ such that } \mathbb{Q}\left(\{\Xi \in \mathcal{C}_d, \Xi \subset B(0, M)\}\right) = 1.$$

Such a condition guarantees that the k-dependence condition (see Condition 1, page 71) is satisfied. It could be interesting to apply this result to investigate extremes of various geometrical characteristics of Boolean models (or germ-grain models) e.g. the maximum of the volumes and the maximum of diameters of connected components.

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