# On quantaloid-enriched categories in general, and sheaves on a quantaloid in particular

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November 6, 2011

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Dissertation présentée en vue de l'obtention de l'Habilitation de Diriger des Recherches, en séance publique, le 12 décembre 2011, au Laboratoire de Mathématiques Pures et Appliquées de l'Université du Littoral-Côte d'Opale, France.

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AMS Subject Classification (2000): 03B60 (Nonclassical logic), 06D22 (Locales), 06F07 (Quantales), 18B10 (Categories of relations), 18B35 (Preorders, orders and lattices viewed as categories), 18D05 (Bicategories), 18D20 (Enriched categories).

# Preface

For this *candidature à l'habilitation à diriger des recherches* I present a selection of my publications that appeared between 2005 and 2011—seven years of extraordinary adventures. After my studies in Brussels (1994–1998) and my doctorate in Louvain-la-Neuve (1998–2005), I spent one year as a post-doc researcher in Coimbra (2005–2006) and then three years in Antwerp (2006–2009), to now find myself as a *maître de conférences* in Calais (since 2009). Inbetween I worked for several months in Sydney, and paid visits to many other countries worldwide. Each of my travels held its surprises, but there is one constant: on each of these occasions I met and collaborated with creative and original mathematicians, several of which I became good friends with. I am very happy that Dirk Hofmann, Steve Lack and Fred Van Oystaeyen unhesitantly accepted to act as *rapporteurs*, and Bob Coecke and Mai Gehrke as *membres de jury*, for this thesis. These mathematicians represent the very best of experiences that I had during my "tour of the world". My post-doc years in Antwerp were particularly marked by my fruitful collaboration with Hans Heymans, at the time a doctorate student of Fred Van Oystaeyen's. If I am now submitting this *candidature à l'habilitation à diriger des recherches*, then I do so too because my co-direction of Hans' doctoral thesis gave me a taste for more.

# Contents

# Complete bibliography

**Publications.**


1. [H. Amira, B. Coecke and I. Stubbe, 1998] *How quantales emerge by introducing induction within the operational approach*, Helv. Phys. Acta **71**, pp. 554–572.

2. [B. Coecke and I. Stubbe, 1999] *Duality of quantales emerging from an operational resolution*, Int. J. Theor. Phys. **38**, pp. 3269–3281.

3. [B. Coecke and I. Stubbe, 1999] *Operational resolutions and state transitions in a categorical setting*, Found. Phys. Letters **12**, pp. 29–49.

4. [B. Coecke and I. Stubbe, 2000] *State transitions as morphisms for complete lattices*, Int. J. Theor. Phys. **39**, pp. 601–610.

5. [F. Borceux and I. Stubbe, 2000] *Short introduction to enriched categories*, chapter in: Fundamental Theories of Physics **111** ("Current research in operational quantum logic: algebras, categories, languages"), Kluwer Academic, pp. 167–194.

6. [B. Coecke, D. Moore and I. Stubbe, 2001] *Quantaloids describing propagation of physical properties*, Found. Phys. Letters **14**, pp. 133–145.

7. [S. Sourbron, I. Stubbe, R. Luypaert and M. Osteaux, 2003] *DCI-MRI based quantification of CBF*, Magnetic Resonance Materials in Physics, Biology and Medicine **16**, pp. s134-s135.

8. [I. Stubbe, 2003] *Categorical structures enriched in a quantaloid: categories and semicategories*, thèse de doctorat, Université Catholique de Louvain.

9. [I. Stubbe, 2005] *Categorical structures enriched in a quantaloid: categories, distributors and functors*, Theory Appl. Categ. **14**, pp. 1–45.

10. [I. Stubbe, 2005] *Categorical structures enriched in a quantaloid: regular presheaves, regular semicategories*, Cahiers Topol. Géom. Differ. Catég. **46**, pp. 99–121.

11. [I. Stubbe, 2005] *Categorical structures enriched in a quantaloid: orders and ideals over a base quantaloid*, Appl. Categ. Structures **13**, pp. 235–255.

12. [I. Stubbe, 2005] *The canonical topology on a meet-semilattice*, Int. J. Theor. Phys. **44**, pp. 2283–2293.

13. [I. Stubbe, 2006] *Categorical structures enriched in a quantaloid: tensored and cotensored categories*, Theory Appl. Categ. **16**, pp. 283–306.

14. [I. Stubbe, 2006] *Towards "dynamic domains": totally continuous cocomplete $\mathcal{Q}$-categories (Extended abstract)*, Elec. Notes Theor. Comp. Sci. **155**, pp. 617–634.

15. [I. Stubbe, 2007] *Towards 'dynamic domains': totally continuous cocomplete Q-categories*, Theor. Comp. Sci. **373**, pp. 142–160.

16. [I. Stubbe, 2007] *Q-modules are Q-suplattices*, Theory Appl. Categ. **19**, pp. 50–60.

17. [I. Stubbe and B. Van Steirteghem, 2007] *Propositional systems, Hilbert lattices and generalised Hilbert spaces*, chapter in: Handbook of Quantum Logic and Quantum Structures: Quantum Structures, Elsevier, pp. 477–524.

18. [M. M. Clementino, D. Hofmann and I. Stubbe, 2009] *Exponentiable functors between quantaloid-enriched categories*, Appl. Categ. Structures **17**, pp. 91–101.

19. [H. Heymans and I. Stubbe, 2009] *On principally generated $\mathcal{Q}$-modules in general, and skew local homeomorphisms in particular*, Ann. Pure Appl. Logic **161**, pp. 43–65.

20. [H. Heymans and I. Stubbe, 2009] *Modules on involutive quantales: canonical Hilbert structure, applications to sheaves*, Order **26**, pp. 177–196.

21. [I. Stubbe, 2010] *'Hausdorff distance' via conical cocompletions*, Cahiers Topol. Géom. Differ. Catég. **51**, pp. 51–76.

22. [D. Hofmann and I. Stubbe, 2010] *Towards Stone duality for topological theories*, Topol. Appl. **158**, pp. 913–925.

23. [H. Heymans and I. Stubbe, 2011] *Symmetry and Cauchy completion of quantaloid-enriched categories*, Theory and Applications of Categories **25**, pp. 276–294.

24. [H. Heymans and I. Stubbe, 2011] *Elementary characterisation of quantaloids of closed cribles*, Journal of Pure and Applied Algebra, to appear, 13 pages.

25. [S. Abramsky, B. Coecke, I. Stubbe and F. Valckenborgh, 2011] *The Faure-Moore-Piron theorem in Chu-spaces*, 14 pages, in preparation.

26. [H. Heymans and I. Stubbe, 2011] *From Grothendieck quantales to allegories: sheaves as enriched categories*, 18 pages, in preparation.

**Talks at conferences with scientific committee.**

27. September 1998: "A duality of quantales describing structure preservance for generalized states", fourth biannual meeting of the International Quantum Structures Association, Liptovski Jan, Slovakia.

28. April 2001: "A corestriction of the Yoneda embedding as frame completion of a bounded poset with finite meets, with an application to quantum logic", fifth biannual meeting of the International Quantum Structures Association, Cesenatico, Italy.

29. September 2003: "Ideals, orders and sets over a base quantaloid", European Category Theory Meeting, Haute-Bodeux, Belgique.

30. July 2004: "Orders and ideals over a base quantaloid", Category Theory 04, Vancouver, Canada.

31. May 2005: "Towards 'dynamic domains': totally continuous cocomplete $\mathcal{Q}$-categories", Twenty-first Conference on the Mathematical Foundations of Programming Semantics (MFPS XXI), University of Birmingham, Birmingham, United Kingdom.

32. June–July 2006: "$\mathcal{Q}$-modules are $\mathcal{Q}$-suplattices", Category Theory 06, Halifax, Canada.

33. June 2008: "Principally generated modules on a quantaloid", Category Theory 08, Calais, France.

34. June 2010: "Symmetry and Cauchy-completion of quantaloid-enriched categories", Category Theory 2010, Genova, Italy.

**Invited talks.**

35. July 2002: "Quantaloid-enriched (semi)categories, distributors and functors: an overview", Meeting on the Theory and Applications of Quantales, Instituto Superior Técnico, Lisboa, Portugal.

36. March 2003: "Causal Duality for processes as (co)tensors in a quantaloid enriched category", Workshop "Seminarie: Logica & Informatica 2003", Brussels, Belgium.

37. June 2004: "Order! Order in the House!", Oxford University's Computing Laboratory, Oxford, United Kingdom.

38. January 2005: "Ordered sheaves on a quantaloid", Workshop on Logic from Quantales, Oxford University's Computing Laboratory, Oxford, United Kingdom.

39. January 2005: "Dynamic domains", Department of Mathematics, Oxford University, Oxford, United Kingdom.

40. February 2005: "$\Omega$-verzamelingen", Departement Wiskunde, Vrije Universiteit Brussel, Bruxelles, België.

41. April 2005: "Theorie des domaines en logique dynamique", Equipe Preuves, Programmes, Systèmes, Institut de mathématiques de Jussieu, Université de Paris 7, Paris, France.

42. December 2005: "$\mathcal{Q}$-modules are $\mathcal{Q}$-suplattices", Workshop on Mathematical Structures in Quantum Computing ("Q-day II"), Institut Henri Poincaré, Paris, France.

43. June 2006: "$\mathcal{Q}$-modules are $\mathcal{Q}$-suplattices", Algebra Seminar, Instituto Superior Técnico, Lisboa, Portugal.

44. July 2006: "Abstract projective geometry and Piron's representation theorem", Cats, Kets and Cloisters, University of Oxford, Oxford, United Kingdom.

45. May 2007: "Exponentiability in Cat(Q)", Algebra Seminar, Department of Algebra and Geometry, Masaryk University, Brno, Czech Republic.

46. July 2007: "Dynamic logic: a noncommutative topology of truth values", Seminar on mathematical aspects of novel approaches to quantum theory and foundations of physics, organized by Basil Hiley and Georg Wikman (SHIRD), Åskloster, Sweden.

47. August 2007: "Sheaves on noncommutative topologies", Categorical Quantum Logic, Oxford, UK.

48. November 2007: "Principally generated Q-modules", Departement Wiskunde, Vrije Universiteit Brussel, Brussel, België.

49. November 2007: "Principally generated Q-modules", Departamento de Matemática, Universidade de Aveiro, Aveiro, Portugal.

50. January 2008: Commentator for Klaas Landsman's lecture on "The principle of general tovariance", Symposium on Logic and Physics, Universiteit Utrecht, Utrecht, The Netherlands.

51. January 2008: "Skew local homeomorphisms", Département de Mathematiques, Université de Louvain, Louvain-la-Neuve, Belgique.

52. September 2008: "Principally generated quantaloid-modules", Workshop on Sheaves in Geometry and Quantum Theory, Radboud Universiteit Nijmegen, the Netherlands.

53. September 2008: "Skew local homeomorphisms", Workshop on Sheaves in Geometry and Quantum Theory, Radboud Universiteit Nijmegen, the Netherlands.

54. October 2008: "Skew local homeomorphisms", Meeting on Categories in Algebra, Geometry and Logic, Royal Academy of Sciences, Brussels, Belgium.

55. January 2009: "Canonical Hilbert structure for modules on involutive quantales", Categories, Logic and Foundations of Physics, Imperial College, London, United Kingdom.

56. March 2009: "'Hausorff distance' via conical cocompletion", Département de Mathématiques, Université de Louvain, Louvain-la-Neuve, Belgium.

57. March 2009: "Enriched colimits vs. suprema", Institute for Mathematics, Astrophysics and Particle Physics, Radboud Universiteit Nijmegen, the Netherlands.

58. July 2009: "Sheaves as modules", Seminar on novel mathematical structures in physics: non-commutative geometry, organized by Basil Hiley and Georg Wikman (SHIRD), Åskloster, Sweden.

59. June 2010: "Symétrie et Cauchy complétion", Séminaire Itinérant des Catégories, Université de Picardie-Jules Verne, Amiens, France.

60. August 2010: "Characterisation of commutative topologies amongst non-commutative topologies", International workshop on noncommutative algebra, Shaoxing, China.

61. May 2011 : "An elementary characterisation of quantaloids of closed cribles", Category Theory, Algebra and Geometry, a conference on the occasion of Ross Street's Chaire de la Vallée Poussin, Département de Mathématiques, Université Catholique de Louvain, Louvain-la Neuve, Belgium.

**Talks at other international events.**

62. June 1998: "Operational resolutions and state transitions in a quantaloid setting", International workshop on current research in operational quantum logic I, Brussels, Belgium.

63. May 1999: "On resolutions, state/property transitions, and the enriched categories they constitute", International workshop on current research in operational quantum logic II, Brussels, Belgium.

64. August 1999: "Why the category of sup-lattices may be of interest to physicists", Australian category theory seminar, Sydney, Australia.

65. April 2000: "Categorical methods in operational quantum logic", 72nd meeting of the Paripathetic Seminar on Sheaves and Logic, Brussels, Belgium.

66. November 2000: "Logique Quantique Opérationelle et Catégories", Séminaire Itinérant des Catégories, Université de Picardie, Amiens, France.

67. November 2001: "What can we certainly say about what may possibly be? The operational resolution, quantales of induced state/property transitions, and causal duality as crux of the biscuit", Oxford University's Computing Laboratory, Oxford, UK.

68. January 2003: "Totally regular semicategories enriched in a quantaloid: from Louvain-la-Neuve to Sydney and back", Australian Category Seminar, Sydney, Australia.

69. January 2003: "The principle of Causal Duality establishes quantum logics with (external) implication as quantaloid-enriched categories", Australian Category Seminar, Sydney, Australia.

70. November 2005: "$\mathcal{Q}$-modules are $\mathcal{Q}$-suplattices", Category Theory Seminar, Coimbra, Portugal.

71. February 2008: "Totally continuous and totally algebraic cocomplete $\mathcal{Q}$-categories", Australian Category Seminar, Sydney, Australia.

72. February 2008: "Locally principally generated $\mathcal{Q}$-modules", Australian Category Seminar, Sydney, Australia.

73. March 2008: "Skew local homeomorphisms", Australian Category Seminar, Sydney, Australia.

74. October 2011: "Faisceaux déguisés en modules", Séminaire Itinérant des Catégories, Université du Littoral-Côte d'Opale, Calais, France.

# Synthesis of selected publications

As is customary for a 'dossier de candidature à l'habilitation à diriger des recherches', I shall provide an introduction to, and a synthesis of, my post-doctoral research *on quantaloid-enriched categories in general, and sheaves on a quantaloid in particular.* The style of this text will be rather informal: explaining the main ideas will be of more importance than going into technical details. I shall not necessarily respect the chronological order of my publications. Notes in this text refer to those on page 37, and citations refer to the references on page 42.

## 1.

In his seminal [1973] paper, William F. Lawvere observed that a generalised metric space $(X, d)$ is, what we call today, a category enriched in the quantale $[0, \infty]$ of extended positive real numbers, and that a distance-decreasing map $f\colon (X, d) \longrightarrow (Y, d)$ is a functor between such enriched categories. Moreover, and more importantly, he proved that a metric space is Cauchy complete (that is, Cauchy sequences converge) precisely when, viewed as a category, it admits all colimits weighted by left adjoint distributors. In [2002] Lawvere commented that "[t]his connection is more fruitful than a mere analogy, because it provides a sequence of mathematical theorems, so that enriched category theory can suggest new directions of research in metric space theory and conversely"; and in a message on the 'categories' mailing list of 28 October 2006, he puts it more strongly: "[t]he whole general theory of enriched categories should in particular be focused on metric spaces and relatives".

Around the same time, Denis Higgs [1973] proved that the topos $\mathsf{Sh}(L)$ of sheaves on a locale $L$ is equivalent to the category $\mathsf{Set}(L)$ of so-called $L$-valued sets and $L$-valued mappings. An $L$-valued set $(A, [\cdot = \cdot])$ consists of a set $A$ (of "local sections") equipped with an $L$-valued relation $A \times A \longrightarrow L\colon (x, y) \mapsto [x = y]$ (giving the "extent to which $x$ equals $y$"), satisfying a number of axioms. Whereas such $L$-valued sets are thus a kind of $L$-enriched structure, they are definitely *not* $L$-enriched *categories*: they lack in particular the unit axiom (corresponding with the sheaf-theoretic fact that local sections are – by definition – not globally defined). However, in [1981] Robert F. C. Walters provided an elegant solution to this problem: instead of keeping the locale $L$ as base for enrichment, he rather considers the quantaloid $L_{\mathsf{si}}$ obtained by splitting the idempotents in $L$ (qua monoid in $\mathsf{Sup}$), and proves that $\mathsf{Sh}(L)$ is equivalent to the category of symmetric and Cauchy complete $L_{\mathsf{si}}$-enriched categories and functors. A year later, Walters [1982] extended this result to Grothendieck topologies, proving that the topos $\mathsf{Sh}(\mathcal{C}, J)$ of sheaves on a small site $(\mathcal{C}, J)$ is equivalent to the category of symmetric and Cauchy complete categories enriched in a suitable quantaloid $\mathcal{R}(\mathcal{C}, J)$ constructed from the site.

That metric spaces on the one hand, and sheaves on the other, can both be described as enriched categories, clearly illustrates the power of expression of enriched category theory.

## 2.

The concept of a category enriched in a bicategory $\mathcal{W}$ was formulated by Jean Bénabou [1967] (who called them "polyads") in the same paper where he first defined bicategories themselves. The hom of a bicategory with only one object is in fact a monoidal category $\mathcal{V}$, and for a complete and cocomplete, symmetric monoidal closed $\mathcal{V}$, the theory of $\mathcal{V}$-enriched categories is well developed, as described in Max Kelly's [1982] magnificent book. When Walters [1981, 1982] used enrichment in bicategories with several objects for his description of sheaves (see above), Ross Street considered this "a major advance which meant that categories enriched in a bicategory needed serious consideration" [Street, 2005], and indeed he himself further developed the theory of $\mathcal{W}$-enriched categories, soon followed by others [Street, 1981, 1983; Betti *et al.*, 1983; Gordon and Power, 1997, 1999].

My preferred flavour of enriched category theory is that where the base for the enrichment is a small quantaloid $\mathcal{Q}$, i.e. a small bicategory whose homs are complete lattices and in which composition distributes on both sides over suprema. Put differently, a quantaloid is precisely a locally posetal, locally cocomplete, closed bicategory. So whenever I speak of $\mathcal{Q}$-enriched categories, functors and distributors, I mean so in the sense of categories enriched in a bicategory. To wit, a $\mathcal{Q}$-category $\mathbb{C}$ is given by a set $\mathbb{C}_0$ of 'objects' together with a 'type'-function $t\colon \mathbb{C}_0 \longrightarrow \mathcal{Q}_0$, and for every pair $(x, y)$ of objects in $\mathbb{C}$ a 'hom'-morphism $\mathbb{C}(y, x)\colon tx \longrightarrow ty$ in $\mathcal{Q}$, such that the composition-inequality and the unit-inequality,

$$\text{resp. } \mathbb{C}(z, y) \circ \mathbb{C}(y, x) \leq \mathbb{C}(z, x) \text{ and } 1_{tx} \leq \mathbb{C}(x, x),$$

hold for every $x, y, z \in \mathbb{C}_0$. (Identifying a quantale with a one-object quantaloid, the type-function $t\colon \mathbb{C}_0 \longrightarrow \mathcal{Q}_0$ becomes obsolete, and quantale-enriched categories are then precisely categories enriched in a posetal monoidal closed category.)

In a quantaloid $\mathcal{Q}$, every diagram of 2-cells commutes trivially. This greatly simplifies certain aspects of $\mathcal{Q}$-enriched categories when compared to $\mathcal{W}$-enriched (or even $\mathcal{V}$-enriched) categories: there are no (or only very few) coherence issues. Indeed, already in the very definition of a $\mathcal{Q}$-category, there is no need to require associativity of composition, nor neutrality of units, let alone coherence of its structural 2-cells, for it is all automatic. On the other hand, $\mathcal{Q}$-enrichment does present new difficulties, not addressed in $\mathcal{V}$-enriched category theory (for $\mathcal{V}$ a symmetric monoidal closed category), stemming from the inherent non-commutativity of the composition of morphisms in $\mathcal{Q}$. This is already illustrated by the simple fact that a $\mathcal{Q}$-category $\mathbb{C}$ need not have a dual: defining $\mathbb{C}^*$ to have the same objects as $\mathbb{C}$, but with hom-morphisms $\mathbb{C}^*(x, y) := \mathbb{C}(y, x)$, does not produce a $\mathcal{Q}$-category (but rather a $\mathcal{Q}^{\mathsf{op}}$-category).

All in all, the study of $\mathcal{Q}$-enriched categories is thus justified, not only by the aforementioned examples (metric spaces, sheaves), but also by the particular phenomena, arising from the non-commutativity of the composition in $\mathcal{Q}$, that it necessarily considers.

## 3.

In ordinary category theory, one usually starts by defining categories, functors and natural transformations, to then go on about co- and contravariant presheaves, Yoneda's Lemma, limits and colimits, adjoint functors, and so on (see e.g. [Borceux, 1994]). In $\mathcal{V}$-enriched category theory, a similar path can be followed if, as Kelly [1982, p. 35] underlines, "our given $\mathcal{V}$ is *symmetric monoidal closed*" because the structure of the 2-category of $\mathcal{V}$-categories, $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations "then becomes rich enough to permit of Yoneda-lemma arguments formally identical to those in [ordinary category theory]". For categories enriched in a quantaloid $\mathcal{Q}$ (or, more generally, in a bicategory $\mathcal{W}$), I have come to appreciate a slightly different approach, which elegantly avoids complications due to the non-commutativity of the composition in $\mathcal{Q}$: by introducing the notion of *distributor* (= module = profunctor) between $\mathcal{Q}$-enriched categories [Bénabou, 1973; Lawvere, 1973; Street, 1981] alongside the usual notion of functor, $\mathcal{Q}$-category theory (with presheaves, Yoneda's Lemma, limits and colimits, and so on) develops from the interplay between functors and distributors.

To illustrate this point, consider co- and contravariant presheaves on a $\mathcal{V}$-category (or, taking $\mathcal{V} = \mathsf{Set}$, on an ordinary category) $\mathbb{C}$: these are defined as functors $F: \mathbb{C} \longrightarrow \mathcal{V}$ and $F: \mathbb{C}^* \longrightarrow \mathcal{V}$ respectively, and thus rely not only on $\mathcal{V}$ being a $\mathcal{V}$-category, but also on each category $\mathbb{C}$ having a dual $\mathbb{C}^*$—*quod non* in the case of enrichment in a quantaloid. However, for $\mathcal{Q}$-categories $\mathbb{C}$ and $\mathbb{C}'$, a distributor $\Phi: \mathbb{C} \multimap \mathbb{C}'$ is easily defined: it consists of a collection $\Phi(x', x): tx \longrightarrow tx'$ of morphisms in $\mathcal{Q}$, one for each pair $(x, x') \in \mathbb{C}_0 \times \mathbb{C}'_0$, such that two action-inequalities,

$$\text{resp. } \Phi(y', y) \circ \mathbb{C}(y, x) \leq \Phi(y', x) \text{ and } \mathbb{C}'(y', x') \circ \Phi(x', x) \leq \Phi(y', x),$$

hold for all $x, y \in \mathbb{C}_0$ and $x', y' \in \mathbb{C}'_0$. Further, for any object $X$ in the base quantaloid $\mathcal{Q}$, there is a trivial one-object $\mathcal{Q}$-category $*_X$ whose hom-arrow is $1_X$. And then it makes perfect sense to define, for any $\mathcal{Q}$-category $\mathbb{C}$, a covariant presheaf of type $X$ on $\mathbb{C}$ to be a distributor $\phi: \mathbb{C} \multimap *_X$ and a contravariant presheaf to be a distributor $\phi: *_X \multimap \mathbb{C}$. The appropriate application of these definitions to $\mathcal{V}$-categories gives back the "usual" definitions.

From the definition of distributor, it is easy to check that, for any two such distributors $\Phi: \mathbb{A} \multimap \mathbb{B}$ and $\Psi: \mathbb{B} \multimap \mathbb{C}$, there is a third distributor $\Psi \otimes \Phi: \mathbb{A} \multimap \mathbb{C}$ whose elements are given by

$$(\Psi \otimes \Phi)(c, a) = \bigvee_{b \in \mathbb{C}_0} \Psi(c, b) \circ \Phi(b, a).$$

Furthermore, parallel distributors can be ordered elementwise: for $\Phi: \mathbb{A} \multimap \mathbb{B}$ and $\Phi': \mathbb{A} \multimap \mathbb{B}$, say that $\Phi \leq \Phi'$ whenever $\Phi(b, a) \leq \Phi'(b, a)$ for all $a \in \mathbb{A}_0$ and $b \in \mathbb{B}_0$. For this composition law and this order, $\mathcal{Q}$-categories and distributors form a quantaloid $\mathsf{Dist}(\mathcal{Q})$. On the other hand, a functor $F: \mathbb{A} \longrightarrow \mathbb{B}$ between $\mathcal{Q}$-categories is – evidently – defined to be an object-map $F: \mathbb{A}_0 \longrightarrow \mathbb{B}_0: a \mapsto Fa$ which preserves types (i.e. $ta = t(Fa)$) and such that the action-inequality $\mathbb{A}(a', a) \leq \mathbb{B}(Fa', Fa)$ holds for all $a, a' \in \mathbb{A}_0$. Functors compose in the obvious way, giving rise to a category $\mathsf{Cat}(\mathcal{Q})$ of $\mathcal{Q}$-categories and functors between them. But each functor $F: \mathbb{A} \longrightarrow \mathbb{B}$ also represents an adjoint pair of distributors: let $F_*: \mathbb{A} \multimap \mathbb{B}$ have elements $F_*(b, a) := \mathbb{B}(b, Fa)$, and let $F^*: \mathbb{B} \multimap \mathbb{A}$ have elements $F^*(a, b) := \mathbb{B}(Fa, b)$, then it is easily seen that $F_*$ is left adjoint to $F^*$ in the quantaloid $\mathsf{Dist}(\mathcal{Q})$. This process is functorial, so that we end up with an inclusion

$$\mathsf{Cat}(\mathcal{Q})\longrightarrow\mathsf{Dist}(\mathcal{Q})\colon \Big(F\colon\mathbb{A}\longrightarrow\mathbb{B}\Big)\mapsto\Big(\mathbb{A}\ \underset{F^*}{\overset{F_*}{\rightleftarrows}}\ \mathbb{B}\Big)$$

of the category of $\mathcal{Q}$-categories and functors in the quantaloid of $\mathcal{Q}$-categories and distributors.

This inclusion – which is an example of a *pro-arrow equipment* [Wood, 1982] – is what I take as starting point for $\mathcal{Q}$-enriched category theory (as summarised in [Stubbe, 2005a]).

As a direct consequence of this take on things, I often rely on various aspects of the distributor calculus (e.g. liftings and extensions in $\mathsf{Dist}(\mathcal{Q})$, representability of a distributor, tabulation of a functor, and so on) to develop arguments in $\mathcal{Q}$-category theory. Also the importance (and significance) of *Cauchy complete* $\mathcal{Q}$-categories is best illustrated from this point of view. Lawvere [1973] defined a category $\mathbb{B}$ to be Cauchy complete when, for any other category $\mathbb{A}$, the order-preserving function

$$\mathsf{Cat}(\mathcal{Q})(\mathbb{A},\mathbb{B})\longrightarrow\mathsf{Map}(\mathsf{Dist}(\mathcal{Q}))(\mathbb{A},\mathbb{B})\colon F\mapsto F_*$$

from functors to left adjoint ("map") distributors between $\mathbb{A}$ and $\mathbb{B}$, is an equivalence of ordered sets. This implies in particular that the inclusion $\mathsf{Cat}(\mathcal{Q})\longrightarrow\mathsf{Dist}(\mathcal{Q})$ (co)restricts to an equivalence $\mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q})\simeq\mathsf{Map}(\mathsf{Dist}(\mathcal{Q}))$ of the category of Cauchy complete categories and functors with the category of (all) categories and left adjoint distributors.

# 4.

In this section I shall comment on my post-doctoral publications that deal with *$\mathcal{Q}$-enriched category theory in general*. To accord with Lawvere's [1973] credo that "enriched categories should in particular be focused on metric spaces and relatives", I shall present these results here as generalisations of metric space theory (categories enriched in $[0,\infty]$) or poset theory (categories enriched in the Boolean algebra $\mathbf{2}=\{0,1\}$). Where I saw fit, I have indicated open problems for further study. In Section 5 on page 25, I shall then pick up the thread of my non-technical exposition.

**4.1. Symmetrisation and Cauchy completion.** It is a matter of fact that the Cauchy completion of an ordinary metric space $(X,d)$ is again an ordinary metric space; that is to say, the canonical way to mesure the distance between Cauchy sequences in $(X,d)$ always provides for a *symmetric* distance function. But this fact is false in general for quantaloid-enriched categories! In [Heymans and Stubbe, 2011a] we analysed this situation (not only with metric spaces in mind, but also motivated by Walters' [1981] description of sheaves on a site as symmetric and Cauchy complete enriched categories), and our main result is the identification of those quantaloids $\mathcal{Q}$ that we call *Cauchy bilateral*: they have the property that the Cauchy completion of a symmetric $\mathcal{Q}$-category is again symmetric. I shall briefly explain this.

If $\mathcal{Q}$ is an involutive[1] quantaloid, with involution $f\mapsto f^{\circ}$, then each $\mathcal{Q}$-category $\mathbb{A}$ can be symmetrised: define $\mathbb{A}_{\mathsf{s}}$ to be the same set of objects (with the same type function) but now with

$$\mathbb{A}_{\mathsf{s}}(x,y) := \mathbb{A}(x,y) \wedge \mathbb{A}(y,x)^{\circ}$$

as hom-morphisms. This procedure straightforwardly determines a symmetrisation functor $(-)_{\mathsf{s}}\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$ on the category of $\mathcal{Q}$-categories; better still, with obvious comultiplication and counit this functor is a comonad. The coalgebras for this comonad are exactly the symmetric $\mathcal{Q}$-categories. On the other hand, as is well-known, each $\mathcal{Q}$-category $\mathbb{A}$ can be Cauchy completed, and this completion process determines a monad $(-)_{\mathsf{cc}}\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$. The algebras for this monad are precisely the Cauchy complete $\mathcal{Q}$-categories. We can now elegantly state the problem of compatiblity between Cauchy completion and symmetry: Is there a distributive law of the Cauchy completion monad $(-)_{\mathsf{cc}}$ over the symmetrisation comonad $(-)_{\mathsf{s}}$?

Because both the monad and the comonad arise from (co)reflective subcategories of $\mathsf{Cat}(\mathcal{Q})$, there is *at most one* distributive law of Cauchy completion over symmetrisation. Therefore, the existence of the distributive law is a property of $\mathcal{Q}$. There are quantaloids for which it exists, and quantaloids for which it doesn't exist. We established a useful elementary criterion for the existence of such a distributive law:

**Theorem.** *Say that a quantaloid $\mathcal{Q}$ is* Cauchy bilateral[2] *if it is involutive (with involution $f \mapsto f^{\circ}$) and when for each family $(f_i\colon X \longrightarrow X_i, g_i\colon X_i \longrightarrow X)_{i\in I}$ of morphisms in $\mathcal{Q}$,*

$$\left.\begin{array}{r}\forall j,k \in I:\ f_k \circ g_j \circ f_j \leq f_k \\ \forall j,k \in I:\ g_j \circ f_j \circ g_k \leq g_k \\ 1_X \leq \bigvee_{i\in I} g_i \circ f_i\end{array}\right\} \Longrightarrow\ 1_X \leq \bigvee_{i\in I}(g_i \wedge f_i^{\circ}) \circ (g_i^{\circ} \wedge f_i).$$

*For each Cauchy bilateral quantaloid $\mathcal{Q}$ there is a distributive law of the Cauchy completion monad $(-)_{\mathsf{cc}}$ over the symmetrisation comonad $(-)_{\mathsf{s}}$ on the category $\mathsf{Cat}(\mathcal{Q})$ of $\mathcal{Q}$-enriched categories.*

Given the importance of symmetry in the theory of (ordinary) metric spaces, it is remarkable that symmetry is an apparently little-studied property for enriched categories. Even more so because there are many interesting questions:

**Problem.** *Prove that the sufficient condition in the above theorem is also necessary, or sharpen it to make it a necessary-and-sufficient condition. For monads $\mathcal{T}\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$ other than Cauchy completion, identify the $\mathcal{T}$-bilateral quantaloids, i.e. those for which there exists a distributive law of the monad $\mathcal{T}$ over the symmetrisation comonad $(-)_{\mathsf{s}}$.*

**4.2. Hausdorff distance.** If $(X,d)$ is a generalised metric space, then it is possible to define a generalised metric on the subsets of $X$:

$$\delta(S,T) := \bigvee_{s\in S}\bigwedge_{t\in T} d(s,t)$$

satisfies all the axioms of a generalised metric on the powerset $\mathcal{P}(X)$; it is called the generalised Hausdorff distance on $\mathcal{P}(X)$. Following the work of [Albert and Kelly, 1988; Kelly and Schmitt, 2005; Schmitt, 2009] on categories enriched in a symmetric monoidal closed category, and that of [Akhvlediani *et al.*, 2010] for categories enriched in a commutative quantale, I analysed this situation in the context of quantaloid-enriched categories [Stubbe, 2010]. It turns out that the Hausdorff construction is an example of a weighted cocompletion (namely, cocompletion with respect to the so-called conical weights, a.k.a. conical cocompletion). More precisely:

**Theorem.** *Each saturated class $\mathcal{C}$ of weights defines, and is defined by, an essentially unique KZ-doctrine*[3] *$\mathcal{T}$ on the category $\mathsf{Cat}(\mathcal{Q})$ of $\mathcal{Q}$-categories and functors; a class $\mathcal{C}$ and a doctrine $\mathcal{T}$ correspond with each other if and only if the $\mathcal{T}$-algebras and their homomorphisms are precisely the $\mathcal{C}$-cocomplete $\mathcal{Q}$-categories and the $\mathcal{C}$-cocontinuous functors between them. Moreover, the KZ-doctrines that arise in this manner are precisely the full sub-KZ-doctrines of the free cocompletion KZ-doctrine; they can be characterised with two simple "fully faithfulness" conditions*[4]*. As an example one finds that conical weights form a saturated class, and the corresponding KZ-doctrine is the 'Hausdorff doctrine': applied to Lawvere's quantale of positive real numbers, it coincides with the Hausdorff construction for generalised metric spaces.*

Related to the notion of Hausdorff distance between subsets of a single metric space, is that of Gromov distance between two metric spaces: it measures how far two such spaces are from being isometric, by computing the infimum of their Hausdorff distance after having been isometrically embedded into any common larger space. This turns the set of all isometry classes of metric spaces into a metric space. Akhvlediani, Clementino and Tholen [2009] provide a generalisation of Gromov distance for categories enriched in a *commutative quantale*, and insist particularly on the fact that Gromov distance is necessarily built up from *symmetrised* Hausdorff distance. As I have indicated above, symmetrisation for quantaloid-enriched categories only makes sense when that quantaloid is involutive. This leaves many open questions:

**Problem.** *Under which conditions on $\mathcal{Q}$ can we symmetrise the Hausdorff doctrine on $\mathsf{Cat}(\mathcal{Q})$, or in other words, which quantaloids are Hausdorff-bilateral? How can we then define the Gromov distance between two quantaloid-enriched categories? And what exactly is then the meaning of "the $\mathcal{Q}$-category of all $\mathcal{Q}$-categories"?*

**4.3. Exponentiability.** It is a mere triviality that any ordered set $A = (A, \leq)$ is exponentiable in the category $\mathsf{Ord}$ of ordered sets and order-preserving functions: that is to say, for any $B = (B, \leq)$ and $C = (C, \leq)$, there is an evident natural bijection between order-preserving functions $C \times A \longrightarrow B$ and order-preserving maps $C \longrightarrow \mathsf{Ord}(A, B)$ (where $\mathsf{Ord}(A, B)$ is the set of order-preserving functions from $A$ to $B$, with pointwise order). The category $\mathsf{Ord}$ being precisely the category $\mathsf{Cat}(\mathbf{2})$ of $\mathbf{2}$-enriched categories ($\mathbf{2}$ being the two-element Boolean algebra), makes one wonder if a similar result holds for other categories of quantaloid-enriched categories too.

The simple answer to this question is negative: in the category of generalised metric spaces, which we know is $\mathsf{Cat}([0, \infty])$, not every object is exponentiable [Clementino and Tholen, 2006]. In [Clementino, Hofmann and Stubbe, 2009] we completely characterise, for any base quantaloid $\mathcal{Q}$, those objects and morphisms in $\mathsf{Cat}(\mathcal{Q})$ which are exponentiable:

**Theorem.** *A functor $F: \mathbb{A} \longrightarrow \mathbb{B}$ between $\mathcal{Q}$-enriched categories is exponentiable, i.e. the functor* *"product with F"*

$$- \times F \colon \mathsf{Cat}(\mathcal{Q})/_{\mathbb{B}} \longrightarrow \mathsf{Cat}(\mathcal{Q})/_{\mathbb{B}}$$

*admits a right adjoint, if and only if the following two conditions hold:*

1. *for every* $a, a' \in \mathbb{A}$ *and* $\bigvee_i f_i \leq \mathbb{B}(Fa', Fa)$,

$$\Big(\bigvee_i f_i\Big) \wedge \mathbb{A}(a', a) = \bigvee_i \Big(f_i \wedge \mathbb{A}(a', a)\Big),$$

2. *for every* $a, a'' \in \mathbb{A}$, $b' \in \mathbb{B}$, $f \leq \mathbb{B}(b', Fa)$ *and* $g \leq \mathbb{B}(Fa'', b')$,

$$(g \circ f) \wedge \mathbb{A}(a'', a) = \bigvee_{a' \in F^{-1}(\{b'\})} \Big((g \wedge \mathbb{A}(a'', a')) \circ (f \wedge \mathbb{A}(a', a))\Big).$$

A $\mathcal{Q}$-category $\mathbb{A}$ is an exponentiable object of $\mathsf{Cat}(\mathcal{Q})$ precisely when the unique functor into the terminal object[5] of $\mathsf{Cat}(\mathcal{Q})$ is exponentiable in the sense of the above theorem: thus it is necessary and sufficient that each hom-arrow $\mathbb{A}(a', a)$ is exponentiable in $\mathcal{Q}(ta, ta')$ and moreover

$$(g \circ f) \wedge \mathbb{A}(a'', a) = \bigvee \Big\{(g \wedge \mathbb{A}(a'', a')) \circ (f \wedge \mathbb{A}(a', a)) \mid a' \in \mathbb{A},\ ta' = Y\Big\}$$

for all $a, a'' \in \mathbb{A}$ and $f \colon ta \longrightarrow Y$, $g \colon Y \longrightarrow ta''$ in $\mathcal{Q}$. If the base of enrichment is a quantal frame (i.e. a quantale whose underlying suplattice is a locale) in which the interchange law $(g \circ f) \wedge (k \circ h) = (g \wedge k) \circ (f \wedge h)$ holds, then it is not difficult to infer from the above that every $\mathcal{Q}$-enriched category is exponentiable. This is in particular true for the one-object suspension of a locale, and *a fortiori* for the one-object suspension of the two-element Boolean algebra, so we recover the initial observation that every ordered set is exponentiable.

**Problem.** *The free quantaloid* $\mathcal{Q}(\mathcal{C})$ *on a category* $\mathcal{C}$ *has the same objects as* $\mathcal{C}$, *the hom-suplattice* $\mathcal{Q}(\mathcal{C})(A, B)$ *is the powerset of* $\mathcal{C}(A, B)$, *composition in* $\mathcal{Q}(\mathcal{C})$ *is done "elementwise" and the identity on* $A$ *is* $\{1_A\}$. *(This construction provides a left adjoint to the forgetful functor from quantaloids to categories.) The interchange law holds in any free quantaloid. Categories enriched in a free quantaloid have been linked with automata theory and process semantics[Betti, 1980; Rosenthal, 1995]: the objects of a* $\mathcal{Q}(\mathcal{C})$*-category* $\mathbb{A}$ *are the (typed)* states *of an automaton, and the arrows of* $\mathcal{C}$ *are its (typed)* labels *or* processes. *To have an* $f \in \mathbb{A}(a', a)$ *is then read as "having a process* $f$ *to produce* $a'$ *from* $a$*"; often this is denoted as* $f \colon a \rightsquigarrow a'$. *It turns out that* $\mathbb{A}$ *is exponentiable if and only if, for any states* $a$ *and* $a''$ *and type* $Y \in \mathcal{C}$, *if* $f \colon a \rightsquigarrow a''$ *and* $f = h \circ g$ *with* $\mathsf{cod}(g) = Y = \mathsf{dom}(h)$ *in* $\mathcal{C}$, *then there is a state* $a'$ *of type* $Y$ *together with* $p \colon a \rightsquigarrow a'$, $q \colon a' \rightsquigarrow a''$ *such that* $f = q \circ p$. *This statement is trivial when* $\mathcal{C}$ *is a one-object category, i.e. in the case of an untyped automaton, but what role does it play in typed automata theory or process semantics?*

**4.4. Tensors and cotensors.** In the theory of ordered sets, complete lattices play of course an important role. Suprema and infima being dual notions, let me single out the former: I write $\mathsf{Sup}$ for the category of suplattices and supmorphisms; it is a subcategory of $\mathsf{Ord}$. From a categorical point of view, a supremum is a colimit, so it comes as no surprise that the correct $\mathcal{Q}$-categorical generalisation of a suplattice is a cocomplete $\mathcal{Q}$-category, that is, one that admits all weigthed colimits. Similarly, supmorphisms between suplattices are generalised by the so-called cocontinuous functors between cocomplete $\mathcal{Q}$-categories, that is, those functors that preserve all weighted colimits; I shall write $\mathsf{Cocont}(\mathcal{Q})$ for the category of cocomplete $\mathcal{Q}$-categories and cocontinuous functors.

For an ordinary category $\mathcal{C}$, it is well known that $\mathcal{C}$ has all colimits if and only if $\mathcal{C}$ has coequalisers and coproducts: every colimit in $\mathcal{C}$ can be constructed as a coequaliser of two morphisms between coproducts. In that same spirit of decomposing a complicated notion (colimit) to simpler ones (coequaliser and coproduct), a $\mathcal{V}$-enriched (or even $\mathcal{W}$-enriched) category is cocomplete (= has all weighted colimits) if and only if it is tensored and has all conical colimits [Kelly, 1982; Gordon and Power, 1999]. Here it must be remarked that both tensors and conical colimits are examples of weighted colimits. In the specific case of quantaloid-enriched categories, I further reduced the notion of weighted colimit to a combination of tensors, cotensors and suprema [Stubbe, 2006], as I shall explain next.

If $\mathbb{C}$ is a $\mathcal{Q}$-enriched category and $X$ an object of $\mathcal{Q}$, then I shall write $\mathbb{C}_X$ for the collection of objects of $\mathbb{C}$ whose type in $\mathcal{Q}$ is $X$; it is the "fibre" of $\mathbb{C}$ over $X$. For any two objects $x, y \in \mathbb{C}_X$ we can then define

$$x \leq y \text{ whenever } 1_X \leq \mathbb{C}(x,y),$$

which makes $(\mathbb{C}_X, \leq)$ an ordered set[6]. It thus makes sense to speak of suprema in these "ordered fibres" of a quantaloid-enriched category $\mathbb{C}$; so define $\mathbb{C}$ to be *order-cocomplete* whenever each of its ordered fibres is a suplattice. And let me insist on the fact that such suprema are *not* weighted colimits[7].

**Theorem.** *If $\mathbb{C}$ is a cotensored $\mathcal{Q}$-category, then the supremum of a family $(x_i)_{i\in I} \in \mathbb{C}_X$ is also its conical colimit in $\mathbb{C}$. It follows that a $\mathcal{Q}$-category $\mathbb{C}$ is cocomplete if and only if it has tensors, cotensors and is ordered-cocomplete. Explicitly, for a distributor $\Phi\colon \mathbb{A} \looparrowright \mathbb{B}$ and a functor $F\colon \mathbb{B} \longrightarrow \mathbb{C}$, the $\Phi$-weighted colimit of $F$ is then the functor*

$$\mathrm{colim}(\Phi, F)\colon \mathbb{A} \longrightarrow \mathbb{C}\colon a \mapsto \bigvee_{b\in\mathbb{B}_0} Fb \otimes \Phi(b,a).$$

*Moreover, a functor $F\colon \mathbb{C} \longrightarrow \mathbb{C}'$ between cocomplete $\mathcal{Q}$-categories is cocontinuous if and only it preserves tensors and suprema in each of the $\mathbb{C}_X$.*

This result indicates that certain concepts from order theory can be "lifted" to $\mathcal{Q}$-category theory (suprema are "lifted" to conical colimits), but the price to pay has to do with the existence of (co)tensors. A similar "lifting" of adjuctions can be done:

**Theorem.** *For any functor $F\colon \mathbb{C} \longrightarrow \mathbb{C}'$ between $\mathcal{Q}$-categories and any object $X$ of $\mathcal{Q}$, the "fibre-wise function" $F_X\colon \mathbb{C}_X \longrightarrow \mathbb{C}'_X\colon x \mapsto Fx$ preserves the order in the "ordered fibres" of $\mathbb{C}$ and $\mathbb{C}'$ over $X$. If $\mathbb{C}$ is tensored, then $F$ is a left adjoint in $\mathsf{Cat}(\mathcal{Q})$ if and only if $F$ preserves tensors and, for all $\mathcal{Q}$-objects $X$, $F\colon \mathbb{C}_X \longrightarrow \mathbb{C}'_X$ is a left adjoint in $\mathsf{Ord}$.*

**4.5. Variation and enrichment.** The two previous theorems (and their corollaries, on which I shall not comment here) now allow for a detailed study of the relation between *enrichment* *in a quantaloid* on the one hand, and *variation over a quantaloid* on the other (to paraphrase [Betti *et al.*, 1983; Gordon and Power, 1997]).

In the case of an ordinary category $\mathcal{C}$, the phrase "variation over $\mathcal{C}$" most often refers to a (contravariant, say) presheaf on $\mathcal{C}$, i.e. a functor $\mathcal{C}^{\mathsf{op}} \longrightarrow \mathsf{Set}$; thus the category of "variable sets over $\mathcal{C}$" is the functor category $\mathsf{Fun}(\mathcal{C}^{\mathsf{op}}, \mathsf{Set})$. When $\mathcal{C}$ is a one-object category, thus a monoid, a presheaf is exactly the same thing as an action of that monoid on a set. Similarly, if $\mathcal{R}$ is an $\mathsf{Ab}$-enriched category (a "ringoid"), "variation over $\mathcal{R}$" refers to the category of $\mathsf{Ab}$-enriched contravariant presheaves on $\mathcal{R}$. When $\mathcal{R}$ has only one object, then it is a ring, and this category is precisely the category of $\mathcal{R}$-modules, so we write it rather as $\mathsf{Mod}(\mathcal{R})$.

As for a quantaloid $\mathcal{Q}$, regarding it as a category enriched in the symmetric monoidal closed category $\mathsf{Sup}$ of suplattices and supmorphisms, it is completely in line with the two previous examples to define $\mathsf{Mod}(\mathcal{Q})$ to be the $\mathsf{Sup}$-enriched presheaf category on $\mathcal{Q}$. By general $\mathcal{V}$-enriched category theory, a $\mathcal{V}$-enriched presheaf category is always a $\mathcal{V}$-category; for a quantaloid $\mathcal{Q}$, this means that $\mathsf{Mod}(\mathcal{Q})$ too is a quantaloid. Explicitly, its objects are the quantaloid homomorphisms (= $\mathsf{Sup}$-enriched functors) from $\mathcal{Q}^{\mathsf{op}}$ to $\mathsf{Sup}$, its morphisms are the $\mathsf{Sup}$-natural transformations, and its 2-cells are given by the pointwise ordering of the natural transformations.

However, and this is an important subtlety, besides the straightforward notion of modules on $\mathcal{Q}$, there are many "weaker" flavours of variation over $\mathcal{Q}$, which I shall now briefly explain. If $\mathcal{A}$ and $\mathcal{B}$ are locally ordered 2-categories, then a *pseudofunctor* $\mathcal{F}\colon \mathcal{A} \longrightarrow \mathcal{B}$ is an action on objects and morphisms that respects the local order and such that functoriality holds up to local isomorphism. For two such pseudofunctors $\mathcal{F}, \mathcal{F}'\colon \mathcal{A} \longrightarrow \mathcal{B}$, a *lax natural transformation* $\varphi\colon \mathcal{F} \Longrightarrow \mathcal{F}'$ is a family of $\mathcal{B}$-morphisms $(\varphi_X\colon \mathcal{F}X \longrightarrow \mathcal{F}'X)_{X \in \mathcal{A}_0}$ satisfying, for any $f\colon X \longrightarrow Y$ in $\mathcal{A}$, $\mathcal{F}'f \circ \varphi_X \leq \varphi_Y \circ \mathcal{F}f$ in $\mathcal{B}(\mathcal{F}X, \mathcal{F}'Y)$. Such a transformation is *pseudonatural* when these inequalities are isomorphisms. Lax natural transformations are ordered componentwise. There are locally ordered 2-categories $\mathsf{Psd}_{\mathsf{lax}}(\mathcal{A}, \mathcal{B})$, resp. $\mathsf{Psd}(\mathcal{A}, \mathcal{B})$, with pseudofunctors as objects and lax natural transformations, resp. pseudonatural transformations, as arrows. Now let $\mathcal{B} = \mathsf{Cat}(\mathbf{2})$; a pseudofunctor $\mathcal{F}\colon \mathcal{A} \longrightarrow \mathsf{Cat}(\mathbf{2})$ is *closed* when, for every $X, Y$ in $\mathcal{A}$ and $x \in \mathcal{F}X$,

$$\mathcal{F}(-)(x)\colon \mathcal{A}(X,Y) \longrightarrow \mathcal{F}Y\colon f \mapsto \mathcal{F}(f)(x)$$

is a left adjoint in $\mathsf{Cat}(\mathbf{2})$. $\mathsf{ClPsd}_{\mathsf{lax}}(\mathcal{A}, \mathsf{Cat}(\mathbf{2}))$ and $\mathsf{ClPsd}(\mathcal{A}, \mathsf{Cat}(\mathbf{2}))$ are the full sub-2-categories of $\mathsf{Psd}_{\mathsf{lax}}(\mathcal{A}, \mathsf{Cat}(\mathbf{2}))$ and $\mathsf{Psd}(\mathcal{A}, \mathsf{Cat}(\mathbf{2}))$ determined by the closed pseudofunctors.

To spell out properly the close relationship between (pseudo)functors on $\mathcal{Q}$ on the one hand, and $\mathcal{Q}$-enriched categories on the other hand, I must introduce some notations concerning $\mathcal{Q}$-categories. I shall write $\mathsf{Cat}_{\otimes}(\mathcal{Q})$ for the full sub-2-category of $\mathsf{Cat}(\mathcal{Q})$ whose objects are tensored categories, and $\mathsf{Tens}(\mathcal{Q})$ for its sub-2-category whose objects are tensored categories and morphisms are tensor-preserving functors. Similarly I use $\mathsf{Cat}_{\langle\rangle}(\mathcal{Q})$ for the full sub-2-category of $\mathsf{Cat}(\mathcal{Q})$ whose objects are cotensored categories, and moreover the obvious combination $\mathsf{Cat}_{\otimes,\langle\rangle}(\mathcal{Q})$. As usual I write $\mathsf{Map}(\mathcal{K})$ for the category of left adjoints ("maps") in any given 2-category $\mathcal{K}$. Recall also that $\mathsf{Cocont}(\mathcal{Q})$ denotes the locally completely ordered 2-category whose objects are cocomplete $\mathcal{Q}$-categories and morphisms are cocontinuous functors. Now I can state the result from [Stubbe, 2006]:

**Theorem.** *A tensored $\mathcal{Q}$-category $\mathbb{C}$ determines a closed pseudofunctor*

$$\mathcal{F}_{\mathbb{C}}\colon \mathcal{Q}^{\mathsf{op}} \longrightarrow \mathsf{Cat}(\mathbf{2})\colon \Big(f\colon X \longrightarrow Y\Big) \mapsto \Big(-\otimes f\colon \mathbb{C}_Y \longrightarrow \mathbb{C}_X\Big),$$

*and a functor $F\colon \mathbb{C} \longrightarrow \mathbb{C}'$ between tensored $\mathcal{Q}$-categories determines a lax natural transformation*

$$\varphi^F\colon \mathcal{F}_{\mathbb{C}} \Longrightarrow \mathcal{F}_{\mathbb{C}'} \text{ with components } \varphi^F_X\colon \mathbb{C}_X \longrightarrow \mathbb{C}'_X\colon x \mapsto Fx.$$

*The action $(F\colon \mathbb{C} \longrightarrow \mathbb{C}') \mapsto (\varphi^F\colon \mathcal{F}_{\mathbb{C}} \Longrightarrow \mathcal{F}_{\mathbb{C}'})$ determines the following biequivalences:*

1. $\mathsf{Cat}_{\otimes}(\mathcal{Q}) \simeq \mathsf{ClPsd}_{\mathsf{lax}}(\mathcal{Q}^{\mathsf{op}}, \mathsf{Cat}_{\otimes}(\mathbf{2}))$,
2. $\mathsf{Tens}(\mathcal{Q}) \simeq \mathsf{ClPsd}(\mathcal{Q}^{\mathsf{op}}, \mathsf{Tens}(\mathbf{2}))$,
3. $\mathsf{Map}(\mathsf{Cat}_{\otimes,\langle\rangle}(\mathcal{Q})) \simeq \mathsf{ClPsd}(\mathcal{Q}^{\mathsf{op}}, \mathsf{Map}(\mathsf{Cat}_{\otimes,\langle\rangle}(\mathbf{2})))$,
4. $\mathsf{Cocont}(\mathcal{Q}) \simeq \mathsf{ClPsd}(\mathcal{Q}^{\mathsf{op}}, \mathsf{Cocont}(\mathbf{2}))$,
5. $\mathsf{Cocont}(\mathcal{Q}) \simeq \mathsf{Mod}(\mathcal{Q})$.

In much of what follows (not only in the following subsection but also in Section 7), the equivalence of cocomplete $\mathcal{Q}$-categories with $\mathcal{Q}$-modules will play a central role.

**4.6. Continuity and algebraicity for categories.** On any suplattice $L$ one may define the so-called way-below relation: $a \ll b$ holds when for every *directed* downset $D \subseteq L$, $b \leq \bigvee D$ implies $a \in D$. A suplattice is said to be continuous when every element is the supremum of all elements way-below it. As a (stronger) variant, one may also define the totally-below relation on a suplattice $L$: $a \lll b$ holds when for *any* downset $D \subseteq L$, $b \leq \bigvee D$ implies $a \in D$. Of course $L$ is now said to be totally continuous when every element is the supremum of all elements totally-below it; in this case $L$ is also continuous. Robert Rosebrugh and Richard Wood [1994] studied the categorical aspects of the latter notion; let me recall some of its features:

(a) A suplattice $L$ is totally continuous if and only if any supmorphism $f\colon L \longrightarrow M$ factors through any surjective supmorphism $g\colon K \twoheadrightarrow M$, i.e. it is a projective object in $\mathsf{Sup}$.

(b) Totally continuous suplattices are precisely those suplattices for which the map sending a downset to its supremum has a left adjoint: the left adjoint to $\bigvee\colon \mathsf{Dwn}(L) \longrightarrow L\colon D \mapsto \bigvee D$ is namely the map $a \mapsto \{x \in L \mid x \lll a\}$. In other words, the supremum-map is required to preserve all infima; and so such a suplattice is also said to be completely distributive.

(c) Given a set equipped with an idempotent binary relation $(X, \prec)$, the subsets $S \subseteq X$ such that $x \in S$ if and only if there exists a $y \in S$ such that $x \prec y$, form a totally continuous suplattice, denoted $\mathcal{R}(X, \prec)$. In fact, every totally continuous suplattice $L$ is isomorphic to such an $\mathcal{R}(X, \prec)$.

(d) Given any ordered set $(X, \leq)$, the construction in (c) implies that the set $\mathsf{Dwn}(X)$ of downsets of $(X, \leq)$ is a totally continuous suplattice. But it distinguishes itself in that every element of $\mathsf{Dwn}(X)$ is the supremum of totally compact elements, i.e. elements that are totally below themselves. Such a suplattice is said to be totally algebraic; and in fact all totally algebraic suplattices are of the form $\mathsf{Dwn}(X)$ for some ordered set $(X, \leq)$.

In [Stubbe, 2007] I proved that the crucial aspects of the theory of totally continuous suplattices recalled above all generalise neatly to cocomplete $\mathcal{Q}$-categories: it is possible to make sense of such notions as 'projectivity', 'complete distributivity', 'total continuity' and 'total algebraicity' in the context of cocomplete $\mathcal{Q}$-categories. By way of illustration, let me spell out how the totally-below relation is defined on an arbitrary cocomplete $\mathcal{Q}$-category $\mathbb{A}$.

First, consider the Yoneda embedding $Y_{\mathbb{A}} \colon \mathbb{A} \longrightarrow \mathcal{P}\mathbb{A}$; it induces a right adjoint distributor $\mathcal{P}\mathbb{A}(Y_{\mathbb{A}}-,-) \colon \mathcal{P}\mathbb{A} \multimap \mathbb{A}$. Because $\mathbb{A}$ is cocomplete, we can compute for any $\phi \in \mathcal{P}\mathbb{A}$ the $\phi$-weighted colimit of the identity functor on $\mathbb{A}$; this provides for a functor $\sup_{\mathbb{A}} \colon \mathcal{P}\mathbb{A} \longrightarrow \mathbb{A}$, which in turn determines a left adjoint distributor $\mathbb{A}(-,\sup_{\mathbb{A}}-) \colon \mathcal{P}\mathbb{A} \multimap \mathbb{A}$. In the quantaloid $\mathsf{Dist}(\mathcal{Q})$ of $\mathcal{Q}$-categories and distributors we can then define a new distributor $\Theta_{\mathbb{A}} \colon \mathbb{A} \multimap \mathbb{A}$ as the right extension of $\mathbb{A}(-,\sup_{\mathbb{A}}-)$ through $\mathcal{P}\mathbb{A}(Y_{\mathbb{A}}-,-)$:

$$\begin{array}{ccc} \mathcal{P}\mathbb{A} & \xrightarrow{\ \mathcal{P}\mathbb{A}(Y_{\mathbb{A}}-,-)\ } & \mathbb{A} \\ {\scriptstyle \mathbb{A}(-,\sup_{\mathbb{A}}-)}\downarrow & \nearrow & \\ \mathbb{A} & & \end{array} \qquad \Theta_{\mathbb{A}} = \Big\{ \mathbb{A}(-,\sup_{\mathbb{A}}-), \mathcal{P}\mathbb{A}(Y_{\mathbb{A}}-,-) \Big\}.$$

Explicitly, this extension in $\mathsf{Dist}(\mathcal{Q})$ can be reduced to an infimum of extensions in $\mathcal{Q}$: it turns out that the elements of $\Theta_{\mathbb{A}} \colon \mathbb{A} \multimap \mathbb{A}$ are, for $x, y \in \mathbb{A}$,

$$\Theta_{\mathbb{A}}(x,y) = \bigwedge_{\phi \in \mathcal{P}\mathbb{A}} \{\mathbb{A}(y, \sup_{\mathbb{A}}\phi), \phi(x)\}.$$

It is precisely this distributor which is the $\mathcal{Q}$-categorical analogue of the "totally-below" relation; indeed, when putting $\mathcal{Q} = \mathbf{2}$, thus working with ordered sets, the above formula encodes precisely the phrase that

$$x \lll y \iff \text{for all downclosed subsets } \phi\text{: } y \leq \sup(\phi) \text{ implies } x \in \phi,$$

that is, we recover the "classical" definition of the totally below relation.

The following theorems give an idea of the many interesting things that can be said about continuity and algebraicity for $\mathcal{Q}$-categories:

**Theorem.** *For a cocomplete $\mathcal{Q}$-category $\mathbb{A}$, the following are equivalent:*

1. *$\mathbb{A}$ is* totally continuous*, i.e. $\sup_{\mathbb{A}}(\Theta_{\mathbb{A}}(-,x)) \cong x$ for every $x \in \mathbb{A}$,*
2. *$\mathbb{A}$ is a projective object in $\mathsf{Cocont}(\mathcal{Q})$,*
3. *$\mathbb{A}$ is* completely distributive*, i.e. the functor $\sup_{\mathbb{A}} \colon \mathcal{P}\mathbb{A} \twoheadrightarrow \mathbb{A}$ has a left adjoint,*
4. *$\mathbb{A}$ is equivalent to the category of regular presheaves[8] on a regular $\mathcal{Q}$-semicategory $\mathbb{B}$.*

*Mapping a regular $\mathcal{Q}$-semicategory $\mathbb{B}$ to the cocomplete $\mathcal{Q}$-category $\mathcal{R}(\mathbb{B})$ of regular presheaves on $\mathbb{B}$ extends to a biequivalence $\mathsf{RSDist}(\mathcal{Q}) \simeq \mathsf{Cocont}_{\mathsf{tc}}(\mathcal{Q})$ between the quantaloid of regular $\mathcal{Q}$-semicategories and regular semidistributors and the full subcategory of $\mathsf{Cocont}(\mathcal{Q})$ determined by the totally continuous cocomplete $\mathcal{Q}$-categories.*

**Theorem.** *Let* $\mathbb{A}$ *be a cocomplete* $\mathcal{Q}$*-category. Say that an object* $a \in \mathbb{A}$ *is* totally compact *when* $1_{ta} \leq \Theta_{\mathbb{A}}(a,a)$*; let* $\mathbb{A}_{\mathsf{c}}$ *be the full subcategory of totally compact objects of* $\mathbb{A}$*. As particular case of the above theorem, the following are equivalent:*

1. $\mathbb{A}$ *is* totally algebraic*, i.e. the left Kan extension of* $i\colon \mathbb{A}_{\mathsf{c}} \longrightarrow \mathbb{A}$ *along itself is the identity,*
2. $\mathbb{A}$ *is equivalent to the category of presheaves on a* $\mathcal{Q}$*-category* $\mathbb{C}$*.*

*Mapping a* $\mathcal{Q}$*-category* $\mathbb{C}$ *to the cocomplete* $\mathcal{Q}$*-category* $\mathcal{P}(\mathbb{C})$ *of presheaves on* $\mathbb{C}$ *extends to a biequivalence* $\mathsf{Dist}(\mathcal{Q}) \simeq \mathsf{Cocont}_{\mathsf{ta}}(\mathcal{Q})$ *between the quantaloid of* $\mathcal{Q}$*-categories and distributors and the full subcategory of* $\mathsf{Cocont}(\mathcal{Q})$ *determined by the totally algebraic cocomplete* $\mathcal{Q}$*-categories.*

In the context of theoretical computer science, [Abramsky and Jung, 1994] argue that a mathematical structure deserves to be called a "domain" when it is an algebraic structure that unites aspects of convergence and of approximation. A totally continuous cocomplete $\mathcal{Q}$-category does exactly that: it is cocomplete ("every presheaf converges") and is equipped with a well-behaved totally-below relation ("approximations from below"). So this result has the flavour of "quantaloid-enriched domain theory"—or "dynamic domains", as I call them.

**Problem.** *Totally continuous (or algebraic) cocomplete* $\mathcal{Q}$*-categories are very strong structures, because in the definition we abandonned the notion of "directedness" (which itself is related to "finiteness"). How can a notion of "directedness" be brought back in again? How should "directed (or filtered) colimits" be defined for categories enriched in a quantaloid? Can we then define (non-totally) continuous and algebraic* $\mathcal{Q}$*-categories? And how about a quantaloid-enriched setting for recursive domain equations then?*

**4.7. Continuity and algebraicity for modules.** Previously I pointed out that $\mathsf{Cocont}(\mathcal{Q})$, the 2-category of cocomplete $\mathcal{Q}$-categories and cocontinuous functors, is biequivalent to $\mathsf{Mod}(\mathcal{Q})$, the quantaloid of modules on $\mathcal{Q}$. Under this equivalence, the projective objects in $\mathsf{Cocont}(\mathcal{Q})$ correspond of course with the projective objects in $\mathsf{Mod}(\mathcal{Q})$. But, somewhat surprisingly, more can be said [Stubbe, 2007]:

**Theorem.** *Let* $\mathbb{A}$ *be a cocomplete* $\mathcal{Q}$*-category and* $\mathcal{F}$ *a* $\mathcal{Q}$*-module that correspond with each other under the equivalence* $\mathsf{Cocont}(\mathcal{Q}) \simeq \mathsf{Mod}(\mathcal{Q})$*. The following statements are equivalent:*

1. $\mathbb{A}$ *is totally continuous,*
2. $\mathcal{F}$ *is projective,*
3. $\mathcal{F}$ *is a retract of a direct sum of representable* $\mathcal{Q}$*-modules,*
4. $\mathcal{F}$ *is small-projective*[9]*.*

In [Heymans and Stubbe, 2009a] we similarly express the total algebraicity of a cocomplete $\mathcal{Q}$-category $\mathbb{A}$ in terms of its associated $\mathcal{Q}$-module $\mathcal{F}$. To explain this, it is useful to introduce some new terminology first. Consider a $\mathcal{Q}$-module $\mathcal{F}\colon \mathcal{Q}^{\mathsf{op}} \longrightarrow \mathsf{Sup}$ and an element $x \in \mathcal{F}(X)$ (for some $X \in \mathcal{Q}$). By the $\mathsf{Sup}$-enriched Yoneda Lemma, $x$ corresponds with a module morphism

$\tau_x\colon \mathcal{Q}(-,X)\Longrightarrow\mathcal{F}$. If $\tau_x$ is a left adjoint in $\mathsf{Mod}(\mathcal{Q})$, then $x$ is, by definition, a *principal element* of $\mathcal{F}$. The set of principal elements of $\mathcal{F}$ is thus

$$\mathcal{F}_{\mathsf{pr}}=\Big\{\tau(1_X)\mid X\in\mathcal{Q}\text{ and }\tau\colon\mathcal{Q}(-,X)\Longrightarrow\mathcal{F}\text{ is a left adjoint}\Big\},$$

and we say that $\mathcal{F}$ is *principally generated* if

$$\mathsf{id}_{\mathcal{F}}=\bigvee\Big\{\tau\circ\tau^*\ \Big|\ X\in\mathcal{Q}\text{ and }\tau\colon\mathcal{Q}(-,X)\Longrightarrow\mathcal{F}\text{ is a left adjoint}\Big\}.$$

**Theorem.** *Let $\mathbb{A}$ be a cocomplete $\mathcal{Q}$-category and $\mathcal{F}$ a $\mathcal{Q}$-module that correspond with each other under the equivalence $\mathsf{Cocont}(\mathcal{Q})\simeq\mathsf{Mod}(\mathcal{Q})$. The following statements are equivalent:*

1. *$\mathbb{A}$ is totally algebraic*[10]*,*
2. *$\mathcal{F}$ is principally generated,*
3. *$\mathcal{F}$ is an adjoint retract of a direct sum of representable $\mathcal{Q}$-modules*[11]*.*

It follows trivially that the biequivalence $\mathsf{Mod}(\mathcal{Q})\simeq\mathsf{Cocont}(\mathcal{Q})$ restricts to a biequivalence between the category $\mathsf{Mod}_{\mathsf{pg}}(\mathcal{Q})$ of principally generated modules and module morphisms and the category $\mathsf{Cocont}_{\mathsf{ta}}(\mathcal{Q})$ of totally algebraic cocomplete $\mathcal{Q}$-categories and cocontinuous functors. From the previous subsection it follows furthermore that $\mathsf{Cocont}_{\mathsf{ta}}(\mathcal{Q})$ is equivalent to $\mathsf{Dist}(\mathcal{Q})$, so taking left adjoints we end up with the equivalence $\mathsf{Map}(\mathsf{Mod}_{\mathsf{pg}}(\mathcal{Q}))\simeq\mathsf{Map}(\mathsf{Dist}(\mathcal{Q}))\simeq\mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q})$. This observation will be useful in Section 7.

## 5.

As recalled in Section 1, Higgs [1973] described a sheaf on $L$ as an $L$-valued set, whereas Walters [1981] described a sheaf as a symmetric and Cauchy complete category enriched in $L_{\mathsf{si}}$, the split-idempotent completion of $L$. In the article [Stubbe, 2005c], drawn from my doctoral thesis, I analysed this situation for a quantaloid $\mathcal{Q}$ instead of a locale $L$. I shall present a very brief overview of my findings, for it plays an essential role in my post-doctoral research too.

Whereas [Higgs, 1973] presents the equivalence of $\mathsf{Sh}(L)$ with the category of $L$-valued sets and $L$-valued functions, Francis Borceux and Rosanna Cruciani [1998] gave a similar presentation of ordered objects and order-preserving maps in $\mathsf{Sh}(L)$: they prove that $\mathsf{Ord}(\mathsf{Sh}(L))$ is equivalent to the category of $L$-valued posets and $L$-valued order-preserving maps. An $L$-valued poset $(A,[\cdot\leq\cdot])$ consists of a set $A$ equipped with an $L$-valued relation $A\times A\longrightarrow L\colon (x,y)\mapsto[x\leq y]$, satisfying a number of axioms. When studying this notion, replacing the locale $L$ by a quantaloid $\mathcal{Q}$, and trying to make sense of the axioms, I realised that the correct, general definition is that of a 'totally regular $\mathcal{Q}$-semicategory' [Stubbe, 2005c].

Precisely, a $\mathcal{Q}$-enriched *totally regular semicategory* $\mathbb{A}$ is a set $\mathbb{A}_0$ of 'objects' together with a 'type'-function $t\colon\mathbb{A}_0\longrightarrow\mathcal{Q}_0$, and a 'hom'-morphism $\mathbb{A}(y,x)\colon tx\longrightarrow ty$ in $\mathcal{Q}$ for each pair $(x,y)$ of objects such that the composition-inequality and two total-regularity-equalities hold:

$$\mathbb{A}(z,y)\circ\mathbb{A}(y,x)\leq\mathbb{A}(z,x)\text{ and }\mathbb{A}(y,y)\circ\mathbb{A}(y,x)=\mathbb{A}(y,x)=\mathbb{A}(y,x)\circ\mathbb{A}(x,x).$$

This notion lies between that of $\mathcal{Q}$-category and that of regular $\mathcal{Q}$-semicategory: every $\mathcal{Q}$-category is a totally regular $\mathcal{Q}$-semicategory and every totally regular $\mathcal{Q}$-semicategory is a regular semicategory, but neither of the converses are true. However, much like for $\mathcal{Q}$-categories, there are appropriate notions of *regular semidistributor* and *regular semifunctor* between totally regular $\mathcal{Q}$-semicategories, making for a quantaloid $\mathsf{TRSDist}(\mathcal{Q})$ of totally regular $\mathcal{Q}$-semicategories and regular semidistributors between them, and a locally ordered category $\mathsf{TRSCat}(\mathcal{Q})$ with the same objects but now with regular semifunctors as morphisms. There is an inclusion $\mathsf{TRSCat}(\mathcal{Q}) \longrightarrow \mathsf{TRSDist}(\mathcal{Q})$ (mapping a semifunctor onto its 'graph'), giving rise to the notion of a *Cauchy complete* totally regular $\mathcal{Q}$-semicategory $\mathbb{B}$: one such that, for each $\mathbb{A}$,

$$\mathsf{TRSCat}(\mathcal{Q})(\mathbb{A},\mathbb{B}) \longrightarrow \mathsf{Map}(\mathsf{TRSDist}(\mathcal{Q}))(\mathbb{A},\mathbb{B})$$

is an equivalence of posets. Writing $\mathsf{TRSCat}_{\mathsf{cc}}(\mathcal{Q})$ for the full sub-2-category of $\mathsf{TRSCat}(\mathcal{Q})$ whose objects are Cauchy complete, it thus follows that $\mathsf{TRSCat}_{\mathsf{cc}}(\mathcal{Q}) \simeq \mathsf{Map}(\mathsf{TRSDist}(\mathcal{Q}))$.

The point is then that, for a locale $L$ viewed as a one-object quantaloid, $\mathsf{Map}(\mathsf{TRSDist}(L))$ is exactly the category of $L$-valued posets *à la* [Borceux and Cruciani, 1998]; it follows that this category is equivalent to $\mathsf{Ord}(\mathsf{Sh}(L))$. Furthermore, it makes perfect sense to define a totally regular $L$-semicategory $\mathbb{A}$ to be *symmetric* whenever $\mathbb{A}(y,x) = \mathbb{A}(x,y)$ for any two objects $x$ and $y$, since both $\mathbb{A}(y,x)$ and $\mathbb{A}(x,y)$ are elements of $L$. These symmetric totally regular $L$-semicategories (i.e. '$L$-valued symmetric posets') are the objects of a full coreflective subcategory $\mathsf{Map}(\mathsf{TRSDist}_{\mathsf{sym}}(L))$ of $\mathsf{Map}(\mathsf{TRSDist}(L))$, which is precisely the category of $L$-valued sets in the sense of [Higgs, 1973]; so it is equivalent to $\mathsf{Sh}(L)$.

Now, the very definition of a totally regular $\mathcal{Q}$-semicategory $\mathbb{A}$ already hints at the important role that idempotents in $\mathcal{Q}$ play: putting $x = y = z$ in either of the two total-regularity-equations shows that each endo-hom-morphism $\mathbb{A}(x,x)\colon tx \longrightarrow tx$ in $\mathbb{A}$ is an idempotent in $\mathcal{Q}$. In fact, each totally regular $\mathcal{Q}$-semicategory $\mathbb{A}$ can be "reshuffled" to obtain a category $\widehat{\mathbb{A}}$ enriched in $\mathcal{Q}_{\mathsf{si}}$, the split-idempotent completion of $\mathcal{Q}$: $\widehat{\mathbb{A}}_0 = \mathbb{A}_0$ is the set of objects, $\widehat{t}\colon \widehat{\mathbb{A}}_0 \longrightarrow (\mathcal{Q}_{\mathsf{si}})_0\colon x \mapsto \mathbb{A}(a,a)$ is the type-function, and $\widehat{\mathbb{A}}(y,x) := \mathbb{A}(y,x)$ is the hom-arrow for a pair $(x,y)$ of objects. A similar "reshuffle" applies to regular distributors between totally regular $\mathcal{Q}$-semicategories too, determining an equivalence $\mathsf{TRSDist}(\mathcal{Q}) \simeq \mathsf{Dist}(\mathcal{Q}_{\mathsf{si}})$. Taking left adjoints in these equivalent quantaloids produces equivalent locally ordered categories $\mathsf{TRSCat}_{\mathsf{cc}}(\mathcal{Q}) \simeq \mathsf{Map}(\mathsf{TRSDist}(\mathcal{Q})) \simeq \mathsf{Map}(\mathsf{Dist}(\mathcal{Q}_{\mathsf{si}})) \simeq \mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q}_{\mathsf{si}})$.

Applying the latter equivalence of locally ordered categories to a locale $L$ viewed as a one-object quantaloid, we find as many equivalent descriptions of $\mathsf{Ord}(\mathsf{Sh}(L))$, the locally ordered category of ordered objects and order-preserving maps in the topos of sheaves on $L$; in particular can an ordered sheaf on $L$ be described as a Cauchy complete category enriched in $L_{\mathsf{si}}$, the split-idempotent completion of $L$. For an $L_{\mathsf{si}}$-category $\mathbb{C}$ it makes moreover sense to say that it is *symmetric* whenever $\mathbb{C}(y,x) = \mathbb{C}(x,y)$ holds for any two objects $x$ and $y$, because both $\mathbb{C}(y,x)$ and $\mathbb{C}(x,y)$ are elements of $L$. The symmetric Cauchy complete $L_{\mathsf{si}}$-categories determine a full subcategory $\mathsf{Cat}_{\mathsf{cc,sym}}(L_{\mathsf{si}})$, which by its equivalence to $\mathsf{Map}(\mathsf{Dist}_{\mathsf{sym}}(L_{\mathsf{si}}) \simeq \mathsf{Map}(\mathsf{TRSDist}_{\mathsf{sym}}(L))$ is further equivalent to the topos $\mathsf{Sh}(L)$ of sheaves on $L$. That is to say, Walters' [1981] description of sheaves on a locale $L$, as symmetric and Cauchy complete $L_{\mathsf{si}}$-categories, follows here from the equivalence $\mathsf{TRSCat}_{\mathsf{cc}}(L) \simeq \mathsf{Cat}_{\mathsf{cc}}(L_{\mathsf{si}})$ by singling out the symmetric objects.

Incidentally, for Lawvere's [1973] quantale $[0,\infty]$ of extended positive real numbers, it is not difficult to prove that $\mathsf{TRSCat}_{\mathsf{cc}}([0,\infty]) \simeq \mathsf{Cat}_{\mathsf{cc}}(\mathcal{R}([0,\infty]))$ is further equivalent to $\mathsf{Cat}_{\mathsf{cc}}([0,\infty])$, thus to the category of Cauchy complete metric spaces and distance decreasing maps.

## 6.

To borrow a phrase from [Reyes, 1977], sheaf theory is all about "algebraic logic". In the case of a locale $L$, it is indeed very well understood that the internal logic of the topos $\mathsf{Sh}(L)$ is an intuitionistic higher-order predicate logic with $L$ as object of truth values; in other words, $\mathsf{Sh}(L)$ is a "set theory" gouverned by $L$-valued logic. A sheaf on a locale $L$ can be described in many different ways (as a $\mathsf{Set}$-valued functor on $L$, or as a set with an $L$-valued equality, or as an $L_{\mathsf{si}}$-enriched category, or as a local homeomorphism into $L$), and each of these formulations puts the versatile concept of "sheaf" in a different perspective.

In the light of developments in algebra, geometry and logic, which tend to get more and more non-commutative (or non-cartesian, or "fuzzy" as some say, or "quantum" as others say), it is natural to study non-commutative topological phenomena too, particularly in connection with sheaf theory. Quantales, and *a fortiori* quantaloids, are a non-commutative generalisation of locales, so they may be regarded as "non-commutative topologies". As a step in the direction of a full-blown theory of "sheaves on a non-commutative topology" (or "non-commutative algebraic logic"), it is thus natural to investigate possible notions of "sheaves" on a quantaloid.

There are about as many different definitions of a "sheaf on a quantale (or quantaloid)" as there are authors writing about them, see e.g. [Borceux and Van den Bossche, 1986; Van den Bossche, 1995; Mulvey and Nawaz, 1995; Ambler and Verity, 1996; Höhle, 1998; Gylys, 2001; Coniglio and Miraglia, 2001; Garraway, 2005; Resende, 2011]. In each of these papers, the proposed definition depends on its author's favourite example (ring theory, linear logic, groupoids), mathematical background (enriched categories, module theory, locale theory), and applies often only to a particular class of quantales or quantaloids (right-sided, involutive, modular). Mainly motivated by the analysis of the localic example in section 5, and aiming at a definition that applies to any quantaloid $\mathcal{Q}$, I proposed [2005c] to define the category of *ordered sheaves on* $\mathcal{Q}$ ($\mathcal{Q}$-*orders* for short) to be (either of) the equivalent categories

$$\mathsf{Ord}(\mathcal{Q}) := \mathsf{TRSCat}_{\mathsf{cc}}(\mathcal{Q}) \simeq \mathsf{Map}(\mathsf{TRSDist}(\mathcal{Q})) \simeq \mathsf{Map}(\mathsf{Dist}(\mathcal{Q}_{\mathsf{si}})) \simeq \mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q}_{\mathsf{si}}).$$

Several of my post-doc publications are devoted to a further study of the category $\mathsf{Ord}(\mathcal{Q})$: to prove further equivalent formulations, to get a better grip on the concept of "sheaf on a quantaloid" itself, to compare this particular definition with other definitions and examples in the literature, and - hopefully - to have learned something about "non-commutative algebraic logic" in the end.

## 7.

In this section I shall comment in more detail on the various aspects of *sheaves on a quantaloid* that I studied in my post-doctoral publications. Most of the results are geared towards a generalisation of localic sheaf theory, but I shall indicate how sheaves on a site fit in the picture too. As before I shall mention some open problems for further study.

**7.1. Quantaloids of closed cribles.** As briefly mentioned in Section 1, Walters [1982] proved that the topos of sheaves on a site $(\mathcal{C}, J)$ is equivalent to the category of symmetric and Cauchy complete categories enriched in a suitable quantaloid $\mathcal{R}(\mathcal{C}, J)$. This quantaloid is constructed as follows: First, for any small category $\mathcal{C}$ and objects $X$ and $Y$ in $\mathcal{C}$, a *crible* $R\colon X \twoheadrightarrow Y$ is a set of spans $(f, g)\colon X \twoheadrightarrow Y$ in $\mathcal{C}$ such that $(f, g) \in R \implies (fh, gh) \in R$. If $J$ is a Grothendieck topology on $\mathcal{C}$, define the *closure* of a crible $R$ to be $\overline{R} = \{(f, g) \mid \exists S \in J : \forall s \in S, (fs, gs) \in R\}$. The closed cribles are then the morphisms of a small quantaloid, denoted by $\mathcal{R}(\mathcal{C}, J)$[12].

By definition, a *small quantaloid of closed cribles* is one that is equivalent to $\mathcal{R}(\mathcal{C}, J)$, for some small site $(\mathcal{C}, J)$[13]. We sought to give an elementary axiomatisation of this notion [Heymans and Stubbe, 2011b]:

**Theorem.** *A small quantaloid $\mathcal{Q}$ is a small quantaloid of closed cribles if and only if*

$$J(X) := \left\{ S \text{ is a sieve on } X \text{ in } \mathsf{Map}(\mathcal{Q}) \;\middle|\; 1_X = \bigvee_{s \in S} ss^* \right\}$$

*defines a topology $J$ on $\mathsf{Map}(\mathcal{Q})$ for which $\mathcal{Q} \cong \mathcal{R}(\mathsf{Map}(\mathcal{Q}), J)$, if and only if $\mathcal{Q}$ has the following properties:*

1. *locally localic: each $\mathcal{Q}(X, Y)$ is a locale,*
2. *map-discrete: for left adjoints, $f \le g$ implies $f = g$,*
3. *weakly tabular: for every morphism $q$, $q = \bigvee\{fg^* \mid f, g \text{ are left adjoints and } fg^* \le q\}$,*
4. *weakly modular[14]: for every parallel pair of spans of left adjoints, say $(f, g)\colon X \twoheadrightarrow Y$ and $(m, n)\colon X \twoheadrightarrow Y$, $fg^* \wedge mn^* \le f(g^*n \wedge f^*m)n^*$.*

*In this case, $\mathcal{Q}$ carries an involution, sending a morphism $q\colon Y \longrightarrow X$ to*

$$q^\circ := \bigvee \left\{ gf^* \;\middle|\; (f, g)\colon Y \twoheadrightarrow X \text{ is a span of left adjoints such that } fg^* \le q \right\},$$

*which makes $\mathcal{Q}$ a modular quantaloid. Moreover, the Grothendieck topology $J$ on $\mathsf{Map}(\mathcal{Q})$ is always subcanonical, and if coreflexives split in $\mathcal{Q}$, then $J$ is the canonical topology.*

The theorem thus spells out how two, at first sight quite different, generalisations of locales, namely Grothendieck topologies on the one hand, and quantaloids on the other, relate: the former can be understood to form an axiomatically described subclass of the latter. Having this correspondence between small sites and small quantaloids of closed cribles, it is natural to ask for:

**Problem.** *Determine the "correct" notion of morphism between small quantaloids of closed cribles to establish an equivalence between the category of small sites and the category of small quantaloids of closed cribles. Study the "Morita equivalence" of sites via the associated quantaloids of closed cribles.*

A locale $L$ carries a canonical topology[15], so determines a small site $(L, J_{\mathsf{can}})$, and the quantaloid of closed cribles $\mathcal{R}(L, J_{\mathsf{can}})$ turns out to be precisely the split-idempotent completion $L_{\mathsf{si}}$ of $L$. Ordered sheaves on $L$ are precisely the same thing as Cauchy complete $L_{\mathsf{si}}$-categories, and the symmetric such categories correspond with the ("symmetrically ordered") sheaves on $L$. In fact, $L_{\mathsf{si}}$ is a Cauchy bilateral quantaloid (cf. subsection 4.1), so that there is a distributive law of the Cauchy completion monad over the symmetrisation comonad on the category of $L_{\mathsf{si}}$-categories. This in turn implies a right adjoint to the inclusion,

$$\mathsf{Sh}(L) \simeq \mathsf{Cat}_{\mathsf{cc,sym}}(L_{\mathsf{si}}) \underset{\longrightarrow}{\overset{\longleftarrow}{\top}} \mathsf{Cat}_{\mathsf{cc}}(L_{\mathsf{si}}) \simeq \mathsf{Ord}(\mathsf{Sh}(L)),$$

so that the topos $\mathsf{Sh}(L)$ is precisely the category of coalgebras for the induced ("symmetrisation") comonad on $\mathsf{Ord}(\mathsf{Sh}(L))$.

It can be verified, either with an explicit computation or from the axiomatic description, that any quantaloid of closed cribles $\mathcal{Q} = \mathcal{R}(\mathcal{C}, J)$ is Cauchy bilateral; and so we get the more general picture

$$\mathsf{Sh}(\mathcal{C}, J) \simeq \mathsf{Cat}_{\mathsf{cc,sym}}(\mathcal{Q}) \underset{\longrightarrow}{\overset{\longleftarrow}{\top}} \mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q})$$

of a right adjoint to the inclusion, inducing a ("symmetrisation") comonad on $\mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q})$ whose category of algebras is the topos $\mathsf{Sh}(\mathcal{C}, J)$. This suggests:

**Problem.** *Prove that* $\mathsf{Cat}_{\mathsf{cc}}(\mathcal{R}(\mathcal{C}, J))$ *is the category of ordered objects in* $\mathsf{Sh}(\mathcal{C}, J)$ *and that there is a symmetrisation comonad on* $\mathsf{Ord}(\mathcal{R}(\mathcal{C}, J))$ *whose category of coalgebras is (equivalent to)* $\mathsf{Sh}(\mathcal{C}, J)$. *Give such a description for any small quantaloid of closed cribles, without reference to its underlying site, and see which part of it can be extrapolated to more general involutive quantaloids (see also subsection 7.5 further on).*

Currently I am writing up a preprint, in collaboration with Hans Heymans, addressing this problem (and other things too). We will submit it for publication in December 2011.

**7.2. Suplattices.** A result of C. J. Mikkelsen's [1976] states that an ordered object in an elementary topos $\mathcal{E}$ is cocomplete, i.e. it is an internal suplattice, if and only if the "principal downset embedding" from that object to its powerobject has a left adjoint in $\mathsf{Ord}(\mathcal{E})$. In the case of a localic topos, it turns out that the internal suplattices in $\mathsf{Sh}(L)$ are precisely the $L$-modules, and supmorphisms are just the module morphisms [Joyal and Tierney, 1984; Pitts, 1988]. This actually holds in the generality of $\mathcal{Q}$-orders [Stubbe, 2007]:

**Theorem.** *The forgetful functor* $\mathcal{U}\colon \mathsf{Cocont}(\mathcal{Q}) \longrightarrow \mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q})$ *is right adjoint to the functor, written* $\mathcal{P}\colon \mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q}) \longrightarrow \mathsf{Cocont}(\mathcal{Q})$, *which sends a (Cauchy complete)* $\mathcal{Q}$*-category* $\mathbb{A}$ *to the cocomplete* $\mathcal{Q}$*-category* $\mathcal{P}(\mathbb{A})$ *of contravariant presheaves on* $\mathbb{A}$*. Composing left and right adjoint produces a KZ-doctrine on* $\mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q})$*, called the 'presheaf doctrine', whose category of algebras is equivalent to* $\mathsf{Cocont}(\mathcal{Q})$*. Applying this to* $\mathcal{Q}_{\mathsf{si}}$ *instead of* $\mathcal{Q}$*, and reckoning that* $\mathsf{Cocont}(\mathcal{Q}_{\mathsf{si}}) \simeq \mathsf{Mod}(\mathcal{Q}_{\mathsf{si}})$ *(as indicated in subsection 4.5) and* $\mathsf{Mod}(\mathcal{Q}_{\mathsf{si}}) \simeq \mathsf{Mod}(\mathcal{Q})$ *(because idempotents split in* $\mathsf{Sup}$*), the diagram*

$$\mathsf{Mod}(\mathcal{Q}) \simeq \mathsf{Mod}(\mathcal{Q}_{\mathsf{si}}) \simeq \mathsf{Cocont}(\mathcal{Q}_{\mathsf{si}}) \underset{\mathcal{U}}{\overset{\mathcal{P}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q}_{\mathsf{si}}) \simeq \mathsf{Ord}(\mathcal{Q})$$

*exhibits the quantaloid* $\mathsf{Mod}(\mathcal{Q})$ *as the category of algebras for the presheaf doctrine on* $\mathsf{Ord}(\mathcal{Q})$*.*

For a locale $L$, consider an ordered object $(F, \leq)$ in the topos $\mathsf{Sh}(L)$; it corresponds, under the equivalence of $\mathsf{Ord}(\mathsf{Sh}(L))$ with $\mathsf{Cat}_{\mathsf{cc}}(L_{\mathsf{si}})$, with a Cauchy complete $L_{\mathsf{si}}$-category $\mathbb{A}$. The presheaf doctrine sends $\mathbb{A}$ to the cocomplete $L_{\mathsf{si}}$-category $\mathcal{P}(\mathbb{A})$, which corresponds in turn with the ordered sheaf of the downclosed subobjects of $(F, \leq)$. Now $(F, \leq)$ is an internal suplattice in $\mathsf{Sh}(L)$ if and only if the "principal downset inclusion" $F \longrightarrow L^F$ has a left adjoint [Mikkelsen, 1976; Johnstone, 2002, B2.3.9], which is constructively equivalent with the existence of a left adjoint to its factorization over the (object of) downsets of $F$. This is the case if and only if the Yoneda embedding $Y_{\mathbb{A}} \colon \mathbb{A} \mapsto \mathcal{P}(\mathbb{A})$, which is the unit at $\mathbb{A}$ of the presheaf doctrine, has a left adjoint; in other words, $\mathbb{A}$ is a cocomplete $L_{\mathsf{si}}$-category. So an ordered sheaf $(F, \leq)$ is an internal suplattice in $\mathsf{Sh}(L)$ if and only if the associated $L_{\mathsf{si}}$-category $\mathbb{A}$ is cocomplete. Cocomplete $L_{\mathsf{si}}$-categories being equivalent to $L$-modules, this provides an independent proof of the fact that $L$-modules are precisely the internal cocomplete objects of $\mathsf{Ord}(\mathsf{Sh}(L))$.

**Problem.** *If* $(\mathcal{C}, J)$ *is a small site, are the* $\mathcal{R}(\mathcal{C}, J)$*-modules the internal suplattices in* $\mathsf{Sh}(\mathcal{C}, J)$*?*

Earlier, in subsection 4.6, I used the term "dynamic domain theory" for a generalisation of notions from classical domain theory (total continuity and total algebraicity of suplattices) to cocomplete $\mathcal{Q}$-categories; and I indicated in subsection 4.7 how these notions can be transported over to $\mathcal{Q}$-modules via the equivalence $\mathsf{Mod}(\mathcal{Q}) \simeq \mathsf{Cocont}(\mathcal{Q})$. In the current subsection I just showed that, in fact, $\mathcal{Q}$-modules (and thus also cocomplete $\mathcal{Q}$-categories) are precisely the 'suplattices' in $\mathsf{Ord}(\mathcal{Q})$: in hindsight it is thus correct to say that "dynamic domain theory" is a study of convergence and approximation in the world of ordered sheaves on $\mathcal{Q}$.

**Problem.** *Can notions from classical domain theory, that do not require the completeness of the underlying ordered set, be generalised to* $\mathsf{Ord}(\mathcal{Q})$*? Can this lead to a better understanding and further development of "dynamic domains", so that applying general results to either a locale L or the quantale* $[0, \infty]$ *gives interesting results for "constructive domains" or "metric domains"?*

**7.3. Locally principally generated modules.** The well-known adjunction between the category $\mathsf{Ord}$ of ordered sets and order-preserving functions on the one hand, and the category $\mathsf{Sup}$ of complete lattices and supremum-preserving functions on the other,

$$\mathsf{Ord} \underset{U}{\overset{F}{\rightleftarrows}} \mathsf{Sup}, \quad (F \dashv U)$$

has the feature that both functors involved are embeddings. This allows us to view $\mathsf{Sup}$ as a part of $\mathsf{Ord}$, but also $\mathsf{Ord}$ as a part of $\mathsf{Sup}$. The first viewpoint corresponds to the common understanding that a complete lattice is an ordered set in which all suprema exist and that a supmorphism is an order-preserving function that preserves suprema. The second point of view corresponds with the fact that the replete image of the left adjoint in the above adjunction is precisely the subcategory of $\mathsf{Sup}$ of totally algebraic objects and left adjoint morphisms.

In the previous subsection I indicated a generalisation to $\mathcal{Q}$-orders: there is a biadjunction

$$\mathsf{Ord}(\mathcal{Q}) \underset{U}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathsf{Mod}(\mathcal{Q})$$

that splits the presheaf construction on $\mathsf{Ord}(\mathcal{Q})$, and $\mathsf{Mod}(\mathcal{Q})$ is the category of algebras for that doctrine. This describes $\mathsf{Mod}(\mathcal{Q})$ as part of $\mathsf{Ord}(\mathcal{Q})$, and for $\mathcal{Q} = \mathbf{2}$ we thus recover exactly half of the situation described in the first paragraph above. An obvious question is whether we can also intrinsically characterise $\mathsf{Ord}(\mathcal{Q})$ as part of $\mathsf{Mod}(\mathcal{Q})$: can we give a module-theoretic condition on objects and morphisms of $\mathsf{Mod}(\mathcal{Q})$ to describe the image of $F$?

In [Heymans and Stubbe, 2009a] we obtained such a module-theoretic description of $\mathsf{Ord}(\mathcal{Q})$ by "tweaking" the equivalence $\mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q}) \simeq \mathsf{Mod}_{\mathsf{pg}}(\mathcal{Q})$ (that I explained in subsection 4.7). Since we can equate $\mathsf{Ord}(\mathcal{Q})$ with $\mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q}_{\mathsf{si}})$, it follows that that $\mathsf{Ord}(\mathcal{Q}) \simeq \mathsf{Map}(\mathsf{Mod}_{\mathsf{pg}}(\mathcal{Q}_{\mathsf{si}}))$. Furthermore, a $\mathcal{Q}$-module $\mathcal{F}\colon \mathcal{Q}^{\mathsf{op}} \longrightarrow \mathsf{Sup}$ determines a $\mathcal{Q}_{\mathsf{si}}$-module $\mathcal{F}_{\mathsf{si}}\colon \mathcal{Q}_{\mathsf{si}}^{\mathsf{op}} \longrightarrow \mathsf{Sup}$, essentially by the splitting of idempotents in $\mathsf{Sup}$, and this correspondence extends to an equivalence $\mathsf{Mod}(\mathcal{Q}) \simeq \mathsf{Mod}(\mathcal{Q}_{\mathsf{si}})$. So it remains to find out what it means for a $\mathcal{Q}$-module $\mathcal{F}$ to have as counterpart $\mathcal{F}_{\mathsf{si}}$ a principally generated $\mathcal{Q}_{\mathsf{si}}$-module. This is what I shall explain next.

First I must introduce some terminology. If $e\colon X \longrightarrow X$ is an idempotent in $\mathcal{Q}$, then the representable module morphism $\mathcal{Q}(-,e)\colon \mathcal{Q}(-,X) \Longrightarrow \mathcal{Q}(-,X)$ is an idempotent in $\mathsf{Mod}(\mathcal{Q})$. All idempotents in $\mathsf{Mod}(\mathcal{Q})$ split, so this one does too. I shall write the splitting as

$$\mathcal{F}_e \underset{\pi_e}{\overset{\sigma_e}{\underset{\Longleftarrow}{\Longrightarrow}}} \mathcal{Q}(-,X),$$

where $\mathcal{F}_e\colon \mathcal{Q}^{\mathsf{op}} \longrightarrow \mathsf{Sup}$ sends an object $A$ of $\mathcal{Q}$ to the suplattice $\{f\colon A \longrightarrow X \mid e \circ f = f\}$; for the obvious reason, I shall call such a module $\mathcal{F}_e$ a *fixpoint module (for the idempotent $e$)*.

Now let $\mathcal{F}\colon \mathcal{Q}^{\mathsf{op}} \longrightarrow \mathsf{Sup}$ be a $\mathcal{Q}$-module, and consider an element $x \in \mathcal{F}(X)$; by the $\mathsf{Sup}$-enriched Yoneda Lemma it corresponds with a module morphism $\tau_x\colon \mathcal{Q}(-,X) \Longrightarrow \mathcal{F}$. This element $x \in \mathcal{F}$ is a *locally principal element (at an idempotent $e\colon A \longrightarrow A$ in $\mathcal{Q}$)*[16] if (there is an idempotent $e\colon X \longrightarrow X$ in $\mathcal{Q}$ such that) $\mathcal{F}(e)(x) = x$ and $\tau_x \circ \sigma_e\colon \mathcal{F}_e \Longrightarrow \mathcal{F}$ is a left adjoint in $\mathsf{Mod}(\mathcal{Q})$. Thus, the set of locally principal elements is

$$\mathcal{F}_{\mathsf{lpr}} = \Big\{ \zeta(e) \;\Big|\; e \text{ is an idempotent in } \mathcal{Q} \text{ and } \zeta\colon \mathcal{F}_e \Longrightarrow \mathcal{F} \text{ is a left adjoint in } \mathsf{Mod}(\mathcal{Q}) \Big\}.$$

Finally, a $\mathcal{Q}$-module $\mathcal{F}$ is said to be *locally principally generated* if

$$\mathsf{id}_{\mathcal{F}} = \bigvee \Big\{ \zeta \circ \zeta^* \;\Big|\; e \text{ is an idempotent in } \mathcal{Q} \text{ and } \zeta\colon \mathcal{F}_e \Longrightarrow \mathcal{F} \text{ is a left adjoint in } \mathsf{Mod}(\mathcal{Q}). \Big\}.$$

With these notions we proved:

**Theorem.** *For a $\mathcal{Q}$-module $\mathcal{F}\colon \mathcal{Q}^{\mathsf{op}} \longrightarrow \mathsf{Sup}$, the following statements are equivalent:*

1. *the corresponding $\mathcal{Q}_{\mathsf{si}}$-module $\mathcal{F}_{\mathsf{si}}\colon \mathcal{Q}_{\mathsf{si}}^{\mathsf{op}} \longrightarrow \mathsf{Sup}$ is principally generated,*
2. *$\mathcal{F}$ is locally principally generated,*

*3. $\mathcal{F}$ is an adjoint retract of a direct sum of fixpoint modules.*

Writing $\mathsf{Mod}_{\mathsf{lpg}}(\mathcal{Q})$ for the full subquantaloid of $\mathsf{Mod}(\mathcal{Q})$ whose objects are the locally principally generated $\mathcal{Q}$-modules, it follows trivially that the biequivalence $\mathsf{Mod}(\mathcal{Q}_{\mathsf{si}}) \simeq \mathsf{Mod}(\mathcal{Q})$ restricts to a biequivalence $\mathsf{Mod}_{\mathsf{pg}}(\mathcal{Q}_{\mathsf{si}}) \simeq \mathsf{Mod}_{\mathsf{lpg}}(\mathcal{Q})$. Furthermore, as indicated in subsection 4.7, we also know that $\mathsf{Mod}_{\mathsf{pg}}(\mathcal{Q}_{\mathsf{si}}) \simeq \mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q}_{\mathsf{si}})$. All this makes the following diagram – in which $\mathcal{P}\colon \mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q}_{\mathsf{si}}) \longrightarrow \mathsf{Cocont}(\mathcal{Q}_{\mathsf{si}})$ is the presheaf functor – commute:

$$\begin{array}{ccccccc} & & \mathsf{Cocont}(\mathcal{Q}_{\mathsf{si}}) & \xrightarrow{\sim} & \mathsf{Mod}(\mathcal{Q}_{\mathsf{si}}) & \xrightarrow{\sim} & \mathsf{Mod}(\mathcal{Q}) \\ & \nearrow^{\mathcal{P}} & \uparrow & & \uparrow & & \uparrow \\ \mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q}_{\mathsf{si}}) & \xrightarrow{\sim} & \mathsf{Map}(\mathsf{Cocont}_{\mathsf{ta}}(\mathcal{Q}_{\mathsf{si}})) & \xrightarrow{\sim} & \mathsf{Map}(\mathsf{Mod}_{\mathsf{pg}}(\mathcal{Q}_{\mathsf{si}})) & \xrightarrow{\sim} & \mathsf{Map}(\mathsf{Mod}_{\mathsf{lpg}}(\mathcal{Q})) \end{array}$$

Up to the identification of $\mathsf{Ord}(\mathcal{Q})$ with $\mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q}_{\mathsf{si}})$, the composition of the 2-functors

$$\mathsf{Ord}(\mathcal{Q}) = \mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q}_{\mathsf{si}}) \xrightarrow{\mathcal{P}} \mathsf{Cocont}(\mathcal{Q}_{\mathsf{si}}) \xrightarrow{\sim} \mathsf{Mod}(\mathcal{Q}_{\mathsf{si}}) \xrightarrow{\sim} \mathsf{Mod}(\mathcal{Q})$$

is precisely the left biadjoint $F\colon \mathsf{Ord}(\mathcal{Q}) \longrightarrow \mathsf{Mod}(\mathcal{Q})$ to the forgetful $U\colon \mathsf{Mod}(\mathcal{Q}) \longrightarrow \mathsf{Ord}(\mathcal{Q})$. Its factorisation over $\mathsf{Map}(\mathsf{Mod}_{\mathsf{lpg}}(\mathcal{Q}))$ gives the sought-after characterisation of $\mathcal{Q}$-orders in terms of $\mathcal{Q}$-modules:

**Theorem.** *The locally ordered category* $\mathsf{Ord}(\mathcal{Q})$ *of ordered sheaves on a small quantaloid* $\mathcal{Q}$ *is biequivalent to* $\mathsf{Map}(\mathsf{Mod}_{\mathsf{lpg}}(\mathcal{Q}))$*, the locally ordered category of locally principally generated $\mathcal{Q}$-modules and left adjoint $\mathcal{Q}$-module morphisms between them.*

**7.4. Skew local homeomorphisms.** As is well known, a sheaf on a locale $L$ can be described as a local homeomorphism into $L$. In the context of my work on *ordered* sheaves on a quantaloid, it is natural to seek for a description of $L$-orders in terms of some kind of locale morphism into $L$: this is what the notion of 'skew local homeomorphism' achieves [Heymans and Stubbe, 2009a]. As I shall explain next, this is an application of the notion of locally principally generated module.

In what follows, $\mathsf{Loc}$ denotes the (2-)category of locales: objects are locales, a locale morphism $f\colon Y \longrightarrow X$ is an adjoint pair

$$Y \underset{f_*}{\overset{f^*}{\leftrightarrows}} X \qquad (f^* \dashv f_*)$$

in the 2-category of partially ordered sets such that the left adjoint preserves finite infima, and for $f, g\colon Y \rightrightarrows X$ in $\mathsf{Loc}$ we define[17] that $f \leq g$ if $f_* \leq g_*$. Given an $f\colon Y \longrightarrow X$ in $\mathsf{Loc}$, it is easily seen that

$$Y \times X \longrightarrow Y\colon (y, x) \mapsto y \circ_f x := y \wedge f^*(x)$$

is in an action[18] of the monoid $(X, \wedge, \top)$ on $Y$ in $\mathsf{Sup}$. In other words, from $f\colon Y \longrightarrow X$ in $\mathsf{Loc}$

we get an object $(Y, \circ_f) \in \mathsf{Mod}(X)$. Moreover, suppose that

$$\begin{array}{ccc} Y & \xrightarrow{\quad h \quad} & Z \\ & {\scriptstyle f}\searrow \qquad \swarrow{\scriptstyle g} & \\ & X & \end{array}$$

commutes in $\mathsf{Loc}$, then $h^*: Z \longrightarrow Y$ is a morphism in $\mathsf{Sup}$ satisfying $h^*(z \circ_f x) = h^*(z) \circ_g x$, for all $x \in X$, $z \in Z$. That is to say, $h^*: (Z, \circ_g) \longrightarrow (Y, \circ_f)$ is a morphism in $\mathsf{Mod}(X)$. All this adds up to an injective and faithful (but not full) 2-functor $(\mathsf{Loc}/_X)^{\mathsf{coop}} \longrightarrow \mathsf{Mod}(X)$.

The general theory for $\mathcal{Q}$-orders says that the locally ordered category $\mathsf{Ord}(\mathcal{Q})$ can be described as $\mathsf{Map}(\mathsf{Mod}_{\mathsf{lpg}}(\mathcal{Q}))$, that is, as locally principally generated $\mathcal{Q}$-modules and left adjoint module morphisms between them. This applies of course to a locale $X$, so our task is to characterise, in elementary terms, those objects and morphisms of $(\mathsf{Loc}/_X)^{\mathsf{coop}}$ which correspond, under the action of the 2-functor $(\mathsf{Loc}/_X)^{\mathsf{coop}} \longrightarrow \mathsf{Mod}(X)$, to $\mathsf{Map}(\mathsf{Mod}_{\mathsf{lpg}}(X))$.

First, say that a morphism $h: f \longrightarrow g$ in $\mathsf{Loc}/_X$ is *skew open* if the corresponding order-preserving function $h^*: Z \longrightarrow Y$ has a left adjoint $h_!: Y \longrightarrow Z$ satisfying the "balanced Frobenius identity[19]":

$$\text{for all } y \in Y \text{ and } x \in X,\ h_!(y \wedge f^*(x)) = h_!(y) \wedge g^*(x).$$

Clearly the identity morphisms in $\mathsf{Loc}/_X$ are skew open, and the composition of skew open morphisms is again skew open; it thus makes sense to speak of the sub-2-category $(\mathsf{Loc}/_X)^{\mathsf{o}}$ of $\mathsf{Loc}/_X$ with the same objects but only its skew open morphisms. Upon inspection it is easily seen that, for any two locale morphisms $f: Y \longrightarrow X$ and $g: Z \longrightarrow X$, there is an isomorphism of ordered sets

$$(\mathsf{Loc}/_X)^{\mathsf{o}}(f, g) \cong \mathsf{Map}(\mathsf{Mod}(X))((Y, \circ_f), (Z, \circ_g))$$

given by sending a skew open morphism $h$ to the $X$-module morphism $h_!$ with right adjoint $h^*$. Sending skew open morphisms in $\mathsf{Loc}/_X$ to their utmost left adjoints (i.e. $h \mapsto h_!$) thus gives rise to an injective and fully faithful 2-functor $(\mathsf{Loc}/_X)^{\mathsf{o}} \longrightarrow \mathsf{Map}(\mathsf{Mod}(X))$. In the codomain category of this functor we are now interested in the locally principally generated objects.

As is customary, for any $u \in X$ we generically write $i: \downarrow u \rightarrowtail X$ for the corresponding open sublocale of $X$; it is also skew open in $\mathsf{Loc}/_X$ as (unique) morphism from $i: \downarrow u \rightarrowtail X$ to the terminal object $1_X: X \longrightarrow X$. For $f: Y \longrightarrow X$ in $\mathsf{Loc}$, we put $S_f^{\mathsf{o}}(u) := (\mathsf{Loc}/_X)^{\mathsf{o}}(i, f)$ and call its elements the *skew open sections of $f$ at $u$*. Now we define $f: Y \longrightarrow X$ to be a *skew local homeomorphism*[20] if

$$1_Y = \bigvee \{ s_! \circ s^* \mid u \in X, s \in S_f^{\mathsf{o}}(u) \}.$$

The formal resemblance between this condition and the condition characterising locally principally generated $\mathcal{Q}$-modules (cf. the previous subsection) already hints at:

**Theorem.** *An $f: Y \longrightarrow X$ in $\mathsf{Loc}$ is a skew local homeomorphism if and only if $(Y, \circ_f)$ is a locally principally generated $X$-module[21]. Conversely, if $(M, \circ)$ is a locally principally generated $X$-module, then the locale morphism $f: M \longrightarrow X$ with inverse image $f^*(x) = \top_M \circ x$ is a skew local homeomorphism that satisfies $(M, \circ) = (M, \circ_f)$. As a result, writing $(\mathsf{Loc}/_X)^{\mathsf{o}}_{\mathsf{slh}}$ for the full* *subcategory of* $(\mathsf{Loc}/X)^{\mathsf{o}}$ *whose objects are the skew local homeomorphisms, it follows that the full embedding* $(\mathsf{Loc}/X)^{\mathsf{o}}\longrightarrow\mathsf{Map}(\mathsf{Mod}(X))$ *(co)restricts to an isomorphism*

$$(\mathsf{Loc}/X)^{\mathsf{o}}_{\mathsf{slh}}\cong\mathsf{Map}(\mathsf{Mod}_{\mathsf{lpg}}(X))$$

*of locally ordered categories, both of which are thus equivalent to the category* $\mathsf{Ord}(X)$ *of ordered sheaves on* $X$*.*

**Problem.** *Is there a notion of "skew local homeomorphism" for quantales (or quantaloids)? That is to say, can a* $\mathcal{Q}$*-order be described as some kind of quantale morphism into* $\mathcal{Q}$*?*

**7.5. Étale modules.** Any local homeomorphism is necessarily a skew local homeomorphism, and any open locale morphism is necessarily skew open too. Writing $\mathsf{LH}$ for the category of locales and local homeomorphisms, it follows that $\mathsf{LH}/X$ is a full subcategory of $(\mathsf{Loc}/X)^{\mathsf{o}}_{\mathsf{slh}}$. Whereas $\mathsf{LH}/X$ is a well-known equivalent of the topos $\mathsf{Sh}(X)$, the previous subsection proved $(\mathsf{Loc}/X)^{\mathsf{o}}_{\mathsf{slh}}$ to be isomorphic to $\mathsf{Map}(\mathsf{Mod}_{\mathsf{lpg}}(X))$. Thus it makes sense to determine those locally principally generated $X$-modules which, under this isomorphism, correspond to local homeomorphisms.

The pertinent definition is the following [Heymans and Stubbe, 2009a]: an *étale* $X$*-module* $(M,\circ)$ is a locally principally generated $X$-module such that, for every $u\in X$, every left adjoint $X$-module morphism $\zeta\colon(\downarrow u,\wedge)\longrightarrow(M,\circ)$ is *open* in the sense that

$$\zeta(v\wedge\zeta^*(m))=\zeta(v)\wedge m$$

holds for all $v\in\downarrow u$ and $m\in M$. With this notion we proved:

**Theorem.** *A skew local homeomorphism* $f\colon Y\longrightarrow X$ *is a local homeomorphism if and only if* $(Y,\circ_f)$ *is an étale* $X$*-module. Letting* $\mathsf{Mod}_{\mathsf{\acute{e}t}}(X)$ *stand for the full sub-2-category of* $\mathsf{Mod}_{\mathsf{lpg}}(X)$ *consisting of étale* $X$*-modules, the isomorphism* $(\mathsf{Loc}/X)^{\mathsf{o}}_{\mathsf{slh}}\cong\mathsf{Map}(\mathsf{Mod}_{\mathsf{lpg}}(X))$ *restricts to an isomorphism* $\mathsf{LH}/X\cong\mathsf{Map}(\mathsf{Mod}_{\mathsf{\acute{e}t}}(X))$ *of categories, both of which are thus equivalent to the topos* $\mathsf{Sh}(X)$ *of sheaves on* $X$*.*

A sheaf on a locale $X$ is thus the same thing as an étale $X$-module; and an étale $X$-module is a locally principally generated $X$-module satisfying an extra openness condition. It turns out (but this is not straightforward) that the openness condition is equivalent to a symmetry condition, which in turn can be generalised to principally generated $\mathcal{Q}$-modules *whenever* $\mathcal{Q}$ *is an involutive quantaloid* [Heymans and Stubbe, 2009b].

Precisely, suppose that $\mathcal{Q}$ is an involutive quantaloid, whose involution I shall write as $f\mapsto f^{\mathsf{o}}$, and let $\mathcal{E}=\{e\colon A\longrightarrow A\text{ in }\mathcal{Q}\mid e^{\mathsf{o}}=e=e^2\}$ be the class of symmetric idempotents in $\mathcal{Q}$. An $\mathcal{E}$*-principal element* of a $\mathcal{Q}$-module $\mathcal{F}\colon\mathcal{Q}^{\mathsf{op}}\longrightarrow\mathsf{Sup}$ is, by definition, a principal element of $\mathcal{F}$ *at an idempotent in* $\mathcal{E}$; so the set of $\mathcal{E}$-principal elements of $\mathcal{F}$ is

$$\mathcal{F}_{\mathcal{E}\mathsf{pr}}=\Big\{\zeta(e)\;\Big|\;e\in\mathcal{E}\text{ and }\zeta\colon\mathcal{F}_e\Longrightarrow\mathcal{F}\text{ is a left adjoint in }\mathsf{Mod}(\mathcal{Q})\Big\}.$$

Naturally, a $\mathcal{Q}$-module $\mathcal{F}\colon\mathcal{Q}^{\mathsf{op}}\longrightarrow\mathsf{Sup}$ is said to be $\mathcal{E}$*-principally generated* when it is generated by its elements which are $\mathcal{E}$-locally principal:

$$\mathsf{id}_{\mathcal{F}}=\bigvee\Big\{\zeta\circ\zeta^*\;\Big|\;e\in\mathcal{E}\text{ and }\zeta\colon\mathcal{F}_e\Longrightarrow\mathcal{F}\text{ is a left adjoint in }\mathsf{Mod}(\mathcal{Q})\Big\}.$$

And $\mathcal{F}$ is said to be *$\mathcal{E}$-principally symmetric* whenever, for any two left adjoint module morphisms $\zeta\colon \mathcal{F}_e \Longrightarrow \mathcal{F}$ and $\eta\colon \mathcal{F}_f \Longrightarrow \mathcal{F}$ with $e, f \in \mathcal{E}$ (and adjoints written as $\zeta \dashv \zeta^*$ and $\eta \dashv \eta^*$), we have

$$(\zeta^*(\eta(f)))^\circ = \eta^*(\zeta(e)).$$

In general not every idempotent in an involutive quantaloid $\mathcal{Q}$ is symmetric, so a $\mathcal{Q}$-module $\mathcal{F}$ has fewer $\mathcal{E}$-principal elements than it has locally principal elements. Each $\mathcal{E}$-principally generated module is therefore necessarily locally principally generated: there is a subquantaloid $\mathsf{Mod}_{\mathcal{E}\mathsf{pg}}(\mathcal{Q}) \subseteq \mathsf{Mod}_{\mathsf{lpg}}(\mathcal{Q})$ of the former in the latter. Among the objects of $\mathsf{Mod}_{\mathcal{E}\mathsf{pg}}(\mathcal{Q})$ we can single out the $\mathcal{E}$-principally symmetric modules, making a further subquantaloid $\mathsf{Mod}_{\mathcal{E}\mathsf{pg},\mathcal{E}\mathsf{sym}}(\mathcal{Q})$.

Regarding a locale $X$ as a one-object quantaloid with identity involution, each idempotent in $X$ is evidently symmetric, so an $X$-module is $\mathcal{E}$-principally generated if and only if it is locally principally generated. And an $\mathcal{E}$-principally symmetric $X$-module is referred to as, simply, a locally principally symmetric one. We then have:

**Theorem.** *For a locale $X$, an $X$-module $(M, \circ)$ is étale if and only if it is locally principally generated and locally principally symmetric. Thus the topos $\mathsf{Sh}(X)$ is further equivalent to the category $\mathsf{Map}(\mathsf{Mod}_{\mathsf{lpg,sym}}(X))$ of locally principally generated, locally principally symmetric $X$-modules and left adjoint module morphisms.*

This result suggests that, for any involutive quantaloid $\mathcal{Q}$ (and $\mathcal{E}$ the set of its symmetric idempotents), the category of $\mathcal{E}$-principally generated, $\mathcal{E}$-principally symmetric $\mathcal{Q}$-modules, with left adjoint module morphisms between them, is a natural candidate to be "the category of (symmetrically ordered) sheaves on $\mathcal{Q}$".

**Problem.** *For an involutive (and probably also Cauchy bilateral) quantaloid $\mathcal{Q}$, identify the subcategories of "symmetric $\mathcal{Q}$-orders" within each of the equivalent descriptions of $\mathsf{Ord}(\mathcal{Q})$, viz. $\mathsf{TRSCat}_{\mathsf{cc}}(\mathcal{Q}) \simeq \mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q}_{\mathsf{si}}) \simeq \mathsf{Map}(\mathsf{Mod}_{\mathsf{lpg}}(\mathcal{Q}))$. For $\mathcal{Q} = \mathcal{R}(\mathcal{C}, J)$, show how this then gives back the sheaves on $(\mathcal{C}, J)$ (see also subsection 7.1).*

In the preprint that I already alluded to at the end of subsection 7.1, Hans Heymans and I address aspects of this problem.

**7.6. Hilbert modules.** Let $Q$ be an involutive quantale (with involution $f \mapsto f^\circ$); I shall think of a $Q$-module $(M, \circ)$ as a suplattice $M$ with an action $M \times Q \longrightarrow M\colon (m, f) \mapsto m \circ f$ (satisfying the obvious conditions). Jan Paseka [1999] defined a *pre-inner product* on $(M, \circ)$ to be a map $M \times M \longrightarrow Q\colon (m, n) \mapsto \langle m, n\rangle$ such that, for all $m, n \in M$, $\langle m, -\rangle\colon M \longrightarrow Q$ is a module morphism and $\langle m, n\rangle^\circ = \langle n, m\rangle$; it is an *inner product* if moreover $\langle -, m\rangle = \langle -, n\rangle$ implies $m = n$. Slightly generalising a notion that appeared in [Resende and Rodrigues, 2010] in the context of modules on a locale, we further say that a subset $\Gamma \subseteq M$ is a *Hilbert basis* (for a given pre-inner product on $M$) if it satisfies, for all $m \in M$, $m = \bigvee_{s\in\Gamma} s \circ \langle s, m\rangle$. It is trivial to check that, if $M$ is a $Q$-module with a pre-inner product $\langle -, -\rangle$ admitting a Hilbert basis $\Gamma$, then $\langle -, -\rangle$ is in fact an inner product. If a $Q$-module $M$ bears a (pre-)inner product admitting a Hilbert basis, then we speak of its *Hilbert structure*; the couple $(M, \langle -, -\rangle)$ is a *Hilbert module*. For starters we proved [Heymans and Stubbe, 2009b]:

**Theorem.** *The quantaloid* $\mathsf{Hilb}(Q)$*, whose objects are* $Q$*-modules with Hilbert structure and whose morphisms are module morphisms, is equivalent to the quantaloid* $\mathsf{Proj}(Q)$ *of* projection matrices[22] *with elements in* $Q$*.*

Now let $\mathcal{E} \subseteq \mathcal{Q}$ be the set of symmetric idempotents in $\mathcal{Q}$. It is a trivial but crucial observation that the formula

$$\langle m, n\rangle_{\mathsf{can}} := \bigvee \Big\{ (\zeta^*(m))^{\circ} \circ \zeta^*(n) \ \Big| \ e \in \mathcal{E},\ \zeta\colon Q^e \longrightarrow M \text{ left adjoint in } \mathsf{Mod}(Q) \Big\}$$

defines a pre-inner product, called the *canonical pre-inner product*, on $M$. It allows us to make a link between Hilbert modules and locally principally generated modules [Heymans and Stubbe, 2009b]:

**Theorem.** *Let* $Q$ *be an involutive quantale,* $\mathcal{E} \subseteq Q$ *the set of symmetric idempotents, and* $M$ *a* $Q$*-module. The following are equivalent:*

1. $M$ *is* $\mathcal{E}$*-principally generated and* $\mathcal{E}$*-principally symmetric,*
2. *the set* $\Gamma_{\mathsf{can}} := \{$*all elements of* $M$ *which are locally principal at some* $e \in \mathcal{E}\}$ *is a Hilbert basis for the canonical pre-inner product on* $M$*, called the* canonical Hilbert basis*.*

*In this case, it follows that the canonical pre-inner product is an inner product; we speak of the* canonical Hilbert structure on $M$*. In general a* $\mathcal{Q}$*-module* $M$ *may be equipped with several different Hilbert structures, but if* $Q$ *is a modular quantal frame*[23]*, then the* only possible *Hilbert structure is the canonical one. Therefore, for a modular quantal frame* $Q$ *there is an equivalence of quantaloids* $\mathsf{Hilb}(Q) \simeq \mathsf{Mod}_{\mathcal{E}\mathsf{pg},\mathcal{E}\mathsf{sym}}(Q)$*.*

For a locale $L$, the above result together with the results in the previous subsection implies in particular that

$$\mathsf{Sh}(L) \simeq \mathsf{Map}(\mathsf{Hilb}(L)).$$

More generally, any small quantaloid of closed cribles $\mathcal{R}(\mathcal{C}, J)$ is Morita-equivalent to a modular quantal frame $Q$, and there is in fact an equivalence

$$\mathsf{Sh}(\mathcal{C}, J) \simeq \mathsf{Map}(\mathsf{Hilb}(Q)).$$

And from Pedro Resende's results [2007, 2008][24] it follows that the classifying topos $BG$ of sheaves on an étale groupoid $G$ is equivalent to a category Hilbert modules,

$$BG \simeq \mathsf{Map}(\mathsf{Hilb}(\mathcal{O}(G))),$$

where $\mathcal{O}(G)$ is the so-called *inverse quantal frame* associated with $G$ (which, in the case of a topological groupoid, is the topology of the space of the groupoid arrows, equipped with pointwise multiplication of arrows). These three examples seem to indicate that, at least for a modular quantal frame $Q$, the category $\mathsf{Map}(\mathsf{Hilb}(Q))$ ought to be (equivalent to) "the category of (symmetrically ordered) sheaves on $Q$".

**Problem.** *Is there a "quantaloidal version" of André Joyal and Myles Tierney's [1984] theorem saying that any Grothendieck topos is the category of continuous* $\mathcal{G}$*-sets for some localic groupoid* $\mathcal{G}$*? That is to say, can we pass back and forth between small quantaloids of closed cribles on the one hand, and the inverse quantal frames on the other hand, in a purely quantaloid-theoretical manner?*

## Notes

1. An involution on a quantaloid $\mathcal{Q}$ is a homomorphism $(-)^{\circ}\colon \mathcal{Q}^{\mathsf{op}} \longrightarrow \mathcal{Q}$ which is the identity on objects and satisfies $f^{\circ\circ} = f$ for any morphism $f$ in $\mathcal{Q}$. An involution thus associates to any morphism $f\colon A \longrightarrow B$ in $\mathcal{Q}$ a morphism $f^{\circ}\colon B \longrightarrow A$ such that in particular the following conditions hold: $(f \circ g)^{\circ} = g^{\circ} \circ f^{\circ}$, if $f \leq g$ then $f^{\circ} \leq g^{\circ}$, and $f^{\circ\circ} = f$. Actually, these three conditions are also sufficient for $f \mapsto f^{\circ}$ to be a homomorphism. Clearly, a quantale is commutative if and only if the identity mapping is an involution; in that sense, an involutive quantaloid is "as symmetric a quantaloid as it gets".

2. It is noteworthy that many an example satisfies a stronger condition. Say that a quantaloid $\mathcal{Q}$ is *strongly Cauchy bilateral* when it is involutive (with involution $f \mapsto f^{\circ}$) and for any family $(f_i\colon X \longrightarrow X_i, g_i\colon X_i \longrightarrow X)_{i \in I}$ of morphisms in $\mathcal{Q}$,

$$1_X \leq \bigvee_i g_i \circ f_i \implies 1_X \leq \bigvee_i (f_i^{\circ} \wedge g_i) \circ (f_i \wedge g_i^{\circ}).$$

For a so-called integral quantaloid – i.e. when the top element of each $\mathcal{Q}(X,X)$ is $1_X$ – Cauchy bilaterality and strong Cauchy bilaterality are equivalent notions, but in general the latter is strictly stronger than the former. This condition is satisfied by the integral and commutative quantale $Q = ([0,\infty], \bigwedge, +, 0)$ with its trivial involution: for any family $(a_i, b_i)_{i \in I}$ of pairs of elements of $[0,\infty]$, if $\bigwedge_i (a_i + b_i) \leq 0$ is assumed then

$$\bigwedge_i (\max\{a_i,b_i\} + \max\{a_i,b_i\}) = 2 \cdot \bigwedge_i \max\{a_i,b_i\} \leq 2 \cdot \bigwedge_i (a_i + b_i) \leq 0.$$

This "explains" the well known fact that the Cauchy completion of a symmetric generalised metric space is again symmetric.

3. A Kock-Zöberlein-doctrine (or KZ-doctrine, for short) $\mathcal{T}$ on a locally ordered category $\mathcal{K}$ is a 2-functor $\mathcal{T}\colon \mathcal{K} \longrightarrow \mathcal{K}$ for which there are a multiplication $\mu\colon \mathcal{T} \circ \mathcal{T} \Longrightarrow \mathcal{T}$ and a unit $\eta\colon 1_{\mathcal{K}} \Longrightarrow \mathcal{T}$ making $(\mathcal{T}, \mu, \eta)$ a 2-monad, and satisfying moreover the "KZ-inequation": $\mathcal{T}(\eta_K) \leq \eta_{\mathcal{T}(K)}$ for all objects $K$ of $\mathcal{K}$. The notion was invented independently by Volker Zöberlein [1976] and Anders Kock [1972] in the more general setting of 2-categories; see [Kock, 1995] for details. A straightforward example is the following. Let $\mathsf{Ord}$ denote the locally ordered 2-category whose objects are ordered sets, morphisms are order-preserving functions, and 2-cells are given by pointwise order. The functor $\mathcal{D}\colon \mathsf{Ord} \longrightarrow \mathsf{Ord}$ that sends an ordered set $(X, \leq)$ to the orderd set $(\mathcal{D}(X), \subseteq)$ of down-closed subsets of $X$ is a KZ-doctrine. In fact, $\mathcal{D}(X)$ is the free complete suplattice on $X$. In the context of $\mathcal{Q}$-enriched categories, for $\mathcal{Q}$ any quantaloid, this is exactly generalised by the free cocompletion functor $\mathcal{P}\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$, sending a $\mathcal{Q}$-category $\mathbb{A}$ to the category $\mathcal{P}(\mathbb{A})$ of contravariant presheaves on $\mathbb{A}$. However, one may be interested, not in computing the *free* cocompletion by adding *all* colimits to a given $\mathcal{Q}$-category $\mathbb{A}$, but in a cocompletion that only adds colimits *of a certain type*. This can be achieved by setting up a so-called (saturated) class of weights, say $\mathcal{C}$, and computing the weighted cocompletion $\mathcal{C}(\mathbb{A})$ of a given $\mathcal{Q}$-category $\mathbb{A}$.

4. Precisely, a KZ-doctrine $(\mathcal{T}, \mu, \eta)$ on $\mathsf{Cat}(\mathcal{Q})$ is a full sub-KZ-doctrine of the free cocompletion doctrine $\mathcal{P}$ (whose unit is formed by the Yoneda embeddings $Y_{\mathbb{A}}: \mathbb{A} \longrightarrow \mathcal{P}(\mathbb{A})$) if and only if all $\eta_{\mathbb{A}}: \mathbb{A} \longrightarrow \mathcal{T}(\mathbb{A})$ are fully faithful and all left Kan extensions $\langle Y_{\mathbb{A}}, \eta_{\mathbb{A}} \rangle: \mathcal{T}(\mathbb{A}) \longrightarrow \mathcal{P}(\mathbb{A})$ are fully faithful.

5. The terminal $\mathcal{Q}$-category $\mathbb{T}$ has as objects precisely the objects of $\mathcal{Q}$, and for two such objects, say $X$ and $Y$, the hom-arrow $\mathbb{T}(Y, X)$ is the top element of the complete lattice $\mathcal{Q}(X, Y)$. The unique functor from a $\mathcal{Q}$-category $\mathbb{A}$ into $\mathbb{T}$ is $!: \mathbb{A} \longrightarrow \mathbb{T}: a \longrightarrow ta$, that is to say, it associates to any object $a$ of $\mathbb{A}$ its type-object in $\mathcal{Q}$.

6. More generally, for two objects $x, y$ in a $\mathcal{Q}$-category $\mathbb{C}$, it makes perfect sense to define $x \leq y$ to mean that $x$ and $y$ have the same type-object in $\mathcal{Q}$, say $X$, and that moreover $1_X \leq \mathbb{C}(x, y)$; this defines an order relation on the set $\mathbb{C}_0$ of objects of $\mathbb{C}$. However, precisely because two elements in the order relation are necessarily of the same type, there can never be a supremum of two (or more) objects of different type. Thus, it makes little sense to require that $(\mathbb{C}_0, \leq)$ be a suplattice, for this requires that all objects of $\mathbb{C}$ be of the same type. That is essentially why suprema can only exist "fibrewise".

7. That is to say, the universal property of a supremum in $\mathbb{C}_X$ is *not* expressed by a weighted colimit. To illustrate this, consider a family of elements $(x_i)_{i \in I}$ in $\mathbb{C}_X$. Next, let $\mathbb{I}$ be the free $\mathcal{Q}(X, X)$-enriched category on the ordered set $(I, \leq)$, where by definition $i \leq j$ holds precisely when $x_i \leq x_j$ holds in $(\mathbb{C}_X, \leq)$. There is then an obvious functor $F: \mathbb{I} \longrightarrow \mathbb{C}: i \longrightarrow x_i$, but to compute its "colimit", we must first append a weight to $F$. The only reasonable choice is to take the distributor $\phi: *_X \mathrel{-\!\!\circ\!\!\to} \mathbb{I}$ whose elements are $\phi(i) = 1_X$ for all $i$ (it is the "trivial" weight). The $\phi$-weighted colimit of $F: \mathbb{I} \longrightarrow \mathbb{C}$ (which may or may not exist) is called the *conical colimit of $F$*, or – because the construction of $F$ is canonical – simply the *conical colimit of the* $(x_i)_i$. If the conical colimit of the $(x_i)_i$ exists, then it turns out to be the supremum of the $(x_i)_i$ in the order $(\mathbb{C}_X, \leq)$; but the supremum of the $(x_i)_i$ may exist even though the conical colimit does not.

8. Let me recall the definitions of regular semicategory and regular presheaf [Stubbe, 2005b]. A regular $\mathcal{Q}$-semicategory $\mathbb{B}$ consists of a set $\mathbb{B}_0$ of 'objects' together with a 'type'-function $t: \mathbb{B}_0 \longrightarrow \mathcal{Q}_0$, and for any couple $(x, y)$ of objects a 'hom'-morphism $\mathbb{B}(y, x): tx \longrightarrow ty$ in $\mathcal{Q}$, such that the *saturated composition inequality* holds:

$$\bigvee_{y \in \mathbb{B}_0} \mathbb{B}(z, y) \circ \mathbb{B}(y, x) = \mathbb{B}(z, x)$$

for all $x, z \in \mathbb{B}_0$. A *regular semidistributor* $\Phi: \mathbb{B} \mathrel{-\!\!\circ\!\!\to} \mathbb{B}'$ between regular semicategories consists of $\mathcal{Q}$-morphisms $\Phi(x', x): tx \longrightarrow tx'$, one for each $(x, x') \in \mathbb{B}_0 \times \mathbb{B}'_0$, such that two *saturated action-inequalities* hold:

$$\bigvee_{x' \in \mathbb{B}'_0} \mathbb{B}'(y', x') \circ \Phi(x', x) = \Phi(y', x) = \bigvee_{y \in \mathbb{B}_0} \Phi(y', y) \circ \mathbb{B}(y, x)$$

for all $(x,y')\in\mathbb{B}_0\times\mathbb{B}'_0$. Regular $\mathcal{Q}$-semicategories and regular semidistributors from a quantaloid $\mathsf{RSDist}(\mathcal{Q})$ (composition of regular semidistributors is done with the usual "matrix"-formula), which contains the quantaloid $\mathsf{Dist}(\mathcal{Q})$ of $\mathcal{Q}$-categories and distributors. A *regular contravariant presheaf* of type $X$ on a regular $\mathcal{Q}$-semicategory $\mathbb{B}$ is, by definition, a regular semidistributor $\phi\colon *_X \longrightarrow\!\!\!\!\!\!\circ\;\; \mathbb{B}$ (where, as before, $*_X$ is the one-object $\mathcal{Q}$-category whose hom-arrow is $1_X$). The collection of such presheaves on $\mathbb{B}$ is written as $\mathcal{R}(\mathbb{B})$; endowed with canonical 'hom'-morphisms, it turns out to be a cocomplete $\mathcal{Q}$-category.

9. A $\mathcal{Q}$-module $\mathcal{F}$ is projective if and only if the representable homomorphism

$$\mathsf{Mod}(\mathcal{Q})(\mathcal{F},-)\colon \mathsf{Mod}(\mathcal{Q})\longrightarrow \mathsf{Sup}$$

preserves epimorphisms; this is really a straightforward reformulation of the definition of projectivity. A related notion is of much importance in the theory of $\mathcal{V}$-enriched categories [Kelly, 1982]: a $\mathcal{V}$-enriched presheaf on a $\mathcal{V}$-category $\mathcal{C}$, say $F\colon\mathcal{C}^{\mathsf{op}}\longrightarrow\mathcal{V}$, is *small-projective* when the representable $\mathcal{V}$-functor

$$\mathcal{V}\text{-}\mathsf{Fun}(\mathcal{C}^{\mathsf{op}},\mathcal{V})(F,-)\colon \mathcal{V}\text{-}\mathsf{Fun}(\mathcal{C}^{\mathsf{op}},\mathcal{V})\longrightarrow\mathcal{V}$$

preserves all small weighted colimits. This definition applies of course to the particular case $\mathcal{V}=\mathsf{Sup}$, so that we can speak of small-projective $\mathcal{Q}$-modules. Clearly a small-projective $\mathcal{Q}$-module is also projective (because a morphism is an epimorphism if and only if its pushout with itself consists of identities); so the surprise here is that the converse also holds.

10. Recall from the equivalence between $\mathsf{Cocont}(\mathcal{Q})$ and $\mathsf{Mod}(\mathcal{Q})$ that the objects of $\mathbb{A}$ are precisely precisely the elements of $\mathcal{F}$. An element $x$ of $\mathcal{F}$ is principal if and only if, as an object of $\mathbb{A}$, it is totally compact. Thus, $\mathcal{F}$ is principally generated ("generated by its principal elements") if and only if $\mathbb{A}$ is totally algebraic ("generated by its totally compact objects").

11. This implies that the principally generated modules on $\mathcal{Q}$ form a *closed class of colimit weights* in the sense of [Albert and Kelly, 1988; Kelly and Schmitt, 2005]; in fact, this class is the *closure* of the class of (weights for) direct sums and adjoint retracts. The general theory explained for $\mathcal{V}$-enriched categories in the cited references implies that, for any small quantaloid $\mathcal{Q}$, $\mathsf{Mod}_{\mathsf{pg}}(\mathcal{Q})$ is precisely the free cocompletion of $\mathcal{Q}$ for direct sums and adjoint retracts, or equivalently, the free cocompletion of $\mathcal{Q}$ for all colimits weighed by a principally generated module. Since $\mathsf{Dist}(\mathcal{Q})\simeq\mathsf{Mod}_{\mathsf{pg}}(\mathcal{Q})$, this at once describes the universal property of the distributor quantaloid too. In [Stubbe, 2005a] it is shown that $\mathsf{Dist}(\mathcal{Q})$ is the universal "direct sum and split monad" completion of $\mathcal{Q}$; but it is trivial that, in a quantaloid, splitting monads are the same thing as adjoint retracts. In the latter reference it is moreover shown that direct sums and splitting monads suffice to admit all lax limits and all lax colimits. Combining all this, it thus follows that the principally generated modules, as a class of weights, describe precisely the lax (co)completion of $\mathcal{Q}$.

12. With the benefit of hindsight [Betti and Carboni, 1983; Rosenthal, 1996], we can summarise Walters' [1982] construction as follows: (i) $\mathcal{R}(\mathcal{C})\subseteq\mathsf{Rel}(\mathsf{Set}^{\mathcal{C}^{\mathsf{op}}})$ is the full subquantaloid whose

objects are the representables, (ii) giving a topology $J$ on $\mathcal{C}$ is equivalent to giving a locally left exact nucleus $j$ on $\mathcal{R}(\mathcal{C})$:

$$j(R) = \Big\{(f,g) \;\Big|\; \exists S \in J : \forall s \in S, (fs, gs) \in R\Big\},$$

$$J(X) = \Big\{S \subseteq \mathcal{C}(-,X) \;\Big|\; \mathsf{id}_X \leq j\Big(\{(s,s) \mid s \in S\}\Big)\Big\},$$

and (iii) $\mathcal{R}(\mathcal{C}, J)$ is the quotient quantaloid $\mathcal{R}(\mathcal{C})_j$ of $j$-closed morphisms in $\mathcal{R}(\mathcal{C})$.

13. Walters originally called $\mathcal{R}(\mathcal{C}, J)$ the 'bicategory of relations' in $(\mathcal{C}, J)$, wrote it as $\mathsf{Rel}(\mathcal{C}, J)$, and called its arrows 'relations'. To avoid confusion with the 'bicategories of relations' that [Carboni and Walters, 1987] and others have since then worked on, I prefer to stick closer to the actual construction and speak of a 'quantaloid of closed cribles'.

14. Weak tabularity and weak modularity are new notions which inherited their name from the stronger notions of tabularity and modularity introduced in [Freyd and Scedrov, 1990].

15. Incidentally, every meet-semilattice $M$ carries a *canonical Grothendieck topology* which can be described explicitly as follows [Stubbe, 2005]: for $x \in M$,

$$J(x) = \left\{(x_i)_{i \in I} \;\middle|\; x = \bigvee_i x_i \text{ and for all } y \in M,\ y \wedge \bigvee_i x_i = \bigvee_i y \wedge x_i\right\}$$

is the set of covers of $x$. In words, the covering families are precisely the *distributive suprema*.

16. Thus, a locally principal element of $\mathcal{F}\colon \mathcal{Q}^{\mathsf{op}} \longrightarrow \mathsf{Sup}$ *at an identity of* $\mathcal{Q}$ is the same thing as, simply, a principal element of $\mathcal{F}$ (as in subsection 4.7): *idempotents* in $\mathcal{Q}$ are viewed as *localities* (or *"opens"*). It follows that a principally generated $\mathcal{Q}$-module is necessarily also locally principally generated; but the converse is not true in general.

17. Here I follow the notational convention of [Johnstone, 1982, p. 40] for morphisms in $\mathsf{Loc}$, but I do *not* follow the convention of [Johnstone, 1982; Mac Lane and Moerdijk, 1992] when it comes to defining an order on the hom-sets in $\mathsf{Loc}$. That is to say: here we have that $\mathsf{Loc} \cong \mathsf{Frm}^{\mathsf{coop}}$ as 2-categories, whereas the cited references have $\mathsf{Loc} \cong \mathsf{Frm}^{\mathsf{op}}$. The reason for this preference is in the first place notational convenience, especially for the 2-functors considered further on. However, there is maybe a deeper reason why this different ordering of locale morphisms is natural here. In the cited references locale morphisms are studied as inducing geometric morphisms between toposes of sheaves; the ordering of locale morphisms is chosen to correspond with the usual notion of natural transformation between geometric morphisms. We however shall study locale morphisms (or rather, morphisms in the slice category $\mathsf{Loc}/_X$) as inducing order-preserving morphisms between the (ordered) sheaves themselves; and the ordering of the locale morphisms is chosen to correspond with the natural ordering of those morphisms between sheaves.

18. Considering a locale $X$ as a monoid $(X, \wedge, \top)$ in $\mathsf{Sup}$ it makes sense to write $\mathsf{Mod}(X)$ for the quantaloid of modules on the locale. Instead of writing these modules as contravariant $\mathsf{Sup}$-enriched presheaves on the one-object suspension of the locale, we rather consider them as

objects of $\mathsf{Sup}$ on which $(X, \wedge, \top)$ acts on the right: we write $(M, \circ)$ for a $\mathsf{Sup}$-object $M$ together with the action $(m, x) \mapsto m \circ x$. In the same vein, an $X$-module morphism $\alpha\colon (M, \circ) \longrightarrow (N, \circ)$ is a $\mathsf{Sup}$-morphism $\alpha\colon M \longrightarrow N$ which is equivariant for the respective actions.

19. Putting $Z = X$ and $g = 1_X$ this reduces to what is called the "Frobenius identity" in [Mac Lane and Moerdijk, 1992, p. 500]; we call this generalisation "balanced" because we get the ("unbalanced") Frobenius identity by plugging in a terminal object. Precisely, an $h\colon Y \longrightarrow Z$ in $\mathsf{Loc}$ is open (according to the "usual" definition of openness as in e.g. [Mac Lane and Moerdijk, 1992, p. 500]) if and only if for any $f\colon Y \longrightarrow X$ and $g\colon Z \longrightarrow X$ in $\mathsf{Loc}$ such that $g \circ h = f$, the morphism $h\colon f \longrightarrow g$ in $\mathsf{Loc}/_X$ is skew open, if and only if for $h\colon Y \longrightarrow Z$ and $1_Z\colon Z \longrightarrow Z$ as objects in $\mathsf{Loc}/_Z$, the (unique) morphism $h\colon h \longrightarrow 1_Z$ in $\mathsf{Loc}/_Z$ is skew open.

20. If $f\colon Y \longrightarrow X$ is in $\mathsf{Loc}$ then the elements of the set $S_f(u) := \mathsf{Loc}/_X(i, f)$ are the *sections of $f$ at $u$* [Mac Lane and Moerdijk, 1992, p. 524]. This defines a sheaf $S_f\colon X^{\mathsf{op}} \longrightarrow \mathsf{Set}$, and this construction extends to a functor $\mathsf{Loc}/_X \longrightarrow \mathsf{Sh}(X)$ whose restriction to local homeomorphisms is an equivalence of categories. A particular feature of local homeomorphisms is that, whenever $f = g \circ h$ in $\mathsf{Loc}$, if $f$ and $g$ are local homeomorphisms then so is $h$; recall also that a local homeomorphism is always open in $\mathsf{Loc}$. Thus, if $f\colon Y \longrightarrow X$ is a local homeomorphism then every $s \in S_f(u)$ is an *open section* in the sense that $s\colon {\downarrow} u \longrightarrow Y$ is an open locale morphism. A morphism $f\colon Y \longrightarrow X$ in $\mathsf{Loc}$ is a local homeomorphism if and only if (the top element of) $Y$ can be covered by its open sections [Johnstone, 2002, vol. 2, p. 503], i.e.

$$\top_Y = \bigvee \left\{ s_!(u) \;\middle|\; u \in X,\ s \in S_f(u) \text{ and } s \text{ is open in } \mathsf{Loc} \right\}.$$

In this case, every $y \in Y$ can be covered by open sections of $f$, by taking the restrictions of the open sections of $f$ to $y$. In fact, every open section $s\colon {\downarrow} u \longrightarrow Y$ of a locale map $f\colon Y \longrightarrow X$ is necessarily skew open too; but the converse need not hold. However, if $f\colon Y \longrightarrow X$ is a local homeomorphism then $S_f(u) = S_f^{\circ}(u)$ for all $u \in X$. It is immediately clear that every local homeomorphism is a skew local homeomorphism. A skew local homeomorphism is a local homeomorphism if and only if its (skew open) sections are all open.

21. If $s \in S_f^{\circ}(u)$ is a skew open section of $f$ at $u \in X$ then $s_!(u) \in Y$ is a locally principal element of the $X$-module $(Y, \circ_f)$ at the idempotent $u \in (X, \wedge, \top_X)$. This actually determines a bijection.

22. A $Q$-matrix $\Lambda\colon S \longrightarrow T$ is an indexed set of elements of $Q$, $(\Lambda(y, x))_{(x,y) \in S \times T} \in Q$. Matrices compose straightforwardly with a "linear algebra formula", and this makes for a quantaloid $\mathsf{Matr}(Q)$. This construction makes sense for any quantale (and even quantaloid), and whenever $Q$ is involutive then so is $\mathsf{Matr}(Q)$: the involute of a matrix is computed elementwise. It then makes sense to define a *projection matrix* with elements in $Q$ to be a symmetric idempotent in $\mathsf{Matr}(Q)$. If $\Sigma\colon S \longrightarrow S$ is such a projection matrix, then

$$\mathcal{R}(\Sigma) := \{ f\colon S \longrightarrow Q \mid \forall s \in S : f(s) = \bigvee_{s \in S} \Sigma(s, x) \circ f(x) \}$$

is a $Q$-module with inner product and Hilbert basis respectively

$$\langle f, g\rangle := \bigvee_{s\in S} (f(s))^{\circ} \circ g(s) \text{ and } \Gamma := \{f_s : S \longrightarrow Q : x \mapsto \Sigma(x,s) \mid s \in S\}.$$

This object correspondence $\Sigma \mapsto \mathcal{R}(\Sigma)$ extends to a $\mathsf{Sup}$-functor from $\mathsf{Proj}(Q)$ to $\mathsf{Hilb}(Q)$ (in fact, it is the restriction to symmetric idempotent matrices of the embedding of the Cauchy completion of $Q$ qua one-object $\mathsf{Sup}$-category into $\mathsf{Mod}(Q)$). Conversely, a module $M$ with inner product $\langle -, - \rangle$ and Hilbert basis $\Gamma$ obviously determines a projection matrix $\Sigma : \Gamma \longrightarrow \Gamma$ with elements $\Sigma(s,t) := \langle s, t\rangle$; this easily extends to a $\mathsf{Sup}$-functor from $\mathsf{Hilb}(Q)$ to $\mathsf{Proj}(Q)$. These two functors set up the equivalence. A notable consequence of this equivalence is the existence of an involution on $\mathsf{Hilb}(Q)$, induced by the obvious involution on $\mathsf{Proj}(Q)$: the involute of a morphism $\phi : M \longrightarrow N$ in $\mathsf{Hilb}(Q)$ is the unique module morphism $\phi^{\circ} : N \longrightarrow M$ characterised by

$$\langle \phi(s), t\rangle = \langle s, \phi^{\circ}(t)\rangle$$

for all basis elements $s$ of $M$ and $t$ of $N$.

23. A *modular quantale* $Q$ is an involutive quantale which satisfies Freyd's modular law [Freyd and Scedrov, 1990]. Resende [2007] speaks of a *quantal frame* $Q$ whenever it is a quantale whose underlying lattice is a frame (= locale). The term *modular quantal frame* then speaks for itself. It is a matter of fact that modular quantal frames are precisely the one-object locally complete distributive allegories of Peter Freyd and Andre Scedrov [1990].

24. Meanwhile, Resende's [2008] preprint has been superseded by the article [Resende, 2011] which fixes a couple of imperfections from the original preprint. It contains in particular a complete proof of the equivalence $BG \simeq \mathsf{Map}(\mathsf{Hilb}(\mathcal{O}(G)))$ [Resende, 2011, p. 62–65], acknowledging the importance of our contributions [Heymans and Stubbe, 2009b, Theorem 4.1, Example 4.7(3)]. Resende [2011, Lemma 4.26, Theorem 4.29] also gives exactly the same proof of the equivalence $\mathsf{Hilb}(Q) \simeq \mathsf{Proj}(Q)$ as we did in [Heymans and Stubbe, 2009b, Example 3.7(4)] (whereas [Resende, 2008] had noted the object correspondence, but not the morphism correspondence, thus not the equivalence of quantaloids.)

## References


1. [Samson Abramsky and Achim Jung, 1994] Domain theory, *Handbook of logic in computer science (volume 3)*, Oxford University Press, pp. 1–168.

2. [Andrei Akhvlediani, Maria Manuel Clementino and Walter Tholen, 2010] On the categorical meaning of Hausdorff and Gromov distances, *Topology Appl.* **157**, pp. 1275–1295.

3. [Simon Ambler and Dominic Verity, 1996] Generalized logic and the representation of rings, *Appl. Categ. Structures* **4**, pp. 283–296.

4. [M. H. Albert and G. M. Kelly, 1988] The closure of a class of colimits, *J. Pure Appl. Algebra* **51**, pp. 1–17.

5. [Jean Bénabou, 1967] Introduction to bicategories, *Lecture Notes in Math.* **47**, pp. 1–77.

6. [Jean Bénabou, 1973] Les distributeurs, *Rapport des Séminaires de Mathématique Pure* **33**, Université Catholique de Louvain, Louvain-la-Neuve.

7. [Renato Betti, 1980] Automata and closed categories, *Boll. Un. Mat. Ital. B Serie V* **17**, pp. 44–58.

8. [Renato Betti and Aurelio Carboni, 1983] Notion of topology for bicategories, *Cahiers Top. et Géom. Diff.* **24**, pp. 19–22.

9. [Renato Betti, Aurelio Carboni, Ross H. Street and Robert F. C. Walters, 1983] Variation through enrichment, *J. Pure Appl. Algebra* **29**, pp. 109–127.

10. [Francis Borceux, 1994] *Handbook of Categorical Algebra (3 volumes)*, Cambridge University Press, Cambridge.

11. [Francis Borceux and Rosanna Cruciani, 1998] Skew Ω-sets coincide with Ω-posets, *Cahiers Topol. Géom. Différ. Catég.* **39**, pp. 205–220.

12. [Francis Borceux and Gilberte Van den Bossche, 1986] Quantales and their sheaves, *Order* **3**, pp. 61–87.

13. [Aurelio Carboni and Robert F. C. Walters, 1987] Cartesian bicategories I, *J. Pure Appl. Algebra* **49**, pp. 11–32.

14. [Maria Manuel Clementino, Dirk Hofmann and Isar Stubbe, 2009] *Exponentiable functors between quantaloid-enriched categories*, Appl. Categ. Structures **17**, pp. 91–101.

15. [Maria Manuel Clementino and Dirk Hofmann, 2006] Exponentiation in $\mathcal{V}$-categories, *Topology Appl.* **153**, pp. 3113–3128.

16. [Marcelo E. Coniglio and Francisco Miraglia, 2001] Modules in the category of sheaves over quantales, *Ann. Pure Appl. Logic* **108**, Issues 1-3, pp. 103–136.

17. [Peter J. Freyd and Andre Scedrov, 1990] *Categories, Allegories*, North-Holland Mathematical Library **39**, Amsterdam.

18. [W. Dale Garraway, 2005] Sheaves for an involutive quantaloid, *Cah. Topol. Géom. Différ. Catég.* **46**, pp. 243–274.

19. [Robert Gordon and A. John Power, 1997] Enrichment through variation, *J. Pure Appl. Algebra* **120**, pp. 167–185.

20. [Robert Gordon and A. John Power, 1999] Gabriel-Ulmer duality for categories enriched in bicategories, *J. Pure Appl. Algebra* **137**, pp. 29–48.

21. [Remigijus P. Gylys, 2001] Sheaves on involutive quantaloids, *Liet. Mat. Rink.* **41**, pp. 44–69.

22. [Hans Heymans and Isar Stubbe, 2009] On principally generated $\mathcal{Q}$-modules in general, and skew local homeomorphisms in particular, *Ann. Pure Appl. Logic* **161**, pp. 43–65.

23. [Hans Heymans and Isar Stubbe, 2009] Modules on involutive quantales: canonical Hilbert structure, applications to sheaves, *Order* **26**, pp. 177–196.

24. [Hans Heymans and Isar Stubbe, 2011] Symmetry and Cauchy completion of quantaloid-enriched categories, *Theory Appl. Categ.* **25**, pp. 276–294.

25. [Hans Heymans and Isar Stubbe, 2011] Elementary characterisation of quantaloids of closed cribles, *J. Pure Appl. Algebra*, to appear, 13 pages.

26. [Denis Higgs, 1973] A category approach to boolean valued set theory, Lecture Notes, University of Waterloo.

27. [Denis Higgs, 1984] Injectivity in the topos of complete Heyting algebra valued sets, *Canad. J. Math.* **36**, pp. 550–568.

28. [Ulrich Höhle, 1998] GL-quantales: $Q$-valued sets and their singletons, *Studia Logica* **61**, pp. 123–148.

29. [Peter T. Johnstone, 1982] *Stone spaces*, Cambridge University Press, Cambridge.

30. [Peter T. Johnstone, 2002] *Sketches of an elephant: a topos theory compendium (2 volumes published, 3rd in preparation)*, Oxford Logic Guides, Oxford University Press, New York.

31. [André Joyal and Myles Tierney, 1984] An extension of the Galois theory of Grothendieck, *Mem. Amer. Math. Soc.* **51**.

32. [Anders Kock, 1972] Monads for which structures are adjoint to units (Version 1), *Aarhus Preprint Series* **35**.

33. [Anders Kock, 1995] Monads for which structures are adjoint to units, *J. Pure Appl. Algebra* **104**, pp. 41–59.

34. [G. Max Kelly, 1982] *Basic concepts of enriched category theory*, Cambridge University Press, Cambridge.

35. [G.M. Kelly and V. Schmitt, 2005] Notes on enriched categories with colimits of some class, *Theory Appl. Categ.* **14**, pp. 399–423.

36. [F. William Lawvere, 1973] Metric spaces, generalized logic and closed categories, *Rend. Sem. Mat. Fis. Milano* **43**, pp. 135–166.

37. [F. William Lawvere, 2002] Reprint of [Lawvere, 1973] with an author commentary: Enriched categories in the logic of geometry and analysis, *Repr. Theory Appl. Categ.* **1**, pp. 1–37.

38. [Saunders Mac Lane and Ieke Moerdijk, 1992] *Sheaves in geometry and logic*, Springer-Verlag, New York.

39. [Christian J. Mikkelsen, 1976] Lattice theoretic and logical aspects of elementary topoi, *Various Publications Series* **25**, Matematisk Institut, Aarhus University, Aarhus.

40. [Christopher J. Mulvey and Mohammed Nawaz, 1995] Quantales: quantal sets, *Non-Classical Logics and their Application to Fuzzy Subsets: A Handbook of the Mathematical Foundations of Fuzzy Set Theory*, Kluwer, pp. 159–217.

41. [Jan Paseka, 1999] Hilbert $Q$-modules and nuclear ideals in the category of $\bigvee$-semilattices with a duality, *Electr. Notes Theor. Comput. Sci.* **29**, pp. 1–19.

42. [Andrew M. Pitts, 1988] Applications of sup-lattice enriched category theory to sheaf theory, *Proc. London Math. Soc.* **57**, pp. 433–480.

43. [Pedro Resende, 2007] Etale groupoids and their quantales, *Adv. Math.* **208**, pp. 147–209.

44. [Pedro Resende, 2008] Groupoid sheaves as Hilbert modules. Preprint available on the arXiv: 0807.4848v1.

45. [Pedro Resende, 2011] Groupoid sheaves as quantale sheaves, *J. Pure Appl. Algebra* **216**, pp. 41–70.

46. [Pedro Resende and Elias Rodrigues, 2010] Sheaves as modules, *Appl. Categ. Structures* **18**, pp. 199–217.

47. [Gonzalo E. Reyes, 1977] Sheaves and concepts: a model-theoretic interpretation of Grothendieck topoi, *Cahiers Topol. Géom. Différ. Catég.* **18**, pp. 105–137.

48. [Robert Rosebrugh and Richard J. Wood, 1994] Constructive complete distributivity IV, *Appl. Categ. Structures* **2**, pp. 119–144.

49. [Kimmo I. Rosenthal, 1995] Quantaloids, enriched categories and automata theory, *Appl. Categ. Structures* **3**, pp. 279–301.

50. [Kimmo I. Rosenthal, 1996] *The theory of quantaloids*, Pitman Research Notes in Mathematics Series **348**, Longman, Harlow.

51. [Vincent Schmitt, 2009] Completions of non-symmetric metric spaces via enriched categories, *Georgian Math. J.* **16**, pp. 157–182.

52. [Ross H. Street, 1981] Cauchy characterization of enriched categories, *Rend. Sem. Mat. Fis. Milano* **51**, pp. 217–233.

53. [Ross H. Street, 1983] Enriched categories and cohomology, *Questiones Math.* **6**, pp. 265–283.

54. [Ross H. Street, 2002] Reprint of [Street, 1983] with an author commentary, *Reprints Theory Appl. Categ.* **14**, p. 1–18.

55. [Isar Stubbe, 2005], Categorical structures enriched in a quantaloid: categories, distributors and functors, *Theory Appl. Categ.* **14**, pp. 1–45.

56. [Isar Stubbe, 2005] Categorical structures enriched in a quantaloid: regular presheaves, regular semicategories, *Cahiers Topol. Géom. Différ. Catég.* **46**, pp. 99–121.

57. [Isar Stubbe, 2005] Categorical structures enriched in a quantaloid: orders and ideals over a base quantaloid, *Appl. Categ. Structures* **13**, pp. 235–255.

58. [Isar Stubbe, 2005] The canonical topology on a meet-semilattice, *Int. J. Theor. Phys.* **44**, pp. 2283–2293.

59. [Isar Stubbe, 2006] Categorical structures enriched in a quantaloid: tensored and cotensored categories, *Theory Appl. Categ.* **16**, pp. 283–306.

60. [Isar Stubbe, 2007] Towards 'dynamic domains': totally continuous cocomplete Q-categories, *Theor. Comp. Sci.* **373**, pp. 142–160.

61. [Isar Stubbe, 2007] Q-modules are Q-suplattices, *Theory Appl. Categ.* **19**, pp. 50–60.

62. [Isar Stubbe, 2010] 'Hausdorff distance' via conical cocompletions, *Cahiers Topol. Géom. Différ. Catég.* **51**, pp. 51–76.

63. [Robert F. C. Walters, 1981] Sheaves and Cauchy-complete categories, *Cahiers Topol. Géom. Différ. Catég.* **22**, pp. 283–286.

64. [Robert F. C. Walters, 1982] Sheaves on sites as cauchy complete categories, *J. Pure Appl. Algebra* **24**, pp. 95–102.

65. [Richard J. Wood, 1982] Proarrows, *Cahiers Topol. Géom. Différ. Catég.* **23**, pp. 279–290.

66. [Volker Zöberlein, 1976] Doctrines on 2-categories, *Math. Z.* **148**, pp. 267–279.

# Selected publications

Hereafter I provide copies of my publications that I referred to in the previous chapter. In chronological order they are:

*Theory and Applications of Categories,* Vol. 14, No. 1, 2005, pp. 1–45.

# CATEGORICAL STRUCTURES ENRICHED IN A QUANTALOID: CATEGORIES, DISTRIBUTORS AND FUNCTORS

ISAR STUBBE

ABSTRACT. We thoroughly treat several familiar and less familiar definitions and results concerning categories, functors and distributors enriched in a base quantaloid $\mathcal{Q}$. In analogy with $\mathcal{V}$-category theory we discuss such things as adjoint functors, (pointwise) left Kan extensions, weighted (co)limits, presheaves and free (co)completion, Cauchy completion and Morita equivalence. With an appendix on the universality of the quantaloid $\mathsf{Dist}(\mathcal{Q})$ of $\mathcal{Q}$-enriched categories and distributors.

## 1. Introduction

The theory of categories enriched in a symmetric monoidal closed category $\mathcal{V}$ is, by now, well known [Bénabou, 1963, 1965; Eilenberg and Kelly, 1966; Lawvere, 1973; Kelly, 1982]. For such a $\mathcal{V}$ with "enough" (co)limits the theory of $\mathcal{V}$-categories, distributors and functors can be pushed as far as needed: it includes such things as (weighted) (co)limits in a $\mathcal{V}$-category, $\mathcal{V}$-presheaves on a $\mathcal{V}$-category, Kan extensions of enriched functors, Morita theory for $\mathcal{V}$-categories, and so on.

Monoidal categories are precisely one-object bicategories [Bénabou, 1967]. It is thus natural to ask how far $\mathcal{V}$-category theory can be generalized to $\mathcal{W}$-category theory, for $\mathcal{W}$ a general bicategory. But, whereas in $\mathcal{V}$-category theory one usually assumes the symmetry of the tensor in $\mathcal{V}$ (which is essential for showing that $\mathcal{V}$ is itself a $\mathcal{V}$-category with hom-objects given by the right adjoint to tensoring), in working over a general bicategory $\mathcal{W}$ we will have to sacrifice this symmetry: tensoring objects in $\mathcal{V}$ corresponds to composing morphisms in $\mathcal{W}$ and in general it simply does not make sense for the composition $g \circ f$ of two arrows $f, g$ to be "symmetric".

On the other hand, we can successfully translate the notion of closedness of a monoidal category $\mathcal{V}$ to the more general setting of a bicategory $\mathcal{W}$: ask that, for any object $X$ of $\mathcal{W}$ and any arrow $f : A \longrightarrow B$ in $\mathcal{W}$, both functors

$$- \circ f : \mathcal{W}(B, X) \longrightarrow \mathcal{W}(A, X) : x \mapsto x \circ f, \tag{1}$$

$$f \circ - : \mathcal{W}(X, A) \longrightarrow \mathcal{W}(X, B) : x \mapsto f \circ x \tag{2}$$

have respective right adjoints

$$\{f, -\} : \mathcal{W}(A, X) \longrightarrow \mathcal{W}(B, X) : y \mapsto \{f, y\}, \tag{3}$$

Received by the editors 2004-05-13 and, in revised form, 2004-01-14.
Transmitted by Jiri Rosicky. Published on 2005-01-18.
2000 Mathematics Subject Classification: 06F07, 18B35, 18D20.
Key words and phrases: Quantales and quantaloids, enriched categories.

CAHIERS DE TOPOLOGIE ET
GEOMETRIE DIFFERENTIELLE CATEGORIQUES

Volume XLVI-2 (2005)

# CATEGORICAL STRUCTURES ENRICHED IN A QUANTALOID: REGULAR PRESHEAVES, REGULAR SEMICATEGORIES

*by Isar STUBBE*

**Résumé.** On étudie les préfaisceaux sur des semicatégories enrichies dans un quantaloïde: cela donne lieu à la notion de préfaisceau régulier. Une semicatégorie est régulière si tous les préfaisceaux représentables sont réguliers, et ses préfaisceaux réguliers forment alors une (co)localisation essentielle de la catégorie de tous ses préfaisceaux. La notion de semidistributeur régulier permet d'établir l'équivalence de Morita des semicatégories régulières. Les ordres continus et les Ω-ensembles fournissent des exemples.

**Mots clés:** quantaloïde, semicatégorie, préfaisceau, régularité, équivalence de Morita, ordre continu, Ω-ensemble

**Keywords:** quantaloid, semicategory, presheaf, regularity, Morita equivalence, continuous order, Ω-set

**AMS Subject Classification (2000):** 06F07, 18B35, 18D05, 18D20

## 1 Introduction

In [Moens *et al.*, 2002] the theory of regular modules on an $R$-algebra without unit, for $R$ a commutative ring, was generalized to a theory of regular presheaves on a $\mathcal{V}$-enriched semicategory, for $\mathcal{V}$ a symmetric monoidal closed base category. As a monoidal category $\mathcal{V}$ is a one-object bicategory, it is natural to ask in how far in the above the base $\mathcal{V}$ can be replaced by a bicategory $\mathcal{W}$ (thus necessarily loosing symmetry of the tensor). Here we present such a theory of regular presheaves on a $\mathcal{Q}$-enriched semicategory, where now $\mathcal{Q}$ is any (small) quantaloid.

A quantaloid is a Sup-enriched category; it is thus in particular a bicategory. There is a theory of categories enriched in a quantaloid $\mathcal{Q}$, as particular case of categories enriched in a bicategory. A presentation thereof is given in [Stubbe, 2004a] which is our reference for all the basic notions and results concerning $\mathcal{Q}$-categories that we may need further on. A $\mathcal{Q}$-semicategory is then simply a "$\mathcal{Q}$-category without unit-inequalities".

A presheaf on a $\mathcal{Q}$-semicategory $\mathbb{A}$ is formally the same thing as a presheaf on the free $\mathcal{Q}$-category on $\mathbb{A}$. Thus the presheaves on $\mathbb{A}$ constitute a $\mathcal{Q}$-category $\mathcal{P}\mathbb{A}$.

Applied Categorical Structures (2005) 13: 235–255
DOI: 10.1007/s10485-004-7421-5


# Categorical Structures Enriched in a Quantaloid: Orders and Ideals over a Base Quantaloid

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(Received: 12 August 2004; accepted: 9 December 2004)

**Abstract.** Applying (enriched) categorical structures we define the notion of ordered sheaf on a quantaloid $\mathcal{Q}$, which we call '$\mathcal{Q}$-order'. This requires a theory of semicategories enriched in the quantaloid $\mathcal{Q}$, that admit a suitable Cauchy completion. There is a quantaloid $\mathsf{Idl}(\mathcal{Q})$ of $\mathcal{Q}$-orders and ideal relations, and a locally ordered category $\mathsf{Ord}(\mathcal{Q})$ of $\mathcal{Q}$-orders and monotone maps; actually, $\mathsf{Ord}(\mathcal{Q}) = \mathsf{Map}(\mathsf{Idl}(\mathcal{Q}))$. In particular is $\mathsf{Ord}(\Omega)$, with $\Omega$ a locale, the category of ordered objects in the topos of sheaves on $\Omega$. In general $\mathcal{Q}$-orders can equivalently be described as Cauchy complete categories enriched in the split-idempotent completion of $\mathcal{Q}$. Applied to a locale $\Omega$ this generalizes and unifies previous treatments of (ordered) sheaves on $\Omega$ in terms of $\Omega$-enriched structures.

**Mathematics Subject Classifications (2000):** 06F07, 18B35, 18D05, 18D20.

**Key words:** quantaloid, quantale, locale, ordered sheaf, enriched categorical structure, Cauchy completion.

## 1. Introduction

An "ordered set" is usually thought of as a set equipped with a reflexive and transitive relation; that is to say, it is an ordered object in $\mathsf{Set}$. But one can also treat an order $(A, \leq)$ by means of the classifying map for its order relation, say $[\cdot \leq \cdot] : A \times A \to \mathbf{2}$, where now $\mathbf{2}$ is the object of truth values. This takes us into the realm of enriched categorical structures, for the reflexivity and transitivity axioms on the order relation translate into unit-inequalities and composition-inequalities for the enrichment $[\cdot \leq \cdot]$ of $A$ over $\mathbf{2}$. So order theory is then a matter of applied (enriched) categorical structures.

More generally, an "ordered sheaf on a locale $\Omega$" is an ordered object in the topos $\mathsf{Sh}(\Omega)$ of sheaves on the locale. Here too one may attempt at describing such an $\Omega$-order $(A, \leq)$ in terms of enriched categorical structures. There are two approaches: Walters [8] (implicitly) treats such $\Omega$-orders as *categories* enriched in $\mathsf{Rel}(\Omega)$; whereas Borceux and Cruciani [2] prefer to work with *semicategories* enriched in $\Omega$. The first option has the advantage that it speaks of categories enriched in a quantaloid, a clear and transparent theory that may be developed along the lines of the well-known theory of $\mathcal{V}$-enriched categories; but it has the disadvantage that

*International Journal of Theoretical Physics, Vol. 44, No. 12, December 2005 (© 2005)*
DOI: 10.1007/s10773-005-8023-1

# The Canonical Topology on a Meet-Semilattice

**Isar Stubbe**[1]

*Received ; accepted*

Considering the lattice of properties of a physical system, it has been argued elsewhere that—to build a calculus of propositions having a well-behaved notion of disjunction (and implication)—one should consider a very particular frame completion of this lattice. We show that the pertinent frame completion is obtained as sheafification of the presheaves on the given meet-semilattice with respect to its canonical Grothendieck topology, an explicit description of which is easily given. Our conclusion is that there is an intrinsic categorical quality to the notion of "disjunction" in the context of property lattices of physical systems.

**KEY WORDS:** meet-semilattice; topology; sheaf; frame completion.

**PACS:** 02.10.Ab; 03.65.Fd.

## 1. DISTRIBUTIVE JOINS IN A PROPERTY LATTICE ARE DISJUNCTIONS

Following Piron (1972, 1976, 1990) in his operational approach to (quantum) physics, a physical system is described by its properties, the actuality of each property being tested by a definite experimental project. The collection of properties (actual or not) of a system forms a complete lattice $(L, \leq)$: the order relation in $L$ is the "implication of actuality" of properties ($a \leq b$ in $L$ means that $b$ is actual whenever $a$ is), and it is a matter of fact that the infimum in $L$ is the conjunction of properties ($\wedge_i a_i$ in $L$ is actual if and only if every $a_i$ is actual). The state of a system is defined as the collection of all of its actual properties; but it is easily seen that one can identify a state $\varepsilon \subset L$ with $p_\varepsilon := \wedge \varepsilon \in L$. When denoting by $S$ the set of all possible states of a given physical system, Aerts (1982) put forward that $S$ is a *space* rather than just a set, its structure coming from the so-called "Cartan map"

$$\mu : L \to \mathcal{P}(S) : a \mapsto \{\varepsilon \in S \mid p_\varepsilon \leq a\}.$$

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0020-7748/05/1200-2283/0 © 2005 Springer Science+Business Media, Inc.

*Theory and Applications of Categories,* Vol. 16, No. 14, 2006, pp. 283–306.

# CATEGORICAL STRUCTURES ENRICHED IN A QUANTALOID: TENSORED AND COTENSORED CATEGORIES

ISAR STUBBE

ABSTRACT. A quantaloid is a sup-lattice-enriched category; our subject is that of categories, functors and distributors enriched in a base quantaloid $\mathcal{Q}$. We show how cocomplete $\mathcal{Q}$-categories are precisely those which are tensored and conically cocomplete, or alternatively, those which are tensored, cotensored and 'order-cocomplete'. In fact, tensors and cotensors in a $\mathcal{Q}$-category determine, and are determined by, certain adjunctions in the category of $\mathcal{Q}$-categories; some of these adjunctions can be reduced to adjuctions in the category of ordered sets. Bearing this in mind, we explain how tensored $\mathcal{Q}$-categories are equivalent to order-valued closed pseudofunctors on $\mathcal{Q}^{\mathsf{op}}$; this result is then finetuned to obtain in particular that cocomplete $\mathcal{Q}$-categories are equivalent to sup-lattice-valued homomorphisms on $\mathcal{Q}^{\mathsf{op}}$ (a.k.a. $\mathcal{Q}$-modules).

## Introduction

The concept of "category enriched in a bicategory $\mathcal{W}$" is as old as the definition of bicategory itself [Bénabou, 1967]; however, J. Bénabou called them "polyads". Taking a $\mathcal{W}$ with only one object gives a monoidal category, and for symmetric monoidal closed $\mathcal{V}$ the theory of $\mathcal{V}$-categories is well developed [Kelly, 1982]. But also categories enriched in a $\mathcal{W}$ with more than one object are interesting. R. Walters [1981] observed that sheaves on a locale give rise to bicategory-enriched categories: "variation" (sheaves on a locale $\Omega$) is related to "enrichment" (categories enriched in $\mathsf{Rel}(\Omega)$). This insight was further developed in [Walters, 1982], [Street, 1983] and [Betti *et al.*, 1983]. Later [Gordon and Power, 1997, 1999] complemented this work, stressing the important rôle of tensors in bicategory-enriched categories.

Here we wish to discuss "variation and enrichment" in the case of a base quantaloid $\mathcal{Q}$ (a small sup-lattice-enriched category). This is, of course, a particular case of the above, but we believe that it is also of particular interest; many examples of bicategory-enriched categories (like Walters') are really quantaloid-enriched. Since in a quantaloid $\mathcal{Q}$ every diagram of 2-cells commutes, many coherence issues disappear, so the theory of $\mathcal{Q}$-enriched categorical structures is very transparent. Moreover, by definition a quantaloid $\mathcal{Q}$ has stable local colimits, hence (by local smallness) it is closed; this is of great help when working with $\mathcal{Q}$-categories. The theory of quantaloids is documented in [Rosenthal, 1996]; examples and applications of quantaloids abound in the literature; and [Stubbe,

Received by the editors 2005-03-14 and, in revised form, 2006-06-14.
Transmitted by Ross Street. Published on 2006-06-19.
2000 Mathematics Subject Classification: 06F07, 18D05, 18D20.
Key words and phrases: quantaloid, enriched category, weighted (co)limit, module.

Available online at www.sciencedirect.com
ScienceDirect
Theoretical Computer Science 373 (2007) 142–160
Theoretical Computer Science
www.elsevier.com/locate/tcs

# Towards "dynamic domains": Totally continuous cocomplete $\mathcal{Q}$-categories

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Received 10 October 2005; received in revised form 28 August 2006; accepted 6 January 2007

Communicated by M.W. Mislove

**Abstract**

It is common practice in both theoretical computer science and theoretical physics to describe the (static) logic of a system by means of a complete lattice. When formalizing the dynamics of such a system, the updates of that system organize themselves quite naturally in a quantale, or more generally, a quantaloid. In fact, we are led to consider cocomplete quantaloid-enriched categories as a fundamental mathematical structure for a dynamic logic common to both computer science and physics. Here we explain the theory of totally continuous cocomplete categories as a generalization of the well-known theory of totally continuous suplattices. That is to say, we undertake some first steps towards a theory of "dynamic domains".


*Keywords:* Quantaloid-enriched category; Module; Projectivity; Small-projectivity; Complete distributivity; Total continuity; Total algebraicity; Dynamic domain; Dynamic logic

## 1. Introduction

**Towards "dynamic domains".** It is common practice in both theoretical computer science and theoretical physics to describe the 'properties' of a 'system' by means of a complete lattice $\mathcal{L}$; this lattice is then thought of as the logic of the system. For example, the lattice of closed subspaces of a Hilbert space is the logic of properties of a quantum system; and, in computer science, a domain is the logics of observables of a computational system.

More recently, also another ordered structure has been recognized to play an important rôle in both physics and computer science: when formalizing the dynamics of a physical or computational system, it turns out that the 'updates' of a system – think of them as programs for a computational system, and property transitions for a physical system – organize themselves quite naturally in a quantale $\mathcal{Q}$ [2,8].

Having a complete lattice $\mathcal{L}$ of properties of a system and a quantale $\mathcal{Q}$ of updates, we give an operational meaning to each $f \in \mathcal{Q}$ by the so-called Principle of Causal Duality (explained in detail in [18] but going back to [10,12] for

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0304-3975/$ - see front matter 
doi:10.1016/j.tcs.2007.01.002

*Theory and Applications of Categories,* Vol. 19, No. 4, 2007, pp. 50–60.

# $\mathcal{Q}$-MODULES ARE $\mathcal{Q}$-SUPLATTICES

ISAR STUBBE

ABSTRACT. It is well known that the internal suplattices in the topos of sheaves on a locale are precisely the modules on that locale. Using enriched category theory and a lemma on KZ doctrines we prove (the generalization of) this fact in the case of ordered sheaves on a small quantaloid. Comparing module-equivalence with sheaf-equivalence for quantaloids and using the notion of centre of a quantaloid, we refine a result of F. Borceux and E. Vitale.

## 1. Introduction

When studying topos theory one inevitably must study order theory too: if only because many advanced features of topos theory depend on order-theoretic arguments using the internal Heyting algebra structure of the subobject classifier in a topos, as C. J. Mikkelsen [1976] illustrates plainly. One of the results of [Mikkelsen, 1976] states that an ordered object in an elementary topos $\mathcal{E}$ is cocomplete, i.e. it is an internal suplattice, if and only if the "principal downset embedding" from that object to its powerobject has a left adjoint in $\mathsf{Ord}(\mathcal{E})$. In the case of a localic topos, it turns out that the internal suplattices in $\mathsf{Sh}(\Omega)$ are precisely the $\Omega$-modules, and supmorphisms are just the module morphisms [Joyal and Tierney, 1984; Pitts, 1988].

Now consider quantaloids (i.e. $\mathsf{Sup}$-enriched categories) as non-commutative, multi-typed generalization of locales. Using the theory of categories enriched in a quantaloid, and building further on results by B. Walters [1981] and F. Borceux and R. Cruciani [1998], I. Stubbe [2005b] proposed the notion of *ordered sheaf on a (small) quantaloid* $\mathcal{Q}$ (or $\mathcal{Q}$*-order* for short): one of several equivalent ways of describing a $\mathcal{Q}$-order is to say that it is a Cauchy complete category enriched in the split-idempotent completion of $\mathcal{Q}$. There is thus a locally ordered category $\mathsf{Ord}(\mathcal{Q})$ of $\mathcal{Q}$-orders and functors between them. If one puts $\mathcal{Q}$ to be the one-object suspension of a locale $\Omega$, then $\mathsf{Ord}(\Omega)$ is equivalent to $\mathsf{Ord}(\mathsf{Sh}(\Omega))$. (And if one puts $\mathcal{Q}$ to be the one-object suspension of the Lawvere reals $[0,\infty]$, then $\mathsf{Ord}([0,\infty])$ is equivalent to the category of Cauchy complete generalized metric spaces.)

In this paper we shall explain how $\mathsf{Mod}(\mathcal{Q})$, the quantaloid of $\mathcal{Q}$-modules, is the category of Eilenberg-Moore algebras for the KZ doctrine on $\mathsf{Ord}(\mathcal{Q})$ that sends a $\mathcal{Q}$-order $\mathbb{A}$ to its free cocompletion $\mathcal{P}\mathbb{A}$. The proof of this fact is, altogether, quite straightforward:

Isar Stubbe is a Postdoctoral Fellow of the Research Foundation Flanders (FWO).

Received by the editors 2007-01-15 and, in revised form, 2007-07-15.
Transmitted by Stephen Lack. Published on 2007-07-25.
2000 Mathematics Subject Classification: 06F07, 18D05, 18D20.
Key words and phrases: Quantaloid, quantale, locale, ordered sheaf, module, centre, KZ doctrine.

Appl Categor Struct (2009) 17:91–101
DOI 10.1007/s10485-007-9104-5

# Exponentiable Functors Between Quantaloid-Enriched Categories

**Maria Manuel Clementino · Dirk Hofmann · Isar Stubbe**

Received: 27 October 2006 / Accepted: 17 August 2007 / Published online: 7 September 2007


**Abstract** Exponentiable functors between quantaloid-enriched categories are characterized in elementary terms. The proof goes as follows: the elementary conditions on a given functor translate into existence statements for certain adjoints that obey some lax commutativity; this, in turn, is precisely what is needed to prove the existence of partial products with that functor; so that the functor's exponentiability follows from the works of Niefield (J. Pure Appl. Algebra 23:147–167, 1982) and Dyckhoff and Tholen (J. Pure Appl. Algebra 49:103–116, 1987).

**Keywords** Quantaloid · Enriched category · Exponentiability · Partial product

**Mathematics Subject Classifications (2000)** 06F07 · 18A22 · 18D05 · 18D20

## 1 Introduction

The study of exponentiable morphisms in a category $\mathcal{C}$, in particular of exponentiable functors between (small) categories (i.e. Conduché fibrations), has a long history; see [7] for a short account. Recently M. M. Clementino and D. Hofmann [3] found simple necessary-and-sufficient conditions for the exponentiability of a functor between $\mathcal{V}$-enriched categories, where $\mathcal{V}$ is a symmetric quantale which has its top element as

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Annals of Pure and Applied Logic 161 (2009) 43–65

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

journal homepage: www.elsevier.com/locate/apal

# On principally generated quantaloid-modules in general, and skew local homeomorphisms in particular

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ARTICLE INFO

*Article history:*
Received 5 March 2008
Received in revised form 2 April 2008
Accepted 3 May 2009
Available online 30 May 2009
Communicated by I. Moerdijk

*MSC:*
06D22
06F07
18C20
18D20

*Keywords:*
Quantaloid
Ordered sheaf
Principally generated module
Skew local homeomorphism

ABSTRACT

Ordered sheaves on a small quantaloid $\mathcal{Q}$ have been defined in terms of $\mathcal{Q}$-enriched categorical structures; they form a locally ordered category $\mathsf{Ord}(\mathcal{Q})$. The free-cocompletion KZ-doctrine on $\mathsf{Ord}(\mathcal{Q})$ has $\mathsf{Mod}(\mathcal{Q})$, the quantaloid of $\mathcal{Q}$-modules, as its category of Eilenberg–Moore algebras. In this paper we give an intrinsic description of the Kleisli algebras: we call them the *locally principally generated $\mathcal{Q}$-modules*. We deduce that $\mathsf{Ord}(\mathcal{Q})$ is biequivalent to the 2-category of locally principally generated $\mathcal{Q}$-modules and left adjoint module morphisms. The example of locally principally generated modules on a locale $X$ is worked out in full detail: relating $X$-modules to objects of the slice category $\mathsf{Loc}/_X$, we show that ordered sheaves on $X$ correspond with *skew local homeomorphisms into $X$* (like sheaves on $X$ correspond with local homeomorphisms into $X$).


## 1. Introduction

### 1.1. Locales and quantales, sheaves and logic

A *locale* $X$ is a complete lattice in which finite infima distribute over arbitrary suprema. A particular class of examples of locales comes from topology: the open subsets of any topological space form a locale. But not every locale arises in this way, whence the slogan that locales are "pointfree topologies" [23]. There is a "pointfree" way to do sheaf theory: a sheaf $F$ on a locale $X$ is a functor $F\colon X^{\mathrm{op}} \longrightarrow \mathsf{Set}$ satisfying gluing conditions. The collection of all such functors, together with natural transformations between them, forms the topos $\mathsf{Sh}(X)$ of *sheaves on* $X$. One of the many close ties between logic and sheaf theory, which is of particular interest to us, is that the *internal logic* of $\mathsf{Sh}(X)$ is an intuitionistic higher-order predicate logic with $X$ as object of truth values [29,8,24]. To borrow a phrase from [38] and others, sheaf theory thus serves as *algebraic logic*.

The definition of locale can be restated: $X$ is a complete lattice and $(X, \wedge, \top)$ is a monoid such that the multiplication distributes on both sides over arbitrary suprema. It is natural to generalise this: a *quantale* $Q = (Q, \circ, 1)$ is, by definition, a monoid structure on a complete lattice such that the multiplication distributes on both sides over arbitrary suprema [31,

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0168-0072/$ – see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.apal.2009.05.001

Order (2009) 26:177–196
DOI 10.1007/s11083-009-9116-x

# Modules on Involutive Quantales: Canonical Hilbert Structure, Applications to Sheaf Theory

**Hans Heymans · Isar Stubbe**

Received: 10 September 2008 / Accepted: 11 June 2009 / Published online: 27 June 2009


**Abstract** We explain the precise relationship between two module-theoretic descriptions of sheaves on an involutive quantale, namely the description via so-called Hilbert structures on modules and that via so-called principally generated modules. For a principally generated module satisfying a suitable symmetry condition we observe the existence of a canonical Hilbert structure. We prove that, when working over a modular quantal frame, a module bears a Hilbert structure if and only if it is principally generated and symmetric, in which case its Hilbert structure is necessarily the canonical one. We indicate applications to sheaves on locales, on quantal frames and even on sites.

**Keywords** Quantale · Module · Principal element · Principal symmetry · Inner product · Sheaf

## 1 Introduction

Jan Paseka [8–10] introduced the notion of *Hilbert module* on an involutive quantale: it is a module equipped with an *inner product*. This provides for an order-theoretic notion of "inner product space", originally intended as a generalisation of complete lattices with a duality. Recently, [13] applied this definition to a locale $X$ and further defined what it means for a Hilbert $X$-module to have a *Hilbert basis*. These Hilbert $X$-modules with Hilbert basis describe, in a module-theoretic way, the sheaves on $X$.

At the same time, the present authors defined the notion of *(locally) principally generated module* on a quantaloid [3]. Our aim too was to describe "sheaves as

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*CAHIERS DE TOPOLOGIE ET GEOMETRIE DIFFERENTIELLE CATEGORIQUES* — *Vol. LI-1 (2010)*

# "HAUSDORFF DISTANCE" VIA CONICAL COCOMPLETION

*by Isar STUBBE**

*À Francis Borceux, qui m'a tant appris et qui m'apprend toujours*

**Résumé.** Dans le contexte des catégories enrichies dans un quantaloïde, nous expliquons comment toute classe de poids saturée définit, et est définie par, une unique sous-KZ-doctrine pleine de la doctrine pour la cocomplétion libre. Les KZ-doctrines qui sont des sous-KZ-doctrines pleines de la doctrine pour la cocomplétion libre, sont caractérisées par deux conditions simples de "pleine fidélité". Les poids coniques forment une classe saturée, et la KZ-doctrine correspondante est exactement (la généralisation aux catégories enrichies dans un quantaloïde de) la doctrine de Hausdorff de [Akhvlediani *et al.*, 2009].

**Abstract.** In the context of quantaloid-enriched categories, we explain how each saturated class of weights defines, and is defined by, an essentially unique full sub-KZ-doctrine of the free cocompletion KZ-doctrine. The KZ-doctrines which arise as full sub-KZ-doctrines of the free cocompletion, are characterised by two simple "fully faithfulness" conditions. Conical weights form a saturated class, and the corresponding KZ-doctrine is precisely (the generalisation to quantaloid-enriched categories of) the Hausdorff doctrine of [Akhvlediani *et al.*, 2009].

**Keywords.** Enriched category, cocompletion, KZ-doctrine, Hausdorff distance

**Mathematics Subject Classification (2010).** 18D20, 18A35, 18C20

## 1. Introduction

At the meeting on "Categories in Algebra, Geometry and Logic" honouring Francis Borceux and Dominique Bourn in Brussels on 10–11 October 2008, Walter Tholen gave a talk entitled "On the categorical meaning of Hausdorff and Gromov distances", reporting on joint work with Andrei Akhvlediani and Maria Manuel Clementino [2009]. The term 'Hausdorff distance' in his title refers to the following construction: if $(X, d)$ is a metric space and $S, T \subseteq X$, then

$$\delta(S,T) := \bigvee_{s\in S} \bigwedge_{t\in T} d(s,t)$$

defines a (generalised) metric on the set of subsets of $X$. But Bill Lawvere [1973] showed that metric spaces are examples of enriched categories, so one can aim at

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*Theory and Applications of Categories,* Vol. 25, No. 11, 2011, pp. 276–294.

# SYMMETRY AND CAUCHY COMPLETION OF QUANTALOID-ENRICHED CATEGORIES

HANS HEYMANS AND ISAR STUBBE

Abstract. We formulate an elementary condition on an involutive quantaloid $\mathcal{Q}$ under which there is a distributive law from the Cauchy completion monad over the symmetrisation comonad on the category of $\mathcal{Q}$-enriched categories. For such quantaloids, which we call Cauchy-bilateral quantaloids, it follows that the Cauchy completion of any symmetric $\mathcal{Q}$-enriched category is again symmetric. Examples include Lawvere's quantale of non-negative real numbers and Walters' small quantaloids of closed cribles.

## 1. Introduction

A quantaloid $\mathcal{Q}$ is a category enriched in the symmetric monoidal closed category $\mathsf{Sup}$ of complete lattices and supremum-preserving functions. Viewing $\mathcal{Q}$ as a bicategory, it is natural to study categories, functors and distributors enriched in $\mathcal{Q}$. If $\mathcal{Q}$ comes equipped with an involution, it makes sense to consider symmetric $\mathcal{Q}$-enriched categories. An important application of quantaloid-enriched categories was discovered by R.F.C. Walters [1981, 1982]: he proved that the topos of sheaves on a small site $(\mathcal{C}, J)$ is equivalent to the category of *symmetric and Cauchy complete* categories enriched in a suitable "small quantaloid of closed cribles" $\mathcal{R}(\mathcal{C}, J)$. A decade earlier, F.W. Lawvere [1973] had already pointed out that the category of generalised metric spaces and non-expansive maps is equivalent to the category of categories enriched in the quantale (that is, a one-object quantaloid) $([0,\infty], \bigwedge, +, 0)$ of extended non-negative real numbers. This is a symmetric quantale, hence it is trivially involutive; and here too the *symmetric and Cauchy complete* $[0,\infty]$-enriched categories are important, if only to connect with the classical theory of metric spaces. Crucial in both examples is thus the use of categories enriched in an involutive quantaloid $\mathcal{Q}$ which are both symmetric and Cauchy complete. R. Betti and R.F.C. Walters [1982] therefore raised the question "whether the Cauchy completion of a symmetric [quantaloid-enriched] category is again symmetric". That is to say, they ask whether it is possible to *restrict* the Cauchy completion functor $(-)_{\mathsf{cc}}\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$ along the embedding $\mathsf{SymCat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$ of symmetric $\mathcal{Q}$-categories. They show that the answer to their question is affirmative for both $\mathcal{R}(\mathcal{C}, J)$ and $[0,\infty]$, by giving an *ad hoc* proof in each case; they also give an example of an involutive quantale for which the answer to their question is negative. Thus, it depends on the base quantaloid $\mathcal{Q}$ whether

Received by the editors 2010-05-04 and, in revised form, 2011-06-23.
Transmitted by R. F. C. Walters. Published on 2011-06-25.
2000 Mathematics Subject Classification: 06F07, 18C15, 18D05, 18D20.
Key words and phrases: Quantaloid, enriched category, symmetry, Cauchy completion.

# Elementary characterisation of small quantaloids of closed cribles

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March 17, 2011

**Abstract**

Each small site $(\mathcal{C}, J)$ determines a small quantaloid of closed cribles $\mathcal{R}(\mathcal{C}, J)$. We prove that a small quantaloid $\mathcal{Q}$ is equivalent to $\mathcal{R}(\mathcal{C}, J)$ for some small site $(\mathcal{C}, J)$ if and only if there exists a (necessarily subcanonical) Grothendieck topology $J$ on the category $\mathsf{Map}(\mathcal{Q})$ of left adjoints in $\mathcal{Q}$ such that $\mathcal{Q} \cong \mathcal{R}(\mathsf{Map}(\mathcal{Q}), J)$, if and only if $\mathcal{Q}$ is locally localic, map-discrete, weakly tabular and weakly modular. If moreover coreflexives split in $\mathcal{Q}$, then the topology $J$ on $\mathsf{Map}(\mathcal{Q})$ is the canonical topology.

## 1. Introduction

A quantaloid $\mathcal{Q}$ is, by definition, a category enriched in the symmetric monoidal closed category $\mathsf{Sup}$ of complete lattices and supremum-preserving functions [Rosenthal, 1996]. Viewing $\mathcal{Q}$ as a bicategory, it is natural to study categories, functors and distributors enriched in $\mathcal{Q}$ [Bénabou, 1967; Street, 1983; Stubbe, 2005a]. A major application of quantaloid-enriched category theory was discovered by B. Walters, and published in this journal: in [1982], he proved that the topos of sheaves on a site $(\mathcal{C}, J)$ is equivalent to the category of symmetric and Cauchy complete categories enriched in the *small quantaloid of closed cribles* $\mathcal{R}(\mathcal{C}, J)$ constructed from the given site.

Given the importance of the *construction* of the quantaloid of closed cribles $\mathcal{R}(\mathcal{C}, J)$ from a small site $(\mathcal{C}, J)$, we provide in this paper an elementary *axiomatisation* of this notion. Precisely, we prove that a small quantaloid $\mathcal{Q}$ is equivalent to $\mathcal{R}(\mathcal{C}, J)$ for some small site $(\mathcal{C}, J)$ if and only if there exists a Grothendieck topology $J$ on the category $\mathsf{Map}(\mathcal{Q})$ of left adjoints in $\mathcal{Q}$ such that $\mathcal{Q} \cong \mathcal{R}(\mathsf{Map}(\mathcal{Q}), J)$, if and only if $\mathcal{Q}$ is locally localic, map-discrete, weakly tabular and weakly modular. (The latter two notions seem to be new, and inherited their name from the stronger notions of tabularity and modularity introduced in [Freyd and Scedrov, 1990].) The Grothendieck topology $J$ on $\mathsf{Map}(\mathcal{Q})$ is always subcanonical, and if coreflexives split in $\mathcal{Q}$, then $J$ is the canonical topology.

This result thus spells out how two, at first sight quite different, generalisations of locales, namely Grothendieck topologies on the one hand, and quantaloids on the other, relate: the former can be understood to form an axiomatically described subclass of the latter. It is hoped

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# On quantaloid-enriched categories in general, and sheaves on a quantaloid in particular

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Dissertation présentée en vue de l'obtention de l'Habilitation de Diriger des Recherches, en séance publique, le 12 décembre 2011, au Laboratoire de Mathématiques Pures et Appliquées de l'Université du Littoral-Côte d'Opale, France.