CT 2008

Injectivity, exponentiability, and completeness

Walter Tholen

York University

Toronto, Canada

CALAIS, 23 JUNE 2008

PLAN:

1. Generalities on injectivity and cogenerators
2. Case study: Nullstellensatz
3. Categorical versions of Birkhoff's and Banaschewski's Theorems
4. Injectivity = (co)completeness
5. Weak factorization systems and (weak) exponentiability, and an exercise in elementary linear algebra

The correspondence governing injectivity:

$\mathcal{C}$: $h \vartriangleright I$ $\mathcal{H} \vartriangleright \mathcal{I}$

$$\begin{tikzcd} \cdot \arrow[r, "\forall"] \arrow[d, "h"'] & I \\ \cdot \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

"I $\mathcal{H}$-injective"

"h $\mathcal{I}$-extendable"

Maranda 1965

Examples:

$\mathcal{C} = \underline{\mathrm{Set}}$ $\mathcal{H} = \mathrm{Mono}$ $\mathcal{H}$-inj. = non-empty

$\mathcal{C} = \underline{\mathrm{Ord}}$ $\mathcal{H} = \mathrm{RegMono}$ " = complete

$\mathcal{C} = \underline{\mathrm{Fld}}$ $\mathcal{H} = \{\text{alg. exts}\}$ " = alg'ly closed

$\mathcal{C} = \underline{\mathrm{Met}}$ $\mathcal{H} = \{\text{dense isom.}\}$ " = Cauchy compl.

⋮

$\leadsto$ 1) Characterization Problem

2) Embeddability Problem

Being fixed under the correspondence:

Assume 1) $\mathcal{H} \triangleleft \mathcal{I}$

2) $\forall X \; \exists \eta_X : X \to TX$ with $\eta_X \in \mathcal{H}, \; TX \in \mathcal{I}$

Then: a) $\mathcal{H} = {}^{\triangleleft}\mathcal{I} = {}^{\triangleleft}\{TX \mid X \in \mathcal{C}\}$

iff $\mathcal{H}$ is left cancellable

$(gh \in \mathcal{H} \Rightarrow h \in \mathcal{H})$

b) $\mathcal{I} = \mathcal{H}^{\triangleleft} = \{\eta_X \mid X \in \mathcal{C}\}^{\triangleleft}$

iff $\mathcal{I}$ is closed under retracts

$(I \underset{r}{\overset{i}{\rightleftarrows}} J \in \mathcal{I} \Rightarrow I \in \mathcal{I})$

The minimalist's approach (Hébert 2007)

(*) Have: $\{\eta_X : X \to TX\}_X$ with $\{\eta_X\}_X \triangleright \{TX\}_X$

Put $\mathcal{H} := \{ h : A \to B \mid \exists g : g \cdot h = \eta_A \}$,

$\mathcal{I} := \{ I \mid \exists r : r \cdot \eta_I = 1 \}$

Then:

$$\mathcal{H} = {}^{\triangleright}\mathcal{I} \text{ and } \mathcal{I} = \mathcal{H}^{\triangleright}$$

How to get (*) ?

The ideal case : ∃ injective cogenerator

$\mathcal{G} \subseteq \mathrm{ob}\,\mathcal{C}$ set with:

- $\mathcal{G}$ is coseparating

$$\forall \underset{f \neq g}{f, g : X \to Y} \ \exists y : Y \to C \in \mathcal{G} : yf \neq yg$$

Better: $\mathcal{Y} : \mathcal{C}^{op} \to \underline{Set}^{\mathcal{G}}$, $A \longmapsto \mathcal{C}(A,-)$ faithful

- $\forall C \in \mathcal{G}$: C is injective ($\mathcal{H} = \mathrm{Mono}$)

Then:

$$A \xrightarrow{\eta_A} \prod_{C \in \mathcal{G}} C^{\mathcal{C}(A,C)} =: TA$$

satisfies (*) (and is functorial/natural).

Relativized versions: $\mathcal{H}$-cogenerator

$\forall A$: $\mathcal{C}(A, \mathcal{G})$ jointly in $\mathcal{H}$

- $\mathcal{H} = \mathrm{Mono}$: $y : \mathcal{C}^{op} \to \underline{\mathrm{Set}}^{\mathcal{G}}$ faithful : cogenerator
- $\mathcal{H} = \mathrm{ExtMono}$: y ff & refl. isos: strong cogenerator
- $\mathcal{H} = \mathrm{StrictMono}$: y fully faithful : codense cogenerator

Examples:

$\mathbb{Q}/\mathbb{Z}$ in $\underline{\mathrm{AbGrp}}$, $\hom_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ in R-$\underline{\mathrm{Mod}}$

$\mathbb{R}, \mathbb{C}$ in $\underline{\mathrm{Ban}}_1$ (Hahn-Banach)

$[0,1]$ in $\underline{\mathrm{Tych}}$, $\underline{\mathrm{CompHaus}}$ (Tietze-Urysohn)

$[0,1]^2$ in $\underline{\mathrm{CompHaus}}$

Nullstellensatz

NSS 1 (Hilbert 1893) $f_i \in k[\underbrace{x_1, \dots, x_n}_{x}]$ $(i = 1, \dots, r)$

$k \leq F$ algebraically closed. Then:

$\exists a \in F^n \ \forall i : f_i(a) = 0 \iff \nexists g_i \in k[x] : \sum_{i=1}^{r} g_i \cdot f_i = 1$

$\Uparrow$ Take $P = (f_1, \dots, f_r)$

NSS 2 $P \ntrianglelefteq k[x]$, $k \leq F$ alg. cl. $\Rightarrow \sqrt{P} = \mathcal{J}(S(P))$

$\sqrt{P} := \{ f \mid \exists m \geq 1 : f^m \in P \}$; $P \longmapsto S(P) := \{ a \in F^n \mid \forall f \in P : f(a) = 0 \}$

$\mathcal{J}(X) = \{ f \in k[x] \mid \forall a \in X : f(a) = 0 \} \longleftarrow X \subseteq F^n$

Note: Hilbert's Basis Theorem: $k[x]$ is Noetherian

$\Uparrow$ Take $A = k[x]/\sqrt{P}$

NSS 3 A finitely generated k-algebra with 1, no nilpotent elts, $0 \neq u \in A$, $k \leq F$ alg. cl.

Then: $\exists \ \varphi : A \to F$ k-homom., $\varphi(u) \neq 0$

$\Uparrow$ 1. K fin gen $\Rightarrow K/k$ alg
2. algebraically closed $=$ inj wrt. {alg. ext's}

NSS4 A finitely gen. k-alg. with 1, no nilp. elts

$0 \neq u \in A \Rightarrow \exists\, K \geq k$, K fin. gen. (as a k-alg.),

$\exists\, \chi : A \to K$, $\chi(u) \neq 0$

$\Uparrow$ routine algebra, maximal ideals

NSS5 A comm. ring, no nilp. elts

$0 \neq u \in A \Rightarrow \exists\, Q \trianglelefteq A$ prime, $u \notin Q$

(easy proof)

Cor. R comm. ring; $P \trianglelefteq I$ radical ideal iff
P is intersection of prime ideals.

= Infinitary version of Noether Decomp. Theorem:
R comm, Noetherian, $P \trianglelefteq I \Rightarrow$
P is finite intersection of irreducible ideals.

1921

Hilbert-Noether-Birkhoff categories:

$\mathcal{C}$ <u>HNB</u> $\Leftrightarrow$
1. $\mathcal{C}$ has (strong epi, mono)-facts
2. $\mathcal{C}$ is weakly cowellpowered
3. $\mathcal{C}$ has a generator consisting of <u>finitary</u> objects

P finitary $\Leftrightarrow$

$$\begin{tikzcd}
P \arrow[d, shift right, "a"'] \arrow[d, shift left, "b"] \arrow[dr, shift left, "\neq"] \arrow[dr, shift right] & \\
A_{i_0} \arrow[r, "f_{i_0 j}"'] \arrow[dr, "h_{i_0}"'] & A_j \arrow[d, "h_j"] \\
& B
\end{tikzcd}$$

$\forall$ a, b: $f_{i_0 j}\, a \neq f_{i_0 j}\, b$

$\exists$ h_{i_0}: $h_{i_0} a \neq h_{i_0} b$

Note: 1) $\mathcal{C}$ with dir. colims, P finitely presentable $\Rightarrow$ P finitary

2) $U : \mathcal{C} \to \underline{\mathrm{Set}}^I$ faithful, right-adj., pres. $\underrightarrow{\lim}$ $\Rightarrow$ $\mathcal{C}$ satisfies 3.

3) $\left.\begin{array}{l}\mathcal{C} \text{ HNB,} \\ \text{finite } \times\text{'s}\end{array}\right\} \Longleftrightarrow \forall A \;\; \mathcal{C}/A$ HNB

Birkhoff's Subdirect Representation Thm

A <u>sdi</u> $\Leftrightarrow \forall (f_i : A \to S_i)_i$ monic $\exists i_0 : f_{i_0}$ monic

$(f_i : A \to S_i)_i$ representation of A (by S_i's) $\Leftrightarrow$
$(f_i)_i$ monic, $\forall i$ f_i strong epi

$(f_i)_i$ rep. of A is <u>trivial</u> $\Leftrightarrow \exists i_0 : f_{i_0}$ iso

Birkhoff 1944, T1979/81 $\mathcal{C}$ HNB, $A \in \mathrm{ob}\,\mathcal{C}$
$\Rightarrow$ A has representation by sdi objects.

Cor. A sdi $\Leftrightarrow$ every rep. of A is trivial
$\Leftrightarrow \exists P \overset{a}{\underset{b}{\rightrightarrows}} A \quad \forall f : A \to B$
$(fa \neq fb \Rightarrow f$ monic$)$

Residually small HNB categories

$\mathcal{C}$ residually small $\Leftrightarrow \{S \mid S \text{ sdi}\}$ ess'ly small

Cor. 1) $\mathcal{C}$ res. small HNB $\Rightarrow \mathcal{C}$ has cogen. consisting of sdi objects

2) $\mathcal{C}$ wellpwd HNB ($\Rightarrow \mathcal{C}$ res. small with a cogenerator)

Cor. $\mathcal{C}$ residually small, HNB. Equivalent:

(i) $\mathcal{C}$ small-complete, large intersections of monos.

(ii) $\mathcal{C}$ <u>total</u> ($y : \mathcal{C} \to \underline{\mathrm{Set}}^{\mathcal{C}^{op}}$ has left adj.).

(iii) $\mathcal{C}$ small-cocomplete, large cointers. of epis.

(iv) $\mathcal{C}$ <u>cototal</u>.

<u>Apply</u> to: $\mathcal{C}^{op}$, $\mathcal{C}/A$; subd. rep. rank.

<u>Questions</u>: How to get sdi's to be injective?

$\mathcal{H}$-injective hulls

(any $\mathcal{H} \supseteq \mathrm{Iso}$, $\mathrm{Iso}\cdot\mathcal{H}\cdot\mathrm{Iso} \subseteq \mathcal{H}$)

Proposition: $h: X \to A$ in $\mathcal{H}$, A $\mathcal{H}$-inj.

(i) $\forall t: A \to A$ $(th = h \Rightarrow t \text{ iso})$ "h is $\mathcal{H}$-inj. hull of A"

(ii) $\forall f: A \to B$ $(fh \in \mathcal{H} \Rightarrow f \text{ split mono})$

(iii) $\forall f: A \to B$ $(fh \in \mathcal{H} \Rightarrow f \in \mathcal{H})$ "h is $\mathcal{H}$-ess."

Then: (i) $\Leftrightarrow$ (ii) $\Leftarrow$ (iii) ; all equiv. if $\mathrm{SplitMono} \subseteq \mathcal{H}$

$\mathcal{H}^* = \{ h \in \mathcal{H} \mid h \ \mathcal{H}\text{-essential} \}$

Idea: Get injective hull if there are not too many ess. extensions

Banaschewski's Theorem (BB 1970, T 1981)

- $\mathcal{C}$ with proper $(\mathcal{E}, \mathcal{H})$-factorization system,
- $\mathcal{C}$ $\mathcal{H}$-wellpowered and $\mathcal{E}$-cowellpowered,
- $\forall$ $\exists$

$$\begin{tikzcd}
X_0 \arrow[r, tail] \arrow[rr, bend left] \arrow[rrr, bend left, "\mathcal{H}"] \arrow[rrrrd, bend right, "\mathcal{H}"'] & X_1 \arrow[r] \arrow[rrrd] & X_2 \arrow[r] \arrow[rrd] & X_3 \arrow[r] \arrow[rd] & \dots \\
 & & & & Y
\end{tikzcd}$$

("amalgamation")

Then:

Every object has $\mathcal{H}$-inj. hull $\Leftrightarrow$ $\begin{cases} 1.\ \mathcal{C} \text{ is } \mathcal{H}^{*}\text{-cowpd} \\ 2. \end{cases}$

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, "\forall\ \mathcal{H}"'] \arrow[dr, phantom, "\exists"] & \cdot \arrow[d, dotted, "\mathcal{H}"] \\
\cdot \arrow[r, dashed] & \cdot
\end{tikzcd}$$

Combine with Birkhoff for $\mathcal{H} = \text{Mono}$.

Existence Thm for inj. cogenerator (T1981)

$\mathcal{C}$ HNB, small compl. & cocompl., wellpd,
that is:
- $\mathcal{C}$ small compl. & cocompl.
- $\mathcal{C}$ wellpd & weakly cowellpd.
- $\mathcal{C}$ has generator consisting of finitary objects.

Equivalent are:

(i) $\mathcal{C}$ has enough injectives.

(ii) Every object has an injective hull.

(iii) $\mathcal{C}$ is residually small, Mono po-stable.

(iv) $\mathcal{C}$ is $\mathrm{Mono}^{\#}$-cowpd, Mono po-stable.

(v) $\mathcal{C}$ has cogenerator consisting of injective sdi objects.

In particular: $\mathcal{C}$ is total and cototal.

A very modest excursion to higher dimension

$\mathcal{V} = (\mathcal{V}, k, \otimes)$ unital, comm. quantale

$\mathcal{V}$-**Rel** $\quad X \overset{r}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y \overset{s}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Z$

$$(s \cdot r)(x,z) = \bigvee_{y} r(x,y) \otimes s(y,z)$$

$\mathcal{V}$-**Cat**

$$\begin{array}{ll} X = (X,a) & \bullet\ 1_X \leq a \quad \text{(refl.)} \\ \Big\downarrow f & \bullet\ a \cdot a \leq a \quad \text{(trans.)} \\ Y = (Y,b) & f \cdot a \leq b \cdot f \end{array}$$

$\mathcal{V}$-**Mod** $\quad X \overset{\varphi}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y \qquad \varphi \cdot a \leq \varphi, \; b \cdot \varphi \leq \varphi$

$$\mathcal{V}\text{-}\mathbf{Cat}^{op} \longrightarrow \mathcal{V}\text{-}\mathbf{Mod} \qquad \mathcal{V}\text{-}\mathbf{Cat}^{co} \longrightarrow \mathcal{V}\text{-}\mathbf{Mod}$$

$$(X \xrightarrow{f} Y) \longmapsto (Y \overset{f^*}{\longrightarrow\!\!\!\!\!\!\!+\;\;} X) \qquad (X \xrightarrow{f} Y) \longmapsto (X \overset{f_*}{\longrightarrow\!\!\!\!\!\!\!+\;\;} Y)$$

$$f^* = f^\circ \cdot b \qquad f_* = b \cdot f$$

$$f_* \dashv f^*$$

A natural closure for $\mathcal{V}$-$\underline{\mathrm{Cat}}$

$$1_X^* \leq f^* \cdot f_* \quad , \quad f_* \cdot f^* \leq 1_Y^*$$

f fully faithful $\iff 1_X^* = f^* \cdot f_*$ $\qquad a(x,x') = b(f(x), f(x'))$

f L-dense $\iff f_* \cdot f^* = 1_Y^*$

$\iff (g \cdot f = h \cdot f \Rightarrow g \cong h)$

"epi up to iso"

$M \subseteq X$

$\overline{M}$ = least subobject of X in which M is L-dense

$$= \{ y \in X \mid k \leq \bigvee_{x \in M} b(y,x) \otimes b(x,y) \}$$

L-separation, L-completion

X L-separated $\iff \Delta_X \subseteq X \times X$ L-closed

$\iff \forall g, h : P \to X$

$(g \cong h \Rightarrow g = h)$

$\iff y : X \to \hat{X} = \mathcal{V}^{X^{op}}$ is 1-1

Lemma: $\mathcal{V} = (V, \multimap)$ is L-separated

and {fully faithful}-injective,

in particular: $\underbrace{\text{\{fff, L-dense\}-}\overset{\text{pseudo}}{\wedge}\text{injective}}_{\text{"L-injective"}}$

Naturally:

$$y : X \longrightarrow \tilde{X} := \overline{y(X)} \subseteq \hat{X} = \mathcal{V}^{X^{op}}$$

should be a "completion"

Theorem (Lawvere 1973,
Clementino, Hofmann, Stubbe, T 04-08)

Equivalent are for a $\mathcal{V}$-category X:

(i) X L-complete (every $\varphi \dashv \psi : X \to\!\!\!\!\!\!\!\!\!\;\;\circ\;\; Y$ in $\mathcal{V}$-Mod is representable as $f_* \dashv f^*$ for some f).

(ii) X L-injective.

(iii) X is a (pseudo-) retract of $\tilde{X}$.

(iv) $y : X \to \tilde{X}$ is onto.

For a considerable* generalization of this theorem → Hofmann's talk

*) • $\mathcal{V} \rightsquigarrow \mathbb{T}$ "topological theory"
• relativized injectivity

Reflection, inj. hull, cogenerator:

- $\mathcal{V}\text{-}\underline{\mathrm{Cat}}_{cpl} \hookrightarrow \mathcal{V}\text{-}\underline{\mathrm{Cat}}_{sep}$ reflective, reflection:

  $y : X \longrightarrow \tilde{X}$ L-dense full embedding

- y is actually {L-dense full emb.}-inj. hull

  Note: Functorial $\mathcal{H}$-inj. hulls are rare since:

  $$\left.\begin{array}{l}\eta : 1 \to T \text{ ptw. } \mathcal{H}\text{-inj. hull} \\ \text{and ptw. monic}\end{array}\right\} \Rightarrow \eta \text{ ptw. epic}$$

- $X \in \mathcal{V}\text{-}\underline{\mathrm{Cat}}_{cpl} \Leftrightarrow \exists\, X \underset{\text{full L-closed emb.}}{\rightarrowtail} \prod_I \mathcal{V}$

Injectivity by slice = Projectivity by coslice

$$\begin{tikzcd} \bullet \arrow[r, "\forall"] \arrow[d, "h"'] & I \\ \bullet \arrow[ur, dotted, "\exists"'] & \end{tikzcd} \qquad\qquad \begin{tikzcd} & \bullet \arrow[d, "q"] \\ P \arrow[ur, dotted, "\exists"] \arrow[r, "\forall"'] & \bullet \end{tikzcd}$$

$$h \,\square\, q \qquad \begin{tikzcd} \bullet \arrow[r, "\forall"] \arrow[d, "h"'] & \bullet \arrow[d, "q"] \\ \bullet \arrow[ur, dotted, "\exists"] \arrow[r, "\forall"'] & \bullet \end{tikzcd}$$

"left/right lifting prop."

Assume: $\mathcal{H} \,\square\, Q$ & $\forall f\, \exists$ $\begin{tikzcd} & \bullet \arrow[dr, "\pi_f \in Q"] & \\ \bullet \arrow[ur, "\mathcal{H} \ni \eta_f"] \arrow[rr, "f"'] & & \bullet \end{tikzcd}$

Then:

$\mathcal{H}^{\square} = Q \iff Q$ closed under retracts by slice:

$q \cdot r \in Q,\ r \cdot i = 1 \Rightarrow q \in Q$

$\mathcal{H} = {}^{\square}Q \iff \mathcal{H}$ closed under coretracts by coslice

$i \cdot h \in \mathcal{H},\ r \cdot i = 1 \Rightarrow h \in \mathcal{H}$

"Economy class" (T-Rosický 2002):

Have: $\left(\begin{tikzcd} & \cdot \arrow[dr, "\pi_f"] & \\ A \arrow[ur, "\eta_f"] \arrow[rr, "f"'] & & B \end{tikzcd} \right)_f$ with $\{\eta_f\}_f \,\square\, \{\pi_f\}_{f \in \mathcal{C}/B}$

Put: $\mathcal{H} = \{ h \mid \exists i : \pi_h \cdot i = 1,\ i \cdot h = \eta_h \}$

$Q = \{ q \mid \exists r : r \cdot \eta_q = 1,\ q \cdot r = \pi_q \}$

Then $\mathcal{H} = {}^{\square}Q$ and $Q = \mathcal{H}^{\square}$.

Theorem $\mathcal{C}$ HNB, small compl. & cocompl., wellpwd, residually small. Then:

$\exists$ (Mono, Q) weak fact. system (functorial)

$\Leftrightarrow$ Mono is stable under pushout.

(Weakly) h^*-couniversal arrows

$\mathcal{C}$ with pbs, $h : C \to B$, hence

$$h^* : \mathcal{C}/B \longrightarrow \mathcal{C}/C$$

If $(h_! \dashv) h^* \dashv h_*$, every h^*-couniversal arrow is iso precisely when h is monic.

Prop. Let h be monic. Then:

1. $\left.\begin{array}{l} w : h^*(\bar{p}) \to p \text{ weakly} \\ h^*\text{-couniversal} \end{array}\right\} \Rightarrow w \text{ split epi in } \mathcal{C}/B$
2. w h^*-couniversal $\Rightarrow$ w iso

"Exponentiability $\Rightarrow$ Injectivity" (T-CT00)

$\mathcal{C}$ with pbs, $\mathcal{H}$ pb stable.

Let $f = (A \xrightarrow[\mathcal{H}^{\square}]{p} C \xrightarrow[\mathcal{H}]{h} B)$ s.th.

- $\exists$ h^*-couniversal arrow $w: h^*(\bar{p}) \to p$, h mono

or

- $\exists$ wkly h^*-couniv. arrow w for p, $\mathcal{H}\cdot\mathrm{SplitMono} \subseteq \mathcal{H}$

Then:

$$\exists \quad f = \begin{tikzcd} (A \arrow[r, "\bar{h}", "\mathcal{H}"'] & D \arrow[loop below, "t"] \arrow[r, "\bar{p}", "\mathcal{H}^{\square}"'] & B) \end{tikzcd}$$

Furthermore:

This factorization is essential if w is iso, in particular, if h is monic.

"Injectivity $\Rightarrow$ Weak Exponentiability" (T-2008)

$\mathcal{C}$ with pbs, $\mathcal{H}$ pb stable, $(gh \in \mathcal{H},\ g \in \mathcal{H} \Rightarrow h \in \mathcal{H})$

(automatic for $\mathcal{H} \subseteq$ strong)

Assume

$$\begin{tikzcd}
 & C \arrow[dr, "h", "\mathcal{H}"'] & \\
A \arrow[ur, "p"', "\mathcal{H}^{\square}"] \arrow[rr, "f"] \arrow[dr, "\bar{h}", "\mathcal{H}"'] & & B \\
 & D \arrow[ur, "\bar{p}", "\mathcal{H}^{\square}"'] &
\end{tikzcd}$$

Then: $\exists\ w : h^*(\bar{p}) \to p$ wkly h^*-couniversal

Furthermore: w iso if fact. $f = \bar{p} \cdot \bar{h}$ is essential,

and then the wkly h^*-couniv. arrow is essential.

Cor. $(\mathcal{E}, \mathcal{H})$ orth. f.s. in cat. with pbs,

$\mathcal{E} \subseteq \mathcal{H}^{\square}$, let every $f : A \to B$ have $\mathcal{H}_B$-inj. hull

$\Rightarrow \forall\, p : A \to C$ in $\mathcal{E}$, $h : C \to B$ in $\mathcal{H}$

$\exists$ weakly h^*-couniv. arrow for p, essential, iso.

Extensive categories, etc

- $\mathcal{C}$ with $+$ $\Rightarrow \exists$ wfs $(\mathcal{H}, \text{Split Epi})$
  $$\mathcal{H} \supseteq \{\text{coproduct injections}\}$$
- $\mathcal{C}$ extensive $= (\{+\text{-inj}\}, \text{Split Epi})$ wfs

$\mapsto$ certain wkly comm. arrows exist. But:
Every coproduct injection is exponentiable.

- $\underline{\text{Vec}}_k$ is not extensive but still has:
  $(\underbrace{\{+\text{-inj}\}}_{\text{Mono}}, \underbrace{\text{Split Epi}}_{\text{Epi}})$ wfs

  Hence: wkly h^*-comm. arrows for p (epi) are obtained from an essential (Mono, Epi)-factorization.

Essential (Mono, Epi)-fact in $\underline{\mathrm{Vec}}_k$

$$\begin{tikzcd}
 & \operatorname{im} f \arrow[dr, tail, "h"] & \\
A \arrow[ur, two heads, "p"] \arrow[rr, "f"] \arrow[dr, tail, "\bar{h}"'] & & B \\
 & A \oplus \tilde{B} \arrow[ur, two heads, "\bar{p}"'] &
\end{tikzcd}$$

$$B = \operatorname{im} f \oplus \tilde{B}$$

<u>Exercise 1</u>: $f = \bar{p} \cdot \bar{h}$ is essential.

<u>Exercise 2</u>: $G = \{ t \in \operatorname{End}_k(A \oplus \tilde{B}) \mid t\,\bar{h} = \bar{h},\ \bar{p}\,t = \bar{p} \}$

Then
$$\begin{aligned} G &\longrightarrow (\operatorname{Hom}_k(\tilde{B}, \ker f), +) \\ t &\longmapsto (b \longmapsto t(b) - b) \end{aligned}$$

is an <u>isomorphism of groups</u>!

Hence: The "inner" automorphism group of $A \oplus \tilde{B}$ carries a k-vector space structure.