

CT 2008

Injectivity, exponentiability,
and completeness

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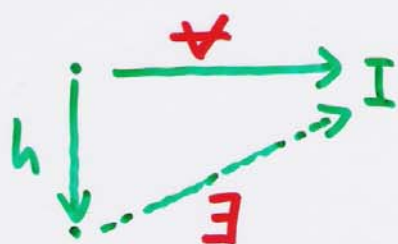
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PLAN:

1. Generalities on injectivity and cogenerators
2. Case study: Nullstellensatz
3. Categorical versions of Birkhoff's and Banaschewski's Theorems
4. Injectivity = (co)completeness
5. Weak factorization systems and (weak) exponentiability, and an exercise in elementary linear algebra

The correspondence governing injectivity:

$\mathcal{C} : \quad \mathcal{H} \triangleright I \quad \mathcal{H} \triangleright J$



"I \mathcal{H} -injective"

"h J -extendable"

Maranda 1965

Examples:

$\mathcal{C} = \underline{\text{Set}}$ $\mathcal{H} = \text{Mono}$ \mathcal{H} -inj. = non-empty

$\mathcal{C} = \underline{\text{Ord}}$ $\mathcal{H} = \text{Reg Mono}$ " = complete

$\mathcal{C} = \underline{\text{Fld}}$ $\mathcal{H} = \{\text{alg. exts}\}$ " = alg'ly closed

$\mathcal{C} = \underline{\text{Met}}$ $\mathcal{H} = \{\text{dense isom.}\}$ " = Gandy compl.

⋮

- 1) Characterization Problem
2) Embeddability Problem

Being fixed under the correspondence:

Assume 1) $\mathcal{H} \triangleright \mathcal{J}$

2) $\forall X \exists \eta_X: X \rightarrow TX$ with
 $\eta_X \in \mathcal{H}, TX \in \mathcal{J}$

Then: a) $\mathcal{H} = \triangleright \mathcal{J} = \triangleright \{TX \mid X \in \mathcal{C}\}$

iff \mathcal{H} is left cancellable
($gh \in \mathcal{H} \Rightarrow h \in \mathcal{H}$)

b) $\mathcal{J} = \mathcal{H} \triangleright = \{\eta_X \mid X \in \mathcal{C}\} \triangleright$

iff \mathcal{J} is closed under retracts

($I \xrightleftharpoons[r]{i} J \in \mathcal{J} \Rightarrow I \in \mathcal{J}$)

The minimalist's approach (Hébert 2007)

(*) Have: $\{\eta_X: X \rightarrow TX\}_X$ with $\{\eta_X\}_X \triangleright \{TX\}_X$

Put $\mathcal{H} := \{h: A \rightarrow B \mid \exists g: g \cdot h = \eta_A\}$,

$\mathcal{J} := \{I \mid \exists \tau: \tau \cdot \eta_I = 1\}$

Then:

$$\mathcal{H} = \triangleright \mathcal{J} \text{ and } \mathcal{J} = \mathcal{H} \triangleright$$

How to get (*) ?

The ideal case : \exists injective cogenerator

$\mathcal{C} \subseteq \text{ob } \mathcal{C}$ set with:

• \mathcal{C} is coseparating

$$\forall f, g: X \rightarrow Y \quad \begin{matrix} f \neq g \\ \exists y: Y \rightarrow C \in \mathcal{C} : yf \neq yg \end{matrix}$$

Better : $\gamma: \mathcal{C}^{\text{op}} \rightarrow \underline{\text{Set}}^{\mathcal{C}}$ faithful
 $A \mapsto \mathcal{C}(A, -)$

• $\forall C \in \mathcal{C} : C$ is injective (I = Mono)

Then:

$$A \xrightarrow{\gamma_A} \prod_{C \in \mathcal{C}} \mathcal{C}(A, C) =: TA$$

satisfies (*) (and is functorial/natural).

Relativized versions: \mathcal{H} -cogenerator

$\forall A: \mathcal{C}(A, \mathcal{G})$ jointly in \mathcal{H}

- $\mathcal{H} = \text{Mono}$: $\gamma: \mathcal{C}^{\text{op}} \rightarrow \underline{\text{Set}}^{\mathcal{G}}$ faithful : cogenerator
- $\mathcal{H} = \text{Ext Mono}$: γ ff & refl. isos : strong cogenerator
- $\mathcal{H} = \text{Strict Mono}$: γ fully faithful : codeuse cogenerator

Examples:

\mathbb{Q}/\mathbb{Z} in AbGrp, $\text{hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ in R-Mod

\mathbb{R}, \mathbb{C} in Ban, (Hahn-Banach)

$[0,1]$ in Tych, CompHaus (Tietze-Urysohn)

$[0,1]^2$ in CompHaus

Nullstellensatz

NSS 1 (Hilbert 1893) $f_i \in k[x_1, \dots, x_n]$ ($i=1, \dots, r$)

$k \subseteq F$ algebraically closed. Then:

$$\exists a \in F^n \quad \forall i: f_i(a) = 0 \iff \nexists g_i \in k[x]: \sum_{i=1}^r g_i f_i = 1$$

↑ Take $P = (f_1, \dots, f_r)$

NSS 2 $P \not\subseteq k[x]$, $k \subseteq F$ alg. cl. $\Rightarrow \sqrt{P} = \mathcal{J}(S(P))$

$$\sqrt{P} := \{f \mid \exists m \geq 1: f^m \in P\}; \quad P \mapsto S(P) := \{a \in F^n \mid \forall f \in P: f(a) = 0\}$$

$$\mathcal{J}(X) = \{f \in k[x] \mid \forall a \in X: f(a) = 0\} \longleftarrow X \subseteq F^n$$

Note: Hilbert's Basis Theorem: $k[x]$ is Noetherian

↑ Take $A = k[x]/\sqrt{P}$

NSS 3 A finitely generated k -algebra with 1, no nilpotent elts, $0 \neq u \in A$, $k \subseteq F$ alg. cl.

Then: $\exists \varphi: A \rightarrow F$ k -homom., $\varphi(u) \neq 0$

- ↑↑ 1. K fin gen $\Rightarrow K/k$ alg
2. algebraically closed = inj wrt. {alg. ext's}

NSS4 A finitely gen. k -alg. with 1, no nilp. elts
 $0 \neq u \in A \Rightarrow \exists K \supseteq k, K$ fin. gen. (as a k -alg.),
 $\exists \chi: A \rightarrow K, \chi(u) \neq 0$

↑↑ routine algebra, maximal ideals

NSS5 A comm. ring, no nilp. elts
 $0 \neq u \in A \Rightarrow \exists Q \subseteq A$ prime, $u \notin Q$
(easy proof)

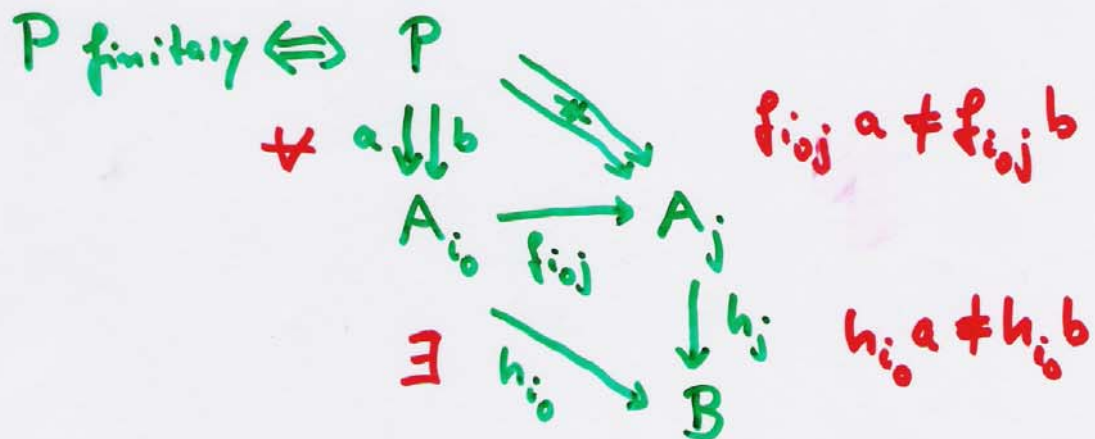
Cor. R comm. ring; $P \subseteq I$ radical ideal iff
 P is intersection of prime ideals.

= Infinitary version of Noether Decomp. Theorem:
 R comm, Noetherian, $P \subseteq I \Rightarrow$
 P is finite intersection of irreducible ideals.

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Hilbert-Noether-Birkhoff categories:

- \mathcal{C} HNB \Leftrightarrow
1. \mathcal{C} has (strong epi, mono)-facts
 2. \mathcal{C} is weakly cocomplete
 3. \mathcal{C} has a generator consisting of finitary objects



Note: 1) \mathcal{C} with dir. colims, P finitely presentable $\Rightarrow P$ finitary

2) $U: \mathcal{C} \rightarrow \underline{\text{Set}}^I$ faithful, right-adj., pres. \varinjlim $\Rightarrow \mathcal{C}$ satisfies 3.

3) \mathcal{C} HNB, $\left(\begin{array}{l} \Leftrightarrow \forall A \in \mathcal{C} \\ \text{finite } x\text{'s} \end{array} \right) \Rightarrow \mathcal{C}/A \text{ HNB}$

Birkhoff's Subdirect Representation Theorem

A sdi $\Leftrightarrow \forall (f_i: A \rightarrow S_i)_i$ monic $\exists i_0: f_{i_0}$ monic

$(f_i: A \rightarrow S_i)_i$ representation of A (by S_i 's) \Leftrightarrow
 $(f_i)_i$ monic, $\forall i$ f_i strong epi

$(f_i)_i$ rep. of A is trivial $\Leftrightarrow \exists i_0: f_{i_0}$ iso

Birkhoff 1944, T1979/81 \in HNB, Acob²
 $\Rightarrow A$ has representation by sdi objects.

Cor. A sdi \Leftrightarrow every rep. of A is trivial
 $\Leftrightarrow \exists P \xrightarrow{a} A \quad \forall f: A \rightarrow B$
 $(fa \neq fb \Rightarrow f \text{ monic})$

Residually small HNB categories

\mathcal{C} residually small $\Leftrightarrow \{S \mid S \text{ sdi}\}$ essentially small

Cor. 1) \mathcal{C} res. small HNB $\Rightarrow \mathcal{C}$ has cogen. consisting of sdi objects

2) \mathcal{C} wellpvd HNB $\Rightarrow \mathcal{C}$ res. small with a cogenerator

Cor. \mathcal{C} residually small, HNB. Equivalent:

(i) \mathcal{C} small-complete, large intersections of monos.

(ii) \mathcal{C} total ($\gamma: \mathcal{C} \rightarrow \underline{\text{Set}}^{\mathcal{C}^{\text{op}}}$ has left adj.).

(iii) \mathcal{C} small-cocomplete, large cointers. of epis.

(iv) \mathcal{C} cototal.

Apply to: \mathcal{C}^{op} , \mathcal{C}/A ; subd. rep. rank.

Questions: How to get sdi's to be injective?

\mathcal{H} -injective hulls

(any $\mathcal{H} \supseteq \text{Iso}$, $\text{Iso} \cdot \mathcal{H} \cdot \text{Iso} \subseteq \mathcal{H}$)

Proposition: $h: X \rightarrow A$ in \mathcal{H} , A \mathcal{H} -inj.

- (i) $\forall t: A \rightarrow A$ ($th = h \Rightarrow t$ iso) "h is \mathcal{H} -inj. hull of A"
- (ii) $\forall f: A \rightarrow B$ ($fh \in \mathcal{H} \Rightarrow f$ split mono)
- (iii) $\forall f: A \rightarrow B$ ($fh \in \mathcal{H} \Rightarrow f \in \mathcal{H}$) "h is \mathcal{H} -ess."

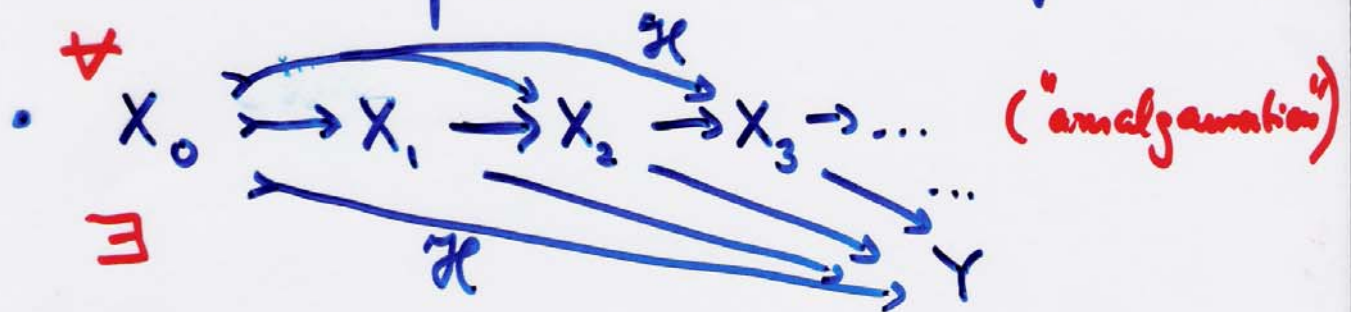
Then: (i) \Leftrightarrow (ii) \Leftrightarrow (iii); all equiv. of $\text{Splitmons} \in \mathcal{H}$

$$\mathcal{H}^* = \{ h \in \mathcal{H} \mid h \text{ } \mathcal{H}\text{-essential} \}$$

Idea: Get injective hull if there are not too many ess. extensions

Banaschewski's Theorem (BB 1970, T1981)

- \mathcal{C} with proper $(\mathcal{E}, \mathcal{H})$ -factorization system,
- \mathcal{C} \mathcal{H} -wellpowered and \mathcal{E} -cowellpowered,



Then:

Every object has \mathcal{H} -inj. hull \Leftrightarrow {

1. \mathcal{C} is \mathcal{H}^* -cowpd



Combine with Birkhoff for $\mathcal{H} = \text{Mono}$.

Existence Thm for inj. cogenerator (T1981)

- \mathcal{C} HNB, small compl. & cocompl., wellpd,
that is:
- \mathcal{C} small compl. & cocompl.
 - \mathcal{C} wellpd & weakly cowellpd.
 - \mathcal{C} has generator consisting of finitary objects.

Equivalent are:

- (i) \mathcal{C} has enough injectives.
- (ii) Every object has an injective hull.
- (iii) \mathcal{C} is residually small, Mono po-stable.
- (iv) \mathcal{C} is Mono^{*}-cowlpd, Mono po-stable.
- (v) \mathcal{C} has cogenerator consisting of injective sdi objects.

In particular: \mathcal{C} is total and cototal.

^{very}
A modest excursion to higher dimension

$U = (U, k, \otimes)$ unital, comm. quantale

U-Rel $X \xrightarrow{r} Y \xrightarrow{s} Z$

$$(s \cdot r)(x, z) = \bigvee_y r(x, y) \otimes s(y, z)$$

U-Cat $X = (X, a)$ $\bullet 1_X \leq a$ (refl.)
 $\bullet a \cdot a \leq a$ (trans.)
 $f \downarrow$
 $Y = (Y, b)$ $f \cdot a \leq b \cdot f$

U-Mod $X \xrightarrow{\varphi} Y$ $\varphi \cdot a \leq \varphi, b \cdot \varphi \leq \varphi$

$U\text{-Cat}^{\text{op}} \rightarrow U\text{-Mod}$
 $(X \xrightarrow{f} Y) \mapsto (Y \xrightarrow{f^*} X)$
 $f^* = f \circ b$

$U\text{-Cat}^{\text{co}} \rightarrow U\text{-Mod}$
 $(X \xrightarrow{f} Y) \mapsto (X \xrightarrow{f_*} Y)$
 $f_* = b \cdot f$

$$f_* = f^*$$

A natural closure for $\underline{\text{U-Cat}}$

$$l_X^* \leq f^* \cdot f_* \quad , \quad f_* \cdot f^* \leq l_Y^*$$

$$f \text{ fully faithful} \Leftrightarrow l_X^* = f^* \cdot f_* \quad a(x, x') = b(f(x), f(x'))$$

$$f \text{ L-dense} \Leftrightarrow f_* \cdot f^* = l_Y^*$$

$$\Leftrightarrow (g \cdot f = h \cdot f \Rightarrow g \cong h)$$

"epi up to iso"

$$M \subseteq X$$

\bar{M} = least subobject of X in which M is L-dense

$$= \left\{ y \in X \mid k \leq \bigvee_{x \in M} b(y, x) \oplus b(x, y) \right\}$$

L-separation, L-completion

X L-separated $\Leftrightarrow \Delta_X \subseteq X \times X$ L-closed

$\Leftrightarrow \forall g, h: P \rightarrow X$
 $(g \approx h \Rightarrow g = h)$

$\Leftrightarrow \gamma: X \rightarrow \hat{X} = U^{X^{op}}$ is 1-1

Lemma: $U = (U, \dashv)$ is L-separated
and {fully faithful} - injective,

in particular: {fff, L-dense} - injective
pseudo

"L-injective"

Naturally:

$\gamma: X \longrightarrow \tilde{X} := \overline{\gamma(X)} \subseteq \hat{X} = U^{X^{op}}$

should be a "completion"

Theorem (Lawvere 1973,
Clementino, Hofmann, Stubbe, T04-08)

Equivalent are for a \mathcal{U} -category X :

(i) X L -complete (every $\varphi + \psi: X \rightarrow Y$
in $\mathcal{U}\text{-Mod}$ is representable
as $f_* + f^*$ for some f).

(ii) X L -injective.

(iii) X is a (pseudo-)retract of \tilde{X} .

(iv) $\gamma: X \rightarrow \tilde{X}$ is onto.

For a considerable* generalization
of this theorem \rightarrow Hofmann's talk

- *)
- $\mathcal{U} \rightsquigarrow \mathcal{J}$ "topological theory"
 - relativized injectivity

Reflection, inj. hull, cogenerator:

- $U\text{-Cat}_{\text{cpt}} \hookrightarrow U\text{-Cat}_{\text{sep}}$ reflective, reflection:

$$\gamma: X \longrightarrow \tilde{X} \quad L\text{-dense full embedding}$$

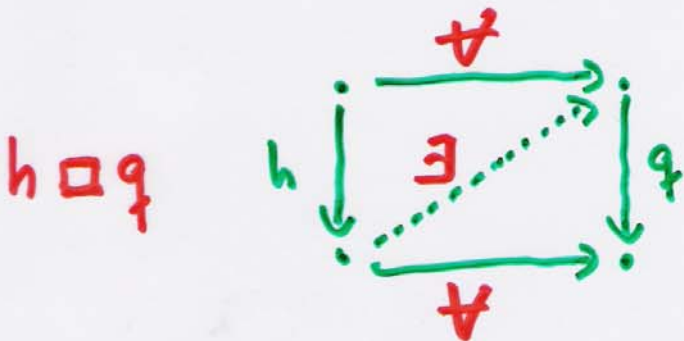
- γ is actually $\{L\text{-dense full emb.}\}$ -inj. hull

Note: Functorial \mathcal{H} -inj. hulls are rare since:

$$\eta: T \rightarrow T \text{ ptw. } \mathcal{H}\text{-inj. hull} \left\{ \begin{array}{l} \text{and ptw. monic} \end{array} \right\} \Rightarrow \eta \text{ ptw. epic}$$

- $X \in U\text{-Cat}_{\text{cpt}} \iff \exists X \xrightarrow{\quad} \prod_I U$
full L -closed emb.

Injectivity by slice = Projectivity by coslice



$h \square q$
 $f \square h$

"left/right lifting prop."

Assume: $\mathcal{H} \square Q$ & $\forall f \in \mathcal{H} \exists \pi_f \in Q$

Then:

$\mathcal{H} \square Q \Leftrightarrow Q$ closed under retracts by slice:
 $q \cdot r \in Q, r \cdot i = 1 \Rightarrow q \in Q$

$\mathcal{H} = \square Q \Leftrightarrow \mathcal{H}$ closed under coretracts by coslice:
 $i \cdot h \in \mathcal{H}, r \cdot i = 1 \Rightarrow h \in \mathcal{H}$

"Economy class" (T-Rosidy' 2002):

Have: $\left(\begin{array}{ccc} & \xrightarrow{\gamma_f} & \\ A & \xrightarrow{f} & B \\ & & \downarrow \pi_f \\ & & \end{array} \right)_f$ with $\{\gamma_f\}_f \square \{\pi_f\}_f \in \mathcal{C}/\mathcal{B}$

Put: $\mathcal{H} = \{h \mid \exists i: \pi_h \cdot i = 1, i \cdot h = \gamma_h\}$

$\mathcal{Q} = \{g \mid \exists r: r \cdot \gamma_g = 1, g \cdot r = \pi_g\}$

Then $\mathcal{H} = \square \mathcal{Q}$ and $\mathcal{Q} = \mathcal{H} \square$.

Theorem $\mathcal{C} \text{ HNB}$, small compl. & cocompl, wellpwd, residually small. Then:

\exists (Mono, \mathcal{Q}) weak fact. system (functorial)

\Leftrightarrow Mono is stable under pullout.

(Weakly) h^* -couniversal arrows

\mathcal{C} with pbs, $h: C \rightarrow B$, hence

$$h^*: \mathcal{C}/B \rightarrow \mathcal{C}/C$$

If $(h_! \dashv) h^* \dashv h_*$, every h^* -couniversal arrow is iso precisely when h is monic.

Prop. Let h be monic. Then:

1. $w: h^*(\bar{p}) \rightarrow p$ weakly h^* -couniversal $\left. \vphantom{w: h^*(\bar{p}) \rightarrow p} \right\} \Rightarrow w$ split epi in \mathcal{C}/B
2. w h^* -couniversal $\Rightarrow w$ iso

"Exponentiability \Rightarrow Injectivity" (T-CT00)

\mathcal{C} with pbs, \mathcal{H} pb stable.

Let $f = (A \xrightarrow[p]{\mathcal{H}^\square} C \xrightarrow[\mathcal{H}]{h} B)$ s.th.

- \exists h^* -cominiversal arrow $w: h^*(\bar{p}) \rightarrow p$, h mono
- or • \exists wkly h^* -cominiv. arrow w for p , \mathcal{H} -Splitting $\subseteq \mathcal{H}$

Then:

$$\exists f = (A \xrightarrow[\mathcal{H}]{\bar{h}} D \xrightarrow[\mathcal{H}^\square]{\bar{p}} B)$$

\uparrow
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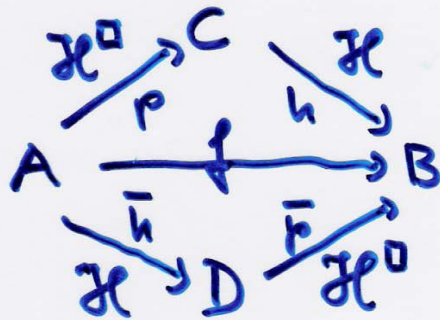
Furthermore:

This factorization is **essential** if w is iso, in particular, if h is monic.

"Injectivity \Rightarrow Weak Exponentiability" (T-2008)

\mathcal{C} with pbs, \mathcal{H} pb stable, $(g \in \mathcal{H}, g \in \mathcal{H} \Rightarrow h \in \mathcal{H})$
 (automatic for \mathcal{H} strong)

Assume



Then: $\exists w: h^*(\bar{p}) \rightarrow p$ w/ h^* -couniversal

Furthermore: w iso if fact. $f = \bar{p} \cdot \bar{h}$ is essential,
 and then the w/ h^* -couniv. arrow is essential.

Cor. $(\mathcal{E}, \mathcal{H})$ orth. f.s. in cat. with pbs,

$\mathcal{E} \subseteq \mathcal{H}^\square$, let every $f: A \rightarrow B$ have \mathcal{H}_B -inj. hull

$\Rightarrow \forall p: A \rightarrow C$ in \mathcal{E} , $h: C \rightarrow B$ in \mathcal{H}

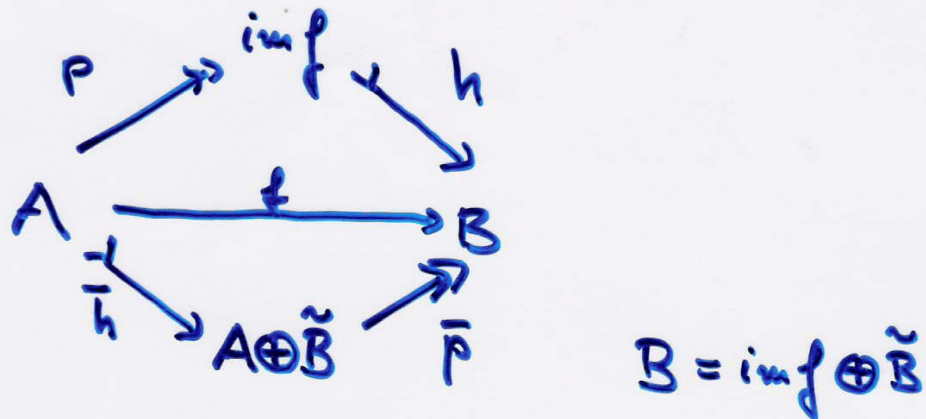
\exists weakly h^* -couniv. arrow for p , essential, iso.

Extensive categories, etc

- \mathcal{C} with $+$ $\Rightarrow \exists$ wfs $(\mathcal{I}l, \text{Split Epi})$
 \cup
 $\{\text{coproduct injections}\}$
- \mathcal{C} extensive = $(\{+-inj\}, \text{Split Epi})$ wfs
 \Rightarrow certain wkly commiv. arrows exist. But:
Every coproduct injection is exponentiable.
- $\underline{\text{Vec}}_k$ is not extensive but still has:
 $(\underbrace{\{+-inj\}}_{\text{Mono}}, \underbrace{\text{Split Epi}}_{\text{Epi}})$ wfs

Hence: wkly h^* -commiv. arrows for $p(\text{epi})$
are obtained from an essential
 $(\text{Mono}, \text{Epi})$ -factorization.

Essential (Mono, Epi)-fact in $\underline{\text{Vec}}_k$



Exercise 1: $f = \bar{p} \cdot \bar{h}$ is essential.

Exercise 2: $G = \{ t \in \text{Ead}_k(A \oplus \tilde{B}) \mid t \bar{h} = \bar{h}, \bar{p} t = \bar{p} \}$

Then $G \xrightarrow{\quad} (\text{Hom}_k(\tilde{B}, \ker f), +)$
 $t \longmapsto (b \longmapsto t(b) - b)$

is an isomorphism of groups!

Hence: The inner automorphism group of $A \oplus \tilde{B}$
 carries a k -vector space structure.