On the hierarchy: monad, theory, operad

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§0. Motivation

§1. Monads w/arities & homogeneous theories

§2. Algebraic theories & symmetric operads.

§3. Globular theories & globular operads.

§0 Motivation

symmetric operads ⇝ algebraic structure of iterated loop spaces

Vogt 68' (categories of operators in standard form)

May 72' operad

Lambek 69' multicategory

globular operads ⇝ algebraic structure of higher categories

Batanin 98'

What is an operad ?

Tentative answer:

operad ⟷ homogeneous theory.

theory ⟷ monad w/arities.

§1. Monads w/arities & symmetric operads

$\mathcal{E}$ cocomplete category

$\Theta_0 \subset \mathcal{E}$ <u>dense</u> subcategory $\qquad \widehat{\Theta}_0 = Sets^{\Theta_0^{op}}$

$$\iff \quad \begin{array}{rcl} N_0 : \mathcal{E} & \longrightarrow & \widehat{\Theta}_0 \\ X & \longmapsto & \mathcal{E}(-,X) \end{array} \quad \text{fully faithful}$$

$$\iff \quad \boxed{\operatorname{colim}_{\Theta_0/X}(-) \xrightarrow{\cong} X \quad \text{for all } X \text{ in } \mathcal{E} \quad (*)}$$

<u>Dfn.</u> (Lack - Weber 07')

$T : \mathcal{E} \to \mathcal{E}$ has <u>arities</u> Θ_0

if $N_0 T$ preserves the colimits $(*)$

Assume (T, μ, η) <u>monad</u> on $\mathcal{E}$

$$\begin{tikzcd} \Theta_T \arrow[r, hook, "\text{dff}"] & \mathcal{E}^T \arrow[r, "N_T"] \arrow[d, shift left, "U_T"] & \widehat{\Theta}_T \arrow[d, shift left, "i^*"] \\ \Theta_0 \arrow[u, "i"] \arrow[r, hook] & \mathcal{E} \arrow[u, shift left, "F_T"] \arrow[r, "N_0"] & \widehat{\Theta}_0 \arrow[u, shift left, "i_!"] \end{tikzcd}$$

bij. on objects

$$\Theta_T(a,b) = \mathcal{E}^T(F_T(a), F_T(b)) \subset Kl(T)$$

<u>Thm (Weber)</u> Let T be a monad on $\mathcal{E}$ (cocomplete)
Then T has arities Θ_0 iff

$$i^* i_! : \hat{\Theta}_0 \to \hat{\Theta}_0 \quad \text{preserves } \underline{\text{essential image of } N_0}$$

and

$$\begin{tikzcd} \mathcal{E}^T \arrow[r, "N_T"] \arrow[d, "U_T"] & \hat{\Theta}_T \arrow[d, "i^*"] \\ \mathcal{E} \arrow[r, "N_0"] & \hat{\Theta}_0 \end{tikzcd}$$

<u>pseudo-pullback</u>
(h-pullback in Cat)

<u>Specialization</u> : Assume $\mathcal{E}$ <u>topos</u>
$\Rightarrow$ $\exists!$ topology on Θ_0 such that

$$\mathcal{E} \xrightarrow{\sim} Sh(\Theta_0) \subset \hat{\Theta}_0 \quad \text{essential image of } N_0.$$

(<u>epimorphic family topology</u>)

<u>Dfn.</u> (Θ_T-models) $\quad Mod_T = \{X \in \hat{\Theta}_T \mid i^* X \in Sh(\Theta_0)\}$

$$\begin{tikzcd} \mathcal{E}^T \arrow[r, "\sim"] \arrow[d, "U_T"] & Mod_T \arrow[r, hook] \arrow[d, "i^*"] & \hat{\Theta}_T \arrow[d, "i^*"] \\ \mathcal{E} \arrow[r, "\sim"] & Sh(\Theta_0) \arrow[r, hook] & \hat{\Theta}_0 \end{tikzcd}$$

$$\boxed{\begin{array}{l} \underline{\text{Thm.}}\ \ T \ \underline{\text{monad w/arities}}\ \Theta_0 \ \text{ iff} \\ \quad i^* i_! : \hat{\Theta}_0 \to \hat{\Theta}_0 \ \text{ preserves sheaves} \\ \text{and } \ \mathcal{E}^T \underset{N_T}{\xrightarrow{\sim}} Mod_T \quad (T\text{-algebras are } \Theta_T\text{-models}) \end{array}}$$

Cor. 1-1 correspondence between

monads w/arities Θ_0 on $\mathcal{E}$ (topos)

and $i : \Theta_0 \hookrightarrow \Theta_T$ bij. on objects

s.th. $i^* i_!$ preserves sheaves

for the "image-topology".

Θ_T "theory" associated to T

$Mod_T \simeq \mathcal{E}^T$.

Dfn. Θ_T homogeneous iff $\Theta_T = (\Theta_{gen}, \Theta_0)$

$\exists$ "generic-free" factorization system

Rmk. For a homogeneous theory Θ_T the associated monad T has a simple form.

$$X \in \widehat{\Theta_0} \qquad i^* i_!(X)_a = \underset{a/\Theta}{\operatorname{colim}} X_{(-)}$$

$$= \left(\coprod_{\substack{a \to b \\ \Theta_{gen}}} \{X_b\} \right) \Big/ Aut(b) = \coprod_b \Theta_{gen}(a,b) \times_{Aut(b)} X_b$$

The theory is already determined by the "generics". The monad T is "operadic".

§2. Algebraic theories and symmetric operads

$\mathcal{E} = Sets \qquad Ob(\Theta_0) = \{\underline{n},\ n \geq 0\} \qquad \underline{n} = \{1, \dots, n\}$

Θ_0 = skeleton of the subcategory of "*finitely presented*" objects.

$\mathcal{E} \longrightarrow \hat{\Theta}_0$ image-topology has minimal covering sieves

$$\begin{tikzcd} \underline{1} \arrow[dr] & \\ \vdots & \underline{n} \\ \underline{1} \arrow[ur] & \end{tikzcd} \qquad \forall n \geq 0.$$

$X \in \hat{\Theta}_0$ *sheaf* iff $X(\underline{n}) \xrightarrow{\cong} X(\underline{1})^n \ \forall n \geq 0$.

Lma. $T : Sets \to Sets$ *has arities* Θ_0
iff T *preserves filtered colimits*

Lma $i : \Theta_0 \hookrightarrow \Theta_T$ (bij. on objects) fulfills

$i^* i_!$ preserves sheaves

iff coproduct in Θ_0 remains coproduct in Θ_T

iff Θ_T^{op} *algebraic theory* in Lawvere's sense

(Lawvere)
$\rightsquigarrow$ *Thm* 1-1 correspondence between
filtered colimit preserving monads on Sets
and *algebraic theories*.

Θ_T homogeneous algebraic theory

$$\begin{tikzcd} \mathrm{Sets}^T \arrow[r, "\sim", "N_T"'] \arrow[d] & \mathrm{Mod}_T \arrow[r, hook] \arrow[d] & \hat{\Theta}_T \arrow[d] \\ X \in \mathrm{Sets} \arrow[r, "\sim", "N_0"'] & \mathrm{Sh}(\Theta_0) \arrow[r, hook] & \hat{\Theta}_0 \end{tikzcd}$$

$$N_0(X)(\underline{n}) = \mathrm{Sets}(\underline{n}, X) = X^n$$

$$TX = (i^* i_! N_0(X))(\underline{1}) = \Big(\coprod_{\underline{1} \xrightarrow[\text{gen}]{} \underline{n}} (N_0(X)(\underline{n}))\Big)\Big/\!\!\Big/ \mathrm{Aut}(\underline{n})$$

$$= \coprod_{n \geq 0} \Theta_T^{\text{gen}}(\underline{1}, \underline{n}) \underset{\mathfrak{S}_n}{\times} X^n$$

<u>Dfn.</u> $\Theta_T(n) = \Theta_T^{\text{gen}}(\underline{1}, \underline{n})$ $(n \geq 0)$ $\mathfrak{S}_n$-set.

<u>Lma</u> $\Theta_T(n) \times \Theta_T(k_1) \times \cdots \times \Theta_T(k_n) \to \Theta_T(k_1 + \cdots + k_n)$

(induced by category structure of Θ_T)

defines a symmetric operad in Sets

<u>sth.</u> T may be identified with the monad associated to Θ_T.

<u>Rmk.</u> The terminal operad $\Theta_T(n) = *$ corresponds to the theory Θ_T for <u>commutative monoids</u>. Each hom. alg theory comes equipped with unique map to this Θ_T.

§3. Globular theories & globular operads.

$$\mathcal{E} = \hat{\mathbb{G}} = \{\text{globular sets}\} = \{\omega\text{-graphs}\}$$

$\mathbb{G}$ $\quad 0 \underset{t}{\overset{s}{\rightrightarrows}} 1 \underset{t}{\overset{s}{\rightrightarrows}} 2 \underset{t}{\overset{s}{\rightrightarrows}} 3 \quad \cdots$ s.th. $ss = ts$ & $st = tt$

representable presheaves

$\mathbb{G}$ is filtered by dimension $\mathbb{G}_0 \subset \mathbb{G}_1 \subset \mathbb{G}_2 \subset \cdots$

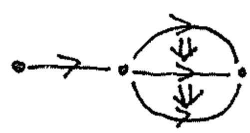
$\hat{\mathbb{G}}_n = \{n\text{-graphs}\}$ $\quad \hat{\mathbb{G}}_1 = \{\text{graphs}\}$

$\hat{\mathbb{G}}_0 = \{\text{sets}\}$.

Dfn. $\Theta_0 \subset \hat{\mathbb{G}}$ "arities" for globular sets.

Θ_0 full subcategory of $\hat{\mathbb{G}}$ on T_* where T runs through finite level trees.

$T \qquad \longleftrightarrow \qquad T_*$



Batanin's star-construction

sectors $\longleftrightarrow$ cells.

Rmk. All globular maps $S_* \to T_*$ are *monic*.

Θ_0 *filtered* by height.

$Gr \subset \Theta_0$ full (*linear* level-trees).

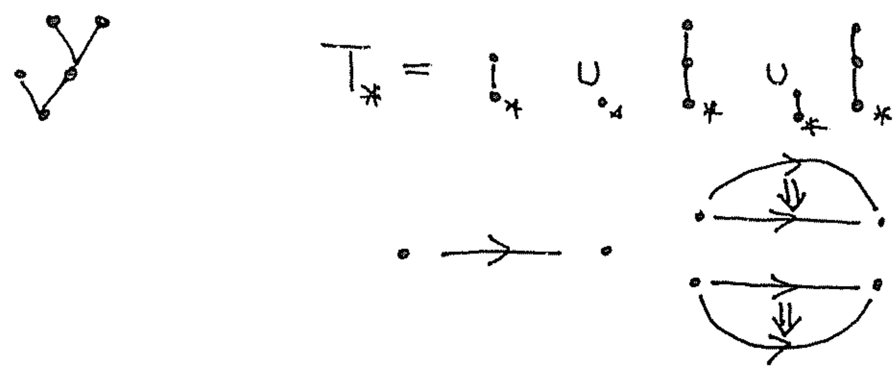
For each *globular set* X and each *level-tree* T, define the

"tree-power" $X^T = \widehat{Gr}(T_*, X)$

$= \{$cell-configurations in X of shape $T_*\}$.

minimal covering sieve of T_*

$T =$ $\qquad$ $T_* =$



Dfn. *globular theory* $i : \Theta_0 \hookrightarrow \Theta_T$ bij. on objects

sth. $i^* i_! : \hat{\Theta}_0 \longrightarrow \hat{\Theta}_0$ preserves sheaves

Dfn. Θ_T *homogeneous* if $\Theta_T = (\Theta_T^{gen}, \Theta_0)$ generic-free fact. system

compatible with height-filtration, i.e.

"generics" lower or preserve height.

$\boxed{\text{Thm. 1-1 correspondence between homogeneous globular theories and globular operads of Batanin.}}$

Rmk. X globular set $\Theta_{\mathbb{T}}$ globular theory

X $\mathbb{T}$-algebra (resp. $\Theta_{\mathbb{T}}$-model) iff equipped with actions

$$\boxed{\Theta_{\mathbb{T}}(S,T) \times X^{T} \longrightarrow X^{S}}$$

If $\Theta_{\mathbb{T}}$ homogeneous (i.e. $\mathbb{T}$ comes from ω-operad) then these actions are determined by

$$\left(\ell_n = \text{linear tree height } n\right) \qquad \Theta_{\mathbb{T}}(\ell_n, T) \times X^{T} \longrightarrow X^{\ell_n} = X_n$$

(evaluations of T-configurations in X)

Rmk. An ω-operad in Batanin's sense is given by a collection of sets $\Theta_{T,n}$ where T is a level-tree of height $\leq n$, together with

$$\Theta_{S_*} \times \Theta_{T_1} \times \cdots \times \Theta_{T_k} \longrightarrow \Theta_{T} \quad \text{multiplications}$$

where T_* is obtained by gluing $(T_1)_*, \ldots (T_k)_*$ along S_*.

Thm. (Makkai-Zawadowski)
The terminal homogeneous globular theory is isomorphic to Joyal's category Θ.