On a Duality of Quantales emerging from an Operational Resolution

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Abstract

We introduce the notion of operational resolution, i.e., an isotone map from a powerset to a poset that meets two additional conditions, which generalizes the description of states as the atoms in a property lattice (Piron, 1976 and Aerts, 1982) or as the underlying set of a closure operator (Aerts, 1994 and Moore, 1995). We study the structure preservance of the related state transitions and show how the operational resolution constitutes an epimorphism between two unitary quantales.

1 Introduction

In Piron (1976), the *states*[1] of a physical entity are defined as the atoms of the (atomistic) property lattice of that entity[2]. A complementary approach, founded in Aerts (1994), takes the collection of states of a physical entity as the underlying set of a closure space[3]. In Coecke (1998a) it is shown that in order to describe individual entities within a compound system, a more general definition for state is needed. In this paper we define a map, referred to as the *operational resolution*, that relates states, which are allowed to be partially ordered, to *operational*

[1]To be interpreted in an ontological sense and not as merely statistical objects.

[2]For a general overview of the physical and operational motives behind this approach we refer to Piron (1976), Aerts (1982, 1994) and Moore (1999).

[3]For details, see Aerts (1994), Moore, (1995, 1997, 1999), Valckenborgh (1997)

properties[4]. For the case of a single entity, the proposed formulation covers both 'states as atoms in a property lattice' and 'states as the underlying set of a closure space'. We show that every operational resolution factors in a closure operator and a poset embedding that is a lattice isomorphism on its image. Further, we identify a condition under which state transitions, to be interpreted along the lines of Amira *et al.* (1998), are structure preserving in the sense that the operational resolution, the state transition and its representation within the image of the operational resolution yield a commuting square. Explicitly, we obtain two unitary quantales[5], one for the state transitions and one for their representation within the image of the operational resolution, between which the operational resolution determines a unitary quantale epimorphism. At the end of this paper we sketch some possible further developments involving aspects of orthocomplementation.

2 Operational resolution

Definition 1 *For a given collection of states Σ, an operational resolution is defined as a map $\mathcal{C}_{pr} : \mathcal{P}(\Sigma) \to \mathcal{L}$, with as codomain a poset*[6] *$(\mathcal{L}, \leq)$, such that the following conditions are met (all $T, T', T_i \in \mathcal{P}(\Sigma)$):*

$$\begin{array}{rcl} T \subseteq T' & \Rightarrow & \mathcal{C}_{pr}(T) \leq \mathcal{C}_{pr}(T') \qquad (1) \\ \forall i : \mathcal{C}_{pr}(T_i) \leq \mathcal{C}_{pr}(T) & \Rightarrow & \mathcal{C}_{pr}(\cup_i T_i) \leq \mathcal{C}_{pr}(T) \qquad (2) \\ T \neq \emptyset & \Rightarrow & \mathcal{C}_{pr}(T) \neq \mathcal{C}_{pr}(\emptyset) \qquad (3) \end{array}$$

In the presence of Eq.(1), one easily verifies that Eq.(2) is equivalent to $\forall i : \mathcal{C}_{pr}(T_i) \leq \mathcal{C}_{pr}(T) \Rightarrow \mathcal{C}_{pr}(T \cup (\cup_i T_i)) = \mathcal{C}_{pr}(T)$. As a first example, we have the following 'minimal' operational resolution: for a poset $\mathcal{L}$ containing $\{0, 1\}$, set $\mathcal{C}_{pr}(\emptyset) = 0$ and, for any $\emptyset \neq T \subseteq \Sigma$, $\mathcal{C}_{pr}(T) = 1$. $\mathcal{L} = \{0, 1\}$ is the 'optimal' codomain for this prescription for $\mathcal{C}_{pr}$ in the sense that it makes $\mathcal{C}_{pr}$ surjective. A 'maximal' example is the following: $\mathcal{L} = \mathcal{P}(\Sigma)$ and $\mathcal{C}_{pr} = id_{\mathcal{L}}$. This prescription for $\mathcal{C}_{pr}$ works for any poset $\mathcal{L}$ that contains $\mathcal{P}(\Sigma)$ with $\mathcal{P}(\Sigma)$ itself as the 'optimal' partner for this particular $\mathcal{C}_{pr}$.

We recall that a set Σ equipped with an operator $\mathcal{C} : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$ is called 'closure space', and $\mathcal{C}$ is called 'closure operator' or 'closure', if the following conditions are met for all $T, T' \in \mathcal{P}(\Sigma)$: (C1): $T \subseteq \mathcal{C}(T)$; (C2): $T \subseteq T' \Rightarrow \mathcal{C}(T) \subseteq \mathcal{C}(T')$; (C3): $\mathcal{C}(\mathcal{C}(T)) = \mathcal{C}(T)$; (C4)[7]: $\mathcal{C}(\emptyset) = \emptyset$. The closure is called 'T_1'

[4] The definition of operational resolution is chosen in such a way that a realist picture (Piron, 1976; Aerts, 1982; Moore, 1999) as well as a somewhat more empiricist picture (Aerts, 1994) can be held for the emerging operational properties.

[5] A quantale is a complete lattice equipped with a not-necessarily commutative product & which distributes over arbitrary joins. They were introduced in Mulvey (1986), for an overview we refer to Rosenthal (1990).

[6] 'Poset' is short for 'partially ordered set'.

[7] Note that this condition is not a standard one.

if in addition the following is met: (C5): $\mathcal{C}(\{t\}) = \{t\}$ for all $t \in \Sigma$. A set $F \subseteq \Sigma$ is called 'closed' if $\mathcal{C}(F) = F$. The collection of closed subsets will be denoted by $\mathcal{F}(\Sigma)$ and constitutes a complete lattice, where $\wedge_i F_i = \cap_i F_i$ and $\vee_i F_i = \mathcal{C}(\cup_i F_i)$. Remark that $\mathcal{F}(\Sigma)$ is a complete atomistic lattice if the closure is T_1: its atoms are exactly the singletons. If $(\Sigma, \mathcal{C})$ is a closure space then for $\mathcal{L} = \mathcal{F}(\Sigma)$ a surjective operational resolution is $\mathcal{C}_{pr} : \mathcal{P}(\Sigma) \to \mathcal{F}(\Sigma) : T \mapsto \mathcal{C}(T)$. More in general: if $\theta : \mathcal{F}(\Sigma) \to \mathcal{L}$ is a poset embedding that is a lattice isomorphism on its image, then $\mathcal{C}_{pr} = \theta \circ \mathcal{C} : \mathcal{P}(\Sigma) \to \mathcal{L}$ is an operational resolution. A type of operational resolution that is 'derived' from this situation is extensively studied in Amira *et al.* (1998): we considered as Σ the states of an entity described by an atomistic property lattice $\mathcal{L}$, and $\mathcal{C}_{pr} = \mu^{-1} \circ \mathcal{C} : \Sigma \to \mathcal{L}$ where $\mathcal{C}$ is the closure on Σ which has $\{F_a := \{p \in \Sigma \mid p \leq a\} \mid a \in \mathcal{L}\}$ as closed subsets and where μ^{-1} is the inverse of the Cartan representation $\mu : a \mapsto F_a$.

As last example we consider the following situation: (i) $\mathcal{C}_{pr_{(1)}} : \mathcal{P}(\Sigma_1) \to \mathcal{L} : T \mapsto \mathcal{C}_{pr_{(1,2)}}(T \times \Sigma_2)$; (ii) $\mathcal{C}_{pr_{(2)}} : \mathcal{P}(\Sigma_2) \to \mathcal{L} : T \mapsto \mathcal{C}_{pr_{(1,2)}}(\Sigma_1 \times T)$; (iii) $\mathcal{C}_{pr_{(1,2)}} : \mathcal{P}(\Sigma_1 \times \Sigma_2) \to \mathcal{L} : T \mapsto \mathcal{C}_{pr_{(1)}}(\pi_1(T)) \wedge \mathcal{C}_{pr_{(2)}}(\pi_2(T))$ with $\pi_1 : \mathcal{P}(\Sigma_1 \times \Sigma_2) \to \mathcal{P}(\Sigma_1)$ and $\pi_2 : \mathcal{P}(\Sigma_1 \times \Sigma_2) \to \mathcal{P}(\Sigma_2)$ the respective cartesian projections. The reader might identify in this an implementation of the notion of coproducts[8] $im(\mathcal{C}_{pr_{(1)}}) \coprod im(\mathcal{C}_{pr_{(2)}}) = im(\mathcal{C}_{pr_{(1,2)}})$ of the category of complete lattices, where the lattice structure of these images is assured by some results that we will prove further in this paper.

The image of $\mathcal{C}_{pr}$ (that is: $im(\mathcal{C}_{pr}) = \{\mathcal{C}_{pr}(T) \mid T \in \mathcal{P}(\Sigma)\}$) is a subset of $\mathcal{L}$, thus it inherits the partial order $\leq$ of $\mathcal{L}$. The next propostion shows that $im(\mathcal{C}_{pr})$ is a complete lattice.

Proposition 1 *The poset $(im(\mathcal{C}_{pr}), \leq)$ is a complete lattice with respect to the following definition for 'join': $\forall \{T_i\}_i \subseteq \mathcal{P}(\Sigma) : \vee_i \mathcal{C}_{pr}(T_i) := \mathcal{C}_{pr}(\cup_i T_i)$. Its bottom element is $\mathcal{C}_{pr}(\emptyset)$ and its top element is $\mathcal{C}_{pr}(\Sigma)$.*

Proof: Due to Eq.(1) we have $\forall i : \mathcal{C}_{pr}(T_i) \leq \mathcal{C}_{pr}(\cup_i T_i)$. Suppose that there exists $T' \subseteq \Sigma$ such that $\forall i : \mathcal{C}_{pr}(T_i) \leq \mathcal{C}_{pr}(T')$. Then, due to Eq.(2), $\mathcal{C}_{pr}(\cup_i T_i) \leq \mathcal{C}_{pr}(T')$, and thus $\vee_i \mathcal{C}_{pr}(T_i)$ is indeed the *lub* of $\{\mathcal{C}_{pr}(T_i)\}_i$. The rest of the claim is evident. •

The poset $\mathcal{L}$ reversely structurizes Σ through $\mathcal{C}_{pr}$. Below we study this structure, and we show how the conditions on $\mathcal{C}_{pr}$ generalize the notion of a closure operator on a set. More in particular, we associate to an operational resolution $\mathcal{C}_{pr}$ a collection of $\mathcal{C}_{pr}$-closed subsets of Σ.

Definition 2 *We call $T \in \mathcal{P}(\Sigma)$ $\mathcal{C}_{pr}$-closed if and only if for any $T' \in \mathcal{P}(\Sigma)$ we have that $T' \supset T \Rightarrow \mathcal{C}_{pr}(T') > \mathcal{C}_{pr}(T)$.*

[8]The coproduct —see for example Borceux (1994)— is by some authors considered as a description for compound physical systems (Aerts, 1984). For more details on the description of compound systems within the context of operational resolutions and state transitions we refer to Coecke and Stubbe (1999).

We denote the collection of $\mathcal{C}_{pr}$-closed subsets of Σ by $\mathcal{F}_{pr}(\Sigma)$. It is evident from Eq.(3) that $\emptyset$ is $\mathcal{C}_{pr}$-closed, Σ is trivially $\mathcal{C}_{pr}$-closed. We will now work towards a characterization of $\mathcal{F}_{pr}(\Sigma)$. We need two lemmas.

Lemma 1 *Define a relation on $\mathcal{P}(\Sigma)$ as follows: $T \sim T' \Leftrightarrow \mathcal{C}_{pr}(T) = \mathcal{C}_{pr}(T')$.*
(i) $\sim$ is an equivalence relation.
Denoting the equivalence class of $T \in \mathcal{P}(\Sigma)$ as $[T]$, then:
(ii) $\cup[T] := \cup\{T' \mid T' \in [T]\} \in [T]$;
(iii) $\cup[T] \in \mathcal{F}_{pr}(\Sigma)$;
(iv) $[T]$ contains no other $\mathcal{C}_{pr}$-closed elements than $\cup[T]$;
(v) $[\emptyset] = \{\emptyset\}$.

Proof: (i) Trivial verification. (ii) $\mathcal{C}_{pr}(T) \leq \mathcal{C}_{pr}(\cup[T])$ is immediate from the application of Eq.(1) on the trivial fact that $T \subseteq \cup[T]$. On the other hand we have that $\forall T' \in [T] : \mathcal{C}_{pr}(T') = \mathcal{C}_{pr}(T)$ from which it follows by Eq.(2) that $\mathcal{C}_{pr}(\cup[T]) \leq \mathcal{C}_{pr}(T)$. Hence we conclude that $\mathcal{C}_{pr}(T) = \mathcal{C}_{pr}(\cup[T])$ and thus $\cup[T] \in [T]$. (iii) For any $T' \supset \cup[T]$ we have by application of Eq.(1) that $\mathcal{C}_{pr}(T') \geq \mathcal{C}_{pr}(\cup[T])$. Suppose that $\mathcal{C}_{pr}(T') = \mathcal{C}_{pr}(\cup[T])$ then using (ii) gives that $\mathcal{C}_{pr}(T') = \mathcal{C}_{pr}(T)$ hence $T' \in [T]$ and $T' \subseteq \cup[T]$, which contradicts with the assumption. We conclude that $T' \supset \cup[T]$ implies $\mathcal{C}_{pr}(T') > \mathcal{C}_{pr}(\cup[T])$, thus $\cup[T]$ is $\mathcal{C}_{pr}$-closed. (iv) Let $F \in [T]$ be $\mathcal{C}_{pr}$-closed, then it follows, using (ii), that $\mathcal{C}_{pr}(F) = \mathcal{C}_{pr}(T) = \mathcal{C}_{pr}(\cup[T])$, and also $\cup[T] \supseteq F$. Suppose that $\cup[T] \supset F$ then the $\mathcal{C}_{pr}$-closedness of F implies $\mathcal{C}_{pr}(\cup[T]) > \mathcal{C}_{pr}(F)$ which leads to a contradiction. Hence $F = \cup[T]$. (v) Immediate from Eq.(3).
•

Lemma 2 *The following maps*
(i) $\phi : \mathcal{P}(\Sigma)/\!\sim\, \to \mathcal{F}_{pr}(\Sigma) : [T] \mapsto \cup[T]$
(ii) $\psi : \mathcal{F}_{pr}(\Sigma) \to im(\mathcal{C}_{pr}) : F \mapsto \mathcal{C}_{pr}(F)$
are bijections with as respective inverses:
(iii) $\phi^{-1} : \mathcal{F}_{pr}(\Sigma) \to \mathcal{P}(\Sigma)/\!\sim\, : F \mapsto [F]$
(iv) $\psi^{-1} : im(\mathcal{C}_{pr}) \to \mathcal{F}_{pr}(\Sigma) : \mathcal{C}_{pr}(T) \mapsto \cup[T]$

Proof: Straightforward verifications.
•

Proposition 1 shows that $im(\mathcal{C}_{pr})$ is a complete lattice, for it inherits the partial order from $\mathcal{L}$ and we constructed a join $\vee$. Also $\mathcal{F}_{pr}(\Sigma)$ can be equipped in a natural way with a join: the join of $\{F_i\}_i \subseteq \mathcal{F}_{pr}(\Sigma)$ is the smallest element of $\mathcal{F}_{pr}(\Sigma)$ that contains all the F_i. Equivalently: the join of $\{F_i\}_i \subseteq \mathcal{F}_{pr}(\Sigma)$ is the smallest element of $\mathcal{F}_{pr}(\Sigma)$ that contains $\cup_i F_i$. In anticipation to the following proposition, we will denote this join in $\mathcal{F}_{pr}(\Sigma)$ by $\vee_i F_i$.

Proposition 2 *For $(\mathcal{F}_{pr}(\Sigma), \vee)$ we have that:*
(i) $\vee_i F_i = \cup[\cup_i F_i]$;
(ii) $\mathcal{F}_{pr}(\Sigma) \cong im(\mathcal{C}_{pr})$.

Proof: (i) Obviously $\cup_i F_i \subseteq \cup[\cup_i F_i]$, and by part (iii) of Lemma 1 we know that $\cup[\cup_i F_i] \in \mathcal{F}_{pr}(\Sigma)$. If $\cup_i F_i$ is $\mathcal{C}_{pr}$-closed then we have by Lemma 1 part (ii) that $\cup[\cup_i F_i] = \cup_i F_i$ and then indeed $\vee_i F_i = \cup_i F_i = \cup[\cup_i F_i]$. Now consider the case where $\cup_i F_i$ is not $\mathcal{C}_{pr}$-closed, and suppose that there is an $F \in \mathcal{F}_{pr}(\Sigma)$ such that $\cup_i F_i \subset F \subset \cup[\cup_i F_i]$. Then by Eq.(1) we have that $\mathcal{C}_{pr}(\cup_i F_i) \leq \mathcal{C}_{pr}(F)$ and by $\mathcal{C}_{pr}$-closedness of F we have that $\mathcal{C}_{pr}(F) < \mathcal{C}_{pr}(\cup[\cup_i F_i]) \stackrel{*}{=} \mathcal{C}_{pr}(\cup_i F_i)$, using Lemma 1 part (ii) for $*$. This leads to a contradiction, thus there cannot be such an F. (ii) It is enough to check whether ψ and ψ^{-1} preserve joins, because then they are order preserving bijections, thus they yield a lattice isomorphism. We have: $\psi(\vee_i F_i) = \psi(\cup[\cup_i F_i]) = \mathcal{C}_{pr}(\cup[\cup_i F_i]) = \mathcal{C}_{pr}(\cup_i F_i) = \vee_i \mathcal{C}_{pr}(F_i) = \vee_i \psi(F_i)$, and reversely: $\psi^{-1}(\vee_i \mathcal{C}_{pr}(T_i)) = \psi^{-1}(\mathcal{C}_{pr}(\cup_i T_i)) = \cup[\cup_i T_i] \stackrel{*}{=} \cup[\cup_i(\cup[T_i])] = \vee_i(\cup[T_i]) = \vee_i \psi^{-1}(\mathcal{C}_{pr}(T_i))$. In both reasonings we used (i) of this proposition, part (ii) of Lemma 1 and the definition for the join in $im(\mathcal{C}_{pr})$ cfr. Proposition 1. The validity of $*$ follows from part (ii) of Lemma 1: $\forall i : \mathcal{C}_{pr}(T_i) = \mathcal{C}_{pr}(\cup[T_i]) \Rightarrow \vee_i \mathcal{C}_{pr}(T_i) = \vee_i \mathcal{C}_{pr}(\cup[T_i]) \Rightarrow \mathcal{C}_{pr}(\cup_i T_i) = \mathcal{C}_{pr}(\cup_i(\cup[T_i])) \Rightarrow [\cup_i T_i] = [\cup_i(\cup[T_i])] \Rightarrow \cup[\cup_i T_i] = \cup[\cup_i(\cup[T_i])]$.
•

In the examples we showed how a closure space $(\Sigma, \mathcal{C})$ and a poset embedding that is a lattice isomorphism on its image, say $\theta : \mathcal{F}(\Sigma) \to \mathcal{L}$, define an operational resolution $\mathcal{C}_{pr} = \theta \circ \mathcal{C} : \mathcal{P}(\Sigma) \to \mathcal{L}$. We are now ready to prove a converse.

Proposition 3 *Every operational resolution $\mathcal{C}_{pr} : \mathcal{P}(\Sigma) \to \mathcal{L}$ 'factorizes' into:*
(i) a closure operator $\mathcal{C}$ on Σ: $\mathcal{C} : \mathcal{P}(\Sigma) \to \mathcal{F}(\Sigma) \subseteq \mathcal{P}(\Sigma) : T \mapsto \cup[T]$
(ii) a poset embedding that is a lattice isomorphism on its image: $\theta : \mathcal{F}(\Sigma) := im(\mathcal{C}) \to \mathcal{L} : F \mapsto \mathcal{C}_{pr}(F)$.

Proof: (i) We check the closure axioms. (C1): $T \subseteq \cup[T]$ is obvious, thus $T \subseteq \mathcal{C}(T)$. (C2): $T \subseteq U \Rightarrow \mathcal{C}_{pr}(T) \leq \mathcal{C}_{pr}(U) \Rightarrow \cup[T] \subseteq \cup[U]$ by Eq.(1) and order preservance of ψ^{-1}, hence $\mathcal{C}(T) \subseteq \mathcal{C}(U)$ follows. (C3): $\cup[\cup[T]] = \cup[T]$ by part (ii) of Lemma 1, hence $\mathcal{C}(\mathcal{C}(T)) = \mathcal{C}(T)$. (C4): $\mathcal{C}(\emptyset) = \cup[\emptyset] = \cup\{\emptyset\} = \emptyset$. (ii) Denoting $\mathcal{F}(\Sigma)$ for the $\mathcal{C}$-closed subsets of Σ, we have by construction and by Proposition 2 that $\mathcal{F}(\Sigma) = \{\mathcal{C}(T) \mid T \in \mathcal{P}(\Sigma)\} = \{\cup[T] \mid T \in \mathcal{P}(\Sigma)\} = \{\cup[T] \mid [T] \in \mathcal{P}(\Sigma)/\sim\} \stackrel{*}{=} \mathcal{F}_{pr}(\Sigma) \cong im(\mathcal{C}_{pr}) \subseteq \mathcal{L}$ where $*$ follows from the bijection $\phi : \mathcal{P}(\Sigma)/\sim \to \mathcal{F}_{pr}(\Sigma)$ and where $im(\mathcal{C}_{pr}) \subseteq \mathcal{L}$ is a poset embedding.
•

In a first corollary we give some specific features of a $\mathcal{C}_{pr} : \mathcal{P}(\Sigma) \to \mathcal{L}$ for which $\mathcal{L}$ is a complete lattice. It should be noted that in general $im(\mathcal{C}_{pr})$ is not a sublattice of $\mathcal{L}$: in particular the join of elements of the poset $im(\mathcal{C}_{pr})$ considered as elements of the lattice $im(\mathcal{C}_{pr})$ does not necessarily coincide with the join of these elements considered as elements of the complete lattice $\mathcal{L}$. To formally distinguish the two joins, we will use $\bigvee$ for the join in $\mathcal{L}$, in contrast to $\vee$ as notation for the join in $im(\mathcal{C}_{pr})$.

Corollary 1 *Consider an operational resolution* $\mathcal{C}_{pr}:\mathcal{P}(\Sigma)\to\mathcal{L}$ *for which* $\mathcal{L}$ *is a complete lattice. Then we have the following:*

(i) In the presence of Eq.(1) we have $\forall\{T_i\}_i\subseteq\mathcal{P}(\Sigma):\bigvee_i\mathcal{C}_{pr}(T_i)\leq\mathcal{C}_{pr}(\cup_iT_i)$*. As such, if* $\forall\{T_i\}_i\subseteq\mathcal{P}(\Sigma):\mathcal{C}_{pr}(\cup_iT_i)\leq\bigvee_i\mathcal{C}_{pr}(T_i)$ *then:*

$$\forall\{T_i\}_i\subseteq\mathcal{P}(\Sigma):\mathcal{C}_{pr}(\cup_iT_i)=\bigvee_i\mathcal{C}_{pr}(T_i)\tag{4}$$

(ii) Conversely, Eq.(4) implies Eq.(1) and Eq.(2).
Consequently, any map $\mathcal{C}_{pr}:\mathcal{P}(\Sigma)\to\mathcal{L}$ *on a complete lattice* $\mathcal{L}$ *with join* $\bigvee$*, that meets the condition of Eq.(4), is an operational resolution.*

In the case where we consider only one Σ, the powerset of which is mapped on a poset $\mathcal{L}$ through an operational resolution $\mathcal{C}_{pr}$, we can formally restrict our attention to the case where $\mathcal{C}_{pr}$ is surjective: the 'relevant' part of $\mathcal{L}$ for determining the entity's operational properties is the complete lattice $im(\mathcal{C}_{pr})$ and thus we can work with the corestriction $\mathcal{C}_{pr}:\mathcal{P}(\Sigma)\to im(\mathcal{C}_{pr})$. In a second corollary we study surjective operational resolutions.

Corollary 2 *If* $\mathcal{C}_{pr}:\mathcal{P}(\Sigma)\to\mathcal{L}$ *is a surjective operational resolution, then* $\mathcal{L}$ *is a complete lattice*[9]*,* $\mathcal{C}_{pr}$ *is join preserving,* $\mathcal{C}_{pr}(\emptyset)$ *is the bottom element of* $\mathcal{L}$ *and* $\mathcal{C}_{pr}(\Sigma)$ *its top element. Moreover,* $\mathcal{C}_{pr}$ *'factors' in a closure* $\mathcal{C}$ *and a lattice isomorphism* θ*, that is* $\mathcal{C}_{pr}=\theta\circ\mathcal{C}$*, where:*
(i) $\mathcal{C}:\mathcal{P}(\Sigma)\to\mathcal{F}(\Sigma)\subseteq\mathcal{P}(\Sigma):T\mapsto\cup\{T'\in\mathcal{P}(\Sigma)\mid\mathcal{C}_{pr}(T')=\mathcal{C}_{pr}(T)\}$*;*
(ii) $\theta:\mathcal{F}(\Sigma)\xrightarrow{\sim}\mathcal{L}:F\mapsto\mathcal{C}_{pr}(F)$*;*
(iii) $\theta^{-1}:\mathcal{L}\xrightarrow{\sim}\mathcal{F}(\Sigma):t\mapsto\cup\{T'\in\mathcal{P}(\Sigma)\mid\mathcal{C}_{pr}(T')=t\}$*.*

To end this paragraph, we give in a third and last corollary a large class of surjective operational resolutions that arise 'naturally' in the particular circumstance that Σ is a 'full set of states' (Piron, 1976; Aerts, 1982) for a complete lattice $\mathcal{L}$ with join $\vee$, i.e., Σ is a subset of $\mathcal{L}$ that does not contain the bottom element, with the property that $\forall t\in\mathcal{L}:t=\vee\{a\in\Sigma\mid a\leq t\}$.

Corollary 3 *Let* Σ *full set of states for a complete lattice* $\mathcal{L}$*. Then:*

$$\mathcal{C}_{pr}:\mathcal{P}(\Sigma)\to\mathcal{L}:T\mapsto\vee T\tag{5}$$

is surjective and 'factors' into $\theta\circ\mathcal{C}$ *where:*
(i) $\mathcal{C}:\mathcal{P}(\Sigma)\to\mathcal{F}(\Sigma)\subseteq\mathcal{P}(\Sigma):T\mapsto\{t\in\Sigma\mid t\leq\vee T\}$*;*
(ii) $\theta:\mathcal{F}(\Sigma)\xrightarrow{\sim}\mathcal{L}:F\mapsto\vee F$*;*
(iii) $\theta^{-1}:\mathcal{L}\xrightarrow{\sim}\mathcal{F}(\Sigma):t\mapsto\{a\in\Sigma\mid a\leq t\}$*.*

Two important examples are: (i) $\Sigma=\mathcal{L}\setminus\{\textit{bottom element}\}$ for any complete lattice $\mathcal{L}$; (ii) if $\mathcal{L}$ is atomistic then $\Sigma=\{\textit{atoms in }\mathcal{L}\}$ is a full set of states in $\mathcal{L}$. The physical motivation for example (i) can be found in Coecke (1998a). Example

[9] We denote the join of $\mathcal{L}=im(\mathcal{C}_{pr})$ by $\vee$.

(ii) is a translation to our context of the equivalence of complete atomistic lattices and T_1-closure spaces (Aerts, 1994; Moore 1995): it can be verified that in the situation of this example the 'factor' $\mathcal{C}$ defines a T_1-closure on Σ. In any case, the map θ^{-1} can be seen as a 'generalized Cartan representation'.

3 State transitions and structure preservance.

In Amira *et al.* (1998) we intensively studied a specific kind of 'state transitions' of a physical system in the particular case where the operational resolution is a T_1-closure on a set Σ of states. Here we intend to give a generalization of those results. We will consider the *not-necessarily deterministic state transitions which respect the operational resolution*. As in Amira *et al.* (1998) we consider a first formalization of this idea by means of a map $f' : \Sigma \to \mathcal{P}(\Sigma) : s \mapsto f'(s)$ where $f'(s)$ stands for "the collection of states that may result after the transition of the physical system from its initial state s", thus $\mathcal{P}(\Sigma)$ as codomain expresses the possible non-determinedness. If Σ is ordered, then obviously f' should be order preserving. Implementing a possible lack of knowledge on the initial state, we equalize domain and codomain:

$$f : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma) : T \mapsto \cup\{f'(s) \mid s \in T\} \tag{6}$$

Such a map has two characterizing properties:
$A_\emptyset$: $\forall T \in \mathcal{P}(\Sigma) : f(T) = \emptyset \Leftrightarrow T = \emptyset$;
$A_\cup$: $\forall \{T_i\}_i \subseteq \mathcal{P}(\Sigma) : f(\cup_i T_i) = \cup_i f(T_i)$.
We denote $\mathcal{Q}(\mathcal{P}(\Sigma)) = \{f : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma) \mid f \text{ meets } A_\emptyset, A_\cup\}$. We can equip $\mathcal{Q}(\mathcal{P}(\Sigma))$ with two natural operations: (i) $f\&f'$ stands for the composition of transitions, first transition f and then transition f'; it corresponds to composition of maps, that is: $(f\&f')(-) = (f' \circ f)(-)$, and (ii) $\bigvee_i f_i$ stands for the transition that represents the choice between the f_i, or formally equivalent, a lack of knowledge on the precise state transition; it corresponds to the pointwise join in $\mathcal{P}(\Sigma)$, that is: $(\bigvee_i f_i)(-) = \cup_i(f_i(-))$. In the next proposition we show that $\mathcal{Q}(\mathcal{P}(\Sigma))$ equipped with these operations $\bigvee$ and $\&$ has a quantale structure, but first we give the exact definitions for quantales and quantale morphisms.

Definition 3 *A quantale Q is a complete join semi-lattice $(Q, \bigvee)$ equipped with an associative product, $\& : Q \times Q \to Q$, which satisfies $\forall a, b_i \in Q$:*
(i) $a\&(\bigvee_i b_i) = \bigvee_i(a\&b_i)$;
(ii) $(\bigvee_i b_i)\&a = \bigvee_i(b_i\&a)$.
A quantale Q is called unitary if there exists a so-called unit element $e \in Q$ which satisfies $\forall a \in Q : e\&a = a = a\&e$. Given two quantales Q and Q', we call $F : Q \to Q'$ a quantale morphism if it preserves $\&$ and $\bigvee$. Given two unitary quantales Q and Q' with respective units e and e', we call $F : Q \to Q'$ a unitary quantale morphism if it is a quantale morphism such that $F(e) = e'$. A quantale

Q' is called subquantale of Q if the injection $I : Q' \hookrightarrow Q : q \mapsto q$ is a quantale morphism. If Q' and Q are both unitary, and I is a unitary quantale morphism, then Q' is called a unitary subquantale of Q.

Proposition 4 *$\mathcal{Q}(\mathcal{P}(\Sigma))$ is a unitary quantale.*

Proof: First we show that the operations are internal. Let all $f, f', f_i \in \mathcal{Q}(\mathcal{P}(\Sigma))$ and all $T, T', T_i, T_j \in \mathcal{P}(\Sigma)$, then:
(i) $(f\&f')(T) = \emptyset \Leftrightarrow f'(f(T)) = \emptyset \Leftrightarrow f(T) = \emptyset \Leftrightarrow T = \emptyset$;
(ii) $(f\&f')(\cup_i T_i) = f'(f(\cup_i T_i)) = f'(\cup_i f(T_i)) = \cup_i(f'(f(T_i))) = \cup_i((f\&f')(T_i))$;
(iii) $(\bigvee_i f_i)(T) = \emptyset \Leftrightarrow \cup_i(f_i(T)) = \emptyset \Leftrightarrow \forall i : f_i(T) = \emptyset \Leftrightarrow T = \emptyset$;
(iv) $(\bigvee_i f_i)(\cup_j T_j) = \cup_i(f_i(\cup_j T_j)) = \cup_i(\cup_j f_i(T_j)) = \cup_j(\cup_i f_i(T_j)) = \cup_j((\bigvee_i f_i)(T_j))$.
Next we show that & distributes over $\bigvee$: $((\bigvee_i f_i)\&f)(T) = f((\bigvee_i f_i)(T)) = f(\cup_i(f_i(T))) = \cup_i(f(f_i(T))) = \cup_i((f_i\&f)(T)) = (\bigvee_i(f_i\&f))(T)$; analogously we have : $(f\&(\bigvee_i f_i))(T) = \bigvee_i(f\&f_i))(T)$. Finally, it is clear that $id_{\mathcal{P}(\Sigma)}$ meets both $A_\emptyset$ and $A_\cup$, and is the unit of the quantale.
•

The correspondence between $\mathcal{P}(\Sigma)$ and $im(\mathcal{C}_{pr})$ through $\mathcal{C}_{pr}$ suggests that a map $f \in \mathcal{Q}(\mathcal{P}(\Sigma)$ is 'seen' through the operational resolution as follows:

$$f_{pr} : im(\mathcal{C}_{pr}) \to im(\mathcal{C}_{pr}) : t \mapsto \mathcal{C}_{pr}(f(T)) \ for \ T \in \mathcal{P}(\Sigma) : \mathcal{C}_{pr}(T) = t \qquad (7)$$

This definition requires that, for any $t \in im(\mathcal{C}_{pr}) \subseteq \mathcal{L}$, we choose a $T \in \mathcal{P}(\Sigma)$ for which $\mathcal{C}_{pr}(T) = t$ and then set $f_{pr}(t) = \mathcal{C}_{pr}(f(T))$. Of course we need that $f_{pr}(t)$ is independent of the choice for T, which is exactly the expression of the idea that the state transition f must respect the operational resolution $\mathcal{C}_{pr}$. We can formulate this condition on an $f : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$ exactly as:
$A_\#$: $T, T' \in \mathcal{P}(\Sigma), \mathcal{C}_{pr}(T) = \mathcal{C}_{pr}(T') \Rightarrow \mathcal{C}_{pr}(f(T)) = \mathcal{C}_{pr}(f(T'))$.
We will denote $\mathcal{Q}^\#(\mathcal{P}(\Sigma)) = \{f \in \mathcal{Q}(\mathcal{P}(\Sigma)) \mid f \text{ meets } A_\#\}$. This is the collection of state transitions that we wanted to describe in the first place. Evidently, $\mathcal{Q}^\#(\mathcal{P}(\Sigma))$ inherits the operations $\bigvee$ and & from $\mathcal{Q}(\mathcal{P}(\Sigma))$, but there is more.

Proposition 5 *$\mathcal{Q}^\#(\mathcal{P}(\Sigma))$ is a unitary subquantale of $\mathcal{Q}(\mathcal{P}(\Sigma))$.*

Proof: First we show that both operations $\bigvee$ and & respect condition $A_\#$. Let $f, f', f_i \in \mathcal{Q}^\#(\mathcal{P}(\Sigma))$ and $T, T' \in \mathcal{P}(\Sigma)$ with $\mathcal{C}_{pr}(T) = \mathcal{C}_{pr}(T')$, then it follows that:
(i) $\mathcal{C}_{pr}(f(T)) = \mathcal{C}_{pr}(f(T')) \Rightarrow \mathcal{C}_{pr}(f'(f(T)) = \mathcal{C}_{pr}(f'(f(T')) \Rightarrow \mathcal{C}_{pr}((f\&f')(T)) = \mathcal{C}_{pr}((f\&f')(T'))$; (ii) $\forall i : \mathcal{C}_{pr}(f_i(T)) = \mathcal{C}_{pr}(f_i(T')) \Rightarrow \vee_i \mathcal{C}_{pr}(f_i(T)) = \vee_i \mathcal{C}_{pr}(f_i(T')) \Rightarrow \mathcal{C}_{pr}(\cup_i f_i(T)) = \mathcal{C}_{pr}(\cup_i f_i(T')) \Rightarrow \mathcal{C}_{pr}((\bigvee_i f_i)(T)) = \mathcal{C}_{pr}((\bigvee_i f_i)(T'))$. Finally, it is trivial that $id_{\mathcal{P}(\Sigma)}$ meets $A_\#$.
•

In the following lemmas we give some crucial properties of the map $F_{pr} : f \mapsto f_{pr}$.

Lemma 3 *Let all* $f_i, f, f' \in \mathcal{Q}^{\#}(\mathcal{P}(\Sigma))$, *then:*
(i) $(f\&f')_{pr} = f_{pr}\&f'_{pr}$ *where* $(f_{pr}\&f'_{pr})(-) = (f'_{pr} \circ f_{pr})(-)$ *(composition of maps);*
(ii) $(\bigvee_i f_i)_{pr} = \bigvee_i f_{i,pr}$ *where* $(\bigvee_i f_{i,pr})(-) = \vee_i(f_{i,pr}(-))$ *(pointwise computation);*
(iii) $(id_{\mathcal{P}(\Sigma)})_{pr} = id_{im(\mathcal{C}_{pr})}$*;*
(iv) f_{pr} *meets* $A_0 : f_{pr}(t) = 0 \Leftrightarrow t = 0$ *where* $0 := \mathcal{C}_{pr}(\emptyset)$ *and* $t \in im(\mathcal{C}_{pr})$*;*
(v) f_{pr} *meets* $A_\vee : f_{pr}(\vee_i t_i) = \vee_i(f_{pr}(t_i))$ *for all* $\{t_i\}_i \subseteq im(\mathcal{C}_{pr})$*.*

Proof: For all $t, t_i \in im(\mathcal{C}_{pr})$ we choose $T, T_i \in \mathcal{P}(\Sigma)$ such that $\mathcal{C}_{pr}(T) = t, \mathcal{C}_{pr}(T_i) = t_i$. Condition $A_\#$ assures that all computations concerning $f_{pr}(t)$ can be done via $f_{pr}(\mathcal{C}_{pr}(T)) = \mathcal{C}_{pr}(f(T))$. Then:
(i)

$$\begin{aligned}
(f\&f')_{pr}(\mathcal{C}_{pr}(T)) &= \mathcal{C}_{pr}((f\&f')(T)) \\
&= \mathcal{C}_{pr}(f'(f(T))) \\
&= f'_{pr}(\mathcal{C}_{pr}(f(T))) \\
&= f'_{pr}(f_{pr}(\mathcal{C}_{pr}(T))) \\
&= (f_{pr}\&f'_{pr})(\mathcal{C}_{pr}(T));
\end{aligned}$$

and
(ii)

$$\begin{aligned}
(\bigvee_i f_i)_{pr}(\mathcal{C}_{pr}(T)) &= \mathcal{C}_{pr}((\bigvee_i f_i)(T)) \\
&= \mathcal{C}_{pr}(\cup_i f_i(T)) \\
&= \vee_i(\mathcal{C}_{pr}(f_i(T))) \\
&= \vee_i(f_{i,pr}(\mathcal{C}_{pr}(T))) \\
&= (\bigvee_i f_{i,pr})(\mathcal{C}_{pr}(T));
\end{aligned}$$

(iii) $(id_{\mathcal{P}(\Sigma)})_{pr}(\mathcal{C}_{pr}(T)) = \mathcal{C}_{pr}(id_{\mathcal{P}(\Sigma)}(T)) = \mathcal{C}_{pr}(T)$;
(iv) $f_{pr}(\mathcal{C}_{pr}(T)) = \mathcal{C}_{pr}(f(T)) = \mathcal{C}_{pr}(\emptyset) \overset{*}{\Leftrightarrow} f(T) = \emptyset \Leftrightarrow T = \emptyset$ where $*$ uses Eq.(3);
(v)

$$\begin{aligned}
f_{pr}(\vee_i \mathcal{C}_{pr}(T_i)) &= f_{pr}(\mathcal{C}_{pr}(\cup_i T_i)) \\
&= \mathcal{C}_{pr}(f(\cup_i T_i)) \\
&= \mathcal{C}_{pr}(\cup_i f(T_i)) \\
&= \vee_i \mathcal{C}_{pr}(f(T_i)) \\
&= \vee_i f_{pr}(\mathcal{C}_{pr}(T_i)).
\end{aligned}$$

•

We denote $\mathcal{Q}(im(\mathcal{C}_{pr})) = \{g : im(\mathcal{C}_{pr}) \to im(\mathcal{C}_{pr}) \mid g \text{ meets } A_\vee, A_0\}$, and equip this set with $\bigvee$ and & defined by pointwise computation and composition of maps respectively.

Lemma 4 *We have:*
(i) $\mathcal{Q}(im(\mathcal{C}_{pr}))$ *is a unitary quantale*
(ii) $\mathcal{Q}(im(\mathcal{C}_{pr})) = \{f_{pr} \mid f \in \mathcal{Q}^{\#}(\mathcal{P}(\Sigma))\}$.

Proof: (i) Straightforward verification analogous to Proposition 4, the unit of $\mathcal{Q}(im(\mathcal{C}_{pr}))$ is $id_{im(\mathcal{C}_{pr})}$;
(ii) Given $g \in \mathcal{Q}(im(\mathcal{C}_{pr}))$ define $f : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$ by setting $f(X) = Y \Leftrightarrow g(\mathcal{C}_{pr}(X)) = \mathcal{C}_{pr}(Y)$. We will prove that $f \in \mathcal{Q}^{\#}(\mathcal{P}(\Sigma))$, and that $f_{pr} = g$.
(a) $f(T) = \emptyset \Leftrightarrow g(\mathcal{C}_{pr}(T)) = \mathcal{C}_{pr}(\emptyset) =: 0 \Leftrightarrow \mathcal{C}_{pr}(T) = 0 \Leftrightarrow T = \emptyset$, where we used Eq.(3) in the last step of the reasoning;
(b) By definition of f we have that

$$\begin{aligned} \forall i : g(\mathcal{C}_{pr}(T_i)) = \mathcal{C}_{pr}(f(T_i)) &\Leftrightarrow \vee_i g(\mathcal{C}_{pr}(T_i)) = \vee_i \mathcal{C}_{pr}(f(T_i)) \\ &\Leftrightarrow g(\vee_i \mathcal{C}_{pr}(T_i)) = \mathcal{C}_{pr}(\cup_i f(T_i)) \\ &\Leftrightarrow g(\mathcal{C}_{pr}(\cup_i T_i)) = \mathcal{C}_{pr}(\cup_i f(T_i)) \\ &\Leftrightarrow f(\cup_i T_i) = \cup_i f(T_i); \end{aligned}$$

(c) $\mathcal{C}_{pr}(T) = \mathcal{C}_{pr}(T') \Rightarrow g(\mathcal{C}_{pr}(T)) = g(\mathcal{C}_{pr}(T')) \Rightarrow \mathcal{C}_{pr}(f(T)) = \mathcal{C}_{pr}(f(T'))$;
(d) $f_{pr}(\mathcal{C}_{pr}(T)) = \mathcal{C}_{pr}(f(T)) = g(\mathcal{C}_{pr}(T))$.
•

Proposition 6 $F_{pr} : \mathcal{Q}^{\#}(\mathcal{P}(\Sigma)) \to \mathcal{Q}(im(\mathcal{C}_{pr})) : f \mapsto f_{pr}$ *is a surjective unitary quantale morphism.*

Proof: Follows from the lemmas above.
•

It is easy to see that the above results are indeed a generalization of the situation described in Amira *et al.* (1998). Consider as operational resolution a T_1-closure $\mathcal{C}_{pr} = \mathcal{C} : \mathcal{P}(\Sigma) \to \mathcal{F}(\Sigma) \subseteq \mathcal{P}(\Sigma)$, that is, $\Sigma = \{atoms\ of\ \mathcal{F}(\Sigma)\}$. Then, according to the above, a state transition is a map $f : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$ that meets $A_\emptyset$, $A_\cup$ and $A_\#$. Moreover, we have that $f_{pr} : \mathcal{F}(\Sigma) \to \mathcal{F}(\Sigma) : F \mapsto \mathcal{C}(f(T))$ where $T \in \mathcal{P}(\Sigma)$ is chosen in such a way that $\mathcal{C}(T) = F$. Exploiting $F = \mathcal{C}(T) = \mathcal{C}(\mathcal{C}(T))$ it follows that $f_{pr}(F) = \mathcal{C}(f(\mathcal{C}(T))) = \mathcal{C}(f(T))$ and thus $f(\mathcal{C}(T)) \subseteq \mathcal{C}(f(T))$. In Amira *et al.* (1998) this condition is given the notation:
$A_* : \forall T \in \mathcal{P}(\Sigma) : f(\mathcal{C}(T)) \subseteq \mathcal{C}(f(T))$.
For a map $f : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$ that meets $A_\emptyset$, $A_\cup$ and A_* it is then argued that it is 'seen' through the operational resolution as $f_{pr}^{bis} : \mathcal{F}(\Sigma) \to \mathcal{F}(\Sigma) : F \mapsto \mathcal{C}(f(F))$. However, it can be easily verified that, concerning a map $f : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$ that meets $A_\emptyset$ and $A_\cup$, it is equivalent to work with either condition $A_\#$ and f_{pr} or condition A_* and f_{pr}^{bis}.

4 Conclusions, remarks and further research

Every operational resolution factors in a closure operator and a lattice isomorphism on its image. As such, it mathematically generalizes the duality [states $\leftrightarrow$ properties], which is also exhibited in the correspondences [underlying set of a closure space $\leftrightarrow$ lattice of closed subsets] and [full set of states $\leftrightarrow$ lattice]. Although the codomain of the operational resolution is a poset, its image has a lattice structure in a natural way. Non-deterministic state transitions are formalized, and a condition for them to preserve the operational resolution is derived. The collection of structure preserving state transitions forms a unitary quantale, so does their image through the operational resolution, and between these quantales the operational resolution suggests a natural surjective quantale morphism.

Within this scheme it is possible to implement aspects of orthogonality, more or less along the lines of the construction in Aerts (1994) and Valckenborgh (1997). Suppose that there exists an orthogonality relation $\perp$ on $\mathcal{L}$. Then we can define an orthogonality on Σ by setting $p \perp q \Leftrightarrow \mathcal{C}_{pr}(p) \perp \mathcal{C}_{pr}(q)$, derive an orthocomplementation ${}^{\perp} : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma) : T \mapsto \{p \in \Sigma \mid \forall q \in T : p \perp q\}$ and relate to this a closure operator $\mathcal{C}_{\perp} : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma) : T \mapsto T^{\perp\perp}$, where the collection of closed subsets $\mathcal{F}_{\perp}(\Sigma) = \{T \subseteq \Sigma \mid \mathcal{C}_{\perp}(T) = T\}$ proves to be orthocomplemented. It can be shown that $\mathcal{F}_{pr}(\Sigma)$, equipped with the above defined orthogonality relation, is orthocomplemented if and only if $\mathcal{F}_{\perp}(\Sigma) = \mathcal{F}_{pr}(\Sigma)$. Obviously, orthocomplementedness of $\mathcal{L}$ does not imply orthocomplementedness of $\mathcal{F}_{pr}(\Sigma)$, not even if $im(\mathcal{C}_{pr})$ is a sublattice of $\mathcal{L}$. An interesting situation demonstrating this is that of three operational resolutions related to the coproduct (cfr. example in section 2), where the orthocomplementations of $im(\mathcal{C}_{pr_{(1)}})$ and $im(\mathcal{C}_{pr_{(2)}})$ do not necessarily imply an orthocomplementation on $im(\mathcal{C}_{pr_{(1,2)}})$, but where the separated product of $im(\mathcal{C}_{pr_{(1)}})$ and $im(\mathcal{C}_{pr_{(2)}})$ as codomain $\mathcal{L}$ does inherit an orthocomplementation (Aerts, 1982). It would be worthwhile to investigate the connection between orthogonality on $\mathcal{F}_{pr}(\Sigma)$ and orthogonality on $\mathcal{L}$, and the implications for the state transitions as we have studied them in this paper[10]. This is of particular interest in the study of descriptions of compound systems where the structure preserving state transitions could play a crucial role (Coecke, 1998a, 1998b).

5 Credits

We are more than happy with the care the referee took in reading this paper, and for his very detailed remarks and suggestions on formulations and proofs.

[10]Quantales of maps with underlying orthocomplemented lattices have already been considered in for example Paseka (1996) and Roman and Zuazua (1996).

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