The double power monad is the composite power monad☆

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Abstract

In the context of quantaloid-enriched categories, we rely essentially on the classifying property of presheaf categories to give a conceptual proof of a theorem due to Höhle: the double power monad and the composite power monad, on the category of quantaloid-enriched categories, are the same. Via the theory of distributive laws, we identify the algebras of this monad to be the completely codistributive complete categories, and the homomorphisms between such algebras are the bicontinuous functors. With these results we hope to contribute to the further development of a theory of Q-valued preorders (in the sense of Pu and Zhang).

Keywords: Category theory, Fuzzy preorders, Non-classical logics

1. Introduction

If $P = (P, \leq)$ is an ordered set, then its downclosed subsets form a sup-lattice $(\mathsf{Dwn}(P), \subseteq)$, and the order-preserving inclusion $P \longrightarrow \mathsf{Dwn}(P)\colon x \mapsto\downarrow x$ has a left adjoint if and only if P has all suprema. Dually, taking the upclosed subsets of P produces an inf-lattice $(\mathsf{Up}(P), \supseteq)$ (note that upsets are ordered by containment, whereas downsets are ordered by inclusion), and the order-preserving inclusion $P \longrightarrow \mathsf{Up}(P)\colon x \mapsto\uparrow x$ has a right adjoint if and only if P has all infima. Of course, P is a sup-lattice if and only if it is an inf-lattice, and then it is said to be a 'complete lattice'.

These two object correspondences can be made functorial in several ways, and the resulting functors interact in at least two ways. For starters, the inverse image of an order-preserving function $f\colon P \longrightarrow Q$ is a new order-preserving function $f^{-1}\colon \mathsf{Dwn}(Q) \longrightarrow \mathsf{Dwn}(P)$. This action on objects and morphisms defines a 2-functor on the locally ordered category Ord of ordered sets which reverses arrows and local order; for the sake of this introduction, let us write it as $\mathcal{L}\colon \mathsf{Ord} \longrightarrow \mathsf{Ord}^{\mathsf{coop}}$. It then so happens that this is a left 2-adjoint, and that the action of its right 2-adjoint $\mathcal{R}\colon \mathsf{Ord}^{\mathsf{coop}} \longrightarrow \mathsf{Ord}$ on objects is $Q \mapsto \mathsf{Up}(Q)$. As a result, the induced 2-monad $\mathcal{T} := \mathcal{R}\mathcal{L}\colon \mathsf{Ord} \longrightarrow \mathsf{Ord}$ acts on objects as $P \mapsto \mathsf{Up}(\mathsf{Dwn}(P))$: it is the **double power monad** on Ord.

On the other hand it is well-known that the locally ordered category Sup of sup-lattices and sup-morphisms is included in Ord by a forgetful 2-functor $\mathcal{U}\colon \mathsf{Sup} \longrightarrow \mathsf{Ord}$, right 2-adjoint to an $\mathcal{F}\colon \mathsf{Ord} \longrightarrow \mathsf{Sup}$ whose action on objects is $P \mapsto \mathsf{Dwn}(P)$; a 2-monad $\mathsf{Dwn} = \mathcal{U}\mathcal{F}\colon \mathsf{Ord} \longrightarrow \mathsf{Ord}$ results, and its action on objects is $P \mapsto \mathsf{Dwn}(P)$. In a similar manner, because the forgetful 2-functor $\mathcal{V}\colon \mathsf{Inf} \longrightarrow \mathsf{Ord}$ admits a left 2-adjoint $\mathcal{G}\colon \mathsf{Ord} \longrightarrow \mathsf{Inf}$, their composition produces a 2-monad $\mathsf{Up} = \mathcal{V}\mathcal{G}\colon \mathsf{Ord} \longrightarrow \mathsf{Ord}$, whose action on objects is $Q \mapsto \mathsf{Up}(Q)$. Now it turns out that the composition of these 2-monads, $\mathcal{S} := \mathsf{UpDwn}\colon \mathsf{Ord} \longrightarrow \mathsf{Ord}$, is again a 2-monad, and its action on objects is thus $P \mapsto \mathsf{Up}(\mathsf{Dwn}(P))$: it is the **composite power monad** on Ord.

In this note we show how **the double power monad and the composite power monad are the same**. We prove this in the generality of **quantaloid-enriched categories**, of which not only ordered sets but also metric spaces [Lawvere, 1973], partial metric spaces [Höhle and Kubiak, 2011; Stubbe, 2014],

☆Submitted to Fuzzy Sets and Systems: February 19, 2015; this revised version: April 21, 2016.
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sheaves [Walters, 1982], fuzzy preorders and sets [Stubbe, 2014; Tao, Lai and Zhang, 2014], and several other structures, are instances. In doing so, the downsets/upsets of an orderd set must be replaced by contravariant/covariant presheaves on a quantaloid-enriched category. This result is due to U. Höhle [2014] (see his Theorem 5.10), for whom it is a stepping stone towards a definition of 'quantaloid-enriched topological spaces', but our (somewhat more conceptual) proof technique is different from his. Putting the **classifying property** of enriched presheaf categories central, our treatment of the double power monad follows easily from R. Street's [2012] characterisation of the **core of an adjunction**; and the theory surrounding the composite power monad is, quite naturally, that of J. Beck's [1969] **distributive laws**. A simple description (in fact, several equivalent descriptions, thanks to [Stubbe, 2007]) of the **algebras** of the double/composite power monad follows from all this: they are precisely the completely codistributive complete categories; and the homomorphisms between such algebras are precisely the bicontinuous functors.

The level of generality offered by quantaloid-enriched categories is, in our opinion, particularly appropriate in fuzzy logic and set theory. To make our point, let us recall Zadeh's definition of **fuzzy preorder** [1971]: it is a set X together with a map $P\colon X \times X \longrightarrow [0,1]$ such that $P(x,y) \wedge P(y,z) \leq P(x,z)$ and $1 \leq P(x,x)$ hold for any $x,y,z \in X$. This $[0,1]$-valued interpretation of the axioms of a preorder is easily generalised to obtain so-called **quantale-enriched categories**: replace $[0,1]$ by any quantale $Q = (Q, \circ, 1)$ (i.e. a complete residuated lattice), and replace $\wedge$ by the multiplication $\circ$ in Q. However, as analysed by [Höhle, 1995; Pu and Zhang, 2012], such fuzzy preorders in the sense of Zadeh are still defined on *crisp* sets—which, besides bringing about certain technical deficiencies, also makes them not quite "fully fuzzy". To amend this, these authors propose the following definition, in the case where Q is a divisible and commutative quantale (e.g. any continuous t-norm on $[0,1]$, or any complete Heyting algebra, or any BL-algebra): a Q-**valued preorder** is a set X together with a map $P\colon X \times X \longrightarrow Q$ such that $P(x,y) \leq P(x,x) \wedge P(y,y)$ and $P(x,y) \circ (P(y,y) \Rightarrow P(y,z)) \leq P(x,z)$ hold for any $x,y,z \in X$ (where $\Rightarrow$ is the residuation for the multiplication in Q). This new notion no longer fits in the theory of *quantale*-enriched categories, but it is still an example of a *quantaloid*-enriched category! Indeed, from any quantale Q one can construct the so-called quantaloid of diagonals $\mathcal{D}(Q)$, so that $\mathcal{D}(Q)$-enriched categories are precisely the same thing as Q-valued preorders [Höhle and Kubiak, 2011; Pu and Zhang, 2012]. Interestingly, such diverse mathematical structures as partial metric spaces and localic sheaves both are examples of Q-preorders in this sense (see [Stubbe, 2014] for an overview and [Hofmann and Stubbe, 2016] specifically for partial metrics); but also for further theoretical developments of fuzzy preorders the quantaloidal point of view has proven beneficial [Tao, Lai and Zhang, 2014; Pu and Zhang, 2014].

Acknowledgement. This work was first made public in September 2013 as a note in the preprint series *Cahiers du LMPA* of the Université du Littoral. It was directly inspired by a preprint that U. Höhle had sent me of [Höhle, 2014], following our participation in the *33rd Linz Seminar on Fuzzy Set Theory* in 2012. Further discussions with D. Zhang, especially during my stay at Sichuan University in 2014, convinced me to submit this paper for publication in this journal for fuzzy logicians and set theorists. The referee reports helped me to improve this paper.

2. Notations

Throughout this note we shall use the definitions and notations for quantaloid-enriched categories, distributors and functors, as in [Stubbe, 2005]. All preliminaries for this note can be found either in that paper or in [Stubbe, 2007]; the reader may find [Stubbe, 2014] a useful introduction, with many examples from fuzzy logic and set theory. For the sake of readability, we quickly recall a few notational conventions.

A **quantaloid** $\mathcal{Q}$ is a category – say with objects and arrows written as $f\colon A \longrightarrow B$, $g\colon B \longrightarrow C$, etc., composition as $g \circ f\colon A \longrightarrow C$ and identities as $1_A\colon A \longrightarrow A$ – such that each set of arrows with the same domain and codomain forms a complete lattice – with suprema written as $\bigvee_i f_i\colon A \longrightarrow B$ – and for which the composition distributes on both sides over these suprema: $g \circ (\bigvee_i f_i) = \bigvee_i (g \circ f_i)$ and $(\bigvee_i f_i) \circ h = \bigvee_i (f_i \circ h)$. The resulting right adjoints to composition with a fixed arrow f (the "residuations") are denoted by $f \circ - \dashv f \searrow -$ and $- \circ f \dashv - \swarrow f$. That is to say,

$$g \circ f \leq h \quad \Longleftrightarrow \quad f \leq g \searrow h \quad \Longleftrightarrow \quad g \leq h \swarrow f$$

holds for any three arrows $f\colon A \longrightarrow B$, $g\colon B \longrightarrow C$ and $h\colon A \longrightarrow C$ in $\mathcal{Q}$. A quantaloid with a single object is a **quantale**: it is precisely a monoidal complete lattice in which multiplication distributes over arbitrary suprema.

In all that follows *we fix a base quantaloid $\mathcal{Q}$ and assume it to be small*, that is, the collection of all arrows of $\mathcal{Q}$ is a set (so in particular the objects of $\mathcal{Q}$ form a set $\mathcal{Q}_0$).

A **$\mathcal{Q}$-category** $\mathbb{A}$ is determined by a set $\mathbb{A}_0$, a so-called type-function $t\colon \mathbb{A}_0 \longrightarrow \mathcal{Q}_0$, and a $(\mathbb{A}_0 \times \mathbb{A}_0)$-indexed matrix of so-called hom-arrows $\mathbb{A}(a',a)\colon ta \longrightarrow ta'$ in $\mathcal{Q}$; this data is subject to transitivity and reflexivity axioms:

$$\mathbb{A}(a'',a') \circ \mathbb{A}(a',a) \leq \mathbb{A}(a'',a) \quad \text{and} \quad 1_{ta} \leq \mathbb{A}(a,a).$$

A **distributor** $\Phi\colon \mathbb{A} \mathrel{-\!\!\circ\!\!\!\rightarrow} \mathbb{B}$ is an $(\mathbb{B}_0 \times \mathbb{A}_0)$-indexed matrix of arrows $\Phi(b,a)\colon ta \longrightarrow tb$ in $\mathcal{Q}$, subject to action axioms:

$$\Phi(b,a) \circ \mathbb{A}(a,a') \leq \Phi(b,a') \quad \text{and} \quad \mathbb{B}(b',b) \circ \Phi(b,a) \leq \Phi(b',a).$$

The composition of $\Phi\colon \mathbb{A} \mathrel{-\!\!\circ\!\!\!\rightarrow} \mathbb{B}$ and $\Psi\colon \mathbb{B} \mathrel{-\!\!\circ\!\!\!\rightarrow} \mathbb{C}$ is written as $\Psi \otimes \Phi\colon \mathbb{A} \mathrel{-\!\!\circ\!\!\!\rightarrow} \mathbb{C}$ and computed with a "matrix multiplication formula":

$$(\Psi \otimes \Phi)(c,a) = \bigvee_{b \in \mathbb{B}_0} \Psi(c,b) \circ \Phi(b,a).$$

Note that the identity distributor on a $\mathcal{Q}$-category $\mathbb{A}$ is $\mathbb{A}\colon \mathbb{A} \mathrel{-\!\!\circ\!\!\!\rightarrow} \mathbb{A}$ itself. As moreover parallel distributors are ordered elementwise,

$$\Phi \leq \Phi'\colon \mathbb{A} \mathrel{-\!\!\circ\!\!\!\rightarrow} \mathbb{B} \quad \stackrel{\text{def.}}{\iff} \quad \forall (a,b) \in \mathbb{A}_0 \times \mathbb{B}_0\colon\ \Phi(b,a) \leq \Phi'(b,a),$$

the supremum of a family $(\Phi_i\colon \mathbb{A} \mathrel{-\!\!\circ\!\!\!\rightarrow} \mathbb{B})_{i \in I}$, written as $\bigvee_i \Phi_i\colon \mathbb{A} \mathrel{-\!\!\circ\!\!\!\rightarrow} \mathbb{B}$, is computed as:

$$(\bigvee_i \Phi_i)(b,a) = \bigvee_i \Phi_i(b,a).$$

A (large) quantaloid $\mathsf{Dist}(\mathcal{Q})$ of $\mathcal{Q}$-enriched categories and distributors results; here too, adjoints to composition are written as $\Phi \otimes - \dashv \Phi \searrow -$ and $- \otimes \Phi \dashv - \swarrow \Phi$. The reader will have no difficulty in verifying that, for distributors $\Phi\colon \mathbb{A} \mathrel{-\!\!\circ\!\!\!\rightarrow} \mathbb{B}$, $\Theta\colon \mathbb{C} \longrightarrow \mathbb{B}$ and $\Sigma\colon \mathbb{C} \mathrel{-\!\!\circ\!\!\!\rightarrow} \mathbb{D}$, the elements of $\Phi \searrow \Theta\colon \mathbb{A} \mathrel{-\!\!\circ\!\!\!\rightarrow} \mathbb{C}$ and $\Sigma \swarrow \Phi\colon \mathbb{C} \mathrel{-\!\!\circ\!\!\!\rightarrow} \mathbb{D}$ are exactly

$$(\Phi \searrow \Theta)(c,a) = \bigwedge_{b \in \mathbb{B}_0} \Phi(b,c) \searrow \Theta(b,a) \quad \text{and} \quad (\Sigma \swarrow \Phi)(d,b) = \bigwedge_{c \in \mathbb{C}_0} \Sigma(d,c) \swarrow \Phi(b,c).$$

Because $\mathsf{Dist}(\mathcal{Q})$ is a particular 2-category (there is a 2-cell between two parallel morphisms Φ and Φ' precisely when $\Phi \leq \Phi'$), we can apply all available notions from general 2-category theory. In particular, an **adjoint pair of distributors**

$$\begin{tikzcd} \mathbb{A} \arrow[r, bend left, "\Phi"] \arrow[r, phantom, "\perp"] & \mathbb{B} \arrow[l, bend left, "\Psi"] \end{tikzcd}$$

means by definition that $\mathbb{A} \leq \Psi \otimes \Phi$ and $\Phi \otimes \Psi \leq \mathbb{B}$; here Φ is the left adjoint to Ψ (and Ψ is the right adjoint to Φ).

A **functor** $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ is a function $F\colon \mathbb{A}_0 \longrightarrow \mathbb{B}_0\colon a \mapsto Fa$ preserving types and satisfying a functoriality axiom:

$$ta = t(Fa) \quad \text{and} \quad \mathbb{A}(a',a) \leq \mathbb{B}(Fa',Fa).$$

Functors compose in the obvious manner, so a category $\mathsf{Cat}(\mathcal{Q})$ results. Every functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ between $\mathcal{Q}$-categories determines an adjunction in $\mathsf{Dist}(\mathcal{Q})$:

$$\begin{tikzcd} \mathbb{A} \arrow[r, bend left, "\mathbb{B}(-,F-)"] \arrow[r, phantom, "\perp"] & \mathbb{B} \arrow[l, bend left, "\mathbb{B}(F-,-)"] \end{tikzcd}.$$

The assignment $F \mapsto \mathbb{B}(-,F-)$ (resp. $F \mapsto \mathbb{B}(F-,-)$) extends to a covariant (contravariant) inclusion of $\mathsf{Cat}(\mathcal{Q})$ in $\mathsf{Dist}(\mathcal{Q})$. In doing so, $\mathsf{Cat}(\mathcal{Q})$ inherits the 2-cells of $\mathsf{Dist}(\mathcal{Q})$: explicitly, for $F,F'\colon \mathbb{A} \longrightarrow \mathbb{B}$ we put

$$F \leq F' \quad \stackrel{\text{def.}}{\iff} \quad \mathbb{B}(-,F-) \leq \mathbb{B}(-,F'-) \quad \iff \quad \forall a \in \mathbb{A}_0 : 1_{ta} \leq \mathbb{B}(Fa,F'a).$$

Note, however, that this order relation on functors is not anti-symmetric: it may very well be that $F \leq F'$ and $F' \leq F$ (a situation that we shall write as $F \cong F'$), but still $F \neq F'$. In particular, a diagram of $\mathcal{Q}$-categories and functors, e.g.

$$\begin{tikzcd} \mathbb{A} \arrow[r, "H'"] \arrow[d, "H"'] & \mathbb{B} \arrow[d, "K'"] \\ \mathbb{C} \arrow[r, "K"'] & \mathbb{D} \end{tikzcd}$$

may not commute *strictly* (so $K \circ H \neq K' \circ H'$) even though it does commute *up to isomorphism* (so $K \circ H \cong K' \circ H'$). Henceforth we use the term "commutative diagram" whenever we have commutativity up to isomorphism.

In the 2-category $\mathsf{Cat}(\mathcal{Q})$, an **adjoint pair of functors**

$$\begin{tikzcd} \mathbb{A} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbb{B} \arrow[l, bend left, "G"] \end{tikzcd},$$

is defined to mean that $1_{\mathbb{A}} \leq G \circ F$ and $F \circ G \leq 1_{\mathbb{B}}$. It is however easily verified that

$$F \dashv G \quad \iff \quad \forall (a,b) \in \mathbb{A}_0 \times \mathbb{B}_0\colon \mathbb{B}(Fa,b) = \mathbb{A}(a,Gb),$$

and the latter expression explains why F is said to be the "left" adjoint, and G the "right" adjoint.

For any object $X \in \mathcal{Q}_0$, we write $\mathbb{1}_X$ for the one-element $\mathcal{Q}$-category whose single hom-arrow is 1_X. Clearly, a functor $\mathbb{1}_X \longrightarrow \mathbb{A}$ into any $\mathcal{Q}$-category $\mathbb{A}$ determines, and is determined by, an element $a \in \mathbb{A}_0$ of type $ta = X$. From the order relation on parallel functors, we now infer the **underlying order** of a $\mathcal{Q}$-category $\mathbb{A}$:

$$a' \leq a \quad \stackrel{\text{def.}}{\iff} \quad 1_{ta} \leq \mathbb{A}(a',a).$$

This order relation is indeed transitive and reflexive, but need not be anti-symmetric[1]. With this natural definition we then find (with notations as above) that

$$F \leq F' \quad \iff \quad \forall a \in \mathbb{A}_0 : Fa \leq Fa'$$

and also these familiar-looking characterisations of adjoint functors:

$$F \dashv G \iff \forall a \in \mathbb{A}_0, b \in \mathbb{B}_0 : a \leq GFa \text{ and } FGb \leq b \iff \forall a \in \mathbb{A}_0, b \in \mathbb{B}_0 : \Big[Fa \leq b \iff a \leq Gb\Big].$$

Furthermore, a functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ is **fully faithful** whenever $\mathbb{A}(a',a) = \mathbb{B}(Fa',Fa)$ for all $a,a' \in \mathbb{A}_0$; it is then necessarily **essentially injective** in the sense that $Fa \cong Fa'$ implies $a \cong a'$. And $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ is

[1] A $\mathcal{Q}$-category $\mathbb{A}$ is said to be **skeletal** if its underlying order is anti-symmetric. Some authors prefer all their quantaloid-enriched categories to be skeletal, thus making all commutative diagrams of functors in fact strictly commutative. Even though this is not our choice here, if one so wishes, one can restrict all that follows to skeletal $\mathcal{Q}$-categories; indeed, our main tool – the formation of $\mathcal{Q}$-categories $\mathcal{P}\mathbb{A}$ and $\mathcal{P}^{\dagger}\mathbb{A}$ of presheaves on a given $\mathcal{Q}$-category $\mathbb{A}$, see Section 3 – always produces skeletal $\mathcal{Q}$-categories.

essentially surjective when for every $b \in \mathbb{B}_0$ there exists an $a \in \mathbb{A}_0$ such that $Fa \cong b$; it is then necessarily **dense** in the sense that $\mathbb{B}(b', F-) \otimes \mathbb{B}(F-, b) = \mathbb{B}(b', b)$ for all $b, b' \in \mathbb{B}_0$.

Finally, a word on duality. When $\mathcal{Q}$ is a quantaloid, then so is its opposite $\mathcal{Q}^{\mathsf{op}}$ obtained by formally reversing the direction of the arrows. Whenever $\mathbb{C}$ is a $\mathcal{Q}$-category, define $\mathbb{C}^{\mathsf{op}}$ to have the same objects and types as $\mathbb{C}$, but put $\mathbb{C}^{\mathsf{op}}(y, x) := \mathbb{C}(x, y)$; this produces a $\mathcal{Q}^{\mathsf{op}}$-category $\mathbb{C}^{\mathsf{op}}$. Extended to distributors and functors in the obvious way, we obtain isomorphic 2-categories

$$\mathsf{Dist}(\mathcal{Q}) \cong \mathsf{Dist}(\mathcal{Q}^{\mathsf{op}})^{\mathsf{op}} \text{ and } \mathsf{Cat}(\mathcal{Q}) \cong \mathsf{Cat}(\mathcal{Q}^{\mathsf{op}})^{\mathsf{co}},$$

where the "co" means that we formally reverse the order between the arrows (but keep the direction of the arrows). This now gives a **duality principle**: when a notion, say a *widget*, is defined for general quantaloid-enriched categories, it has an incarnation in $\mathcal{Q}$-categories and an incarnation in $\mathcal{Q}^{\mathsf{op}}$-categories; translating the latter back in terms of $\mathcal{Q}$-categories via the above isomorphisms, produces the dual notion to the original widget, usually (but not always) called *cowidget*. We shall encounter several such situations further on.

Example 2.1 To relate the above definitions to the introductory section of this paper, we shall work out the simplest of examples. Let $\mathbf{2}$ be the quantaloid with a single object, two arrows 0 and 1, one 2-cell $0 \leq 1$, and composition given by infimum (making 1 the identity arrow). In other words, $\mathbf{2}$ is (the "suspension" of) the two-element boolean algebra $\{0 \leq 1\}$. Writing out the definition of a $\mathbf{2}$-category $\mathbb{A}$, the type function is obsolete (because its codomain is a singleton), and we are left with a binary predicate

$$\mathbb{A}\colon \mathbb{A}_0 \times \mathbb{A}_0 \longrightarrow \{0, 1\}$$

which encodes a transitive and reflexive relation; in other words, we may regard $\mathbb{A} = (A, \leq)$ as an ordered set. (Requiring $\mathbb{A}$ to be skeletal would make it an anti-symmetric order.)

A distributor $\Phi\colon (A, \leq) \mathrel{\circ\!\!\!\longrightarrow} (B, \leq)$ is, in the same vein, a relation $\Phi \subseteq B \times A$ which is downclosed in B ($b \in \Phi$ and $b' \leq b$ implies $b' \in \Phi$) and upclosed in A ($a' \in \Phi$ and $a' \leq a$ implies $a \in \Phi$); its composite with another distributor $\Psi\colon (B, \leq) \mathrel{\circ\!\!\!\longrightarrow} (C, \leq)$ is

$$\Psi \otimes \Phi = \{(c, a) \in C \times A \mid \exists b \in B : (c, b) \in \Psi \text{ and } (b, a) \in \Phi\},$$

whereas the identity distributor on $(A, \leq)$ is

$$\mathbb{A} = \{(a', a) \in A \times A \mid a' \leq a\}.$$

The order-relation between parallel distributors is given by inclusion, the supremum of parallel distributors is thus their union.

A functor $F\colon (A, \leq) \longrightarrow (B, \leq)$ is an order-preserving function from A to B. Written as relations, the adjoint pair of distributors determined by F is

$$\mathbb{B}(-, F-) = \{(b, a) \in B \times A \mid b \leq Fa\} \quad \text{and} \quad \mathbb{B}(F-, -) = \{(a, b) \in A \times B \mid Fa \leq b\}.$$

The order on functors is pointwise, and an adjoint pair of functors is a (monotone) Galois connection.

As we develop, in the course of this paper, some very general quantaloid-enriched category theory, we shall always come back to ordered sets as our running example. Whereas several other examples – including *sheaves*, *fuzzy sets* and *fuzzy relations*, *fuzzy orders*, and *partial metric spaces* – are worked out in the papers mentioned at the end of the Introduction, we want to indicate the expressive power of quantaloid-enriched categories by briefly describing the case of *fuzzy orders*.

Example 2.2 A **left-continuous** t**-norm** is a binary operator

$$[0, 1] \times [0, 1] \longrightarrow [0, 1]\colon (x, y) \mapsto x * y$$

which provides the multiplication for a commutative ($x * y = y * x$) and integral ($1 * x = x = x * 1$) quantale structure on the sup-lattice $([0, 1], \bigvee)$. Examples include the *product t-norm* ($x * y = xy$), the *Lukasiewicz* *t-norm* ($x*y = \max\{x+y-1, 0\}$), and the *minimum t-norm* ($x*y = \min\{x, y\}$); these are in fact *continuous* t-norms, and it is well-known that *any* continuous t-norm is – in a precise manner – an amalgamation of these three. An example of a left-continuous non-right-continuous t-norm is obtained by setting $x * y$ to $\min(x, y)$ if $x + y > 1$ and to 0 otherwise. Such t-norms are important structures in fuzzy logic: taking the real unit interval as a model of possibly vague statements, conjunction is interpreted by a t-norm. A standard reference on this (and much more) is [Gottwald, 2001].

Let us now fix a **continuous t-norm** $([0, 1], *)$. Regarding it as a one-object quantaloid, it makes perfect sense to consider a $[0, 1]$-enriched category $\mathbb{A}$: it consists of a set $\mathbb{A}_0$ (with obsolete type function, since its codomain is a singleton) together with a predicate

$$\mathbb{A}\colon \mathbb{A}_0 \times \mathbb{A}_0 \longrightarrow [0, 1]$$

which satisfies $\mathbb{A}(z, y) * \mathbb{A}(y, x) \leq \mathbb{A}(z, x)$ and $1 \leq \mathbb{A}(x, x)$. Reading $\mathbb{A}(y, x)$ as "the extent to which y precedes x", the "fuzzy transitivity" seems to indicate that we can think of the predicate $\mathbb{A}$ as a "fuzzy order relation". However, the reflexivity axiom requires that $1 \leq \mathbb{A}(x, x)$, that is, the extent to which any $x \in \mathbb{A}_0$ *exists*, is necessarily equal to 1. In still other words, a $[0, 1]$-enriched $\mathbb{A}$ is a *crisp set* endowed with a *reflexive and fuzzy transitive relation.*

An elegant manner to completely "fuzzify" the notion of ordered set, goes as follows. Let $\mathcal{D}[0, 1]$ be the quantaloid with

- objects: the elements of $[0, 1]$,
- arrows: an arrow $a\colon u \longrightarrow v$ is an $a \leq u \wedge v$ in $[0, 1]$, the composite with $b\colon v \longrightarrow w$ is $b*(v \Rightarrow a)\colon u \longrightarrow w$, and the identity arrow on $u \in [0, 1]$ is $u\colon u \longrightarrow u$ itself,
- local suprema: $\bigvee_{i\in I}(a_i\colon u \longrightarrow v) = (\bigvee_i a_i\colon u \longrightarrow v)$ is computed as in $[0, 1]$.

(We wrote $\Rightarrow$ for the residuation in $[0, 1]$ wrt. the given t-norm.) Taken literally, a $\mathcal{D}[0, 1]$-category $\mathbb{A}$ is a set $\mathbb{A}_0$ together with functions

$$t\colon \mathbb{A}_0 \longrightarrow [0, 1] \quad \text{and} \quad \mathbb{A}\colon \mathbb{A}_0 \times \mathbb{A}_0 \longrightarrow [0, 1]$$

which satisfy

$$\mathbb{A}(y, x)\colon tx \longrightarrow ty \text{ in } \mathcal{D}[0, 1], \quad 1_{tx} \leq \mathbb{A}(x, x) \quad \text{and} \quad \mathbb{A}(z, y) * (\mathbb{A}(y, y) \Rightarrow \mathbb{A}(y, x)) \leq \mathbb{A}(z, x).$$

Translating this back into inequations in the quantale $[0, 1]$, and using that $ta \in [0, 1]$ is the identity arrow on the object ta of $\mathcal{D}[0, 1]$, this means that

$$\mathbb{A}(y, x) \leq tx \wedge ty, \quad tx \leq \mathbb{A}(x, x) \quad \text{and } \mathbb{A}(z, y) * (\mathbb{A}(y, y) \Rightarrow \mathbb{A}(y, x)) \leq \mathbb{A}(z, x).$$

Clearly now, the unary predicate is implicit in the binary predicate, and the reflexivity axiom is obsolete, so we are actually left with

$$\mathbb{A}(y, x) \leq \mathbb{A}(x, x) \wedge \mathbb{A}(y, y) \quad \text{and} \quad \mathbb{A}(z, y) * (\mathbb{A}(y, y)) \Rightarrow \mathbb{A}(y, x)) \leq \mathbb{A}(z, x).$$

If we now read $\mathbb{A}(y, x)$ as "the extent to which y precedes x", and in particular $\mathbb{A}(x, x)$ as "the extent to which x exists", then we find here a fuzzy order relation on a set of fuzzy elements. We happily follow (amongst others) [Höhle and Kubiak, 2011; Pu and Zhang, 2012] in promoting this notion to be taken as the definition of a **fuzzy order**. With a little more care this procedure carries over to left-continuous t-norms too, in fact even to general quantales and quantaloids—see [Stubbe, 2013, 2.14].

3. Presheaves

A **contravariant presheaf** of type $X \in \mathcal{Q}_0$ on a $\mathcal{Q}$-category $\mathbb{A}$ is defined to be a distributor $\phi\colon \mathbb{1}_X \mathrel{-\!\!\circ\!\!\rightarrow} \mathbb{A}$. We shall write the elements of such a distributor as $\phi(a)\colon X \longrightarrow ta$ in $\mathcal{Q}$ (one for each $a \in \mathbb{A}_0$), thus avoiding the more cumbersome notation $\phi(a,*)\colon t* \longrightarrow ta$ where the $*$ would denote the single object of $\mathbb{1}_X$. To support the idea that such a $\phi\colon \mathbb{1}_X \mathrel{-\!\!\circ\!\!\rightarrow} \mathbb{A}$ indeed merits to be called a "contravariant presheaf" on $\mathbb{A}$, it is useful think of it as a function

$$\phi\colon \mathbb{A}_0 \longrightarrow \{\text{arrows in } \mathcal{Q} \text{ with domain } X\}\colon a \mapsto \phi(a)$$

satisfying

$$\forall a, a' \in \mathbb{A}_0 : \mathbb{A}(a', a) \le \phi(a') \swarrow \phi(a).$$

This function taking values in the set of arrows in $\mathcal{Q}$ (the "truth values" in the world of $\mathcal{Q}$-categories), makes it a *presheaf* on $\mathbb{A}$; and the "backward residuation" in the right hand side of the above inequation accounts for its *contravariance*.

The collection of all contravariant presheaves on $\mathbb{A}$ quite naturally forms a (skeletal) $\mathcal{Q}$-category $\mathcal{P}\mathbb{A}$, as follows. For two presheaves, say $\phi\colon \mathbb{1}_X \mathrel{-\!\!\circ\!\!\rightarrow} \mathbb{A}$ and $\psi\colon \mathbb{1}_Y \mathrel{-\!\!\circ\!\!\rightarrow} \mathbb{A}$, we may compute the lifting in the following diagram:

$$\begin{tikzcd}
 & \mathbb{A} & \\
\mathbb{1}_X \arrow[ur, "\phi", "\circ" marking] \arrow[rr, "\circ" marking, "\psi \searrow \phi"'] & & \mathbb{1}_Y \arrow[ul, "\psi"', "\circ" marking]
\end{tikzcd}$$

The distributor $\psi \searrow \phi\colon \mathbb{1}_X \mathrel{-\!\!\circ\!\!\rightarrow} \mathbb{1}_Y$ is completely determined by a single arrow $X \longrightarrow Y$ in $\mathcal{Q}$; only slightly abusing notation[2] we shall use $\psi \searrow \phi\colon X \longrightarrow Y$ as notation for that $\mathcal{Q}$-arrow too, which we can compute explicitly as

$$\psi \searrow \phi = \bigwedge_{a \in \mathbb{A}_0} \psi(a) \searrow \phi(a).$$

This suggests the construction of a new $\mathcal{Q}$-category $\mathcal{P}\mathbb{A}$ as follows:

- $(\mathcal{P}\mathbb{A})_0$ is the collection of *all* contravariant presheaves on $\mathbb{A}$,
- the type of an element $\phi\colon \mathbb{1}_X \mathrel{-\!\!\circ\!\!\rightarrow} \mathbb{A}$ of $(\mathcal{P}\mathbb{A})_0$ is X,
- for any two elements $\phi\colon \mathbb{1}_X \mathrel{-\!\!\circ\!\!\rightarrow} \mathbb{A}$ and $\psi\colon \mathbb{1}_Y \mathrel{-\!\!\circ\!\!\rightarrow} \mathbb{A}$ of $(\mathcal{P}\mathbb{A})_0$, the hom-arrow $\mathcal{P}\mathbb{A}(\psi, \phi)\colon t\phi \longrightarrow t\psi$ is $\psi \searrow \phi\colon X \longrightarrow Y$.

Remark that, for any two $\phi, \psi \in \mathcal{P}\mathbb{A}$, their order *qua elements of* $\mathcal{P}\mathbb{A}$ coincides with their order *qua distributors*; that is to say,

$$\psi \le \phi \text{ wrt. the underlying order of } \mathcal{P}\mathbb{A} \quad \Longleftrightarrow \quad \forall a \in \mathbb{A}_0 : \psi(a) \le \phi(a).$$

It follows in particular that $\mathcal{P}\mathbb{A}$ is a skeletal $\mathcal{Q}$-category (even when $\mathbb{A}$ is not). The **Yoneda embedding** is the functor $Y_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathcal{P}\mathbb{A}\colon a \mapsto \mathbb{A}(-, a)$. The well-known Yoneda lemma says that, for any $a \in \mathbb{A}$ and $\phi \in \mathcal{P}\mathbb{A}$, $\mathcal{P}\mathbb{A}(Y_{\mathbb{A}}a, \phi) = \phi(a)$ (from which it follows in particular that $Y_{\mathbb{A}}$ is indeed an essentially injective, fully faithful functor.)

Dual to the above, a **covariant presheaf** of type X on $\mathbb{A}$ is a distributor $\phi\colon \mathbb{A} \mathrel{-\!\!\circ\!\!\rightarrow} *_X$; as before, we shall write its elements as $\phi(a)\colon ta \longrightarrow X$. When thought of as a function

$$\phi\colon \mathbb{A}_0 \longrightarrow \{\text{arrows in } \mathcal{Q} \text{ with codomain } X\}\colon a \mapsto \phi(a)$$

[2] There is a full embedding of $\mathcal{Q}$ in $\mathsf{Dist}(\mathcal{Q})$ by sending any $f\colon X \longrightarrow Y$ in $\mathcal{Q}$ to the distributor $(f)\colon \mathbb{1}_X \mathrel{-\!\!\circ\!\!\rightarrow} \mathbb{1}_Y$ whose single element is f. We may thus identify $\mathcal{Q}$ with a full subquantaloid of $\mathsf{Dist}(\mathcal{Q})$, which is precisely what we do here.

it satisfies the inequality

$$\forall a,a'\in\mathbb{A}_0:\mathbb{A}(a',a)\leq\phi(a')\searrow\phi(a),$$

which clearly exhibits the covariance of ϕ. The (skeletal) $\mathcal{Q}$-category $\mathcal{P}^\dagger\mathbb{A}$ of covariant presheaves on $\mathbb{A}$ is:

- $(\mathcal{P}^\dagger\mathbb{A})_0$ is the collection of *all* covariant presheaves on $\mathbb{A}$,
- the type of an element $\phi\colon\mathbb{A}\to\mathbb{1}_X$ of $(\mathcal{P}^\dagger\mathbb{A})_0$ is X,
- for any two elements $\phi\colon\mathbb{A}\to\mathbb{1}_X$ and $\psi\colon\mathbb{A}\to\mathbb{1}_Y$ of $(\mathcal{P}^\dagger\mathbb{A})_0$, the hom-arrow $\mathcal{P}^\dagger\mathbb{A}(\psi,\phi)\colon t\phi\longrightarrow t\psi$ is $\psi\swarrow\phi\colon X\longrightarrow Y$.

Note that the underlying order of $\mathcal{P}^\dagger\mathbb{A}$ is "odd":

$$\psi\leq\phi \text{ wrt. the underlying order of } \mathcal{P}^\dagger\mathbb{A}\quad\iff\quad\forall a\in\mathbb{A}_0:\phi(a)\leq\psi(a);$$

it is best thought of as an ordering-by-containment. The Yoneda embedding is $Y^\dagger_{\mathbb{A}}\colon\mathbb{A}\longrightarrow\mathcal{P}^\dagger\mathbb{A}\colon a\mapsto\mathbb{A}(a,-)$, and the Yoneda lemma now says that $\mathcal{P}^\dagger\mathbb{A}(\phi,Y^\dagger_{\mathbb{A}}a)=\phi(a)$ (which implies fully faithfulness of $Y^\dagger_{\mathbb{A}}$).

Proposition 3.1 *Every functor $F\colon\mathbb{A}\longrightarrow\mathbb{B}$ determines adjunctions*

$$\begin{tikzcd} \mathcal{P}\mathbb{A} \arrow[r, bend left, "\mathcal{P}_lF"] \arrow[r, bend right, "\mathcal{P}_*F"'] & \mathcal{P}\mathbb{B} \arrow[l, "\mathcal{P}_rF"', "\perp" above, "\perp" below] \end{tikzcd} \quad\textit{and}\quad \begin{tikzcd} \mathcal{P}^\dagger\mathbb{A} \arrow[r, bend left, "\mathcal{P}^\dagger_*F"] \arrow[r, bend right, "\mathcal{P}^\dagger_rF"'] & \mathcal{P}^\dagger\mathbb{B} \arrow[l, "\mathcal{P}^\dagger_lF"', "\perp" above, "\perp" below] \end{tikzcd} \tag{1}$$

in $\mathsf{Cat}(\mathcal{Q})$, *where*

$$\begin{array}{ll} \mathcal{P}_lF(\phi):=\mathbb{B}(-,F-)\otimes\phi & \mathcal{P}^\dagger_*F(\phi):=\phi\swarrow\mathbb{B}(-,F-)\\ \mathcal{P}_rF(\psi):=\mathbb{B}(F-,-)\otimes\psi & \mathcal{P}^\dagger_lF(\psi):=\psi\otimes\mathbb{B}(-,F-)\\ \mathcal{P}_*F(\phi):=\mathbb{B}(F-,-)\searrow\phi & \mathcal{P}^\dagger_rF(\phi):=\phi\otimes\mathbb{B}(F-,-) \end{array}$$

These actions on arrows in $\mathsf{Cat}(\mathcal{Q})$ *determine 2-functors*

$$\begin{array}{ll} \mathcal{P}_l\colon\mathsf{Cat}(\mathcal{Q})\longrightarrow\mathsf{Cat}(\mathcal{Q}) & \mathcal{P}^\dagger_*\colon\mathsf{Cat}(\mathcal{Q})\longrightarrow\mathsf{Cat}(\mathcal{Q})\\ \mathcal{P}_r\colon\mathsf{Cat}(\mathcal{Q})\longrightarrow\mathsf{Cat}(\mathcal{Q})^{\mathsf{coop}} & \mathcal{P}^\dagger_l\colon\mathsf{Cat}(\mathcal{Q})\longrightarrow\mathsf{Cat}(\mathcal{Q})^{\mathsf{coop}}\\ \mathcal{P}_*\colon\mathsf{Cat}(\mathcal{Q})\longrightarrow\mathsf{Cat}(\mathcal{Q}) & \mathcal{P}^\dagger_r\colon\mathsf{Cat}(\mathcal{Q})\longrightarrow\mathsf{Cat}(\mathcal{Q}) \end{array}$$

(where "coop*" means taking formally opposite arrows* and *2-cells) that satisfy the following equalities:*

$$\begin{array}{lll} \mathcal{P}_l\mathcal{P}_r=\mathcal{P}_r\mathcal{P}_l & \mathcal{P}_l\mathcal{P}_*=\mathcal{P}_r\mathcal{P}_r=\mathcal{P}_*\mathcal{P}_l & \mathcal{P}_r\mathcal{P}_*=\mathcal{P}_*\mathcal{P}_r\\ \mathcal{P}_l\mathcal{P}^\dagger_l=\mathcal{P}_r\mathcal{P}^\dagger_* & \mathcal{P}_l\mathcal{P}^\dagger_r=\mathcal{P}_*\mathcal{P}^\dagger_*=\mathcal{P}_r\mathcal{P}^\dagger_l & \mathcal{P}_r\mathcal{P}^\dagger_r=\mathcal{P}_*\mathcal{P}^\dagger_l\\ \mathcal{P}^\dagger_l\mathcal{P}_l=\mathcal{P}^\dagger_*\mathcal{P}_r & \mathcal{P}^\dagger_r\mathcal{P}_l=\mathcal{P}^\dagger_*\mathcal{P}_*=\mathcal{P}^\dagger_l\mathcal{P}_r & \mathcal{P}^\dagger_r\mathcal{P}_r=\mathcal{P}^\dagger_l\mathcal{P}_*\\ \mathcal{P}^\dagger_l\mathcal{P}^\dagger_r=\mathcal{P}^\dagger_r\mathcal{P}^\dagger_l & \mathcal{P}^\dagger_*\mathcal{P}^\dagger_r=\mathcal{P}^\dagger_l\mathcal{P}^\dagger_l=\mathcal{P}^\dagger_r\mathcal{P}^\dagger_* & \mathcal{P}^\dagger_*\mathcal{P}^\dagger_l=\mathcal{P}^\dagger_l\mathcal{P}^\dagger_* \end{array}$$

The Yoneda embeddings provide for natural transformations $Y\colon 1_{\mathsf{Cat}(\mathcal{Q})}\Longrightarrow\mathcal{P}_l$ and $Y^\dagger\colon 1_{\mathsf{Cat}(\mathcal{Q})}\Longrightarrow\mathcal{P}^\dagger_r$, that is, for any functor $F\colon\mathbb{A}\longrightarrow\mathbb{B}$ the following diagrams commute:

$$\begin{tikzcd} \mathbb{A} \arrow[r, "F"] \arrow[d, "Y_{\mathbb{A}}"'] & \mathbb{B} \arrow[d, "Y_{\mathbb{B}}"] \\ \mathcal{P}\mathbb{A} \arrow[r, "\mathcal{P}_lF"'] & \mathcal{P}\mathbb{B} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{A} \arrow[r, "F"] \arrow[d, "Y^\dagger_{\mathbb{A}}"'] & \mathbb{B} \arrow[d, "Y^\dagger_{\mathbb{B}}"] \\ \mathcal{P}^\dagger\mathbb{A} \arrow[r, "\mathcal{P}^\dagger_rF"'] & \mathcal{P}^\dagger\mathbb{B} \end{tikzcd} \tag{2}$$

Proof : First we perform some generally valid computations with liftings/extensions through left/right adjoints, in the quantaloid $\mathsf{Dist}(\mathcal{Q})$. Given $\phi \in \mathcal{P}\mathbb{A}$ and $\psi \in \mathcal{P}\mathbb{B}$, we have that

$$\mathcal{P}_l F(\phi) = \mathbb{B}(-, F-) \otimes \phi \leq \psi \iff \phi \leq \mathbb{B}(-, F-) \searrow \psi.$$

Because $\mathbb{B}(-, F-) \dashv \mathbb{B}(F-, -)$ in $\mathsf{Dist}(\mathcal{Q})$, it furthermore follows that

$$\mathbb{B}(-, F-) \searrow \psi = \mathbb{B}(F-, -) \otimes \psi = \mathcal{P}_r F(\psi),$$

and also

$$\mathcal{P}_r F(\psi) = \mathbb{B}(F-, -) \otimes \psi \leq \phi \iff \psi \leq \mathbb{B}(F-, -) \searrow \phi = \mathcal{P}_* F(\phi).$$

This proves exactly that $\mathcal{P}_l F \dashv \mathcal{P}_r F \dashv \mathcal{P}_* F$. The results for the covariant presheaves are dual.

The 2-functoriality of $\mathcal{P}_l$ etc. is easy to see; one must only be a bit careful with the direction of 2-cells and remember that $\mathcal{P}^\dagger\mathbb{A}$ and $\mathcal{P}^\dagger\mathbb{B}$ have an "odd" underlying order. Each of these 2-functors therefore preserves adjunctions. Applying $\mathcal{P}_l$ to $\mathcal{P}_l F \dashv \mathcal{P}_r F \dashv \mathcal{P}_* F$ gives $\mathcal{P}_l\mathcal{P}_l F \dashv \mathcal{P}_l\mathcal{P}_r F \dashv \mathcal{P}_l\mathcal{P}_* F$; but on the other hand we know that $\mathcal{P}_l\mathcal{P}_l F \dashv \mathcal{P}_r\mathcal{P}_l F \dashv \mathcal{P}_*\mathcal{P}_l F$ too; so uniqueness of adjoints implies that $\mathcal{P}_l\mathcal{P}_r F = \mathcal{P}_r\mathcal{P}_l F$ and $\mathcal{P}_l\mathcal{P}_* F = \mathcal{P}_*\mathcal{P}_l F$. Similar arguments work for the other equations.

The naturality of the Yoneda embeddings is easy to check, e.g.

$$\mathcal{P}_l F(Y_{\mathbb{A}} a) = \mathbb{B}(-, F-) \otimes \mathbb{A}(-, a) = \mathbb{B}(-, Fa) = Y_{\mathbb{B}}(Fa),$$

and similar for the other commutative square. $\square$

Note how both Yoneda lemmas can be written in terms of the 2-functors from Proposition 3.1:

Lemma 3.2 *For any $\mathcal{Q}$-category $\mathbb{A}$ we have that $\mathcal{P}_r Y_{\mathbb{A}} \circ Y_{\mathcal{P}\mathbb{A}} = 1_{\mathcal{P}\mathbb{A}}$ and $\mathcal{P}_l^\dagger Y_{\mathbb{A}}^\dagger \circ Y_{\mathcal{P}^\dagger\mathbb{A}}^\dagger = 1_{\mathcal{P}^\dagger\mathbb{A}}$.*

Example 3.3 To pick up with Example 2.1, we apply the above generalities to the particular case of **2**-enriched categories, i.e. ordered sets. A contravariant presheaf ϕ on an order $\mathbb{A} = (A, \leq)$ is a subset $\phi \subseteq A$ which is downclosed: $a \leq a'$ and $a' \in \phi$ implies $a \in \phi$. The **2**-category $\mathcal{P}\mathbb{A}$ corresponds with the set of all downclosed subsets of $(A, \leq)$ ordered by inclusion, usually written as $(\mathsf{Dwn}(A), \subseteq)$. The Yoneda embedding $Y_{\mathbb{A}} \colon (A, \leq) \longrightarrow (\mathsf{Dwn}(A), \subseteq)$ sends $a \in A$ to the principal downset $\downarrow a = \{a' \in A \mid a' \leq a\}$, and the Yoneda lemma says that

$$\forall a \in A, \phi \in \mathsf{Dwn}(A) : \quad a \in \phi \quad \iff \quad \downarrow a \subseteq \phi.$$

Dually, $\mathcal{P}^\dagger\mathbb{A}$ is the set of all upclosed subsets of $(A, \leq)$ ordered by containment, usually written as $(\mathsf{Up}(A), \supseteq)$, the Yoneda embedding is $Y_{\mathbb{A}}^\dagger \colon (A, \leq) \longrightarrow (\mathsf{Up}(A), \supseteq) \colon a \mapsto \uparrow a$ and satisfies

$$\forall a \in A, \phi \in \mathsf{Up}(A) : \quad a \in \phi \quad \iff \quad \uparrow a \supseteq \phi.$$

Given an order-preserving function $F \colon (A, \leq) \longrightarrow (B, \leq)$ it is not difficult to compute explicitly the order-preserving functions of Proposition 3.1—to find in particular the familiar notions of direct and inverse image:

- for $\phi \in \mathsf{Dwn}(A)$, $\mathcal{P}_l F(\phi) = \{b \in B \mid \exists a \in A : b \leq Fa \text{ and } a \in \phi\} =$ downclosure of $F(\phi)$,
- for $\psi \in \mathsf{Dwn}(B)$, $\mathcal{P}_r F(\psi) = \{a \in A \mid \exists b \in B : Fa \leq b \text{ and } b \in \psi\} = F^{-1}(\psi)$,
- for $\phi \in \mathsf{Up}(A)$, $\mathcal{P}_r^\dagger F(\phi) = \{b \in B \mid \exists a \in A : a \in \phi \text{ and } Fa \leq b\} =$ upclosure of $F(\phi)$,
- for $\psi \in \mathsf{Up}(B)$, $\mathcal{P}_l^\dagger F(\psi) = \{a \in A \mid \exists b \in B : b \in \psi \text{ and } b \leq Fa\} = F^{-1}(\psi)$.

The naturality of the Yoneda embeddings now says that $\downarrow Fa$ is the downclosure of $F(\downarrow a)$, and that $\uparrow Fa$ is the upclosure of $F(\uparrow a)$.

4. Double power monad

Both $\mathcal{P}\mathbb{A}$ and $\mathcal{P}^\dagger\mathbb{A}$ enjoy a universal property: they classify distributors, but each in a different manner. Precisely:

1. $\mathsf{Dist}(\mathcal{Q})(\mathbb{B},\mathbb{A}) \longrightarrow \mathsf{Cat}(\mathcal{Q})(\mathbb{B},\mathcal{P}\mathbb{A})\colon (\Phi\colon \mathbb{B} \mathrel{\circ\!\!\!\longrightarrow} \mathbb{A}) \mapsto (F_\Phi\colon \mathbb{B}\longrightarrow\mathcal{P}\mathbb{A}\colon b\mapsto \Phi(-,b))$ is an isomorphism of ordered sets, with inverse $(F\colon \mathbb{B}\longrightarrow\mathcal{P}\mathbb{A}) \mapsto (\Phi_F\colon \mathbb{B} \mathrel{\circ\!\!\!\longrightarrow} \mathbb{A}, \Phi(a,b)=F(b)(a))$,
2. $\mathsf{Dist}(\mathcal{Q})(\mathbb{B},\mathbb{A}) \longrightarrow \mathsf{Cat}(\mathcal{Q})(\mathbb{A},\mathcal{P}^\dagger\mathbb{B})^{\mathsf{op}}\colon (\Psi\colon \mathbb{B} \mathrel{\circ\!\!\!\longrightarrow} \mathbb{A}) \mapsto (G_\Psi\colon \mathbb{A}\longrightarrow\mathcal{P}^\dagger\mathbb{B}\colon a\mapsto \Psi(a,-))$ is an isomorphism of ordered sets, with inverse $(G\colon \mathbb{A}\longrightarrow\mathcal{P}^\dagger\mathbb{B}) \mapsto (\Psi_G\colon \mathbb{B} \mathrel{\circ\!\!\!\longrightarrow} \mathbb{A}, \Psi(a,b)=G(a)(b))$.

Composing (and turning upside down) these isomorphisms of ordered sets, we get an isomorphism

$$\pi_{\mathbb{A},\mathbb{B}}\colon \mathsf{Cat}(\mathcal{Q})^{\mathsf{coop}}(\mathcal{P}\mathbb{A},\mathbb{B}) = \mathsf{Cat}(\mathcal{Q})(\mathbb{B},\mathcal{P}\mathbb{A})^{\mathsf{op}} \cong \mathsf{Dist}(\mathcal{Q})(\mathbb{B},\mathbb{A})^{\mathsf{op}} \cong \mathsf{Cat}(\mathcal{Q})(\mathbb{A},\mathcal{P}^\dagger\mathbb{B});$$

explicitly,

$$\pi_{\mathbb{A},\mathbb{B}}(F\colon \mathbb{B}\longrightarrow\mathcal{P}\mathbb{A}) = (\pi(F)\colon \mathbb{A}\longrightarrow\mathcal{P}^\dagger\mathbb{B}\colon a\mapsto F(-)(a))$$
$$\pi^{-1}_{\mathbb{A},\mathbb{B}}(G\colon \mathbb{A}\longrightarrow\mathcal{P}^\dagger\mathbb{B}) = (\pi^{-1}(G)\colon \mathbb{B}\longrightarrow\mathcal{P}\mathbb{A}\colon b\mapsto G(-)(b)).$$

Proposition 4.1 *The object functions $\mathbb{A}\mapsto\mathcal{P}\mathbb{A}$ and $\mathbb{A}\mapsto\mathcal{P}^\dagger\mathbb{A}$ together with the family of order-isomorphisms $\pi_{\mathbb{A},\mathbb{B}}\colon \mathsf{Cat}(\mathcal{Q})^{\mathsf{coop}}(\mathcal{P}\mathbb{A},B)\longrightarrow\mathsf{Cat}(\mathcal{Q})(\mathbb{A},\mathcal{P}^\dagger\mathbb{B})$ consitute the* **core of an adjunction** *in the sense of [Street, 2012]. This means that these object functions uniquely extend to a pair of 2-functors, making the family of order-isomorphisms natural (in $\mathbb{A}$ and $\mathbb{B}$) so that it expresses the 2-adjunction of this pair of 2-functors. In fact, this 2-adjunction turns out to be precisely*

$$\begin{tikzcd} \mathsf{Cat}(\mathcal{Q}) \arrow[rr, bend left, "\mathcal{P}_r"] \arrow[rr, phantom, "\perp"] & & \mathsf{Cat}(\mathcal{Q})^{\mathsf{coop}} \arrow[ll, bend left, "\mathcal{P}^\dagger_l"] \end{tikzcd},$$

with units $\beta_\mathbb{A}\colon \mathbb{A}\longrightarrow\mathcal{P}^\dagger\mathcal{P}\mathbb{A}\colon a\mapsto 1_{\mathcal{P}\mathbb{A}}(-)(a)$ and counits $\alpha_\mathbb{A}\colon \mathbb{A}\longrightarrow\mathcal{P}\mathcal{P}^\dagger\mathbb{A}\colon a\mapsto 1_{\mathcal{P}^\dagger\mathbb{A}}(-)(a)$.

Proof : We begin with some definitions (using the notations of [Street, 2012]):

1. $\beta_\mathbb{A}\in\mathsf{Cat}(\mathcal{Q})(\mathbb{A},\mathcal{P}^\dagger\mathcal{P}\mathbb{A})$ is $\pi_{\mathbb{A},\mathcal{P}\mathbb{A}}(1_{\mathcal{P}\mathbb{A}})$,
2. $\alpha_\mathbb{A}\in\mathsf{Cat}(\mathcal{Q})^{\mathsf{coop}}(\mathcal{P}\mathcal{P}^\dagger\mathbb{A},\mathbb{A})$ is $\pi^{-1}_{\mathcal{P}^\dagger\mathbb{A},\mathbb{A}}(1_{\mathcal{P}^\dagger\mathbb{A}})$,
3. $\mathcal{U}_{\mathbb{A},\mathbb{B}}\colon \mathsf{Cat}(\mathcal{Q})^{\mathsf{coop}}(\mathbb{A},\mathbb{B})\longrightarrow\mathsf{Cat}(\mathcal{Q})(\mathcal{P}^\dagger\mathbb{A},\mathcal{P}^\dagger\mathbb{B})$ sends $F\colon\mathbb{B}\longrightarrow\mathbb{A}$ to $\pi_{\mathcal{P}^\dagger\mathbb{A},\mathbb{B}}(\alpha_\mathbb{A}\circ F)$,
4. $\mathcal{F}_{\mathbb{A},\mathbb{B}}\colon \mathsf{Cat}(\mathcal{Q})(\mathbb{A},\mathbb{B})\longrightarrow\mathsf{Cat}(\mathcal{Q})^{\mathsf{coop}}(\mathcal{P}\mathbb{A},\mathcal{P}\mathbb{B})$ sends $F\colon\mathbb{A}\longrightarrow\mathbb{B}$ to $\pi^{-1}_{\mathbb{A},\mathcal{P}\mathbb{B}}(\beta_\mathbb{B}\circ F)$.

It is fairly easy to compute that, for any $G\colon\mathbb{C}\longrightarrow\mathbb{B}$ and $F\colon\mathbb{B}\longrightarrow\mathcal{P}\mathbb{A}$,

$$\pi_{\mathbb{A},\mathbb{C}}(F\circ G) = \mathcal{U}_{\mathbb{C},\mathbb{B}}(G)\circ\pi_{\mathbb{A},\mathbb{B}}(F),$$

and (equivalently), for any $F\colon\mathbb{A}\longrightarrow\mathbb{B}$ and $G\colon\mathbb{B}\longrightarrow\mathcal{P}^\dagger\mathbb{C}$,

$$\pi^{-1}_{\mathbb{A},\mathbb{C}}(G\circ F) = \mathcal{F}_{\mathbb{A},\mathbb{B}}(F)\circ\pi^{-1}_{\mathbb{B},\mathbb{B}}(G).$$

Street's [2012] general theorem then implies that there is a 2-adjunction $\mathcal{F}\dashv\mathcal{U}$ with unit β and counit α. Explicit computations furthermore show that, in the case at hand, $\mathcal{F}=\mathcal{P}_r$ and $\mathcal{U}=\mathcal{P}^\dagger_l$. $\square$

The unit and counit of the 2-adjunction in the above Proposition can be understood in terms of the Yoneda embeddings:

Lemma 4.2 *With notations as in the proof of Proposition 4.1 we have:*

- $\beta_\mathbb{A} = Y^\dagger_{\mathcal{P}\mathbb{A}}\circ Y_\mathbb{A} = \mathcal{P}^\dagger_r Y_\mathbb{A}\circ Y^\dagger_\mathbb{A}$,

- $\alpha_{\mathbb{A}} = Y_{\mathcal{P}^\dagger\mathbb{A}} \circ Y^\dagger_{\mathbb{A}} = \mathcal{P}_l Y^\dagger_{\mathbb{A}} \circ Y_{\mathbb{A}}$.

Proof: The composite functor $Y^\dagger_{\mathcal{P}\mathbb{A}} \circ Y_{\mathbb{A}} \colon \mathbb{A} \longrightarrow \mathcal{P}^\dagger\mathcal{P}\mathbb{A}$ has the same domain and codomain as $\beta_{\mathbb{A}}$, and it maps $a \in \mathbb{A}$ to the covariant presheaf $Y^\dagger_{\mathcal{P}\mathbb{A}}(Y_{\mathbb{A}}(a)) = \mathcal{P}\mathbb{A}(-, Y_{\mathbb{A}}(a))$ on $\mathcal{P}\mathbb{A}$. In $\phi \in \mathcal{P}\mathbb{A}$ the latter takes the value $\phi(a)$ (by the Yoneda Lemma), which is exactly what $\beta_{\mathbb{A}}(a)$ does too. Therefore we find that $\beta_{\mathbb{A}} = Y^\dagger_{\mathcal{P}\mathbb{A}} \circ Y_{\mathbb{A}}$ as functors; and the equality $Y^\dagger_{\mathcal{P}\mathbb{A}} \circ Y_{\mathbb{A}} = \mathcal{P}^\dagger_r Y_{\mathbb{A}} \circ Y^\dagger_{\mathbb{A}}$ follows from naturality of $Y^\dagger$. (Similar for $\alpha_{\mathbb{A}}$.) $\square$

By "abstact nonsense", the above 2-adjunction determines a 2-monad like so:

Theorem 4.3 (Double power monad) *There is a 2-monad* $(\mathcal{T}, m, u)$ *on* $\mathsf{Cat}(\mathcal{Q})$ *as follows:*

- *the 2-functor is* $\mathcal{T} := \mathcal{P}^\dagger_l \mathcal{P}_r$,
- *the multiplications are* $m_{\mathbb{A}} := \mathcal{P}^\dagger_l \alpha_{\mathcal{P}\mathbb{A}}$,
- *the units are* $u_{\mathbb{A}} := \beta_{\mathbb{A}}$.

Example 4.4 We work out the above Theorem 4.3 for **2**-categories, i.e. ordered sets, as begun in Examples 2.1 and 3.3. Writing $\mathsf{Ord} = \mathsf{Cat}(\mathbf{2})$ for the 2-category of ordered sets, order-preserving functions, and pointwise order on those functions, the 2-functor $\mathcal{T} \colon \mathsf{Ord} \longrightarrow \mathsf{Ord}$ sends $F \colon (A, \leq) \longrightarrow (B, \leq)$ to

$$\mathcal{P}^\dagger_l \mathcal{P}_r F \colon \mathsf{Up}(\mathsf{Dwn}(A)) \longrightarrow \mathsf{Up}(\mathsf{Dwn}(B)) \colon \Gamma \mapsto \{\psi \in \mathsf{Dwn}(B) \mid F^{-1}(\psi) \in \Gamma\}.$$

The multiplications and units that make $\mathcal{T}$ a monad are given by

- $m_A \colon \mathsf{Up}(\mathsf{Dwn}(\mathsf{Up}(\mathsf{Dwn}(A)))) \longrightarrow \mathsf{Up}(\mathsf{Dwn}(A)) \colon \Omega \mapsto \{\phi \in \mathsf{Dwn}(A) \mid \{\Gamma \in \mathsf{Up}(\mathsf{Dwn}(A)) \mid \phi \in \Gamma\} \in \Omega\}$,
- $u_A \colon A \longrightarrow \mathsf{Up}(\mathsf{Dwn}(A)) \colon a \mapsto \{\phi \in \mathsf{Dwn}(A) \mid a \in \phi\}$.

This description of the double power monad will be simplified in the next half of this paper, where we shall show it to be the composite of two simpler ("single power") monads, from which a straightforward characterisation of its algebras will be possible.

5. Composite of power monads

Suppose given a distributor $\Phi \colon \mathbb{A} \xrightarrow{\circ} \mathbb{B}$ and a functor $D \colon \mathbb{B} \longrightarrow \mathbb{C}$, all between $\mathcal{Q}$-categories. Using the adjoints to composition in $\mathsf{Dist}(\mathcal{Q})$ we may compute the distributor $\Phi \searrow \mathbb{C}(D-, -) \colon \mathbb{C} \xrightarrow{\circ} \mathbb{A}$. If there is a (necessarily essentially unique) functor $\mathrm{colim}(\Phi, D) \colon \mathbb{A} \longrightarrow \mathbb{C}$ such that $\Phi \searrow \mathbb{C}(D-, -) = \mathbb{C}(\mathrm{colim}(\Phi, D)-, -)$, then we call it the **$\Phi$-weighted colimit of** D. A functor $F \colon \mathbb{C} \longrightarrow \mathbb{C}'$ is said to **preserve** $\mathrm{colim}(\Phi, D)$ if $F \circ \mathrm{colim}(\Phi, D)$ coincides (up to isomorphism) with $\mathrm{colim}(\Phi, F \circ D)$. All this is illustrated in the diagrams below:

$$\begin{tikzcd} \mathbb{B} \arrow[r, "D"] & \mathbb{C} \\ \mathbb{A} \arrow[u, "\Phi", "\circ" marking] & \end{tikzcd} \qquad \begin{tikzcd} \mathbb{B} & \mathbb{C} \arrow[l, "\mathbb{B}(D-,-)"', "\circ" marking] \arrow[dl, dotted, "\Phi \searrow \mathbb{C}(D-,-)", "\circ" marking] \\ \mathbb{A} \arrow[u, "\Phi", "\circ" marking] & \end{tikzcd} \qquad \begin{tikzcd} \mathbb{B} \arrow[r, "D"] & \mathbb{C} \arrow[r, "F"] & \mathbb{C}' \\ \mathbb{A} \arrow[u, "\Phi", "\circ" marking] \arrow[ur, dotted, "\mathrm{colim}(\Phi, D)"] \arrow[urr, dotted, "\mathrm{colim}(\Phi, F \circ D)"'] & & \end{tikzcd}$$

A $\mathcal{Q}$-category $\mathbb{C}$ is **cocomplete** if it has all colimits (for all weights Φ and all diagrams D); and a functor $F \colon \mathbb{C} \longrightarrow \mathbb{C}'$ is **cocontinuous** if it preserves all colimits that happen to exist in its domain. It is furthermore well known [Stubbe, 2005, 6.8] that:

Proposition 5.1 *For a cocomplete $\mathcal{Q}$-category $\mathbb{A}$, a functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ is cocontinous if and only if it has a right adjoint; and the right adjoint in case is exactly $F^*\colon \mathbb{B} \longrightarrow \mathbb{A}\colon b \mapsto \mathrm{colim}(\mathbb{B}(F-,b), 1_{\mathbb{A}})$.*

Writing $\mathsf{Cocont}(\mathcal{Q})$ for the 2-category of *cocomplete $\mathcal{Q}$-categories and cocontinuous functors*, the forgetful 2-functor $\mathcal{U}\colon \mathsf{Cocont}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$ is monadic, with its left 2-adjoint given by $\mathcal{P}_l$. The unit for this 2-adjunction is the Yoneda embedding $Y_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathcal{P}\mathbb{A}$. The counit is given by "supremum": if $\mathbb{B}$ is a cocomplete category, then $\sup_{\mathbb{B}}\colon \mathcal{P}\mathbb{B} \longrightarrow \mathbb{B}$ is the map that sends $\phi \in \mathcal{P}\mathbb{B}$ to the element of $\mathbb{B}$ picked out by the functor $\mathrm{colim}(\phi, 1_{\mathbb{B}})\colon \mathbb{1}_{t\phi} \longrightarrow \mathbb{B}$. The 2-monad on $\mathsf{Cat}(\mathcal{Q})$ determined by this 2-adjunction is thus:

- the 2-functor $\mathcal{P}_l$,
- with multiplications $\sup_{\mathcal{P}\mathbb{A}}\colon \mathcal{P}\mathcal{P}\mathbb{A} \longrightarrow \mathcal{P}\mathbb{A}$,
- and units $Y_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathcal{P}\mathbb{A}$.

It is a so-called KZ-doctrine [Zöberlein, 1976; Kock, 1995]: the inequality $\mathcal{P}_l Y_{\mathbb{A}} \leq Y_{\mathcal{P}\mathbb{A}}$ holds. This implies in particular that:

Proposition 5.2 *Concerning a $\mathcal{Q}$-category $\mathbb{B}$, the following statements are equivalent*[3]*:*

1. *$\mathbb{B}$ is cocomplete (in the sense recalled above),*
2. *$\mathbb{B}$ is injective wrt. fully faithful functors in $\mathsf{Cat}(\mathcal{Q})$,*
3. *$Y_{\mathbb{B}}$ has a left inverse in $\mathsf{Cat}(\mathcal{Q})$,*
4. *$Y_{\mathbb{B}}$ has a left adjoint in $\mathsf{Cat}(\mathcal{Q})$,*
5. *$\mathbb{B}$ is a $\mathcal{P}_l$-algebra;*

in particular is the left adjoint/left inverse to $Y_{\mathbb{B}}$ then exactly $\sup_{\mathbb{B}}$.

Proof: The equivalence $(1 \Leftrightarrow 4)$ is in [Stubbe, 2005, 6.10], and the equivalence $(4 \Leftrightarrow 5)$ is the whole point of $\mathcal{P}_l$ being a KZ-doctrine (as [Kock, 1995] puts it, "structures are adjoint to units"), so here we shall only prove that $(1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4)$. First, assume $\mathbb{B}$ to be cocomplete, and consider functors $F\colon \mathbb{A} \longrightarrow \mathbb{C}$ and $G\colon \mathbb{A} \longrightarrow \mathbb{B}$, with F fully faithful; we must exhibit a (not necessarily unique) factorisation (up to isomorphism) of G through F. The $\mathbb{C}(F-,-)$-weighted colimit of G provides a functor $H\colon \mathbb{C} \longrightarrow \mathbb{B}$, and from the general rules for computing a weighted colimit it follows – with the aid of F's fully faithfulness in the second equality – that, for any $a \in \mathbb{A}$, $H(Fa) = \mathrm{colim}(\mathbb{C}(F-,Fa), G) = \mathrm{colim}(\mathbb{A}(-,a), G) \cong Ga$. So $\mathbb{B}$ is injective wrt. fully faithful functors. Next, from the mere injectivity of $\mathbb{B}$ and $Y_{\mathbb{B}}$'s fully faithfulness, the diagram

$$\begin{tikzcd} \mathbb{B} \arrow[rr, "1_{\mathbb{B}}"] \arrow[dr, "Y_{\mathbb{B}}"'] & & \mathbb{B} \\ & \mathcal{P}\mathbb{B} \arrow[ur, dotted] & \end{tikzcd}$$

exhibits a left inverse to $Y_{\mathbb{B}}$. Finally, suppose that $L \circ Y_{\mathbb{B}} \cong 1_{\mathbb{B}}$. For any $\phi \in \mathcal{P}\mathbb{B}$ we then have that $\phi = \mathcal{P}\mathbb{B}(Y_{\mathbb{B}}-, \phi) \leq \mathbb{B}(LY_{\mathbb{B}}-, L\phi) = \mathbb{B}(-, L\phi) = Y_{\mathbb{B}} L\phi$, hence $Y_{\mathbb{B}} \circ L \geq 1_{\mathcal{P}\mathbb{B}}$, which suffices to show that $L \dashv Y_{\mathbb{B}}$. $\square$

The dual version of the above goes as follows. A $\mathcal{Q}$-category $\mathbb{C}$ is **complete** if, for any $\Phi\colon \mathbb{B} \mathrel{-\!\!\circ\!\!\rightarrow} \mathbb{A}$ and any functor $D\colon \mathbb{B} \longrightarrow \mathbb{C}$, there exists a (necessarily essentially unique) functor $\lim(\Phi, D)\colon \mathbb{A} \longrightarrow \mathbb{C}$ such that $\mathbb{C}(-, \lim(\Phi, D)-) = \mathbb{C}(-, D-) \swarrow \Phi$. A functor $F\colon \mathbb{C} \longrightarrow \mathbb{C}'$ is said to **preserve** $\lim(\Phi, D)$ if $F \circ \lim(\Phi, D)$ coincides with $\lim(\Phi, F \circ D)$; and a functor is **continuous** if preserves all limits that happen to exist in its domain. Continuous functors into a complete category coincide with right adjoint functors.

[3] This is the content of a two-page note by the author, entitled "Cocomplete $\mathcal{Q}$-categories are precisely the injectives wrt. fully faithful functors", distributed privately in June 2006; for completeness' sake we repeat the argument here. For an extensive study of the link between monads and injectives we refer to [Escardó, 1998].

Writing $\mathsf{Cont}(\mathcal{Q})$ for the 2-category of *complete $\mathcal{Q}$-categories and continuous functors*, the forgetful 2-functor $\mathcal{U}^\dagger\colon \mathsf{Cont}(\mathcal{Q})\longrightarrow \mathsf{Cat}(\mathcal{Q})$ is monadic, with left 2-adjoint $\mathcal{P}^\dagger_r$: the unit is given by Yoneda embeddings $Y^\dagger_{\mathbb{A}}\colon \mathbb{A}\longrightarrow \mathcal{P}^\dagger\mathbb{A}$ and the counit is given by "infimum", $\mathrm{inf}_{\mathbb{B}}\colon \mathcal{P}^\dagger\mathbb{B}\longrightarrow\mathbb{B}\colon \phi\mapsto \mathrm{lim}(\phi,1_{\mathbb{B}})$. The resulting 2-monad on $\mathsf{Cat}(\mathcal{Q})$ is:

- the 2-functor $\mathcal{P}^\dagger_r$,
- with multiplications $\mathrm{inf}_{\mathcal{P}^\dagger\mathbb{A}}\colon \mathcal{P}^\dagger\mathcal{P}^\dagger\mathbb{A}\longrightarrow\mathcal{P}^\dagger\mathbb{A}$,
- and units $Y^\dagger_{\mathbb{A}}\colon \mathbb{A}\longrightarrow\mathcal{P}^\dagger\mathbb{A}$.

It is a co-KZ-doctrine, because $\mathcal{P}^\dagger_r Y^\dagger_{\mathbb{A}} \geq Y^\dagger_{\mathcal{P}^\dagger\mathbb{A}}$. This gives many characterisations of complete categories, entirely dual to Proposition 5.2:

1. $\mathbb{B}$ is complete,
2. $\mathbb{B}$ is injective wrt. fully faithful functors in $\mathsf{Cat}(\mathcal{Q})$,
3. $Y^\dagger_{\mathbb{B}}$ has a left inverse in $\mathsf{Cat}(\mathcal{Q})$,
4. $Y^\dagger_{\mathbb{B}}$ has a right adjoint in $\mathsf{Cat}(\mathcal{Q})$,
5. $\mathbb{B}$ is a $\mathcal{P}^\dagger_r$-algebra;

and the right adjoint/left inverse to $Y^\dagger_{\mathbb{B}}$ is exactly $\mathrm{inf}_{\mathbb{B}}$.

As is immediately clear now, a $\mathcal{Q}$-category $\mathbb{B}$ is complete if and only if it is cocomplete—for both reduce to the self-dual notion of injectivity wrt. fully faithful functors in $\mathsf{Cat}(\mathcal{Q})$. (This was observed in [Stubbe, 2005, 5.10] too, but with a different proof.)

The Yoneda lemmas, as stated in Lemma 3.2, now have the following incarnation:

Lemma 5.3 *For any $\mathcal{Q}$-category $\mathbb{A}$, $\mathrm{sup}_{\mathcal{P}\mathbb{A}}=\mathcal{P}_r Y_{\mathbb{A}}$ and $\mathrm{inf}_{\mathcal{P}^\dagger\mathbb{A}}=\mathcal{P}^\dagger_l Y^\dagger_{\mathbb{A}}$.*

Proof : The free $\mathcal{P}$-algebra $\mathcal{P}\mathbb{A}$ is necessarily cocomplete, so $\mathrm{sup}_{\mathcal{P}\mathbb{A}}$ is the left adjoint/inverse to $Y_{\mathcal{P}\mathbb{A}}$. The statement in Lemma 3.2, saying that $\mathcal{P}_r Y_{\mathbb{A}}$ is left inverse to $Y_{\mathcal{P}\mathbb{A}}$, is thus equivalent to the statement that $\mathrm{sup}_{\mathcal{P}\mathbb{A}}=\mathcal{P}_r Y_{\mathbb{A}}$. Similar for the other statement. $\Box$

Lemma 5.4 *For any (co)complete $\mathbb{B}$,*

$$\begin{array}{llll}
\mathcal{P}^\dagger_l Y_{\mathbb{B}}=\mathcal{P}^\dagger_r\mathrm{sup}_{\mathbb{B}} & \mathcal{P}_l Y_{\mathbb{B}}=\mathcal{P}_r\mathrm{sup}_{\mathbb{B}} & \mathcal{P}_r Y_{\mathbb{B}}=\mathcal{P}_*\mathrm{sup}_{\mathbb{B}} & \mathcal{P}^\dagger_* Y_{\mathbb{B}}=\mathcal{P}^\dagger_l\mathrm{sup}_{\mathbb{B}} \\
\mathcal{P}^\dagger_* Y^\dagger_{\mathbb{B}}=\mathcal{P}^\dagger_l\mathrm{inf}_{\mathbb{B}} & \mathcal{P}_r Y^\dagger_{\mathbb{B}}=\mathcal{P}_l\mathrm{inf}_{\mathbb{B}} & \mathcal{P}_* Y^\dagger_{\mathbb{B}}=\mathcal{P}_r\mathrm{inf}_{\mathbb{B}} & \mathcal{P}^\dagger_l Y_{\mathbb{B}}=\mathcal{P}^\dagger_*\mathrm{inf}_{\mathbb{B}}
\end{array}$$

Proof : We know that $\mathrm{sup}_{\mathbb{B}}\dashv Y_{\mathbb{B}}$, and by 2-functoriality of (say) $\mathcal{P}_l$ it follows that $\mathcal{P}_l\mathrm{sup}_{\mathbb{B}}\dashv\mathcal{P}_l Y_{\mathbb{B}}$ too. However, $\mathcal{P}_l\mathrm{sup}_{\mathbb{B}}\dashv\mathcal{P}_r\mathrm{sup}_{\mathbb{B}}$ too, so by uniqueness of adjoints we find that $\mathcal{P}_r\mathrm{sup}_{\mathbb{B}}=\mathcal{P}_l Y_{\mathbb{B}}$. Similar for the other equations. $\Box$

Suppose now that $F\colon\mathbb{A}\longrightarrow\mathbb{B}$ is any cocontinuous functor between cocomplete $\mathcal{Q}$-categories: it has a right adjoint $F^*\colon\mathbb{B}\longrightarrow\mathbb{A}$ (by Proposition 5.1). Applying the 2-functor $\mathcal{P}^\dagger_r\colon\mathsf{Cat}(\mathcal{Q})\longrightarrow\mathsf{Cat}(\mathcal{Q})$ produces again an adjunction $\mathcal{P}^\dagger_r(F)\dashv\mathcal{P}^\dagger_r(F^*)$, exhibiting $\mathcal{P}^\dagger_r F\colon\mathcal{P}^\dagger\mathbb{A}\longrightarrow\mathcal{P}^\dagger\mathbb{B}$ to be a cocontinuous functor between cocomplete categories. This shows that the 2-functor $\mathcal{P}^\dagger_r$ *lifts* to the full subcategory $\mathsf{Cocont}(\mathcal{Q})$ of $\mathsf{Cat}(\mathcal{Q})$:

$$\begin{tikzcd}
\mathsf{Cat}(\mathcal{Q}) \arrow[r, "\mathcal{P}^\dagger_r"] & \mathsf{Cat}(\mathcal{Q}) \\
\mathsf{Cocont}(\mathcal{Q}) \arrow[u, "\mathcal{U}"] \arrow[r, dotted, "\mathcal{P}^\dagger_r"'] & \mathsf{Cocont}(\mathcal{Q}) \arrow[u, "\mathcal{U}"']
\end{tikzcd} \tag{3}$$

Because $\mathcal{P}_l$ is a KZ-doctrine, the forgetful 2-functor $\mathcal{U}\colon \mathsf{Cocont}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$ is injective on objects and morphisms, so there can be at most one lifting of $\mathcal{P}^\dagger_r$ to $\mathsf{Cocont}(\mathcal{Q})$. In the speak of the theory of *distributive laws* [Beck, 1969], the existence of this (necessarily unique) lifting is equivalent to the existence of a (necessarily unique) distributive law of the *monad* $\mathcal{P}_l$ over the *endofunctor* $\mathcal{P}^\dagger_r$: a natural transformation $\lambda\colon \mathcal{P}_l \circ \mathcal{P}^\dagger_r \Longrightarrow \mathcal{P}^\dagger_r \circ \mathcal{P}_l$ satisfying the commutativity of

$$\begin{tikzcd}
 & \mathcal{P}\mathcal{P}^\dagger\mathbb{A} \arrow[dd, "\lambda_{\mathbb{A}}"] & & \mathcal{P}\mathcal{P}\mathcal{P}^\dagger\mathbb{A} \arrow[ll, "\sup_{\mathcal{P}\mathcal{P}^\dagger\mathbb{A}}"'] \arrow[d, "\mathcal{P}_l\lambda_{\mathbb{A}}"] \\
\mathcal{P}^\dagger\mathbb{A} \arrow[ur, "Y_{\mathcal{P}^\dagger\mathbb{A}}"] \arrow[dr, "\mathcal{P}^\dagger_r Y_{\mathbb{A}}"'] & & & \mathcal{P}\mathcal{P}^\dagger\mathcal{P}\mathbb{A} \arrow[d, "\lambda_{\mathcal{P}\mathbb{A}}"] \\
 & \mathcal{P}^\dagger\mathcal{P}\mathbb{A} & & \mathcal{P}^\dagger\mathcal{P}\mathcal{P}\mathbb{A} \arrow[ll, "\mathcal{P}^\dagger_r\sup_{\mathcal{P}\mathbb{A}}"]
\end{tikzcd} \tag{4}$$

Indeed, the commutative diagram in (3) determines (via the "calculus of mates") the distributive law as follows[4]:

$$\mathcal{P}\mathcal{P}^\dagger\mathbb{A} \xrightarrow{\mathcal{P}_l\mathcal{P}^\dagger_r Y_{\mathbb{A}}} \mathcal{P}\mathcal{P}^\dagger\mathcal{P}\mathbb{A} \xrightarrow{\sup_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}}} \mathcal{P}^\dagger\mathcal{P}\mathbb{A}$$

And conversely, the distributive law λ determines the lifting of $\mathcal{P}^\dagger_r$ to $\mathsf{Cocont}(\mathcal{Q})$ in the following sense: for any cocomplete $\mathbb{A}$, i.e. $\mathcal{P}_l$-algebra $\sup_{\mathbb{A}}\colon \mathcal{P}\mathbb{A} \longrightarrow \mathbb{A}$, the $\mathcal{P}_l$-algebra structure on the (free) $\mathcal{P}^\dagger_r$-algebra $\inf_{\mathcal{P}^\dagger\mathbb{A}}\colon \mathcal{P}^\dagger\mathcal{P}^\dagger\mathbb{A} \longrightarrow \mathcal{P}^\dagger\mathbb{A}$ is

$$\mathcal{P}\mathcal{P}^\dagger\mathbb{A} \xrightarrow{\lambda_{\mathbb{A}}} \mathcal{P}^\dagger\mathcal{P}\mathbb{A} \xrightarrow{\mathcal{P}^\dagger_r\sup_{\mathbb{A}}} \mathcal{P}^\dagger\mathbb{A}$$

In other words, if $\mathbb{A}$ is cocomplete then $\sup_{\mathcal{P}^\dagger\mathbb{A}} = \mathcal{P}^\dagger_r\sup_{\mathbb{A}} \circ \lambda_{\mathbb{A}}$; in still other words, if $\mathbb{A}$ is cocomplete then $Y_{\mathcal{P}^\dagger\mathbb{A}}$ is right adjoint to $\mathcal{P}^\dagger_r\sup_{\mathbb{A}} \circ \lambda_{\mathbb{A}}$.

So, with a minimal effort, we exhibited a distributive law λ of the *monad* $\mathcal{P}_l$ over the *endofunctor* $\mathcal{P}^\dagger_r$: λ is compatible with the multiplication and the unit of the monad $\mathcal{P}_l$, as expressed by the commutativity of the diagram in (4). It takes a bit more effort to show that λ is also compatible with the multiplication and unit of the monad $\mathcal{P}^\dagger_r$.

Lemma 5.5 *The transformation λ satisfies the commutativity of*

$$\begin{tikzcd}
 & \mathcal{P}\mathcal{P}^\dagger\mathbb{A} \arrow[dd, "\lambda_{\mathbb{A}}"] & & \mathcal{P}\mathcal{P}^\dagger\mathcal{P}^\dagger\mathbb{A} \arrow[ll, "\mathcal{P}_l\inf_{\mathcal{P}^\dagger\mathbb{A}}"'] \arrow[d, "\lambda_{\mathcal{P}^\dagger\mathbb{A}}"] \\
\mathcal{P}\mathbb{A} \arrow[ur, "\mathcal{P}_l Y^\dagger_{\mathbb{A}}"] \arrow[dr, "Y^\dagger_{\mathcal{P}\mathbb{A}}"'] & & & \mathcal{P}^\dagger\mathcal{P}\mathcal{P}^\dagger\mathbb{A} \arrow[d, "\mathcal{P}^\dagger_r\lambda_{\mathbb{A}}"] \\
 & \mathcal{P}^\dagger\mathcal{P}\mathbb{A} & & \mathcal{P}^\dagger\mathcal{P}^\dagger\mathcal{P}\mathbb{A} \arrow[ll, "\inf_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}}"]
\end{tikzcd}$$

Proof : Using naturality of $Y^\dagger$ in the upper trapezoid, cocontinuity[5] of $Y^\dagger_{\mathcal{P}\mathbb{A}}$ in the lower trapezoid, (one of) the unit-counit axioms for the 2-adjunction $\mathcal{P}_l \dashv \mathcal{U}$ in the left-hand triangle, and the definition of λ in the

[4]Note that $\lambda_{\mathbb{A}}$ is left adjoint to $\mathcal{P}_r\mathcal{P}^\dagger_r Y_{\mathbb{A}} \circ Y_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}}$.

[5]Because $\mathcal{P}\mathbb{A}$ is (co)complete, $Y^\dagger_{\mathcal{P}\mathbb{A}}$ has a right adjoint, namely $\inf_{\mathcal{P}\mathbb{A}}$.

right-hand triangle, the following diagram is seen to commute:

$$\begin{tikzcd}
\mathcal{P}\mathbb{A} \arrow[rrr, "\mathcal{P}_l Y^\dagger_{\mathbb{A}}"] \arrow[dr, "\mathcal{P}_l Y_{\mathbb{A}}"] \arrow[dd, "1_{\mathcal{P}\mathbb{A}}"'] & & & \mathcal{P}\mathcal{P}^\dagger\mathbb{A} \arrow[dl, "\mathcal{P}_l \mathcal{P}^\dagger_r Y_{\mathbb{A}}"'] \arrow[dd, "\lambda_{\mathbb{A}}"] \\
& \mathcal{P}\mathcal{P}\mathbb{A} \arrow[r, "\mathcal{P}_l Y^\dagger_{\mathcal{P}\mathbb{A}}"'] \arrow[dl, "\mathrm{sup}_{\mathcal{P}\mathbb{A}}"] & \mathcal{P}\mathcal{P}^\dagger\mathcal{P}\mathbb{A} \arrow[dr, "\mathrm{sup}_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}}"'] & \\
\mathcal{P}\mathbb{A} \arrow[rrr, "Y^\dagger_{\mathcal{P}\mathbb{A}}"'] & & & \mathcal{P}^\dagger\mathcal{P}\mathbb{A}
\end{tikzcd}$$

The outer square is exactly the triangle in the statement of the proposition.

As for the rectangle in the statement of the proposition, we break it down in two parts. First we use the definition of λ in the left-hand triangle, continuity of (the right adjoint) $\mathcal{P}^\dagger_r Y_{\mathbb{A}}$ in the upper trapezoid and cocontinuity[6] of $\mathrm{inf}_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}}$ in the lower trapezoid to check the commutativity of the following diagram:

$$\begin{tikzcd}
\mathcal{P}^\dagger\mathcal{P}\mathbb{A} \arrow[dr, "\mathcal{P}_l \mathcal{P}^\dagger_r Y_{\mathbb{A}}"] \arrow[dd, "\lambda_{\mathbb{A}}"'] & & \mathcal{P}\mathcal{P}^\dagger\mathcal{P}^\dagger\mathbb{A} \arrow[ll, "\mathcal{P}_l \mathrm{inf}_{\mathcal{P}^\dagger\mathbb{A}}"'] \arrow[d, "\mathcal{P}_l \mathcal{P}^\dagger_r \mathcal{P}^\dagger_r Y_{\mathbb{A}}"] \\
& \mathcal{P}\mathcal{P}^\dagger\mathcal{P}\mathbb{A} \arrow[dl, "\mathrm{sup}_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}}"] & \mathcal{P}\mathcal{P}^\dagger\mathcal{P}^\dagger\mathcal{P}\mathbb{A} \arrow[l, "\mathcal{P}_l \mathrm{inf}_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}}"] \arrow[d, "\mathrm{sup}_{\mathcal{P}^\dagger\mathcal{P}^\dagger\mathcal{P}\mathbb{A}}"] \\
\mathcal{P}^\dagger\mathcal{P}\mathbb{A} & & \mathcal{P}^\dagger\mathcal{P}^\dagger\mathcal{P}\mathbb{A} \arrow[ll, "\mathrm{inf}_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}}"]
\end{tikzcd}$$

Next, by naturality of Y in the upper-left square, cocontinuity[7] of $\mathcal{P}^\dagger_r \mathcal{P}_l \mathcal{P}^\dagger_r Y_{\mathbb{A}}$ in the lower-left square, cocontinuity[8] of $\mathcal{P}^\dagger_r \mathrm{sup}_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}}$ in the lower-right square, (one of) the unit-counit axioms for the 2-adjunction $\mathcal{P}_l \dashv \mathcal{U}$ in the upper-right triangle, and finally the definition of λ for the remaining bent arrows, we obtain

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[6]Because $\mathrm{inf}_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}} = \mathcal{P}^\dagger_l Y^\dagger_{\mathcal{P}\mathbb{A}}$ by Lemma 5.3, it is a left adjoint, with right adjoint $\mathcal{P}^\dagger_r Y^\dagger_{\mathcal{P}\mathbb{A}}$.

[7]Because $\mathcal{P}^\dagger_r \mathcal{P}_l \mathcal{P}^\dagger_r Y_{\mathbb{A}} = \mathcal{P}^\dagger_l \mathcal{P}_r \mathcal{P}^\dagger_r Y_{\mathbb{A}}$ by Proposition 3.1, it is a left adjoint, with right adjoint $\mathcal{P}^\dagger_r \mathcal{P}_r \mathcal{P}^\dagger_r Y_{\mathbb{A}}$.

[8]Because $\mathcal{P}^\dagger\mathcal{P}\mathbb{A}$ is (co)complete, $\mathcal{P}^\dagger_r \mathrm{sup}_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}} = \mathcal{P}^\dagger_l Y_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}}$ by Lemma 5.4, so it is a left adjoint, with right adjoint $\mathcal{P}^\dagger_r Y_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}}$.

the commutativity of:

$$\begin{tikzcd}
\mathcal{P}\mathcal{P}^\dagger\mathcal{P}^\dagger\mathbb{A} \arrow[r, "\mathcal{P}_l\mathcal{P}_r^\dagger\mathcal{P}_r^\dagger Y_\mathbb{A}"] \arrow[d, "\mathcal{P}_l\mathcal{P}_r^\dagger Y_{\mathcal{P}^\dagger\mathbb{A}}"] \arrow[dd, bend right, "\lambda_{\mathcal{P}^\dagger\mathbb{A}}"'] & \mathcal{P}\mathcal{P}^\dagger\mathcal{P}^\dagger\mathcal{P}\mathbb{A} \arrow[d, "\mathcal{P}_l\mathcal{P}_r^\dagger Y_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}}"] \arrow[dr, "1_{\mathcal{P}\mathcal{P}^\dagger\mathcal{P}^\dagger\mathcal{P}\mathbb{A}}"] & \\
\mathcal{P}\mathcal{P}^\dagger\mathcal{P}\mathcal{P}^\dagger\mathbb{A} \arrow[r, "\mathcal{P}_l\mathcal{P}_r^\dagger\mathcal{P}_l\mathcal{P}_r^\dagger Y_\mathbb{A}"] \arrow[d, "\sup_{\mathcal{P}^\dagger\mathcal{P}\mathcal{P}^\dagger\mathbb{A}}"] & \mathcal{P}\mathcal{P}^\dagger\mathcal{P}\mathcal{P}^\dagger\mathcal{P}\mathbb{A} \arrow[r, "\mathcal{P}_l\mathcal{P}_r^\dagger\sup_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}}"] \arrow[d, "\sup_{\mathcal{P}^\dagger\mathcal{P}\mathcal{P}^\dagger\mathcal{P}\mathbb{A}}"] & \mathcal{P}\mathcal{P}^\dagger\mathcal{P}^\dagger\mathcal{P}\mathbb{A} \arrow[d, "\sup_{\mathcal{P}^\dagger\mathcal{P}^\dagger\mathcal{P}\mathbb{A}}"] \\
\mathcal{P}^\dagger\mathcal{P}\mathcal{P}^\dagger\mathbb{A} \arrow[r, "\mathcal{P}_r^\dagger\mathcal{P}_l\mathcal{P}_r^\dagger Y_\mathbb{A}"'] \arrow[rr, bend right, "\mathcal{P}_r^\dagger\lambda_\mathbb{A}"'] & \mathcal{P}^\dagger\mathcal{P}\mathcal{P}^\dagger\mathcal{P}\mathbb{A} \arrow[r, "\mathcal{P}_r^\dagger\sup_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}}"'] & \mathcal{P}^\dagger\mathcal{P}^\dagger\mathcal{P}\mathbb{A}
\end{tikzcd}$$

Pasting these two commutative diagrams together along their common side $\sup_{\mathcal{P}^\dagger\mathcal{P}^\dagger\mathcal{P}\mathbb{A}} \circ \mathcal{P}_l\mathcal{P}_r^\dagger\mathcal{P}_r^\dagger Y_\mathbb{A}$ produces a big diagram whose contour is precisely the rectangle in the statement of the proposition. $\square$

The above lemma establishes the distributive law λ between the monads $\mathcal{P}_l$ and $\mathcal{P}_r^\dagger$; in the next one we shall compute the so-called λ-algebras, i.e. those that will turn out to be the algebras of the composite monad.

Lemma 5.6 *If $\mathbb{B}$ is a (co)complete category, then the diagram*

$$\begin{tikzcd}
\mathcal{P}\mathcal{P}^\dagger\mathbb{B} \arrow[r, "\mathcal{P}_l\inf_\mathbb{B}"] \arrow[d, "\lambda_\mathbb{B}"'] & \mathcal{P}\mathbb{B} \arrow[dr, "\sup_\mathbb{B}"] & \\
 & & \mathbb{B} \\
\mathcal{P}^\dagger\mathcal{P}\mathbb{B} \arrow[r, "\mathcal{P}_r^\dagger\sup_\mathbb{B}"'] & \mathcal{P}^\dagger\mathbb{B} \arrow[ur, "\inf_\mathbb{B}"'] &
\end{tikzcd}$$

commutes if and only if $\inf_\mathbb{B} \colon \mathcal{P}^\dagger\mathbb{B} \longrightarrow \mathbb{B}$ *is cocontinuous.*

Proof : It was indicated before that $\mathcal{P}_r^\dagger\sup_\mathbb{B} \circ\ \lambda_\mathbb{B} = \sup_{\mathcal{P}^\dagger\mathbb{B}}$, so the diagram in the statement of the lemma is identical to

$$\begin{tikzcd}
\mathcal{P}\mathcal{P}^\dagger\mathbb{B} \arrow[r, "\mathcal{P}_l\inf_\mathbb{B}"] \arrow[d, "\sup_{\mathcal{P}^\dagger\mathbb{B}}"'] & \mathcal{P}\mathbb{B} \arrow[d, "\sup_\mathbb{B}"] \\
\mathcal{P}^\dagger\mathbb{B} \arrow[r, "\inf_\mathbb{B}"'] & \mathbb{B}
\end{tikzcd}$$

Its commutativity amounts to the cocontinuity of $\inf_\mathbb{B}$. $\square$

Dualizing results from [Stubbe, 2007, 4.1, 5.4] provides us with several equivalent characterisations of the objects encountered in Lemma 5.6:

Proposition 5.7 *For a (co)complete category $\mathbb{B}$, the following conditions are equivalent:*

1. $\inf_{\mathbb{B}}$ *is cocontinuous,*
2. $\inf_{\mathbb{B}}$ *has a right adjoint in* $\mathsf{Cat}(\mathcal{Q})$,
3. $\inf_{\mathbb{B}}$ *has a continuous right inverse,*
4. $\mathbb{B}$ *is projective wrt. essentially surjective morphisms in* $\mathsf{Cont}(\mathcal{Q})$,
5. *for the distributor* $\Delta_{\mathbb{B}} := \mathbb{B}(\inf_{\mathbb{B}}-,-)\searrow\mathcal{P}^\dagger\mathbb{B}(-,Y^\dagger_{\mathbb{B}}-)\colon \mathbb{B} \mathrel{-\!\!\circ\!\!\to} \mathbb{B}$ *we have that*

$$\forall b \in \mathbb{B}_0 : \ b = \inf_{\mathbb{B}}(\Delta_{\mathbb{B}}(b,-));$$

and then $D_{\mathbb{B}}\colon \mathbb{B} \longrightarrow \mathcal{P}^\dagger\mathbb{B}\colon b \mapsto \Delta_{\mathbb{B}}(b,-)$ *is the right adjoint/continuous splitting of* $\inf_{\mathbb{B}}$*. Such a (co)complete* $\mathcal{Q}$*-category is said to be* **completely codistributive** *(or also* **totally cocontinuous***).*

The theory of distributive laws now has the following conclusion for us:

Theorem 5.8 (Composite power monad) *The natural transformation* $\lambda\colon \mathcal{P}_l\mathcal{P}^\dagger_r \Longrightarrow \mathcal{P}^\dagger_r\mathcal{P}_l$ *with components*

$$\lambda_{\mathbb{A}} = \sup_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}} \circ \ \mathcal{P}_l\mathcal{P}^\dagger_r Y_{\mathbb{A}}$$

is a distributive law of the 2-monad $\mathcal{P}_l$ over the 2-monad $\mathcal{P}^\dagger_r$. *This means in particular that we can speak of their* composite 2-monad*:*

- *the 2-functor is* $\mathcal{S} := \mathcal{P}^\dagger_r\mathcal{P}_l$,
- *the multiplications*[9] *are* $\mu_{\mathbb{A}} := (\inf_{\mathcal{P}^\dagger\mathbb{A}} * \sup_{\mathcal{P}\mathbb{A}}) \circ (1 * \lambda_{\mathbb{A}} * 1)$,
- *and the units are* $\eta := \mathcal{P}^\dagger_r Y_{\mathbb{A}} \circ Y^\dagger_{\mathbb{A}}$.

A $\mathcal{Q}$*-category* $\mathbb{A}$ *is an* $\mathcal{S}$*-algebra if and only if it is (co)complete and completely codistributive; and an* $\mathcal{S}$*-homomorphism between* $\mathcal{S}$*-algebras* $\mathbb{A}$ *and* $\mathbb{B}$ *is a bicontinuous (i.e. both cocontinuous and continuous) functor* $F\colon \mathbb{A} \longrightarrow \mathbb{B}$.

6. Double power monad = composite power monad

With notations as in the previous two sections we may now conclude:

Theorem 6.1 *The double power monad* $(\mathcal{T}, m, u)$ *is equal to the composite power monad* $(\mathcal{S}, \mu, \eta)$.

Proof: The underlying 2-functors of $\mathcal{T} = \mathcal{P}^\dagger_l\mathcal{P}_r$ and $\mathcal{S} = \mathcal{P}^\dagger_r\mathcal{P}_l$ are identical thanks to Proposition 3.1, and the units of both monads are trivially identical. The only thing to verify, is the equality of the multiplications. But recall that

$$m_{\mathbb{A}} = \mathcal{P}^\dagger_l\alpha_{\mathcal{P}\mathbb{A}} = \mathcal{P}^\dagger_l Y^\dagger_{\mathcal{P}\mathbb{A}} \circ \mathcal{P}^\dagger_l Y_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}}$$

whereas

$$\mu_{\mathbb{A}} = \inf_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}} \circ \mathcal{P}^\dagger_r\mathcal{P}^\dagger_r\sup_{\mathcal{P}\mathbb{A}} \circ \mathcal{P}^\dagger_r\lambda_{\mathcal{P}\mathbb{A}} = \inf_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}} \circ \mathcal{P}^\dagger_r\sup_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}} = \inf_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}} \circ \mathcal{P}^\dagger_l Y_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}},$$

so it suffices to see that $\inf_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}} = \mathcal{P}^\dagger_l Y^\dagger_{\mathcal{P}\mathbb{A}}$. This is an incarnation of the Yoneda lemma, cf. Lemma 5.3. $\Box$

Note in particular that each $m_{\mathbb{A}} = \mu_{\mathbb{A}}$ is a right adjoint, and so is each functor in the image of $\mathcal{T}$. It follows that Höhle's [2014] 'Condition (R)' is satisfied (see his Definition 4.3 and Theorem 5.13), which is important for his study of 'regular topological $\mathcal{T}$-spaces'.

[9] Here the operation '$*$' is the Godement product (i.e. the horizontal composition) of natural transformations.

Example 6.2 As in Examples 2.1, 3.3 and 4.4, we shall specify the above results to the case of **2**-categories, *viz.* ordered sets. The 2-functor $\mathcal{P}_l\colon \mathsf{Ord}\longrightarrow\mathsf{Ord}$ sends an order-preserving function $F\colon (A,\leq)\longrightarrow(B,\leq)$ to

$$\mathcal{P}_l F\colon \mathsf{Dwn}(A)\longrightarrow\mathsf{Dwn}(B)\colon \phi\mapsto \text{downclosure of } F(\phi).$$

The multiplications and units that make $\mathcal{P}_l$ a monad are:

- $\sup_{\mathsf{Dwn}(A)}\colon \mathsf{Dwn}(\mathsf{Dwn}(A))\longrightarrow\mathsf{Dwn}(A)\colon \Gamma\mapsto\bigcup\Gamma$,
- $Y_A\colon A\longrightarrow\mathsf{Dwn}(A)\colon a\mapsto\downarrow a$.

The category of $\mathcal{P}_l$-algebras, which we wrote as $\mathsf{Cocont}(\mathbf{2})$, has as objects the ordered sets having all suprema, and as morphisms those (order-preserving) functions that preserve all suprema. If we would restrict our attention to anti-symmetric orders (i.e. to skeletal **2**-categories), this is exactly the "usual" category Sup of sup-lattices and sup-morphisms.

Dually, the monad $\mathcal{P}_r^\dagger\colon \mathsf{Ord}\longrightarrow\mathsf{Ord}$ sends $F\colon (A,\leq)\longrightarrow(B,\leq)$ to

$$\mathcal{P}_r^\dagger F\colon \mathsf{Up}(A)\longrightarrow\mathsf{Up}(B)\colon \psi\mapsto \text{upclosure of } F(\psi)$$

and comes with multiplications and units

- $\inf_{\mathsf{Up}(A)}\colon \mathsf{Up}(\mathsf{Up}(A))\longrightarrow\mathsf{Up}(A)\colon \Gamma\mapsto\bigcup\Gamma$,
- $Y_A^\dagger\colon A\longrightarrow\mathsf{Up}(A)\colon a\mapsto\uparrow a$.

The category $\mathsf{Cont}(\mathbf{2})$ of $\mathcal{P}_r^\dagger$-algebras is, essentially, the category Inf of inf-lattices and inf-morphisms.

The literal definition of the distributive law $\lambda\colon \mathcal{P}_l\mathcal{P}_r^\dagger\Longrightarrow\mathcal{P}_r^\dagger\mathcal{P}_l$ says that it has components

$$\begin{aligned}\lambda_A &: \mathsf{Dwn}(\mathsf{Up}(A))\longrightarrow\mathsf{Up}(\mathsf{Dwn}(A))\\ &: \Gamma\mapsto\bigcap\{\Sigma\in\mathsf{Up}(\mathsf{Dwn}(A))\mid \exists\gamma\in\Gamma:\Sigma\supseteq\{\phi\in\mathsf{Dwn}(A)\mid\exists a\in A: a\in\gamma \text{ and } \downarrow a\subseteq\phi\}\}\end{aligned}$$

but – luckily – this complicated expression reduces to $\lambda_A(\Gamma)=\{\phi\in\mathsf{Dwn}(A)\mid\forall\gamma\in\Gamma:\phi\cap\gamma\neq\emptyset\}$, an expression sometimes referred to as the *cross-cut* of Γ.

Indeed, let us for the sake of the argument write, for $\Gamma\in\mathsf{Dwn}(\mathsf{Up}(A))$,

$$\Lambda_\Gamma=\{\Sigma\in\mathsf{Up}(\mathsf{Dwn}(A))\mid\exists\gamma\in\Gamma:\Sigma\supseteq\{\phi\in\mathsf{Dwn}(A)\mid\exists a\in A:a\in\gamma\text{ and }\downarrow a\subseteq\phi\}\}.$$

For any $\gamma\in\mathsf{Up}(A)$ and $\phi\in\mathsf{Dwn}(A)$ we have that

$$\exists a\in A: a\in\gamma\text{ and }\downarrow a\subseteq\phi\quad\iff\quad\phi\cap\gamma\neq\emptyset$$

so that already

$$\Lambda_\Gamma=\{\Sigma\in\mathsf{Up}(\mathsf{Dwn}(A))\mid\exists\gamma\in\Gamma:\Sigma\supseteq\{\phi\in\mathsf{Dwn}(A)\mid\phi\cap\gamma\neq\emptyset\}\}.$$

Furthermore, given any $\Sigma\in\Lambda_\Gamma$ we know (by assumption) that there is an element $\gamma_\Sigma\in\Gamma$ for which

$$\Sigma\supseteq\{\phi\in\mathsf{Dwn}(A)\mid\phi\cap\gamma_\Sigma\neq\emptyset\};$$

and conversely, given any $\gamma\in\Gamma$, we may define

$$\Sigma_\gamma=\{\phi\in\mathsf{Dwn}(A)\mid\phi\cap\gamma\neq\emptyset\}\in\Lambda_\Gamma.$$

Finally, we also put

$$\Sigma_0=\{\phi\in\mathsf{Dwn}(A)\mid\forall\gamma\in\Gamma:\phi\cap\gamma\neq\emptyset\}\in\Lambda_\Gamma.$$

From the inclusions $\Sigma_0\subseteq\Sigma_{\gamma_\Sigma}\subseteq\Sigma$ in Λ_Γ we then have

$$\Sigma_0\supseteq\bigcap\Lambda_\Gamma=\bigcap_{\gamma\in\Gamma}\Sigma_\gamma\supseteq\Sigma_0,$$

so we conclude that $\lambda_A(\Gamma)=\Sigma_0$, as claimed.

The distributive law guarantees in particular that the composite 2-functor $\mathcal{S}=\mathcal{P}_r^\dagger\mathcal{P}_l\colon \mathsf{Ord}\longrightarrow\mathsf{Ord}$, which sends an $F\colon(A,\leq)\longrightarrow(B,\leq)$ to

$$\mathcal{P}_r^\dagger\mathcal{P}_l F\colon \mathsf{Up}(\mathsf{Dwn}(A))\longrightarrow\mathsf{Up}(\mathsf{Dwn}(B))\colon \Gamma\mapsto\{\psi\in\mathsf{Dwn}(B)\mid\exists\gamma\in\Gamma: F\gamma\subseteq\psi\},$$

is again a monad, with multiplications and units given by

- $\mu_A\colon \mathsf{Up}(\mathsf{Dwn}(\mathsf{Up}(\mathsf{Dwn}(A))\longrightarrow\mathsf{Up}(\mathsf{Dwn}(A))\colon \Omega\mapsto\bigcup\{\bigcap\omega\in\mathsf{Up}(\mathsf{Dwn}(A))\mid\omega\in\Omega\}$,
- $\eta_A\colon A\longrightarrow\mathsf{Up}(\mathsf{Dwn}(A))\colon a\mapsto\{\phi\in\mathsf{Dwn}(A)\mid a\in\phi\}$.

Theorem 6.1 shows this monad to be *identical* to the monad $\mathcal{T}=\mathcal{P}_l^\dagger\mathcal{P}_r$ from Example 4.4.

The diagram in Lemma 5.6 can be written out to state that a complete lattice $(B,\leq)$ is an algebra for the composite power monad if and only if

$$\forall\Gamma\in\mathsf{Dwn}(\mathsf{Up}(B)):\bigvee\{\bigwedge\gamma\mid\gamma\in\Gamma\}=\bigwedge\{\bigvee\phi\mid\phi\in\mathsf{Dwn}(B)\text{ is such that }\forall\gamma\in\Gamma:\phi\cap\gamma\neq\emptyset\},$$

which is a familiar (choice-free) expression for the complete codistributivity of B. Via Proposition 5.7 we find several equivalent characterisations, particularly that

$$\forall b\in B:\ b=\bigwedge\{b'\in B\mid\forall\psi\in\mathsf{Up}(B):\bigwedge\psi\leq b\Rightarrow b'\in\psi\}$$

(in words: each $b\in B$ is the infimum of the elements *totally above* b). The homomorphisms between such algebras are the sup-and-inf-preserving functions.

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