Extensivity, rig geometry and core varieties

F. W. Lawvere

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An axiomatic theory often captures more examples than originally intended. In the present case not only the smooth generalization (SDG) but also some semi-combinatorial ones are of interest. Some of these can be approached via sites of definition that can be handled in ways very closely analogous to Grothendieck's algebraic geometry constructions. Even ultra-basic properties of cohesive categories, such as extensivity, may not be true in the sites themselves; a modest step toward the treatment of that problem is the recognition of some algebraic categories as *core varieties* within others. But first I recall another remarkable feature of many cohesive toposes that, although discovered through its relevance to differential equations, nevertheless points toward the power of map spaces in building combinatorial objects from simple ingredients.

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1. Euler's principle

Focussing on the role of map spaces in generating spaces reveals a distinct feature of algebraic geometry over rigs (a feature shared by smooth geometry where the category of algebras consists of C^∞ rings) that I call *Euler's principle.* He used the fact that macro-quantities are representable as ratios of infinitesimals. We can objectify this by exploiting the fact that the category of spaces has exponentiation. Considering the theory of K-rigs, let

$$D = \text{spec } K[\varepsilon] \quad \varepsilon^2 = 0$$

$$R = \text{spec } K[t]$$

where $K[\varepsilon]$ is the finitely-generated module $K \times K$ with the Leibniz multiplication and where $K[t]$ is the monoid rig over $\{t^n\}$. Then the point-preserving maps D ----> D are indeed ratios, and R, the space of these, is a retract.

$$\begin{array}{c} R \hookrightarrow D^D \longrightarrow R \\ \xrightarrow[\text{identity}]{\hspace{8em}} \end{array}$$

Note that this is true not only for K = real numbers but also, for example, for K = 2; even in combinatorial settings the infinitesimals D have a definite role, the tangent bundles (special map spaces) of affine spaces being still affine. Euler's principle recovers the finitely-generated algebras from the infinitesimal ones precisely, and so is much stronger than the Birkhoff Nullstellensatz which only represents them up to a monomorphism.

For a combinatorial example of a closely analogous phenomenon, let D denote instead the reflexive graph with just one non-degenerate loop. The self-exponential of D contains the generic arrow, which is a generator for the entire topos of reflexive graphs.

2. Extensivity and the disjoint covering topology

A small category C with coproducts is extensive if and only if the category of product-preserving presheaves is a topos $G(C)$. If C is the opposite of the category of finitely-presented algebras for an algebraic theory, we may briefly refer to the whole category of algebras as *co-extensive*; in case C has finite limits we say for short that C is *lextensive* and note that such categories satisfy the distributivity of products over coproducts. The topos $G(C)$ is the classifying topos for a certain strengthening of the equational theory. To understand what sort of strengthening, we consider some examples.

3. Rigs and generalizations

The category of rigs is co-extensive; that is seen concretely via partitions of unity by disjoint idempotents: if $s_i\ i \varepsilon I$ is a finite family of elements of a rig for which $s_i s_j = 0$ for $i \neq j$ while $\Sigma\, s_i = 1$, then the rig is uniquely a cartesian product of smaller rigs such that s_i becomes 1 in the ith factor.

There are many concrete generalizations based on the example of rigs. If K is a rig, then for any commutative monoid M, the distributive law yields a monoid K-rig

$$M \xrightarrow{t(\,)} K[M]$$

which is a polynomial rig if M is free. This theory of K-rigs is also co-extensive by the same calculation. More generally, the category of K-rigs acted on by a given group G (or a monoid, or a Lie algebra) is extensive. However, there are subexamples of these, not quite as obvious, that I will call *core varieties*.

4. Algebraic geometry where 1 + 1 = 1

Algebraic geometry over rigs (not only in the traditional cases $K = Z$, or K = a field) is a suggested aid to understanding not only $K = N$, but also $K = 2$ (wherein $1 + 1 = 1$). Schanuel has shown that the simple objects in the category of rigs are the fields in the usual sense, plus the single additional example 2 itself. Note that the example that launched ring theory was not a ring but a 2-rig, namely the system of ideals of a ring. Schanuel's [3] classification of the spaces in certain lextensive categories required a pair of invariants: an 'Euler characteristic' in the ring $\mathbf{Z} \otimes B$ but also a 'dimension' in the 2-rig $2 \otimes B$ derived from the rig B of isomorphism classes of spaces. Because

$$\dim(X+Y) = \max(\dim X, \dim Y)$$

addition is idempotent in the dimension rig, but multiplication is not. (In simple cases the usual dimension numbers can be read off logarithmically.) At the level of abstract algebra, a monoid rig over 2

$$2[M]$$

just consists of all finite subsets of the monoid, with the usual set-wise product and with union as addition.

Algebraic geometry constructs categories $\mathcal{X}$ of spaces from categories $\mathcal{A}$ of algebras in such a way that every space has an algebra of functions and every algebra has a spectral space, a contravariant adjointness; experience shows that it is not a duality in the naive sense of equivalence. More specifically, the finitely-presented algebras

$$C \subset \mathcal{A}^{op}$$

permit representing $\mathcal{A}$ as the left exact functors on C and $\mathcal{X}$ as a subtopos of the classifying topos (consisting of all contravariant functors on C), whereas the Yoneda embedding Kan-extends to spec: $\mathcal{A}^{op}$ --->$\mathcal{X}$. The need for a subtopos (i.e. for Grothendieck topology) is due to the fact that neither the 'affine schemes' C *nor* the presheaves have the geometrically correct colimits!

5. The inexactness of affine schemes

The co-extensivity of the algebra permits a first step toward resolving the colimit problem, because it suggests that we can assume that $\mathcal{X} \hookrightarrow G(C)$. Because of this trivial disposal of finite disjoint families the study of deeper choices of Grothendieck topology can focus on coverings by single maps.

While a category of algebras is Barr-exact, its opposite C is not (even when it is extensive). Yet that opposite category is confronted with the pretopos exactness of spaces via the spec construction. I conjecture that in a suitable sense lextensivity is all the exactness these two categories have in common, so that lextensive categories form the smallest reasonable common umbrella for this basic construction. Though pretoposes have exactness properties that are clearly 'good', one should not conclude that C is 'bad'; the category of pointed objects in C supports a whole category of modules as abelian group objects in its opposite (whereas that construction would fail in a pretopos).

6. Core varieties

I follow Birkhoff in calling a *variety* any subcategory of a given algebraic category which is in its own right also an algebraic category, in such a way that the inclusion is induced by a surjective map of theories (there are typically non-varieties that are full algebraic subcategories, but with the inclusion induced by introducing inverses for given operations, rather than by merely imposing identities on given operations). When is a variety in an extensive algebraic category also extensive? In general, extensivity is not hereditary in that sense, but there is a specific approach to certain of those subvarieties.

In an extensive category the summands of X induced by a given map to a coproduct are in fact pullbacks. In an opposite algebraic category these pullbacks are of course amalgamated coproducts (amalgamated tensor products in the case of K-rigs). Thus co-extensivity of a variety is easily verified if finite colimits there agree with the colimits in the ambient co-extensive algebraic category. Of course that also often fails, but there is a standard way to guarantee it.

If a functor has a right adjoint, then it preserves coproducts. Thus a further extensive theory arises if we can find, inside a given co-extensive algebraic category, a *core variety*, meaning a variety whose inclusion has a further right adjoint, for then those algebras satisfying the additional identities will be closed under the ambient $\otimes$. Gavin Wraith [4] studied algebraic functors that have (besides the usual left adjoint relatively free functor) also a right adjoint, but because we are limiting ourselves here to full inclusions, in the single-sorted case a 'core' of this kind is given simply by a set of unary equations such that, for any algebra, the subset satisfying them is an algebra. Because $K[t]$ is free, its core will typically be only K (even if we assume that K itself satisfies the additional identities); the new algebraic theory is as usual given by the reflection, not the coreflection.

7. Fixed points of Frobenius in arbitrary toposes

A particular way to obtain a core variety is to find a central unary operation of a theory, i.e. an operation whose application is an algebra endomorphism, and then define a core to consist of its fixed points. For example, in the category of K-algebras where K is a given finite field, suitable powers are homomorphisms and hence define cores. Because these algebraic adjoints also induce additional operations in the corresponding geometric toposes, the Frobenius operations of algebraic geometry are seen to have a rather precise analog also in general algebra. There is a formulation independent of the particular algebraic theory: the center of a topos E (or any category) is the commutative monoid of natural endomorphisms of the identity functor; if F is the poset of its submonoids, the Frobenius analysis of each object is the E-valued presheaf on F obtained by extracting fixed points.

8. Distributive lattices, tropical geometry, dimension rigs

Of course, the equations defining a core need not involve homomorphisms, it is only the closure properties of the result that are required. For example, among 2-rigs we might consider those where (not only addition but) multiplication is idempotent; however, $x+1$ is not necessarily idempotent even if x is, so this variety is not a core variety. But if we adjoin a second equation $x + 1 = 1$, we find that the set of elements in a 2-rig satisfying

$$x^2 = x$$

$$x + 1 = 1$$

is closed under addition and multiplication and that, moreover, multiplication is the infimum in the semilattice ordering given by addition. This core variety is the algebraic category of distributive lattices.

What is algebraic geometry over distributive lattices?

The full classifying topos for the algebraic theory is the category of presheaves on the category C of finite partially-ordered sets. The subtopos $G(C)$ classifies those distributive lattices which are 'connected' in that

$st = 0$ & $s + t = 1$ entails $s = 1$ or $t = 1$.

Of course there are still smaller subtoposes (though not of the core type). The simplicial sets classify the totally-ordered distributive lattices; in that context the spectrum of a distributive lattice is the nerve of the corresponding poset. Intermediate is the analog of the Zariski topos that classifies 'local' distributive lattices; it is distinguished by the fact that the characteristic map of the unit is a homomorphism of rigs from the generic local lattice to the truth value object in its topos.

Consider the equation $x + 1 = 1$ by itself (i.e.the condition $x \leq 1$); in a 2-rig the elements satisfying that equation are closed under addition and multiplication. This identity implies that whenever b divides a one has also $a \leq b$; thus the free algebra on one generator x for the subcategory can be calculated to consist just of $0 = x^{-\infty}, \ldots, x^n, \ldots x^2, x, x^0 = 1$; its spectrum can be visualized as an interval I and the spectra of the free algebras can be visualized as cubes I^n. (One can alternatively view I as an extended positive line by reading the structure logarithmically; this example suggests a topos simplification of the recently-popular 'tropical geometry'.)

The 'opposite' inequality $1 \leq x$ would exclude 0, but if we consider a slight generalization

$$x \leq x^2, \quad (\text{i.e. } x + x^2 = x^2)$$

we find again a core variety in 2-rigs and hence a potentially important algebraic geometry. The Schanuel dimension rig of a lextensive category will lie in this core variety in case all objects are separable, because the complement of the diagonal yields, under the map from B to $2 \otimes B$, a proof that $x \leq x^2$.

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References

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