Propositional systems, Hilbert lattices and generalized Hilbert spaces

Isar Stubbe* and Bart Van Steirteghem†

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Abstract. With this chapter we provide a compact yet complete survey of two most remarkable "representation theorems": every arguesian projective geometry is represented by an essentially unique vector space, and every arguesian Hilbert geometry is represented by an essentially unique generalized Hilbert space. C. Piron's original representation theorem for propositional systems is then a corollary: it says that every irreducible, complete, atomistic, orthomodular lattice satisfying the covering law and of rank at least 4 is isomorphic to the lattice of closed subspaces of an essentially unique generalized Hilbert space. Piron's theorem combines abstract projective geometry with lattice theory. In fact, throughout this chapter we present the basic lattice theoretic aspects of abstract projective geometry: we prove the categorical equivalence of projective geometries and projective lattices, and the triple categorical equivalence of Hilbert geometries, Hilbert lattices and propositional systems.

Keywords: Projective geometry, projective lattice, Hilbert geometry, Hilbert lattice, propositional system, equivalence of categories, coproduct decomposition in irreducible components, Fundamental Theorems of projective geometry, Representation Theorem for propositional systems.

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1. Introduction

Description of the problem. The definition of a Hilbert space H is all about a perfect marriage between linear algebra and topology: H is a vector space together with an inner product such that the norm associated to the inner product turns H into a complete metric space. As is well-known for any vector space, the one-dimensional linear subspaces of H are the points of a *projective geometry*, the collinearity relation being coplanarity. In other words, the set $\mathcal{L}(H)$ of linear subspaces, ordered by inclusion, forms a so-called *projective lattice*.

*Postdoctoral Fellow of the Research Foundation Flanders (FWO), Departement Wiskunde en Informatica, Universiteit Antwerpen. E-mail: `isar.stubbe@ua.ac.be`

†Partially supported by FCT/POCTI/FEDER, Departamento de Matemática, Instituto Superior Técnico, Lisboa. E-mail: `bvans@math.ist.utl.pt`

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$$\begin{tikzcd}
\mathsf{GenHilb} \arrow[r] \arrow[d] & \mathsf{HilbGeom} \arrow[r, leftrightarrow, "\sim"] \arrow[d] & \mathsf{HilbLat} \arrow[r, leftrightarrow, "\sim"] \arrow[d] & \mathsf{PropSys} \\
\mathsf{Vec} \arrow[r] & \mathsf{ProjGeom} \arrow[r, leftrightarrow, "\sim"] & \mathsf{ProjLat} &
\end{tikzcd}$$

Figure 1: A diagrammatic summary

Using the metric topology on H we can distinguish, amongst all linear subspaces, the closed ones: we will note the set of these as $\mathcal{C}(H)$. In fact, the inner product on H induces an *orthogonality operator* on $\mathcal{L}(H)$ making it a *Hilbert lattice*, and the map

$$(\ )^{\perp\perp}\colon \mathcal{L}(H) \longrightarrow \mathcal{L}(H)\colon A \mapsto A^{\perp\perp}$$

is a closure operator on $\mathcal{L}(H)$ whose fixpoints are precisely the elements of $\mathcal{C}(H)$. For many reasons, explained in detail elsewhere in this volume, it is the substructure $\mathcal{C}(H) \subseteq \mathcal{L}(H)$ – and not $\mathcal{L}(H)$ itself – which plays an important rôle in quantum logic; it is called a *propositional system*.

In this survey paper we wish to explain the lattice theoretic axiomatization of such a propositional system: we study necessary and sufficient conditions for an ordered set $(C, \leq)$ to be isomorphic to $(\mathcal{C}(H), \subseteq)$ for some (real, complex, quaternionic or generalized) Hilbert space H. As the above presentation suggests, this matter is intertwined with some deep results on projective geometry.

Overview of contents. Section 2 of this paper presents the relevant definitions of, and some basic results on, abstract (also called 'modern' or 'synthetic') projective geometry. Following Cl.-A. Faure and A. Frölicher's [2000] reference on the subject, we define a 'projective geometry' as a set together with a ternary collinearity relation (satisfying suitable axioms). The one-dimensional subspaces of a vector space are an example of such a projective geometry, with coplanarity as the ternary relation. After discovering some particular properties of the ordered set of 'subspaces' of such a projective geometry, we make an abstraction of this ordered set and call it a 'projective lattice'. We then speak of 'morphisms' between projective geometries, resp. projective lattices, and show that the category $\mathsf{ProjGeom}$ of projective geometries and the category $\mathsf{ProjLat}$ of projective lattices are equivalent. Vector spaces and 'semilinear maps' form a third important category Vec, and there is a functor $\mathsf{Vec} \longrightarrow \mathsf{ProjGeom}$. The bottom row in figure 1 summarizes this.

A projective geometry for which every line contains at least three points, is said to be 'irreducible'. We deal with these in section 3, for this geometric fact has an important categorical significance [Faure and Frölicher, 2000]: a projective geometry is irreducible precisely when it is not a non-trivial coproduct in $\mathsf{ProjGeom}$, and every projective geometry is the coproduct of irreducible ones. By the categorical equivalence between $\mathsf{ProjGeom}$ and $\mathsf{ProjLat}$, the "same" result holds for projective lattices. The projective geometries in the image of the functor $\mathsf{Vec} \longrightarrow \mathsf{ProjGeom}$ are always irreducible.

Having set the scene, we deal in section 4 with the linear representation of projective geometries (of dimension at least 2) and their morphisms, i.e. those objects and morphisms that lie in the image of the functor $\mathsf{Vec} \longrightarrow \mathsf{ProjGeom}$. The First Fundamental Theorem, which is by now part of mathematical folklore, says that precisely the 'arguesian' geometries (which include all geometries of dimension at least 3) are "linearizable". The Second Fundamental Theorem characterizes the "linearizable" morphisms. [Holland, 1995, §3] and [Faure, 2002] have some comments on the history of these results. We outline the proof of the First Fundamental Theorem as given in [Beutelspacher and Rosenbaum, 1998]; for a short proof of the Second Fundamental Theorem we refer to [Faure, 2002].

Again following [Faure and Frölicher, 2000], we turn in section 5 to projective geometries that come with a binary orthogonality relation which satisfies certain axioms: so-called 'Hilbert geometries'. The key example is given by the projective geometry of one-dimensional subspaces of a 'generalized Hilbert space' (a notion due to C. Piron [1976]), with the orthogonality induced by the inner product. The projective lattice of subspaces of such a Hilbert geometry inherits an orthogonality operator which satisfies some specific conditions, and this leads to the notion of 'Hilbert lattice'. The elements of a Hilbert lattice that equal their biorthogonal are said to be '(biorthogonally) closed'; they form a 'propositional system' [Piron, 1976]: a complete, atomistic, orthomodular lattice satisfying the covering law. Considering Hilbert geometries, Hilbert lattices and propositional systems together with suitable ('continuous') morphisms, we obtain a triple equivalence of the categories $\mathsf{HilbGeom}$, $\mathsf{HilbLat}$ and $\mathsf{PropSys}$. And there is a category $\mathsf{GenHilb}$ of generalized Hilbert spaces and continuous semilinear maps, with a functor $\mathsf{GenHilb} \longrightarrow \mathsf{HilbGeom}$. Since a Hilbert geometry is a projective geometry with extra structure, and a continuous morphism between Hilbert geometries is a particular morphism between (underlying) projective geometries, there is a faithful functor $\mathsf{HilbGeom} \longrightarrow \mathsf{ProjGeom}$. Similarly there are faithful functors $\mathsf{HilbLat} \longrightarrow \mathsf{ProjLat}$ and $\mathsf{GenHilb} \longrightarrow \mathsf{Vec}$ too, and the resulting (commutative) diagram of categories and functors is sketched in figure 1.

Then we show in section 6 that a Hilbert geometry is irreducible (as a projective geometry, i.e. each line contains at least three points) if and only if it is not a non-trivial coproduct in $\mathsf{HilbGeom}$; and each Hilbert geometry is the coproduct of irreducible ones. By categorical equivalence, the "same" is true for Hilbert lattices and propositional systems.

In section 7 we present the Representation Theorem for propositional systems or, equivalently, Hilbert geometries (of dimension at least 2): the arguesian Hilbert geometries constitute the image of the functor $\mathsf{GenHilb} \longrightarrow \mathsf{HilbGeom}$. For finite dimensional geometries this result is due to G. Birkhoff and J. von Neumann [1936] while the more general (infinite-dimensional) version goes back to C. Piron's [1964, 1976] representation theorem: every irreducible propositional system of rank at least 4 is isomorphic to the lattice of closed subspaces of an essentially unique generalized Hilbert space. We provide an outline of the proof given in [Holland, 1995, §3].

The final section 8 contains some comments and remarks on various interesting points that we did not address or develop in the text.

Required lattice and category theory. Throughout this chapter we use quite a few notions and (mostly straightforward) facts from lattice theory. For completeness' sake we have added a short appendix in which we explain the words marked with a "†" in our text. The standard references on lattice theory are [Birkhoff, 1967; Grätzer, 1998], but [Maeda and Maeda, 1970; Kalmbach, 1983] have everything we need too. Finally, we also use some very basic category theory: we speak of an 'equivalence of categories', compute some 'coproducts', and talk about 'full' and 'faithful' functors. Other categorical notions that we need, are explained in the text. The classic [Mac Lane, 1971] or the first volume of [Borceux, 1994] contain all this (and much more).

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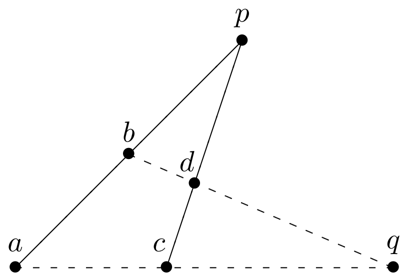

Figure 2: Illustration of (G3)

surrounded by some of the field's most outstanding researchers—Dirk Aerts, Bob Coecke, Frank Valckenborgh. Moreover, the quantum physics group in Brussels being next of kin to Constantin Piron's group in Geneva, we also had the chance to interact with the members of the latter—Claude-Alain Faure, Constantin Piron, David Moore. It is with great pleasure that we dedicate this chapter to all those who made that period unforgettable. We thank Mathieu Dupont, Claude-Alain Faure, Chris Heunen and Frank Valckenborgh for their comments and suggestions.

2. Projective geometries, projective lattices

It is a well-known slogan in mathematics that "the lines of a vector space are the points of a projective geometry". To make this statement precise, we must introduce the abstract notion of a 'projective geometry'.

Definition 2.1 A **projective geometry** (G, l) is a set G of **points** together with a ternary **collinearity relation** $l \subseteq G \times G \times G$ such that

(G1) for all $a, b \in G$, $l(a, b, a)$,
(G2) for all $a, b, p, q \in G$, if $l(a, p, q)$, $l(b, p, q)$ and $p \neq q$, then $l(a, b, p)$,
(G3) for all $a, b, c, d, p \in G$, if $l(p, a, b)$ and $l(p, c, d)$ then there exists a $q \in G$ such that $l(q, a, c)$ and $l(q, b, d)$.

Often, since no confusion will arise, we shall speak of "a projective geometry G", without explicitly mentioning its collinearity relation l. The axioms for the collinearity relation – as well as many of the calculations further on – are best understood by means of a simple picture, in which one draws "dots" for the points of G, and a "line" through any three points a, b, c such that $l(a, b, c)$. With this intuition (which will be made exact further on), (G1) and (G2) say that two distinct points determine one and only one line, and (G3) is depicted in figure 2.

Example 2.2 Let V be a (left) vector space over a (not necessarily commutative) field K. The set of lines of V endowed with the coplanarity relation forms a projective geometry; it will be denoted further on as $\mathcal{P}(V)$. Note that the collinearity relation is trivial when $\mathsf{dim}(V) \leq 2$.

The example $\mathcal{P}(V)$ is very helpful for sharpening the intuition on abstract projective geometry. For example, it is clear that the collinearity relation in $\mathcal{P}(V)$ is symmetric; but in fact this property holds also in the general case.

Lemma 2.3 *A ternary relation l on a set G satisfying (G1–2) is symmetric, meaning that for $a_1, a_2, a_3 \in G$, if $l(a_1, a_2, a_3)$ then also $l(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)})$ for any permutation σ on $\{1, 2, 3\}$.*

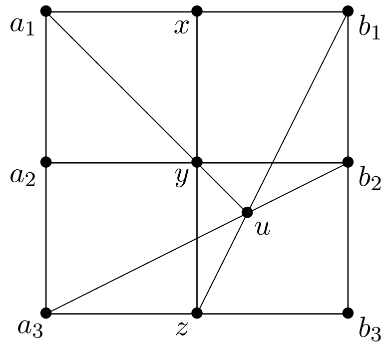

Figure 3: Illustration of the proof of 2.4

Proof. The group of permutations on $\{1, 2, 3\}$ being generated by its elements $(123) \mapsto (132)$ and $(123) \mapsto (312)$, we only need to check two cases. This is a simple exercise. $\square$

It is not hard to show that (G3) follows from (G1–2) when $\mathsf{card}\{a, b, c, d, p\} \neq 5$ or when $\{a, b, c, d\}$ contains three (different) points that belong to l.

For a projective geometry (G, l), any two distinct points $a, b \in G$ determine the **projective line** $a \star b := \{x \in G \mid l(x, a, b)\}$. For notational convenience, we also put that $a \star a := \{a\}$. It is a useful corollary of 2.3 that for $a, b, c \in G$, if $a \neq c$ then $a \in b \star c$ implies $b \in a \star c$.

Now we define a **subspace** S of G to be a subset $S \subseteq G$ with the property that

$$\text{if } a, b \in S \text{ then } a \star b \subseteq S.$$

Trivially, any projective geometry G has the empty subspace $\emptyset \subseteq G$ and the total subspace $G \subseteq G$. Moreover, all $a \star b \subseteq G$ are subspaces; these include all singletons $\{a\} = a \star a$.

The set of all subspaces of G will be denoted $\mathcal{L}(G)$. Since subspaces of G are particular subsets of G, $\mathcal{L}(G)$ is ordered by inclusion. The following proposition collects some features of the ordered set† $(\mathcal{L}(G), \subseteq)$, but first we shall record a key lemma.

Lemma 2.4 *In the lattice*† $\mathcal{L}(G)$ *of subspaces of a projective geometry* G*,*

i. for any family of subspaces $(S_i)_{i \in I}$*,* $\bigcap_i S_i$ *is a subspace,*
ii. for a directed† *family of subspaces* $(S_i)_{i \in I}$*,* $\bigcup_i S_i$ *is a subspace,*
iii. for two non-empty subspaces S *and* T*,* $\bigcup\{a \star b \mid a \in S, b \in T\}$ *is a subspace.*

Proof. The proofs of the first two statements are straightforward. As for the third statement, we must prove that, if $l(x, a_1, b_1)$, $l(y, a_2, b_2)$ and $l(z, x, y)$, with $a_1, a_2 \in S$ and $b_1, b_2 \in T$, then $l(z, a_3, b_3)$ for some $a_3 \in S$ and $b_3 \in T$. The picture in figure 3 suggests how to do this, using the symmetry of the collinearity relation and applying (G3) over and over again. $\square$

It follows that for a subset $A \subseteq G$ of a projective geometry G

$$\mathsf{cl}(A) := \bigcap\{S \in \mathcal{L}(G) \mid A \subseteq S\}$$

is the smallest subspace of G that contains A: it is its so-called **projective closure**[1]. The third statement in 2.4 is often referred to as the **projective law**. In terms of the projective closure it

[1]This terminology is well-chosen, for the mapping $A \mapsto \mathsf{cl}(A)$ does indeed define a closure operator† on the set of subsets of G; see also 8.1.

may be stated as: for non-empty subspaces S and T of G,

$$\mathsf{cl}(S \cup T) = \bigcup \{a \star b \mid a \in S, b \in T\}.$$

Proposition 2.5 *For any projective geometry G, $(\mathcal{L}(G), \subseteq)$ is a complete†, atomistic†, continuous†, modular† lattice.*

Proof. The order on $\mathcal{L}(G)$ is complete, because the intersection of subspaces is their infimum†; thus the supremum† of a family $(S_i)_{i \in I} \in \mathcal{L}(G)$ is $\bigvee_i S_i = \mathsf{cl}(\bigcup_i S_i)$. This makes it at once clear that any subspace $S \in \mathcal{L}(G)$ is the supremum of its points: $S = \mathsf{cl}(S) = \bigvee_{a \in S} \{a\}$; and singleton subspaces being exactly the atoms† of $\mathcal{L}(G)$ this also shows that $\mathcal{L}(G)$ is atomistic. The continuity of $\mathcal{L}(G)$ follows trivially from the fact that directed suprema in $\mathcal{L}(G)$ are simply unions. Finally, to show that $\mathcal{L}(G)$ is modular, it suffices to verify that for non-empty subspaces $S, T, U \subseteq G$, if $S \subseteq T$ then $(S \vee U) \cap T \subseteq S \vee (U \cap T)$. We are going to use the projective law a couple of times. Suppose that $x \in (S \vee U) \cap T$; so $x \in T$, but also $x \in S \vee U$, which means that $x \in a \star b$ for some $a \in S$ and $b \in U$. If $x = a$ then $x \in S \subseteq S \vee (U \cap T)$; if $x \neq a$ then $x \in a \star b$ implies that $b \in a \star x \subseteq S \vee T = T$ (using that $S \subseteq T$) so that $x \in a \star b \subseteq S \vee (U \cap T)$ in this case too. $\square$

Definition 2.6 An ordered set $(L, \leq)$ is a **projective lattice** if it is a complete, atomistic, continuous, modular lattice.

There are equivalent formulations for the definition of 'projective lattice'; we shall encounter some further on in this section. Here we shall already give one alternative for the continuity condition, which is sometimes easier to handle and will be used in the proofs of 2.16 and 3.8.

Lemma 2.7 *A complete atomistic lattice L is continuous if and only if its atoms are* **compact***, i.e. if a is an atom and $(x_i)_{i \in I}$ is a directed family in L, then $a \leq \bigvee_i x_i$ implies $a \leq x_k$ for some $k \in I$.*

Proof. If L is continuous and (with notations as in the statement of the lemma) $a \leq \bigvee_i x_i$, then $a = a \wedge (\bigvee_i x_i) = \bigvee_i (a \wedge x_i)$, so there must be a $k \in I$ for which $a \wedge x_k \neq 0$, whence $a \leq x_k$ (for a is an atom). Conversely, $\bigvee_i (y \wedge x_i) \leq y \wedge (\bigvee_i x_i)$ holds for any element $y \in L$. Suppose that this inequality is *strict*. By atomisticity of L there must exist an atom a such that $a \not\leq \bigvee_i (y \wedge x_i)$ and $a \leq y \wedge (\bigvee_i x_i)$. This implies in particular that $a \leq y$ and $a \leq \bigvee_i x_i$, and by hypothesis a is compact so that $a \leq x_k$ for some $k \in I$. But then $a \leq y \wedge x_k \leq \bigvee_i (y \wedge x_i)$ is a contradiction. Thus necessarily $\bigvee_i (y \wedge x_i) = y \wedge (\bigvee_i x_i)$ for every $y \in L$. $\square$

Example 2.8 For a K-vector space V, the set $\mathcal{L}(V)$ of linear subspaces, ordered by set-inclusion, is isomorphic to the projective lattice $\mathcal{L}(\mathcal{P}(V))$ of subspaces of the projective geometry $\mathcal{P}(V)$ of 2.2: the mappings

$$\mathcal{L}(V) \longrightarrow \mathcal{L}(\mathcal{P}(V)) \colon W \mapsto \mathcal{P}(W)$$
$$\mathcal{L}(\mathcal{P}(V)) \longrightarrow \mathcal{L}(V) \colon S \mapsto \{x \in V \mid Kx \in S\} \cup \{0\}$$

are well-defined, preserve order† and are each other's inverse. With slight abuse of notation we shall write $\mathcal{L}(V)$ even when we actually mean $\mathcal{L}(\mathcal{P}(V))$.

So 2.5 states that a projective geometry G determines a projective lattice $\mathcal{L}(G)$; but the converse is also true. First we prove a lattice theoretical lemma that exhibits the strength of the modularity condition.

Lemma 2.9 *Let L be a complete atomistic lattice, and $\mathcal{G}(L)$ its set of atoms.*

i. If L is modular, then it is both upper semimodular† and lower semimodular†.
ii. L is upper semimodular if and only if it satisfies the covering law†.
iii. If L is lower semimodular and satisfies the covering law then it has the **intersection property***: for any $x \in L$ and $p, q \in \mathcal{G}(L)$ with $p \neq q$, if $p \leq q \vee x$ then $(p \vee q) \wedge x \neq 0$.*
iv. If L has the intersection property, then for $a, b, c \in \mathcal{G}(L)$ with $a \neq b$, if $a \leq b \vee c$ then also $c \leq a \vee b$.
v. If L has the intersection property, then $\mathcal{G}(L)$ forms a projective geometry for the ternary relation

$$l(a, b, c) \text{ if and only if } a \leq b \vee c \text{ or } b = c. \tag{1}$$

Proof. (i) For any lattice L, the maps

$$\varphi\colon [u \wedge v, v] \longrightarrow [u, u \vee v]\colon x \mapsto x \vee u \text{ and } \psi\colon [u, u \vee v] \longrightarrow [u \wedge v, v]\colon y \mapsto y \wedge v$$

are well-defined and preserve order. If L is modular then moreover $\psi(\varphi(x)) = (x \vee u) \wedge v = x \vee (u \wedge x) = x$; similarly $\varphi(\psi(y)) = y$. So the two segments are isomorphic lattices. Now clearly $u \wedge v \lessdot v \Leftrightarrow \mathsf{card}[u \wedge v, v] = 2 \Leftrightarrow \mathsf{card}[u, u \vee v] = 2 \Leftrightarrow u \lessdot u \vee v$, which proves both upper and lower semimodularity of L (resp. $\Rightarrow$ and $\Leftarrow$ in this equivalence).

(ii) The covering law is a special case of upper semimodularity. Conversely, in an atomistic lattice satisfying the covering law it is the case that

$$x \lessdot y \text{ if and only if there exists } a \in \mathcal{G}(L) : a \not\leq x, x \vee a = y. \tag{2}$$

(Indeed, the "only if" follows from atomisticity, and the "if" is the covering law.) So if now $u \wedge v \lessdot v$ then there is an atom $a \in \mathcal{G}(L)$ such that $a \not\leq u \wedge v$ and $(u \wedge v) \vee a = v$. But then $u \vee v = u \vee [(u \wedge v) \vee a)] = [u \vee (u \wedge v)] \vee a = u \vee a$; and $a \not\leq u$ (for $a \leq v$ but $a \not\leq u \wedge v$) so by (2) we conclude that $u \lessdot u \vee v$.

(iii) Let $p \neq q \in \mathcal{G}(L)$ and $x \in L$ be such that $p \leq q \vee x$. If $q \leq x$ then trivially $q \leq (p \vee q) \wedge x$; if $q \not\leq x$ then $x \lessdot q \vee x = (p \vee q) \vee x$ by the covering law and the hypothesis $p \leq q \vee x$. This in turn implies $x \wedge (p \vee q) \lessdot p \vee q$ by lower semimodularity. Now $x \wedge (p \vee q) \neq 0$ because it is covered by $p \vee q$ which is not an atom.

(iv) From the assumptions and the intersection property it follows that $(a \vee b) \wedge c \neq 0$, so that necessarily $c \leq a \vee b$, for c is an atom.

(v) We shall check the axioms in 2.1, using the notations introduced there and keeping in mind that the collinearity relation is as in (1). Axiom (G1) is trivial. For (G2) we may suppose that $b \neq p$. Then, by (iv), $b \leq p \vee q$ implies $q \leq b \vee p$ and hence $a \leq p \vee q \leq p \vee (b \vee p) = p \vee b$, as wanted. As for (G3), we may suppose that a, b, c, d, and p are different points; then $p \leq a \vee b$ implies $a \leq p \vee b$, hence $a \leq b \vee c \vee d$, and therefore, by the intersection property, $(b \vee d) \wedge (a \vee c) \neq 0$, which means (by atomisticity) that $q \leq (b \vee d) \wedge (a \vee c)$ for some $q \in \mathcal{G}(L)$, as wanted. $\square$

The lemma above is not stated as "sharply" as possible. In fact, the 'intersection property' in (iii) can be rephrased for an arbitrary lattice L with 0 as

$$\text{if } p \leq q \vee x \text{ then there exists an atom } r \text{ such that } r \leq (p \vee q) \wedge x$$

for atoms $p \neq q$ and an arbitrary x; this is how it first appeared in [Faure and Frölicher, 1995]. Then statement (i), sufficiency in (ii), and statement (iii) are true for any lattice with 0 (not necessarily complete nor atomistic), while the converse of (iii) holds for an atomistic L. We shall only need these results for a complete atomistic lattice L (in 2.10 below and also in 5.11 further on).

We may now state the following as a simple corollary of 2.9.

Proposition 2.10 *The set $\mathcal{G}(L)$ of atoms of a projective lattice L forms a projective geometry for the ternary relation in (1).*

If L is a lattice with 0 for which $\mathcal{G}(L)$ is a projective geometry for the collinearity in (1), then $\mathcal{L}(\mathcal{G}(L))$ is a projective lattice, according to 2.5. Would L be isomorphic to $\mathcal{L}(\mathcal{G}(L))$ then necessarily L must be a projective lattice too: completeness, atomisticity, continuity and modularity are transported by isomorphism. From the work in the rest of this section it will follow that L being projective is also sufficient for it to be naturally isomorphic to $\mathcal{L}(\mathcal{G}(L))$. Similarly it is also true that a projective geometry G may be identified with $\mathcal{G}(\mathcal{L}(G))$. More precisely, we shall show that projective geometries and projective lattices are categorically equivalent notions. So we better start building categories!

Recall first that a **partial map** f between sets A and B is a map from a subset $D_f \subseteq A$ to B. The set D_f is the **domain** of f, and the set-complement $K_f = (D_f)^{\mathsf{c}}$ is its **kernel**. Most of the time we write such a partial map as $f\colon A \dashrightarrow B$ instead of $f\colon A \setminus K_f \longrightarrow B$ or $f\colon D_f \subseteq A \longrightarrow B$. Partial maps compose: for $f\colon A \dashrightarrow B$ with kernel K_f and $g\colon B \dashrightarrow C$ with kernel K_g, $g \circ f\colon A \dashrightarrow C$ has kernel $K_f \cup f^{-1}(K_g)$ and maps an element a of its domain to $g(f(a))$. This composition law is associative, and the identity map on a set (viewed as partial map with empty kernel) is a two-sided identity for this composition. That is to say, there is a perfectly good category ParSet of sets and partial maps.

Definition 2.11 Given two projective geometries G_1 and G_2, a partial map $g\colon G_1 \dashrightarrow G_2$ is a **morphism of projective geometries** if, for any subspace T of G_2,

$$g^*(T) := K_g \cup g^{-1}(T)$$

is a subspace of G_1.

Since $\emptyset \subseteq G_2$ is a subspace, the kernel K_g of $g\colon G_1 \dashrightarrow G_2$ must be a subspace of G_1. In the proof of 3.7 we shall show that a morphism $g\colon G_1 \dashrightarrow G_2$ maps any line $a \star b$ in G_1, with $a, b \notin K_g$, either to a single point of G_2 (in case $g(a) = g(b)$) or injectively to the line $g(a) \star g(b)$ of G_2 (in case $g(a) \neq g(b)$). This provides a geometric interpretation, in terms of points and lines, of the notion of 'morphism between projective geometries'. (As a matter of fact, these latter conditions are also sufficient for g to be a morphism, provided that $K_g = \emptyset$.)

With composition of two morphisms of projective geometries defined as the composition of the underlying partial maps, we obtain a category $\mathsf{ProjGeom}$. An isomorphism in $\mathsf{ProjGeom}$ is, as in any category, a morphism $g\colon G_1 \dashrightarrow G_2$ with a two-sided inverse $g'\colon G_2 \dashrightarrow G_1$. But it can easily be seen that such is the same as a bijection (with empty kernel) $g\colon G_1 \longrightarrow G_2$ which preserves and reflects the collinearity relation: $l_1(a, b, c)$ if and only if $l_2(g(a), g(b), g(c))$ for all $a, b, c \in G_1$.

Example 2.12 By definition, a **semilinear map** between a K_1-vector space V_1 and a K_2-vector space V_2 is an additive map $f\colon V_1 \longrightarrow V_2$ for which there exists a homomorphism of fields

$\sigma\colon K_1 \longrightarrow K_2$ such that $f(\alpha x) = \sigma(\alpha) f(x)$ for every $\alpha \in K_1$ and $x \in V_1$. Sometimes we call this a **σ-linear map** $f\colon V_1 \longrightarrow V_2$ too. The σ is uniquely determined by f whenever f is non-zero; and the zero-map is semilinear if and only if there exists a homomorphism $\sigma\colon K_1 \longrightarrow K_2$. There is a category Vec of vector spaces and semilinear maps. A semilinear map $f\colon V_1 \longrightarrow V_2$ determines a morphism of projective geometries

$$\mathcal{P}(f)\colon \mathcal{P}(V_1) \dashrightarrow \mathcal{P}(V_2)\colon K_1 x \mapsto K_2 f(x) \text{ with kernel } \mathcal{P}(\mathsf{ker}(f)).$$

This, in fact, defines a functor

$$\mathcal{P}\colon \mathsf{Vec} \longrightarrow \mathsf{ProjGeom}\colon \Big(f\colon V_1 \longrightarrow V_2\Big) \mapsto \Big(\mathcal{P}(f)\colon \mathcal{P}(V_1) \dashrightarrow \mathcal{P}(V_2)\Big).$$

The following example [Faure and Frölicher, 2000, 6.3.9–11] shows that semilinear maps can behave surprisingly when the associated field homomorphism is not an isomorphism.

Example 2.13 Let F be a commutative field, let $K := F(x)$ be the field of rational functions and let n be a positive integer. Consider the field homomorphism $\sigma\colon K \longrightarrow K\colon q(x) \mapsto q(x^n)$. One can show that $\sigma(K) \subseteq K$ is an extension of fields of degree n: putting $\alpha_i := x^{i-1}$, the set $\{\alpha_1, \ldots, \alpha_n\}$ forms a basis of K over $\sigma(K)$. It follows that $\varphi\colon K^n \longrightarrow K\colon (a_1, \ldots, a_n) \mapsto \sigma(a_1)\alpha_1 + \ldots + \sigma(a_n)\alpha_n$ is a σ-linear form with zero kernel. Moreover, picking any nonzero $b \in K^n$ we obtain a σ-linear map $f\colon K^n \longrightarrow K^n\colon x \mapsto \varphi(x) b$ for which $\mathcal{P}(f)$ is constant and which has empty kernel.

We now turn to projective lattices.

Definition 2.14 Given projective lattices L_1 and L_2, a map $f\colon L_1 \longrightarrow L_2$ is a **morphism of projective lattices** if it preserves arbitrary suprema and sends atoms in L_1 to atoms or to the bottom element in L_2.

We thus get a category $\mathsf{ProjLat}$. Note that an isomorphism in $\mathsf{ProjLat}$ is indeed the same thing as an order-preserving and reflecting bijection, so that in 2.8 there is no doubt about the meaning of the word.

We know from 2.5 that any projective geometry G determines a projective lattice $\mathcal{L}(G)$; and any projective lattice L determines a projective geometry $\mathcal{G}(L)$ according to 2.10. For morphisms we can play a similar game.

Proposition 2.15 *If $g\colon G_1 \dashrightarrow G_2$ is a morphism of projective geometries, then*

$$\mathcal{L}(g)\colon \mathcal{L}(G_1) \longrightarrow \mathcal{L}(G_2)\colon S \mapsto \bigcap \{T \in \mathcal{L}(G_2) \mid S \subseteq g^*(T)\}$$

is a morphism of projective lattices. And if $f\colon L_1 \longrightarrow L_2$ is a morphism of projective lattices, then

$$\mathcal{G}(f)\colon \mathcal{G}(L_1) \dashrightarrow \mathcal{G}(L_2)\colon a \mapsto f(a) \textit{ with kernel } \{a \in \mathcal{G}(L_1) \mid f(a) = 0\}$$

is a morphism of projective geometries.

Proof. First note that $g\colon G_1 \dashrightarrow G_2$ defines the "inverse image" map

$$g^*\colon \mathcal{L}(G_2) \longrightarrow \mathcal{L}(G_1)\colon T \mapsto K_g \cup g^{-1}(T),$$

which preserves arbitrary intersections. Intersections of subspaces being their infima, g^* must have a left adjoint†. This left adjoint is precisely $\mathcal{L}(g)$, which proves that $\mathcal{L}(g)$ preserves arbitrary suprema. The atoms of $\mathcal{L}(G_1)$ and $\mathcal{L}(G_2)$ corresponding to their respective singleton subspaces, $\mathcal{L}(g)$ sends atoms to atoms or to the bottom element.

Because $f\colon L_1 \longrightarrow L_2$ sends atoms of L_1 to atoms or the bottom element of L_2, $\mathcal{G}(f)$ is a well-defined partial map. Now let $T \subseteq \mathcal{G}(L_2)$ be a subspace of the projective geometry $\mathcal{G}(L_2)$; if $a, b \in \mathcal{G}(f)^*(T)$ and $c \leq a \vee b$ then $f(c) \leq f(a \vee b) = f(a) \vee f(b)$, showing that either $f(c) = 0$ or $f(c) \in T$ (by T being a subspace). That is to say, $\mathcal{G}(f)^*(T)$ is a subspace of G_1. $\square$

The above proposition explains the requirement in 2.14 that a morphism of projective lattices preserve *arbitrary* suprema: such a morphism must be thought of as the left adjoint to an inverse image.

Now we are ready to state and prove the result promised a while ago.

Theorem 2.16 *The categories* $\mathsf{ProjGeom}$ *and* $\mathsf{ProjLat}$ *are equivalent. To wit, the assignments*

$$\mathcal{L}\colon \mathsf{ProjGeom} \longrightarrow \mathsf{ProjLat}\colon \Big(g\colon G_1 \dashrightarrow G_2\Big) \mapsto \Big(\mathcal{L}(g)\colon \mathcal{L}(G_1) \longrightarrow \mathcal{L}(G_2)\Big)$$
$$\mathcal{G}\colon \mathsf{ProjLat} \longrightarrow \mathsf{ProjGeom}\colon \Big(f\colon L_1 \longrightarrow L_2\Big) \mapsto \Big(\mathcal{G}(f)\colon \mathcal{G}(L_1) \dashrightarrow \mathcal{G}(L_2)\Big)$$

are functorial, and for a projective geometry G and a projective lattice L there are natural isomorphisms

$$\alpha_G\colon G \xrightarrow{\sim} \mathcal{G}(\mathcal{L}(G))\colon a \mapsto \{a\},$$
$$\beta_L\colon L \xrightarrow{\sim} \mathcal{L}(\mathcal{G}(L))\colon x \mapsto \{a \in \mathcal{G}(L) \mid a \leq x\}.$$

Proof. It is a matter of straightforward calculations to see that $\mathcal{L}$ and $\mathcal{G}$ are functorial. We shall prove that α_G and β_L are isomorphisms, and leave the verification of their naturality to the reader.

First, the map α_G is obviously a well-defined bijection (with empty kernel): the atoms of $\mathcal{L}(G)$ are precisely the singleton subsets of G, i.e. the points of G. We need to show that a, b, c are collinear in G if and only if $\{a\}, \{b\}, \{c\}$ are collinear in $\mathcal{G}(\mathcal{L}(G))$; but this comes down to showing that $a \in b \star c$ in G if and only if $\{a\} \subseteq \{b\} \vee \{c\}$ in $\mathcal{L}(G)$, which is an instance of the projective law.

Next, it is easy to see that β_L is a well-defined map, i.e. that any $\beta_L(x) \subseteq \mathcal{G}(L)$ is indeed a subspace (for the collinearity relation on $\mathcal{G}(L)$ as in 2.10). We claim now that the map

$$\gamma_L\colon \mathcal{L}(\mathcal{G}(L)) \mapsto L\colon S \mapsto \bigvee S$$

is the inverse of β_L in $\mathsf{ProjLat}$. In fact, it is clear that both β_L and γ_L preserve order; thus it suffices to show that they are mutually inverse maps to prove that they constitute an isomorphism in $\mathsf{ProjLat}$. That $\gamma_L \circ \beta_L$ is the identity, is the atomisticity of L. Conversely, for a subspace $S \subseteq \mathcal{G}(L)$ we have that $(\beta_L \circ \gamma_L)(S) = \{a \in \mathcal{G}(L) \mid a \leq \bigvee S\}$; so it suffices to prove that $a \leq \bigvee S \Leftrightarrow a \in S$ to find that $\beta_L \circ \gamma_L$ is the identity on $\mathcal{L}(\mathcal{G}(L))$. But $\bigvee S = \bigvee\{\bigvee S' \mid S' \subseteq_{\mathsf{f}} S\}$ – where we write $S' \subseteq_{\mathsf{f}} S$ for a *finite* subset – which expresses $\bigvee S$ as a directed join of finite joins. Because the atoms of L are compact (by continuity and 2.7), $a \leq \bigvee S$ if and only if $a \leq \bigvee S'$ for some $S' \subseteq_{\mathsf{f}} S$. Using the intersection property of L (cf. 2.9 and 2.10) and using the subspace property of S for the collinearity relation on $\mathcal{G}(L)$, we shall prove by induction on the number of elements of $S' = \{s_1, ..., s_n\}$ that $a \in S$. The case $n = 1$ is trivial, so let the case $n-1$ be true by induction hypothesis, and let $a \leq s_1 \vee ... \vee s_{n-1} \vee s_n$ with $s_n \not\leq s_1 \vee ... \vee s_{n-1}$. If $a \leq s_n$ then $a = s_n \in S$ and

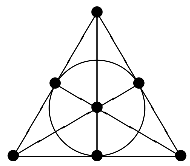

Figure 4: The smallest non-trivial irreducible geometry

we are done. If $a \not\leq s_n$ then $a \neq s_n$ and by the intersection property $(s_1 \vee ... \vee s_{n-1}) \wedge (s_n \vee a) \neq 0$, so (by atomisticity) there is an atom $r \in \mathcal{G}(L)$ such that $r \leq (s_1 \vee ... \vee s_{n-1}) \wedge (s_n \vee a)$. But then $r \leq s_1 \vee ... \vee s_{n-1}$ thus $r \in S$ by the induction hypothesis. And since $r \neq s_n$ (for otherwise $s_n \leq s_1 \vee ... \vee s_{n-1}$), $r \leq s_n \vee a$ implies $a \leq r \vee s_n$ by (iv) of 2.9, so that $a \in S$ by S being a subspace. $\square$

In the proof for $\beta_L \colon L \longrightarrow \mathcal{L}(\mathcal{G}(L))$ being an isomorphism, modularity of L was not explicitly used, except for the fact that it implies the intersection property as in 2.9. Therefore, since $L \cong \mathcal{L}(\mathcal{G}(L))$ and the latter is a projective lattice, it follows that a complete, atomistic, continuous lattice is modular if and only if it has the intersection property.

Until now we have considered the following diagram of categories and functors:

$$\mathsf{Vec} \longrightarrow \mathsf{ProjGeom} \xleftrightarrow{\sim} \mathsf{ProjLat}.$$

In section 4 we shall discuss a converse to the functor $\mathcal{P} \colon \mathsf{Vec} \longrightarrow \mathsf{ProjGeom}$, but thereto we need to deal with another issue first.

3. Irreducible components

However trivial it may seem that every plane in a vector space V contains at least three lines, this is actually not automatic for abstract projective geometries.

Definition 3.1 A projective geometry (G, l) is **irreducible** if for every $a, b \in G$, $\mathsf{card}(a \star b) \neq 2$; otherwise it is **reducible**.

Since we defined that $a \star a = \{a\}$, this definition says that G is an irreducible projective geometry precisely when every line contains at least three points. This definition is clearly invariant under isomorphism.

Example 3.2 For any vector space V, $\mathcal{P}(V)$ is an irreducible projective geometry: if $Kx \neq Ky$ then $K(x+y)$ is a third point on the line $Kx \star Ky$. Taking V to be the cube of the field with two elements, one gets the smallest irreducible projective geometry with three non-collinear points; it is pictured in figure 4 (all straight segments *and* the circle in the picture designate projective lines).

Example 3.3 Any set G becomes a **discrete projective geometry** when putting $l(a, b, c)$ to mean that $\mathsf{card}\{a, b, c\} \leq 2$. A discrete projective geometry is irreducible if and only if G is a singleton.

The following construction is in a precise sense a generalization of 3.3; it is important enough to record it as a lemma saying in particular that the category $\mathsf{ProjGeom}$ has coproducts.

Lemma 3.4 *Given a family $(G_i, l_i)_{i\in I}$ of projective geometries, the disjoint union $\biguplus_i G_i$ equipped with the relation*

$$l(a,b,c) \text{ if either } \mathsf{card}\{a,b,c\} \leq 2 \text{ or } l_k(a,b,c) \text{ in } G_k \text{ for some } k \in I,$$

together with the inclusions

$$s_k \colon G_k \longrightarrow \biguplus_i G_i \colon a \mapsto a \tag{3}$$

is a coproduct in $\mathsf{ProjGeom}$.

Proof. First we check that $\biguplus_i G_i$ forms a projective geometry for the indicated collinearity relation; we shall verify axioms (G1–3) in 2.1, keeping the notations used there. For (G1) there is nothing to prove. Axiom (G2) is trivial when we have $\mathsf{card}\{a,b,p\} \neq 3$; from now on we assume the contrary. If $a = q$ then $l(b,p,q)$ means that $b,p,a \in G_k$ and $l_k(b,p,a)$ for some $k \in I$ because of the previous assumption, but then $l_k(a,b,p)$ by symmetry of l_k and hence $l(a,b,p)$ as wanted; so suppose $a \neq q$. The hypothesis $l(a,p,q)$ now implies that $a,p,q \in G_k$ and $l_k(a,p,q)$ for some $k \in I$; but then also $b \in G_k$ and $l_k(b,p,q)$ by $l(b,p,q)$; so (G2) for $(\biguplus_i G_i, l)$ follows from (G2) for (G_k, l_k). Finally, it suffices to check (G3) in the case where $\mathsf{card}\{a,b,c,d,p\} = 5$; but by the hypotheses $l(p,a,b)$ and $l(p,c,d)$ these points must then all lie in the same G_k and satisfy $l_k(p,a,b)$ and $l_k(p,c,d)$; so applying (G3) to (G_k, l_k) proves (G3) for $(\biguplus_i G_i, l)$.

From the definition of the projective geometry $(\biguplus_i G_i, l)$ it follows directly that, for $a, b \in \biguplus_i G_i$, $a \star b = a \star_k b$ if $a, b \in G_k$ and $a \star b = \{a,b\}$ otherwise. From this it follows in turn that a sub*set* $S \subseteq \biguplus_i G_i$ is a sub*space* if and only if, for every $k \in I$, $S \cap G_k$ is a subspace of G_k. But then, referring to the maps in (3), since $s_k^*(S) = S \cap G_k$ these maps are morphisms (with empty kernels) of projective geometries, forming a cocone in $\mathsf{ProjGeom}$.

Suppose finally that $(g_k \colon G_k \dashrightarrow G)_{k\in I}$ is another cocone in $\mathsf{ProjGeom}$; we claim that

$$g \colon \biguplus_i G_i \dashrightarrow G \colon a \mapsto g_k(a) \text{ if } a \in G_k \setminus K_{g_k}, \text{ with kernel } \biguplus_i K_{g_i}$$

is the unique morphism of projective geometries satisfying $g \circ s_k = g_k$ for all $k \in I$. To see this, note first that $g^*(S) = \biguplus_i g_i^*(S)$ for a subspace $S \subseteq G$; so $g^*(S) \cap G_k = g_k^*(S)$ for $k \in I$, and since these are subspaces of the respective G_k's, it follows that $g^*(S)$ is a subspace of $\biguplus_i G_i$. Hence g is a morphism of projective geometries; and obviously $g \circ s_k = g_k$ for all $k \in I$. If $\overline{g} \colon \biguplus_i G_i \dashrightarrow G$ is another such morphism, then necessarily $K_{\overline{g}} = \overline{g}^*(\emptyset) = \biguplus_i K_{g_i} = K_g$; and for $a \in G_k \setminus K_{g_k}$, $\overline{g}(a) = \overline{g}(s_k(a)) = g_k(a) = g(s_k(a)) = g(a)$. That is to say, $\overline{g} = g$, and we thus verified the universal property of the cocone in (3). □

Clearly, any coproduct of two or more (non-empty) projective geometries is reducible. In fact, a discrete projective geometry G as in 3.3 is nothing but the coproduct of the singleton projective geometries $(\{a\})_{a\in G}$.

As the terminology suggests, every projective geometry G can be "reduced" to a coproduct of irreducible ones. Note first that a subspace $S \subseteq G$ of a projective geometry (G, l) is a projective geometry for the inherited collinearity relation; and the inclusion $S \hookrightarrow G$ is then a morphism (with empty kernel) of projective geometries. We say that $S \subseteq G$ is an **irreducible subspace** when it is irreducible as projective geometry in its own right; and S is a **maximal irreducible subspace** if moreover it is not strictly contained in any other irreducible subspace. This terminology is consistent: a projective geometry G is irreducible if and only if $G \subseteq G$ is a (trivially maximal) irreducible subspace.

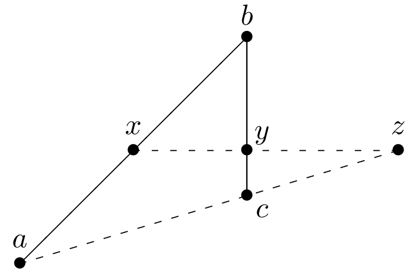

Figure 5: Illustration for the proof of 3.5

Theorem 3.5 *Any projective geometry G is the coproduct in* $\mathsf{ProjGeom}$ *of its maximal irreducible subspaces, which are precisely the equivalence classes of the equivalence relation on G defined as: $a \sim b$ if $\mathsf{card}(a \star b) \neq 2$.*

Proof. If G has no points, then there is nothing to prove, so in the rest of this proof we suppose that $G \neq \emptyset$.

First we check that the binary relation $\sim$ on G is an equivalence relation. Reflexivity and symmetry are trivial; for the transitivity we argue as follows (see also figure 5). Suppose that $a, b, c \in G$ are different non-collinear points, that x is a third point on the line $a \star b$ and that y is a third point on $b \star c$; then there exists a third point z on $a \star c$. Indeed, we have from $l(b, a, x)$ and $l(b, c, y)$ that $l(z, a, c)$ and $l(z, x, y)$ for some $z \in G$. Would $z = a$ then $l(y, a, x)$, $l(b, a, x)$ and $l(y, b, c)$ imply that $y \in (a \star x) \cap (b \star c) = (a \star b) \cap (b \star c) = \{b\}$, which is in contradiction with the hypothesis that $y \neq b$; so $z \neq a$, and similarly $z \neq c$.

An equivalence class $[a]$ for this relation is a subspace of G: if $x \neq y \in [a]$ and $z \in x \star y$, then either $z = x \in [a]$, or $z = y \in [a]$, or $\mathsf{card}\{x, y, z\} = 3$ and $l(x, y, z)$ so $z \sim x$ thus $z \in [a]$ by transitivity. In fact, $[a]$ is an irreducible subspace, because $x, y \in [a]$ implies that $x \sim y$ so that, when $x \neq y$, automatically $\mathsf{card}(x \star y) \neq 2$. Would $[a]$ be contained in another irreducible subspace $S \subseteq G$, then $a \star x \subseteq S$ for every $x \in S$ but at the same time $\mathsf{card}(a \star x) \neq 2$; so in fact $x \sim a$, whence $x \in [a]$. This means that $[a]$ is a maximal irreducible subspace. Further, if $S \subseteq G$ is *any* non-empty irreducible subspace, then all of its elements are equivalent, hence S is contained in one of the equivalence classes. This proves that the latter are *precisely* all the maximal irreducible subspaces of G.

Finally, as for any equivalence relation, the equivalence classes of $\sim$ form a covering by disjoint subsets of G. As any subspace, $[a]$ becomes a projective geometry in its own right for the inherited collinearity and the inclusions $[a] \hookrightarrow G$ are morphisms (with empty kernels) of projective geometries. Moreover, it is straightforward that, for $x, y, z \in G$, $l(x, y, z)$ if and only if *either* x, y, z are collinear in $[x]$ *or* $\mathsf{card}\{x, y, z\} \leq 2$. By 3.4, G is thus the coproduct of the equivalence classes of $\sim$. $\square$

The following categorical characterization of irreducible projective geometries is now an easy corollary.

Corollary 3.6 *A projective geometry G is irreducible if and only if it is not a coproduct in* $\mathsf{ProjGeom}$ *of two or more non-empty projective geometries.*

Interestingly, morphisms of projective geometries behave well with respect to irreducible components, as the next proposition shows.

Proposition 3.7 *If a morphism of projective geometries $g: G_1 - \to G_2$ has an irreducible domain, then its image lies in a maximal irreducible subspace of G_2.*

Proof. If $g(a) \neq g(b)$ for $a, b \in G_1 \setminus K_g$, then $a \neq b$ so there exists a $c \in G_1$, different from a and b, such that $c \in a \star b$. We shall show that $g(c)$ is different from $g(a)$ and $g(b)$ and lies on $g(a) \star g(b)$, so that $g(a)$ and $g(b)$ indeed lie in the same maximal irreducible subspace of G_2.

Suppose first that $c \in K_g$; then $b, c \in g^*(\{g(b)\})$ thus $a \in b \star c \subseteq g^*(\{g(b)\})$: this is in contradiction with $a \notin K_g$ and $g(a) \neq g(b)$. Hence we know that $c \notin K_g$, and therefore $c \in a \star b \subseteq g^*(g(a) \star g(b))$ implies that $g(c) \in g(a) \star g(b)$. Would now $g(c) = g(a)$, then $a, c \in g^*(\{g(a)\})$ and this implies a contradiction in the same way as before; so $g(c) \neq g(a)$. Similarly one shows that $g(c) \neq g(b)$. □

A morphism $g: G_1 - \to G_2$ of projective geometries is, by the universal property of the coproduct, the same thing as a family $(g^i: G_1^i - \to G_2)_{i \in I}$ of morphisms, where $(G_1^i)_{i \in I}$ denotes the family of maximal irreducible subspaces of G_1. Writing $(G_2^j)_{j \in J}$ for the family of maximal irreducible subspaces of G_2, we know by 3.7 that each image $g^i(G_1^i)$ lies in some $G_2^{j_i}$. Hence g can be "reduced" to a family $(g^i: G_1^i - \to G_2^{j_i})_{i \in I}$ of morphisms between irreducible projective geometries. This goes to show that, when studying projective geometry, we can limit our attention to irreducible geometries and morphisms between them; after all, the reducible ones can be "regenerated by taking coproducts".

Since the categories $\mathsf{ProjGeom}$ and $\mathsf{ProjLat}$ are equivalent, the previous results on projective geometries have twin siblings for projective lattices. We shall go through the translation from geometries to lattices. First a word on the construction of coproducts of projective lattices.

Lemma 3.8 *For a family of projective lattices $(L_i)_{i \in I}$, the cartesian product of sets $\times_i L_i$ equipped with componentwise order, together with the inclusion maps*

$$s_k: L_k \longrightarrow \times_i L_i: x \mapsto (x_i)_{i \in I} \tag{4}$$

where $x_k = x$ and $x_i = 0$ for $i \neq k$, is a coproduct in $\mathsf{ProjLat}$.

Proof. By the categorical equivalence $\mathsf{ProjGeom} \simeq \mathsf{ProjLat}$ and 3.4, we already know that coproducts exist in $\mathsf{ProjLat}$; we shall quickly verify their explicit construction as given in the statement of the lemma.

First we check that the cartesian product $\times_i L_i$ is a projective lattice whenever the L_i's are[2]. Since $\times_i L_i$ has the componentwise structure, it is clear that it is complete and modular; in particular is the zero tuple $0 = (0_i)_i$ its least element. An atom in $\times_i L_i$ is precisely an element $a = (a_i)_i$ with all components zero except for one a_k which is an atom in L_k; thus it follows easily that $\times_i L_i$ is atomistic too. As for the continuity of $\times_i L_i$, it now suffices by 2.7 to show that its atoms are compact: but if $a \leq \bigvee_\alpha x^\alpha$ for some atom a and a directed family $(x^\alpha)_{\alpha \in A}$ in $\times_i L_i$, then (supposing that the non-zero component of $a = (a_i)_i$ is the atom $a_k \in L_k$) necessarily $a_k \leq \bigvee_\alpha x_k^\alpha$ in L_k. Since a_k is compact in L_k, we have $a_k \leq x_k^\beta$ for some $\beta \in A$, and thus also $a \leq x^\beta$ because the components of a other than a_k are zero.

It is a consequence of these observations that the maps in (4) preserve suprema and send atoms onto atoms; thus they indeed constitute a cocone in $\mathsf{ProjLat}$. This cocone is universal, for

[2] The converse is also true; see 8.8.

if $(f_k\colon L_k \longrightarrow L)_{k\in I}$ is another cocone in $\mathsf{ProjLat}$, then the map

$$f\colon \times_i L_i \longrightarrow L\colon (x_i)_i \mapsto \bigvee_i f_i(x_i)$$

is clearly the unique morphism of projective lattices satisfying $f \circ s_k = f_k$ for all $k \in I$. $\square$

Since 3.6 tells us "in categorical terms" what the irreducibility of a projective geometry is all about, the following is entirely natural (given that under a categorical equivalence coproducts in one category correspond to coproducts in the other).

Definition 3.9 A projective lattice L is **irreducible** if it is not a coproduct in $\mathsf{ProjLat}$ of two (or more) non-trivial projective lattices.

Proposition 3.10 *Let G be a projective geometry and L a projective lattice that correspond to each other under the categorical equivalence* $\mathsf{ProjGeom} \simeq \mathsf{ProjLat}$. *Then G is irreducible if and only if L is irreducible.*

One can now deduce, again from the equivalence of projective geometries and projective lattices, the following statement.

Theorem 3.11 *Each projective lattice L can be written as a coproduct in* $\mathsf{ProjLat}$ *of irreducible projective lattices.*

We could have given a much more precise statement of the previous theorem: it would speak of "maximal irreducible segments" of a projective lattice as analogs for the maximal irreducible subspaces of a projective geometry, and so forth. But we do not really need this precision and detail further on, so we shall leave it to the interested reader to figure out the *exact* analog of 3.5.

On the other hand, in references on lattice theory such as G. Birkhoff's [1967] or F. Maeda and S. Maeda's [1970], the previous theorem is often given for a vastly larger class of lattices. Thereto one typically makes use of the very general notion of 'central element' of a (bounded) lattice. This highly interesting subject falls outside the scope of this chapter (but see also 8.8).

4. The Fundamental Theorems of projective geometry

In this section we will explain to what extent the functor $\mathcal{P}\colon \mathsf{Vec} \longrightarrow \mathsf{ProjGeom}$ can be "inverted": we will describe linear representations of projective geometries and the morphisms between them. It is a very nice result that the objects and morphisms in the image of $\mathcal{P}$ can indeed be characterized geometrically; this is the content of the age-old First Fundamental Theorem of projective geometry (for the objects) and the more recent[3] Second Fundamental Theorem (for the morphisms). Moreover, it turns out that $\mathcal{P}$ is "injective up to scalar" on so-called 'non-degenerate' semilinear maps (as stated explicitly in 4.19) and "injective up to isomorphism" on vector spaces of dimension at least 3 (as in 4.21).

We will only provide a brief sketch of the proof of the First Fundamental Theorem: we essentially outline the proof of R. Baer [1952, chapter VII] following the pleasant [Beutelspacher

[3]Calling the Second Fundamental Theorem "recent" for the case of isomorphisms would be quite a stretch: it can be found in [Baer, 1952, chapter III, §1] or [Artin, 1957, chapter II, §10] for example. However, the more general case we present here is due to due to Cl.-A. Faure and A. Frölicher [1994].

and Rosenbaum, 1998, chapter 3] (see also [Maeda and Maeda, 1970, §33–34]). For the morphisms we refer to the very short [Faure, 2002], which is inspired by and generalizes [Baer, 1952, §III.1]. All the details can be found in these references or in [Faure and Frölicher, 2000, chapters 8, 9 and 10].

In order to state the Fundamental Theorems of projective geometry properly, we need to introduce the 'dimension' of a projective geometry. We refer to [Baer, 1952, VII.2] and [Faure and Frölicher, 2000, chapter 4] for more on this.

Definition 4.1 A projective lattice L is **of finite rank** if its top element $1 \in L$ is a supremum of a finite number of atoms; the **rank** of L is then the minimum number of atoms required to write 1 as their supremum, written as $\mathsf{rk}(L)$. Otherwise L is **of infinite rank**, written $\mathsf{rk}(L) = \infty$. The **dimension** of a projective geometry G is $\mathsf{dim}(G) := \mathsf{rk}(\mathcal{L}(G)) - 1$ (which can be ∞).

Example 4.2 If V is a vector space of dimension n, then $\mathsf{dim}(\mathcal{P}(V)) = n - 1$. If V is of infinite dimension, then so is $\mathcal{P}(V)$.

Viewing a subspace of G as a projective geometry in its own right we may also speak of "the dimension of a subspace", which – as to be expected – is at most the dimension of G.

Example 4.3 The subspaces of dimension -1, 0 and 1 of a projective geometry G are respectively the empty subspace, the points and the projective lines. A projective geometry (or a subspace) of dimension 2 is called a **projective plane**. With this terminology we may say that the geometry in figure 4 (cf. 3.2) is the smallest irreducible projective plane: it is the so-called Fano plane.

Next we introduce some standard terminology for the projective geometries which are in the image of the functor $\mathcal{P}\colon \mathsf{Vec} \longrightarrow \mathsf{ProjGeom}$.

Definition 4.4 A projective geometry G **admits homogeneous coordinates** if there exists a vector space V such that $G \cong \mathcal{P}(V)$ in $\mathsf{ProjGeom}$.

For the rest of this section, we will assume all projective geometries to be **irreducible** *and to have* **dimension at least 2**. The first condition is obviously necessary if we are to construct homogeneous coordinates for a given projective geometry, cf. 3.2; and the latter excludes the trivial empty geometry, singletons and projective lines ("freak cases", as E. Artin [1957] calls them).

The following notion characterizes, as we will see, the "linearizable" projective geometries.

Definition 4.5 A projective geometry G is **arguesian** if it is irreducible, has dimension at least 2, and satisfies **Desargues' property**: for any choice of points $a_1, a_2, a_3, b_1, b_2, b_3 \in G$ for which

i. there is a $c \in G$ such that $l(c, a_j, b_j)$ and $c \neq a_j \neq b_j \neq c$ hold for $j = 1, 2, 3$,
ii. no three of the points c, a_1, a_2, a_3 and no three of the points c, b_1, b_2, b_3 are collinear,

we have that the three points $(a_i \star a_k) \cap (b_i \star b_k)$ with $i < k \in \{1, 2, 3\}$ are collinear (see figure 6).

As we will point out below, a theorem by R. Baer says that Desargues' property guarantees the existence of certain automorphisms of the geometry (4.14), a key result in the proof of the First Fundamental Theorem. It turns out that once a geometry is big enough, it is arguesian.

Proposition 4.6 *Every projective geometry of dimension at least* 3 *(including* ∞*) is arguesian.*

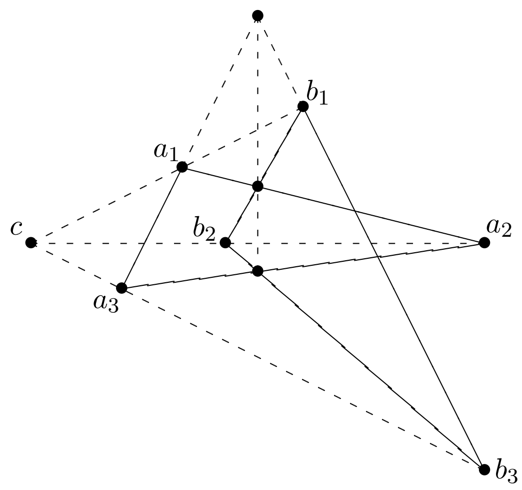

Figure 6: Desargues' property

Proof. See [Faure and Frölicher, 2000, 8.4.6] or [Beutelspacher and Rosenbaum, 1998, 2.7.1]. The idea here is that if a projective plane is strictly embedded in another projective geometry, then it satisfies Desargues' property. Artin [1957, p. 101] "suggests viewing the configuration [in figure 6] as the projection of a three-dimensional configuration onto the plane. The three-dimensional configuration is easily proved [from the axioms]." □

Example 4.7 A simple example of a projective plane that does not satisfy Desargues' property is the "Moulton plane" (after the mathematician F.R. Moulton [1902]). See e.g. [Beutelspacher and Rosenbaum, 1998, §2.6] for a description.

Example 4.8 It is an exercise in linear algebra that for a vector space V of dimension at least 3, $\mathcal{P}(V)$ is arguesian [Beutelspacher and Rosenbaum, 1998, 2.2.1].

The content of the First Fundamental Theorem is that Desargues' property is also sufficient for an irreducible projective geometry to admit homogeneous coordinates.

Theorem 4.9 (First Fundamental Theorem) *Every arguesian projective geometry admits homogeneous coordinates.*

We start the sketch of the proof by giving its general idea. We then proceed by a series of lemmas highlighting the key ingredients.

Definition 4.10 A **hyperplane** H of a projective geometry G is a maximal strict subspace $H \subset G$: it is thus a coatom† of $\mathcal{L}(G)$.

The example to keep in mind for the proof of the existence of homogeneous coordinates is the following well-known construction of a projective geometry by "adding a hyperplane at infinity" to a vector space.

Example 4.11 [Faure and Frölicher, 2000, 2.1.7] For a vector space V over K we can give the disjoint union $V \uplus \mathcal{P}(V)$ the structure of a projective geometry using the bijection

$$\mathcal{P}(V \times K) \longrightarrow V \uplus \mathcal{P}(V) \colon K(x,\xi) \mapsto \begin{cases} \xi^{-1}x \in V \text{ if } \xi \neq 0 \\ Kx \in \mathcal{P}(V) \text{ if } \xi = 0 \end{cases}$$

Given an arguesian geometry G, we construct homogeneous coordinates $G \cong \mathcal{P}(V \times K) \cong V \uplus \mathcal{P}(V)$ by "recovering" the group of translations of the desired vector space V and its group of homotheties as certain **collineations** of G (i.e. isomorphisms from G to itself in $\mathsf{ProjGeom}$) that fix a chosen hyperplane (which will be $\mathcal{P}(V)$).

Definition 4.12 A collineation α of G is called a **central collineation** if there is a hyperplane H of G (the **axis** of α) and a point $c \in G$ (the **center** of α) such that α fixes every point of H and every line through c.

Central collineations are very rigid, as the following lemma shows.

Lemma 4.13 *Let α be a central collineation of G with axis H and center c. Let p be a point in $G \setminus (\{c\} \cup H)$. Then α is uniquely determined by $\alpha(p)$. In particular, for every $x \in G$ not on H nor on $c \star p$, we have*

$$\alpha(x) = (c \star x) \cap (f \star \alpha(p)) \tag{5}$$

where $f := (p \star x) \cap H$.

Proof. See [Beutelspacher and Rosenbaum, 1998, 3.1.3] Note that $\alpha(x) \in c \star x$ because α has center c and $\alpha(x) \in f \star \alpha(p)$ because $x \in p \star x$ and α has axis H. These two lines intersect in a point because they are distinct (as $x \notin c \star p = c \star \alpha(p)$) and they lie in the plane spanned by p, x and c. $\square$

Here is the announced theorem where Desargues' property comes into play.

Lemma 4.14 (Baer's existence theorem of central collineations) *Let G be arguesian. If H is a hyperplane and c, p, p' are distinct collinear points of G with $p, p' \notin H$, then there is exactly one collineation of G with center c and axis H mapping p to p'.*

Proof. See [Beutelspacher and Rosenbaum, 1998, 3.1.8] or [Faure and Frölicher, 2000, 8.4.11]. The basic idea is to use (5) to define the map. Desargues' property is then used to show that it is well-defined and a collineation, through rather lengthy geometric verifications. $\square$

We now fix a hyperplane H of G and a point $o \in G \backslash H$. Let T be the set of collineations with axis H and center *on* H. We call an element of T a **translation**.

Lemma 4.15 *T is an abelian group (under composition) which acts simply transitively on $G \backslash H$. Translating this action of T into an addition on $V := G \setminus H$, the latter also becomes an abelian group.*

Proof. [Beutelspacher and Rosenbaum, 1998, 3.2.2] Simple transitivity of the action means that if $p \neq q$ in $G \backslash H$ then there is a unique collineation of G with axis H and center $(p \star q) \cap H$ sending p to q; it is a consequence of 4.14. The fact that T is a subgroup of the group of collineations uses the fact that if a collineation has an axis, it has a center, and the rigidity of lemma 4.13. The commutativity of the group requires a little more work.

The simple transitivity allows us to transport the group structure from T to V. Indeed, for $p \in V$, denote τ_p the unique element in T such that $\tau_p(o) = p$. For $p, q \in V$, we then put

$$p + q := \tau_p(q) = \tau_p(\tau_q(o)).$$

$\square$

Next, let $K^\times$ be the group (under composition) of all collineations of G with center o and axis H. We call an element of $K^\times$ a **homothety**. It is an immediate consequence of 4.14 that $K^\times$ acts simply transitively on $L \setminus \{o\}$ for every line L through o. Let α_o be the constant morphism $G \longrightarrow G: p \mapsto o$.

Lemma 4.16 *If on the set $K := K^\times \cup \{\alpha_o\}$, we define addition by*

$$(\sigma_1 + \sigma_2)(x) := \sigma_1(x) + \sigma_2(x) \textit{ for every } x \in V$$

and multiplication by composition, then K becomes a field.

Proof. [Beutelspacher and Rosenbaum, 1998, 3.3.4] The main difficulty here is showing that K is closed under this addition. □

This multiplication is not commutative in general. *Pappus' Theorem* geometrically characterizes those arguesian geometries for which it is (see [Artin, 1957, chapter II, §7] for example).

Lemma 4.17 *The action of K on V by*

$$K \times V \longrightarrow V: (\sigma, x) \mapsto \sigma \cdot x := \sigma(x)$$

is a "scalar multiplication" making V a (left) vector space over K.

Proof. [Beutelspacher and Rosenbaum, 1998, 3.3.5] Showing that $\sigma(-x) = -\sigma(x)$ is what requires the most (but not that much) work. □

The next lemma then finishes the proof of 4.9.

Lemma 4.18 *G is isomorphic as a projective geometry to $\mathcal{P}(V \times K)$.*

Proof. [Beutelspacher and Rosenbaum, 1998, 3.4.2] Remember that $V = G \backslash H$. The isomorphism $\varphi: G \longrightarrow \mathcal{P}(V \times K)$ is defined by

$$\varphi(x) := \begin{cases} K(x, 1) \text{ if } x \in G \backslash H \\ K(y, 0) \text{ if } x \in H, \text{ where } y \neq x \text{ is any point of } o \star x \end{cases}$$

which identifies H with $\mathcal{P}(V)$ as expected. □

Having dealt with objects, we move on to the linear representation of (some of) the morphisms of $\mathsf{ProjGeom}$. Cl.-A. Faure [2002] has provided a short and elementary proof of the next theorem, which originally appeared in [Faure and Frölicher, 1994].

Theorem 4.19 (Second Fundamental Theorem) *Let V_1 and V_2 be vector spaces. Every* **non-degenerate morphism** *$g: \mathcal{P}(V_1) \dashrightarrow \mathcal{P}(V_2)$, meaning that its image contains three non-collinear points, is of the form $\mathcal{P}(f)$ for some semilinear map $f: V_1 \longrightarrow V_2$. Moreover f is unique up to scalar multiplication.*

The uniqueness is an immediate consequence of the following fact (see e.g. [Faure, 2002, 2.4]), which we record here for future reference. Its proof is an exercise in linear algebra.

Proposition 4.20 *Let $f_1, f_2 \colon V_1 \longrightarrow V_2$ be two additive maps between vector spaces (K_1, V_1) and (K_2, V_2). Assume that $f_2(x) \in K_2 f_1(x)$ for every $x \in V_1$ and that $f_1(V_1)$ contains two linearly independent vectors. Then there exists a $\mu \in K_2$ such that $f_2 = \mu f_1$.*

The Second Fundamental Theorem implies that the vector space whose existence was guaranteed by the First Fundamental Theorem is essentially unique.

Corollary 4.21 *If $\varphi \colon \mathcal{P}(V_1) \longrightarrow \mathcal{P}(V_2)$ is an isomorphism in $\mathsf{ProjGeom}$ where V_1 is a K_1-vector space of dimension at least 3 and V_2 is a K_2-vector space (with K_1, K_2 fields), then there exists a field isomorphism $\sigma \colon K_1 \longrightarrow K_2$ and a bijective σ-linear map $f \colon V_1 \longrightarrow V_2$ such that $\varphi = \mathcal{P}(f)$.*

Remark that the uniqueness of homogeneous coordinates holds for projective dimension at least 2 while existence needs dimension at least 3 (or Desargues' property).

A composition of two non-degenerate morphisms need not be non-degenerate (think of the composition $f \circ g$ of two linear maps with $\mathsf{im} g \subseteq \mathsf{ker} f$). A morphism of projective geometries is called **arguesian** when it is the composite of finitely many non-degenerate morphisms. The following proposition [Faure and Frölicher, 2000, 10.3.1] says that these are exactly the morphisms induced by semilinear maps.

Proposition 4.22 *For a partial map $g : \mathcal{P}(V_1) \dashrightarrow \mathcal{P}(V_2)$ between arguesian geometries, the following conditions are equivalent:*

i. g is induced by a semilinear map $f \colon V_1 \longrightarrow V_2$,
ii. g is the composite of two non-degenerate morphisms between arguesian geometries,
iii. g is the composite of finitely many non-degenerate morphisms between arguesian geometries.

Proof. The only nontrivial implication is (i $\Rightarrow$ ii). Let σ be the field homomorphism associated to f. By hypothesis $\mathsf{dim}(V_2) \geq 3$ and we can pick three linearly independent vectors $y_1, y_2, y_3 \in V_2$. Put $W := V_1 \times K_1^3$. Now define the maps

$$f_1 \colon V_1 \longrightarrow W : x \mapsto (x, 0, 0, 0)$$
$$f_2 \colon W \longrightarrow V_2 : (x, k_1, k_2, k_3) \mapsto f(x) + \sigma(k_1) y_1 + \sigma(k_2) y_2 + \sigma(k_3) y_3$$

Then $f = f_2 \circ f_1$ and thus $g = \mathcal{P}(f) = \mathcal{P}(f_2) \circ \mathcal{P}(f_1)$, where $\mathcal{P}(f_1)$ and $\mathcal{P}(f_2)$ are clearly non-degenerate. $\square$

We define the category Arg of arguesian projective geometries and arguesian morphisms. The Fundamental Theorems may then be summarized in the following statement, which is as powerful as one could hope.

Theorem 4.23 *The functor $\mathcal{P} \colon \mathsf{Vec}_{\mathsf{dim} \geq 3} \longrightarrow \mathsf{Arg}$ is essentially surjective and essentially injective on objects, full, and only identifies semilinear maps when they are a nonzero scalar multiple of each other.*

5. Hilbert geometries, Hilbert lattices, propositional systems

A (real, complex, quaternionic or generalized) Hilbert space H is in particular a vector space, so by 2.8 its one-dimensional linear subspaces form a projective geometry $\mathcal{P}(H)$. But the orthogonality relation on the elements of H, defined as $x \perp y$ if and only if the inner product of x and y is zero, obviously induces an *orthogonality relation* on $\mathcal{P}(H)$: $A \perp B$ in $\mathcal{P}(H)$ when $a \perp b$ for some $a \in A \setminus \{0\}$ and $b \in B \setminus \{0\}$. We make an abstraction of this.

Definition 5.1 Given a binary relation $\perp \subseteq G \times G$ on a projective geometry G and a subset $A \subseteq G$, we put $A^{\perp} := \{b \in G \mid \forall a \in A : b \perp a\}$. A **Hilbert geometry** G is a projective geometry together with an **orthogonality relation** $\perp \subseteq G \times G$ such that, for all $a, b, c, p \in G$,

(O1) if $a \perp b$ then $a \neq b$,
(O2) if $a \perp b$ then $b \perp a$,
(O3) if $a \neq b$, $a \perp p$, $b \perp p$ and $l(a, b, c)$ then $c \perp p$,
(O4) if $a \neq b$ then there is a $q \in G$ such that $l(q, a, b)$ and $q \perp a$,
(O5) if $S \subseteq G$ is a subspace such that $S^{\perp\perp} = S$, then $S \vee S^{\perp} = G$.

Very often we shall simply speak of a "Hilbert geometry G", leaving both the collinearity l and the orthogonality $\perp$ understood. A subspace $S \subseteq G$ is said to be **(biorthogonally) closed** if $S^{\perp\perp} = S$.

Axioms (O1–4) in the above definition say in particular that a Hilbert geometry is a 'state space' in the sense of [Moore, 1995] as we explain in 8.2. The fifth axiom could have been written as: $S = S^{\perp\perp}$ if and only if $S \vee S^{\perp} = G$, because (as we shall show in 5.6 (iv) in a more abstract setting) the necessity is always true. We make some more comments on these axioms in section 8.

The term 'Hilbert geometry' is well-chosen, as C. Piron's now famous example [1964, 1976] shows.

Definition 5.2 A **generalized Hilbert space** (also called **orthomodular space**) $(H, K, {}^*, \langle\ ,\ \rangle)$ is a vector space H over a field K together with an involutive anti-automorphism $K \longrightarrow K : \alpha \mapsto \alpha^*$ and an **orthomodular Hermitian form** $H \times H \longrightarrow K : (x, y) \mapsto \langle x, y\rangle$, that is, a form satisfying

(S1) $\langle \lambda x_1 + x_2, y\rangle = \lambda\langle x_1, y\rangle + \langle x_2, y\rangle$ for all $x_1, x_2, y \in H, \lambda \in K$,
(S2) $\langle y, x\rangle = \langle x, y\rangle^*$ for all $x, y \in H$,

and such that, when putting $S^{\perp} := \{x \in H \mid \forall y \in S : \langle x, y\rangle = 0\}$ for a linear subspace $S \subseteq H$,

(S3) $S = S^{\perp\perp}$ implies $S \oplus S^{\perp} = H$.

Note that an orthomodular Hermitian form is automatically **anisotropic**,

(S4) $\langle x, x\rangle \neq 0$ for all $x \in H \setminus \{0\}$,

and that in the finite dimensional case the converse is true too. I. Amemiya and H. Araki [1966] proved that when K is one of the "classical" fields equipped with its "classical" involution ($\mathbb{R}$ with identity, $\mathbb{C}$ and $\mathbb{H}$ with their respective conjugations), the definition of 'generalized Hilbert space' is equivalent to the "classical" definition of a Hilbert space as inner-product space which is complete for the metric induced by the norm. While H. Keller [1980] was the first to construct a "nonclassical" generalized Hilbert space, M. Solèr [1995] proved that an infinite dimensional generalized Hilbert space H is "classical" precisely when H contains an orthonormal sequence. We refer to [Holland, 1995] for a nice survey, and to A. Prestel's [2006] contribution to this handbook for a complete and historically annotated proof of Solèr's theorem. For a comment on the lattice-theoretic meaning of Solèr's theorem, see 8.10.

Example 5.3 For a generalized Hilbert space H, the projective geometry $\mathcal{P}(H)$ together with the obvious orthogonality relation forms a Hilbert geometry: axioms (O1–3) are immediate, (O4) follows from a standard Gram-Schmidt trick and (O5) is also immediate since $S \vee S^\perp = S \oplus S^\perp$ for any linear subspace S of H.

From our work in section 2 we know that, since a Hilbert geometry is in particular a projective geometry, the lattice of subspaces $\mathcal{L}(H)$ is a projective lattice. Because of the orthogonality relation on G, there is some extra structure on $\mathcal{L}(G)$; the following proposition identifies it.

Proposition 5.4 *If G is a Hilbert geometry with orthogonality relation $\perp$, then the operator ${}^\perp: \mathcal{L}(G) \longrightarrow \mathcal{L}(G): S \mapsto S^\perp$ satisfies, for all $S, T \in \mathcal{L}(G)$,*

i. $S \subseteq S^{\perp\perp}$,
ii. *if* $S \subseteq T$ *then* $T^\perp \subseteq S^\perp$,
iii. $S \cap S^\perp = \emptyset$,
iv. *if* $S = S^{\perp\perp}$ *and* $a \in G$ *then* $\{a\} \vee S = (\{a\} \vee S)^{\perp\perp}$.
v. *if* $S = S^{\perp\perp}$ *then* $S \vee S^\perp = G$.

Proof. All is straightforward, except for (iv). We need to prove that $(\{a\} \vee S)^{\perp\perp} \subseteq \{a\} \vee S$ for $S = S^{\perp\perp}$. If $a \in S$ then this is trivial so we suppose from now on that $a \notin S = S^{\perp\perp}$, i.e. there exists $p \in S^\perp$ such that $a \not\perp p$. Let $b \in (\{a\} \vee S)^{\perp\perp}$; if $b = a$ or $b \in S$ then obviously $b \in \{a\} \vee S$. If $b \neq a$ and $b \notin S$ then we claim that $(a \star b) \cap \{p\}^\perp$ is a singleton and moreover that its single element, call it q, belongs to S. This then proves the assertion, for $q \perp p$ implies $q \neq a$, which makes $q \in a \star b$ imply that $b \in a \star q \subseteq \{a\} \vee S$.

Now $(a \star b) \cap \{p\}^\perp$ is non-empty[4], because in case that $a \not\perp p \not\perp b$ we can always pick $x \in p \star a$ and $y \in p \star b$ such that $x, y \in \{p\}^\perp$ by (O4); then $x \neq a$ and $y \neq b$ so $p \in (a \star x) \cap (b \star y)$ and (G3) thus gives a $q \in (a \star b) \cap (x \star y) \subseteq (a \star b) \cap \{p\}^\perp$ (for $\{p\}^\perp$ is a subspace by (O3)). Would $q_1 \neq q_2 \in (a \star b) \cap \{p\}^\perp$, then $l(q_1, q_2, a)$ by (G2) hence $a \in \{p\}^\perp$, a contradiction. So we conclude that $(a \star b) \cap \{p\}^\perp = \{q\}$.

We shall show that $q \in S = S^{\perp\perp}$, i.e. for any $r \in S^\perp$ we have $q \perp r$. For $r = p$ this is true by construction; for $r \neq p$ we may determine, by the "same" argument as above, a (unique) point $s \in \{a\}^\perp \cap (p \star r) \subseteq (\{a\}^\perp \cap S^\perp) = (\{a\} \vee S)^\perp$. The latter equality can be shown with a simple calculation, but we also give a more abstract proof in 5.6 (iii). Because $a, b \in (\{a\} \vee S)^{\perp\perp}$ it follows that $a \perp s$ and $b \perp s$; hence we get $q \perp s$ from $q \in a \star b$ and (O3). But also $s \neq p$ follows, thus $s \in p \star r$ implies $r \in p \star s$, and because we know that $q \perp p$ too, we finally obtain $q \perp r$, again from (O3). $\square$

This proposition calls for a new definition.

Definition 5.5 A projective lattice L is a **Hilbert lattice** if it comes with an **orthogonality operator** ${}^\perp: L \longrightarrow L: x \mapsto x^\perp$ satisfying, for all $x, y \in L$,

(H1) $x \leq x^{\perp\perp}$,
(H2) if $x \leq y$ then $y^\perp \leq x^\perp$,
(H3) $x \wedge x^\perp = 0$,

[4]What we *really* prove here is that for a, b, p with $a \neq b$ in a Hilbert geometry G there always exists some $q \in a \star b$ such that $q \perp p$. This statement, which is obviously stronger than (O4), is often used instead of (O4). See 8.3 for a relevant comment.

(H4) if $x = x^{\perp\perp}$ and a is an atom of L then $a \vee x = (a \vee x)^{\perp\perp}$,
(H5) if $x = x^{\perp\perp}$ then $x \vee x^{\perp} = 1$.

Usually we shall simply speak of "a Hilbert lattice L", and leave the orthogonality operator understood.

The crux of 5.4 is thus that the projective lattice of subspaces of a Hilbert geometry is a Hilbert lattice. Having 2.10 in mind, it should not come as a surprise that there is a converse to this. However, we shall not give a direct proof of such a statement, for we wish to involve yet another mathematical structure. Again the source of inspiration is the concrete example of Hilbert spaces: the subspaces $S \subseteq H$ for which $S = S^{\perp\perp}$ are particularly important, for they are precisely the subspaces which are closed for the norm topology on H (see [Schwartz, 1970, p. 392] for example). Also in the abstract case they are worth a closer look.

By a **(biorthogonally) closed element** of a Hilbert lattice $(L, {}^{\perp})$ we shall of course mean an $x \in L$ for which $x = x^{\perp\perp}$. We write $\mathcal{C}(L) \subseteq L$ for the ordered set of closed elements, with order inherited from L. We shall now discuss some features of this ordered set that – as it will turn out – describe it completely. First we prove a technical lemma.

Lemma 5.6 *For a Hilbert lattice L,*

i. if $x \in L$ then $x^{\perp} \in \mathcal{C}(L)$,
ii. $0^{\perp\perp} = 0$, $0^{\perp} = 1$ and $1^{\perp} = 0$,
iii. $(\bigvee_i x_i)^{\perp} = \bigwedge_i x_i^{\perp}$ for $(x_i)_{i \in I} \in L$,
iv. if $x \vee x^{\perp} = 1$ then $x = x^{\perp\perp}$,
v. the map ${}^{\perp\perp}\colon L \longrightarrow L\colon x \mapsto x^{\perp\perp}$ is a closure operator with fixpoints $\mathcal{C}(L)$.

Proof. Statements (i) and (ii) are almost trivial. For (iii) one uses (H1–2) over and again to verify $\geq$ and $\leq$ as follows:

$\forall k \in I : \bigwedge_i x_i^{\perp} \leq x_k^{\perp}$	$\forall k \in I : x_k \leq \bigvee_i x_i$
$\Rightarrow \forall k \in I : x_k \leq x_k^{\perp\perp} \leq (\bigwedge_i x_i^{\perp})^{\perp}$	$\Rightarrow \forall k \in I : (\bigvee_i x_i)^{\perp} \leq x_k^{\perp}$
$\Rightarrow \bigvee_i x_i \leq (\bigwedge_i x_i^{\perp})^{\perp}$	$\Rightarrow (\bigvee_i x_i)^{\perp} \leq \bigwedge_i x_i^{\perp}$
$\Rightarrow \bigwedge_i x_i^{\perp} \leq (\bigwedge_i x_i^{\perp})^{\perp\perp} \leq (\bigvee_i x_i)^{\perp}$	

As for (iv), the assumption together with (H3) and modularity in L (for $x \leq x^{\perp\perp}$) give $x = x \vee 0 = x \vee (x^{\perp} \wedge x^{\perp\perp}) = (x \vee x^{\perp}) \wedge x^{\perp\perp} = 1 \wedge x^{\perp\perp} = x^{\perp\perp}$. Finally, it straightforwardly follows from (H2) that

$$\varphi\colon L \longrightarrow \mathcal{C}(L)\colon x \mapsto x^{\perp\perp} \quad \text{and} \quad \psi\colon \mathcal{C}(L) \longrightarrow L\colon y \longrightarrow y$$

are maps that preserve order, and they satisfy $\varphi(x) \leq y \Leftrightarrow x \leq \psi(y)$ for any $x \in L$ and $y \in \mathcal{C}(L)$. So these maps are adjoint, $\varphi \dashv \psi$, and since moreover φ is surjective and ψ injective, the composition $\psi \circ \varphi\colon L \longrightarrow L\colon x \mapsto x^{\perp\perp}$ is a closure operator with fixpoints $\mathcal{C}(L)$, as claimed in (v). $\square$

For closed elements $(x_i)_{i \in I} \in L$ we shall write $\nabla_i x_i$ for $(\bigvee_i x_i)^{\perp\perp}$, and in particular $x \nabla y$ for $(x \vee y)^{\perp\perp}$.

Proposition 5.7 *For any Hilbert lattice L, the ordered set $(\mathcal{C}(L), \leq)$ together with the restricted operator ${}^{\perp}\colon \mathcal{C}(L) \longrightarrow \mathcal{C}(L)\colon x \mapsto x^{\perp}$ is a complete, atomistic, orthomodular[†] lattice satisfying the covering law.*

Proof. By (v) of 5.6, $\mathcal{C}(L)$ is a complete lattice inheriting infima from L, and with suprema given by ∇. Moreover, (H2–3) assert that $x \mapsto x^\perp$ is an orthocomplementation[†] on $\mathcal{C}(L)$. It is straightforward from (H4) and 5.6 (ii) that the atoms of L are closed; and conversely is it clear that the atoms of $\mathcal{C}(L)$ are atoms of L too. So $\mathcal{C}(L)$ is atomistic, because L is. In the same way, since L has the covering law (cf. 2.9) and the atoms of L are precisely those of $\mathcal{C}(L)$, again (H4) assures that $\mathcal{C}(L)$ has the covering law too. Finally, if $x \leq y$ in $\mathcal{C}(L)$ then by the modular law in L and (H5)

$$x \nabla (x^\perp \wedge y) = (x \vee (x^\perp \wedge y))^{\perp\perp} = ((x \vee x^\perp) \wedge y)^{\perp\perp} = (1 \wedge y)^{\perp\perp} = y^{\perp\perp} = y;$$

i.e. the orthomodular law holds in $\mathcal{C}(L)$. $\square$

Example 5.8 By (H4) it follows that, if $a_1, ..., a_n$ are atoms of a Hilbert lattice L, then (each one of them is closed and) $a_1 \nabla ... \nabla a_n = a_1 \vee ... \vee a_n$. If L is a Hilbert lattice of finite rank, then every $x \in L$ can be written as a *finite* supremum of atoms (this is true for any atomistic lattice satisfying the covering law of finite rank, see e.g. [Maeda and Maeda, 1970, section 8]), hence $x = x^{\perp\perp}$; i.e. $L \cong \mathcal{C}(L)$. So if G is a Hilbert geometry of finite dimension, then every subspace of G is biorthogonally closed; in particular is this true for $\mathcal{P}(H)$ when H is a (generalized) Hilbert space of finite dimension.

Inspired by the result in 5.7 we now give another definition due to C. Piron [1964, 1976].

Definition 5.9 An ordered set $(C, \leq)$ with an operator $^\perp : C \longrightarrow C : x \mapsto x^\perp$ is a **propositional system** if it is a complete, atomistic, orthomodular lattice that satisfies the covering law (with $x \mapsto x^\perp$ as orthocomplementation).

We shall speak of "a propositional system C", always using $x^\perp$ as notation for the orthocomplement of $x \in C$. And we shall continue to write $\nabla_i x_i$ for the supremum in C, and $\bigwedge_i x_i$ for the infimum.

Example 5.10 The closed subspaces of a generalized Hilbert space H form a propositional system, that we shall write as $\mathcal{C}(H)$ instead of $\mathcal{C}(\mathcal{L}(H))$.

According to 5.7 and 5.9, the closed elements of a Hilbert lattice form a propositional system. Earlier we proved (cf. 5.4 and 5.5) that the subspaces of a Hilbert geometry form a Hilbert lattice. It is now time to come full circle: we want to associate a Hilbert geometry to a given propositional system. The lattice-theoretical results in 2.9 will be useful here too.

Proposition 5.11 *The set $\mathcal{G}(C)$ of atoms of a propositional system C form a Hilbert geometry for collinearity and orthogonality given by*

$$l(a, b, c) \text{ if } a \leq b \nabla c \text{ or } b = c, \qquad a \perp b \text{ if } a \leq b^\perp. \tag{6}$$

Proof. By definition C is complete, atomistic and satisfies the covering law; therefore it is upper semimodular by (ii) of 2.9. But C is also orthocomplemented[†], so it is isomorphic to its opposite[†] (by $C \longrightarrow C^{\mathsf{op}} : x \mapsto x^\perp$): upper semimodularity thus implies lower semimodularity. Then C must have the intersection property by (iii) of 2.9, and so its atoms form a projective geometry for the indicated collinearity.

Now we must check the axioms for the orthogonality relation; the first three are (almost) trivial. For (O4), if $a \neq b$ in $\mathcal{G}(C)$ then $b \leq a \nabla a^\perp = 1$ hence, by the intersection property,

$(a \triangledown b) \wedge a^\perp \neq 0$; the atomisticity of C gives us a $q \in \mathcal{G}(C)$ such that $q \leq a \triangledown b$ and $q \leq a^\perp$, as wanted. Finally, (O5) requires some more sophisticated calculations. First note that, for any subspace $S \subseteq \mathcal{G}(C)$ and element $a \in \mathcal{G}(C)$,

$$a \in S^\perp \Leftrightarrow \forall b \in S : b \leq a^\perp \Leftrightarrow \bigvee S \leq a^\perp \Leftrightarrow a \leq (\bigvee S)^\perp.$$

Thus we always have that $S^\perp = \{a \in \mathcal{G}(C) \mid a \leq (\bigvee S)^\perp\}$, which by atomisticity of C means that $\bigvee(S^\perp) = (\bigvee S)^\perp$; in particular is S closed, $S = S^{\perp\perp}$, if and only if $S = \{a \in \mathcal{G}(C) \mid a \leq \bigvee S\}$. If S is a trivial subspace, then it is clear that $S \vee S^\perp = \mathcal{G}(C)$; so from now on, let $S = S^{\perp\perp}$ be non-trivial. By the projective law, valid in $\mathcal{L}(\mathcal{G}(C))$ as in any other projective lattice, $S \vee S^\perp = \mathcal{G}(C)$ just means that for any $p \in \mathcal{G}(C)$ there exist $a, b \in \mathcal{G}(C)$ such that $a \leq \bigvee S$, $b \leq (\bigvee S)^\perp$ and $p \leq a \triangledown b$. And this is indeed true in the propositional system C; to simplify notations we shall write $x := \bigvee S$ in the argument that follows[5]. Suppose first that $x \wedge (x^\perp \triangledown p) \not\leq x^\perp$, then (by C's atomisticity) there must exist an $a \in \mathcal{G}(C)$ such that $a \leq x \wedge (x^\perp \triangledown p)$ and $a \not\leq x^\perp$. If $a = p$ then $p \leq x$ and we can pick any atom $b \leq x^\perp$ to show that $p \leq a \triangledown b$ as wanted. If $a \neq p$ then from $a \leq x^\perp \triangledown p$ and the intersection property we get an atom $b \leq x^\perp \wedge (a \triangledown p)$; but certainly is $a \neq b$ (because $a \leq x$ and $b \leq x^\perp$) so $b \leq a \triangledown p$ is equivalent to $p \leq a \triangledown b$ by 2.9 (iv), as wanted. Next suppose that $x \wedge (x^\perp \triangledown p) \leq x^\perp$; this means that $x^\perp = x^\perp \triangledown (x \wedge (x^\perp \triangledown p)) = x^\perp \triangledown p$ by orthomodularity in the second equality, so $p \leq x^\perp$. Picking any atom $a \leq x$ and putting $b := p$ we have $p \leq a \triangledown b$ as wanted. □

Note that, for a given Hilbert lattice L, the atoms of $\mathcal{C}(L)$ are exactly those of L, and the supremum of two atoms in $\mathcal{C}(L)$ is equal to their supremum in L. Thus it follows that $\mathcal{G}(\mathcal{C}(L))$ (as in 5.11) is the *same* projective geometry as $\mathcal{G}(L)$ (as in 2.10).

Our aim is to build a *triple* categorical equivalence between Hilbert geometries, Hilbert lattices and propositional systems. We must therefore define an appropriate notion of 'morphism between propositional systems'. And then it turns out that we must restrict the morphisms between Hilbert geometries, resp. Hilbert lattices, if we want to establish such a triple equivalence.

Definition 5.12 Let C_1 and C_2 be propositional systems. A map $h: C_1 \longrightarrow C_2$ is a **morphism of propositional systems** if it preserves arbitrary suprema and maps atoms of C_1 to atoms or the bottom element of C_2.

It is a simple observation that propositional systems and their morphisms form a category $\mathsf{PropSys}$.

We shall now adapt the definition of 'morphism' between Hilbert geometries, resp. Hilbert lattices: since these structures come with their respective closure operators, it is natural to consider 'continuous morphisms'.

Definition 5.13 Let G_1 and G_2 be Hilbert geometries. A morphism of projective geometries $g: G_1 \dashrightarrow G_2$ (as in 2.11) is **continuous** when, for every closed subspace F of G_2, $g^*(F)$ is a closed subspace of G_1.

Let L_1 and L_2 be Hilbert lattices. A morphism of projective lattices $f: L_1 \longrightarrow L_2$ (as in 2.14) is **continuous** when $f(x^{\perp\perp}) \leq f(x)^{\perp\perp}$ for every $x \in L_1$.

Hilbert geometries and continuous morphisms form a category $\mathsf{HilbGeom}$, and there is a faithful functor $\mathsf{HilbGeom} \longrightarrow \mathsf{ProjGeom}$ that "forgets" the orthogonality relation on a Hilbert geometry.

[5]This argument *actually* shows that for any $p \in \mathcal{G}(C)$ and any $x \in C$ which is not 0 nor 1, there exist $a, b \in \mathcal{G}(C)$ such that $a \leq x$, $b \leq x^\perp$ and $p \leq a \triangledown b$; see also [Maeda and Maeda, 1970, 30.7].

Similarly Hilbert lattices and continuous morphisms form a category $\mathsf{HilbLat}$ with a forgetful functor to $\mathsf{ProjLat}$.

Example 5.14 We say that a semilinear map $f\colon H_1 \longrightarrow H_2$ between generalized Hilbert spaces is **continuous** when it is so in the usual sense of the word with respect to biorthogonal closure. This is precisely saying that the induced morphism $\mathcal{P}(f)\colon \mathcal{P}(H_1) \dashrightarrow \mathcal{P}(H_2)$ is continuous in the sense of 5.13. There is then a category $\mathsf{GenHilb}$ of generalized Hilbert spaces and continuous semilinear maps, and also a functor

$$\mathcal{P}\colon \mathsf{GenHilb} \longrightarrow \mathsf{HilbGeom}\colon \Big(f\colon H_1 \longrightarrow H_2\Big) \mapsto \Big(\mathcal{P}(f)\colon \mathcal{P}(H_1) \dashrightarrow \mathcal{P}(H_2)\Big).$$

The Second Fundamental Theorem 4.19 implies that every non-degenerate continuous morphism between Hilbert geometries $\mathcal{P}(H_1)$ and $\mathcal{P}(H_2)$ is induced by a continuous semilinear map.

In passing we note that the forgetful functor $\mathsf{HilbGeom} \longrightarrow \mathsf{ProjGeom}$ is not full: there exist non-continuous linear maps between Hilbert spaces, and these induce non-continuous morphisms between (the underlying projective geometries of) Hilbert geometries.

The following is then the expected amendment of 2.15.

Proposition 5.15 *If $g\colon G_1 \dashrightarrow G_2$ is a continuous morphism between Hilbert geometries then*

$$\mathcal{L}(g)\colon \mathcal{L}(G_1) \longrightarrow \mathcal{L}(G_2)\colon S \mapsto \bigcap\{T \in \mathcal{L}(G_2) \mid S \subseteq g^*(T)\}$$

is a continuous morphism between Hilbert lattices. If $f\colon L_1 \longrightarrow L_2$ is a continuous morphism between Hilbert lattices then its restriction to closed elements

$$\mathcal{C}(f)\colon \mathcal{C}(L_1) \longrightarrow \mathcal{C}(L_2)\colon x \mapsto f(x)^{\perp\perp}$$

is a morphism of propositional systems. And if $h\colon C_1 \longrightarrow C_2$ is a morphism between propositional systems then

$$\mathcal{G}(h)\colon \mathcal{G}(C_1) \dashrightarrow \mathcal{G}(C_2)\colon a \mapsto h(a) \text{ with kernel } \{a \in \mathcal{G}(C_1) \mid h(a) = 0\}$$

is a continuous morphism between Hilbert geometries.

Proof. For a continuous morphism $g\colon G_1 \dashrightarrow G_2$ between Hilbert geometries and $S \in \mathcal{L}(G_1)$ we can compute, with notations as in 2.15, that

$$\begin{aligned}
&S \subseteq g^*\Big(\mathcal{L}(g)(S)\Big) \subseteq g^*\Big(\Big(\mathcal{L}(g)(S)\Big)^{\perp\perp}\Big)\\
&\Rightarrow S^{\perp\perp} \subseteq \Big(g^*\Big(\Big(\mathcal{L}(g)(S)\Big)^{\perp\perp}\Big)\Big)^{\perp\perp} = g^*\Big(\Big(\mathcal{L}(g)(S)\Big)^{\perp\perp}\Big)\\
&\Rightarrow \mathcal{L}(g)(S^{\perp\perp}) \subseteq \Big(\mathcal{L}(g)(S)\Big)^{\perp\perp}
\end{aligned}$$

because we know that $\mathcal{L}(g) \dashv g^*$ (used in the first and last line) and continuity of $g\colon G_1 \dashrightarrow G_2$ assures the equality in the above argument. So $\mathcal{L}(g)$ is a continuous morphism of Hilbert lattices.

Given $f\colon L_1 \longrightarrow L_2$, a continuous morphism of Hilbert lattices, $\mathcal{C}(f)$ is precisely defined as the unique map that makes the square in figure 7 commute. By continuity of f, its right adjoint $f^*\colon L_2 \longrightarrow L_1$ maps closed elements to closed elements; thus the restriction of f^* to closed elements provides a right adjoint to $\mathcal{C}(f)$, showing that the latter preserves suprema. It is merely an

$$\begin{tikzcd}
L_1 \arrow[r, "f"] \arrow[d, "(\ )^{\perp\perp}"'] & L_2 \arrow[d, "(\ )^{\perp\perp}"] \\
\mathcal{C}(L_1) \arrow[r, "\mathcal{C}(f)"'] & \mathcal{C}(L_2)
\end{tikzcd}$$

Figure 7: The definition of $\mathcal{C}(f)$

observation that, because f sends atoms to atoms or 0, so does $\mathcal{C}(f)$. Thus $\mathcal{C}(f)$ is a morphism of propositional systems.

Finally, let $h: C_1 \longrightarrow C_2$ be a morphism of propositional systems. From the proof of 5.11 we know that a subspace $S \subseteq \mathcal{G}(C_2)$ is closed if and only if $S = \{b \in \mathcal{G}(C_2) \mid b \leq \bigvee S\}$; so in this case

$$\begin{aligned}
\mathcal{G}(h)^*(S) &= \{a \in \mathcal{G}(C_1) \mid h(a) = 0\} \cup \mathcal{G}(h)^{-1}(S) \\
&= \{a \in \mathcal{G}(C_1) \mid h(a) \leq \bigvee S\}.
\end{aligned}$$

Writing $h^*: C_2 \longrightarrow C_1$ for the right adjoint to h, this can be written as

$$\mathcal{G}(h)^*(S) = \{a \in \mathcal{G}(C_1) \mid a \leq h^*(\bigvee S)\}$$

which shows that $\mathcal{G}(h)^*(S)$ is closed, since $\bigvee(\mathcal{G}(h)^*(S)) = h^*(\bigvee S)$ by atomisticity. $\square$

Now we can conclude this section with the following result.

Theorem 5.16 *The categories* $\mathsf{HilbGeom}$, $\mathsf{HilbLat}$ *and* $\mathsf{PropSys}$ *are equivalent: the assignments*

$$\begin{aligned}
&\mathcal{L}: \mathsf{HilbGeom} \longrightarrow \mathsf{HilbLat}: \Big(g: G_1 \dashrightarrow G_2\Big) \mapsto \Big(\mathcal{L}(g): \mathcal{L}(G_1) \longrightarrow \mathcal{L}(G_2)\Big) \\
&\mathcal{C}: \mathsf{HilbLat} \longrightarrow \mathsf{PropSys}: \Big(f: L_1 \longrightarrow L_2\Big) \mapsto \Big(\mathcal{C}(f): \mathcal{C}(L_1) \longrightarrow \mathcal{C}(L_2)\Big) \\
&\mathcal{G}: \mathsf{PropSys} \longrightarrow \mathsf{HilbGeom}: \Big(h: C_1 \longrightarrow C_2\Big) \mapsto \Big(\mathcal{G}(h): \mathcal{G}(C_1) \longrightarrow \mathcal{G}(C_2)\Big)
\end{aligned}$$

are functorial, and for a Hilbert geometry G, a Hilbert lattice L and a propositional system C there are natural isomorphisms

$$\begin{aligned}
&\kappa_G: G \xrightarrow{\sim} \mathcal{G}(\mathcal{C}(\mathcal{L}(G))): a \mapsto \{a\}, \\
&\lambda_L: L \xrightarrow{\sim} \mathcal{L}(\mathcal{G}(\mathcal{C}(L))): x \mapsto \{a \in \mathcal{G}(L) \mid a \leq x\}, \\
&\mu_C: C \xrightarrow{\sim} \mathcal{C}(\mathcal{L}(\mathcal{G}(C))): x \mapsto \{a \in \mathcal{G}(C) \mid a \leq x\}.
\end{aligned}$$

Proof. We shall leave some verifications to the reader: the functoriality of $\mathcal{G}$, $\mathcal{L}$ and $\mathcal{C}$, and the naturality of κ, λ and μ. But we shall prove that the latter are indeed isomorphisms.

Right after 5.11 we had already remarked that $\mathcal{G}(\mathcal{C}(\mathcal{L}(G))) = \mathcal{G}(\mathcal{L}(G))$, i.e. $\mathcal{C}(\mathcal{L}(G))$ and $\mathcal{L}(G)$ have the *same* atoms and induce the *same* collinearity relation, so we already know by 2.16 that κ_G is an isomorphism of projective geometries (it was called α_G in 2.16); we only need to prove that κ_G and its inverse are continuous. A sufficient condition thereto is that two points a and

b of G are orthogonal if and only if $\kappa_G(a)$ and $\kappa_G(b)$ are orthogonal in $\mathcal{G}(\mathcal{C}(\mathcal{L}(G)))$. Indeed this fact is true:

$$a \perp b \Leftrightarrow a \in \{b\}^\perp \Leftrightarrow \{a\} \subseteq \{b\}^\perp \Leftrightarrow \kappa_G(a) \perp \kappa_G(b).$$

Similarly as above, we already know by 2.16 that λ_L is an isomorphism of projective lattices (it was called β_L before). Now let a be an atom of $\mathcal{C}(L)$ (or of L) and $x \in L$, then

$$a \in \lambda_L(x^\perp) \Leftrightarrow a \leq x^\perp \overset{*}{\Leftrightarrow} x \leq a^\perp \Leftrightarrow \forall b \in \lambda_L(x) : b \leq a^\perp \Leftrightarrow a \in \lambda_L(x)^\perp$$

(the equivalence in $(*)$ uses that the atom a is closed and that $x \leq x^{\perp\perp}$). This proves that $\lambda_L(x^\perp) = \lambda_L(x)^\perp$ for all $x \in L$, from which it follows that λ_L and its inverse are continuous morphisms of Hilbert lattices.

Finally, it is clear that each $\mu_C(x) \subseteq \mathcal{G}(C)$ is a subspace (for the collinearity on $\mathcal{G}(C)$ as in 5.11). Like before we have that $\mu_C(x^\perp) = \mu_C(x)^\perp$ for $x \in C$. It then follows easily that any such $\mu_C(x)$ is biorthogonally closed in $\mathcal{L}(\mathcal{G}(C))$, so at least is μ_C a well-defined, and obviously order-preserving, map between ordered sets. The order-preserving map

$$\eta_C : \mathcal{C}(\mathcal{L}(\mathcal{G}(C))) \longrightarrow C : S \mapsto \bigvee S$$

satisfies $\eta_C \circ \mu_C = \mathsf{id}_C$ by atomisticity of C; and in the proof of 5.11 we had already shown that a subspace $S \subseteq \mathcal{G}(C)$ is closed if and only if $S = \mu_C(\eta_C(S))$, so $\mu_C \circ \eta_C$ is the identity too. Thus we find that μ_C and η_C constitute an isomorphism of lattices and hence μ_C is an isomorphism in $\mathsf{PropSys}$, with inverse η_C. $\square$

The natural maps κ_G, λ_L and μ_C are actually *more* than isomorphisms in their respective categories: they are examples of 'ortho-isomorphisms'. For completeness' sake, we shall very quickly make this precise.

Definition 5.17 A continuous morphism $g: G_1 \dashrightarrow G_2$ between Hilbert geometries is an **ortho-morphism** if $a \perp b$ implies $g(a) \perp g(b)$ for every a, b in the domain of g. A continuous morphism $f: L_1 \longrightarrow L_2$ between Hilbert lattices is an **ortho-morphism** if $f(x^\perp) \leq f(x)^\perp$ for all $x \in L$. And a morphism $h: C_1 \longrightarrow C_2$ between propositional systems is an **ortho-morphism** if $h(x^\perp) \leq h(x)^\perp$.

It obviously makes sense to consider the subcategories $\mathsf{HilbGeom}_\perp$, $\mathsf{HilbLat}_\perp$ and $\mathsf{PropSys}_\perp$ of $\mathsf{HilbGeom}$, $\mathsf{HilbLat}$ and $\mathsf{PropSys}$ with the same objects but with ortho-morphisms. An isomorphism in one of those categories is called a **ortho-isomorphism**. It turns out that the functors in 5.16 restrict to these smaller categories, and we have actually shown in 5.16 that the maps κ_G, λ_L and μ_C are ortho-isomorphisms: so the three categories $\mathsf{HilbGeom}_\perp$, $\mathsf{HilbLat}_\perp$ and $\mathsf{PropSys}_\perp$ are equivalent too.

It is also possible to consider 'non-continuous ortho-morphisms' between Hilbert geometries G_1 and G_2, resp. Hilbert lattices L_1 and L_2: such is a morphism $g: G_1 \dashrightarrow G_2$ in $\mathsf{ProjGeom}$, resp. $f: L_1 \longrightarrow L_2$ in $\mathsf{ProjLat}$, satisfying the appropriate orthogonality condition of 5.17. Two equivalent categories are obtained, but it is not known if there is a third equivalent category of propositional systems, i.e. if there is a suitable notion of morphism between propositional systems to correspond with that of 'non-continuous ortho-morphism' between Hilbert geometries, resp. Hilbert lattices. See [Faure and Frölicher, 2000, 14.3] for a discussion.

6. Irreducible components again

By "forgetting" about its orthogonality relation, we may view a Hilbert geometry G as an object of $\mathsf{ProjGeom}$ and consider its decomposition in maximal irreducible subspaces, cf. 3.5. We shall show that this coproduct actually "lives" in $\mathsf{HilbGeom}$, but we must start off with a couple of lemmas. First an adaptation of 3.4: we prove that the functor $\mathsf{HilbGeom} \longrightarrow \mathsf{ProjGeom}$ "creates" coproducts, from which it follows that $\mathsf{HilbGeom}$ has all coproducts and that the functor preserves these.

Lemma 6.1 *Given a family $(G_i, l_i, \perp_i)_{i\in I}$ of Hilbert geometries, the coproduct of the underlying projective geometries $(\uplus_i G_i, l)$ in $\mathsf{ProjGeom}$ becomes a Hilbert geometry for the orthogonality relation*

$$a \perp b \text{ if either } a \in G_k, b \in G_l, k \neq l \text{ or } a \perp_k b \text{ in some } G_k,$$

and the inclusion morphisms $(s_k : G_k \longrightarrow \uplus_i G_i)_{k\in I}$ become continuous morphisms that form a coproduct in $\mathsf{HilbGeom}$.

Proof. First we verify axioms (O1–5) of 5.1, using the notations introduced there. Clearly (O1–2) are trivial. For (O3), if a, b both belong to some G_k, then also $c \in G_k$ by the hypothesis $l(a, b, c)$; if $p \notin G_k$ then the conclusion holds trivially, and if $p \in G_k$ then we may use (O3) in $(G_k, l_k, \perp_k)$. If a, b belong to different components then $c = a$ or $c = b$ so that (O3) is trivially satisfied. The argument for (O4): if $a \perp b$ then $q := b$ does the job; if $a \not\perp b$ then a and b necessarily belong to the same component G_k and we can apply (O4) to $(G_k, l_k, \perp_k)$. The verification of (O5) is a bit more tricky. We already know from the proof of 3.4 that $S \subseteq \uplus_i G_i$ is a subspace of $(\uplus_i G_i, l)$ if and only if, for all k, $S \cap G_k$ is a subspace of (G_k, l_k). Now an easy calculation shows that $S^\perp \cap G_k = (S \cap G_k)^{\perp_k}$, whence $S^\perp = \uplus_i (S^\perp \cap G_i) = \uplus_i (S \cap G_i)^{\perp_i}$ and in particular $S^{\perp\perp} = \uplus_i (S \cap G_i)^{\perp_i\perp_i}$. From this it follows that S is biorthogonally closed in G if and only if, for all $k \in I$, $S \cap G_k$ is biorthogonally closed in G_k. In this case we can thus compute that:

$$\begin{aligned} S \vee S^\perp &= \Big(\uplus_i\,(S\cap G_i)\Big) \vee \Big(\uplus_i\,(S\cap G_i)^{\perp_i}\Big) \\ &\overset{*}{=} \uplus_i \Big((S\cap G_i) \vee (S\cap G_i)^{\perp_i}\Big) \\ &= \uplus_i G_i \\ &= G. \end{aligned}$$

The equation $(*)$ follows because the coproduct $\mathcal{L}(\uplus_i G_i) \cong \times_i \mathcal{L}(G_i)$ has the componentwise order.

Since $S \subseteq G$ is closed if and only if each $S \cap G_k$ is closed, it is clear that each $s_k : G_k \longrightarrow \uplus_i G_i$ as in (3) is continuous: because $s_k^*(S) = S \cap G_i$. And for the same reason, the unique factorization of a family of *continuous* morphisms $(g_k : G_k \dashrightarrow G)$ is easily seen to be continuous too: if $S \subseteq G$ is closed, then each $g^*(S) \cap G_k = g_k^*(S) \subseteq G_k$ is closed, so $g^*(S) = \uplus_i g_i^*(S) \subseteq \uplus_i G_i$ is closed. $\square$

Recall that a subspace $S \subseteq G$ of a projective geometry is always a projective geometry in its own right (for the inherited collinearity), and that the inclusion $S \longrightarrow G$ is always a morphism of projective geometries (with empty kernel). In the case of a Hilbert geometry G we can prove that the closure by biorthocomplement on G can be "relativized" to any of its *closed* subspaces.

Lemma 6.2 *If $S \subseteq G$ is a closed subspace of a Hilbert geometry, then S is a Hilbert geometry for the inherited collinearity and orthogonality, and the inclusion $S \longrightarrow G$ is a continuous morphism of Hilbert geometries.*

Proof. In this proof we shall use the following notation: if $T \subseteq S \subseteq G$ are subspaces with S closed, we put $T' := T^\perp \cap S$. We can prove a little trick:

$$T'' = (T^\perp \cap S)^\perp \cap S \overset{*}{=} (T^{\perp\perp} \vee S^\perp) \cap S \overset{**}{=} T^{\perp\perp} \vee (S^\perp \cap S) = T^{\perp\perp}. \tag{7}$$

We used $S = S^{\perp\perp}$ and 5.6 (iii) in $(*)$, and modularity of $\mathcal{L}(G)$ (or orthomodularity of $\mathcal{C}(\mathcal{L}(G))$, if one wishes) in $(**)$.

The verification of (O1–4) for the projective geometry S with inherited orthogonality is entirely straightforward. As for (O5), let $T \subseteq S$ be a subspace. Then (7) says that $T'' = T$ implies $T^{\perp\perp} = T$ which implies $T \vee T^\perp = G$, so that by modularity of $\mathcal{L}(G)$,

$$T \vee T' = T \vee (T^\perp \cap S) = (T \vee T^\perp) \cap S = S.$$

To verify the continuity of the morphism of projective geometries $S \hookrightarrow G$, let $T \subseteq G$ be a closed subspace. The intersection of closed subspaces is always a closed subspace, so $s^*(T) = T \cap S$ is closed in G. But by (7) this is the same as being closed in S. $\square$

S. Holland [1995, 3.3] explains how this lemma can be seen as motivation for axiom (O5) in the definition of 'Hilbert geometry'.

Finally we show a (remarkably strong) converse to 6.1.

Lemma 6.3 *Given a family $(G_i, l_i)_{i\in I}$ of projective geometries, if their coproduct $(\uplus_i G_i, l)$ in* ProjGeom *is in fact a Hilbert geometry for some orthogonality relation $\perp$, then each (G_k, l_k) becomes a Hilbert geometry for the induced orthogonality, and the inclusion morphisms $(s_k \colon G_k \longrightarrow \uplus_i G_i)_{k\in I}$ become continuous morphisms that form a coproduct in* HilbGeom.

Proof. By the above lemmas we only need to prove that the $(G_i)_i$ form a pairwise orthogonal family of closed subspaces of $\uplus_i G_i$. But let $a \in G_j$ and $b \in G_k$ for some $j \neq k$, then, by the coproduct construction in ProjGeom, the line $a \star b$ can only contain the points a and b, which thus by (O4) of 5.1 must be orthogonal in the Hilbert geometry $\uplus_i G_i$. From this and (O1) it is then easily seen that $G_j^\perp = \uplus_{i\neq j} G_i$, so that $G_j^{\perp\perp} = G_j$ follows. $\square$

Having these technical results, we shall come to the point: we begin with a corollary of 3.5 and the above lemmas.

Theorem 6.4 *A Hilbert geometry G is the coproduct in* HilbGeom *of its maximal irreducible subspaces.*

This now allows us to turn the elementary notion of 'irreducibility' for a Hilbert geometry into a categorical one by refining the result given in 3.6 for more general projective geometries.

Corollary 6.5 *A Hilbert geometry G is irreducible (in the sense of 3.1 or equivalently 3.6) if and only if it is not a coproduct in* HilbGeom *of two (or more) non-empty Hilbert geometries.*

Proof. The coproduct-decompositions of a Hilbert geometry $(G, l, \perp)$ in HilbGeom correspond to those of the underlying projective geometry (G, l) in ProjGeom, by 6.1 and 6.3. $\square$

What the above really says, is that there is only one meaning for the term 'irreducible Hilbert geometry' G, namely: all projective lines in G have at least three points, or equivalently: G is not coproduct-decomposable in $\mathsf{ProjGeom}$, or equivalently: G is not coproduct-decomposable in $\mathsf{HilbGeom}$.

It is a matter of exploiting the categorical equivalence of Hilbert geometries and Hilbert lattices to deduce the following statements from the above and the results in section 3.

Lemma 6.6 *The category* $\mathsf{HilbLat}$ *has coproducts. Explicitly, if* $(L_i)_{i\in I}$ *are Hilbert lattices with respective orthogonality operators* $x \mapsto x^{\perp_i}$, *then the coproduct* $\times_i L_i$ *in* $\mathsf{ProjLat}$ *becomes a Hilbert lattice for the orthogonality operator*

$$(x_i)_i \mapsto (x_i^{\perp_i})_i,$$

and the inclusion morphisms $s_k\colon L_k \longrightarrow \times_i L_i$ *become continuous morphisms that form a universal cocone in* $\mathsf{HilbLat}$.

Proposition 6.7 *A Hilbert lattice* L *is irreducible (in the sense of 3.9) if and only if it is not a coproduct in* $\mathsf{HilbLat}$ *of two (or more) non-trivial Hilbert lattices.*

Proposition 6.8 *Let* G *be a Hilbert geometry and* L *a Hilbert lattice that correspond to each other under the equivalence* $\mathsf{HilbGeom} \simeq \mathsf{HilbLat}$. *Then* L *is irreducible if and only if* G *is.*

Theorem 6.9 *Every Hilbert lattice* L *is a coproduct in* $\mathsf{HilbLat}$ *of irreducible Hilbert lattices.*

Finally we can state everything in terms of propositional systems; surely the reader is by now familiar with our way of doing this, so we shall once again skip all the details.

Lemma 6.10 *The category* $\mathsf{PropSys}$ *has coproducts. In fact, given a family* $(C_i)_{i\in I}$ *of propositional systems with respective orthogonality operators* $x \mapsto x^{\perp_i}$, *the cartesian product* $\times_i C_i$ *with componentwise structure and the map*

$$\times_i C_i \longrightarrow \times_i C_i\colon (x_i)_i \mapsto (x_i^{\perp_i})_i$$

as orthocomplementation, is a propositional system; and the maps

$$s_k\colon C_k \longrightarrow \times_i C_i\colon x \mapsto (x_i)_i$$

where $x_k = x$ *and* $x_i = 0$ *if* $i \neq k$, *form a universal cocone in* $\mathsf{PropSys}$.

Definition 6.11 A propositional system C is **irreducible** if it is not a coproduct in $\mathsf{PropSys}$ of two (or more) non-trivial propositional systems.

Proposition 6.12 *Let* G *be a Hilbert geometry,* L *a Hilbert lattice and* C *a propositional system that correspond to each other under the triple equivalence* $\mathsf{HilbGeom} \simeq \mathsf{HilbLat} \simeq \mathsf{PropSys}$. *Then* G *is an irreducible Hilbert geometry if and only if* L *is an irreducible Hilbert lattice, if and only if* C *is an irreducible propositional system.*

Theorem 6.13 *Every propositional system* C *is the coproduct in* $\mathsf{PropSys}$ *of irreducible propositional systems.*

The remark that we have made at the end of section 3, can be repeated here: on the one hand can the statements in 6.9 and 6.13 be made more precise by saying exactly which are the "irreducible components" of a Hilbert lattice, resp. propositional system; on the other hand are these theorems particular cases of a more general principle involving 'central elements' of lattices. Again we refer to 8.8 for a comment on this matter.

7. The Representation Theorem for propositional systems

This beautiful result is due to C. Piron [1964, 1976], who generalized the finite-dimensional version of G. Birkhoff and J. von Neumann [1936].

Theorem 7.1 (Piron's Representation Theorem) *Every irreducible propositional system of rank at least 4 is ortho-isomorphic to the lattice of (biorthogonally) closed subspaces of a generalized Hilbert space.*

We can obviously put this theorem in geometric terms.

Theorem 7.2 *For every arguesian Hilbert geometry* $(G, \perp)$ *there exists a generalized Hilbert space* $(H, K, {}^*, \langle\ ,\ \rangle)$ *such that* $(G, \perp)$ *is ortho-isomorphic to* $(\mathcal{P}(H), \perp)$*, where* $\mathcal{P}(H)$ *is given the orthogonality relation induced by* $\langle\ ,\ \rangle$*.*

Moreover, the Hermitian form of which this theorem speaks, is essentially unique.

Proposition 7.3 (Uniqueness of Hermitian form) *Let* H *be a vector space over* K*. Let* $\alpha \mapsto \alpha^*$ *and* $\alpha \mapsto \alpha^\#$ *be two involutions on* K*, let* $(x, y) \mapsto \langle x, y\rangle$ *be a* $*$*-Hermitian form and* $(x, y) \mapsto [x, y]$ *a* $\#$*-Hermitian form on* H*. If both Hermitian forms induce the same orthogonality on* $\mathcal{P}(H)$*, then there exists* $0 \neq \lambda = \lambda^* \in K$ *such that* $\rho^\# = \lambda^{-1}\rho^*\lambda$ *for all* $\rho \in K$ *and* $[x, y] = \langle x, y\rangle\lambda$ *for all* $x, y \in H$*.*

The existence and uniqueness of a vector space with an anisotropic Hermitian form inducing the Hilbert geometry $(G, \perp)$ only require axioms (O1) through (O4) of the definition of a Hilbert geometry (cf. 5.1). Axiom (O5) makes the Hermitian form orthomodular (see 8.4 and 8.5 for related comments). The field K in Theorem 7.2 cannot be finite [Eckmann and Zabey, 1969; Ivert and Sjödin, 1978]; see also [Faure and Frölicher, 2000, 14.1.12].

We shall now present S. Holland's [1995, §3] proof of the geometric version of Piron's theorem: it is essentially a smart application of the Second Fundamental Theorem to the isomorphism induced by the orthogonality between G and its **opposite geometry** G^{op}. The latter is by definition $\mathcal{G}(C^{\mathsf{op}})$, where C^{op} is the propositional system opposite to $C := \mathcal{C}(\mathcal{L}(G))$ and it is the isomorphism $C \cong C^{\mathsf{op}}$ given by the orthocomplementation which induces the isomorphism $G \cong G^{\mathsf{op}}$. Geometrically, G^{op} has the closed hyperplanes $\{p^\perp \mid p \in G\}$ as its points and its collinearity satisfies, for $p, q, r \in G$,

$$l(p^\perp, q^\perp, r^\perp) \text{ in } G^{\mathsf{op}} \Leftrightarrow l(p, q, r) \text{ in } G \Leftrightarrow q = r \text{ or } p^\perp \supseteq q^\perp \cap r^\perp \text{ in } \mathcal{L}(G). \tag{8}$$

Given an arguesian Hilbert geometry G, we can by the First Fundamental Theorem assume that $G = \mathcal{P}(H)$ for a K-vector space H. The dual

$$H^* := \{f\colon V \longrightarrow K \mid f \text{ linear}\}$$

is then a right K-vector space, thus a left vector space over the opposite field K^{op} (in which the multiplication, written with a centered dot, is reversed: $\rho \cdot \lambda = \lambda\rho$). For every closed hyperplane $M = (Kx)^{\perp}$ of $\mathcal{P}(H)$ there exists a linear functional $f_x \in H^*$, unique up to scalar multiple, which has M as its kernel. Now let F be the subspace of H^* spanned by $\{f_x \mid x \in H\}$.

Lemma 7.4 *The map $\varphi\colon \mathcal{P}(H) \longrightarrow \mathcal{P}(F)\colon Kx \mapsto K^{\mathsf{op}} \cdot f_x$ is an isomorphism of projective geometries.*

Sketch of the proof. With (8) the lemma is proved by verifying that, for linear functionals $f, g, h \in H^*$, f belongs to the K^{op}-span of g and h if and only if $\mathsf{ker}(f) \supset \mathsf{ker}(g) \cap \mathsf{ker}(h)$; which is an exercise in linear algebra [Faure and Frölicher, 2000, 11.1.11]. □

Applying the Second Fundamental Theorem to the isomorphism φ, there exists a field isomorphism $\sigma\colon K \longrightarrow K^{\mathsf{op}}$ and a bijective σ-linear map $A\colon H \longrightarrow F$ such that $\varphi(Kx) = K^{\mathsf{op}} \cdot A(x)$ for all $x \in H$. Note that $\mathsf{ker}(A(y)) = (Ky)^{\perp}$.

Lemma 7.5 *The map $[\ ,\ ]\colon H \times H \longrightarrow K\colon (x, y) \mapsto [x, y] := A(y)(x)$ is* **sesquilinear***: it is additive in both factors and satisfies for all $x, y \in H$ and $\lambda \in K$*

(Q1) $[\lambda x, y] = \lambda[x, y]$,
(Q2) $[x, \lambda y] = [x, y]\sigma(\lambda)$.

Moreover, the orthogonality $\perp$ on $G = \mathcal{P}(H)$ corresponds to the one induced by $[\ ,\ ]$, that is, $Kx \perp Ky \Leftrightarrow [x, y] = 0$. Consequently, $[\ ,\ ]$ is anisotropic (i.e. it satisfies (S4) right after 5.2).

All that is left is to find the involution on K and to rescale $[\ ,\ ]$ to make it Hermitian. Because $[\ ,\ ]$ is anisotropic we can choose a $z \in H$ such that $\varepsilon := [z, z] \neq 0$. Define another sesquilinear form on H by putting $\langle\ ,\ \rangle := [\ ,\ ]\varepsilon^{-1}$ and set $\rho^* := \varepsilon\sigma(\rho)\varepsilon^{-1}$ for all $\rho \in K$. Then $\langle\ ,\ \rangle$ induces the same orthogonality as $[\ ,\ ]$ and it still satisfies (Q1) and (Q2) with σ replaced by the anti-automorphism $\rho \mapsto \rho^*$ of K. To satisfy all requirements for $(H, K, {}^*, \langle\ ,\ \rangle)$ to be a generalized Hilbert space (cf. 5.2) we now only need to prove a last result.

Lemma 7.6 *For all $x, y \in H$ and all $\rho \in K$, we have $\langle x, y\rangle = \langle y, x\rangle^*$ and $\rho^{**} = \rho$.*

Proof. Apply 4.20 to the two maps $H \longrightarrow F$ given by $y \mapsto \langle \cdot, y\rangle$ and $y \mapsto \langle y, \cdot\rangle^{\#}$, where $\alpha \mapsto \alpha^{\#}$ is the inverse automorphism of $\alpha \mapsto \alpha^*$. Remembering that F is a K^{op}-vector space, we obtain a nonzero $\xi \in K$ such that $\langle x, y\rangle^* = \xi^*\langle y, x\rangle$ for all $x, y \in H$. Then $1 = \langle z, z\rangle^* = \xi^*\langle z, z\rangle = \xi^*$ because $\langle z, z\rangle = 1$. Moreover $\rho = \langle \rho z, z\rangle = \langle z, \rho z\rangle^* = \langle \rho z, z\rangle^{**} = \rho^{**}\langle z, z\rangle^{**} = \rho^{**}$, for every $\rho \in K$, proving that $\alpha \mapsto \alpha^*$ is an involution on K. □

This finishes the proof of 7.1 and 7.2. Uniqueness of the Hermitian form up to scaling (as stated in 7.3) is obtained by an application of 4.20.

In [Faure and Frölicher, 2000, §14.3] the reader can find representation theorems for morphisms between projective geometries preserving orthogonality, among which Wigner's theorem which geometrically characterizes isometries for real, complex or hamiltonian Hilbert spaces. See also [Faure, 2002]. We state here a slightly more general but also well-known version.

Definition 7.7 Let H_1 and H_2 be orthomodular spaces over a field K. An isomorphism $f\colon H_1 \longrightarrow H_2$ in Vec is called a **semi-unitary** if there exists a $\lambda \in K$ such that for all $x, y \in H_1$, we have $\langle f(x), f(y)\rangle = \sigma(\langle x, y\rangle)\lambda$ where $\sigma\colon K \longrightarrow K$ is the automorphism associated to f. Moreover, f is called **unitary** when $\langle f(x), f(y)\rangle = \langle x, y\rangle$ for all $x, y \in H_1$.

Theorem 7.8 (Wigner) *Let H_1 and H_2 be orthomodular spaces of dimension at least 3 over a field K. Then every ortho-isomorphism $\mathcal{C}(H_1) \longrightarrow \mathcal{C}(H_2)$ is induced by a semi-unitary map $H_1 \longrightarrow H_2$.*

8. From here on

We shall end with some comments on the material that we presented in this chapter, and with some hints for further study.

8.1 Projective closure. In a remark following 2.4 we have hinted at the fact that to any sub*set* $A \subseteq G$ of a projective geometry we can associate the smallest sub*space* $\mathsf{cl}(A) \subseteq G$ that contains A. It is easily verified that this operation $A \mapsto \mathsf{cl}(A)$ satisfies the following conditions:

i. it is monotone and satisfies $\mathsf{cl}(\mathsf{cl}(A)) \subseteq \mathsf{cl}(A) \supseteq A$,
ii. $a \in \mathsf{cl}(A)$ implies $a \in \mathsf{cl}(B)$ for some finite subset $B \subseteq A$,
iii. $x \notin \mathsf{cl}(A)$ and $x \in \mathsf{cl}(A \cup \{b\})$ imply $b \in \mathsf{cl}(A \cup \{x\})$,
iv. $\mathsf{cl}(\emptyset) = \emptyset$ and $\mathsf{cl}(\{a\}) = \{a\}$ for all $a \in G$,
v. for non-empty $A, B \subseteq G$, $\mathsf{cl}(A \cup B) = \bigcup\{\mathsf{cl}(\{a, b\}) \mid a \in \mathsf{cl}(A), b \in \mathsf{cl}(B)\}$.

A set G together with an operation $\mathsf{cl}\colon 2^G \longrightarrow 2^G$ satisfying (i) is called a **closure space**; if on top of that it satisfies (ii–iii) then it is a **matroid**. A closure space that also satisfies (iv) is a **simple closure space**; and a simple matroid is often called a **geometry**. If $A \mapsto \mathsf{cl}(A)$ satisfies the whole lot (i–v) then it is a **projective closure**, and one can prove that any projective closure space (G, cl) is necessarily provided by a projective geometry. That is to say, there is an equivalence of categories $\mathsf{ProjGeom} \simeq \mathsf{ProjClos}$ of projective geometries on the one hand and projective closure spaces on the other (with appropriate morphisms). But also the 'weaker' structures (matroids, geometries) are interesting in their own right; in particular can a whole deal of "dimension theory" for projective geometries (cf. 4.1) be carried out for structures as basic as matroids. This is the subject of Cl.-A. Faure and A. Frölicher's [1996], see also their [2000, chapters 3 and 4].

8.2 State spaces and property lattices. In the definition 5.1 of Hilbert geometry, it follows from (O1–4) that a Hilbert geometry is a **state space** in the sense of [Moore, 1995]: if $a \neq b$ then $l(q, a, b)$ and $q \perp a$ for some $q \in G$ by (O4), but would $q \perp b$ as well then $q \perp q$ by (O2–3) (and the symmetry of l) which is excluded by (O1). That is to say, the relation $\perp$ is irreflexive, symmetric and separating (in the sense that $a \neq b$ implies the existence of some q such that $a \perp q \not\perp b$). Moore [1995] proves that the biorthogonally closed subspaces of a state space form a so-called **property lattice**: a complete, atomistic and orthocomplemented lattice. Of course, a propositional system (cf. 5.9) is a particular example of such a 'property lattice'. More precisely, state spaces and property lattices are the objects of equivalent categories State and Prop of which the equivalence of $\mathsf{HilbGeom}$ with $\mathsf{PropSys}$ is a restriction. For the relevance of State and Prop in theoretical physics see [Moore, 1999].

8.3 Fewer axioms for geometries with orthogonality. F. Buekenhout [1993] explains how A. Parmentier and he showed that, remarkably, (G3) of 2.1 is automatically true for a set G with a collinearity l satisfying just (G1–2) and an orthogonality $\perp$ satisfying (cf. 5.1)

(O2) if $a \perp b$ then $b \perp a$,
(O3) if $a \neq b$, $a \perp p$, $b \perp p$ and $c \in a \star b$ then $c \perp p$,

(O6) if $a, b, p \in G$ and $a \neq b$ then there is a $q \in a \star b$ with $q \perp p$,
(O7) for all $a \in G$ there is a $b \in G$ with $a \not\perp b$.

Clearly, (O1) implies (O7), and in the proof of 5.4 we have shown that (O6) too is valid in any Hilbert geometry.

8.4 Geometries "with extra structure". A Hilbert geometry is, by 5.1, a projective geometry with extra structure—a lot of extra structure, actually. There are many notions of 'projective geometry with extra structure' that are weaker than Hilbert geometries but still have many interesting properties. A large part of [Faure and Frölicher, 2000] is devoted to the study of such things as **Mackey geometries**, **regular Mackey geometries**, **orthogeometries** and **pure orthogeometries**: structures that lie between projective geometries and Hilbert geometries. Several of these 'geometries with extra structure' can be represented by appropriate 'vector spaces with extra structure.' In that spirit, [Holland, 1995, 3.6] and [Faure and Frölicher, 2000, 14.1.8] are slight generalizations of Piron's representation theorem (cf. 7.1 and 7.2) which include skew-symmetric forms.

8.5 On orthomodularity. Given a vector space V with an anisotropic Hermitian form (i.e. a form satisfying (S1, S2 and S4) in 5.2) and induced orthogonality $\perp$ we have that $(\mathcal{P}(V), \perp)$ satisfies (O5) if and only if $\mathcal{L}(V)$ satisfies (H5) if and only if the lattice of closed subspace $\mathcal{C}(V)$ is orthomodular if and only if the Hermitian form satisfies (S3). In other words, and this is a key insight of Piron's [1964], orthomodularity of $\mathcal{C}(V)$ is what distinguishes the generalized Hilbert spaces among the (anisotropic) Hermitian spaces. This is one of the reasons why orthomodular lattices have been heavily studied; the standard reference on the subject is [Kalmbach, 1983].

8.6 Projectors. For a projective geometry G together with a binary relation $\perp$ on G that satisfies (O1–4) in 5.1, a (necessarily closed, cf 5.6) subspace $S \subseteq G$ satisfies $S \vee S^{\perp} = G$ if and only if for every $a \in G \setminus S^{\perp}$ the subspace $(\{a\} \vee S^{\perp}) \cap S$ is non-empty. In this case, $(\{a\} \vee S^{\perp}) \cap S$ is a singleton, and writing $r(a)$ for its single element gives a partial map

$$r: G \dashrightarrow S : a \mapsto r(a) \text{ with kernel } S^{\perp}$$

which is a retract to the inclusion $i: S \longrightarrow G$.

Proof. Suppose that $S \vee S^{\perp} = G$ and that $a \in G \setminus (S \cup S^{\perp})$ (if $a \in S$ then all is trivial). By the projective law, $a \in x \star y$ for some $x \in S$ and $y \in S^{\perp}$, whence $x \in a \star y \subseteq \{a\} \vee S^{\perp}$, so $x \in (\{a\} \vee S^{\perp}) \cap S \neq \emptyset$. Conversely, suppose that $a \in G \setminus (S \cup S^{\perp})$. Pick any $x \in (\{a\} \vee S^{\perp}) \cap S$: thus $x \in S$ and $x \in a \star y$ for some $y \in S^{\perp}$, whence $a \in x \star y \subseteq S \vee S^{\perp}$ (using the projective law twice), which proves that $S \vee S^{\perp} = G$.

Would x_1, x_2 be different elements of $(\{a\} \vee S^{\perp}) \cap S$ for some $a \notin S^{\perp}$, then $x_1, x_2 \in S$ and there exist $y_1, y_2 \in \{a\} \vee S^{\perp}$ such that $x_i \in a \star y_i$ for $i = 1, 2$. Because $a \notin S^{\perp}$, a is necessarily different from the y_i's. But a is also different from the x_i's: if $a = x_1$ for example, then $x_2 \in x_1 \star y_2$ from which $y_2 \in x_1 \star x_2 \subseteq S$, which is impossible since $S \cap S^{\perp} = \emptyset$ by (O1). So we can equivalently write that $a \in (x_1 \star y_1) \cap (x_2 \star y_2)$; and by axiom (G3) of 2.1 we get a point $b \in (x_1 \star x_2) \cap (y_1 \star y_2)$. But such b lies in both S and $S^{\perp}$, which is impossible. Hence the non-empty set $(\{a\} \vee S^{\perp}) \cap S$ is a singleton. In particular does this argument imply that $\{a\} = (\{a\} \vee S^{\perp}) \cap S$ if $a \in S$: so the partial map $r: G \dashrightarrow S$ sending $a \notin S^{\perp}$ to the single element of $(\{a\} \vee S^{\perp}) \cap S$ is a retract to the inclusion $i: S \longrightarrow G$. $\square$

Now $i: S \longrightarrow G$ is a morphism of projective geometries when we let S inherit the collinearity from G, but in fact so is $r: G \dashrightarrow S$. Therefore $\mathsf{pr} := i \circ r: G \dashrightarrow G$ is an idempotent morphism of projective geometries with kernel $S^{\perp}$ and image S. It is moreover true that $\mathsf{pr}(a) \perp b \Leftrightarrow a \perp \mathsf{pr}(b)$ for $a, b \notin S^{\perp}$ (the morphism is "self-adjoint"), and so we have every reason to speak of the **projector** with image S and kernel $S^{\perp}$. Much more on this can be found in [Faure and Frölicher, 2000, section 14.4].

8.7 More on projectors. Interestingly, there is a lattice-theoretic analog of 8.6: A complete orthocomplemented lattice C is orthomodular if and only if for each $x \in C$ the map $\varphi_x: C \longrightarrow C: y \mapsto x \wedge (x^{\perp} \vee y)$ has a right adjoint, which then is the map $\psi_x: C \longrightarrow C: y \mapsto x^{\perp} \vee (x \wedge y)$. If C is moreover atomistic and satisfies the covering law, then φ_x is a morphism of propositional systems.

Proof. Clearly the maps φ_x and ψ_x preserve order. Now let C be a complete orthocomplemented orthomodular lattice, then

$$\psi_x(\varphi_x(y)) = x^{\perp} \vee \Big(x \wedge (x \wedge (x^{\perp} \vee y))\Big) = x^{\perp} \vee \Big(x \wedge (x^{\perp} \vee y)\Big) \overset{*}{=} x^{\perp} \vee y \geq y$$

where orthomodularity was used in $(*)$. Similarly one shows $\varphi_x(\psi_x(y)) \leq y$, so we get the adjunction $\varphi_x \dashv \psi_x$. Conversely, if $x \leq y$ in a complete orthocomplemented lattice C, then using this information in $(**)$ gives

$$\begin{aligned} \varphi_{x^{\perp}}(y) &= x^{\perp} \wedge (x^{\perp\perp} \vee y) = x^{\perp} \wedge (x \vee y) \overset{**}{=} x^{\perp} \wedge y \leq y, \\ \psi_{x^{\perp}}(y) &= x^{\perp\perp} \vee (x^{\perp} \wedge y) = x \vee (x^{\perp} \wedge y) \overset{**}{\leq} y \vee y = y. \end{aligned}$$

Assuming that $\varphi_{x^{\perp}} \dashv \psi_{x^{\perp}}$ we get $y \leq \psi_{x^{\perp}}(y)$ from the first line, thus $y = \psi_{x^{\perp}}(y)$ if we combine it with the second line, which is the orthomodular law.

Next suppose that C is a propositional system, let $a \in C$ be an atom and $x \in C$. If $a \leq x^{\perp}$ then $\varphi_x(a) = 0$. If $a \not\leq x^{\perp}$ then $a \wedge x^{\perp} = 0$ so $x^{\perp} \lessdot a \vee x^{\perp}$. By lower semimodularity of C (see 5.11 and use [Faure and Frölicher, 2000, 1.5.7]) it follows that either $x \wedge x^{\perp} = x \wedge (a \vee x^{\perp})$ or $x \wedge x^{\perp} \lessdot x \wedge (a \vee x^{\perp})$; in any case we have shown that $\varphi_x(a)$ is 0 or covers 0. $\square$

A map like the $\varphi_x: C \longrightarrow C$ in the statement above, is called a **Sasaki projector**, and its right adjoint is a **Sasaki hook**. These maps were introduced by U. Sasaki [1954], and extensively used in [Piron, 1976, 4–1] to describe lattice-theoretically the effect of an "ideal measurement of the first kind" on a (quantum) physical system. See also [Coecke and Smets, 2004] for a discussion of the (quantum logical) meaning of the adjunction of Sasaki projection and Sasaki hook.

8.8 Another irreducibility criterion. A **bounded lattice** is, by definition, a lattice with a smallest element 0 and a greatest element 1. If L_1 and L_2 are bounded lattices then, with componentwise lattice structure, the cartesian product $L_1 \times L_2$ is a bounded lattice too. An element $z \in L$ of a bounded lattice is **central** if there exist bounded lattices L_1, L_2 and an isomorphism (i.e. a bijection that preserves and reflects order) $\varphi: L_1 \times L_2 \xrightarrow{\sim} L$ such that $z = \varphi(1, 0)$. The set $\mathcal{Z}(L)$ of central elements, called the **center** of L, is an ordered subset of L that contains at least 0 and 1. A wealth of information on this topic can be found in [Maeda and Maeda, 1970, sections 4 and 5] or any other standard reference on lattice theory.

One can easily figure out that the cartesian product $L_1 \times L_2$ of bounded lattices is a projective lattice if and only if L_1 and L_2 are projective lattices (just view such an L_i as a segment in L);

$$\begin{tikzcd} \mathcal{R} \arrow[rr, "\sigma"] \arrow[dr, "\mathcal{M}"'] & & \mathcal{S} \arrow[dl, "\mathcal{N}"] \\ & \mathsf{Ab} \arrow[u, phantom, "\overset{f}{\Rightarrow}"] & \end{tikzcd}$$

Figure 8: Categorical definition of 'semilinear map'

and $L_1 \times L_2$ is then a coproduct in $\mathsf{ProjLat}$ (see also 3.8). Hence L is an irreducible projective lattice if and only if $\mathcal{Z}(L) = \{0,1\}$ ("L has a trivial center"). One can moreover show that $\mathcal{Z}(L)$ forms a complete atomistic Boolean (i.e. complemented and distributive) sublattice of L; and the segments $[0,\alpha] \subseteq L$, with α an atom of $\mathcal{Z}(L)$, are precisely the 'maximal irreducible segments' of L (a notion that we did not bother defining in section 3); so L is the coproduct in $\mathsf{ProjLat}$ of these segments. Details are in [Maeda and Maeda, 1970, 16.6] for example, where the term 'modular matroid lattice' is used synonymously for 'projective lattice'.

For a propositional system C, one can work along the same lines to prove that C is irreducible if and only if $\mathcal{Z}(C) = \{0,1\}$; the center $\mathcal{Z}(C)$ is again always a complete atomistic Boolean sublattice of C; and C is the coproduct in $\mathsf{PropSys}$ of the segments $[0,\alpha]$ with α an atom of $\mathcal{Z}(C)$. C. Piron [1976, p. 29] has called the atoms of $\mathcal{Z}(C)$ the **superselection rules** of the propositional system C. In geometric terms, a subspace $S \subseteq G$ of a projective geometry is a central element in $\mathcal{L}(G)$ if and only if also the set-complement $S^{\mathsf{c}} := G \setminus S$ is a subspace of G. And if G is a Hilbert geometry then S is central if and only if $S^{\mathsf{c}} = S^{\perp}$, in which case S is necessarily a closed subspace. So $\mathcal{Z}(\mathcal{L}(G)) \cong \mathcal{Z}(\mathcal{C}(\mathcal{L}(G)))$ for a Hilbert geometry G, proving at once that the center of a Hilbert lattice L is the same Boolean algebra as the center of the propositional system $\mathcal{C}(L)$ of closed elements in L.

8.9 Modules on a ring. Vector spaces on fields are very particular examples of modules on rings; and modules on rings are very "categorical" objects: consider a (not necessarily commutative) ring R as a one-object Ab-enriched category $\mathcal{R}$, then a (left) module (M,R) is an Ab-presheaf $\mathcal{M}\colon \mathcal{R} \longrightarrow \mathsf{Ab}$. (As usual, Ab denotes the category of abelian groups.) In the same vein, also semilinear maps between vector spaces are instances of an intrinsically categorical notion: viewing ring-modules (M,R) and (N,S) as Ab-presheaves $\mathcal{M}\colon \mathcal{R} \longrightarrow \mathsf{Ab}$ and $\mathcal{N}\colon \mathcal{S} \longrightarrow \mathsf{Ab}$, a "semilinear map" $(f,\sigma)\colon (M,R) \longrightarrow (N,S)$ ought to be defined as an Ab-functor $\sigma\colon \mathcal{R} \longrightarrow \mathcal{S}$ together with and Ab-natural transformation $f\colon \mathcal{M} \Longrightarrow \mathcal{N} \circ \sigma$, cf. figure 8. It is thus natural to investigate whether and how one can associate a (suitably adapted notion of) 'projective geometry' to a general module, and a morphism of projective geometries to a semilinear map as defined above. M. Greferath and S. Schmidt's Appendix E in [Grätzer, 1998] and Faure's [2004] deal with aspects of this; in our opinion it would be enlightening to both algebraists and geometers to study the (Ab-enriched) categorical side of this.

8.10 Lattice-theoretic equivalents to Solèr's condition. Recall that M.P. Solèr [1995] proved that an infinite dimensional generalized Hilbert space is a "classical" Hilbert space (over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$) exactly when it has an orthonormal sequence (see [Prestel, 2006] in this volume for much more on this theorem). As Solèr pointed out in the same paper, the "angle bisecting" axiom of R.P. Morash [1973] provides an equivalent but lattice-theoretic condition. S. Holland [1995, §4] used “harmonic conjugates” to formulate another lattice-theoretic alternative. He also proposed [Holland, 1995, §5] a (non lattice-theoretic) “ample unitary group axiom”: an infinite dimensional orthomodular space H over K is a “classical” Hilbert space if and only if for any two orthogonal nonzero vectors $a, b \in H$ there exists a unitary map $U : H \longrightarrow H$ (see 7.7) such that $U(Ka) = Kb$. R. Mayet [1998] has proved the following lattice-theoretic alternative: an orthomodular space H is an infinite dimensional Hilbert space over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ if and only if there exist $a, b \in \mathcal{C}(H)$ where $\mathsf{dim}\ b \geq 2$ and an ortho-isomorphism $f : \mathcal{C}(H) \longrightarrow \mathcal{C}(H)$ such that $f|_{[0,b]}$ is the identical map and $f(a) \gneqq a$. The condition on $f|_{[0,b]}$ guarantees that the semi-unitary map inducing f (by Wigner's theorem, see 7.8) is unitary. Similar characterizations using “symmetries” of the lattice $\mathcal{C}(H)$ were proposed in [Aerts and Van Steirteghem, 2000] and in [Engesser and Gabay, 2002]. The question whether the transitivity of the whole group of ortho-isomorphisms of $\mathcal{C}(H)$ still characterizes the “classical” Hilbert spaces among the infinite dimensional orthomodular spaces seems to be unanswered.

9. Appendix: notions from lattice theory

Mostly to fix terminology, we recall the notions from lattice theory we have used in this chapter; in the previous sections these are marked with a “†” when they are used for the first time.

A **partially ordered set**, also called simply **ordered set** or **poset**, is a set P together with a binary relation $\leq$ which is reflexive, antisymmetric and transitive. We also use the standard notation $x < y$ for $x \leq y$ and $x \neq y$. The **opposite** ordered set P^{op} has the same elements as P but with its order relation $\preccurlyeq$ reversed: for $x, y \in P$ we have $x \leq y \Leftrightarrow y \preccurlyeq x$.

For a subset X of P we say that $p \in P$ is an **upper bound** of X if $x \leq p$ for every $x \in X$, we say that p is a **least upper bound**, or a **supremum**, or a **join**, of X if for every other upper bound q we have $p \leq q$. By antisymmetry a least upper bound is unique if it exists. The concept of **greatest lower bound** (also called **infimum** or **meet**) is defined dually. If the supremum of X exists and lies in X we call it the **maximum** of X, denoted by $\max X$. Dually, we can define $\min X$, the **minimum** of X.

A **lattice** is a poset L any two of whose elements $x, y \in L$ have a **meet** denoted by $x \wedge y$ and a **join** denoted by $x \vee y$. It is **complete** if any subset $X \subseteq L$ has a join, then denoted by $\bigvee X$, and a meet $\bigwedge X$. (In fact, if all joins exist in an ordered set L then so do all meets, and *vice versa*; thus an ordered set L is a complete lattice if and only if it has all joins, if and only if it has all meets.) Putting $X = L$ we see that a complete lattice has a **bottom element** 0 and a **top element** 1, that is, elements satisfying $0 \leq x \leq 1$ for every $x \in L$. For $a \leq b$ in a lattice L, the **interval** or **segment** $[a, b]$ is the lattice $\{x \in L \mid a \leq x \leq b\}$.

A map $f\colon P_1 \longrightarrow P_2$ between two ordered sets is said to **preserve order**, or is called **monotone**, if for any $x, y \in P_1$,

$$x \leq y \text{ implies } f(x) \leq f(y).$$

It is an **isomorphism** of ordered sets (or of lattices when appropriate) if it moreover has an order-preserving inverse.

For two elements x, y of P we say that y **covers** x and we write $x \lessdot y$ when $x < y$ but never $x < p < y$ for $p \in P$. If P is a poset with bottom element 0, we call $a \in P$ an **atom** if a covers 0. If P is a poset with top element 1 then c is a **coatom** if 1 covers c. A lattice L with bottom element 0 is called[6] **atomistic** if every element $x \in L$ is the join of the atoms it contains:

[6] G. Birkhoff [1967] calls these ‘atomic’ or ‘point lattices’.

$x = \bigvee\{a \in L \mid a \text{ atom}, a \leq x\}$.

A nonempty subset $D \subseteq P$ of a poset is called **directed** if for any $x, y \in D$, there exists $z \in D$ such that $x \leq z$ and $y \leq z$. A complete lattice L is called **continuous** (some say **meet-continuous**) if for any directed set $D \subseteq L$ and any $a \in L$ we have $a \wedge (\bigvee D) = \bigvee\{a \wedge d \mid d \in D\}$.

A lattice L is called **modular** if, for every $x, y, z \in L$,

$$x \leq z \text{ implies } x \vee (y \wedge z) = (x \vee y) \wedge z.$$

The following are weaker notions: L is **upper semimodular** if $u \wedge v \lessdot v$ implies $u \lessdot u \vee v$; and it is **lower semimodular** if $u \lessdot u \wedge v$ implies $u \wedge v \lessdot v$.

A lattice L with 0 satisfies the **covering law** if for any $x \in L$ and any atom $a \in L$ we have

$$a \wedge x = 0 \text{ implies } x \lessdot a \vee x.$$

An **orthocomplementation** on a lattice L with 0 and 1 is a map $L \longrightarrow L\colon x \mapsto x^{\perp}$ which satisfies, for all $x, y \in L$,

i. $x \leq y$ implies $y^{\perp} \leq x^{\perp}$,
ii. $(x^{\perp})^{\perp} = x$,
iii. $x \vee x^{\perp} = 1$ and $x \wedge x^{\perp} = 0$.

A lattice is called **orthocomplemented** if it is equipped with an orthocomplementation. Such a lattice L is called **orthomodular** if moreover, for all $x, y \in L$,

$$x \leq y \text{ implies } x \vee (x^{\perp} \wedge y) = y.$$

Since the orthocomplementation induces an isomorphism $L \longrightarrow L^{\mathrm{op}}$ this is equivalent to

$$x \leq y \text{ implies } y \wedge (y^{\perp} \vee x) = x.$$

Given two order-preserving maps $f\colon P_1 \longrightarrow P_2$ and $g\colon P_2 \longrightarrow P_1$ in opposite directions, we say that f is a **left adjoint** of g, and g a **right adjoint** of f, written $f \dashv g$, if they satisfy one, and hence all, of the following equivalent conditions:

i. for all $x \in P_1$ and $y \in P_2$ we have $f(x) \leq y \Leftrightarrow x \leq g(y)$,
ii. $f(x) = \min\{y \in P_2 \mid x \leq g(y)\}$ for all $x \in P_1$,
iii. $g(y) = \max\{x \in P_1 \mid f(x) \leq y\}$ for all $y \in P_2$,
iv. $x \leq g(f(x))$ and $f(g(y)) \leq y$ for all $x \in P_1$ and all $y \in P_2$.

The pair (f, g) is called a **Galois connection** or said to form an **adjunction** (between the ordered sets P_1 and P_2.) It follows from conditions (ii) and (iii) above that adjoints determine each other uniquely (when they exist). And one can check that f is surjective if and only if g is injective, if and only if $f(g(y)) = y$ for all $y \in P_2$.

Still considering such an adjunction $f \dashv g$, f preserves all joins that exist in P_1; similarly, g preserves all meets that exist in P_2. Conversely, for an ordered set L the following conditions are equivalent:

i. L is a complete lattice,
ii. every map $h\colon L \longrightarrow P$ preserving all joins has a right adjoint,
iii. every map $h\colon L \longrightarrow Q$ preserving all meets has a left adjoint.

A **closure operator** on a poset P is a monotone map $\mathsf{cl}\colon P \longrightarrow P$ satisfying, for all $x \in P$,

i. $\mathsf{cl}(\mathsf{cl}(x)) \leq \mathsf{cl}(x)$,
ii. $x \leq \mathsf{cl}(x)$.

It is obvious that $\mathsf{cl}(\mathsf{cl}(x)) = \mathsf{cl}(x)$, i.e. that a closure operator is an idempotent map. Its fixpoints are often said to be the **closed elements** of P (w.r.t. cl): they form a sub-poset $\mathsf{cl}(P) \subseteq P$. The surjection $\mathsf{cl}\colon P \longrightarrow \mathsf{cl}(P)$ and the inclusion $i\colon \mathsf{cl}(P) \longrightarrow P$ form an adjunction $\mathsf{cl} \dashv i$. Conversely, for any adjunction $f \dashv g$ between posets P_1 and P_2, $g \circ f$ is a closure operator on P_1; and if moreover $f \circ g$ is the identity on P_2 then P_2 is isomorphic to the poset of fixpoints of $g \circ f$.

One now easily deduces that, for a closure operator cl on a complete lattice L, also $\mathsf{cl}(L)$ is a complete lattice for the order inherited from L: it has the "same" meets as L (since i preserves meets) and the joins are given by $\bigtriangledown S = \mathsf{cl}(\bigvee S)$ where $S \subseteq \mathsf{cl}(L)$ and $\bigvee$ is the join in L.

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