Introduction to

"Categorical structures enriched in a quantaloid: categories and semicategories"

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Preface (added July 26, 2007)

On November 12, 2003, I obtained a doctorate on my dissertation entitled "Categorical Structures enriched in a quantaloid: categories and semicategories", at the Université Catholique de Louvain in Louvain-la-Neuve, Belgium. The promotor of my doctoral research was Francis Borceux, and the other members of the jury were Bob Coecke, Yves Félix, Jean-Roger Roisin, Jiří Rosický and Enrico Vitale.

The aim of my doctoral dissertation was to bring together the existing material on quantaloid-enriched categories, and then to use this formalism, adapting it where necessary, to analyze in detail the concept of, loosely speaking, "sheaves on a quantaloid". Besides the Introduction, my dissertation consisted of five chapters: 1. Preliminaries on quantaloids; 2. Categories enriched in a quantaloid; 3. Some categorical algebra; 4. Regular semicategories; and 5. Totally regular semicategories. These have now been published in three articles: "Categorical structures enriched in a quantaloid: categories, distributors, functors", in Theory and Applications of Categories **14**, pp. 1–45 (2005); "Categorical structures enriched in a quantaloid: regular presheaves, regular semicategories", in Cahiers de Topologie et Gomtrie Diffrentielle Catgoriques **46**, pp. 99–121 (2005); and "Categorical structures enriched in a quantaloid: orders and ideals over a base quantaloid", in Applied Categorical Structures **13**, pp. 235–255 (2005). Chopping up my dissertation in these three articles, meant that its Introduction got lost (even though parts of it made it to the introductory sections of the respective articles). It seemed useful to reproduce it as a "stand-alone" paper and make it available via the world wide web, for it provides a not-so-technical overview of the material covered in my thesis (and *a fortiori* in those three articles).

Apart from twiddling with the lay-out, correcting some spelling mistakes and changing a couple of mathematical notations (to make them exactly the same as in the three articles), I didn't change one *iota* of the original Introduction to my doctoral dissertation. In particular are the references included here precisely those of my doctorate.

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1. A quantaloid as base for enriched categories

The theory of categories enriched in a symmetric monoidal closed category $\mathcal{V}$ is, by now, well known [Bénabou, 1963, 1965; Eilenberg and Kelly, 1966; Lawvere, 1973; Kelly, 1982]. For such a $\mathcal{V}$ with "enough" (co)limits the theory of $\mathcal{V}$-categories, distributors and functors can be pushed as far as needed: it includes such things as (weighted) (co)limits in a $\mathcal{V}$-category, $\mathcal{V}$-presheaves on a $\mathcal{V}$-category, Kan extensions of enriched functors, Morita theory for $\mathcal{V}$-categories, and so on. We will argue below that "$\mathcal{V}$-category theory" can be generalized to "$\mathcal{Q}$-category theory", where now $\mathcal{Q}$ denotes a quantaloid. A quantaloid is a kind of bicategory, so this is really a particular case of the theory of "$\mathcal{W}$-category theory", with $\mathcal{W}$ a bicategory, as pioneered by [Bénabou, 1967; Walters, 1981; Street, 1983] and later further developed [Betti *et al.*, 1983; Gordon and Power, 1997].

1.1. From monoidal categories to bicategories. Monoidal categories are precisely one-object bicategories [Bénabou, 1967]. To wit, for a monoidal category $\mathcal{V}$, denote $\mathsf{susp}(\mathcal{V})$ – and call it the suspension of $\mathcal{V}$ – for the following bicategory: it has one object; its arrows are the objects of $\mathcal{V}$, their composition in $\mathsf{susp}(\mathcal{V})$ is given by the tensor in $\mathcal{V}$, and in particular is the identity arrow on the single object of $\mathsf{susp}(\mathcal{V})$ precisely the identity object for the tensor in $\mathcal{V}$; and the 2-cells in $\mathsf{susp}(\mathcal{V})$ correspond precisely to the arrows of $\mathcal{V}$. That $\mathsf{susp}(\mathcal{V})$ is in general a bicategory, and not a 2-category, is due to the fact that the tensor on $\mathcal{V}$ is associative, and has left and right identity, only up to natural isomorphism. (Viewing a monoidal category $\mathcal{V}$ as a one-object bicategory has some analogues that may be more familiar: for instance, a monoid is a one-object category, or more in particular, a group is a one-object groupoid.)

Let $\mathcal{W}$ now denote a general bicategory. Given the above it is natural to ask in how far $\mathcal{V}$-category theory can be generalized to $\mathcal{W}$-category theory. Let us therefore first try to make sense of the usual conditions on a base monoidal category $\mathcal{V}$ – symmetry, closedness, (co)completeness – in the case of a bicategory $\mathcal{W}$.

In $\mathcal{V}$-category theory one usually assumes the symmetry of the tensor in $\mathcal{V}$; this is essential to show that $\mathcal{V}$ is itself a $\mathcal{V}$-category with hom-objects given by the right adjoint to tensoring. But the definitions for $\mathcal{V}$-category, distributor or functor do not require the symmetry of the tensor in $\mathcal{V}$, and moreover, many important features of $\mathcal{V}$-category theory can still be developed in the non-symmetric case (sometimes one has to work a little harder though). This is encouraging, because in passing from a monoidal category $\mathcal{V}$ to a general bicategory $\mathcal{W}$ we will have to sacrifice this symmetry: tensoring objects in $\mathcal{V}$ corresponds to composing morphisms in $\mathcal{W}$ and in general it simply does not make sense for the composition $g \circ f$ of two arrows f, g

to be "symmetric" (the domain of f may be different from the codomain of g)[1]!

On the other hand, we can successfully translate the notion of closedness of a monoidal category $\mathcal{V}$ to the more general setting of a bicategory $\mathcal{W}$. The tensor on objects in $\mathcal{V}$ corresponds to the composition of morphisms in $\mathcal{W}$, so we simply ask that, for any object X of $\mathcal{W}$ and any arrow $f: A \longrightarrow B$ in $\mathcal{W}$, both functors[2]

$$- \circ f : \mathcal{W}(B,X) \longrightarrow \mathcal{W}(A,X) : x \mapsto x \circ f, \tag{1}$$
$$f \circ - : \mathcal{W}(X,A) \longrightarrow \mathcal{W}(X,B) : x \mapsto f \circ x \tag{2}$$

have right adjoints. We'll denote these as

$$\{f,-\} : \mathcal{W}(A,X) \longrightarrow \mathcal{W}(B,X) : y \mapsto \{f,y\}, \tag{3}$$
$$[f,-] : \mathcal{W}(X,B) \longrightarrow \mathcal{W}(X,A) : y \mapsto [f,y]. \tag{4}$$

(Indeed, we have to ask for both adjoints!) Such a bicategory $\mathcal{W}$ is said to be closed. Some call an arrow such as $\{f,y\}$ a (right) extension and $[f,y]$ a (right) lifting (of y through f).

Finally, by saying that $\mathcal{V}$ has "enough limits and colimits" is in practice often meant that $\mathcal{V}$ has small limits and small colimits. (In particular are the colimits required to compose distributors between (small) $\mathcal{V}$-categories. And the limits are crucial if one wants to speak of a $\mathcal{V}$-object of $\mathcal{V}$-natural transformations between two given parallel functors.) In a bicategory $\mathcal{W}$ the analogue for these (co)completeness conditions is straightforward: recalling that in passing from a monoidal category $\mathcal{V}$ to a bicategory $\mathcal{W}$ the rôle of the objects of $\mathcal{V}$ is played by the arrows of $\mathcal{W}$, we must now ask for $\mathcal{W}$ to have in its hom-categories small limits and small colimits (i.e. $\mathcal{W}$ is locally complete and cocomplete).

So, to summarize, when trying to develop category theory over a base bicategory $\mathcal{W}$ rather than a base monoidal category $\mathcal{V}$, it seems reasonable to work with a base bicategory which is closed, locally complete and locally cocomplete. Note that in such a bicategory $\mathcal{W}$, due to its closedness, composition always distributes on both sides over colimits of morphisms:

$$f \circ (\mathrm{colim}_{i\in I}\, g_i) \cong \mathrm{colim}_{i\in I}(f \circ g_i), \tag{5}$$
$$(\mathrm{colim}_{j\in J}\, f_j) \circ g \cong \mathrm{colim}_{j\in J}(f_j \circ g). \tag{6}$$

That is to say, the local colimits are stable under composition. (But this does not hold in general for local limits!)

[1]It may be argued however that the symmetry of the tensor in $\mathcal{V}$ is adequately generalized by an involution on $\mathcal{W}$.

[2]Composition of arrows $f: A \to B$ and $g: B \to C$ is written in the usual manner, as $g \circ f: A \to C$. And $\mathcal{W}(A,B)$ denotes the hom-category whose objects are arrows with domain A and codomain B.

1.2. Quantales and quantaloids. We will focus on a special case of these closed, locally complete and locally cocomplete bicategories: namely, we study such bicategories whose hom-categories are moreover small and skeletal. Thus the hom-categories are simply complete lattices: a small, skeletal complete (cocomplete) category is precisely an ordered set with arbitrary infima (suprema), and the suprema (infima) then come for free. We will write the local structure as an order, and local limits and colimits of morphisms as their infimum, resp. supremum—so for arrows with same domain and codomain we have things like $f \leq f'$, $\bigvee_{i \in I} f_i$, $\bigwedge_{j \in J} g_j$, etc. In particular (5) and (6) become

$$f \circ (\bigvee_i g_i) = \bigvee_i (f \circ g_i), \tag{7}$$

$$(\bigvee_j f_j) \circ g = \bigvee_j (f_j \circ g). \tag{8}$$

The adjoint functor theorem says that the existence of the adjoints (3) and (4) to the composition functors (1) and (2) (not only implies but also) is implied by their distributing over suprema of morphisms as in (7) and (8). Such bicategories – whose hom-categories are complete lattices and whose composition distributes on both sides over arbitrary suprema – are called quantaloids. A one-object quantaloid is a quantale[3]. So a quantaloid $\mathcal{Q}$ is a Sup-enriched category and a quantale is monoid in Sup. (Sup denotes the symmetric monoidal closed category of complete lattices and morphisms that preserve suprema). This at once fixes the notion of a homomorphism between quantaloids: it is a Sup-functor between the quantaloids viewed as Sup-categories. (But due to the local structure of quantaloids, this notion of homomorphism can be "relaxed"; in particular does J. Bénabou's notion of "morphism of bicategories", i.e. the Australian's "lax functor", make sense too.)

To gain some intuition about quantales and quantaloids it may be useful to put them in some perspective. Let us therefore look at some examples.

Of course Sup is the example *par excellence* of a quantaloid. And it follows from standard arguments for symmetric monoidal closed categories that, for any sup-lattice L, $\mathsf{Sup}(L, L)$ is a monoid in Sup, i.e. a quantale, and that any quantale is canonically a subquantale of some $\mathsf{Sup}(L, L)$.

Thinking of Sup as some sort of "infinitary version" of Ab, a quantale (a monoid in Sup) is then much like a ring (a monoid in Ab): instead of a finitary sum $a_1 + a_2 + ... + a_n$ it comes with an infinitary one, namely the supremum $\bigvee_{i \in I} a_i$. (Of course, $\bigvee$ is an idempotent operation—so it is more than just an "infinitary sum".) In particular,

[3]It was C. Mulvey [1986] who introduced the word 'quantale' in his work on (non-commutative) C^*-algebras to contrast with the word 'locale'—see further. In today's literature though the word 'quantale' often means different things to different authors. Let us therefore underline that throughout this text a quantale always has a unit for its multiplication ([Rosenthal, 1990] and [Kruml, 2002] do not ask this), but that on the contrary it should not necessarily be "idempotent" nor "right-sided" (as [Borceux, Rosický and Van den Bossche, 1989] ask).

the distributivity of product over sums in a ring generalizes to the distributivity of composition over suprema in a quantale. On the other hand, thinking of Sup as some "simplified version" of Cat confirms the fact that a quantaloid (a Sup-category) is a simple kind of 2-category (a Cat-category); note that indeed every diagram of 2-cells commutes in a quantaloid (and this is obviously not the case in any 2-category).

Quantales can be viewed as "non-commutative locales". Recall that a locale Ω is a complete lattice in which finitary infima distribute over arbitrary suprema: $\forall x, (y_i)_{i \in I} \in \Omega$,

$$x \wedge (\bigvee_{i \in I} y_i) = \bigvee_{i \in I} (x \wedge y_i).$$

For a topological space $(X, \mathcal{T})$ the topology $\mathcal{T}$ is a locale with infimum given by intersection and supremum by union. Although not every locale can be obtained in this way, it is for this reason that they are often thought of as "pointless topologies" [Johnstone, 1982, 1983]. A locale is in particular a complete lattice for which binary infimum and the top element define a compatible structure of (commutative) monoid. A quantale is, in the same terms, a complete lattice equipped with a compatible monoid structure—but not necessarily with the infimum as multiplication nor the top element as unit. Therefore we may think of a quantale as a "(pointless) non-commutative topology". (This is easier said than done; the "spatiality" of a quantale is highly non-trivial! It is one of the research themes of the Brno school [Rosický, 1995; Paseka and Rosický, 2000; Kruml, 2002].)

Let us, to be a bit more concrete, briefly sketch how quantales and quantaloids quite naturally arise in ring theory as such strange "topologies". (This is in some sense (a modern account of) the "historical example" of quantales and quantaloids. Indeed, the importance of ordered sets with a compatible multiplication as structure *an sich* was first recognized in the study of ideals in a ring [Krull, 1924; Ward and Dilworth, 1939].) Thereto, let R denote a (not necessarily commutative) ring (with unit). The ring R is canonically a right module over itself, and a right ideal I is by definition just a (right) submodule[4]. Similarly one defines left-sided ideals, and two-sided ideals, in R. These form respective sets $\mathsf{Ridl}(R)$, $\mathsf{Lidl}(R)$ and $\mathsf{Tidl}(R)$. (For a commutative R these sets obviously coincide, and we may simply denote $\mathsf{Idl}(R)$ for the ideals in R.) Any two additive subgroups $I, J \subseteq R$ can be multiplied, by the formula

$$I \cdot J = \{\text{finite sums } i_1 j_1 + \ldots + i_n j_n \text{ with all } i_k \in I \text{ and all } j_k \in J\}, \tag{9}$$

and a new additive subgroup of R is obtained. It is easily seen that this multiplication is internal on each of the sets $\mathsf{Ridl}(R)$, $\mathsf{Lidl}(R)$ and $\mathsf{Tidl}(R)$, and that the latter is then even a monoid with unit R. Moreover, for any family $(I_k)_{k \in K}$ of additive subgroups

[4] So R is a "space", and I is a "subspace".

of R we may consider the subgroup generated by (the set-theoretic union of) this family: it is the sum

$$\sum_{k\in K} I_k = \{\text{finite sums of elements in } \bigcup_{k\in K} I_k\}. \tag{10}$$

In particular, the ordered sets $(\mathsf{Ridl}(R), \subseteq)$, $(\mathsf{Lidl}(R), \subseteq)$ and $(\mathsf{Tidl}(R), \subseteq)$ are complete lattices with this sum as supremum (and top element R). The multiplication in (9) distributes on both sides over the arbitrary sums in (10); so $\mathsf{Tidl}(R)$ is a quantale, and $\mathsf{Ridl}(R)$ and $\mathsf{Lidl}(R)$ are "almost" quantales (they lack a two-sided unit for the multiplication).

At first sight it thus seems that the separate structures of right-sided, left-sided and two-sided ideals in a ring R each provide some kind of "(pointless) non-commutative topology". But actually it is not necessary to keep these different kinds of ideals separated, as G. Van den Bossche [1995] points out (but she credits B. Lawvere for this idea): they organize themselves quite naturally in one single "quantaloid of ideals" of R. Indeed, observe that:

when $I \in \mathsf{Lidl}(R)$ and $J \in \mathsf{Ridl}(R)$, then $I \cdot J \in \mathsf{Tidl}(R)$;
when $I \in \mathsf{Ridl}(R)$ and $J \in \mathsf{Tidl}(R)$, then $I \cdot J \in \mathsf{Ridl}(R)$;
when $I \in \mathsf{Tidl}(R)$ and $J \in \mathsf{Lidl}(R)$, then $I \cdot J \in \mathsf{Lidl}(R)$.

When $I \in \mathsf{Ridl}(R)$ and $J \in \mathsf{Lidl}(R)$, then $I \cdot J$ still makes sense, but it is not any longer an ideal in R. However, it is an additive subgroup of R which is a module on the center $\mathcal{Z}(R)$ of R. The collection of these still forms a quantale with multiplication as in (9) (and $\mathcal{Z}(R)$ as unit) and suprema as in (10) (with top R). (Note that any additive subgroup of R which is a left $\mathcal{Z}(R)$-module is also a right $\mathcal{Z}(R)$-module, and conversely. And that for a commutative R, the center being the whole of R, these coincide with the ideals.) This leads to the consideration of the following quantaloid: $\mathcal{Q}_R$ has two objects, 0 and 1; the hom-objects are

$$\begin{aligned}&\mathcal{Q}_R(0,0) = \text{additive subgroups of } R \text{ which are } \mathcal{Z}(R)\text{-modules},\\ &\mathcal{Q}_R(0,1) = \mathsf{Lidl}(R), \quad \mathcal{Q}_R(1,0) = \mathsf{Ridl}(R), \quad \mathcal{Q}_R(1,1) = \mathsf{Tidl}(R)\end{aligned}$$

(which are all complete lattices with suprema as in (10) and, in particular, top element R); composition in $\mathcal{Q}_R$ is the multiplication as in (9) which we read from right to left (so $I \circ J = I \cdot J$); and the identity arrow on 1 is R and that on 0 is $\mathcal{Z}(R)$. (Note that, for a commutative R, this quantaloid $\mathcal{Q}_R$ is equivalent as Sup-category to the quantale $\mathsf{Idl}(R)$ of ideals in R!) This quantaloid is then a "(multi-typed, non-commutative) topology" for the ring R[5].

[5] K. Rosenthal [1996] has a slightly different construction for the "quantaloid of ideals" of a ring R: it is like Van den Bossche's except for the hom-lattice $\mathcal{Q}_R(0,0)$ which he takes to be all additive subgroups of R. He too attributes this construction to Lawvere. But this construction has the drawback that, for a commutative R, it is not equivalent to $\mathsf{Idl}(R)$.

There are many more examples of quantales and quantaloids arising in different branches of mathematics, but this is not the place for an extensive discussion. But just to give an idea of the range of subjects where quantales and quantaloids show up, let us mention that:

- Quantales arise naturally when studying rings and algebras by ordered structures of ideals or other substructures. In particular can at least certain classes of C^*-algebras be recovered from a suitable quantale of ideals [Akemann, 1970; Giles and Kummer, 1971; Borceux, Rosický and Van den Bossche, 1989; Mulvey and Pelletier, 2001, 2002].

- When thinking of a quantale as the order-theoretic version of a ring, it makes sense to develop a theory of modules over a quantale. One may for example investigate conditions on such modules to make them behave like (an order-theoretic version of) a Hilbert space [Paseka, 2002].

- The category of relations in a (Grothendieck) topos is a quantaloid, and difficult calculations with toposes can be reduced to simpler ones by considering precisely those quantaloids of relations [Pitts, 1988].

- Abstract categories of relations have been studied (under different names: 'allegories', 'distributive categories of relations') and they are if not quantaloids then at least locally ordered categories [Carboni and Walters, 1987; Freyd and Scedrov, 1990].

- Certain classes of quantales have been recognized as semantics for propositional linear logic [Girard, 1987; Rosenthal, 1990].

- In computer science quantaloids can be regarded as algebras of typed experimental observations and are as such of use in the theory of process semantics [Abramsky and Vickers, 1993; Resende, 2000].

- More recently quantales and quantaloids have been recognized to play an important rôle in dynamic operational quantum logic [Amira *et al.*, 1998; Coecke and Stubbe, 1999; Coecke *et al.*, 2001]. More generally, a "process logic" may be viewed as a category enriched in a "quantaloid of processes", the enrichment thus providing an external implication to the logic [Coecke and Smets, 2001; Stubbe and Sourbron, 2002; Stubbe, 2003].

In 2000 the AMS decided to consider the theory of quantales as a subject in its own right, and to classify it henceforth under MSC 06F07.

1.3. Enriched categories, distributors and functors. Now that we have agreed on the kind of bicategory that we would like to use as base for enriched categories, namely quantaloids, let us see how the familiar definitions for $\mathcal{V}$-enriched category, distributor and functor – as found in [Kelly, 1982; Borceux, 1994] for example – can be generalized.

Let us first recall that, for $\mathcal{V}$ a symmetric monoidal closed category, a small $\mathcal{V}$-enriched category $\mathbb{A}$ consists of:

- a set $\mathbb{A}_0$ of objects $a, a', a'', ...$;
- for every two objects $a, a' \in \mathbb{A}_0$ a hom-object[6] $\mathbb{A}(a', a)$ in $\mathcal{V}$;
- for every three objects $a, a', a'' \in \mathbb{A}_0$ a composition morphism in $\mathcal{V}$

$$\mu_{a'',a',a}\colon \mathbb{A}(a'', a') \otimes \mathbb{A}(a', a) \longrightarrow \mathbb{A}(a'', a);$$

- for every object $a \in \mathbb{A}_0$ an unit morphism in $\mathcal{V}$

$$\eta_a\colon I \longrightarrow \mathbb{A}(a, a);$$

and these data have to satisfy some diagrammatic axioms. For two $\mathcal{V}$-categories $\mathbb{A}$ and $\mathbb{B}$, a functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ is

- an object mapping $\mathbb{A}_0 \longrightarrow \mathbb{B}_0\colon a \mapsto Fa$;
- for any two objects $a, a' \in \mathbb{A}_0$ a $\mathcal{V}$-morphism

$$F_{a',a}\colon \mathbb{A}(a', a) \longrightarrow \mathbb{B}(Fa', Fa);$$

satisfying axioms that express precisely the preservation of composition and units in $\mathbb{A}$. A distributor (which some also call module or profunctor) $\Phi\colon \mathbb{A} \mathrel{-\!\!\circ\!\!\rightarrow} \mathbb{B}$ is usually[7] defined to be a functor $\Phi\colon \mathbb{B}^\star \otimes \mathbb{A} \longrightarrow \mathcal{V}$. This requires the notion of dual $\mathcal{V}$-category $\mathbb{B}^\star$, which itself requires the symmetry of the tensor in $\mathcal{V}$. In view of the generalization we wish to make, where we no longer dispose of a "symmetric tensor" in the base (bi)category, it is therefore appropriate to note that a distributor $\Phi\colon \mathbb{A} \mathrel{-\!\!\circ\!\!\rightarrow} \mathbb{B}$ is equivalently given by:

- for every two objects $a \in \mathbb{A}_0$, $b \in \mathbb{B}_0$, an object $\Phi(b, a)$ in $\mathcal{V}$;

[6] As for the notation: in the base category $\mathcal{V}$ a hom-object like $\mathcal{V}(A, B)$ contains arrows from A to B; but in a $\mathcal{V}$-category $\mathbb{A}$ the hom-object $\mathbb{A}(a', a)$ is thought of as containing arrows which go from a to a'. This may seem awkward at first, but it will make perfect sense when we will replace $\mathcal{V}$ by a quantaloid $\mathcal{Q}$...

[7] It should be noted that a distributor $\Phi\colon \mathbb{A} \mathrel{-\!\!\circ\!\!\rightarrow} \mathbb{B}$ is equivalently given by a $\mathcal{V}$-functor from $\mathbb{A}$ into $\widehat{\mathbb{B}}$, the latter being the $\mathcal{V}$-category of contravariant $\mathcal{V}$-presheaves on $\mathbb{B}$. This is actually J. Bénabou's original concept behind the notion of distributor that he gave for ordinary categories [Bénabou, 1973]. However, in the context of our generalization to a base quantaloid $\mathcal{Q}$, the problem remains that symmetry of the tensor in $\mathcal{V}$ is required to define the $\mathcal{V}$-category of contravariant $\mathcal{V}$-presheaves on $\mathbb{B}$.

- for every three objects $a, a' \in \mathbb{A}_0$, $b \in \mathbb{B}_0$, a $\mathcal{V}$-morphism

$$\Phi^b_{a',a}: \Phi(b, a') \otimes \mathbb{A}(a', a) \longrightarrow \Phi(b, a);$$

- for every three objects $a \in \mathbb{A}_0$, $b, b' \in \mathbb{B}_0$, a $\mathcal{V}$-morphism

$$\Phi^{b,b'}_a: \mathbb{B}(b, b') \otimes \Phi(b', a) \longrightarrow \Phi(b, a);$$

where these morphisms $\Phi^b_{a',a}$ and $\Phi^{b,b'}_a$ – expressing a right action of $\mathbb{A}$, and a left action of $\mathbb{B}$, on Φ – satisfy certain compatibility conditions with composition and units in $\mathbb{A}$ and $\mathbb{B}$.

How can these familiar definitions be understood when we work over a base quantaloid $\mathcal{Q}$ instead of a symmetric monoidal closed $\mathcal{V}$? We should keep in mind that it are the arrows of $\mathcal{Q}$ that will play the rôle of the objects of $\mathcal{V}$, with the composition of arrows being like the tensor of $\mathcal{V}$-objects, and that the 2-cells in $\mathcal{Q}$ must be thought of as the morphisms between objects in $\mathcal{V}$. It may also be recalled that, due to the hom-categories of a quantaloid being ordered sets, every diagram of 2-cells in a quantaloid $\mathcal{Q}$ commutes.

A small $\mathcal{Q}$-enriched category $\mathbb{A}$ will surely have a set of objects, let us denote $a, a', a'', ... \in \mathbb{A}_0$. For reasons that will soon be clear, we must associate to each $a \in \mathbb{A}_0$ some object of $\mathcal{Q}$; we will systematically denote for an $\mathbb{A}$-object a this associated $\mathcal{Q}$-object by ta, and we shall refer to it as the type[8] of a in $\mathcal{Q}$. (Conversely, given an object A of $\mathcal{Q}$, we say that an object $a \in \mathbb{A}_0$ lies over A precisely when $ta = A$.) So we have that

$$\mathbb{A}_0 \text{ is a } \mathcal{Q}_0\text{-typed set: to each } a \in \mathbb{A}_0 \text{ is associated some } ta \in \mathcal{Q}_0. \tag{11}$$

Such a $\mathcal{Q}_0$-typed set $\mathbb{A}_0$ can now be enriched in $\mathcal{Q}$: for every two objects $a, a' \in \mathbb{A}_0$, ask for an arrow $\mathbb{A}(a', a)$ in $\mathcal{Q}$ to play the rôle of "hom from a to a'". Again, for reasons that will soon be clear, we shall ask that this arrow has domain ta and codomain ta':

$$\text{for all } a, a' \in \mathbb{A}_0, \text{ a hom-arrow } \mathbb{A}(a', a): ta \longrightarrow ta' \text{ in } \mathcal{Q}. \tag{12}$$

Now suppose that we have three objects a, a', a''; we can consider Diagram 1 in $\mathcal{Q}$ of the hom-arrows between these objects. To say that there is a "composition from $\mathbb{A}(a'', a') \circ \mathbb{A}(a', a)$ to $\mathbb{A}(a'', a)$ in the $\mathcal{Q}$-enriched category $\mathbb{A}$", we must ask for a composition 2-cell in $\mathcal{Q}$:

$$\text{for all } a, a' \in \mathbb{A}_0 : \mathbb{A}(a'', a') \circ \mathbb{A}(a', a) \leq \mathbb{A}(a'', a) \text{ in } \mathcal{Q}(ta, ta''). \tag{13}$$

[8]So when $\mathcal{Q}$ has a (small) set of objects, the "typing" of objects is simply a function $t: \mathbb{A}_0 \to \mathcal{Q}_0$.

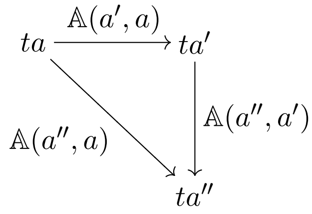

Diagram 1: composition law

Likewise, on any object $a \in \mathbb{A}_0$ there is an endo-hom-arrow $\mathbb{A}(a,a)\colon ta \longrightarrow ta$ in $\mathcal{Q}$, but there is also the identity arrow $1_{ta}\colon ta \longrightarrow ta$ in $\mathcal{Q}$. To say that there is a "unit on a in the $\mathcal{Q}$-enriched category $\mathbb{A}$", we must ask for a unit 2-cell in $\mathcal{Q}$:

$$\text{for all } a \in \mathbb{A}_0 : 1_{ta} \leq \mathbb{A}(a,a) \text{ in } \mathcal{Q}(ta,ta). \tag{14}$$

We do not need to impose any axioms about "composition being associative with left and right units": since "composition" and "units" in $\mathbb{A}$ are determined by 2-cells in $\mathcal{Q}$, the appropriate (diagrammatic) axioms automatically hold! So the above is the appropriate definition of a $\mathcal{Q}$-enriched category $\mathbb{A}$. Note that, quite unlike for $\mathcal{V}$-categories, the composition and unit 2-cells in a $\mathcal{Q}$-category $\mathbb{A}$ are uniquely determined. More precisely, given a $\mathcal{Q}_0$-typed set $\mathbb{A}_0$ as in (11) together with hom-arrows $\mathbb{A}(a',a)\colon ta \longrightarrow ta'$ in $\mathcal{Q}$ as in (12), it is a property, rather than an extra structure, that these objects and hom-arrows form (or not) a $\mathcal{Q}$-category $\mathbb{A}$, i.e. that they satisfy (or not) the composition and unit inequalities of (13) and (14).

For two $\mathcal{Q}$-enriched categories $\mathbb{A}$ and $\mathbb{B}$, a functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ should somehow consist of a mapping of objects together with an action on the hom-arrows. For such an action on hom-arrows to make sense, it will be required that the mapping of objects preserves the types of the objects:

$$\text{an object mapping } \mathbb{A}_0 \longrightarrow \mathbb{B}_0\colon a \mapsto Fa \text{ such that } ta = t(Fa). \tag{15}$$

To express the action of F on the hom-arrows of $\mathbb{A}$, we must ask for action 2-cells in $\mathcal{Q}$:

$$\text{for all } a, a' \in \mathbb{A}_0 : \mathbb{A}(a',a) \leq \mathbb{B}(Fa',Fa) \text{ in } \mathcal{Q}(ta,ta'). \tag{16}$$

The axioms for the "functoriality" of F are certain diagrams of 2-cells in $\mathcal{Q}$, so they are always satisfied. Therefore, given an object mapping as in (15), it is again a property, rather than extra structure, that this mapping determines a functor, i.e. that it satisfies the functor inequalities of (16).

Finally, let us consider a distributor between $\mathcal{Q}$-categories $\mathbb{A}$ and $\mathbb{B}$. In analogy with the above, such a $\Phi\colon \mathbb{A} \mathrel{-\!\!\circ\!\!\!\rightarrow} \mathbb{B}$ consists of a collection of $\mathcal{Q}$-arrows:

$$\text{for all } a \in \mathbb{A}_0, b \in \mathbb{B}_0, \text{ an arrow } \Phi(b,a)\colon ta \longrightarrow tb \text{ in } \mathcal{Q}. \tag{17}$$

And to express the “action of $\mathbb{A}$ on the right, and $\mathbb{B}$ on the left, on Φ” we must ask for action 2-cells in $\mathcal{Q}$:

$$\text{for all } a, a' \in \mathbb{A}_0, b \in \mathbb{B}_0 : \Phi(b, a') \circ \mathbb{A}(a', a) \leq \Phi(b, a) \text{ in } \mathcal{Q}(ta, tb); \tag{18}$$

$$\text{for all } a \in \mathbb{A}_0, b, b' \in \mathbb{B}_0 : \mathbb{B}(b, b') \circ \Phi(b', a) \leq \Phi(b, a) \text{ in } \mathcal{Q}(ta, tb). \tag{19}$$

Once more, the compatibility of these action 2-cells with the composition and unit 2-cells of both $\mathbb{A}$ and $\mathbb{B}$ is automatic, for these compatibilities are expressed by diagrams of 2-cells in $\mathcal{Q}$. Thus, given a family of $\mathcal{Q}$-arrows as in (17), it is a property, rather than extra structure, that this family determines a distributor $\Phi\colon \mathbb{A} \multimap \mathbb{B}$, i.e. satisfies the action inequalities of (18) and (19).

Before we go any further, it is probably useful to discuss briefly some examples.

(a) The extended non-negative reals $\mathbb{R}^+ \cup \{\infty\}$ form a (symmetric) quantale for the opposite order and addition as binary operation. In his famous paper on “Metric spaces, generalized logic and closed categories” B. Lawvere recognized that a category enriched in (the one-object suspension of) this quantale is a “generalized metric space”: the enrichment itself is a binary distance function taking values in the positive reals. In particular is the composition-inequality in such an enriched category the triangular inequality. A functor between such generalized metric spaces is a distance decreasing application. (These metric spaces are “generalized” in that the distance function is not symmetric, that the distance between two points being zero doesn’t imply their being identical, and that the distance between two points may be infinite.)

(b) By **2** we will denote (the one-object suspension of) the two-element Boolean algebra—obviously a particular example of a quantale (or one-object quantaloid). Writing its elements as 0 and 1, the composition in **2** is the conjunction of these truth values, and their disjunction is the supremum. A **2**-enriched category $\mathbb{A}$ is precisely a preordered set: its elements are the objects of $\mathbb{A}$, and the enrichment classifies the preorder:

$$\text{for } x, x' \in \mathbb{A}_0 : \mathbb{A}(x, x') = 1 \text{ if and only if } x \leq x'.$$

The transitivity of the preorder corresponds to the composition-inequality in $\mathbb{A}$, and its reflexivity to the unit-inequality. In the same way, a distributor $\Phi\colon \mathbb{A} \multimap \mathbb{B}$ between **2**-categories is a so-called ideal relation between preorders: the relation $\Phi_0 \subseteq \mathbb{A}_0 \times \mathbb{B}_0$ defined by

$$\text{for } (x, y) \in \mathbb{A}_0 \times \mathbb{B}_0 : \Phi(y, x) = 1 \text{ if and only if } (x, y) \in \Phi_0$$

satisfies – due to the action-inequalities for the distributor – the condition that

$$\text{if } x \leq x' \text{ in } \mathbb{A}_0,\ y' \leq y \text{ in } \mathbb{B}_0 \text{ and } (x, y) \in \Phi_0, \text{ then } (x', y') \in \Phi_0.$$

That is to say, the relation Φ_0 between the preorders $(\mathbb{A}_0, \leq)$ and $(\mathbb{B}_0, \leq)$ is "up-closed" in $\mathbb{A}_0$ and "down-closed" in $\mathbb{B}_0$. Finally, a functor $F: \mathbb{A} \longrightarrow \mathbb{B}$ between **2**-categories is exactly a preorder-preserving map.

(c) Let R be a commutative ring (with unit), then the ideals in R form a quantale $\mathsf{Idl}(R)$. The elements of the ring itself are the objects of an $\mathsf{Idl}(R)$-enriched category whose homs are given by annihilators: for $r, s \in R$ put the hom from r to s to be $\mathsf{Ann}(r-s) = \{x \in R \mid (r-s)x = 0\}$. For a non-commutative R we have seen on p. 6 how the quantale of ideals generalizes to a quantaloid $\mathcal{Q}_R$; and G. Van den Bossche [1995] points out how such a non-commutative ring R determines a $\mathcal{Q}_R$-enriched category by generalizing annihilators to commutators: $\mathsf{Comm}(r, s) = \{x \in R \mid rx = xs\}$. Note first that if r is an element of $\mathcal{Z}(R)$ (the center of R) then $\mathsf{Comm}(r, s)$ is a left ideal in R; if s is an element of $\mathcal{Z}(R)$ then $\mathsf{Comm}(r, s)$ is a right ideal; if both $r, s \in \mathcal{Z}(R)$ then obviously $\mathsf{Comm}(r, s)$ is a two-sided ideal; and if neither r nor s is in $\mathcal{Z}(R)$ then $\mathsf{Comm}(r, s)$ is just an additive subgroup of R which is a $\mathcal{Z}(R)$-module. That is to say, R determines a $\mathcal{Q}_R$-enriched structure – which we denote as Comm_R – whose objects of type 0 are the elements of R and whose objects of type 1 are the elements of the center of R, and the homs of which are given by commutators.

(d) A quantaloid $\mathcal{Q}$ itself determines $\mathcal{Q}$-enriched categories of arrows with common codomain, as follows. Fix an object $A \in \mathcal{Q}_0$, and consider all $\mathcal{Q}$-arrows with codomain A; this already defines a $\mathcal{Q}_0$-typed set

$$(\mathbb{Q}_A)_0 = \coprod_{X \in \mathcal{Q}_0} \mathcal{Q}(X, A) \text{ with types } tf = \mathsf{dom}(f).$$

For two such arrows $f: X \longrightarrow A$ and $g: Y \longrightarrow A$ the lifting $[g, f]: X \longrightarrow Y$ is a $\mathcal{Q}$-arrow from $\mathsf{dom}(f)$ to $\mathsf{dom}(g)$, and actually

$$\mathbb{Q}_A(g, f) = [g, f]$$

defines a $\mathcal{Q}$-category on $(\mathbb{Q}_A)_0$. Similarly, the data

$$(\mathbb{Q}^A)_0 = \coprod_{X \in \mathcal{Q}_0} \mathcal{Q}(A, X), \ \ tf = \mathsf{cod}(f), \ \ \mathbb{Q}^A(g, f) = \{g, f\}$$

give a $\mathcal{Q}^{\mathsf{op}}$-category (but not a $\mathcal{Q}$-category!).

(e) By the "opposite" of a $\mathcal{Q}$-category $\mathbb{A}$ is of course meant the new structure "$\mathbb{A}^{\mathsf{op}}$" with the same objects as $\mathbb{A}$, but whose hom-arrows are reversed:

$$\text{for objects } a, a' : \mathbb{A}^{\mathsf{op}}(a', a) := \mathbb{A}(a, a').$$

But note that this does not define a $\mathcal{Q}$-category, but rather a $\mathcal{Q}^{\mathsf{op}}$-category!

1.4. Matrices, monads and bimodules. To see how $\mathcal{Q}$-categories and distributors organize themselves in a categorical structure, it is useful to reconsider once more their respective definitions. In particular we will make a very clear distinction between two different aspects of these definitions. A first observation is that both a $\mathcal{Q}$-category $\mathbb{A}$ and a distributor $\Phi: \mathbb{A} \to \mathbb{B}$ consist in some way of a "matrix whose elements are arrows in $\mathcal{Q}$": the hom-arrows of a $\mathcal{Q}$-category $\mathbb{A}$ as in (12) form a square matrix, with $|\mathbb{A}_0|$ rows and $|\mathbb{A}_0|$ columns, whose entry at position $(a', a) \in \mathbb{A}_0 \times \mathbb{A}_0$ is precisely $\mathbb{A}(a', a)$; and the elements of a distributor $\Phi: \mathbb{A} \to \mathbb{B}$ as in (17) form a matrix, with $|\mathbb{B}_0|$ rows and $|\mathbb{A}_0|$ columns, whose entry at position $(b, a) \in \mathbb{B}_0 \times \mathbb{A}_0$ is $\Phi(b, a)$. A second and quite separate aspect is then the translation of the composition and unit inequalities (13) and (14) (for a category) and the action inequalities (18) and (19) (for a distributor) to these "matrices". It turns out that categories are monads, and distributors bimodules, in an appropriate quantaloid of matrices whose elements are arrows in $\mathcal{Q}$ [Betti *et al.*, 1983; Carboni *et al.*, 1987].

More precisely, let us write (X, t), (Y, t), (Z, t), ... for $\mathcal{Q}_0$-typed sets: simply sets to every element of which is associated a type in $\mathcal{Q}_0$ (as in (11)). Given two such $\mathcal{Q}_0$-typed sets (X, t) and (Y, t), a matrix with elements in $\mathcal{Q}$ with $|Y|$ rows and $|X|$ columns – written as $\mathbb{M}: (X, t) \longrightarrow (Y, t)$ – is a collection of $\mathcal{Q}$-arrows like so:

$$\Big(\mathbb{M}(y, x): tx \longrightarrow ty\Big)_{(x,y)\in X\times Y}. \tag{20}$$

Two matrices $\mathbb{M}: (X, t) \longrightarrow (Y, t)$ and $\mathbb{N}: (Y, t) \longrightarrow (Z, t)$ can be composed: the new matrix $\mathbb{N} \circ \mathbb{M}: (X, t) \longrightarrow (Z, t)$ has as element on its zth row and xth column

$$(\mathbb{N} \circ \mathbb{M})(z, x) = \bigvee_{y\in Y} \mathbb{N}(z, y) \circ \mathbb{M}(y, x). \tag{21}$$

Reading the supremum of arrows in $\mathcal{Q}$ as their "infinitary sum", and the composition as their "multiplication", this is the formula for matrix multiplication as in linear algebra! The identity matrix on some $\mathcal{Q}_0$-typed set (X, t) will be denoted as $\Delta_X: (X, t) \longrightarrow (X, t)$, because its elements are "Kronecker deltas":

$$\Delta_X(x', x) = \begin{cases} 1_{tx}: tx \longrightarrow tx \text{ when } x = x', \\ 0_{tx',tx}: tx \longrightarrow tx' \text{ otherwise.} \end{cases} \tag{22}$$

The arrow $0_{tx',tx}: tx \longrightarrow tx'$ denotes the smallest element of the sup-lattice $\mathcal{Q}(tx, tx')$. It is easily verified that $\mathcal{Q}_0$-typed sets and matrices with elements in $\mathcal{Q}$ form a category—which we denote by $\mathsf{Matr}(\mathcal{Q})$. Actually, this category is a quantaloid for the "elementwise" ordering of parallel matrices: for

$$\mathbb{M}, \mathbb{M}': (X, t) \rightrightarrows (Y, t)$$

put $\mathbb{M} \leq \mathbb{M}'$ when for all $X \in X$ and $y \in Y$, $\mathbb{M}(y, x) \leq \mathbb{M}'(y, x)$ as $\mathcal{Q}$-arrows.

There is an obvious embedding of the given quantaloid $\mathcal{Q}$ into the quantaloid $\mathsf{Matr}(\mathcal{Q})$ of matrices with elements in $\mathcal{Q}$: every arrow $f\colon A \longrightarrow B$ in $\mathcal{Q}$ can be seen as a one-element matrix between one-element $\mathcal{Q}_0$-typed sets. This embedding has the universal property that it is the direct-sum completion of $\mathcal{Q}$ in the (illegitimate) category of (possibly large) quantaloids and homomorphisms[9]. From this fact it is then easy to characterize those quantaloids that arise as quantaloids of matrices.

Let us now study the inequalities (13) and (14) (for $\mathcal{Q}$-categories), and (18) and (19) (for distributors) in the context of the quantaloid $\mathsf{Matr}(\mathcal{Q})$ of matrices with elements in $\mathcal{Q}$. Rewriting (13) by the equivalent

$$\text{for all } a, a'' \in \mathbb{A}_0 : \bigvee_{a' \in \mathbb{A}_0} \mathbb{A}(a'', a') \circ \mathbb{A}(a', a) \leq \mathbb{A}(a'', a) \tag{23}$$

it is readily seen that this inequality simply says that $\mathbb{A} \circ \mathbb{A} \leq \mathbb{A}$ in $\mathsf{Matr}(\mathcal{Q})$, where now $\mathbb{A}$ is viewed as a square matrix on $(\mathbb{A}_0, t)$. Likewise, equation (14) can be rewritten as

$$\text{for all } a, a' \in \mathbb{A}_0 : \Delta_{\mathbb{A}_0}(a', a) \leq \mathbb{A}(a', a) \tag{24}$$

or even as $\Delta_{\mathbb{A}_0} \leq \mathbb{A}$ in $\mathsf{Matr}(\mathcal{Q})$. These inequalities precisely say that a $\mathcal{Q}$-enriched category is a monad in the quantaloid $\mathsf{Matr}(\mathcal{Q})$! And, much in the same way, (18) and (19) say that a distributor $\Phi\colon \mathbb{A} \multimap \mathbb{B}$ is really a matrix $\Phi\colon (\mathbb{A}_0, t) \longrightarrow (\mathbb{B}_0, t)$ such that

$$\text{for all } a, a' \in \mathbb{A}_0, b \in \mathbb{B}_0 : \textstyle\bigvee_{a' \in \mathbb{A}_0} \Phi(b, a') \circ \mathbb{A}(a', a) \leq \Phi(b, a); \tag{25}$$

$$\text{for all } a \in \mathbb{A}_0, b, b' \in \mathbb{B}_0 : \textstyle\bigvee_{b' \in \mathbb{B}_0} \mathbb{B}(b, b') \circ \Phi(b', a) \leq \Phi(b, a). \tag{26}$$

In other words, $\Phi \circ \mathbb{A} \leq \Phi$ and $\mathbb{B} \circ \Phi \leq \Phi$ in the quantaloid $\mathsf{Matr}(\mathcal{Q})$—thus a distributor $\Phi\colon \mathbb{A} \multimap \mathbb{B}$ is a bimodule between the monads $\mathbb{A}$ and $\mathbb{B}$ in the quantaloid $\mathsf{Matr}(\mathcal{Q})$!

Indeed, an endo-arrow $A \circlearrowleft t$ in a quantaloid $\mathcal{Q}$ is a monad[10] when it satisfies $t \circ t \leq t$ and $1_A \leq t$. And given two monads $A \circlearrowleft t$ and $B \circlearrowleft s$ in $\mathcal{Q}$, a bimodule $b\colon t \multimap s$ is by definition a $\mathcal{Q}$-arrow $b\colon A \longrightarrow B$ satisfying $b \circ t \leq b$ and $s \circ b \leq b$. For monads $A \circlearrowleft t$, $B \circlearrowleft s$ and $C \circlearrowleft r$, two bimodules $b\colon t \multimap s$ and $c\colon s \multimap r$ can be composed: the composition is written $c \otimes_s b\colon t \multimap r$ but is actually computed simply as $c \circ b$, the composition in $\mathcal{Q}$. We use this "tensor notation" for the composition of bimodules to stress that the left and right identities for composition are not inherited from $\mathcal{Q}$: it is indeed easily

[9]The quantaloid $\mathsf{Matr}(\mathcal{Q})$ always has a class of objects, no matter how "small" $\mathcal{Q}$ is! This is our main motivation for not *a priori* restricting our attention to small quantaloids: the large ones arise as important universal constructions on small ones.

[10]The notions of monad and bimodule make sense in any bicategory [Street, 1972], but applied to a quantaloid they take these very simple forms because a quantaloid is locally ordered. In particular any monad in a quantaloid is idempotent. The notation with a "$\multimap$" for bimodules is meant to distinguish them from arrows—and is already anticipating the notation for distributors.

verified that the identity bimodule on a monad $A \overset{t}{\circlearrowleft}$ is the $\mathcal{Q}$-arrow t itself (and not 1_A). So in particular we have, for a bimodule $b\colon t \mathrel{-\!\!\circ\!\!\to} r$, that $b \otimes_t t = b$ and $r \otimes_r b = b$. For any quantaloid $\mathcal{Q}$ we can thus construct a category $\mathsf{Mnd}(\mathcal{Q})$ whose objects are monads in $\mathcal{Q}$ and whose arrows are bimodules. Actually, $\mathsf{Mnd}(\mathcal{Q})$ is a quantaloid: its local structure is inherited from $\mathcal{Q}$ in the sense that for two parallel bimodules $b, b'\colon t \rightrightarrows r$ we put that $b \leq b'$ in $\mathsf{Mnd}(\mathcal{Q})$ precisely when $b \leq b'$ in $\mathcal{Q}$—so for a family $(b_i\colon t \mathrel{-\!\!\circ\!\!\to} r)_{i \in I}$ their supremum as bimodules is calculated "as in $\mathcal{Q}$".

There is then an obvious embedding of $\mathcal{Q}$ into $\mathsf{Mnd}(\mathcal{Q})$, because any morphism $f\colon A \longrightarrow B$ in $\mathcal{Q}$ determines a bimodule between trivial monads. This embedding has a universal property in the (illegitimate) category of quantaloids and homomorphisms[11]: it is the universal splitting of monads in $\mathcal{Q}$. The object over which a monad in a quantaloid splits, is both the object of its algebras (Eilenberg-Moore algebras) and the object of its free algebras (Kleisli algebras). (These notions are due to [Street, 1972].) This is due to the idempotency of any such monad. Thanks to this universal property it is easy to characterize those quantaloids which arise as quantaloids of monads and bimodules.

The point now is the following: (small) $\mathcal{Q}$-enriched categories and distributors are precisely monads and bimodules in the quantaloid of $\mathcal{Q}_0$-typed sets and matrices with elements in $\mathcal{Q}$:

$$\mathsf{Dist}(\mathcal{Q}) = \mathsf{Mnd}(\mathsf{Matr}(\mathcal{Q})). \tag{27}$$

This at once dictates the rules for composition of distributors (the composition $\Psi \otimes_{\mathbb{B}} \Phi\colon \mathbb{A} \mathrel{-\!\!\circ\!\!\to} \mathbb{C}$ of two distributors $\Phi\colon \mathbb{A} \mathrel{-\!\!\circ\!\!\to} \mathbb{B}$ and $\Psi\colon \mathbb{B} \mathrel{-\!\!\circ\!\!\to} \mathbb{C}$ is computed as a matrix composition) and identity distributors (the identity distributor on a $\mathcal{Q}$-category $\mathbb{A}$ is $\mathbb{A}\colon \mathbb{A} \mathrel{-\!\!\circ\!\!\to} \mathbb{A}$ itself). As a result then, $\mathsf{Dist}(\mathcal{Q})$ is a (large) quantaloid in which $\mathcal{Q}$ can be embedded. This embedding has a particular universal property, and therefore those quantaloids arising as quantaloids of $\mathcal{Q}$-enriched categories and distributors can be characterized without much difficulty.

Moreover, $\mathsf{Dist}(\mathcal{Q})$ being a quantaloid, i.e. a particular kind of bicategory, we dispose of all kinds of bicategorical notions that are pertinent for categories and distributors: think of adjoints, liftings and extensions, etc. This will be particularly important for the development of "$\mathcal{Q}$-categorical algebra".

1.5. Categorical algebra over a quantaloid. We've seen how (small) $\mathcal{Q}$-enriched categories and distributors organize themselves in a new quantaloid $\mathsf{Dist}(\mathcal{Q})$. Functors between $\mathcal{Q}$-categories too give rise to a category: it is quite obvious how one computes the composition of two functors $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ and $G\colon \mathbb{B} \longrightarrow \mathbb{C}$:

$$G \circ F\colon \mathbb{A} \longrightarrow \mathbb{C} : a \mapsto G(F(a)), \tag{28}$$

[11] The quantaloid $\mathsf{Mnd}(\mathcal{Q})$ is small whenever $\mathcal{Q}$ is.

and also the identity functor on a $\mathcal{Q}$-category $\mathbb{A}$ is easily defined:

$$1_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathbb{A} : a \mapsto a. \tag{29}$$

We'll write $\mathsf{Cat}(\mathcal{Q})$ for the category of (small) $\mathcal{Q}$-enriched categories and functors.

A crucial lemma is that every functor determines an adjoint pair of distributors. Indeed, let $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ be a functor between $\mathcal{Q}$-enriched categories. Then it is easily seen that both matrices

$$\Big(\Phi(b,a) = \mathbb{B}(b, Fa)\Big)_{(a,b)\in\mathbb{A}_0\times\mathbb{B}_0} \text{ and } \Big(\Psi(a,b) = \mathbb{B}(Fa, b)\Big)_{(b,a)\in\mathbb{B}_0\times\mathbb{A}_0}$$

are in fact distributors, thus $\Phi\colon \mathbb{A} \multimap \mathbb{B}$ and $\Psi\colon \mathbb{B} \multimap \mathbb{A}$. And a calculation shows that Φ is left adjoint to Ψ in the quantaloid $\mathsf{Dist}(\mathcal{Q})$; that is to say, $\mathbb{A} \leq \Psi \otimes_{\mathbb{B}} \Phi$ and $\Psi \otimes_{\mathbb{A}} \Phi \leq \mathbb{B}$. Not to introduce too many unnecessary notations, we shall simply write

$$\mathbb{A} \underset{\mathbb{B}(-,F-)}{\overset{\mathbb{B}(F-,-)}{\rightleftarrows}} \mathbb{B}$$

for these adjoint distributors represented by the functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$.

Sending a functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ to the distributor $\mathbb{B}(-,F-)\colon \mathbb{A} \multimap \mathbb{B}$, which is a left adjoint, determines a functor from $\mathsf{Cat}(\mathcal{Q})$ to $\mathsf{Dist}(\mathcal{Q})$ which is the identity on objects:

$$\mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Dist}(\mathcal{Q})\colon \Big(F\colon \mathbb{A} \longrightarrow \mathbb{B}\Big) \mapsto \Big(\mathbb{B}(-,F-)\colon \mathbb{A} \multimap \mathbb{B}\Big). \tag{30}$$

Sending F to $\mathbb{B}(F-,-)$ of course determines a contravariant such functor. These functors provide the key to the development of $\mathcal{Q}$-categorical algebra. To get a feeling for this, let us look at some important examples.

(i) $\mathsf{Cat}(\mathcal{Q})$ as 2-category: Because of the functor in (30) the category $\mathsf{Cat}(\mathcal{Q})$ inherits the local structure of the quantaloid $\mathsf{Dist}(\mathcal{Q})$: for functors $F, G\colon \mathbb{A} \rightrightarrows \mathbb{B}$ we put

$$F \leq G \text{ in } \mathsf{Cat}(\mathcal{Q})(\mathbb{A},\mathbb{B}) \iff \mathbb{B}(-,F-) \leq \mathbb{B}(-,G-) \text{ in } \mathsf{Dist}(\mathcal{Q})(\mathbb{A},\mathbb{B}). \tag{31}$$

Thus $\mathsf{Cat}(\mathcal{Q})$ becomes a locally preordered category (indeed the local order is not anti-symmetric in general, and certainly not complete), and we dispose of all possible 2-categorical notions for $\mathcal{Q}$-categories and functors too: adjoint functors, equivalent categories, etc. It may by the way be noted that (31) is – as one would expect – the correct translation of the idea of a $\mathcal{V}$-natural transformation to the context of $\mathcal{Q}$-categories, for it may be rewritten as

$$\text{for all } a \in \mathbb{A}_0 : 1_{ta} \leq \mathbb{B}(Fa, Ga) \text{ in } \mathcal{Q}(ta, ta). \tag{32}$$

There is now an evident way in which every $\mathcal{Q}$-enriched category has an "underlying preordered set" (or rather an indexed family of preordered sets), precisely

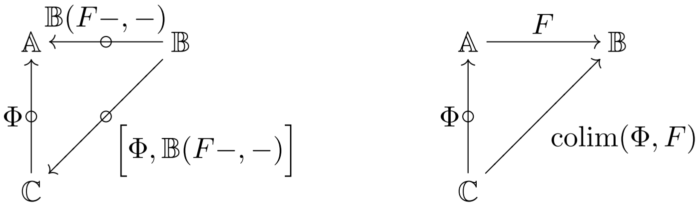

Diagram 2: weighted colimit

like every $\mathcal{V}$-enriched category has an underlying (ordinary) category. Indeed, every object $a \in \mathbb{A}_0$ of a $\mathcal{Q}$-category $\mathbb{A}$ may be identified with a "constant functor"

$$\Delta a\colon *_A \longrightarrow \mathbb{A}\colon * \mapsto a. \tag{33}$$

(For an object A of $\mathcal{Q}$, $*_A$ denotes the one-object $\mathcal{Q}$-category whose single hom-arrow is the identity on A[12]; in the above we must necessarily put $A = ta$.) We may thus order the objects of $\mathbb{A}$ as we order the corresponding functors:

$$a \leq a' \text{ in } \mathbb{A}_0 \iff ta = ta' =: A \text{ and } \Delta a \leq \Delta a' \text{ in } \mathsf{Cat}(\mathcal{Q})(*_A, \mathbb{A}). \tag{34}$$

Hence $(\mathbb{A}_0, \leq)$ is a ($\mathcal{Q}_0$-indexed family of) preordered sets.

A little calculation shows that

$$a \leq a' \text{ in } \mathbb{A}_0 \iff ta = ta' =: A \text{ and } 1_A \leq \mathbb{A}(a, a') \text{ in } \mathcal{Q}(A, A). \tag{35}$$

Thus the local preorder in $\mathsf{Cat}(\mathcal{Q})$ proves to be "pointwise": (32) just says that

$$\text{for all } a \in \mathbb{A}_0 : Fa \leq Ga \text{ in } (\mathbb{B}_0, \leq). \tag{36}$$

So every $\mathcal{Q}$-category determines an underlying $\mathcal{Q}_0$-indexed family of preorders, and every functor determines a $\mathcal{Q}_0$-indexed family of order-preserving maps—we obtain a functor $\mathcal{U}\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{PreOrd}_{\mathcal{Q}_0}$ between locally (pre)ordered categories. This functor admits a left adjoint.

(ii) Weighted colimits in a $\mathcal{Q}$-category: Let $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ be a functor, and $\Phi\colon \mathbb{C} \multimap \mathbb{A}$ a distributor, between $\mathcal{Q}$-enriched categories; we shall say that

$$\mathbb{C} \xrightarrow{\ \Phi\ } \mathbb{A} \xrightarrow{\ F\ } \mathbb{B} \tag{37}$$

is a Φ-weighted diagram in $\mathbb{B}$. Since F determines a right adjoint distributor $\mathbb{B}(F-,-)\colon \mathbb{B} \multimap \mathbb{A}$, we may consider an associated diagram of distributors of which, by closedness of the quantaloid $\mathsf{Dist}(\mathcal{Q})$, we can calculate the universal lifting—see the left hand side of Diagram 2. If there exists a functor $G\colon \mathbb{C} \longrightarrow \mathbb{B}$ for which

[12]In other words, $*_A$ is the image of $A \in \mathcal{Q}_0$ by the canonical inclusion $\mathcal{Q} \longrightarrow \mathsf{Dist}(\mathcal{Q})$.

$\mathbb{B}(G-,-) = \Big[\Phi, \mathbb{B}(F-,-)\Big]$ (i.e. G represents the distributor), then such a G is essentially unique: would G' be another such functor, then G and G' are isomorphic elements in (the preordered set) $\mathsf{Cat}(\mathcal{Q})(\mathbb{C},\mathbb{B})$. Therefore it makes sense to define the Φ-weighted colimit of F as the (essentially unique, when it exists) functor[13] $\mathrm{colim}(\Phi,F)\colon \mathbb{C}\longrightarrow\mathbb{B}$ representing the universal lifting $\Big[\Phi,\mathbb{B}(F-,-)\Big]\colon\mathbb{B}\longrightarrow\!\!\!\!\!\!\circ\;\;\mathbb{C}$. The right hand side of Diagram 2 resumes the situation.

A $\mathcal{Q}$-category $\mathbb{B}$ is said to be cocomplete when for every (small) weighted diagram in $\mathbb{B}$ (cf. (37)) the colimit exists. Such a cocomplete $\mathcal{Q}$-category has at least as many objects as the base quantaloid $\mathcal{Q}$; in particular, if $\mathcal{Q}$ is a "large" quantaloid, then any cocomplete $\mathcal{Q}$-category will be "large" too. In principle we should thus carefully distinguish, in everything that follows, between "small" and "large" $\mathcal{Q}$-categories. To avoid this rather unpleasant – and not very instructive – technical complication, we will *from now on suppose that $\mathcal{Q}$ is a small quantaloid* unless we explicitly state otherwise.

Given data as in

$$\mathbb{A}\xrightarrow{\ \Phi\ }\mathbb{B}\xrightarrow{\ F\ }\mathbb{C}\xrightarrow{\ F'\ }\mathbb{C}', \tag{38}$$

and supposing that the Φ-weighted colimit of F exists, then F' is said to preserve this colimit if the Φ-weighted colimit of $F'\circ F$ exists too, and moreover

$$\mathrm{colim}(\Phi, F'\circ F)\cong F'\circ\mathrm{colim}(\Phi,F). \tag{39}$$

A colimit is absolute if it is preserved by any functor, and a functor is cocontinuous if is preserves any colimit. Colimits whose weight is a left adjoint distributor are always absolute, and functors which are left adjoint are always cocontinuous.

(iii) Presheaves on a $\mathcal{Q}$-category: Let $\mathbb{A}$ be some (small) $\mathcal{Q}$-enriched category. The $\mathcal{Q}$-enriched category $\mathcal{P}\mathbb{A}$ of presheaves on $\mathbb{A}$ is defined by the following universal property: for every $\mathcal{Q}$-category $\mathbb{C}$ there is a natural equivalence of preorders[14]

$$\mathsf{Dist}(\mathcal{Q})(\mathbb{C},\mathbb{A})\simeq\mathsf{Cat}(\mathcal{Q})(\mathbb{C},\mathcal{P}\mathbb{A}). \tag{40}$$

That is to say, $\mathcal{P}\mathbb{A}$ ought to "classify" the distributors with codomain $\mathbb{A}$.

It is not hard to see that the objects of $\mathcal{P}\mathbb{A}$ are exactly distributors with codomain $\mathbb{A}$ and whose domain is a trivial one-object $\mathcal{Q}$-enriched category. We will always denote these with small Greek letters, like $\phi\colon *_A\longrightarrow\!\!\!\!\!\!\circ\;\;\mathbb{A}$, $\psi\colon *_B\longrightarrow\!\!\!\!\!\!\circ\;\;\mathbb{A}$, etc. In elementary terms, such a presheaf $\phi\colon *_A\longrightarrow\!\!\!\!\!\!\circ\;\;\mathbb{A}$ is thus a collection of $\mathcal{Q}$-arrows

$$\Big(\phi(x)\colon A\longrightarrow tx\Big)_{x\in\mathbb{A}_0} \tag{41}$$

[13]Some would probably be happier to see $\Phi\star F$ as notation instead of $\mathrm{colim}(\Phi,F)$.

[14]It turns out that $\mathsf{Cat}(\mathcal{Q})(\mathbb{C},\mathcal{P}\mathbb{A})$ is a partial order rather than a preorder. As $\mathsf{Dist}(\mathcal{Q})(\mathbb{C},\mathbb{A})$ is a complete lattice – it is the "hom" of a quantaloid – this equivalence is actually an isomorphism of complete lattices.

satisfying the inequalities expressing the action by $\mathbb{A}$. (The action by $*_A$ is trivial!) The collection of such contravariant presheaves on $\mathbb{A}$ thus forms a $\mathcal{Q}_0$-typed set[15]: the type in $\mathcal{Q}_0$ of $\phi\colon *_A \to \mathbb{A}$ is $t\phi = A$. For two such presheaves $\phi\colon *_A \to \mathbb{A}$ and $\psi\colon *_B \to \mathbb{A}$ the hom-arrow $\mathcal{P}\mathbb{A}(\psi,\phi)\colon A \longrightarrow B$ is the single element of the distributor $[\psi,\phi]\colon *_A \to *_B$ (i.e. the lifting of ϕ through ψ).

Putting $\mathbb{C} = \mathbb{A}$ in (40) there must correspond to the identity distributor $\mathbb{A}\colon\mathbb{A} \to \mathbb{A}$ some functor from $\mathbb{A}$ to $\mathcal{P}\mathbb{A}$: it is of course the Yoneda embedding $Y_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathcal{P}\mathbb{A}$. It can be figured out that $Y_{\mathbb{A}}$ sends an object $a \in \mathbb{A}_0$ onto the representable presheaf $\mathbb{A}(-,a)\colon *_{ta} \to \mathbb{A}$, the elements of which – as its notation suggests – are

$$\Big(\mathbb{A}(x,a)\colon ta \longrightarrow tx\Big)_{x\in\mathbb{A}_0}. \tag{42}$$

From here on one can prove such things (familiar from $\mathcal{V}$-category theory) as:

- the "Yoneda lemma" says that for any $\phi \in \mathcal{P}\mathbb{A}$ and $a \in \mathbb{A}$,

$$\mathcal{P}\mathbb{A}(Y_{\mathbb{A}}a, \Phi) = \phi(a); \tag{43}$$

- as a consequence, the Yoneda embedding $Y_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathcal{P}\mathbb{A}$ is a fully faithful functor in the sense that, for any $a, a' \in \mathbb{A}_0$,

$$\mathbb{A}(a',a) = \mathcal{P}\mathbb{A}(Y_{\mathbb{A}}a', Y_{\mathbb{A}}a); \tag{44}$$

- every presheaf is a "colimit of representables": for a presheaf $\phi \in \mathcal{P}\mathbb{A}$ (of type $t\phi = A$, say) consider the weighted diagram

$$*_A \xrightarrow{\ \phi\ } \mathbb{A} \xrightarrow{\ Y_{\mathbb{A}}\ } \mathcal{P}\mathbb{A};$$

its colimit exists and is the functor which "picks out" exactly the object $\phi \in \mathcal{P}\mathbb{A}$, so we write with some abuse of notation that

$$\mathrm{colim}(\phi, Y_{\mathbb{A}}) = \phi; \tag{45}$$

- $Y_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathcal{P}\mathbb{A}$ is the free cocompletion in $\mathsf{Cat}(\mathcal{Q})$: the inclusion of the category $\mathsf{Cocont}(\mathcal{Q})$ of cocomplete $\mathcal{Q}$-categories and cocontinuous functors (i.e. functors that preserve colimits) in $\mathsf{Cat}(\mathcal{Q})$ admits a left adjoint which is precisely the presheaf construction:

$$\mathcal{P} \dashv i\colon \mathsf{Cocont}(\mathcal{Q}) \leftrightarrows \mathsf{Cat}(\mathcal{Q}). \tag{46}$$

(iv) Cauchy completion of a $\mathcal{Q}$-category: We've seen that every functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ determines a left adjoint distributor $\mathbb{B}(-,F-)\colon \mathbb{A} \to \mathbb{B}$. But the converse need not

[15] Note that there too it is quite essential that $\mathcal{Q}$ is a *small* quantaloid: if not, $\mathcal{P}\mathbb{A}$ would have a proper class of objects. Similar remarks apply to the Cauchy completion $\mathbb{A}_{\mathsf{cc}}$ encountered further on.

hold: not every left adjoint distributor $\Phi\colon \mathbb{A} \mathrel{\circ\!\!\!\longrightarrow} \mathbb{B}$ is determined by a functor! Following [Lawvere, 1973; Street, 1983] we shall use the term 'Cauchy distributor' as synonym for 'left adjoint distributor', and when a Cauchy distributor $\Phi\colon \mathbb{A} \mathrel{\circ\!\!\!\longrightarrow} \mathbb{B}$ is determined by a functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ we will say that Φ converges to F. (Such an F is necessarily essentially unique.) If all Cauchy distributors with codomain $\mathbb{B}$ converge, then $\mathbb{B}$ is said to be Cauchy complete.

For a small $\mathcal{Q}$-enriched category $\mathbb{A}$ a new $\mathcal{Q}$-category $\mathbb{A}_{\mathsf{cc}}$ is defined by the following universal property: for every $\mathcal{Q}$-category $\mathbb{C}$ there is a natural equivalence of preorders[16]

$$\mathsf{Map}(\mathsf{Dist}(\mathcal{Q}))(\mathbb{C}, \mathbb{A}) \simeq \mathsf{Cat}(\mathcal{Q})(\mathbb{C}, \mathbb{A}_{\mathsf{cc}}). \tag{47}$$

In other words, $\mathbb{A}_{\mathsf{cc}}$ is supposed to universally classify the Cauchy distributors into $\mathbb{A}$. (The left hand side of this equation denotes the left adjoints – the "maps" – amongst the distributors from $\mathbb{C}$ to $\mathbb{A}$.) It can be shown that this new $\mathcal{Q}$-category $\mathbb{A}_{\mathsf{cc}}$ is Cauchy complete; it is called the Cauchy completion of $\mathbb{A}$.

It turns out that $\mathbb{A}_{\mathsf{cc}}$ is really a full subcategory of $\mathcal{P}\mathbb{A}$: it is determined by those presheaves on $\mathbb{A}$ which are left adjoint [Betti and Carboni, 1983; Street, 1983]. So suppose that $\phi\colon *_A \mathrel{\circ\!\!\!\longrightarrow} \mathbb{A}$ and $\psi\colon *_B \mathrel{\circ\!\!\!\longrightarrow} \mathbb{A}$ (contravariant presheaves on $\mathbb{A}$) are actually Cauchy distributors (say $\phi \dashv \phi^*$ and $\psi \dashv \psi^*$ in $\mathsf{Dist}(\mathcal{Q})$), then $\mathbb{A}_{\mathsf{cc}}(\psi, \phi)\colon A \longrightarrow B$ is the $\mathcal{Q}$-arrow which is the single element of the lifting $[\psi, \phi]\colon *_A \mathrel{\circ\!\!\!\longrightarrow} *_B$ in $\mathsf{Dist}(\mathcal{Q})$. (It is – because of the adjunction $\psi \dashv \psi^*$ – equivalently given by the single element of the composition $\psi^* \otimes_{\mathbb{A}} \phi\colon *_A \mathrel{\circ\!\!\!\longrightarrow} *_B$.)

Of course the identity distributor $\mathbb{A}\colon \mathbb{A} \mathrel{\circ\!\!\!\longrightarrow} \mathbb{A}$ itself is Cauchy; thus by (47) it must correspond to some functor $i_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathbb{A}_{\mathsf{cc}}$. This functor is precisely the factorization of the Yoneda embedding $Y_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathcal{P}\mathbb{A}$ over the full subcategory $\mathbb{A}_{\mathsf{cc}} \subseteq \mathcal{P}\mathbb{A}$. Some results – familiar in $\mathcal{V}$-category theory – can now be obtained:

- $i_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathbb{A}_{\mathsf{cc}}$ is the absolute-colimit completion of $\mathbb{A}$ in $\mathsf{Cat}(\mathcal{Q})$: the Cauchy construction is left adjoint to the full inclusion of the Cauchy complete categories in $\mathsf{Cat}(\mathcal{Q})$:

$$(-)_{\mathsf{cc}} \dashv i\colon \mathsf{Cat}_{\mathsf{cc}}(\mathcal{Q}) \leftrightarrows \mathsf{Cat}(\mathcal{Q}); \tag{48}$$

- for two $\mathcal{Q}$-categories $\mathbb{A}$ and $\mathbb{B}$ we have that

$$\mathbb{A} \cong \mathbb{B} \text{ in } \mathsf{Dist}(\mathcal{Q}) \iff \mathbb{A}_{\mathsf{cc}} \simeq \mathbb{B}_{\mathsf{cc}} \text{ in } \mathsf{Cat}(\mathcal{Q}) \iff \mathcal{P}\mathbb{A} \simeq \mathcal{P}\mathbb{B} \text{ in } \mathsf{Cat}(\mathcal{Q}),$$

and $\mathbb{A}$ and $\mathbb{B}$ are said to be Morita equivalent if (any of) these conditions hold;

- any $\mathcal{Q}$-category $\mathbb{A}$ is isomorphic in $\mathsf{Dist}(\mathcal{Q})$ to its Cauchy completion $\mathbb{A}_{\mathsf{cc}}$, so there is an equivalence of quantaloids

$$\mathsf{Dist}_{\mathsf{cc}}(\mathcal{Q}) \simeq \mathsf{Dist}(\mathcal{Q}) \tag{49}$$

[16] It turns out that this is actually an isomorphism of partial orders.

(where $\mathsf{Dist_{cc}}(\mathcal{Q})$ denotes the obvious full subcategory);

- and as a result – in combination with (47) – there are equivalences of locally preordered categories

$$\mathsf{Map}(\mathsf{Dist}(\mathcal{Q})) \simeq \mathsf{Map}(\mathsf{Dist_{cc}}(\mathcal{Q})) \simeq \mathsf{Cat_{cc}}(\mathcal{Q}). \tag{50}$$

So, if one really only cares about "Cauchy invariant" (i.e. "Morita equivalent") constructions involving categories and functors (as one usually does), then one may as well forget about functors altogether and work with $\mathsf{Map}(\mathsf{Dist}(\mathcal{Q}))$ instead.

This is a lot of theory—and it may be useful to look at some examples here.

(a) Consider again Lawvere's generalized metric spaces (see p. 12); let $\mathbb{A}$ denote such a space. In [Lawvere, 1973] it is proved that a left adjoint distributor from a one-point space into $\mathbb{A}$ is precisely (an equivalence class of) a Cauchy sequence in $\mathbb{A}$—and it is therefore that left adjoint distributors are also called Cauchy distributors. Cauchy completeness of $\mathbb{A}$ as enriched category coincides with its Cauchy completeness as metric space ("all Cauchy sequences/distributors converge").

(b) For an object A of a quantaloid $\mathcal{Q}$ we denote $*_A$ for the one-object $\mathcal{Q}$-category whose single object is of type A and whose single hom-arrow is 1_A. A contravariant presheaf on $*_A$ is – applying (41) – just an arrow $f\colon X \longrightarrow A$ in $\mathcal{Q}$; and the hom-arrow between two such presheaves $f\colon X \longrightarrow A$ and $g\colon Y \longrightarrow A$ is the lifting $[g, f]$. That is to say, with notations as introduced in the examples on p. 12, $\mathcal{P}(*_A) = \mathbb{Q}_A$.

(c) Let A be an object of the quantaloid $\mathcal{Q}$, and $\mathbb{A}$ a $\mathcal{Q}$-category. A functor $F\colon \mathbb{A}^{\mathsf{op}} \longrightarrow \mathbb{Q}^A$ between these $\mathcal{Q}^{\mathsf{op}}$-categories (which were defined in the examples on p. 12) is a mapping of objects, associating to any $a \in \mathbb{A}_0$ some arrow $Fa\colon A \longrightarrow ta$ in $\mathcal{Q}$, such that $\mathbb{A}^{\mathsf{op}}(a', a) \leq \{Fa', Fa\}$ for any two objects a, a' in $\mathbb{A}^{\mathsf{op}}$. The latter is equivalent to $\mathbb{A}(a, a') \circ Fa' \leq Fa$, so F is *precisely* the same thing as a distributor $\phi\colon *_A \,{-\!\!\circ\!\!\rightarrow}\, \mathbb{A}$ between $\mathcal{Q}$-categories as in (41)—i.e. a contravariant presheaf on $\mathbb{A}$. (So a contravariant presheaf on $\mathbb{A}$ is a "functor on the opposite of $\mathbb{A}$ with values in $\mathcal{Q}$" after all!)

(d) As indicated previously, $\mathbf{2}$-enriched categories are actually preordered sets, functors between $\mathbf{2}$-categories are just order-preserving maps, and distributors are ideal relations. More precisely we can state that $\mathsf{Cat}(\mathbf{2}) \cong \mathsf{PreOrd}$ and $\mathsf{Dist}(\mathbf{2}) \cong \mathsf{Idl}$. It now follows quite easily that, for a $\mathbf{2}$-category $\mathbb{A}$, a presheaf on $\mathbb{A}$ is exactly a "down-closed subset" of the preorder $\mathbb{A}_0$; and the Yoneda embedding $Y_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathcal{P}\mathbb{A}$ maps an object $a \in \mathbb{A}_0$ to the "principal down-closed subset" that it determines. This is precisely the sup-completion of $\mathbb{A}_0$! But this is entirely consistent with the general fact that the presheaf-category is the free cocompletion, for a $\mathbf{2}$-category $\mathbb{A}$ has all weighted colimits if and only if $\mathbb{A}_0$ has all suprema. To see this, consider a weighted diagram of $\mathbf{2}$-categories

$$*_C \xrightarrow{\ \phi\ } \mathbb{B} \xrightarrow{\ F\ } \mathbb{A}$$

(and without loss of generality we let the weight be a presheaf on $\mathbb{B}$, i.e. ϕ_0 is a down-closed subset of $\mathbb{B}_0$). Then actually the lifting

$$\Big[\phi, \mathbb{B}(F-,-)\Big]\colon \mathbb{A} \mathrel{-\!\!\circ\!\!\!\rightarrow} *_C$$

corresponds to the up-closed subset of $\mathbb{A}_0$ which contains all the upper bounds of the image through F of the elements of ϕ_0:

$$\Big[\phi, \mathbb{B}(F-,-)\Big]_0 = \{a \in \mathbb{A}_0 \mid \text{for all } b \in \phi_0 : Fb \leq a\}.$$

The ϕ-weighted colimit of F exists if and only if this lifting is representable—which is the case precisely when there is a least upper bound for $\{Fb \in \mathbb{A}_0 \mid b \in \phi_0\}$, because this supremum is then precisely the representing object for the colimit! Similar calculations show that every **2**-category is Cauchy complete: any Cauchy presheaf into a 2-category $\mathbb{A}$ is necessarily a principle down-closed set in $\mathbb{A}_0$.

Especially this latter example seems to indicate that the theory of $\mathcal{Q}$-enriched categories, functors and distributors is like an "order-theory with truth-values in $\mathcal{Q}$". This is surely a helpful intuition, but things are a bit more complicated than that. The second part of the thesis is precisely devoted to a thorough study of these complications.

2. Ideals, orders and sets over a base quantaloid

As quantales may be viewed as "non-commutative locales", and quantales themselves are only quantaloids with one single object, one may wonder about "sheaves on a quantaloid" as generalization of sheaves on a locale. Given that sheaves on a locale Ω can be described in terms of so-called Ω-sets, and the latter are in some sense easier to manipulate, several authors have rather tried to generalize those Ω-sets to "$\mathcal{Q}$-sets" for a quantale or quantaloid $\mathcal{Q}$. This is not at all straightforward! Starting from the work of [Van den Bossche, 1995; Borceux and Cruciani, 1998; van der Plancke, 1998] we will analyze the situation using the formalism of quantaloid-enriched categorical structures. As it turns out, we need a theory of "categories without units" enriched in a base quantaloid—so we generalize and further develop the theory of "categories without units" enriched in a symmetric monoidal closed $\mathcal{V}$ of [Moens, 2000; Moens *et al.*, 2002].

But before we plunge ourselves in technical developments of all kinds, we should probably first ask ourselves...

2.1. Do we really need "sheaves on a quantaloid"? To give some motivation (other than curiosity) for the study of "sheaves on a quantaloid" it may be good to quickly recall the basic idea behind so-called sheaf representations of rings; let us do this with a simple example (quoted from [Borceux, 1994]). Thereto let R denote a commutative ring (with unit). An ideal $I \subseteq R$ is radical when, for every $r \in R$ and $n \in \mathbb{N}$, if $r^n \in I$ then also $r \in I$. The radical ideals of R form a localic quotient $\mathsf{Rad}(R)$ of the quantale $\mathsf{Idl}(R)$:

$$\mathsf{Idl}(R) \longrightarrow \mathsf{Rad}(R) \colon I \mapsto \sqrt{I} = \{r \in R \mid \exists n \in \mathbb{N} : r^n \in I\}.$$

In the topos of sheaves on $\mathsf{Rad}(R)$ there is a ring-object $\mathcal{R}$ whose ring of global sections is isomorphic to R; therefore $\mathcal{R}$ is said to be a sheaf representation for R. But this sheaf $\mathcal{R}$ of rings has the interesting feature that it is a local ring in the topos of sheaves on $\mathsf{Rad}(R)$: for any $I \in \mathsf{Rad}(R)$ and $r \in \mathcal{R}(I)$ there exists a covering $I = \bigvee_{k \in K} I_k$ in $\mathsf{Rad}(R)$ such that for every $k \in K$ either $r\mid_{I_k}$ or $(1-r)\mid_{I_k}$ is invertible in $\mathcal{R}(I_k)$. This explains the interest of the sheaf representation: even though R is only a commutative ring (in the topos of sets), $\mathcal{R}$ is a local ring (in the topos of sheaves on $\mathsf{Rad}(R)$).

There are many other examples of interesting sheaf representations, and most of them make use of one or another locale of ideals of a given ring R to serve as "topology" on which R is then represented as a more-or-less interesting sheaf—and so there is absolutely no confusion about the notion of "sheaf" in these cases. But we have argued on p. 6 that the so-to-speak "generic topology" for a ring R is its quantaloid of ideals $\mathcal{Q}_R$ (which, for a commutative R, is equivalent to the quantale

$\mathsf{Idl}(R)$). So it would certainly be interesting to discuss sheaf representations of R over this quantaloid. (We already know of a way in which R determines a $\mathcal{Q}_R$-enriched category – see p. 13 – but in what sense is this a "sheaf" or anything of that kind?)

This is the place to mention that several authors have put forward – sometimes quite different – ideas on this topic in the context of sheaf representations for rings and algebras:

- C. Mulvey and M. Nawaz [1995] work with an "idempotent right-sided quantale[17]", and for them a quantale-set is a sheaf on the sublocale of two-sided elements in the quantale. In the localic topos of these quantale-sets they recognize an extra "quantalic subobject classifier". (See also Nawaz' [1985] doctoral thesis.)

- F. Borceux and B. Van den Bossche [1989] construct, for a given "idempotent right-sided quantale" Q, a category whose objects are the principal down-closed subsets of Q and whose morphisms are suitable "restriction maps". On this category they recognize a Grothendieck topology, and so they define a "sheaf on Q" as a sheaf on this site. In this Grothendieck topos the locales of subobjects can be provided with the additional structure of a quantale.

- U. Berni-Canani *et al.* [1989] too start from such an "idempotent right-sided quantale" and develop a theory of quantale-sets, but now these are the objects of a fibration of localic toposes over the given quantale. And the localic topos of [Mulvey and Nawaz, 1995] is then actually the stalk at the top element of the quantale.

- G. Van den Bossche [1995] takes any quantaloid $\mathcal{Q}$ and defines a "$\mathcal{Q}$-set" to be an idempotent matrix with elements in $\mathcal{Q}$. She shows how, when equipped with suitable morphisms, the category of $\mathcal{Q}_R$-sets (where $\mathcal{Q}_R$ is the "quantaloid of ideals" for a – not necessarily commutative – ring R, as mentioned on p. 7) contains the object Comm_R (see p. 13) whose "global sections" correspond to the elements of $\mathcal{Z}(R)$, and certain other "sections" correspond to the elements of R itself.

- F. van der Plancke [1998] further investigates these "$\mathcal{Q}$-sets" that [Van den Bossche, 1995] introduced, taking the point of view that such an object is a "set with an equality-predicate that takes its values in $\mathcal{Q}$". With this intuition a suitable notion of "subset of a $\mathcal{Q}$-set" is given, and – amongst other things – it is shown how the "$\mathcal{Q}$-sets of subsets" determine, and are determined by, projective objects in the category $\mathsf{Quant}(\mathcal{Q},\mathsf{Sup})$ of Sup-presheaves on $\mathcal{Q}$.

- R. Gylys [2001] combines elements from Van den Bossche's [1995] "matrix-approach" with Borceux and Van den Bossche's [1986] "restriction-and-gluing" approach. For [Gylys, 2001] "$\mathcal{Q}$-sets" are sets with a suitable $\mathcal{Q}$-valued equality; such

[17]Such an "idempotent right-sided quantale" is a complete lattice Q equipped with a binary product $-\&-\colon Q\times Q\to Q$ which distributes on both sides over suprema, for which every element of the lattice is idempotent ($x\&x=x$), and for which the top element of the lattice is a unit-on-the-right ($x\&\top=x$). As the product is not required to have a (two-sided) unit, this is *not* a quantale in our sense!

a $\mathcal{Q}$-set is "complete" if it allows "restriction and gluing" of its "singletons". Therefore, a $\mathcal{Q}$-set is to be thought of as a "presheaf", and a complete $\mathcal{Q}$-set as a "sheaf", on $\mathcal{Q}$.

- Given a commutative ring R, S. Ambler and D. Verity [1996] consider enrichments in the split-idempotent completion of the quantale $\mathsf{Idl}(R)$. They show how the ring R is "represented" by an enriched category – to be thought of as a "presheaf" on $\mathsf{Idl}(R)$ – and how the Cauchy completion of this category corresponds to its "sheafification". Their formalism is thus a generalization of Walters' [1981] description of sheaves on a locale as enriched categories. They also recover the results in [Borceux and Van den Bossche, 1991].

Especially the works of G. Van den Bossche, F. van der Plancke and M.-A. Moens have been of great influence on the present work. I only recently "discovered" [Ambler and Verity, 1996; Gylys, 2001]—and was pleased to find out that their results are particular cases of the more general theory.

2.2. Ordered sheaves on a locale. The topos $\mathsf{Sh}(\Omega)$ of sheaves on a locale Ω can be described in terms of so-called Ω-sets [Fourman and Scott, 1979; Higgs, 1984; Fourman and Scott, 1979; Borceux, 1994; Johnstone, 2002]. An Ω-set $(A, [\,\cdot = \cdot\,])$ consists of a set A equipped with an "Ω-valued equality"

$$[\,\cdot = \cdot\,]: A \times A \longrightarrow \Omega$$

satisfying for all $a, a', a'' \in A$,

$$[a = a'] \wedge [a' = a''] \leq [a = a''], \tag{51}$$

$$[a = a'] = [a' = a]. \tag{52}$$

Clearly (51) is a transitivity axiom, and (52) is symmetry. There is no reflexivity axiom, for it would read

$$\forall a \in A: \top \leq [a = a] \tag{53}$$

which would force all elements of $(A, [\,\cdot = \cdot\,])$ to be "global" ($\top$ denotes the top of the locale). Together with a "good" notion of morphism, these Ω-sets form a category equivalent to $\mathsf{Sh}(\Omega)$.

With the intuition that a "non-symmetric equality" is an "inequality", F. Borceux and R. Cruciani [1998] have worked out a theory of "Ω-orders" (or Ω-posets as those authors call them). Such an Ω-order $(A, [\,\cdot \leq \cdot\,])$ consists of a set A together with an "Ω-valued inequality"

$$[\,\cdot \leq \cdot\,]: A \times A \longrightarrow \Omega$$

satisfying for all $a, a', a'' \in A$,

$$[a \le a'] \wedge [a' \le a''] \le [a \le a''], \tag{54}$$

$$[a \le a'] \le [a \le a], \tag{55}$$

$$[a \le a'] \le [a' \le a']. \tag{56}$$

Here again (54) says that the "inequality" is transitive, and symmetry is gotten rid off (and there is still no reflexivity)—but two new axioms (55) and (56) have to be imposed "to make things work" (to quote F. Borceux). Together with a "good" notion of morphism, Ω-orders form a category equivalent to $\mathsf{Ord}(\mathsf{Sh}(\Omega))$, the category of ordered objects in the topos $\mathsf{Sh}(\Omega)$.

It may now be observed that both Ω-sets and Ω-orders are categorical structures enriched in (the one-object suspension of) the locale Ω. Take an Ω-order $(A, [\,\cdot \le \cdot\,])$; it quite naturally determines an object set $\mathbb{A}_0 = A$ and an enrichment in Ω:

$$\Big(\mathbb{A}(a, a') = [a \le a']\Big)_{(a,a') \in \mathbb{A}_0 \times \mathbb{A}_0}.$$

The transitivity axiom (54) then says that there are composition-inequalities cf. (13), but the lack of reflexivity (53) means that there are no unit-inequalities cf. (14): $\mathbb{A}$ is not a category but only a *semi*category enriched in (the one-object suspension of) Ω. Thus an Ω-order (and *a fortiori* an Ω-set) is an Ω-semicategory satisfying some extra axioms ((55) and (56) for orders, (52) for sets).

The point is that – when rewritten in a suitable manner – the axioms for Ω-orders still make sense when we replace the locale Ω by a quantaloid $\mathcal{Q}$. The axioms for Ω-sets do not make sense for a base quantaloid $\mathcal{Q}$ (because of the symmetry in (52)), unless it is endowed with an involution (see p. 48). Our aim here is to give a very precise analysis of these matters by means of the formalism of categorical structures enriched in a quantaloid.

2.3. Categories without units over a base quantaloid. By a $\mathcal{Q}$-enriched semicategory we will mean precisely a "category without units" enriched in the quantaloid $\mathcal{Q}$: a $\mathcal{Q}$-semicategory $\mathbb{A}$ thus consists of a $\mathcal{Q}_0$-typed set $\mathbb{A}_0$, as in (11), which is enriched in $\mathcal{Q}$, as in (12), satisfying composition-inequalities, as in (13)—and that's it. In terms of matrices, $\mathbb{A}$ is an endo-arrow in the quantaloid $\mathsf{Matr}(\mathcal{Q})$ that satisfies $\mathbb{A} \circ \mathbb{A} \le \mathbb{A}$; it is a "monad without unit" in $\mathsf{Matr}(\mathcal{Q})$.

As neither the definition for "functor between $\mathcal{Q}$-categories" nor that for "distributor between $\mathcal{Q}$-categories" (see p. 11) refer in any way to the units of the involved $\mathcal{Q}$-categories, they still make sense for $\mathcal{Q}$-semicategories. (Essentially this is due to the fact that both the "functoriality" of a functor and the "compatibility of action with unit and composition axioms" of a distributor are diagrammatic axioms of two-cells in the base quantaloid—and any such diagram is commutative!) Thus, for two $\mathcal{Q}$-semicategories $\mathbb{A}$ and $\mathbb{B}$, a semifunctor $F: \mathbb{A} \longrightarrow \mathbb{B}$ is by definition an object mapping $\mathbb{A}_0 \longrightarrow \mathbb{B}_0: a \mapsto Fa$ that preserves types, as in (15), satisfying the action-inequalities for the hom-arrows, as in (16). And a semidistributor $\Phi: \mathbb{A} \multimapdotinv \mathbb{B}$ is just a $|\mathbb{B}_0| \times |\mathbb{A}_0|$ matrix with elements in $\mathcal{Q}$, as in (17), satisfying the action-inequalities of (18) and (19). In other words, such a semidistributor is just a matrix $\Phi: \mathbb{A}_0 \longrightarrow \mathbb{B}_0$ together with action-inequalities $\Phi \circ \mathbb{A} \leq \Phi$ and $\mathbb{B} \circ \Phi \leq \Phi$; it is a "bimodule" between "monads without units" in $\mathsf{Matr}(\mathcal{Q})$.

It is quite trivial to verify that $\mathcal{Q}$-semicategories and semifunctors form a category; the composition and identities are as for functors (see (28) and (29)). We will denote this category as $\mathsf{SCat}(\mathcal{Q})$, and from the definitions it is clear there is a (non-trivial[18]) embedding

$$\mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{SCat}(\mathcal{Q}). \tag{57}$$

This embedding is full: the reason, ultimately, is that in the definition for "functor" there is no reference whatsoever to the unit-inequalities for the domain and codomain $\mathcal{Q}$-categories. Moreover, this embedding has a left adjoint. Explicitly, given a $\mathcal{Q}$-semicategory $\mathbb{A}$, one defines a $\mathcal{Q}$-category $\overline{\mathbb{A}}$ by freely adding units: $\overline{\mathbb{A}}$ has the same objects as $\mathbb{A}$, but for the hom-arrows we put

$$\overline{\mathbb{A}}(a', a) = \begin{cases} \mathbb{A}(a', a) \text{ if } a \neq a', \\ \mathbb{A}(a, a) \bigvee 1_{ta} \text{ if } a = a'. \end{cases}$$

In other terms, viewing $\mathbb{A}$ as endo-matrix on $\mathbb{A}_0$, $\overline{\mathbb{A}}$ is the supremum with the identity matrix on $\mathbb{A}_0$ (22): $\overline{\mathbb{A}} = \mathbb{A} \bigvee \Delta_{\mathbb{A}_0}$. It is now a fact that, for a semicategory $\mathbb{A}$ and a category $\mathbb{C}$, an object mapping $\mathbb{A}_0 \longrightarrow \mathbb{C}_0: a \mapsto Fa$ is a semifunctor between $\mathbb{A}$ and $\mathbb{C}$ if and only if the *same* object mapping is a functor between $\overline{\mathbb{A}}$ and $\mathbb{C}$; so we obtain that

$$\mathsf{Cat}(\mathcal{Q})(\overline{\mathbb{A}}, \mathbb{C}) = \mathsf{SCat}(\mathcal{Q})(\mathbb{A}, \mathbb{C}). \tag{58}$$

In sum we have described the left adjoint

$$\overline{(-)}: \mathsf{SCat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q}). \tag{59}$$

In particular, for a semifunctor $F: \mathbb{A} \longrightarrow \mathbb{C}$ – with $\mathbb{A}$ a semicategory and $\mathbb{C}$ a category – we will write $\overline{F}: \overline{\mathbb{A}} \longrightarrow \mathbb{C}$ for the corresponding functor.

Let us now look at the semidistributors between $\mathcal{Q}$-semicategories. For two $\mathcal{Q}$-semicategories $\mathbb{A}$ and $\mathbb{B}$ it is certainly true that the collection of semidistributors from $\mathbb{A}$ to $\mathbb{B}$ forms a complete lattice:

$$\mathsf{Distrib}(\mathbb{A}, \mathbb{B}) = \{\Phi \in \mathsf{Matr}(\mathcal{Q})(\mathbb{A}, \mathbb{B}) \mid \Phi \circ \mathbb{A} \leq \Phi \text{ and } \mathbb{B} \circ \Phi \leq \Phi\}$$

[18] For example, a strictly ordered set $(P, <)$ determines a **2**-semicategory which is not a category. More examples on p. 34.

is indeed a sublattice of $\mathsf{Matr}(\mathcal{Q})(\mathbb{A},\mathbb{B})$, inheriting the suprema. But it is not true that these complete lattices are the hom-objects of some quantaloid of "$\mathcal{Q}$-semicategories and semidistributors": although the formula for the composition of distributors (21) still makes sense, and although a $\mathcal{Q}$-semicategory $\mathbb{A}$ is a semidistributor $\mathbb{A}\colon\mathbb{A}\mathrel{-\!\!\!\circ\!\!\!\rightarrow}\mathbb{A}$, it is not true in general that $\mathbb{A}$ is the *identity* semidistributor on itself. On the other hand, it follows from elementary calculations in the quantaloid $\mathsf{Matr}(\mathcal{Q})$ that, for any two $\mathcal{Q}$-semicategories $\mathbb{A}$ and $\mathbb{B}$, a matrix from $\mathbb{A}_0$ to $\mathbb{B}_0$ is a semidistributor between the semicategories $\mathbb{A}$ and $\mathbb{B}$ if and only if the *same* matrix is a distributor between the free categories $\overline{\mathbb{A}}$ and $\overline{\mathbb{B}}$:

$$\mathsf{Distrib}(\mathbb{A},\mathbb{B})=\mathsf{Dist}(\mathcal{Q})(\overline{\mathbb{A}},\overline{\mathbb{B}}). \tag{60}$$

When working with $\mathcal{Q}$-semicategories, it are especially (58) and (60) that are of great importance. To illustrate this, let us consider the presheaf-construction for semicategories—we will see how it reduces to the known presheaf construction for categories.

Indeed, let $\mathbb{A}$ and $\mathbb{B}$ be $\mathcal{Q}$-semicategories. It is then a matter of pasting together (58), (60) and (40) to see that

$$\mathsf{Distrib}(\mathbb{A},\mathbb{B})=\mathsf{Dist}(\mathcal{Q})(\overline{\mathbb{A}},\overline{\mathbb{B}})\cong\mathsf{Cat}(\mathcal{Q})(\overline{\mathbb{A}},\mathcal{P}\overline{\mathbb{B}})=\mathsf{SCat}(\mathcal{Q})(\mathbb{A},\mathcal{P}\overline{\mathbb{B}}). \tag{61}$$

This indicates that $\mathcal{P}\overline{\mathbb{B}}$ – the $\mathcal{Q}$-category of (contravariant) presheaves on the free $\mathcal{Q}$-category $\overline{\mathbb{B}}$ – is the correct $\mathcal{Q}$-categorical structure whose objects are the (contravariant) presheaves on the $\mathcal{Q}$-semicategory $\mathbb{B}$: it "classifies" the semidistributors with codomain $\mathbb{B}$ (and as domain any $\mathcal{Q}$-semicategory $\mathbb{A}$). For a $\mathcal{Q}$-category $\mathbb{C}$ we know that $\overline{\mathbb{C}}=\mathbb{C}$ so it follows that $\mathcal{P}\mathbb{C}=\mathcal{P}\overline{\mathbb{C}}$ and there can thus be no confusion about the notion of "presheaf" on a $\mathcal{Q}$-category.

Since $\mathbb{B}$ itself is a semidistributor $\mathbb{B}\colon\mathbb{B}\mathrel{-\!\!\!\circ\!\!\!\rightarrow}\mathbb{B}$, there is a corresponding semifunctor $Y_{\mathbb{B}}\colon\mathbb{B}\longrightarrow\mathcal{P}\overline{\mathbb{B}}$—which we of course will call the Yoneda semifunctor[19]. For an object $b\in\mathbb{B}_0$ there is quite evidently a semidistributor $\mathbb{B}(-,b)\colon *_{tb}\mathrel{-\!\!\!\circ\!\!\!\rightarrow}\mathbb{B}$ with as elements the $\mathcal{Q}$-arrows

$$\Big(\mathbb{B}(x,b)\colon tb\longrightarrow tx\Big)_{x\in\mathbb{B}_0} \tag{62}$$

and by (60) there is a corresponding distributor $\mathbb{B}(-,b)\colon *_{tb}\mathrel{-\!\!\!\circ\!\!\!\rightarrow}\overline{\mathbb{B}}$ (which *as matrix* is identical, but note that the codomain has changed!). The latter is the image of b by $Y_{\mathbb{B}}$: it is the presheaf represented by b. But – unfortunately – two major properties for presheaves on a $\mathcal{Q}$-category fail for presheaves on a $\mathcal{Q}$-semicategory. For one thing, there is no "Yoneda lemma" for all presheaves on a given $\mathcal{Q}$-semicategory $\mathbb{B}$. And a second important difference is that a presheaf on a $\mathcal{Q}$-semicategory is not necessarily a "colimit of representables".

[19] One must be careful not to confuse the Yoneda semifunctor $Y_{\mathbb{A}}\colon\mathbb{A}\to\mathcal{P}\overline{\mathbb{A}}$ with the Yoneda functor $Y_{\overline{\mathbb{A}}}\colon\overline{\mathbb{A}}\to\mathcal{P}\overline{\mathbb{A}}$; in particular is the latter quite different from the functor $\overline{Y_{\mathbb{A}}}\colon\overline{\mathbb{A}}\to\mathcal{P}\overline{\mathbb{A}}$!

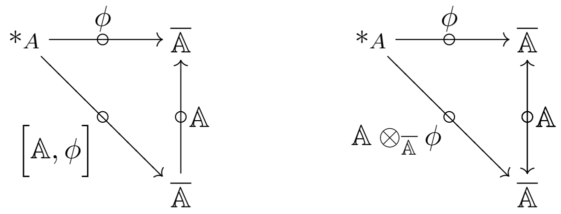

Diagram 3: Yoneda vs. regular presheaf

Still, for certain presheaves on a $\mathcal{Q}$-semicategory the "Yoneda lemma" may hold (we call these appropriately 'Yoneda presheaves'), and certain presheaves may be "colimit of representables" (and we call these 'regular presheaves'). Moens *et al.* [2002] have recognized the importance of – in particular – regular presheaves in the context of semicategories enriched in a symmetric monoidal closed category $\mathcal{V}$. In the next few paragraphs we will generalize their insights to $\mathcal{Q}$-enriched semicategories[20].

(i) Yoneda presheaves: Let $\mathbb{A}$ be a $\mathcal{Q}$-semicategory, and $\phi \in \mathcal{P}\overline{\mathbb{A}}$ a presheaf on $\mathbb{A}$. We'll say that ϕ is a Yoneda presheaf if for all $a \in \mathbb{A}_0$

$$\mathcal{P}\overline{\mathbb{A}}(Y_{\mathbb{A}}a, \phi) = \phi(a) \text{ in } \mathcal{Q}. \tag{63}$$

That is to say, we explicitly ask the validity of the "Yoneda lemma" (43) for $\phi \in \mathcal{P}\overline{\mathbb{A}}$. Another way of putting this, is as follows: consider $\mathbb{A}$ as distributor from $\overline{\mathbb{A}}$ to $\overline{\mathbb{A}}$, then it is required that

$$\left[\mathbb{A}, \phi\right] = \phi \text{ in } \mathsf{Dist}(\mathcal{Q}); \tag{64}$$

see the left-hand side of Diagram 3.

The Yoneda presheaves on a given $\mathcal{Q}$-semicategory $\mathbb{A}$ define a full subcategory of the category of all presheaves; let us denote it as $\mathcal{Y}\mathbb{A}$. It is then an easy corollary that the Yoneda embedding $Y_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathcal{P}\overline{\mathbb{A}}$ factors over the full embedding $\mathcal{Y}\mathbb{A} \longrightarrow \mathcal{P}\overline{\mathbb{A}}$ if and only if $\mathbb{A}$ is a category rather than a semicategory—that is to say, precisely when all representable presheaves on $\mathbb{A}$ are Yoneda presheaves.

(ii) Regular presheaves: We will say that a presheaf $\phi\colon *_A \multimap \overline{\mathbb{A}}$ on a semicategory $\mathbb{A}$ is regular if it is "a colimit of representables". More precisely, given ϕ we may consider the weighted diagram

$$*_A \xrightarrow{\ \phi\ } \overline{\mathbb{A}} \xrightarrow{\ \overline{Y_{\mathbb{A}}}\ } \mathcal{P}\overline{\mathbb{A}} \tag{65}$$

in the $\mathcal{Q}$-category $\mathcal{P}\overline{\mathbb{A}}$ (note that the functor involved is determined – through the universal property of the free category on $\mathbb{A}$ – by $Y_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathcal{P}\overline{\mathbb{A}}$). This diagram must have a colimit, since $\mathcal{P}\overline{\mathbb{A}}$ is cocomplete! This colimit is thus a functor

[20]In [Van den Bossche, 1995; van der Plancke, 1998] the "regularity" of presheaves plays an important rôle too, but this "regularity" is not recognized and studied *as such*.

$\mathrm{colim}(\phi, \overline{Y_{\mathbb{A}}})\colon *_A \longrightarrow \mathcal{P}\overline{\mathbb{A}}$ which "picks out" exactly one object of $\mathcal{P}\overline{\mathbb{A}}$. When this object is precisely ϕ itself, we shall say that the presheaf $\phi \in \mathcal{P}\overline{\mathbb{A}}$ is regular: thus we explicitly ask explicitly that

$$\mathrm{colim}(\phi, \overline{Y_{\mathbb{A}}}) = \phi. \tag{66}$$

Straightforward calculations show that the colimit of the weighted diagram (65) in $\mathcal{P}\overline{\mathbb{A}}$ is actually equal to the composition $\mathbb{A} \otimes_{\overline{\mathbb{A}}} \phi$ ($\mathbb{A}$ is regarded as endo-distributor on $\overline{\mathbb{A}}$), so that the requirement for ϕ to be a regular presheaf on $\mathbb{A}$ becomes quite simply that

$$\mathbb{A} \otimes_{\overline{\mathbb{A}}} \phi = \phi \text{ in } \mathsf{Dist}(\mathcal{Q}); \tag{67}$$

see the right-hand side of Diagram 3.

The regular presheaves on a $\mathcal{Q}$-semicategory $\mathbb{A}$ define a full subcategory of $\mathcal{P}\overline{\mathbb{A}}$; we will denote it as $\mathcal{R}\mathbb{A}$. We will say that the semicategory $\mathbb{A}$ is regular when the Yoneda embedding $Y_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathcal{P}\overline{\mathbb{A}}$ factors over the full inclusion $\mathcal{R}\mathbb{A} \longrightarrow \mathcal{P}\overline{\mathbb{A}}$. In other words, $\mathbb{A}$ is a regular $\mathcal{Q}$-semicategory if and only if the representable presheaves are regular [Moens *et al.*, 2002]. It is clear that every $\mathcal{Q}$-category is a regular semicategory, but the converse is not true[21]. So 'regular semicategory' is a strictly weaker notion than 'category', but still strong enough to admit a "good behavior" of presheaves[22].

(It is true that our definitions for 'Yoneda presheaf' and 'regular presheaf' on a $\mathcal{Q}$-semicategory are different from – but equivalent to! – those in [Moens, 2000; Moens *et al.*, 2002]. Indeed, in discussing presheaves on a $\mathcal{Q}$-semicategory $\mathbb{A}$ we always passed over to the free category $\overline{\mathbb{A}}$: we used the theory of $\mathcal{Q}$-categories to say something about semicategories. But there are completely equivalent descriptions for presheaves, Yoneda presheaves and regular presheaves on semicategories that do not refer in any way to free categories—i.e. a more *ad hoc* treatment of the matter. The original definitions of M.-A. Moens [2000] – given for $\mathcal{V}$-enriched categories – are then precisely these alternative ones.)

In a sense, the requirement for a presheaf ϕ on a semicategory $\mathbb{A}$ to be 'Yoneda' (64) is "adjoint" to the requirement for it to be 'regular' (67)—compare both sides of Diagram 3. Actually, for a regular $\mathcal{Q}$-semicategory $\mathbb{A}$ the inclusion

$$i\colon \mathcal{R}\mathbb{A} \longrightarrow \mathcal{P}\overline{\mathbb{A}}\colon \phi \mapsto \phi$$

is left adjoint to

$$j\colon \mathcal{P}\overline{\mathbb{A}} \longrightarrow \mathcal{R}\mathbb{A}\colon \psi \mapsto \mathbb{A} \otimes_{\overline{\mathbb{A}}} \phi$$

[21] Example of a regular semicategory which is not a category: see p. 34.

[22] In fact, these regular $\mathcal{Q}$-enriched semicategories are exactly what van der Plancke [1998] calls "$\mathcal{Q}$-sets". A regular presheaf is then what he calls a "part" (or "subset") of a $\mathcal{Q}$-set, and he observes that "every part is the union of singletons"—that is to say, every regular presheaf is a colimit of representables.

which in turn is left adjoint to

$$k\colon \mathcal{R}\mathbb{A} \longrightarrow \mathcal{P}\overline{\mathbb{A}}\colon \theta \mapsto [\mathbb{A}, \theta].$$

Both i and k are fully faithful; and the image of $k\colon \mathcal{R}\mathbb{A} \longrightarrow \mathcal{P}\overline{\mathbb{A}}$ is precisely $\mathcal{Y}\mathbb{A}$. (So we have here what B. Lawvere [1996] calls a situation of "unity and identity of opposites".) As a result, the $\mathcal{Q}$-category $\mathcal{R}\mathbb{A}$ is cocomplete. A similar reasoning holds to prove that $\mathcal{Y}\mathbb{A}$ is cocomplete too.

2.4. Regular semicategories: a distributor calculus. A $\mathcal{Q}$-enriched semicategory $\mathbb{A}$ is precisely a "monad without units" in the quantaloid $\mathsf{Matr}(\mathcal{Q})$: it is an endo-matrix $\mathbb{A}\colon \mathbb{A}_0 \longrightarrow \mathbb{A}_0$ for which $\mathbb{A}\circ\mathbb{A} \leq \mathbb{A}$. We decided to call such a semicategory 'regular' when all representable presheaves on $\mathbb{A}$ are regular. In terms of matrices this comes down to asking for the endo-matrix $\mathbb{A}\colon \mathbb{A}_0 \longrightarrow \mathbb{A}_0$ to be idempotent[23].

Indeed, let $\mathbb{A}$ be a $\mathcal{Q}$-semicategory. For an object $a \in \mathbb{A}_0$ the representable presheaf $\mathbb{A}(-,a)\colon *_{ta} \looparrowright \overline{\mathbb{A}}$ is the matrix whose elements are the $\mathcal{Q}$-arrows

$$\Big(\mathbb{A}(x,a)\colon ta \longrightarrow tx\Big)_{x\in\mathbb{A}_0}.$$

And – cf. (67) – this representable is regular when $\mathbb{A} \otimes_{\overline{\mathbb{A}}} \mathbb{A}(-,a) = \mathbb{A}(-,a)$, or, when computing this composition of semidistributors,

$$\forall x \in \mathbb{A}_0 : \bigvee_{y\in\mathbb{A}_0} \mathbb{A}(x,y) \circ \mathbb{A}(y,a) = \mathbb{A}(x,a)$$

as $\mathcal{Q}$-arrows. Thus $\mathbb{A}$ is a regular $\mathcal{Q}$-semicategory if and only if

$$\forall a, a' \in \mathbb{A}_0 : \bigvee_{x\in\mathbb{A}_0} \mathbb{A}(a',x) \circ \mathbb{A}(x,a) = \mathbb{A}(a',a),$$

i.e. the matrix $\mathbb{A}\colon \mathbb{A}_0 \longrightarrow \mathbb{A}_0$ is an idempotent in $\mathsf{Matr}(\mathcal{Q})$.

The theory of regular $\mathcal{Q}$-semicategories is thus the theory of idempotents in $\mathsf{Matr}(\mathcal{Q})$. Let us therefore first consider – quite generally – idempotents in any quantaloid (small or large, that doesn't matter here), to afterwards specify our findings to the quantaloid $\mathsf{Matr}(\mathcal{Q})$.

For two idempotents $A \circlearrowleft^{e}$ and $B \circlearrowleft^{f}$ in a quantaloid $\mathcal{Q}$ we say that a $\mathcal{Q}$-arrow $b\colon A \longrightarrow B$ is a regular bimodule if $b \circ e = b$ and $f \circ b = b$, and we shall write this as $b\colon e \looparrowright f$. Such bimodules can be composed: given $b\colon e \looparrowright f$ and $c\colon f \looparrowright g$ we denote $c \otimes_f b\colon e \looparrowright g$ for the composition which is computed simply as $c \circ b$ in $\mathcal{Q}$. An idempotent e is a regular bimodule on itself, indeed it is the identity bimodule:

[23] In [Van den Bossche, 1995] this is precisely the definition of "$\mathcal{Q}$-set". F. van der Plancke [1995] was apparently aware of the fact that his "$\mathcal{Q}$-sets" – which he defined in more elementary terms, see a previous footnote – are precisely these idempotent matrices, but he didn't exploit this fact.

$e\colon e \to e$. We thus obtain a category $\mathsf{Idm}(\mathcal{Q})$ whose objects are idempotents and whose arrows are regular bimodules; it is in fact a quantaloid because the supremum of regular bimodules $(b_i\colon e \to f)_{i\in I}$ may be calculated as supremum of (the underlying) $\mathcal{Q}$-arrows. (Note that $\mathsf{Idm}(\mathcal{Q})$ is small whenever $\mathcal{Q}$ is.)

Recall that $\mathsf{Mnd}(\mathcal{Q})$ denoted the quantaloid of monads and bimodules in $\mathcal{Q}$ (see p. 16). Quite evidently, every bimodule between monads is a regular bimodule between idempotents. That is to say, there are full embeddings

$$\mathcal{Q} \longrightarrow \mathsf{Mnd}(\mathcal{Q}) \longrightarrow \mathsf{Idm}(\mathcal{Q}). \tag{68}$$

We have seen that $\mathcal{Q} \longrightarrow \mathsf{Mnd}(\mathcal{Q})$ is the universal splitting of monads, and it is thus not too surprising that $\mathcal{Q} \longrightarrow \mathsf{Idm}(\mathcal{Q})$ is the universal splitting of idempotents [Freyd, 1964]. This universal property of $\mathsf{Idm}(\mathcal{Q})$ makes it easy to characterize those quantaloids which arise as quantaloids of idempotents and regular bimodules: they are the quantaloids that are Cauchy complete *as category* [Borceux and Dejean, 1986].

We can now perform the splitting of idempotents in the quantaloid $\mathsf{Matr}(\mathcal{Q})$: we obtain a new quantaloid

$$\mathsf{RSDist}(\mathcal{Q}) = \mathsf{Idm}(\mathsf{Matr}(\mathcal{Q})) \tag{69}$$

whose objects are precisely the regular $\mathcal{Q}$-enriched semicategories, and whose morphisms we will obviously call 'regular semidistributors'. Explicitly, a regular semidistributor $\Phi\colon \mathbb{A} \to \mathbb{B}$ between regular $\mathcal{Q}$-semicategories is a matrix $\Phi\colon (\mathbb{A}_0, t) \longrightarrow (\mathbb{B}_0, t)$ such that

$$\text{for all } a \in \mathbb{A}_0, b \in \mathbb{B}_0 : \textstyle\bigvee_{a'\in\mathbb{A}_0} \Phi(b, a') \circ \mathbb{A}(a', a) = \Phi(b, a); \tag{70}$$

$$\text{for all } a \in \mathbb{A}_0, b \in \mathbb{B}_0 : \textstyle\bigvee_{b'\in\mathbb{B}_0} \mathbb{B}(b, b') \circ \Phi(b', a) = \Phi(b, a). \tag{71}$$

That is to say, a *regular* semidistributor is one for which the action-inequalities of (25) and (26) "saturate" to equalities. This quantaloid $\mathsf{RSDist}(\mathcal{Q})$ is the appropriate "distributor calculus" for (regular) $\mathcal{Q}$-enriched semicategories. The base quantaloid $\mathcal{Q}$ can obviously be embedded in $\mathsf{RSDist}(\mathcal{Q})$; more precisely, there are full embeddings

$$\mathcal{Q} \longrightarrow \mathsf{Dist}(\mathcal{Q}) \longrightarrow \mathsf{RSDist}(\mathcal{Q}) \tag{72}$$

in analogy with (68). For a small $\mathcal{Q}$ the embedding $\mathcal{Q} \longrightarrow \mathsf{RSDist}(\mathcal{Q})$ has a particularly interesting universal property: it is the Cauchy completion of $\mathcal{Q}$ *as quantaloid* [van der Plancke, 1998].

Since regular $\mathcal{Q}$-semicategories and regular semidistributors form a quantaloid $\mathsf{RSDist}(\mathcal{Q})$, it now makes sense to speak of 'adjoint regular semidistributors'—and we may ask ourselves whether "semifunctors between regular semicategories induce adjoints pairs of regular semidistributors" (as is the case for functors between categories). Unfortunately, the answer is negative: let $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ be a semifunctor between regular $\mathcal{Q}$-semicategories, then surely it determines two semidistributors

$$\mathbb{B}(-,F-)\colon \mathbb{A} \relbar\!\!\!\!\circ\!\!\!\rightarrow \mathbb{B} \text{ and } \mathbb{B}(F-,-)\colon \mathbb{B} \relbar\!\!\!\!\circ\!\!\!\rightarrow \mathbb{A} \tag{73}$$

but these needn't be regular[24]! However, would these semidistributors be regular, then they prove to be adjoint:

$$\mathbb{B}(-,F-) \dashv \mathbb{B}(F-,-) \text{ in } \mathsf{RSDist}(\mathcal{Q}).$$

We will thus say that $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ is regular when both semidistributors in (73) are regular—in elementary terms, F should satisfy

$$\text{for all } a \in \mathbb{A}_0, b \in \mathbb{B}_0 : \bigvee_{a' \in \mathbb{A}_0} \mathbb{B}(b, Fa') \circ \mathbb{A}(a', a) = \mathbb{B}(b, Fa); \tag{74}$$

$$\text{for all } a \in \mathbb{A}_0, b \in \mathbb{B}_0 : \bigvee_{a' \in \mathbb{A}_0} \mathbb{A}(a, a') \circ \mathbb{B}(Fa', a) = \mathbb{B}(Fa, b). \tag{75}$$

(The other two equations that one might expect, are trivial.) It is a fact that regular $\mathcal{Q}$-semicategories and regular semifunctors form a category, and we denote it as $\mathsf{RSCat}(\mathcal{Q})$. This category is on the one hand a (non-full) subcategory of $\mathsf{SCat}(\mathcal{Q})$ (the category of *all* $\mathcal{Q}$-semicategories and *all* semifunctors), and on the other hand it contains $\mathsf{Cat}(\mathcal{Q})$ ($\mathcal{Q}$-categories and functors) as (full) subcategory:

$$\mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{RSCat}(\mathcal{Q}) \longrightarrow \mathsf{SCat}(\mathcal{Q}). \tag{76}$$

Further, sending a regular $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ to the left adjoint $\mathbb{B}(-,F-)\colon \mathbb{A} \relbar\!\!\!\!\circ\!\!\!\rightarrow \mathbb{B}$ preserves composition and identities, so that

$$\mathsf{RSCat}(\mathcal{Q}) \longrightarrow \mathsf{RSDist}(\mathcal{Q}) : (F\colon \mathbb{A} \longrightarrow \mathbb{B}) \mapsto (\mathbb{B}(-,F-)\colon \mathbb{A} \relbar\!\!\!\!\circ\!\!\!\rightarrow \mathbb{B}) \tag{77}$$

is a functor (which is the identity on objects). (Sending F to $\mathbb{B}(F-,-)$ determines a contravariant such functor.) The various inclusions and embeddings of (30), (72), (76) and (77) commute as in Diagram 4.

After all this theory, let us look at some examples of such regular semicategories.

(a) To see that 'regular semicategory' is a strictly weaker notion than that of 'category', we may consider the strict ordering on the real numbers as a **2**-semicategory: that is, we think of $\mathbb{R}$ as the **2**-enriched structure whose objects are the real numbers, and whose hom-arrows are given by

$$\mathbb{R}(y,x) = 1 \text{ if and only if } y < x.$$

The transitivity of the order is then precisely the composition-inequality in $\mathbb{R}$, and the order being strict means that the unit-inequality cannot hold. So this is not a

[24]For a counterexample, see a bit further.

$$
\begin{array}{ccc}
\mathsf{Cat}(\mathcal{Q}) & \longrightarrow & \mathsf{RSCat}(\mathcal{Q}) \\
\downarrow & & \downarrow \\
\mathsf{Dist}(\mathcal{Q}) & \longrightarrow & \mathsf{RSDist}(\mathcal{Q})
\end{array}
$$

Diagram 4: categories and regular semicategories

2-category, but a semicategory. However, it is a regular **2**-semicategory, due to the "density[25]" of the reals:

$$\bigvee_{y\in\mathbb{R}} \mathbb{R}(z,y)\wedge\mathbb{R}(y,x)=\mathbb{R}(z,x)$$

holds because for every $z<x$ there exists a y such that $z<y<x$!

(b) More about this example. The free category on the semicategory $(\mathbb{R},<)$ is simply $(\mathbb{R},\leq)$. A presheaf on $(\mathbb{R},<)$ is by definition a presheaf on $(\mathbb{R},\leq)$, i.e. a down-closed subset. But to illustrate the difference between the Yoneda embeddings, note that:

$$
\begin{array}{l}
Y_{(\mathbb{R},<)}\colon\mathbb{R}\longrightarrow\mathcal{P}\mathbb{R}\colon r\mapsto\twoheaddownarrow r=\{x\in\mathbb{R}\mid x<r\}\text{ for }\mathbb{R}\text{-as-semicategory,}\\
Y_{(\mathbb{R},\leq)}\colon\mathbb{R}\longrightarrow\mathcal{P}\mathbb{R}\colon r\mapsto\downarrow r=\{x\in\mathbb{R}\mid x\leq r\}\text{ for }\mathbb{R}\text{-as-category.}
\end{array}
$$

A down-closed set $D\subseteq\mathbb{R}$ corresponds to a regular presheaf on $(\mathbb{R},<)$ whenever for every $x\in D$ there exists a $y\in D$ for which $x<y$ (so D has no maximum); and D is a Yoneda presheaf on $(\mathbb{R},<)$ whenever $\twoheaddownarrow x\subseteq D$ implies that $x\in D$ too. Clearly every $\twoheaddownarrow r$ is a regular presheaf on $(\mathbb{R},<)$, which is consistent with the previous observation that $(\mathbb{R},<)$ is a regular semicategory.

(c) More generally, a continuously ordered set $(P,\leq)$ with "way-below" relation $\ll$ (see for example [Gierz *et al.*, 1980] for a precise definition) may be viewed as a regular **2**-semicategory $\mathbb{P}$ when putting $\mathbb{P}_0=P$ as object-set and

$$\text{for } x,y\in P:\mathbb{P}(y,x)=1\text{ if and only if } y\ll x.$$

The point is that "$\ll$" is an idempotent relation on P. And quite surprisingly (the underlying order of) the **2**-category $\mathcal{R}\mathbb{P}$ of regular presheaves on $\mathbb{P}$ is (isomorphic to) the Scott topology on P [van der Plancke, 1998].

(d) Inspired by the previous example, we may now give an example of a semifunctor between regular semicategories which is not regular. Thereto, consider on the one hand the strict ordering of the reals $(\mathbb{R},<)$ and on the other the (partial) ordering

[25] J. Koslowski [1997] and R. Rosebrugh and R. Wood [1994] would probably prefer to see the word "interpolation property" here.

on an interval, say $([-1,1],\leq)$. They are both examples of regular **2**-semicategories (the latter is even a **2**-category). The mapping

$$F\colon\mathbb{R}\longrightarrow[-1,1]\colon x\mapsto\begin{cases}1 \text{ if } x>0\\ -1 \text{ otherwise}\end{cases}$$

is surely a semifunctor between regular **2**-semicategories: if $x<y$ in $\mathbb{R}$ then $Fx\leq Fy$. For F to be a regular semifunctor, it is required that

- for all $x\in\mathbb{R}$ and $a\in[-1,1]$ satisfying $a\leq Fx$ there exists $x'\in\mathbb{R}$ such that $x'<x$ and $a\leq Fx'$;

- for all $x\in\mathbb{R}$ and $a\in[-1,1]$ satisfying $Fx\leq a$ there exist $x'\in\mathbb{R}$ such that $x<x'$ and $Fx'\leq a$.

The first of these sentences is true, but the second is false (take $x=0$ and $a=-1$), so F cannot be a regular semifunctor!

(e) We have observed (see p. 26) that for a locale Ω, an Ω-set $(A,[\,\cdot=\cdot\,])$ may be viewed as an Ω-enriched semicategory whose objects are the elements of A, and for two objects a,a' the hom is $[a'=a]$. Actually, it is a regular semicategory[26], because for any $a,a'\in A$ we have due to symmetry of the Ω-valued equality that $[a'=a]=[a'=a]\wedge[a=a']\leq[a'=a']$ and therefore

$$[a'=a]=[a'=a']\wedge[a'=a]\leq\bigvee_{x\in A}\Big([a'=x]\wedge[x=a]\Big)\leq[a'=a]$$

(using the transitivity axiom, i.e. the composition inequality). Rewriting the definition of 'regular presheaf' on such an Ω-set in terms of the Ω-valued equality, it is easily seen to be precisely a mapping $\phi\colon A\longrightarrow\Omega$ satisfying for all $a\in A$ that

$$\bigvee_{x\in A}[a=x]\wedge\phi(x)=\phi(a).$$

This is precisely the correct notion of (the "characteristic function" of) an 'Ω-subset' of the given Ω-set: thinking of the Ω-set as a sheaf on Ω, these Ω-subsets correspond to subsheaves [Borceux, 1994]. In particular is the (underlying order of the) Ω-category of regular presheaves on a given Ω-set (viewed as regular semicategory) exactly the locale of subsheaves of that Ω-set (viewed as sheaf).

2.5. Cauchy completion of semicategories: total regularity. 'Regular $\mathcal{Q}$-semicategory', 'regular semidistributor' and 'regular semifunctor' are strictly weaker notions than '$\mathcal{Q}$-category', 'distributor' and 'functor', but they are just strong enough to still allow for a distributor quantaloid $\mathsf{RSDist}(\mathcal{Q})$ and for every regular semifunctor $F\colon\mathbb{A}\longrightarrow\mathbb{B}$ in $\mathsf{RSCat}(\mathcal{Q})$ to induce an adjoint pair of regular semidistributors.

[26] It is even more than that, as we will see further on...

As the similar functor (30) was the key to the development of the theory of $\mathcal{Q}$-categories, we may now ask ourselves in how far (77) allows for a useful theory of regular $\mathcal{Q}$-semicategories. Particularly, it is quite essential to find a decent notion of 'Cauchy completion' for such semicategories; after all, any categorical structure is just a presentation of its Cauchy completion and so in principle a semicategory too should be 'Morita equivalent' to its 'Cauchy completion'.

Recall that for a $\mathcal{Q}$-category $\mathbb{C}$ its Cauchy completion $\mathbb{C}_{\mathsf{cc}}$ is the category classifying the Cauchy distributors with codomain $\mathbb{C}$; it is the full subcategory of the presheaf category $\mathcal{P}\mathbb{C}$ determined by the Cauchy presheaves. The Yoneda embedding $Y_{\mathbb{C}}\colon\mathbb{C}\longrightarrow\mathcal{P}\mathbb{C}$ factors fully over the full inclusion $\mathbb{C}_{\mathsf{cc}}\subseteq\mathcal{P}\mathbb{C}$: every object $c\in\mathbb{C}_0$ can be "pointed at" by means of a constant functor $\Delta c\colon *_{tc}\longrightarrow\mathbb{C}$, which in turn determines the (Cauchy) representable presheaf $\mathbb{C}(-,c)\colon *_{tc}\mathrel{-\!\!\circ\!\!\!\rightarrow}\mathbb{C}$ (adjoint to $\mathbb{C}(c,-)\colon\mathbb{C}\mathrel{-\!\!\circ\!\!\!\rightarrow} *_{tc}$) that thus converges to Δc itself. That is to say, $\mathbb{C}$ can be thought of as a full subcategory of $\mathbb{C}_{\mathsf{cc}}$ because "representable presheaves on $\mathbb{C}$ converge to constant functors into $\mathbb{C}$". (See p. 21 and further.)

Unfortunately, for a (regular) $\mathcal{Q}$-semicategory $\mathbb{A}$ it is no longer true that "representables converge to constants"! Indeed, even though any object $a\in\mathbb{A}_0$ does indeed determine a representable presheaf $\mathbb{A}(-,a)\colon *_{ta}\mathrel{-\!\!\circ\!\!\!\rightarrow}\overline{\mathbb{A}}$, it is simply not true that this representable is left adjoint to $\mathbb{A}(a,-)\colon\overline{\mathbb{A}}\mathrel{-\!\!\circ\!\!\!\rightarrow} *_{ta}$: for then necessarily $1_{ta}\leq\mathbb{A}(a,a)$ by the unit of the adjunction! Put differently, it is impossible in general to "point at" an object of $\mathbb{A}$ with a (regular) semifunctor like $F\colon *_A\longrightarrow\mathbb{A}$: this would imply that $1_A\leq\mathbb{A}(a,a)$, by the action of the semifunctor F on the hom-arrows! If we *per se* want an object $a\in\mathbb{A}_0$ of a (regular) $\mathcal{Q}$-semicategory $\mathbb{A}$ to be "pointed at" by a constant (regular) semifunctor, so that this semifunctor determines a left adjoint "representable presheaf" that in turn converges to the constant, then we will explicitly have to ask this right from the start—and we will have to be willing to consider a constant semifunctor whose domain is not a category.

All this motivates the following definition: an object a of a regular $\mathcal{Q}$-semicategory[27] $\mathbb{A}$ is *stable* if there exists a regular semifunctor from a one-object regular $\mathcal{Q}$-semicategory to $\mathbb{A}$ "pointing at" $a\in\mathbb{A}_0$.

Note that such a one-object regular $\mathcal{Q}$-semicategory is, simply, the "suspension" of an idempotent in $\mathcal{Q}$; for an idempotent $A\overset{e}{\circlearrowleft}$ we will denote $*_e$ for the regular semicategory whose single object $*$ is of type $t* = \mathsf{dom}(e)$ and whose single hom-arrow is precisely e. In the above definition we thus ask, for the object $a\in\mathbb{A}_0$, the existence of some idempotent e_a in $\mathcal{Q}$ and a regular semifunctor

$$F_a\colon *_{e_a}\longrightarrow\mathbb{A}\colon *\mapsto a. \tag{78}$$

As a consequence of the regularity of this semifunctor we then have an adjoint pair

[27]Since we want to speak of "adjoint semidistributors", it is clear that this will only make sense for a regular $\mathcal{Q}$-semicategory $\mathbb{A}$: then we can work in the quantaloid $\mathsf{RSDist}(\mathcal{Q})$.

of semidistributors

$$\mathbb{A}(-,a) \dashv \mathbb{A}(a,-) \colon \mathbb{A} \overset{\longleftarrow\!\!\!\!\circ}{\underset{\circ\!\!\!\!\longrightarrow}{}} *_{e_a}\ . \tag{79}$$

This is precisely what we wanted: to "point at" an object a with a constant semifunctor, and to have a left adjoint "representable" that converges to this constant semifunctor.

This definition can be expressed "equationally": indeed, an object a in $\mathbb{A}$ is stable if and only if

$$\text{for all } a' \in \mathbb{A}_0 : \begin{cases} \mathbb{A}(a',a) \circ \mathbb{A}(a,a) = \mathbb{A}(a',a), \\ \mathbb{A}(a,a) \circ \mathbb{A}(a,a') = \mathbb{A}(a,a'). \end{cases} \tag{80}$$

These equations imply immediately that $\mathbb{A}(a,a) \colon ta \longrightarrow ta$ is an idempotent which itself may play the rôle of the idempotent e_a in (78) and (79) above.

We are now obviously interested in regular semicategories all objects of which are stable: we will call such a semicategory 'totally regular'. Using (80) we may thus spell out that a regular $\mathcal{Q}$-semicategory $\mathbb{A}$ is totally regular if and only if

$$\text{for all } a, a' \in \mathbb{A}_0 : \begin{cases} \mathbb{A}(a',a) \circ \mathbb{A}(a,a) = \mathbb{A}(a',a), \\ \mathbb{A}(a,a) \circ \mathbb{A}(a,a') = \mathbb{A}(a,a'). \end{cases} \tag{81}$$

(These conditions make sense for any – not necessarily regular – $\mathcal{Q}$-semicategory $\mathbb{A}$: they actually imply $\mathbb{A}$'s regularity.) Every $\mathcal{Q}$-category is a totally regular semicategory, but the converse is not true; and not every regular semicategory is totally regular (see p. 41 for examples).

It is easily verified that a semidistributor $\Phi \colon \mathbb{A} \circ\!\!\!\!\longrightarrow \mathbb{B}$ between totally regular $\mathcal{Q}$-semicategories is regular – sensu (70) and (71) – if and only if

$$\text{for all } a \in \mathbb{A}_0, b \in \mathbb{B}_0 : \begin{cases} \Phi(b,a) \circ \mathbb{A}(a,a) = \Phi(b,a), \\ \mathbb{B}(b,b) \circ \Phi(b,a) = \Phi(b,a). \end{cases} \tag{82}$$

Denoting henceforth $\mathsf{TRSDist}(\mathcal{Q})$ for the full sub-quantaloid of $\mathsf{RSDist}(\mathcal{Q})$ determined by the totally regular $\mathcal{Q}$-semicategories, we thus have full embeddings of quantaloids like

$$\mathsf{Dist}(\mathcal{Q}) \longrightarrow \mathsf{TRSDist}(\mathcal{Q}) \longrightarrow \mathsf{RSDist}(\mathcal{Q}). \tag{83}$$

We can apply (82) to semifunctors: for totally regular semicategories $\mathbb{A}$ and $\mathbb{B}$, a semifunctor $F \colon \mathbb{A} \longrightarrow \mathbb{B}$ is regular – sensu (74) and (75) – if and only if

$$\text{for all } a \in \mathbb{A}_0, b \in \mathbb{B}_0 : \begin{cases} \mathbb{B}(b,Fa) \circ \mathbb{A}(a,a) = \mathbb{B}(b,Fa), \\ \mathbb{A}(a,a) \circ \mathbb{B}(Fa,b) = \mathbb{B}(Fa,b). \end{cases} \tag{84}$$

$$
\begin{array}{ccccc}
\mathsf{Cat}(\mathcal{Q}) & \longrightarrow & \mathsf{TRSCat}(\mathcal{Q}) & \longrightarrow & \mathsf{RSCat}(\mathcal{Q}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathsf{Dist}(\mathcal{Q}) & \longrightarrow & \mathsf{TRSDist}(\mathcal{Q}) & \longrightarrow & \mathsf{RSDist}(\mathcal{Q})
\end{array}
$$

Diagram 5: totally regular semicategories

As we denoted $\mathsf{TRSDist}(\mathcal{Q})$ for the full sub-quantaloid of $\mathsf{RSDist}(\mathcal{Q})$ determined by the totally regular semicategories, we will denote $\mathsf{TRSCat}(\mathcal{Q})$ for the full sub-2-category of $\mathsf{RSCat}(\mathcal{Q})$ with those objects. It is quite obvious that we have (strict) embeddings

$$\mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{TRSCat}(\mathcal{Q}) \longrightarrow \mathsf{RSCat}(\mathcal{Q}) \tag{85}$$

(all of which are full), and that Diagram 5 commutes: the arrows from left to right are full embeddings, those from top to bottom indicate that a regular semifunctor, and in particular a functor, induces an adjoint pair of regular semidistributors.

The point is now that a totally regular $\mathcal{Q}$-semicategory is indeed 'Morita equivalent' to its 'Cauchy completion' in a way that is very similar to, but at the same time quite different from, what we know for $\mathcal{Q}$-categories. More precisely we can proceed as follows.

- Say that a totally regular $\mathcal{Q}$-semicategory $\mathbb{B}$ is *Cauchy complete* if every left adjoint regular semidistributor with codomain $\mathbb{B}$, like $\Phi\colon \mathbb{A} \mathrel{-\!\!\circ\!\!\!\rightarrow} \mathbb{B}$, is determined by a (necessarily essentially unique) regular semifunctor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$. That is to say, "all Cauchy semidistributors into $\mathbb{B}$ converge".

- For any given totally regular $\mathcal{Q}$-semicategory $\mathbb{B}$ there exists a new $\mathcal{Q}$-categorical structure $\mathbb{B}_{\mathsf{cc}}$ with the universal property that, for all totally regular $\mathbb{A}$,

$$\mathsf{Map}(\mathsf{TRSDist}(\mathcal{Q}))(\mathbb{A}, \mathbb{B}) \simeq \mathsf{TRSCat}(\mathcal{Q})(\mathbb{A}, \mathbb{B}_{\mathsf{cc}}) \tag{86}$$

(natural in $\mathbb{A}$). This $\mathbb{B}_{\mathsf{cc}}$ can be constructed as follows: its objects are the left adjoint (regular) semidistributors into $\mathbb{B}$ whose domain is a one-object totally regular semicategory, i.e. like $\phi\colon *_e \mathrel{-\!\!\circ\!\!\!\rightarrow} \mathbb{B}$ with $\phi \dashv \phi^*$ in $\mathsf{TRSDist}(\mathcal{Q})$; the type of such an object is $t(\phi:^{\star}_{e} \mathrel{-\!\!\circ\!\!\!\rightarrow} \mathbb{B}) = \mathsf{dom}(e)$; and for two objects $\psi\colon *_e \mathrel{-\!\!\circ\!\!\!\rightarrow} \mathbb{B}$ and $\psi\colon *_f \mathrel{-\!\!\circ\!\!\!\rightarrow} \mathbb{B}$ the hom-arrow $\mathbb{B}_{\mathsf{cc}}(\psi, \phi)\colon \mathsf{dom}(e) \longrightarrow \mathsf{dom}(f)$ is the single element of the regular semidistributor $[\psi, \phi] = \psi^* \otimes_{\mathbb{B}} \phi$ (lifting and composition in the quantaloid $\mathsf{TRSDist}(\mathcal{Q})$). This $\mathbb{B}_{\mathsf{cc}}$ is a Cauchy complete totally regular $\mathcal{Q}$-semicategory.

- Since $\mathbb{B}\colon \mathbb{B} \mathrel{-\!\!\circ\!\!\!\rightarrow} \mathbb{B}$ is a (regular) left adjoint semidistributor, there must be a corresponding (regular) semifunctor $j_{\mathbb{B}}\colon \mathbb{B} \longrightarrow \mathbb{B}_{\mathsf{cc}}$: this is the *Cauchy completion* of $\mathbb{B}$, it sends an object $b \in \mathbb{B}_0$ onto the left adjoint regular semidistributor

$$\mathbb{B}(-, b)\colon *_{\mathbb{B}(b,b)} \mathrel{-\!\!\circ\!\!\!\rightarrow} \mathbb{B}.$$

$$\begin{array}{ccc} \mathsf{TRSCat}(\mathcal{Q}) & \xrightarrow{(-)_{\mathsf{cc}}} & \mathsf{TRSCat}(\mathcal{Q}) \\ \uparrow & \neq & \uparrow \\ \mathsf{TRSCat}(\mathcal{Q}) & \xrightarrow[(-)_{\mathsf{cc}}]{} & \mathsf{TRSCat}(\mathcal{Q}) \end{array} \qquad \begin{array}{ccc} \mathsf{TRSDist}(\mathcal{Q}) & \xrightarrow{(-)_{\mathsf{cc}}} & \mathsf{TRSDist}(\mathcal{Q}) \\ \uparrow & \neq & \uparrow \\ \mathsf{Dist}(\mathcal{Q}) & \xrightarrow[(-)_{\mathsf{cc}}]{} & \mathsf{Dist}(\mathcal{Q}) \end{array}$$

Diagram 6: non-commuting squares!

By the naturality of (86) it is the unit of the left adjoint to the obvious full embedding:

$$(-)_{\mathsf{cc}} \dashv i \colon \mathsf{TRSCat}_{\mathsf{cc}}(\mathcal{Q}) \leftrightarrows \mathsf{TRSCat}(\mathcal{Q}). \tag{87}$$

- The (regular) semifunctor $j_{\mathbb{B}}$ in turn determines (an adjoint pair of) semidistributors $\mathbb{B}_{\mathsf{cc}}(-, j_{\mathbb{B}}-) \colon \mathbb{B} \to \mathbb{B}_{\mathsf{cc}}$ and $\mathbb{B}_{\mathsf{cc}}(j_{\mathbb{B}}-, -) \colon \mathbb{B}_{\mathsf{cc}} \to \mathbb{B}$, which are actually each others' inverse; so $\mathbb{B}$ is isomorphic to $\mathbb{B}_{\mathsf{cc}}$ in $\mathsf{TRSDist}(\mathcal{Q})$. That is to say, $\mathbb{B}$ is *Morita equivalent* to its Cauchy completion. And it follows that

$$\mathsf{TRSDist}_{\mathsf{cc}}(\mathcal{Q}) \simeq \mathsf{TRSDist}(\mathcal{Q}) \tag{88}$$

is an equivalence of quantaloids (where the former is the obvious full subquantaloid of the latter).

- It now follows from (86) and (88) that

$$\mathsf{Map}(\mathsf{TRSDist}(\mathcal{Q})) \simeq \mathsf{Map}(\mathsf{TRSDist}_{\mathsf{cc}}(\mathcal{Q})) \simeq \mathsf{TRSCat}_{\mathsf{cc}}(\mathcal{Q}); \tag{89}$$

so, if one is only interested in "Cauchy invariant" properties of (or "Morita equivalent" constructions on) totally regular $\mathcal{Q}$-semicategories, one may take the left adjoint (regular) semidistributors for "morphisms".

A warning is appropriate here: for a $\mathcal{Q}$-category $\mathbb{C}$ (which is trivially a totally regular $\mathcal{Q}$-semicategory), its Cauchy completion *as category* is very different from its Cauchy completion *as totally regular semicategory*. In fact, the latter isn't even a $\mathcal{Q}$-category anymore (but a totally regular semicategory)! Our notations do not distinguish between $\mathbb{C}$'s "Cauchy completion as category" and its "Cauchy completion as totally regular semicategory"; we write both as $\mathbb{C}_{\mathsf{cc}}$ and hope that the context makes clear which of both is meant. But – to make things perfectly clear – the squares in Diagram 6 are not commutative! The theory of (Cauchy complete) totally regular $\mathcal{Q}$-semicategories is thus not at all "merely a straightforward generalization" of the theory of $\mathcal{Q}$-enriched categories. (Another such situation that makes this apparent, is that for a totally regular $\mathcal{Q}$-semicategory $\mathbb{A}$, its Cauchy completion is not a full substructure of $\mathcal{R}\mathbb{A}$ (or even $\mathcal{P}\overline{\mathbb{A}}$), whereas for a category its Cauchy completion (as category) is a full subcategory of its presheaf category.)

It may be useful to mention that the construction of "$\mathbb{B}_{\mathsf{cc}}$" as outlined below (86) makes perfectly sense for any (not necessarily *totally*) regular $\mathcal{Q}$-semicategory $\mathbb{B}$ (although it may not have the desired properties). But $\mathbb{B}_{\mathsf{cc}}$ is always totally regular, and if $\mathbb{B}$ is fully faithfully embedded in $\mathbb{B}_{\mathsf{cc}}$ (a property that one certainly wants from a "Cauchy completion"!) then $\mathbb{B}$ must be totally regular too. This justifies – in retrospect – why a workable notion of "Cauchy completion" only applies to totally regular $\mathcal{Q}$-semicategories, and not to the merely regular ones.

Here are some examples.

(a) Recall that the strictly ordered reals $(\mathbb{R}, <)$ may be viewed as a regular $\mathbf{2}$-semicategory. But it is not totally regular: for $x < y$ it is clear that $\mathbb{R}(x, y) \wedge \mathbb{R}(y, y) \neq \mathbb{R}(x, y)$. So not every regular semicategory is totally regular. On the other hand it is obvious that not every totally regular semicategory is a category.

(b) Every totally regular $\mathbf{2}$-semicategory is Morita-equivalent to a true and honest $\mathbf{2}$-category, in the following way. Suppose that $\mathbb{A}$ is a totally regular $\mathbf{2}$-semicategory, and that $a \in \mathbb{A}_0$ is an object for which $1 \not\leq \mathbb{A}(a, a)$. Then it follows by total regularity that

$$\text{for all } a' \in \mathbb{A}_0 : \mathbb{A}(a', a) = 0 = \mathbb{A}(a, a'),$$

that is to say, those objects are "isolated" in $\mathbb{A}$. Consider now the full substructure of $\mathbb{A}$ determined by those objects which are not isolated; let $i\colon \widehat{\mathbb{A}} \longrightarrow \mathbb{A}$ denote the full inclusion. Then $\widehat{\mathbb{A}}$ is a $\mathbf{2}$-category, and the adjoint semidistributors $\mathbb{A}(-, i-)\colon \widehat{\mathbb{A}} \looparrowright \mathbb{A}$ and $\mathbb{A}(i-, -)\colon \mathbb{A} \looparrowright \widehat{\mathbb{A}}$ are actually each others' inverse in $\mathsf{TRSDist}(\mathbf{2})$. So the full embedding $\mathsf{Dist}(\mathbf{2}) \longrightarrow \mathsf{TRSDist}(\mathbf{2})$ is an equivalence, and it follows that $\mathsf{Cat}_{\mathsf{cc}}(\mathbf{2}) \simeq \mathsf{TRSCat}_{\mathsf{cc}}(\mathbf{2})$ too. (Recalling that every $\mathbf{2}$-category is Cauchy complete, we may even state that $\mathsf{Cat}(\mathbf{2}) \simeq \mathsf{TRSCat}_{\mathsf{cc}}(\mathbf{2})$.)

(c) A similar remark applies to the structures enriched in the extended non-negative reals (see p. 12). If $\mathbb{A}$ is a totally regular semicategory enriched in $\mathbb{R}^+ \cup \{\infty\}$, and $a \in \mathbb{A}_0$ is an object for which $0 \not\geq \mathbb{A}(a, a)$, then necessarily

$$\text{for all } a' \in \mathbb{A}_0 : \mathbb{A}(a', a) = \infty = \mathbb{A}(a, a').$$

That is to say, a is an "isolated" object ("infinitely far" from every other object—even from itself!). But $\mathbb{A}$ is Morita-equivalent to its full subcategory determined by the non-isolated objects.

(d) About the $\mathcal{Q}_R$-enriched category Comm_R determined by some ring R, introduced on p. 13. On the one hand this $\mathcal{Q}_R$-category may be Cauchy completed *as category*, but on the other hand we may consider it to be a totally regular $\mathcal{Q}_R$-semicategory, and Cauchy complete it *as semicategory*. The outcomes of both procedures are different! In the case of a commutative ring R, [Ambler and Verity, 1996] argue – in a setting that we will shortly prove to be equivalent to ours – that precisely the Cauchy completion of Comm_R as totally regular semicategory gives the correct "(generic) sheaf representation of R over $\mathcal{Q}_R$"; their paper contains the result that $(\mathsf{Comm}_R)_{\mathsf{cc}}$ is a "ring-object" in $\mathsf{TRSCat}_{\mathsf{cc}}(\mathcal{Q}_R)$. The generalization to a non-commutative R has not yet been studied in this fashion (and it must be noted that Van den Bossche's [1995] result is quite different in nature from Ambler and Verity's [1996] work!).

The most important example – that which motivated us to consider semicategories in the first place – deserves a subsection of its own: totally regular semicategories enriched in a locale Ω.

2.6. Finally, the answer to the initial question. We are now – finally! – at the point where we can answer our initial question: "what precisely are those ordered objects in $\mathsf{Sh}(\Omega)$ as Borceux and Cruciani [1998] describe them?"

We observed that the ordered objects in $\mathsf{Sh}(\Omega)$ are those semicategories enriched in (the one-object suspension of) the locale Ω that satisfy two extra ("mysterious") conditions (55) and (56). But these conditions can be rewritten as

$$[a \leq a] \wedge [a \leq a'] = [a \leq a'], \tag{90}$$

$$[a \leq a'] \wedge [a' \leq a'] = [a \leq a']. \tag{91}$$

And then it is clear that an "Ω-order" is exactly a totally regular Ω-semicategory! Further, the "morphisms between $\mathcal{Q}$-orders" of which Borceux and Cruciani [1998] speak, are exactly left adjoint regular semidistributors between totally regular semicategories. So the (2-)category that they prove to be equivalent to $\mathsf{Ord}(\mathsf{Sh}(\Omega))$ is precisely what we denote by $\mathsf{Map}(\mathsf{TRSDist}(\Omega))$.

The proof of this result relies totally on the theory of Cauchy complete totally regular Ω-semicategories. In fact,

- F. Borceux and R. Cruciani first define what we now recognize to be the category $\mathsf{Map}(\mathsf{TRSDist}(\Omega))$;

- then they prove that "every Ω-order is isomorphic to a complete Ω-order", i.e. every totally regular semicategory is Morita equivalent to its Cauchy completion, so it follows that $\mathsf{Map}(\mathsf{TRSDist}(\Omega)) \simeq \mathsf{Map}(\mathsf{TRSDist}_{\mathsf{cc}}(\Omega))$;

- the authors then show that "for those complete Ω-orders the morphisms are equivalently given by certain object-mappings", which is precisely the equivalence $\mathsf{Map}(\mathsf{TRSDist}_{\mathsf{cc}}(\Omega)) \simeq \mathsf{TRSCat}_{\mathsf{cc}}(\Omega)$;

- and finally, by a direct argument, $\mathsf{TRSCat}_{\mathsf{cc}}(\Omega) \simeq \mathsf{Ord}(\mathsf{Sh}(\Omega))$.

Interpreting (87) in this context we have explicitly that

$$(-)_{\mathsf{cc}} \dashv i \colon \mathsf{TRSCat}_{\mathsf{cc}}(\Omega) \leftrightarrows \mathsf{TRSCat}(\Omega)$$

behaves like a "sheafification": the objects of $\mathsf{TRSCat}(\Omega)$ are to be understood as "ordered presheaves on Ω", the Cauchy complete such semicategories are "ordered sheaves", and the Cauchy completion of a semicategory is then like the sheafification of a presheaf.

Encouraged by this example, we now propose to define the "quantaloid of $\mathcal{Q}$-orders and ideals" precisely as

$$\mathsf{Idl}(\mathcal{Q}) := \mathsf{TRSDist}_{\mathsf{cc}}(\mathcal{Q}), \tag{92}$$

and the "locally (pre)ordered category of $\mathcal{Q}$-orders and order-preserving maps" as

$$\mathsf{Ord}(\mathcal{Q}) := \mathsf{TRSCat}_{\mathsf{cc}}(\mathcal{Q}). \tag{93}$$

That is to say, Cauchy complete totally regular $\mathcal{Q}$-semicategories are "ordered sheaves on $\mathcal{Q}$", a regular semidistributor is an "ideal relation" between ordered sheaves, and a regular semifunctor an "order-preserving map". It is then a result that

$$\mathsf{Ord}(\mathcal{Q}) = \mathsf{Map}(\mathsf{Idl}(\mathcal{Q})),$$

i.e. that "left adjoint ideals are (the graphs of) order-preserving maps".

At this point we must make a comment on categorical structures enriched in $\mathbf{2}$. We have argued earlier that $\mathbf{2}$-*categories* are (pre)ordered sets (and functors are order-preserving maps and distributors are ideal relations). But the statements above imply that we should rather take the *Cauchy complete totally regular* $\mathbf{2}$-*semicategories* (and the regular semifunctors and semidistributors) to be the correct understanding of preorders in this context. So which of both is it? Well, as indicated in an example on p. 41, it is "accidentally" so that every totally regular $\mathbf{2}$-semicategory is Morita-equivalent to a true and honest $\mathbf{2}$-category; and moreover every $\mathbf{2}$-category is Cauchy complete. Therefore the possible confusion is easily solved:

$$\mathsf{Ord}(\mathbf{2}) \simeq \mathsf{Cat}(\mathbf{2}) \text{ and } \mathsf{Idl}(\mathbf{2}) \simeq \mathsf{Dist}(\mathbf{2}).$$

(And for similar reasons, Cauchy complete generalized metric spaces are precisely "orders over the quantale of extended reals".)

From now on we use "$\mathcal{Q}$-order" as synonym for "Cauchy complete totally regular $\mathcal{Q}$-semicategory" (which is quite a mouthful).

2.7. Restrictions and gluing. To support the idea that $\mathcal{Q}$-orders are indeed "sheaves", let us indicate in what sense a "family of pairwise compatible elements admits a unique gluing".

Let $\mathbb{A}$ denote a totally regular $\mathcal{Q}$-semicategory. Thinking of $\mathbb{A}$ as a "sheaf" on $\mathcal{Q}$, we read a hom-arrow like $\mathbb{A}(a', a)$ as the "greatest level at which a is smaller than a'". In particular, an object $a \in \mathbb{A}_0$ is a "section of $\mathbb{A}$ over $\mathbb{A}(a, a)$". (This is entirely consistent with the case of Ω-sets: thinking of such an $(A, [\,\cdot = \cdot\,])$ as a sheaf on Ω,

$[a = a]$ is precisely the element of Ω over which a is a section.) Note that (81) says in particular that any $\mathbb{A}(a,a)$ is an idempotent, and any $\mathbb{A}(a',a)$ a regular bimodule between such idempotents, in $\mathcal{Q}$. We thus have to try to generalize such things as restrictions, coverings, compatible families and gluings – which are well known for sheaves on a locale Ω – to this situation where idempotents in $\mathcal{Q}$ seem to play the rôle that the elements of a locale Ω play in the theory of sheaves on Ω.

Whenever for two idempotents $A \circlearrowleft e$ and $A \circlearrowleft f$ in $\mathcal{Q}$ there exists a left adjoint regular bimodule $x\colon e \multimap f$ (with right adjoint denoted $x^*\colon f \multimap e$) that expresses e as a section of f in $\mathsf{Idm}(\mathcal{Q})$ (i.e. $x^* \otimes_f x = e$ and $x \otimes_e x^* \leq f$), then we think of $x\colon e \multimap f$ as a "restriction of idempotents in $\mathcal{Q}$". Suppose now that for an object $a \in \mathbb{A}_0$ in a totally regular $\mathcal{Q}$-semicategory $\mathbb{A}$, $x\colon e \multimap \mathbb{A}(a,a)$ is such a restriction of idempotents. The composition of left adjoint regular semidistributors

$$*_e \xrightarrow{(x)} *_{\mathbb{A}(a,a)} \xrightarrow{\mathbb{A}(-,a)} \mathbb{A}$$

(where (x) denotes the one-element semidistributor between one-object semicategories in the obvious way) gives a new left adjoint in $\mathsf{TRSDist}(\mathcal{Q})$. If this semidistributor converges, then the object ("picked out" by the regular semifunctor) to which it converges (i.e. the representing object for this left adjoint) is the "restriction of a through $x\colon e \multimap \mathbb{A}(a,a)$". Clearly, if $\mathbb{A}$ is Cauchy complete then all such restrictions exist.

Suppose now that for some idempotent $A \circlearrowleft e$ in $\mathcal{Q}$ there is a family of objects $(a_i)_{i \in I}$ in $\mathbb{A}$ for which

(i) for each $i \in I$ there is a restriction of idempotents $x_i\colon \mathbb{A}(a_i,a_i) \multimap e$ such that $e = \bigvee_{i \in I} x_i \circ x_i^*$, and

(ii) for each $i,j \in I$, $x_j^* \circ x_i \leq \mathbb{A}(a_j,a_i)$.

We may then think of the $\mathbb{A}(a_i,a_i)$'s as a "covering" of e (because of the first sentence above), and the a_i's as a "family of pairwise compatible elements" (because of the second sentence). A "gluing" of such a family is then – whenever it exists – an object $a \in \mathbb{A}_0$ such that $\mathbb{A}(a,a) = e$ and the restriction of this a through each of the $x_i\colon \mathbb{A}(a_i,a_i) \multimap \mathbb{A}(a,a)$ (exists and) equals precisely the corresponding a_i—see Diagram 7.

Another way of expressing this, is as follows: let $\mathbb{E}$ denote the $\mathcal{Q}$-enriched structure with

- objects: $\mathbb{E}_0 = I$ and types $ti = A$ (for all $i \in I$),

- hom-arrows: for $i,j \in I$, $\mathbb{E}(j,i) = x_j^* \circ x_i$.

Due to condition (i) above, $\mathbb{E}$ is a totally regular semicategory which is Morita-equivalent to the one-object totally regular semicategory $*_e$: the semidistributor $\Theta\colon *_e \multimap \mathbb{E}$ with elements $\Theta(i,*) = x_i$ has as inverse $\Theta^{-1}\colon \mathbb{E} \multimap *_e$ with elements

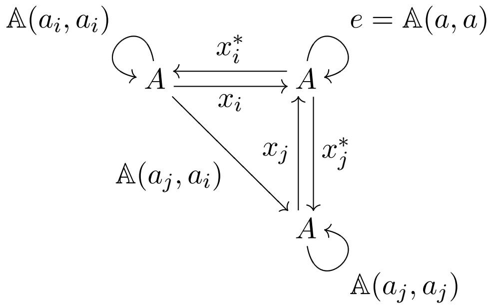

Diagram 7: a "covering"

$\Theta^{-1}(*,i)=x_i^*$. Due to condition (ii) the object-mapping

$$F\colon \mathbb{E}_0 \longrightarrow \mathbb{A}_0\colon i \mapsto Fi := a_i$$

is a regular semifunctor; it induces a left adjoint regular semidistributor $\mathbb{B}(-,F-)$. The composition of the left adjoints

$$*_e \xrightarrow{\ \Theta\ } \mathbb{E} \xrightarrow{\ \mathbb{B}(-,F-)\ } \mathbb{A}$$

yields a new left adjoint. If it is representable (i.e. if it converges) then the representing object is the gluing of the compatible family $(a_i)_i$. So for a Cauchy complete $\mathbb{A}$ all families of pairwise compatible objects admit an essentially unique gluing!

These definitions of "restriction", "covering", "gluing" etc. can of course be applied to Ω-sets (viewed as totally regular Ω-enriched semicategories): they then coincide with the usual notions for the Ω-sets (viewed as sheaves on Ω).

2.8. Cauchy completion of, and enriching in, a base quantaloid. Concerning the calculus of regular $\mathcal{Q}$-semicategories and regular semidistributors, i.e. the quantaloid $\mathsf{RSDist}(\mathcal{Q})$, it requires a simple calculation – using the definitions (27) and (69), and the universal properties of the matrix-construction, the idempotent-construction, and the monad-construction – to see that

$$\mathsf{RSDist}(\mathcal{Q}) \simeq \mathsf{Dist}(\mathsf{RSDist}(\mathcal{Q})). \tag{94}$$

That is to say, "regular semicategories are categories"... over a different base quantaloid[28]!

It is natural to ask wether such a property also holds for the totally regular $\mathcal{Q}$-semicategories—and the answer to that question is affirmative! To see what is

[28]It would however be too easy to do away with the theory of regular semicategories altogether, because clearly $\mathsf{Cat}(\mathsf{RSDist}(\mathcal{Q})) \not\simeq \mathsf{RSCat}(\mathcal{Q})$.

happening it helps to first underline the important rôle that idempotents and regular bimodules in $\mathcal{Q}$ play in definitions (81) and (82):

- a $\mathcal{Q}$-semicategory is totally regular if and only if every endo-hom-arrow is an idempotent and every hom-arrow is a regular bimodule between these idempotents;

- a semidistributor between totally regular semicategories is regular if and only if every element of the semidistributor is a regular bimodule between the endo-hom-arrows.

As idempotents and regular bimodules in $\mathcal{Q}$ are themselves the objects and arrows of the quantaloid $\mathsf{Idm}(\mathcal{Q})$ (see p. 33), it is now maybe not too surprising that totally regular $\mathcal{Q}$-semicategories can be described in terms of an enrichment in $\mathsf{Idm}(\mathcal{Q})$.

Indeed, let $\mathbb{A}$ be a totally regular $\mathcal{Q}$-semicategory, define then an $\mathsf{Idm}(\mathcal{Q})$-category $\widehat{\mathbb{A}}$ as follows: the object-set is $(\widehat{\mathbb{A}})_0 = \mathbb{A}_0$ but the types of the objects are now $\hat{t}a = \mathbb{A}(a,a)$ (which is indeed an object of $\mathsf{Idm}(\mathcal{Q})$); and for the hom-arrows put $\widehat{\mathbb{A}}(a',a) = \mathbb{A}(a',a)$ in $\mathcal{Q}$ (which is indeed an arrow in $\mathsf{Idm}(\mathcal{Q})$ between $\hat{t}a$ and $\hat{t}a'$). Similarly, for a regular semidistributor between totally regular semicategories like $\Phi\colon \mathbb{A} \mathrel{-\!\!\circ\!\!\to} \mathbb{B}$ it is easily seen that $\widehat{\Phi}(b,a) = \Phi(b,a)$ defines a distributor $\widehat{\Phi}\colon \widehat{\mathbb{A}} \mathrel{-\!\!\circ\!\!\to} \widehat{\mathbb{B}}$ between $\mathsf{Idm}(\mathcal{Q})$-categories. This defines a fully faithful homomorphism of quantaloids

$$(\widehat{-})\colon \mathsf{TRSDist}(\mathcal{Q}) \longrightarrow \mathsf{Dist}(\mathsf{Idm}(\mathcal{Q})).$$

This homomorphism proves to be essentially surjective on objects[29], so it is an equivalence of quantaloids:

$$\mathsf{TRSDist}(\mathcal{Q}) \simeq \mathsf{Dist}(\mathsf{Idm}(\mathcal{Q})). \tag{95}$$

Therefore, "totally regular semicategories are categories"... but obviously over a quite different base quantaloid[30]!

On p. 33 we mentioned that $\mathsf{Idm}(\mathcal{Q})$ is the quantaloid which is the Cauchy completion of $\mathcal{Q}$ *as category*, and that $\mathsf{RSDist}(\mathcal{Q})$ is the Cauchy completion of $\mathcal{Q}$ *as quantaloid*. And of course there are full embeddings

$$\mathcal{Q} \longrightarrow \mathsf{Idm}(\mathcal{Q}) \longrightarrow \mathsf{RSDist}(\mathcal{Q}).$$

To stress that these Cauchy completions of $\mathcal{Q}$ (be it as category or as quantaloid) are only determined up to equivalence, we may rather use notations

$$\mathcal{Q} \longrightarrow \mathcal{Q}_{\mathsf{cc}} \longrightarrow \mathcal{Q}_{\mathsf{CC}}$$

[29] This is elementary, but depends on the fact that monads split in the base quantaloid $\mathsf{Idm}(\mathcal{Q})$: the point is to prove that every $\mathsf{Idm}(\mathcal{Q})$-category is Morita-equivalent to one whose endo-hom-arrows are identities.

[30] It follows that $\mathsf{TRSCat}_{\mathsf{cc}}(\mathcal{Q}) \simeq \mathsf{Cat}_{\mathsf{cc}}(\mathsf{Idm}(\mathcal{Q}))$ are biequivalent locally (pre)ordered categories. But again it must be noted that $\mathsf{TRSCat}(\mathcal{Q}) \not\simeq \mathsf{Cat}(\mathsf{Idm}(\mathcal{Q}))$, which underlines the importance of 'Morita equivalence' and 'Cauchy completions'.

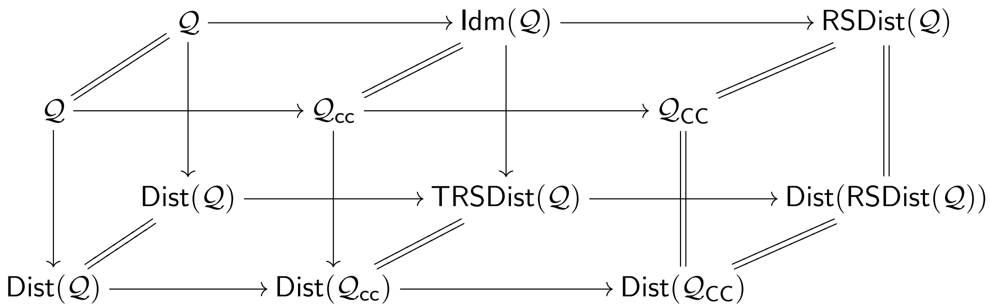

Diagram 8: comparing various enrichments

for any choice of Cauchy completion of $\mathcal{Q}$ as category, resp. as quantaloid. Enriching in these quantaloids then gives rise to embeddings

$$\mathsf{Dist}(\mathcal{Q}) \longrightarrow \mathsf{Dist}(\mathcal{Q}_{\mathsf{cc}}) \longrightarrow \mathsf{Dist}(\mathcal{Q}_{\mathsf{CC}})$$

and in particular can we now bring the various base quantaloids and enrichments together in one big Diagram 8 in which all arrows are quantaloid homomorphisms, and the equalities should be read as equivalences. In this diagram, on the foreground, in the top row $\mathcal{Q}$ gets Cauchy completed and in the bottom row $\mathcal{Q}$ and its Cauchy completions are the base for enriched categories and distributors. In the background we then have a restatement of this scheme "in elementary terms".

As a result, we may now state several equivalent expressions for the quantaloid of $\mathcal{Q}$-orders and ideals, cf. (92):

$$\mathsf{Idl}(\mathcal{Q}) \simeq \mathsf{Dist}_{\mathsf{cc}}(\mathcal{Q}_{\mathsf{cc}}) \simeq \mathsf{Dist}(\mathcal{Q}_{\mathsf{cc}}).$$

By taking left adjoints one obtains as many equivalent expressions for $\mathsf{Ord}(\mathcal{Q})$.

The observation made here actually bridges the gap between the description of (ordered) sheaves on a locale Ω of, on the one hand, [Borceux and Cruciani, 1998] (see p. 26 and 42), and on the other hand [Walters, 1981]. Indeed, B. Walters identified sheaves on Ω as symmetric[31] Cauchy complete categories enriched in – as he described it – $\mathsf{Rel}(\Omega)$, leaving it understood that a (not-necessarily symmetric) $\mathsf{Rel}(\Omega)$-category should be thought of as an "ordered sheaf" on Ω. But $\mathsf{Rel}(\Omega)$ is precisely the split-idempotent completion of Ω, because every element of Ω is idempotent of course, so Walters is working within $\mathsf{Cat}_{\mathsf{cc}}(\Omega_{\mathsf{cc}})$. This latter 2-category is up to equivalence precisely $\mathsf{TRSCat}_{\mathsf{cc}}(\Omega)$, i.e. what we have shown to be Borceux and Cruciani's description of ordered sheaves on Ω.

[31] The symmetry just means that in such an enriched category $\mathbb{A}$ one has for every pair of objects a, a' that $\mathbb{A}(a', a) = \mathbb{A}(a, a')$—which makes sense over $\mathsf{Rel}(\Omega)$ but not in general, see further.

2.9. Symmetric orders: towards sets. Having understood what "orders over a base quantaloid" are, it would be interesting to know more about "symmetric orders", i.e. "sets over a base quantaloid". Without digging deeply in this matter (which is not really part of this thesis) we can still try to outline the subject for future study.

First of all, let $\mathbb{A}$ be a $\mathcal{Q}$-order for some quantaloid $\mathcal{Q}$. For this order to be "symmetric" it would have to satisfy the condition on its hom-arrows that

$$\text{for all } a, a' \in \mathbb{A}_0 : \mathbb{A}(a', a) = \mathbb{A}(a, a')...$$

which clearly doesn't make sense in general: $\mathbb{A}(a', a)$ and $\mathbb{A}(a, a')$ are arrows in the base quantaloid $\mathcal{Q}$ that go in opposite directions! Therefore we will need an extra ingredient that allows us to compare arrows in $\mathcal{Q}$ that go in opposite directions: an involution on $\mathcal{Q}$.

An involution on a quantaloid $\mathcal{Q}$ is a homomorphism $\tau\colon \mathcal{Q}^{\mathsf{op}} \longrightarrow \mathcal{Q}$ which is the identity on objects and its own inverse on arrows. For simplicity we write $f^\tau\colon B \longrightarrow A$ for the effect of τ on some $\mathcal{Q}$-arrow $f\colon A \longrightarrow B$, and read f^τ as the "transpose" of f. Having a quantaloid with involution $(\mathcal{Q}, \tau)$ we may now define a 'τ-symmetric $\mathcal{Q}$-order' $\mathbb{A}$ as a $\mathcal{Q}$-order for which

$$\text{for all } a, a' \in \mathbb{A}_0 : \mathbb{A}(a', a) = \mathbb{A}(a, a')^\tau. \tag{96}$$

As a particular example note that for a locale Ω the identity id_Ω is an involution (viewing Ω as a one-object quantaloid!), and that the id_Ω-symmetric Ω-orders are exactly the Ω-sets. Therefore we speak in general of a '$(\mathcal{Q}, \tau)$-set' instead of a 'τ-symmetric $\mathcal{Q}$-order'.

More generally, if a quantaloid $\mathcal{Q}$ is endowed with an involution τ, then the quantaloid $\mathsf{Matr}(\mathcal{Q})$ of $\mathcal{Q}_0$-typed sets and matrices with elements in $\mathcal{Q}$ (see p. 14) inherits this involution: for a matrix $\mathbb{M}\colon (X, t) \longrightarrow (Y, t)$ one simply puts $\mathbb{M}^\tau\colon (Y, t) \longrightarrow (X, t)$ to be the matrix whose elements are $\mathbb{M}^\tau(x, y) := \mathbb{M}(y, x)^\tau$. A matrix $\mathbb{M}$ is then said to be 'τ-symmetric' whenever $\mathbb{M}^\tau = \mathbb{M}$—in which case $\mathbb{M}$ has to be an endo-matrix. It is then obvious that each $\mathcal{Q}$-enriched structure (category, semicategory, etc.) has a "τ-symmetric variant". It would thus be interesting to understand how those "τ-symmetric variants" organize themselves in substructures of $\mathsf{Dist}(\mathcal{Q})$, $\mathsf{Cat}(\mathcal{Q})$, $\mathsf{RSDist}(\mathcal{Q})$ and so on, and what (universal) properties those new structures have.

Of course, of particular interest are the τ-symmetric totally regular $\mathcal{Q}$-semicategories, i.e. the $(\mathcal{Q}, \tau)$-sets. It is noteworthy that for such a $(\mathcal{Q}, \tau)$-set $\mathbb{A}$, every endo-hom-arrow is a τ-symmetric idempotent in $\mathcal{Q}$. Recalling that a $\mathcal{Q}$-order can be seen as a $\mathcal{Q}_{\mathsf{cc}}$-enriched category (cf. p. 46), it thus seems that the τ-symmetric ones can be seen as τ-symmetric $\mathcal{Q}_{\tau\mathsf{cc}}$-enriched categories, where by $\mathcal{Q}_{\tau\mathsf{cc}}$ we mean the universal splitting of τ-symmetric idempotents in $\mathcal{Q}$ (as opposed to $\mathcal{Q}_{\mathsf{cc}}$ which is the universal splitting of *all* idempotents).

Another point that should be cleared out is: what is the appropriate Cauchy completion of a $(\mathcal{Q},\tau)$-set? This is not as obvious as it may seem at first sight, because the Cauchy completion of a τ-symmetric totally regular $\mathcal{Q}$-semicategory as explained on p. 39 and further is not necessarily τ-symmetric[32]! But it may be observed that for a regular semifunctor $F\colon\mathbb{A}\longrightarrow\mathbb{B}$ between two $(\mathcal{Q},\tau)$-sets, $\mathbb{B}(-,F-)^\tau=\mathbb{B}(F-,-)$ (due to the τ-symmetry of $\mathbb{B}$): the transpose of the left adjoint semidistributor induced by F is actually its right adjoint. It therefore seems reasonable to think that the Cauchy completion of a $(\mathcal{Q},\tau)$-set $\mathbb{A}$ should not classify *all* left adjoint regular semidistributors into $\mathbb{A}$, but only those regular semidistributors that are left adjoint to their transpose.

Ideally these considerations should lead to full substructures

$$\mathsf{TRSDist}(\mathcal{Q},\tau)\longrightarrow\mathsf{TRSDist}(\mathcal{Q})\quad\text{and}\quad\mathsf{TRSCat}(\mathcal{Q},\tau)\longrightarrow\mathsf{TRSCat}(\mathcal{Q})$$

determined by the τ-symmetric objects; and there should be equivalences

$$\mathsf{TRSCat}_{\mathsf{cc}}(\mathcal{Q},\tau)\simeq\mathsf{Map}_\tau(\mathsf{TRSDist}_{\mathsf{cc}}(\mathcal{Q},\tau))\simeq\mathsf{Map}_\tau(\mathsf{TRSDist}(\mathcal{Q},\tau))$$

where by "Map_τ" is meant: taking those left adjoints whose right adjoint is their transpose. It would then make sense to define "sets, relations and functions over a base quantaloid endowed with an involution" as

$$\mathsf{Rel}(\mathcal{Q},\tau):=\mathsf{TRSDist}_{\mathsf{cc}}(\mathcal{Q},\tau)\quad\text{and}\quad\mathsf{Set}(\mathcal{Q},\tau):=\mathsf{Map}_\tau(\mathsf{Rel}(\mathcal{Q},\tau)).$$

And it would then seem that $\mathsf{Rel}(\mathcal{Q},\tau)$ is a full subquantaloid of $\mathsf{Idl}(\mathcal{Q})$, but $\mathsf{Set}(\mathcal{Q},\tau)$ would not be a full subcategory of $\mathsf{Ord}(\mathcal{Q})$.

[32] A similar observation holds for τ-symmetric $\mathcal{Q}$-categories [Betti and Walters, 1982; Labella and Schmitt, 2002].

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