Spans for 2-Categories and Comma Objects

$$\begin{tikzcd} & D \arrow[dr] & \\ R \arrow[ur] \arrow[rr] & & P \end{tikzcd}$$

R. Dawson, R. Paré, D. Pronk

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SPANS FOR A 2-CATEGORY $\underline{\underline{A}}$

CONSTRUCT AN OPLAX DOUBLE CATEGORY

(C.F. LEINSTER'S FC-CATEGORIES) $\mathbb{S}\underline{\underline{A}}$

- SAME OBJECTS AS $\underline{\underline{A}}$
- HORIZONTAL ARROWS - SPANS OF $\underline{\underline{A}}$
- VERTICAL ARROWS - THE ARROWS OF $\underline{\underline{A}}$
- THE CELLS

$$\begin{tikzcd}
A \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & & & & S \arrow[llll] \arrow[rrrr] \arrow[dlll] \arrow[dl] \arrow[dr] \arrow[drrr] \arrow[dll, phantom, "\Rightarrow"] \arrow[d, phantom, "\Rightarrow"] & & & & B \arrow[d] \arrow[dl, phantom, "\Rightarrow"] \\
C_0 & T_1 \arrow[l] \arrow[r] & C_1 & T_2 \arrow[l] \arrow[r] & C_2 & T_3 \arrow[l] \arrow[r] & \cdots & T_n \arrow[l] \arrow[r] & C_n
\end{tikzcd}$$

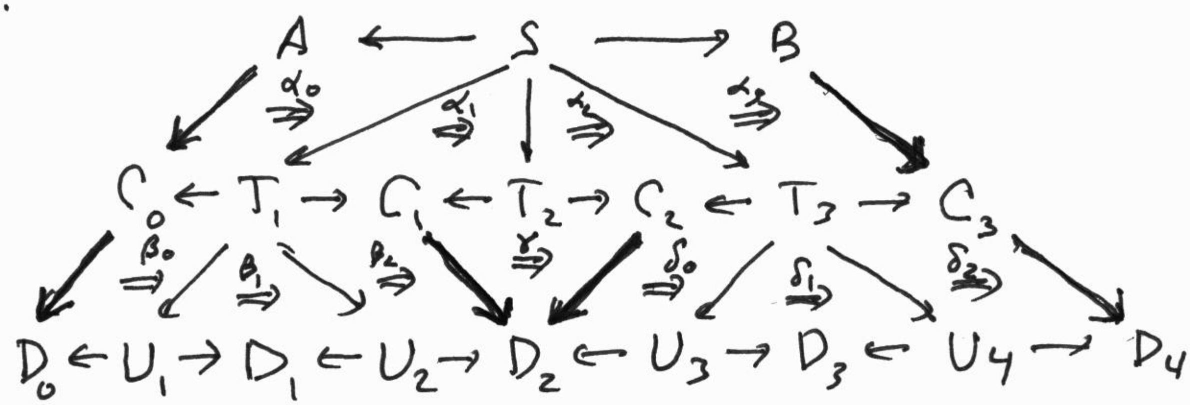
$(n \geq 0)$

- IDENTITIES

$$\begin{tikzcd}
A \arrow[d, equal] \arrow[r, phantom, shift right=6, "\xRightarrow{1}"] & S \arrow[l] \arrow[r] \arrow[d, equal] \arrow[r, phantom, shift right=6, "\xRightarrow{1}"] & B \arrow[d, equal] \\
A & S \arrow[l] \arrow[r] & B
\end{tikzcd}$$

COMPOSITION (VERTICAL)

E.X.

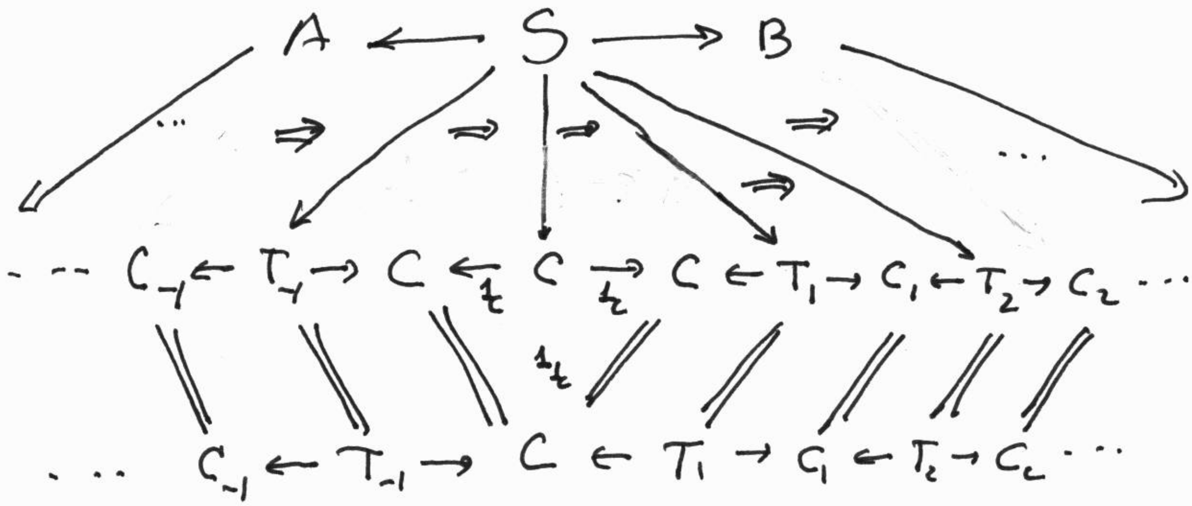


(THE COMPOSITE $(\delta, \gamma, \beta)\cdot\alpha$)

PROP: $\$\underline{\underline{A}}$ IS AN OPLAX DOUBLE CATEGORY.

$\$\underline{\underline{A}}$ IS NOT NORMAL – IDENTITIES NOT REP.

I.E. COMPOSING



IS NOT A BIJECTION.

THE 3-DIMENSIONAL STRUCTURE OF SPANS

AUGMENT $\mathcal{S}\underline{\underline{A}}$ to $\mathcal{S}\mathrm{PAN}\,\underline{\underline{A}}$ BY ADDING 3-CELLS:

$$\begin{tikzcd}
& A \arrow[dl, "f_0"'] \arrow[dr, phantom, "\Rightarrow\alpha_0"] & & & S \arrow[lll] \arrow[rrr] \arrow[dlll, "f_1"'] \arrow[dl, "f_2"'] \arrow[dr, "f_3"] \arrow[drrr, "f_n"] & & & B \arrow[dr, "f_{n+1}"] \arrow[d, phantom, "\Rightarrow\alpha_n"] & \\
C_0 & T_1 \arrow[l] \arrow[r] & C_1 \arrow[r, phantom, "\Rightarrow\alpha_1"] & T_2 \arrow[l] \arrow[r] & C_2 \arrow[r, phantom, "\Rightarrow\alpha_2"] & T_3 \arrow[l] \arrow[r] & \cdots & T_n \arrow[l] \arrow[r] & C_n
\end{tikzcd}$$

$$\begin{tikzcd}
& A \arrow[dl, "f_0 = g_0"'] \arrow[dr, phantom, "\Rightarrow\beta_0"] & & & S \arrow[lll] \arrow[rrr] \arrow[dlll, "g_1"'] \arrow[dl, "g_2"'] \arrow[dr, "g_3"] \arrow[drrr, "g_n"] & & & B \arrow[dr, "g_{n+1} = f_{n+1}"] \arrow[d, phantom, "\Rightarrow\beta_n"] & \\
C_0 & T_1 \arrow[l] \arrow[r] & C_1 \arrow[r, phantom, "\Rightarrow\beta_1"] & T_2 \arrow[l] \arrow[r] & C_2 \arrow[r, phantom, "\Rightarrow\beta_2"] & T_3 \arrow[l] \arrow[r] & \cdots & T_n \arrow[l] \arrow[r] & C_n
\end{tikzcd}$$

(SAME BOUNDARY)

A 3-CELL IS $\langle \lambda_i : f_i \to g_i \rangle_{i=1,\dots,n}$

SUCH THAT

$$\begin{tikzcd}
& S \arrow[dl, bend right, "f_i"'] \arrow[dl, "g_i"] \arrow[dr, "g_{i+1}"] \arrow[d, phantom, "\Rightarrow\beta_i"] & \\
T_i \arrow[r] \arrow[r, phantom, ""] & C_i & T_{i+1} \arrow[l]
\end{tikzcd}
\quad
\begin{tikzcd}
& S \arrow[dl, "f_i"'] \arrow[dr, "f_{i+1}"'] \arrow[dr, bend left, "g_{i+1}"] \arrow[d, phantom, "\Rightarrow\alpha_i"] & \\
T_i \arrow[r] & C_i & T_{i+1} \arrow[l]
\end{tikzcd}$$

I.E. A CELL

$$\begin{tikzcd}
& A \arrow[dl] \arrow[dr, phantom, "\Rightarrow"] & & 2\times S \arrow[ll] \arrow[rr] \arrow[dll] \arrow[d] \arrow[drr] \arrow[dl, phantom, "\Rightarrow"] \arrow[dr, phantom, "\Rightarrow"] & & B \arrow[dr] \arrow[d, phantom, "\Rightarrow"] & \\
C_0 & T_1 \arrow[l] \arrow[r] & C_1 & T_2 \arrow[l] \arrow[r] & \cdots & T_n \arrow[l] \arrow[r] & C_n
\end{tikzcd}$$

(IF $2\times S$ EXISTS)

THE OPLAX DOUBLE CATEGORY $\mathbb{S}\mathrm{PAN}\,\underline{\underline{A}}$

COLLAPSE THE 3-CELLS BY TAKING π_0, CONNECTED COMPONENTS. THIS GIVES THE OPLAX DOUBLE CATEGORY $\mathbb{S}\mathrm{PAN}\,\underline{\underline{A}}$ WITH THE RIGHT PROPERTIES.

AN OPLAX DOUBLE CATEGORY IS <u>NORMAL</u> IF FOR EVERY OBJECT B THERE IS AN ARROW $1_B : B \to B$ AND A CELL 2_B S.T.

$\forall \alpha$

$$\begin{tikzcd}
 & & A \arrow[rrrr, "a"] \arrow[rrrr, phantom, "\alpha"', yshift=-4ex] \arrow[dll] & & & & A' \arrow[drr] & & \\
B_{-k} \arrow[r] & \cdots B_{-2} \arrow[r] & B_{-1} \arrow[r] & B \arrow[r] & B_1 \arrow[r] & B_2 \arrow[r] & \cdots \arrow[r] & B_n
\end{tikzcd}$$

$\exists ! \, \bar{\alpha}$

$$\begin{tikzcd}
 & & A \arrow[rrrrr, "a"] \arrow[rrrrr, phantom, "\bar{\alpha}"', yshift=-4ex] \arrow[dll] & & & & & A' \arrow[drr] & & \\
B_{-k} \arrow[r] \arrow[d, equal] & \cdots B_{-2} \arrow[r] \arrow[d, equal] & B_{-1} \arrow[r] \arrow[d, equal] & B \arrow[r, "1_B"] \arrow[d] \arrow[r, phantom, "2_B"', yshift=-5ex] & B \arrow[r] \arrow[dl] & B_1 \arrow[r] \arrow[d, equal] & B_2 \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & B_n \arrow[d, equal] \\
B_{-k} \arrow[r] & \cdots B_{-2} \arrow[r] & B_{-1} \arrow[r] & B \arrow[rr] & & B_1 \arrow[r] & B_2 \arrow[r] & \cdots \arrow[r] & B_n
\end{tikzcd} \quad = \alpha$$

PROP: $\mathcal{S}\mathrm{PAN}\,\underline{\underline{A}}$ IS NORMAL.

PROOF: GIVEN

$$\begin{tikzcd}
 & A \arrow[dl, "f_0"'] & & S \arrow[ll] \arrow[dll, "f_1"'] \arrow[drr, "f_2"] \arrow[rr] & & A' \arrow[dr, "f_3"] & \\
B_{-1} \arrow[rr, phantom, "\overset{\alpha_0}{\Rightarrow}"] & & T_{-1} \arrow[ll] \arrow[rr, "t'"'] \arrow[r, phantom, "\overset{\alpha_1}{\Rightarrow}"] & & B & T_1 \arrow[l, "t"] \arrow[r] \arrow[r, phantom, "\overset{\alpha_2}{\Rightarrow}"] & B_1
\end{tikzcd} \tag{1}$$

LIFTS TO

$$\begin{tikzcd}
 & A \arrow[dl, "f_0"'] & & S \arrow[ll] \arrow[dll, "f_1"'] \arrow[d, "tf_2"] \arrow[drr, "f_2"] \arrow[rr] & & A' \arrow[dr, "f_3"] & \\
B_{-1} \arrow[rr, phantom, "\overset{\alpha_0}{\Rightarrow}"] & T_{-1} \arrow[l] \arrow[r, "t'"'] \arrow[rr, phantom, "\overset{\alpha_1}{\Rightarrow}"] & B & B \arrow[l, "1"'] \arrow[r, "1"] \arrow[r, phantom, "="] & B & T_1 \arrow[l, "t"] \arrow[r] \arrow[r, phantom, "\overset{\alpha_2}{\Rightarrow}"] & B_1
\end{tikzcd}$$

.

GIVEN ANOTHER LIFTING

$$\begin{tikzcd}
 & A \arrow[dl, "g_0"'] & & S \arrow[ll] \arrow[dll, "g_1"'] \arrow[d, "g_2"] \arrow[drr, "g_3"] \arrow[rr] & & A' \arrow[dr, "g_4"] & \\
B_{-1} \arrow[dr, equal] \arrow[rr, phantom, "\overset{\beta_0}{\Rightarrow}"] & T_{-1} \arrow[l] \arrow[r] \arrow[dr, equal] \arrow[rr, phantom, "\overset{\beta_1}{\Rightarrow}"] & B \arrow[dr, equal] & B \arrow[l] \arrow[r] \arrow[rr, phantom, "\overset{\beta_2}{\Rightarrow}"] & B \arrow[dl, equal] & T_1 \arrow[l] \arrow[r] \arrow[dl, equal] \arrow[r, phantom, "\overset{\beta_3}{\Rightarrow}"] & B_1 \arrow[dl, equal] \\
 & B_{-1} & T_{-1} \arrow[l] \arrow[r] & B & T_1 \arrow[l] \arrow[r] & B_1 &
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[d, "g_0"'] \arrow[r, phantom, "\overset{\beta_0}{\Rightarrow}"] & & S \arrow[ll] \arrow[dl, "g_1"'] \arrow[dr, "g_3"] \arrow[rr] & & A' \arrow[d, "g_4"] \\
B_{-1} & T_{-1} \arrow[l] \arrow[r] \arrow[rr, phantom, "\overset{\beta_2\cdot\beta_1}{\Rightarrow}"] & B & T_1 \arrow[l] \arrow[r] \arrow[r, phantom, "\overset{\beta_3}{\Rightarrow}"] & B_1
\end{tikzcd}$$

IF THIS IS EQUAL TO (1) THEN $f_0 = g_0$, $f_3 = g_4$

AND THERE IS A ZIG-ZAG SEQUENCE

$\langle \lambda_{1i}, \lambda_{2i} \rangle$ RELATING f_1 TO g_1, f_2 TO g_3

α_0 TO β_0, α_1 TO $\beta_2 \cdot \beta_1$, α_2 TO β_3.

THEN $\langle \lambda_{1i}, t\lambda_{2i}, \lambda_{2i} \rangle$ RELATES

$$\begin{tikzcd}
 & A \arrow[dl] & & S \arrow[ll] \arrow[rr] \arrow[dll] \arrow[d] \arrow[drr] & & A' \arrow[dr] & \\
B_{-1} & T_{-1} \arrow[l] \arrow[r] \arrow[u, phantom, "\Rightarrow^{\alpha_0}"] & B \arrow[ur, phantom, "\Rightarrow^{g_1}"] & B \arrow[l] \arrow[r] \arrow[ur, phantom, "="] & B & T_1 \arrow[l] \arrow[r] \arrow[u, phantom, "\Rightarrow^{\alpha_2}"] & B_1
\end{tikzcd}$$

To

$$\begin{tikzcd}
 & A \arrow[dl, "g_0"'] & & S \arrow[ll] \arrow[rr] \arrow[dll, "g_1"'] \arrow[d, "tg_3"] \arrow[dr, "g_3"] & & A' \arrow[dr, "g_4"] & \\
B_{-1} & T_{-1} \arrow[l] \arrow[r] \arrow[u, phantom, "\Rightarrow^{\beta_0}"] & B \arrow[ur, phantom, "\Rightarrow^{\beta_2 \cdot \beta_1}"] & B \arrow[l] \arrow[r] \arrow[ur, phantom, "="] & B & T_1 \arrow[l] \arrow[r] \arrow[u, phantom, "\Rightarrow^{\beta_3}"] & B_1
\end{tikzcd}$$

WHICH IS RELATED TO

$$\begin{tikzcd}
 & A \arrow[dl, "g_0"'] & & S \arrow[ll] \arrow[rr] \arrow[dll, "g_1"'] \arrow[d, "g_2"] \arrow[dr, "g_3"] & & A' \arrow[dr, "g_4"] & \\
B_{-1} & T_{-1} \arrow[l] \arrow[r] \arrow[u, phantom, "\Rightarrow^{\beta_0}"] & B \arrow[ur, phantom, "\Rightarrow^{\beta_1}"] & B \arrow[l] \arrow[r] \arrow[ur, phantom, "\Rightarrow^{\beta_2}"] & B & T_1 \arrow[l, "t"] \arrow[r] \arrow[u, phantom, "\Rightarrow^{\beta_3}"] & B_1
\end{tikzcd}$$

BY

$$\begin{tikzcd}
S \arrow[d, bend right, "g_1"'] \arrow[d, bend left, "g_1"] \arrow[d, phantom, "="] \\
T_{-1}
\end{tikzcd}
\qquad
\begin{tikzcd}
S \arrow[d, bend right, "g_2"'] \arrow[d, bend left, "tg_3"] \arrow[d, phantom, "\Rightarrow^{\beta_2}"] \\
B
\end{tikzcd}
\qquad
\begin{tikzcd}
S \arrow[d, bend right, "g_3"'] \arrow[d, bend left, "g_3"] \\
T_1
\end{tikzcd}$$

∎

NORMALITY IS IMPORTANT!

PROP: EVERY VERTICAL ARROW $f : A \to B$ IN $\mathbb{S}\mathrm{PAN}\,\underline{\underline{A}}$

HAS A <u>COMPANION</u> $f_* = (A \xleftarrow{1} A \xrightarrow{f} B)$

AND A <u>CONJOINT</u> $f^* = (B \xleftarrow{f} A \xrightarrow{1} A)$.

"<u>PROOF</u>": THE BINDING CELLS FOR f_*

ARE

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, phantom, "="] & A \arrow[l, "1"'] \arrow[r, "f"] \arrow[d, "f"] \arrow[dr, phantom, "="] & B \arrow[d, equal] \\ B & B \arrow[l, "1"] \arrow[r, "1"'] & B \end{tikzcd}$$

AND

$$\begin{tikzcd} A \arrow[d, equal] \arrow[dr, phantom, "="] & A \arrow[l, "1"'] \arrow[r, "1"] \arrow[d, equal] \arrow[dr, phantom, "="] & A \arrow[d, "f"] \\ A & A \arrow[l, "1"] \arrow[r, "f"'] & B \end{tikzcd}$$

$\square$

PROP: IN $\mathbb{S}\mathrm{PAN}\,\underline{\underline{A}}$ COMPOSITES $f_* S$

$$A \xleftarrow[s]{} S \xrightarrow[s']{} B \xleftarrow[1]{} B \xrightarrow[f]{} C$$

EXIST (STRONGLY REPRESENTABLE)

AND ARE EQUAL TO $A \xleftarrow[s]{} S \xrightarrow[fs']{} C$.

"<u>PROOF</u>": THE UNIVERSAL CELL IS

$$\begin{tikzcd} & A \arrow[dl, equal] \arrow[d, phantom, "="] & S \arrow[l, "s"'] \arrow[r, "fs'"] \arrow[dl, equal] \arrow[dr, "s'"] \arrow[d, phantom, "="] & C \arrow[dr, equal] \arrow[d, phantom, "="] & \\ A & S \arrow[l, "s"] \arrow[r, "s'"'] & B & B \arrow[l, "1"] \arrow[r, "f"'] & C \end{tikzcd}$$

$\square$

<u>DUALITY</u> $\mathrm{SPAN}(\underline{\underline{A}}^{co}) \cong (\mathrm{SPAN}\,\underline{\underline{A}})^{op}$

<u>COR</u>: IN $\mathrm{SPAN}\,\underline{\underline{A}}$ COMPOSITES Sg^* EXIST.

<u>PROP</u>: IN $\mathrm{SPAN}\,\underline{\underline{A}}$, $f_* \dashv f^*$.

UNIT

$$\begin{tikzcd}[column sep=small] & & A \arrow[dll, no head, double] \arrow[dl, no head, double] \arrow[dr, no head, double] \arrow[drr, no head, double] & & \\ A & A \arrow[l] \arrow[r, "f"'] & B & A \arrow[l, "f"] \arrow[r, "1"'] & A \end{tikzcd}$$

$A \xleftarrow{1} A \xrightarrow{1} A$ (USES NORMAL)

COUNIT

$$\begin{tikzcd} B \arrow[d, equal] & A \arrow[l, "f"'] \arrow[r, "f"] \arrow[d, "f"] & B \arrow[d, equal] \\ B & B \arrow[l, "1"] \arrow[r, "1"'] & B \end{tikzcd}$$

(USES $B \xleftarrow{f} A \xrightarrow{f} B$

$= B \xleftarrow[f]{} A \xrightarrow[1]{} A \xleftarrow[1]{} A \xrightarrow[f]{} B$

$= f_* f^*$)

<u>PROP</u>: If $f \dashv u$ in $\underline{\underline{A}}$, THEN

$$A \xleftarrow{u} S \xrightarrow{v} B \quad \cong \quad A \xleftarrow{1} A \xrightarrow{vf} B.$$

<u>"PROOF"</u>:

$$\begin{tikzcd} A \arrow[d, equal] & S \arrow[l, "u"'] \arrow[r, "v"] \arrow[d, "u"] & B \arrow[d, equal] \\ A & A \arrow[l, "1"] \arrow[r, "vf"'] & B \end{tikzcd} \quad \& \quad \begin{tikzcd} A \arrow[d, equal] & A \arrow[l, "1"'] \arrow[r, "vf"] \arrow[d, "f"] & B \arrow[d, equal] \\ A & S \arrow[l, "u"] \arrow[r, "v"'] & B \end{tikzcd}$$

ARE INVERSE.

$\square$

QUINTETS $\underline{\underline{Q}}\underline{\underline{A}}$ CELLS

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & B \arrow[d, "k"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

PROP:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & B \arrow[d, "k"] \\ C \arrow[r, "g"'] & D \end{tikzcd} \longmapsto \begin{tikzcd} A \arrow[d, "h"'] \arrow[dr, phantom, "="] & A \arrow[l, "1"'] \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & B \arrow[d, "k"] \\ C & C \arrow[l, "1"] \arrow[r, "g"'] & D \end{tikzcd}$$

DEFINES A PSEUDO MORPHISM

$$\underline{\underline{Q}}\underline{\underline{A}} \xrightarrow{(\ )_*} \mathbf{Span}\,\underline{\underline{A}}$$

WHICH IS LOCALLY FULL & FAITHFUL.

"PROOF":

$$\begin{tikzcd} A \arrow[d, "h"'] \arrow[dr, phantom, "\overset{\alpha_0}{\Rightarrow}"] & A \arrow[l, "1"'] \arrow[r, "f"] \arrow[d, "l"'] \arrow[dr, phantom, "\overset{\alpha_1}{\Rightarrow}"] & B \arrow[d, "k"] \\ C & C \arrow[l, "1"] \arrow[r, "g"'] & D \end{tikzcd} = \begin{tikzcd} A \arrow[d, "h"'] \arrow[dr, phantom, "="] & A \arrow[l, "1"'] \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\overset{\alpha_1\cdot\alpha_0}{\Rightarrow}"] & B \arrow[d, "k"] \\ C & C \arrow[l, "1"] \arrow[r, "g"'] & D \end{tikzcd}$$

$\square$

CONSIDER A CELL

$$
\begin{tikzcd}
 & & A \arrow[dll, "f_0"'] & S \arrow[l, "s"'] \arrow[r, "s'"] \arrow[dll, "f_1"'] \arrow[d, "f_2"] \arrow[drr, "f_n"] & A' \arrow[drr, "f_{n+1}"] & & \\
B_0 & T_1 \arrow[l, "t_1"] \arrow[r, "t_1'"'] & B_1 & T_2 \arrow[l, "t_2"] \arrow[r, "t_2'"'] & \cdots & T_n \arrow[l, "t_n"] \arrow[r, "t_n'"'] & B_n
\end{tikzcd}
$$

with 2-cells $\alpha_0, \alpha_1, \ldots, \alpha_n$.

IN WHICH $t_i : T_i \longrightarrow B_{i-1}$ IS A FIBRATION. THEN WE CAN MAKE α_{i-1} INTO AN IDENTITY BY CHANGING f_i AND α_i BUT NOTHING ELSE.

- IF ALL t_i ARE FIBRATIONS WE CAN MAKE ALL THE α_i IDENTITIES EXCEPT α_n.
- IF FURTHERMORE $B_{n-1} \xleftarrow{t_n} T_n \xrightarrow{t_n'} B_n$ IS A BIFIBRATION WE CAN MAKE α_n AN IDENTITY TOO.
- EQUIVALENCE OF CELLS CAN BE REALIZED BY THIS KIND OF CELL.

COMMA OBJECTS

PROP: IF THE COMMA OBJECT

$$\begin{tikzcd} (s',t) \arrow[r, "p_1"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\Rightarrow"] & T \arrow[d, "t"] \\ S \arrow[r, "s'"'] & B \end{tikzcd}$$

EXISTS,

THEN THE COMPOSITE OF

$$A \xleftarrow[s]{} S \xrightarrow[s']{} B \xleftarrow[t]{} T \xrightarrow[t']{} C$$

IS STRONGLY REPRESENTED BY

$$A \xleftarrow[sp_0]{} (s',t) \xrightarrow[t'p_1]{} C .$$

COR: (BECK CONDITION)

IF

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "\Rightarrow"] & B \arrow[d, "k"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

IS A COMMA OBJECT,

THEN $f_* h^* \cong k^* g_*$,

$$\begin{tikzcd} A \arrow[r, "f_*"] \arrow[dr, phantom, "\overset{\cong}{\Rightarrow}"] & B \\ C \arrow[u, "h^*"] \arrow[r, "g_*"'] & D \arrow[u, "k^*"'] \end{tikzcd}$$

PROOF:

$$C \xleftarrow{1} C \xrightarrow{g} D \xleftarrow{k} B \xrightarrow{1} B \;\cong\; C \xleftarrow{h} A \xrightarrow{f} B$$

$$\cong\; C \xleftarrow{h} A \xrightarrow{1} A \xleftarrow{1} A \xrightarrow{f} B \qquad \square$$

ASSUME $\underline{\underline{A}}$ HAS COMMA OBJECTS.

- $$\begin{tikzcd} A^2 \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\Rightarrow"] & A \arrow[d, "1"] \\ A \arrow[r, "1"'] & A \end{tikzcd}$$ COMMA OBJECT $\Rightarrow$ $$\begin{tikzcd} & A^2 \arrow[dl, "d_0"'] \arrow[dr, "d_1"] & \\ A & & A \end{tikzcd} \cong \begin{tikzcd} & A \arrow[dl, "1"'] \arrow[dr, "1"] & \\ A & & A \end{tikzcd}.$$

- $\mathcal{S}$PAN $\underline{\underline{A}}$ "IS" A (PSEUDO) DOUBLE CATEGORY.

- EVERY SPAN S HAS AN ASSOCIATED BIFIB

$I_B S I_A \cong S$

$$\begin{tikzcd}[column sep=small]
 & & & {} \arrow[dl] \arrow[ddrr] & & & \\
 & & {} \arrow[dl] \arrow[dr] & & \Rightarrow & & \\
 & A \arrow[dl, "1"'] \arrow[dr, "1"] & \Rightarrow & S \arrow[dl, "s"'] \arrow[dr, "s'"] & & B \arrow[dl, "1"'] \arrow[dr, "1"] & \\
A & & A & & B & & B
\end{tikzcd}$$

SO WE CAN TAKE $\mathcal{S}$PAN $\underline{\underline{A}}$ TO HAVE BIFIBRATIONS AS HORIZONTAL ARROWS AND AS CELLS <u>EQUIVALENCE CLASSES</u> OF DIAGRAMS

$$\begin{tikzcd} A \arrow[d] \arrow[dr, phantom, "="] & S \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "="] & B \arrow[d] \\ C & T \arrow[l] \arrow[r] & D \end{tikzcd}.$$

SPANS IN $\underline{\underline{\mathrm{CAT}}}$ VS PROFUNCTORS

$\mathbb{C}$AT

$$\begin{tikzcd} \underline{A} \arrow[r, "P", "\shortmid" marking] \arrow[d, "F"'] & \underline{B} \arrow[d, "G"] \\ \underline{C} \arrow[r, "Q"', "\shortmid" marking] \arrow[ur, phantom, "\Downarrow\varphi"] & \underline{D} \end{tikzcd}$$

$P : \underline{A}^{op} \times \underline{B} \longrightarrow \underline{\mathrm{SET}}$

(PROFUNCTOR)

$\varphi : P \to Q(F-, G-)$ NAT. TRANSF.

$\mathbb{E}\mathrm{L} : \mathbb{C}\mathrm{AT} \longrightarrow \mathbb{S}\mathrm{PAN}\ \underline{\underline{\mathrm{CAT}}}$

$\underline{A} \overset{P}{\nrightarrow} \underline{B} \longmapsto \underline{A} \leftarrow El(P) \rightarrow \underline{B}$

$\mathbb{P}\mathrm{O} : \mathbb{S}\mathrm{PAN}\ \underline{\underline{\mathrm{CAT}}} \longrightarrow \mathbb{C}\mathrm{AT}$

$\underline{A} \xleftarrow{U} \underline{S} \xrightarrow{V} \underline{B} \longmapsto \underline{A} \overset{V_* \otimes U^*}{\nrightarrow} \underline{B}$

<u>PROP</u>: • $\mathbb{E}\mathrm{L}$ LAX NORMAL,

• $\mathbb{P}\mathrm{O}$ PSEUDO,

• $\mathbb{P}\mathrm{O}$ IS LEFT (UPPER?) ADJOINT TO $\mathbb{E}\mathrm{L}$. ∎

<u>COMPARE</u>: $\mathrm{SPAN(SET)} \underset{\mathrm{INC}}{\overset{\mathrm{FACT}}{\rightleftarrows}} \mathrm{REL(SET)}$.