QUANTALOIDS DESCRIBING CAUSATION AND PROPAGATION OF PHYSICAL PROPERTIES

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A general principle of 'causal duality' for physical systems, lying at the base of representation theorems for both compound and evolving systems, is proved; formally it is encoded in a quantaloidal setting. Other particular examples of quantaloids and quantaloidal morphisms appear naturally within this setting; as in the case of causal duality, they originate from primitive physical reasonings on the lattices of properties of physical systems. Furthermore, an essentially dynamical operational foundation for studying physical systems is outlined; complementary as it is to the existing static operational foundation, it leads to the natural axiomatization of 'causal duality' in operational quantum logic.

Key words: causal duality, property lattice, galois adjoint, quantaloid.

1. INTRODUCTION

The starting point for our research program is the fact, already observed in Eilenberg and Mac Lane's seminal paper [12], that preordered

1

sets may be considered as small thin categories. One can then not only reformulate a large part of the theory of order structures in categorical terms, but also apply general categorical techniques to specific order theoretic problems. In particular, the notion of an adjunction reduces to that of a residuation [19] §4.5, whereas the notion of a monad reduces to that of a closure operator [19] §6.1-2. Now the above categorical notions have direct physical interpretations in the context of axiomatic quantum theory, the order relation in the property lattice being semantic implication and the meet being operational conjunction [1,2,18,25,26,27]. In particular, the equivalence between suitable categories of closure spaces and complete lattices determined by the existence of monadic comparison functors manifests the primitive duality between the state and property descriptions of a physical system [8,20,21,22]. Further, this static approach can be dynamically generalized by interpreting morphisms as transition structures [3,9,10,11], thereby providing an explicit physical realization of enrichment. Finally, far from being of merely aesthetic interest, the categorical approach to operational quantum theory allows the recovery of concrete representations of abstract notions in the Hilbertian context via the fundamental theorems of projective geometry [14,15,16]. In our opinion, then, category theory is as much a tool for the theoretical physicist as for the working mathematician.

In this paper we derive a physical principle to which we refer as 'causal duality'. Explicitly, we shall present a common extension of the representation theories of deterministic flows [17] and compound systems [6] to the dual notions of causation and propagation, construed as a physical polarity in the property lattice, and that lifts to a quantaloidal duality. Further on, we derive other examples of quantaloids that emerge naturally in this setting, thus providing an essentially dynamical operational foundation for studying physical systems — the study of such dynamics goes back to [20], the approach of which was then conceptualized in [3,10] by means of so-called *inductions*, and here we go one step further by explicitly imposing (or axiomatizing) 'causal duality'.

Quantales, first introduced in [23], are complete lattices $(L, \leq)$ equipped with a not-necessarily commutative binary operation (a "multiplication" of elements of L) that distributes on both sides over arbitrary suprema; a frame is a quantale in which the "multiplication" coincides with the binary infimum. A quantaloid is a category $\underline{Q}$ in which every hom-set is a complete lattice and where composition of morphisms distributes on both sides over arbitrary suprema of morphisms; thus a quantale is precisely a quantaloid with one object. The pertinent functors between two quantaloids, called quantaloidal morphisms, are of course those functors that preserve suprema of morphisms. For a survey of the theory of quantales and quantaloids we refer to [24,28,29]; for a quick introduction to (enriched) category theory – of which the theory of quantaloids is an instance – we refer to

[5]; as minimal preliminaries to this paper the appendices of [3,10] may suffice.

Let us now recall some of the basic notions of the operational approach to physics [1,26,27], and fix some notation. Given a (well defined) *physical system* Ξ we define a *test* α as a real experimental procedure relative to the system where we have defined in advance the so-called positive response. We call such a test *certain* for a particular realization of Ξ iff we *would* obtain the positive response *should* we perform the experiment. For two such tests α and β relative to Ξ, we set $\alpha \preceq \beta$ iff β is certain whenever α is certain. The expression $\alpha \preceq \beta$ then reveals a *physical law* for Ξ, and of course $\preceq$ defines a preorder on $\mathcal{T}(\Xi)$, the collection of all possible tests for Ξ. By standard quotienting techniques we can now work with the collection $L(\Xi)$ of equivalence classes of tests, two tests α and β being equivalent iff both $\alpha \preceq \beta$ and $\beta \preceq \alpha$ (which will be denoted as $\alpha \approx \beta$), that then comes equipped with a partial order $\leq$ derived from the preorder $\preceq$. The key point of this setup is that to any such equivalence class $[\alpha]$ corresponds an "element of physical reality" [13], called a *property* of Ξ [26]. If a test α is certain for a particular realization of the physical system Ξ, then the corresponding property $a = [\alpha]$ is said to be *actual* for this realization — otherwise it is *potential*. Under the (working) hypothesis that $L(\Xi)$ constitutes a set – although in principle all the following holds also for it being a thin category – it can then be proved that $(L(\Xi), \leq)$ is a complete lattice: given a subcollection $\mathcal{A}$ of $\mathcal{T}(\Xi)$, defining $\prod \mathcal{A}$ as "choose any α in $\mathcal{A}$ as you wish and effectuate it", provides $L(\Xi)$ with a meet induced by $\bigwedge \{[\alpha] | \alpha \in \mathcal{A}\} = [\prod \mathcal{A}]$. By its construction, this meet is a physical *conjunction* (an aspect to which we will refer to as [con]) — but the corresponding join has no *a priori* physical significance, so it cannot be treated as a disjunction, *e.g.* orthodox quantum mechanics. It also clearly follows that $a \leq b$ in $L(\Xi)$ can be treated as an *implication relation* (referred to as [imp]), where we say that a is *stronger* than b. Finally, for each particular realization of a system Ξ, we can write ε for the subset of $L(\Xi)$ that contains precisely all the properties that are actual for this particular realization. As any such ε is a complete co-ideal, *i.e.*, closed under meets [con] and upperbounds [imp], it can be characterized by its strongest element $p_\varepsilon = \bigwedge \varepsilon \in L(\Xi)$. Therefore, for each realization of Ξ there exists a strongest actual property, which is appropriately called *state* of the system [26].

2. CATEGORICAL DUALITY INDUCED BY CAUSALITY

In this section we aim to give a common extension of the operational theory of, on the one hand, deterministic flows [17] and, on the other, compound systems [6]; the result of our analysis will be that the deeper structural ingredient in both situations is that "causation is adjoint to

propagation".

Considering an evolving physical system Ξ, any test α relative to the system at time t_2 defines a test $\phi(\alpha)$ relative to the system at an earlier time t_1 as "evolve Ξ from t_1 to t_2 and effectuate α". The property $[\phi(\alpha)]$ has a clear interpretation, namely "guaranteeing actuality of $[\alpha]$". The assignment $L(\Xi) \to L(\Xi) : [\alpha] \mapsto [\phi(\alpha)]$, as we will see below, describes the evolution of Ξ. On the other hand, considering two interacting physical systems Ξ_1 and Ξ_2, any test α_2 on Ξ_2 defines a test $\phi(\alpha_2)$ on Ξ_1 by "let the systems interact and effectuate α_2". The assignment $L(\Xi_2) \to L(\Xi_1) : [\alpha_2] \mapsto [\phi(\alpha_2)]$ now encodes the interaction of Ξ_1 on Ξ_2.

Keeping these two cases in mind, for any two property lattices L_1 and L_2 we dispose of a *causal relation* $\leadsto \subseteq L_1 \times L_2$ where:

$$a_1 \leadsto a_2 \Leftrightarrow \text{"actuality of } a_1 \text{ guarantees actuality of } a_2\text{"}. \tag{1}$$

Lemma 1. *By the operational significance of $\leadsto$ the following holds:*

$$b_1 \leq a_1, a_1 \leadsto a_2, a_2 \leq b_2 \quad \Rightarrow \quad b_1 \leadsto b_2 \tag{2}$$

$$\forall a_2 \in A_2 : a_1 \leadsto a_2 \quad \Rightarrow \quad a_1 \leadsto \bigwedge A_2 \tag{3}$$

where A_2 is a non-empty subset of L_2.

Proof: The proof of eq.(2) relies on [imp], eq.(3) follows by [con]. $\square$

(From an axiomatic point of view the conditions in the previous lemma are 'axioms' for a causal relation.)

Now consider the following map prescription:

$$f^* : L_1 \setminus K \to L_2 : a_1 \mapsto \bigwedge \{a_2 \in L_2 \mid a_1 \leadsto a_2\} \tag{4}$$

with $K = \{a_1 \in L_1 \mid \nexists a_2 \in L_2 : a_1 \leadsto a_2\}$, as such avoiding non-empty meets.

Lemma 2. *By lemma 1 and the explicit definition of f^* we have:*

$$a_1 \leq a_1' \quad \Rightarrow \quad f^*(a_1) \leq f^*(a_1') \tag{5}$$

$$a_1 \leadsto a_2 \quad \Leftrightarrow \quad f^*(a_1) \leq a_2 \tag{6}$$

where it is understood that $a_1' \notin K$.

Proof: For eq.(5), remark that $a_1' \notin K \Rightarrow a_1 \notin K$ by lemma 1, so both $f^*(a_1)$ and $f^*(a_1')$ are defined; then computation shows that indeed $f^*(a_1) \leq f^*(a_1')$. In eq.(6), the sufficiency is trivial; to prove necessity is, by [imp], to prove that $a_1 \leadsto f^*(a_1)$, which is true by [con]. $\square$

4

Now it is clear that $f^*(a_1)$ is the strongest property of L_2 the actuality of which is guaranteed by the actuality of a_1, i.e., f^* describes the *propagation of (strongest actual) properties*. Next, set:

$$f_* : L_2 \to L_1 : a_2 \mapsto \bigvee \{a_1 \in L_1 \mid a_1 \rightsquigarrow a_2\}. \tag{7}$$

Lemma 3. *By eq.(2) and the explicit definition of f_* we have:*

$$a_2 \le a_2' \quad \Rightarrow \quad f_*(a_2) \le f_*(a_2') \tag{8}$$
$$a_1 \rightsquigarrow a_2 \quad \Rightarrow \quad a_1 \le f_*(a_2) \tag{9}$$

Proof: By computation. $\square$

If moreover the condition $1_1 \rightsquigarrow 1_2$ can be derived from the physical particularity of the system under consideration (or formally, if it is an 'axiom' on $\rightsquigarrow$), then $K = \emptyset$ in eq.(4), and thus:

$$f^*(a_1) \le a_2 \Leftrightarrow a_1 \rightsquigarrow a_2 \Rightarrow a_1 \le f_*(a_2) \tag{10}$$

so it remains to show that eq.(11) below is valid to obtain adjointness of f^* and f_*.

Lemma 4. *By the operational significance of f_* (via that of $\rightsquigarrow$) we have:*

$$a_1 \le f_*(a_2) \quad \Rightarrow \quad a_1 \rightsquigarrow a_2. \tag{11}$$

Proof: Since $[\phi(\alpha_2)] \rightsquigarrow [\alpha_2]$ by the definition of ϕ, and since $a_1 \rightsquigarrow a_2$ implies that $a_1 \le [\phi(\alpha_2)]$ we obtain that $f_*([\alpha_2]) = [\phi(\alpha_2)]$. Since eq.(11) is equivalent to $f_*(a_2) \rightsquigarrow a_2$ this completes the proof. $\square$

(Note that formally eq.(11) is an additional 'axiom' on $\rightsquigarrow$.) Physically, lemma 4 states that there exists a well defined "weakest cause" $f_*(a_2)$ in L_1 of any a_2 in L_2, so f_* describes the *assignment of (weakest) causes (for actuality)*. We can now read that f^* is left adjoint to f_* (denoted as $f^* \dashv f_*$) or, in words, that "propagation is adjoint to causation". By general theory on adjoint pairs of morphisms (see for example [8]) we have the following.

Corollary 1. *The propagation $f^* : L_1 \to L_2$ is a join preserving map whereas the causation $f_* : L_2 \to L_1$ is a meet preserving map.*

In case that $1_1 \not\rightsquigarrow 1_2$, one can always extend the domain and codomain of f^* and f_* to the upper pointed extensions $L_1 \dot\cup \underline{1}$ and $L_2 \dot\cup \underline{1}$ of L_1 and L_2, obtained by freely adjoining a new "top" element, and then put $f_*(\underline{1}) = \underline{1}$, $f^*(\underline{1}) = \underline{1}$ and $\forall a_1 \in K : f^*(a_1) = \underline{1}$. Physically,

an interpretation of $\underline{1}$ follows from that of 1_1 and 1_2, respectively being existence of Ξ_1 and Ξ_2; see also [31]. From a technical point of view this situation is now not any different from the case discussed above, and so we obtain again that $f^* \dashv f_*$. Henceforth we develop the case $1_1 \rightsquigarrow 1_2$; it is understood that in the examples where $1_1 \not\rightsquigarrow 1_2$ we have freely adjoined a new "top" so as to reduce this case to the former by the procedure outlined above.

When considering three property lattices L_1, L_2 and L_3 and respective propagations of properties $f_{1,2}^* : L_1 \to L_2$ and $f_{2,3}^* : L_2 \to L_3$, what can be said about $f_{1,3}^* : L_1 \to L_3$?

Lemma 5. *With obvious notations $f_{i,j}^* \dashv f_*^{i,j}$, we have by the operational significance of the causations $f_*^{i,j}$ that $f_*^{1,3} = f_*^{1,2} \circ f_*^{2,3}$.*

Proof: Denoting the corresponding tests for $f_*^{i,j}(a_j)$ by $\phi_{i,j}(\alpha_j)$ we clearly have $\phi_{1,3}(\alpha_3) = \phi_{1,2}(\phi_{2,3}(\alpha_3))$, so $[\phi_{1,3}(\alpha_3)] = [\phi_{1,2}(\phi_{2,3}(\alpha_3))]$ and thus it follows that $f_*^{1,3}(a_3) = f_*^{1,2}([\phi_{2,3}(\alpha_3)]) = f_*^{1,2}(f_*^{2,3}(a_3))$ for all $a_1 \in L_1$. $\qquad\square$

Because adjunctions compose – that is, if $f^* \dashv f_*$ and $g^* \dashv g_*$ then also $f^* \circ g^* \dashv g_* \circ f_*$ – we also obtain $f_{1,3}^* = f_{2,3}^* \circ f_{1,2}^*$. We can read off that the composition of causations stands for "chaining causal assignments" of properties, whereas the composition of propagations then must stand for "consecutive propagation" of properties.

More technically speaking, from corollary 1 and lemma 5 it is now obvious that the property lattices and the causations organize themselves in (a subcategory of) $\underline{\text{MCLat}}$, and the same property lattices equipped with the propagations organize themselves in (a subcategory of) $\underline{\text{JCLat}}$ — where $\underline{\text{MCLat}}$ (resp. $\underline{\text{JCLat}}$) is the category of complete lattices and meet preserving (resp. join preserving) maps. As is well-known, the assignment of adjoints as in

$$\underline{\text{JCLat}}(L_1, L_2) \to \underline{\text{MCLat}}(L_2, L_1) : f \mapsto f_* \qquad (12)$$

is an anti-isomorphism (a "duality") between the complete lattices of respectively join and meet preserving maps, ordered pointwisely. In particular, the conjunction of properties is "lifted" to the "conjunction for causal assignments" in the hom-sets of $\underline{\text{MCLat}}$ and, dually, the superposition of properties is "lifted" to the "superposition of propagations" in the hom-sets of $\underline{\text{JCLat}}$. Rewritten more conveniently, this gives

$$\underline{\text{MCLat}}^{\text{coop}}(L_1, L_2) \cong \underline{\text{MCLat}}(L_2, L_1)^{\text{op}} \cong \underline{\text{JCLat}}(L_1, L_2). \qquad (13)$$

Since both $\underline{\text{MCLat}}^{\text{coop}}$ and $\underline{\text{JCLat}}$ are quantaloids for the pointwise ordering of their hom-sets, and because adjoints compose, we have a

representation of our setting in the category of quantaloids and quantaloid morphisms, denoted as $\underline{\text{QUANT}}$:

$$\underline{\text{MCLat}}^{\text{coop}} \xleftarrow{\;\sim\;} \underline{\text{JCLat}} \tag{14}$$

We can conclude all this by:

Theorem 1. *Causal assignment and propagation of properties are dualized by a quantaloidal isomorphism* $F : \underline{\text{MCLat}}^{\text{coop}} \xrightarrow{\;\sim\;} \underline{\text{JCLat}}$.

We will now briefly discuss some examples of this general setting. The adjunction

$$[f^* : 0_1 \mapsto 0_2, \ \text{rest} \mapsto 1_2] \dashv [f_* : 1_2 \mapsto 1_1, \ \text{rest} \mapsto 0_1] \tag{15}$$

describes 'separation' of the systems described by L_1 and L_2, a situation that previously could not be described in a consistent way within quantum theory [1,6]. By way of contrast, for L_1 and L_2 atomistic the maps that send atoms to atoms or the bottom represent the strongest types of interaction, or analogously, maximally deterministic evolution. When considering lattices of closed subspaces of Hilbert spaces this setting yields representational theorems for the description of compound quantum systems by the Hilbert space tensor product [6] and description of evolution by Schrödinger flows [17], so it is exactly the enrichment that allows a joint consideration of the types of entanglement encountered in classical and quantum physics.

3. PHYSICAL ORIGIN OF CATEGORICAL CONCEPTS

Action of inductions on properties

With the consideration of the map ϕ in section 2 we've introduced a new "dynamical ingredient" to say something more about the physical system than a merely static description could ever do. This can be pushed even further: one can trace the origins of causal duality, i.e. "propagation is adjoint to causation", back to a dynamic operational foundation complementary to the static operational foundation. To that end, one uses as counterpart of the operational notion of test that of "induction" [3]. Whereas giving a fully detailed exposition of this development would lead us too far, we still think that it is useful to at least outline the most important ideas; we plan to dig deeper into this matter in a future work.

By an *induction* on a physical system Ξ is meant an externally imposed change of Ξ. Such an induction can as such for example be an imposed evolution or measurement, or, the action of a system on another in case of so called entanglement. The collection $\mathcal{E}(\Xi)$ of all inductions on the system Ξ (also written $\mathcal{E}$ if no confusion is possible) is naturally equipped with two operations:

1. for $e_1, e_2 \in \mathcal{E}$ two inductions, $e_1 \& e_2$ is the induction that consists of effectuating first e_1 and second e_2;
2. for $\{e_i \mid i \in I\} \subseteq \mathcal{E}$ with I a set, $\bigvee_i e_i$ is the induction that consists of effectuating an arbitrarily chosen element of $\{e_i \mid i \in I\}$.

When focussing on an induction's action on the physical system rather than the physical procedure associated to such an induction, it seems reasonable to suppose that $\mathcal{E}$ is a set; then these operations give $\mathcal{E}$ the structure of a quantale, for clearly we have that $\&$ acts as a product that distributes on both sides over the join $\bigvee$ (whence the notations). The unit for the multiplication can be thought of as the induction *freeze*, denoted as $*$.

The crucial link between inductions and tests is now given by an action of the former on the latter. Namely, for any $e \in \mathcal{E}$ and $\alpha \in \mathcal{T}$, we define a "multiplication" $e \cdot \alpha \in \mathcal{T}$ as follows:

$e \cdot \alpha$ is the test consisting of "first executing the induction e and then performing the test α", the outcome of the test $e \cdot \alpha$ being the one thus obtained for α.

Then the operationality of the notions involved assures that:

(o) $\alpha \preceq \beta$ in $\mathcal{T}$ implies that $e \cdot \alpha \preceq e \cdot \beta$
(i) $* \cdot \alpha \approx \alpha$
(ii) $e \cdot (\Pi_i \alpha_i) \approx \Pi_i(e \cdot \alpha_i)$
(iii) $(\bigvee_i e_i) \cdot \alpha \approx \Pi_i(e_i \cdot \alpha)$
(iv) $e_1 \cdot (e_2 \cdot \alpha) \approx (e_1 \& e_2) \cdot \alpha$

where $\approx$ denotes the equivalence relation derived from the preorder $\preceq$ on $\mathcal{T}$, and the products Π in (i) and (ii) are product tests. When reducing the class $\mathcal{T}$ of tests to the set L of properties, by quotienting $\mathcal{T} \twoheadrightarrow L : \alpha \mapsto [\alpha]$, this "multiplication" boils down to an action

$$\mathcal{E} \times L^{op} \to L^{op} : (e, a := [\alpha]) \mapsto e \cdot a \ := [e \cdot \alpha] \qquad (16)$$

such that L^{op} exhibits itself as a module of the monoid $\mathcal{E}$ in the monoidal category <u>JCLat</u> (for the theory of quantale modules, consult for instance [28]).

At this point it is handy to re-introduce the "causal relation" of the previous section, albeit adapted to this situation in which we want to consider many inductions at once: for $\alpha, \beta \in \mathcal{T}$ and $e \in \mathcal{E}$, we put

$$\alpha \overset{e}{\rightsquigarrow} \beta \Longleftrightarrow \alpha \preceq e \cdot \beta \qquad (17)$$

and with a slight abuse of notation we will also use $a \overset{e}{\rightsquigarrow} b$ for $a, b \in L$. The latter means thus precisely that the actuality of the property a

before the induction e guarantees the actuality of b after the induction. It is now a consequence that to any $e \in \mathcal{E}$ one can associate two mappings,

$$e_* : L \to L : a \mapsto e \cdot a \tag{18}$$

$$e^* : L \to L : a \mapsto \bigwedge \{b \in L \mid a \overset{e}{\rightsquigarrow} b\} \tag{19}$$

for which it is clear that $a \leq e_*(b) \iff a \overset{e}{\rightsquigarrow} b \iff e^*(a) \leq b$ and therefore $e^* \dashv e_* : L \to L$. Furthermore, the unital quantale structure that $\mathcal{E}_* = \{e_* \mid e \in \mathcal{E}\} \subseteq \underline{\text{MCLat}}(L, L)$ is naturally endowed with, corresponds to the one suggested by $\mathcal{E}$; in particular $(e \& f)_* = e_* \circ f_*$, $(\bigvee_i e_i)_* = \bigvee_i (e_i)_*$ and $*_* = id_L$. Likewise for the evident $\mathcal{E}^* \subseteq \underline{\text{JCLat}}(L, L)$. When considering both as one object quantaloids, it is true that $\mathcal{E}^* \cong \mathcal{E}_*^{\text{coop}}$, and as such we recover in this setting the causal duality, as formalized by the theorem in the previous section.

Operational resolutions and quantaloids

In order to introduce aspects of 'uncertainty' and 'arbitrary choice' in our general setup, we want to extend a property lattice $(L, \bigwedge)$ by introducing 'propositions' that represent disjunctions of properties [dis]. We may realize this within $PL := 2^{L \setminus \{0\}}$ *grosso modo* as follows: The embedding of the lattice L of properties into the boolean algebra PL of propositions,

$$L \hookrightarrow PL : x \mapsto \{y \in L \setminus \{0\} \mid y \leq x\}, \tag{20}$$

preserves arbitrary infima such that [con] and [imp] are preserved. Consequently, the embedding has a left adjoint which turns out to be

$$PL \to L : A \mapsto \bigvee A. \tag{21}$$

Such a map was dubbed 'operational resolution' in [3,9,10], for the following reason: this map physically stands for the *verifiability* of collections of properties in the sense that, if we define an 'actuality set' to be an $A \subseteq L$ of which at least one element $a \in A$ is actual but we don't know which one, then by $\bigvee A = \bigwedge \{b \in L \mid \forall a \in A : b \geq a\}$, [con] and [imp], we have that $\bigvee A$ is the strongest property whose actuality is guaranteed for an actuality set A. By its construction, it is clear that in the ambient boolean algebra PL an actuality set A plays the role of the 'disjunction' of its elements.

How should one now describe the propagation of actuality sets? The answer to this question is given by the more general results in [10]; here is what it comes down to:

Lemma 6. *Given two lattices L_1 and L_2, and a map $g \colon PL_1 \to PL_2$ that preserves arbitrary unions, the following are equivalent:*

1. *for all $A, B \in PL_1$,*

$$\bigvee A = \bigvee B \Rightarrow \bigvee g(A) = \bigvee g(B) \qquad (22)$$

2. *there exists a (necessarily unique) morphism $f : L_1 \to L_2$ that preserves arbitrary suprema making the following square, in which the vertical uparrows are the resolutions, commute:*

$$\begin{array}{ccc} L_1 & \xrightarrow{\;f\;} & L_2 \\ \uparrow & & \uparrow \\ PL_1 & \xrightarrow{\;g\;} & PL_2 \end{array}$$

Indeed, such maps $g : PL_1 \to PL_2$ are the appropriate expressions for the propagation of actuality sets: requiring g to preserve arbitrary unions is in accordance with [dis], and requiring the continuity condition of eq.(22) is, via lemma 6, saying that the verification of the propagation of an actuality set through the operational resolution must result in a propagation of properties.

Further application of the general theory of resolutions and their morphisms [10] shows that, when defining the bicategory $Q^{\#}\underline{\mathrm{JCLat}}$ with objects the $PL = 2^{L\setminus\{0\}}$ for all complete lattices L, morphisms the maps g as above, and the local structure being the evident pointwise order of such maps, this bicategory is a quantaloid; furthermore, the action $F_{\#} : Q^{\#}\underline{\mathrm{JCLat}} \to \underline{\mathrm{JCLat}} : PL \mapsto L; g \mapsto f$ (notation of f refers to lemma 6) proves to be a full quantaloidal morphism. Note that $Q^{\#}\underline{\mathrm{JCLat}}$ neither coincides with the categories with the same objects and on the one hand all union preserving maps, and on the other hand pointwise unions of direct image maps of $\bigvee$-preserving maps — a precise characterization can be found in [11]. Together with the theorem of section 2 we have the following scheme in $\underline{\mathrm{QUANT}}$:

$$Q^{\#}\underline{\mathrm{JCLat}} \xrightarrow{\;F_{\#}\;}\!\!\!\!\to \underline{\mathrm{JCLat}} \xleftarrow{\;\cong\;} \underline{\mathrm{MCLat}}^{\mathsf{coop}} \qquad (23)$$

that expresses how the propagation of actuality sets is related to causal assignments.

Our point now is that the quantaloidal nature of $F_{\#}$ reveals that the enrichment of the collection of causal assignments $\underline{\mathrm{MCLat}}^{\mathsf{coop}}$ originates – physically – from the presence of an underlying uncertainty encoded in the local structure of $Q^{\#}\underline{\mathrm{JCLat}}$.

Actuality sets and frame completions

Surely the formal disjunction of properties $a, b, c, ... \in L$ may be expressed in the complete boolean algebra PL of propositions as their union $\{a, b, c, ...\} \in PL$; and consequently the disjunction of $a, b, c, ...$ is actual iff at least one of them is — which could indicate that the

'calculus of actuality sets' as the appropriate 'logic of the propositions' is encoded in PL. That the properties can be, on the one hand, embedded in the propositions without loss of conjunctivity (a primitive notion!) and, on the other hand, be 'recuperated' from them by an operational resolution is a confirmation of these ideas. However, the meets in PL of elements that do not represent properties are in no way to be seen as conjunctions: for $a < b$ we have $\{a\} \cap \{b\} = \emptyset$ where the conjunction $\{a\}$ *and* $\{b\}$ clearly is $\{a\}$.

It turns out that the origin of this lack of general conjunctivity traces back to the fact that the inclusion $L \hookrightarrow PL$ is in general not the most "economical" way of extending L to be able to handle those actuality sets that are necessary to express disjunction of properties. Indeed, consider the case where L is already a complete boolean algebra: the extension of $L \hookrightarrow PL$ is redundant whenever the join in L is already to be understood as a disjunction. For certain classes of property lattices – among which all physically relevant ones – a "most economical extension", i.e. a completion which is universal in an appropriate ambient category, does exist; the construction is subtle but straightforward. A profound discussion can be found in [7] and we will not go into details here; let us however quickly sketch the crucial point.

It is necessary to somehow characterize, for a given property lattice L, those joins of subsets of L that are disjunctions in the sense that the join is actual iff at least one member in this subset is. That is to say, we need a lattice theoretic criterion to decide whether, for example, a binary join $a \vee b$ in L is to be understood as disjunction of a and b or as superposition (the criterion should work also for arbitrary joins). The following result, quoted from [7], provides such a criterion.

Lemma 7. *If the property lattice L "fully represents the physical system with respect to superpositions", the join of a subset $A \subseteq L$ is to be understood as the disjunction of A iff $\bigvee A$ is distributive — that is, for all $x \in L$: $x \wedge \bigvee A = \bigvee(x \wedge A)$.*

This makes at once clear that the 'logic of propositions' (or the 'calculus of actuality sets') must take place in a complete lattice in which every join is distributive – because we want the join of propositions to be their disjunction – hence by definition in a frame. So a frame completion of L is what we're looking for. A detailed discussion – with appropriate references – of such completions and their validity, consequences and applications in the field of quantum logic, and alternative constructions can be found in [7,30].

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