Terminal coalgebras

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Plan

1. Introduction to terminal coalgebras
2. Some theory of terminal coalgebras
3. Trimble-like \( n \)-categories
4. Trimble-like \( \omega \)-categories via terminal coalgebras
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1. Introduction to terminal coalgebras
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3. Trimble-like $n$-categories
4. Trimble-like $\omega$-categories via terminal coalgebras
1. Introduction to terminal coalgebras

A coalgebra for an endofunctor $F : C \to C$ consists of

- an object $A \in C$
- a morphism $F_A : A \to A$

satisfying no axioms.
1. Introduction to terminal coalgebras

A coalgebra for an endofunctor $F : \mathcal{C} \to \mathcal{C}$ consists of

• an object $A \in \mathcal{C}$
• a morphism $F(A) \to A$

satisfying no axioms.
1. Introduction to terminal coalgebras

A coalgebra for an endofunctor $F : \mathcal{C} \rightarrow \mathcal{C}$ consists of

- an object $A \in \mathcal{C}$
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A coalgebra for an endofunctor $F : \mathcal{C} \to \mathcal{C}$ consists of

- an object $A \in \mathcal{C}$
- a morphism $A \to FA$
1. Introduction to terminal coalgebras

A coalgebra for an endofunctor $F : \mathcal{C} \rightarrow \mathcal{C}$ consists of

- an object $A \in \mathcal{C}$

\[ A \quad \Downarrow \quad FA \]

- a morphism satisfying no axioms.

satisfying no axioms.
1. Introduction to terminal coalgebras

Coalgebras for $F$ form a category with the obvious morphisms
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Coalgebras for $F$ form a category with the obvious morphisms

\[
\begin{array}{ccc}
A & \xrightarrow{h} & B \\
\downarrow & & \downarrow \\
FA & \xrightarrow{Fh} & FB
\end{array}
\]
1. Introduction to terminal coalgebras

Coalgebras for $F$ form a category with the obvious morphisms

\[
\begin{array}{ccc}
A & \xrightarrow{h} & B \\
\downarrow & & \downarrow \\
FA & \xrightarrow{Fh} & FB
\end{array}
\]

so we can look for terminal coalgebras.
1. Introduction to terminal coalgebras

Example 1
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Given a set $M$ we have an endofunctor
1. Introduction to terminal coalgebras

Example 1

Given a set $M$ we have an endofunctor

$$\begin{align*}
\text{Set} & \xrightarrow{M \times -} \text{Set} \\
A & \mapsto M \times A
\end{align*}$$
1. Introduction to terminal coalgebras

Example 1

Given a set $M$ we have an endofunctor

$$
\text{Set} \xrightarrow{M \times -} \text{Set}
$$

$$A \mapsto M \times A$$

The terminal coalgebra is given by the set $M^\mathbb{N}$ of “infinite words” in $M$
1. Introduction to terminal coalgebras

Example 1
Given a set $M$ we have an endofunctor

$$\text{Set} \xrightarrow{M \times -} \text{Set}$$

$$A \mapsto M \times A$$

The terminal coalgebra is given by the set $M^\mathbb{N}$ of “infinite words” in $M$

$$(m_1, m_2, m_3, \ldots)$$
1. Introduction to terminal coalgebras

The structure map of this coalgebra:
1. Introduction to terminal coalgebras

The structure map of this coalgebra:

\[ M^\mathbb{N} \rightarrow M \times M^\mathbb{N} \]
1. Introduction to terminal coalgebras

The structure map of this coalgebra:

\[ \begin{array}{c}
M^\mathbb{N} \\
\downarrow \\
M \times M^\mathbb{N}
\end{array} \]

is a canonical isomorphism.
1. Introduction to terminal coalgebras

To see that this is terminal:
1. Introduction to terminal coalgebras

To see that this is terminal:
Given any coalgebra

\[ A \]

\[ M \times A \]
1. Introduction to terminal coalgebras

To see that this is terminal:

Given any coalgebra

\[
A \\
\downarrow \\
M \times A
\]

we need to produce an infinite word in \( M \).
1. Introduction to terminal coalgebras

screen memory
1. Introduction to terminal coalgebras

screen memory

\( a \)
1. Introduction to terminal coalgebras

\[
\text{screen memory}
\]

\[
\begin{array}{ccc}
\text{a} & \rightarrow & (m_1, m_2, m_3, m_4, \ldots) \\
\text{m}_1 & a_1 \\
\end{array}
\]
1. Introduction to terminal coalgebras

<table>
<thead>
<tr>
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1. Introduction to terminal coalgebras

\[
\begin{align*}
\text{screen} & \quad \text{memory} \\
 & \quad a \\
& \quad m_1 \quad a_1 \\
& \quad m_2 \quad a_2 \\
& \quad m_3 \quad a_3 \\
& \quad m_4 \quad a_4 \\
\vdots \\
a \mapsto (m_1, m_2, m_3, m_4, \ldots)
\end{align*}
\]
2. Some theory of terminal coalgebras

Lemma (Lambek)

If \( FA \) is a terminal coalgebra for \( F \) then \( f \) is an isomorphism.
2. Some theory of terminal coalgebras

Lemma (Lambek)
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Lemma (Lambek)

If \( A \) is a terminal coalgebra for \( F \)

\[
\begin{array}{c}
A \\
\downarrow f \\
FA
\end{array}
\]
2. Some theory of terminal coalgebras

Lemma (Lambek)

If $A$ is a terminal coalgebra for $F$

\[
\begin{array}{ccc}
A & \xrightarrow{f} & FA \\
\end{array}
\]

then $f$ is an isomorphism.
2. Some theory of terminal coalgebras

Theorem (Adámek)
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We can construct the terminal coalgebra as the limit of the following diagram:
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We can construct the terminal coalgebra as the limit of the following diagram:

\[
\cdots \xrightarrow{F^3!} F^3 1 \xrightarrow{F^2!} F^2 1 \xrightarrow{F!} F 1 \xrightarrow{!} 1
\]
2. Some theory of terminal coalgebras

**Theorem (Adámek)**

We can construct the terminal coalgebra as the limit of the following diagram:

\[ \cdots \xrightarrow{F^3!} F^3 \xrightarrow{F^2!} F^2 \xrightarrow{F!} F \xrightarrow{!} 1 \]

provided there is a terminal object 1, the limit exists, \( F \) preserves it.
2. Some theory of terminal coalgebras

Example 1 revisited

Given a set $M$ we considered the endofunctor

$$\text{Set} \xrightarrow{\times M} \text{Set}$$

$A \mapsto M \times A$
Example 1 revisited

Given a set $M$ we considered the endofunctor

$$
\text{Set} \xrightarrow{M \times -} \text{Set} \\
A \quad \mapsto \quad M \times A
$$

We can construct a terminal coalgebra as the limit of

$$
\cdots \xrightarrow{F^3!} F^31 \xrightarrow{F^2!} F^21 \xrightarrow{F!} F1 \xrightarrow{!} 1
$$
2. Some theory of terminal coalgebras

**Example 1 revisited**

Given a set $M$ we considered the endofunctor

\[
\begin{array}{rcl}
\text{Set} & \xrightarrow{M \times -} & \text{Set} \\
A & \mapsto & M \times A
\end{array}
\]

We can construct a terminal coalgebra as the limit of

\[
\cdots \rightarrow M^3 \xrightarrow{M \times !} M^2 \xrightarrow{M \times !} M \xrightarrow{!} \{0\}
\]
2. Some theory of terminal coalgebras

Example 1 revisited

Given a set $M$ we considered the endofunctor

$$\begin{align*}
\text{Set} & \xrightarrow{\times M} \text{Set} \\
A & \mapsto M \times A
\end{align*}$$

We can construct a terminal coalgebra as the limit of

$$
\cdots \xrightarrow{M^3 \times !} M^3 \xrightarrow{M^2 \times !} M^2 \xrightarrow{M \times !} M \xrightarrow{!} 1
$$

which does indeed give infinite words in $M$. 
2. Some theory of terminal coalgebras

Example 2 (Simpson)
2. Some theory of terminal coalgebras

Example 2 (Simpson)

There is an endofunctor

\[
\begin{array}{ccc}
\text{SymMonCat} & \longrightarrow & \text{SymMonCat} \\
\forall & \mapsto & \forall\text{-Cat}
\end{array}
\]
2. Some theory of terminal coalgebras

Example 2 (Simpson)

There is an endofunctor

\[
\text{SymMonCat} \to \text{SymMonCat}
\]

\[
\forall \to \forall\text{-Cat}
\]

The terminal coalgebra is given by
2. Some theory of terminal coalgebras

Example 2 (Simpson)

There is an endofunctor

\[
\text{SymMonCat} \longrightarrow \text{SymMonCat}
\]

\[
\forall \quad \longmapsto \quad \forall\text{-Cat}
\]

The terminal coalgebra is given by the category \( \omega\text{-Cat} \) of strict \( \omega \)-categories.
2. Some theory of terminal coalgebras

Example 2 (Simpson)

There is an endofunctor

\[
\text{SymMonCat} \longrightarrow \text{SymMonCat}
\]

\[
\mathcal{V} \mapsto \mathcal{V}\text{-Cat}
\]

The terminal coalgebra is given by the category \( \omega\text{-Cat} \) of strict \( \omega\)-categories.

We note that Lambeck’s Lemma holds:

\[
\omega\text{-Cat} \cong (\omega\text{-Cat})\text{-Cat}.
\]
2. Some theory of terminal coalgebras

Using Adámek’s construction
2. Some theory of terminal coalgebras

Using Adámek’s construction

- $F 1 \cong \text{Set}$
2. Some theory of terminal coalgebras

Using Adámek’s construction

- $F^1 \cong \text{Set}$
- $F^n 1 = n\text{-Cat}$
Using Adámek’s construction

- $F \mathbb{1} \cong \text{Set}$
- $F^n \mathbb{1} = n\text{-Cat}$

The limit diagram

$$
\cdots \xrightarrow{F^3!} F^3 \mathbb{1} \xrightarrow{F^2!} F^2 \mathbb{1} \xrightarrow{F!} F \mathbb{1} \xrightarrow{!} \mathbb{1}
$$
2. Some theory of terminal coalgebras

Using Adámek’s construction

- $F1 \cong \text{Set}$
- $F^n1 = n\text{-Cat}$

The limit diagram becomes

$$\cdots \rightarrow 2\text{-Cat} \rightarrow 1\text{-Cat} \rightarrow 0\text{-Cat} \rightarrow ! \rightarrow \mathbb{1}$$
Using Adámek’s construction

- $F1 \cong \text{Set}$
- $F^n1 = n\text{-Cat}$

The limit diagram becomes

$$\cdots \longrightarrow 2\text{-Cat} \longrightarrow 1\text{-Cat} \longrightarrow 0\text{-Cat} \longrightarrow 1$$

where each morphism here is truncation.
2. Some theory of terminal coalgebras

Idea

This gives us a way of constructing infinite versions of gadgets whose finite versions we can construct simply by induction.
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This gives us a way of constructing infinite versions of gadgets whose finite versions we can construct simply by induction.

Aim

—to apply this to Trimble’s version of weak $n$-categories.
2. Some theory of terminal coalgebras

Problem
2. Some theory of terminal coalgebras

Problem

- a strict $\omega$-category is built from its $n$-truncations, which are strict $n$-categories
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- a strict $\omega$-category is built from its $n$-truncations, which are strict $n$-categories
- however if we truncate a weak $\omega$-category we do not get a weak $n$-category
Problem

- A strict \(\omega\)-category is built from its \(n\)-truncations, which are strict \(n\)-categories.
- However, if we truncate a weak \(\omega\)-category, we do not get a weak \(n\)-category.
  
We get something incoherent at dimension \(n\).
2. Some theory of terminal coalgebras

Problem

- a strict $\omega$-category is built from its $n$-truncations, which are strict $n$-categories
- however if we truncate a weak $\omega$-category we do not get a weak $n$-category

— we get something incoherent at dimension $n$

So we need to build weak $\omega$-categories from

“incoherent $n$-categories”
3. Trimble-like weak $n$-categories

Trimble's idea for weak $n$-categories:

• enrich in $(n-1)$-$\text{Cat}$, and
• weaken the composition using an operad.

What does "weak" mean?
3. Trimble-like weak $n$-categories

Trimble’s idea for weak $n$-categories:
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3. Trimble-like weak \(n\)-categories

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3. Trimble-like weak $n$-categories

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Trimble’s idea for weak $n$-categories:

- enrich in $(n - 1)$-$\text{Cat}$, and
- weaken the composition using an operad.

What does “weak” mean?
3. Trimble-like weak $n$-categories

Given a diagram

\[
\begin{array}{cccccccc}
  a_0 & \xrightarrow{f_1} & a_1 & \xrightarrow{f_2} & \cdots & \xrightarrow{} & a_{k-1} & \xrightarrow{f_k} & a_k \\
\end{array}
\]
3. Trimble-like weak $n$-categories

Given a diagram

$$a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} \cdots \xrightarrow{} a_{k-1} \xrightarrow{f_k} a_k$$

we have
3. Trimble-like weak $n$-categories

Given a diagram

\[
\begin{array}{c}
  a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} \cdots \xrightarrow{f_k} a_{k-1} \xrightarrow{f_k} a_k
\end{array}
\]

we have many composites.
3. Trimble-like weak $n$-categories

Given a diagram

\[
a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} \cdots \rightarrow a_{k-1} \xrightarrow{f_k} a_k
\]

we have many composites.

Given a diagram
3. Trimble-like weak $n$-categories

Given a diagram

$$a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} \ldots \xrightarrow{} a_{k-1} \xrightarrow{f_k} a_k$$

we have many composites.

Given a diagram

we have
3. Trimble-like weak $n$-categories

Given a diagram

$$a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} \cdots \xrightarrow{} a_{k-1} \xrightarrow{f_k} a_k$$

we have many composites.

Given a diagram

we have very many composites.
3. Trimble-like weak $n$-categories
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Idea
3. Trimble-like weak $n$-categories

Idea

Fix an operad $P$ in a symmetric monoidal category $\mathcal{V}$. 
3. Trimble-like weak $n$-categories

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Fix an operad $P$ in a symmetric monoidal category $\mathcal{V}$.

A $(\mathcal{V}, P)$-category
3. Trimble-like weak \( n \)-categories

Idea

Fix an operad \( P \) in a symmetric monoidal category \( \mathcal{V} \).

A \((\mathcal{V}, P)\)-category is defined to be a cross between
3. Trimble-like weak $n$-categories

Idea

Fix an operad $P$ in a symmetric monoidal category $\mathcal{V}$.

A $(\mathcal{V}, P)$-category is defined to be a cross between

- a $\mathcal{V}$-category, and
3. Trimble-like weak $n$-categories

Idea

Fix an operad $P$ in a symmetric monoidal category $\mathcal{V}$.

A $(\mathcal{V}, P)$-category is defined to be a cross between

- a $\mathcal{V}$-category, and
- a $P$-algebra.
3. Trimble-like weak $n$-categories

Idea

Fix an operad $P$ in a symmetric monoidal category $\mathcal{V}$.

A $(\mathcal{V}, P)$-category is defined to be a cross between

- a $\mathcal{V}$-category, and
- a $P$-algebra.

—The underlying data is a $\mathcal{V}$-graph
3. Trimble-like weak $n$-categories

Idea

Fix an operad $P$ in a symmetric monoidal category $\mathcal{V}$.

A $(\mathcal{V}, P)$-category is defined to be a cross between

- a $\mathcal{V}$-category, and
- a $P$-algebra.

—The underlying data is a $\mathcal{V}$-graph but composition is like a $P$-algebra action.
3. Trimble-like weak $n$-categories

- Composition in an ordinary $\mathcal{V}$-category:

$$A(a_{k-1}, a_k) \times \cdots \times A(a_0, a_1) \longrightarrow A(a_0, a_k)$$
3. Trimble-like weak $n$-categories

- Composition in an ordinary $\mathcal{V}$-category:
  \[
  A(a_{k-1}, a_k) \times \cdots \times A(a_0, a_1) \longrightarrow A(a_0, a_k)
  \]

- $P$-algebra action:
  \[
  P(k) \times A \times \cdots \times A \longrightarrow A
  \]
3. Trimble-like weak $n$-categories

- Composition in an ordinary $\mathcal{V}$-category:
  \[ A(a_{k-1}, a_k) \times \cdots \times A(a_0, a_1) \longrightarrow A(a_0, a_k) \]

- $P$-algebra action:
  \[ P(k) \times A \times \cdots \times A \longrightarrow A \]

- Composition in a $(\mathcal{V}, P)$-category:
  \[ P(k) \times A(a_{k-1}, a_k) \times \cdots \times A(a_0, a_1) \longrightarrow A(a_0, a_k) \]
3. Trimble-like weak $n$-categories

We can then build weak $n$-categories:
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- $0$-$\text{Cat} := \text{Set}$
3. Trimble-like weak $n$-categories

We can then build weak $n$-categories:

- $0\text{-Cat} := \text{Set}$
- $(n + 1)\text{-Cat} := (n\text{-Cat}, P_n)\text{-Cat}$
3. Trimble-like weak $n$-categories

We can then build weak $n$-categories:

- $0\text{-Cat} := \text{Set}$
- $(n + 1)\text{-Cat} := (n\text{-Cat}, P_n)\text{-Cat}$

But what operads $P_n$ are we going to use?
3. Trimble-like weak $n$-categories

Trimble’s method
3. Trimble-like weak $n$-categories

Trimble’s method

- start with just one operad $E \in \text{Top}$
3. Trimble-like weak $n$-categories

Trimble’s method

- start with just one operad $E \in \text{Top}$
- take each $P_n(k)$ to be the fundamental $n$-groupoid of $E(k)$
3. Trimble-like weak $n$-categories

Trimble’s method

- start with just one operad $E \in \text{Top}$
- take each $P_n(k)$ to be the fundamental $n$-groupoid of $E(k)$

So instead of picking one operad $P_n$ for each $n$, we just have to construct for each $n$

$$\Pi_n : \text{Top} \longrightarrow n\text{-Cat}$$
3. Trimble-like weak $n$-categories

Trimble’s method

- start with just one operad $E \in \text{Top}$
- take each $P_n(k)$ to be the fundamental $n$-groupoid of $E(k)$

So instead of picking one operad $P_n$ for each $n$, we just have to construct for each $n$

$$\Pi_n : \text{Top} \longrightarrow n\text{-Cat}$$

and this turns out to be easy by induction.
3. Trimble-like weak $n$-categories

Induction step for $\Pi$
3. Trimble-like weak $n$-categories

Induction step for $\Pi$

Given a finite product preserving functor

$$\Pi : \mathbf{Top} \longrightarrow \mathcal{V}$$
3. Trimble-like weak $n$-categories

Induction step for $\Pi$

Given a finite product preserving functor

$$\Pi : \text{Top} \longrightarrow \mathcal{V}$$

we induce a functor

$$\Pi^+ : \text{Top} \longrightarrow (\mathcal{V}, \Pi E)\text{-Cat}$$
3. Trimble-like weak $n$-categories

Induction step for $\Pi$

Given a finite product preserving functor

$$\Pi : \text{Top} \longrightarrow \mathcal{V}$$

we induce a functor

$$\Pi^+ : \text{Top} \longrightarrow (\mathcal{V}, \Pi E)\text{-Cat}$$

“do $\Pi$ locally on the hom objects”
3. Trimble-like weak $n$-categories

Trimble $n$-categories by induction
3. Trimble-like weak $n$-categories

Trimble $n$-categories by induction

- $0$-$\text{Cat} = \text{Set}$
3. Trimble-like weak $n$-categories

Trimble $n$-categories by induction

- $0$-$\text{Cat} = \text{Set}$
  \[ \Pi_0 : \text{Top} \longrightarrow \text{Set} \]
3. Trimble-like weak $n$-categories

Trimble $n$-categories by induction

- $0$-$\text{Cat} = \text{Set}$
  \[ \Pi_0 : \text{Top} \rightarrow \text{Set} \]
  \[ X \mapsto \text{the set of connected components of } X \]
3. Trimble-like weak $n$-categories

Trimble $n$-categories by induction

- $0$-Cat = Set
  \[ \Pi_0 : \text{Top} \rightarrow \text{Set} \]
  \[ X \mapsto \text{the set of connected components of } X \]

- $(n + 1)$-Cat = $(n$-Cat, $\Pi_nE)$-Cat
3. Trimble-like weak $n$-categories

Trimble $n$-categories by induction

- $0$-$\text{Cat} = \text{Set}$
  
  $\Pi_0 : \text{Top} \rightarrow \text{Set}$
  
  $X \mapsto$ the set of connected components of $X$

- $(n + 1)$-$\text{Cat} = (n$-$\text{Cat}, \Pi_n E)$-$\text{Cat}$
  
  $\Pi_{n+1} = \Pi_n^+$
3. Trimble-like weak $n$-categories

Incoherent

Trimble $n$-categories by induction

- $0$-$\text{iCat} = \text{Set}$
  
  \[ \Pi_0 : \text{Top} \longrightarrow \text{Set} \]
  
  \[ X \mapsto \text{the set of connected components of} \ X \]

- $(n + 1)$-$\text{iCat} = (n$-$\text{iCat}, \Pi_n E)$-$\text{Cat}$

  \[ \Pi_{n+1} = \Pi_n^+ \]
3. Trimble-like weak $n$-categories

Incoherent

Trimble $n$-categories by induction

- $0\text{-iCat} = \text{Set}$
  \[
  \Pi_0 : \text{Top} \longrightarrow \text{Set} \quad \text{points}
  \]
  \[
  X \longmapsto \text{the set of connected components of } X
  \]

- $(n+1)\text{-iCat} = (n\text{-iCat}, \Pi_n E)\text{-Cat}$
  \[
  \Pi_{n+1} = \Pi_n^+
  \]
4. Trimble-like weak $\omega$-categories
4. Trimble-like weak $\omega$-categories

For $\omega$-categories we take the following limit

$$
\cdots \longrightarrow 2\text{-iCat} \longrightarrow 1\text{-iCat} \longrightarrow 0\text{-iCat} \overset{!}{\longrightarrow} 1
$$
4. Trimble-like weak $\omega$-categories

For $\omega$-categories we take the following limit

$$
\cdots \rightarrow 2\text{-iCat} \rightarrow 1\text{-iCat} \rightarrow 0\text{-iCat} \rightarrow \mathbb{1}
$$

where each morphism is truncation.
4. Trimble-like weak $\omega$-categories

For $\omega$-categories we take the following limit

$$
\cdots \rightarrow 2\text{-iCat} \rightarrow 1\text{-iCat} \rightarrow 0\text{-iCat} \rightarrow \mathbb{1}
$$

where each morphism is truncation.

Finally: can we get this as
4. Trimble-like weak $\omega$-categories

For $\omega$-categories we take the following limit

$$\cdots \rightarrow 2\text{-iCat} \rightarrow 1\text{-iCat} \rightarrow 0\text{-iCat} \rightarrow \mathbb{1}$$

where each morphism is truncation.

Finally: can we get this as

$$\cdots \xrightarrow{F^3!} F^3\mathbb{1} \xrightarrow{F^2!} F^2\mathbb{1} \xrightarrow{F!} F\mathbb{1} \rightarrow \mathbb{1}$$
4. Trimble-like weak $\omega$-categories

For $\omega$-categories we take the following limit

$$\cdots \longrightarrow 2\text{-iCat} \longrightarrow 1\text{-iCat} \longrightarrow 0\text{-iCat} \longrightarrow 1$$

where each morphism is truncation.

Finally: can we get this as

$$\cdots \xrightarrow{F^3!} F^3 \mathbb{1} \xrightarrow{F^2!} F^2 \mathbb{1} \xrightarrow{F!} F \mathbb{1} \longrightarrow 1$$
4. Trimble-like weak $\omega$-categories

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so we use the obvious category with these objects.
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\[ F : (\mathcal{V}, \Pi) \mapsto ((\mathcal{V}, \Pi E)\text{-Cat}, \Pi^+) \]

so we use the obvious category with these objects.

Objects are pairs \((\mathcal{V}, \Pi)\) where

- \(\mathcal{V}\) is a category with finite products
- \(\Pi\) is a functor \(\text{Top} \rightarrow \mathcal{V}\) preserving finite products.

Morphisms are the obvious commuting triangles.
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Then the limit

$$\cdots \xrightarrow{F^3!} F^3 \mathbb{1} \xrightarrow{F^2!} F^2 \mathbb{1} \xrightarrow{F!} F \mathbb{1} \xrightarrow{!} \mathbb{1}$$
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Then the limit

$$\cdots \xrightarrow{F^3!} F^3 \mathbb{1} \xrightarrow{F^2!} F^2 \mathbb{1} \xrightarrow{F!} F \mathbb{1} \xrightarrow{!} \mathbb{1}$$

becomes
4. Trimble-like weak $\omega$-categories

Then the limit

$$
\cdots \xrightarrow{F^3!} F^31 \xrightarrow{F^2!} F^21 \xrightarrow{F!} F1 \xrightarrow{!} 1
$$

becomes

$$
\cdots \rightarrow 2\text{-iCat} \rightarrow 1\text{-iCat} \rightarrow 0\text{-iCat} \xrightarrow{!} 1
$$
4. Trimble-like weak $\omega$-categories

Then the limit

$$\cdots \xrightarrow{F^3!} F^3 \mathbb{1} \xrightarrow{F^2!} F^2 \mathbb{1} \xrightarrow{F!} F \mathbb{1} \xrightarrow{!} \mathbb{1}$$

becomes

$$\cdots \longrightarrow 2\text{-}i\text{Cat} \longrightarrow 1\text{-}i\text{Cat} \longrightarrow 0\text{-}i\text{Cat} \xrightarrow{!} \mathbb{1}$$

The terminal coalgebra is indeed the limit we were looking for.