RANDOM SERIES OF FUNCTIONS AND APPLICATIONS

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Abstract. In this article we study the continuity properties of trajectories for some random series of functions, \( \sum_{k=0}^{\infty} a_k f(\alpha X_k(\omega)) \) where \((a_k)_{k \geq 0}\) is a complex sequence, \((X_k)_{k \geq 0}\) is a sequence of real independent random variables, \(f\) is a real valued function with period one and summable Fourier coefficients. We obtain almost sure continuity results for these periodic or almost periodic series for a large class of functions \(f\), where the “almost sure” does not depend on the function. The proof relies on gaussian randomization. We show optimality of the results in some cases.

SERIES DE FONCTIONS ALEATOIRES ET APPLICATIONS

Abstract. (Résumé) Dans ce travail, nous étudions des propriétés de continuité de trajectoires de séries de fonctions aléatoires du type \( \sum_{k=0}^{\infty} a_k f(\alpha X_k(\omega)) \) où \((a_k)_{k \geq 0}\) est une suite de nombres complexes, \((X_k)_{k \geq 0}\) une suite de variables aléatoires réelles et indépendantes, \(f\) une fonction 1-périodique à coefficients de Fourier sommables. Nous montrons que, presque sûrement, ces séries de fonctions aléatoires (périodiques ou presque périodiques) sont à trajectoires continues pour une grande classe de fonctions \(f\). Le “presque sûr” est indépendant de \(f\). Les preuves s’appuient sur un procédé de randomisation gaussien. Dans certains cas, nous montrerons l’optimalité des résultats obtenus.

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1. Introduction

The almost sure convergence of series defined by
\[ \sum_{k \geq 1} a_k f(\alpha n) \]
as well as lacunary series of the type
\[ \sum_{k \geq 1} a_k f(\alpha n_k) \]
where \( f \) verifies
\[
\begin{align*}
\int_0^1 f(x) \, dx &= 0 \\
\int_0^1 f^2(x) \, dx &= 1
\end{align*}
\]
has been studied especially when \( (n_k) \) grows rapidly, meaning that \( (n_k) \) satisfies the Hadamard gap condition
\[
\frac{n_{k+1}}{n_k} > q > 1
\]
In this case, in [5], Kac proves that \( \sum_{k \geq 1} a_k f(\alpha n_k) \) converges a.e. if \( \sum |a_k| < +\infty \) and \( f \in \text{Lip}(\gamma) \) with \( \gamma > 0 \). In another direction, in [1], Berkes proves that there exists a function \( f \in \text{Lip}(\frac{1}{2}) \) satisfying (1) and for any \( \varepsilon_k > 0 \), there exists a sequence of integers \( n_k \) with
\[
\frac{n_{k+1}}{n_k} \geq 1 + \varepsilon_k
\]
such that the series \( \sum_{k \geq 1} a_k f(\alpha n_k) \) is a.e. divergent for some \( (a_k) \) with \( \sum |a_k|^2 < +\infty \).

We can naturally adress the question whether the convergence still holds when \( (n_k) \) is polynomial or subexponential, and for which class of functions.

We are going to answer the question when the sequence \( (n_k) \) is randomly generated.

Let us mention that the result exists when \( (n_k) \) is a deterministic polynomial sequence and \( (a_k) \) is randomly distributed (see [6] and [7]).

We want to study the convergence properties of series of functions sampled by a random process. More precisely, consider the torus \( T = \mathbb{R}/\mathbb{Z} \) and define \( A(T) \) as the set of complex valued functions whose Fourier coefficients are absolutely summable:
\[
A(T) = \{ f : T \rightarrow \mathbb{C}, f(\alpha) = \sum_{j \in \mathbb{Z}} \hat{f}(j) \exp(2i\pi \alpha j), \sum_{j \in \mathbb{Z}} |\hat{f}(j)| < +\infty \}
\]
\( (a_k)_{k \geq 0} \) will denote a sequence of complex numbers and \( (X_k)_{k \geq 0} \) a sequence of independent real random variables defined on the probability space \( (\Omega, \mathcal{A}, \mathbb{P}) \).
Our aim is to study the convergence, when $\omega \in \Omega$ is fixed, of the series of functions

$$F(\alpha, \omega) = \sum_{k=0}^{\infty} a_k f(\alpha X_k(\omega))$$

Is it possible to give conditions on the sequence $(a_k)_{k \geq 0}$ and (or) on the law of $X_k$ in order to find a $\mathcal{A}$–measurable set $\Omega_0$ independent of the function $f$, such that $\mathbb{P}(\Omega_0) = 1$, on which the series uniformly converges?

Note that when $X_k$ is not integer valued, $F$ is not a periodical function of the torus. We will deal with this "almost periodical" case. That is why $\alpha$ will have to vary in a compact set $[-M, M]$ and not only $[0, 1]$ (see for example [3])

Define

$$||f|| := \sum_{j \in \mathbb{Z}} |\hat{f}(j)|$$

Define also, for $f : \mathbb{T} \to \mathbb{C}$ and $\eta \geq 1$

$$B(\mathbb{T}) = \{f : \mathbb{T} \to \mathbb{C}, \sum_{j \in \mathbb{Z}} |\hat{f}(j)| \sqrt{\log (|j| + 2)} < +\infty\}$$

$$||f||_B = \sum_{j \in \mathbb{Z}} |\hat{f}(j)| \sqrt{\log (|j| + 2)}$$

$$C_\eta(\mathbb{T}) = \{f : \mathbb{T} \to \mathbb{C}, \sum_{j \in \mathbb{Z}} |\hat{f}(j)||j|^\eta < +\infty\}$$

Remark that $\forall \eta \geq 1, C_\eta(\mathbb{T}) \subset B(\mathbb{T}) \subset A(\mathbb{T})$.

We will denote by $\varphi_X$ the characteristic function of the random variable $X$

$$\forall t \in \mathbb{R}, \varphi_X(t) = \mathbb{E}(e^{2\pi i t X})$$

We will distinguish two cases depending whether $X(\Omega) \subset \mathbb{Z}$ or $G(X(\Omega)) = \mathbb{R}$, where $G(X(\Omega))$ is the additive group generated by the support of the random variable $X$. The more fruitful results will be obtained when $G(X(\Omega)) = \mathbb{R}$.

Remark that in the first case, $\varphi_X$ is periodic whereas in the second case, it is not ; note that there are also cases where $\varphi_X$ is periodic and $X(\Omega) \not\subset \mathbb{Z}$.

Still we will call (with a slight abuse of language):

- **periodic case** : for all $k$, $X_k(\Omega) \subset \mathbb{Z}$
- **non periodic case** : $\exists K, \forall k \geq K, G(X_k(\Omega)) = \mathbb{R}$

We will also distinguish two types of growth for $\mathbb{E}|X_k|$ :

- **polynomial case**: there exist $\beta > 0$ and $d > 0$ with $\mathbb{E}|X_k|^\beta = \mathcal{O}(k^d)$
- **subexponential case**: there exists $\beta > 0$ and $\gamma \in [0, 1]$ with $\mathbb{E}|X_k|^\beta = \mathcal{O}(2^{k\gamma})$

Define also the following sequence :

$$c_n = \begin{cases} 1 + \sqrt{\log n} & \text{in the polynomial case} \\ n^{-\frac{3}{2}} & \text{in the subexponential case} \end{cases}$$
The key ingredient in the following will be estimates on trigonometric polynomials like
\[ \sum_{k=1}^{\Lambda} a_k \left( e^{2i\pi \alpha_j X_k(\omega)} - \mathbb{E}(e^{2i\pi \alpha_j X_k}) \right) \]
The proof of our convergence or continuity results will start by looking separately at the "centered part" 
\[ F(\alpha, \omega) - \mathbb{E}(F(\alpha, .)) := \sum_{k} a_k [f(\alpha X_k(\omega)) - \mathbb{E}(f(\alpha X_k))] \]
and the "expectation part" \( \mathbb{E}(F(\alpha, .)) \).

Plan of the paper
In section 2., we state the main results : centered part in the periodic and non periodic case, estimates on the trigonometric polynomials (in the periodic and non periodic cases) and expectation part in the periodic and non periodic cases.
In section 3., we prove the results concerning the centered part and the expectation part in the periodic and non periodic cases.
Section 4. is devoted to the proof of the core result (th. 2.2) on trigonometric polynomials using gaussian techniques.
Finally, in section 5., we give examples in the non periodic and periodic cases.

2. Main results

2.1. The centered part.

**Theorem 2.1.** Let \((X_k)_{k \geq 0}\) be a sequence of independent real valued random variables Assume we are in the polynomial case or in the subexponential case. Let \((a_k)_{k \geq 1}\) be a sequence of complex numbers enjoying the following property
\[ \sum_{n \geq 1} \frac{\sqrt{\sum_{k \geq n} |a_k|^2}}{nc_n} < +\infty \]
then there exists a measurable set \(\Omega_0\) with \(\mathbb{P}(\Omega_0) = 1\) such that for all \(\omega \in \Omega_0\),
- in the non periodic case, for any \(f \in B(T)\) such that \(\int_T f(t)dt = 0\): for \(\alpha \in \mathbb{R}\), \(F(\alpha, \omega) - \mathbb{E}(F(\alpha, .))\) is well defined, \(\alpha \mapsto F(\alpha, \omega) - \mathbb{E}(F(\alpha, .))\) is continuous, the series defining \(F - \mathbb{E}(F)\) converges uniformly on every compact set and there exists \(C_\omega > 0\) random variable with finite expectation such that for all \(\alpha \in \mathbb{R}\):
  \[ |F(\alpha, \omega) - \mathbb{E}(F(\alpha, .))| \leq C_\omega \|f\| \sqrt{\log(1 + |\alpha|)} \]
- in the periodic case, for any \(f \in A(T)\) such that \(\int_T f(t)dt = 0\) : for \(\alpha \in \mathbb{R}\), \(F(\alpha, \omega) - \mathbb{E}(F(\alpha, .))\) is well defined, \(\alpha \mapsto F(\alpha, \omega) - \mathbb{E}(F(\alpha, .))\) is continuous, the series defining \(F - \mathbb{E}(F)\) converges uniformly on
every compact set and there exists \( C_\omega > 0 \) random variable with finite expectation such that for all \( \alpha \in \mathbb{R} \):
\[
|F(\alpha, \omega) - \mathbb{E}(F(\alpha, .))| \leq C_\omega \|f\|
\]

Remark 2.1.

1. We also have:
\[
\mathbb{E} \sup_{T>1} \frac{\int_0^T |F(t, \omega) - \mathbb{E}(F(t, .))|^2 dt}{\sqrt{T \log T}} < \infty
\]
in the non periodic case.

2. For example, when \( |a_k| = \mathcal{O}(k^{-\delta}) \), in the polynomial case, if \( \delta > 1/2 \), then condition (2) holds and in the subexponential case, if \( \delta > \frac{\gamma + 1}{\gamma} \), then condition (2) holds.

3. Concerning the subexponential case, if \( \gamma \geq 1 \) \( (\mathbb{E}|X_k|^\beta \) grows exponentially) and if the sequence \( |a_k| \) is decreasing, then condition (2) implies the convergence of the series \( \sum |a_k| \). Namely (see step 1. of the proof in section 3.) condition (2) implies
\[
\sum_{k=1}^{2k} \left( \sum_{l=2^{k+1}}^{2k+1} |a_l|^2 \right)^{\frac{1}{2}} < +\infty
\]
and consequently
\[
\sum_{k} 2^k |a_{2k}| < +\infty
\]
(see for example [6] p.85). We now use the fact that \( |a_k| \) is decreasing to get, for any integer \( m > 1 \):
\[
\sum_{n=1}^{m} |a_n| \leq \sum_{n=1}^{2k} |a_n| \leq \sum_{j=0}^{k-1} \sum_{l=2^{j+1}}^{2^{j+1}-1} |a_l| \leq \sum_{j=0}^{k-1} 2^j |a_{2j}|
\]
where \( k \) is such that \( 2^k > m \).
In this case, the function \( F \) is obviously well defined.

This result rely on uniform estimations of the size of some trigonometric polynomials, more precisely on the following:
Recall that \( \log^+ = \max(\log, 0) \).

Theorem 2.2. Let \( \lambda \) and \( \Lambda \) be two integers with \( \lambda \leq \Lambda \), \( (X_k)_{k \geq 0} \) be a sequence of independent real valued random variables for which there exists \( \beta > 0 \) such that, \( \forall N \geq 0, \mathbb{E}|X_N|^\beta < \infty \). Define
\[
\forall N \geq 0, \Phi_{\beta}(N) = 2 + \max(N, \mathbb{E}|X_N|^\beta)
\]
Let \( M \geq 1 \) and \( I_M = [-M, M] \). Let \( (a_k)_{k \geq 1} \) be a sequence of real or complex numbers.
Define

\[ A_{\Lambda,M} = \sqrt{\log (M \Phi_{\beta}(\Lambda)) \sum_{k=\Lambda}^{\Lambda} |a_k|^2}, \]

then

\[ \mathbb{E} \sup_{j \in \mathbb{Z}} \sup_{\lambda \geq 1} \sup_{\lambda \geq \Lambda} \sup_{\alpha \in I_M} \left| \sum_{k=\Lambda}^{\Lambda} a_k [\exp 2i\pi \alpha j X_k(\omega) - \mathbb{E} \exp 2i\pi \alpha j X_k] \right| \sqrt{A^2_{\Lambda,\lambda,M} \log (|j| + 2)} \] \[ < \infty \]

**Remark 2.2.** In the periodic case, the proof of theorem 2.2 is easier. Namely, using the fact that \( \alpha \mapsto j\alpha \pmod{1} \) is onto for \( j \neq 0 \), we get

\[ \mathbb{E} \sup_{j \in \mathbb{Z}} \sup_{\alpha \in \mathbb{T}} \left| \sum_{k=\Lambda}^{\Lambda} a_k [\exp 2i\pi \alpha j X_k(\omega) - \mathbb{E} \exp 2i\pi \alpha j X_k] \right| \] \[ = \mathbb{E} \sup_{\alpha \in \mathbb{T}} \left| \sum_{k=\Lambda}^{\Lambda} a_k [\exp 2i\pi \alpha X_k(\omega) - \mathbb{E} \exp 2i\pi \alpha X_k] \right| \]

The conclusion of theorem 2.2 becomes then:

\[ \mathbb{E} \sup_{\lambda \geq 1} \sup_{\lambda \geq \Lambda} \sup_{\alpha \in I_M} \left| \sum_{k=\Lambda}^{\Lambda} a_k [\exp 2i\pi \alpha X_k(\omega) - \mathbb{E} \exp 2i\pi \alpha X_k] \right| \sqrt{A^2_{\Lambda,\lambda,M}} \] \[ < \infty \]

In the non periodic case, the proof is more tedious. It relies on a fine inequality about decoupling gaussian random functions (see section 3.). We can see here why, for integer-valued \( X_k \), we can work with the functional space \( A(\mathbb{T}) \), whereas for real-valued \( X_k \), we need to introduce the space \( B(\mathbb{T}) \).

We can naturally wonder whether condition (2) in theorem 2.1 is necessary or not. Here is a partial answer. If \( \sum |a_k|^2 \) diverges, then we can construct a stochastic process \((X_k)_{k \geq 1}\) satisfying the hypothesis of theorem 2.1 and find \( f \in A(\mathbb{T}) \) such that the convergence of the series is not uniform on any compact set. In that sense, the conditions imposed to the sequence \((a_k)_{k \geq 1}\) are optimal. Remark that in this case, condition (2) is not fulfilled.

Namely consider a sequence of independent random variables with disjoint supports. For all \( k \geq 1 \) the support of \( X_k \) is the set of integers belonging to \([k^2, (k+1)^2 - 1]\) therefore, the hypothesis on the moment is satisfied and we are in the periodic case. We will come back to the law of \( X_k \) later. Now choose \( f \) in the following way : for all \( \alpha \in \mathbb{T}, f(\alpha) = \exp (2i\pi \alpha), f \in A(\mathbb{T}) \).

As a consequence, if the convergence of the series defining \( F \) was uniform in
$\alpha$ on $\mathbb{T}$, then we would have:

$$\sqrt{\int_0^1 \sum_{k=1}^{\infty} a_k |\exp 2i\pi \alpha X_k(\omega) - \mathbb{E} \exp 2i\pi \alpha X_k|^2} \ d\alpha$$

$$\leq \sup_{\alpha \in [0,1]} \left| \sum_{k=1}^{\infty} a_k |\exp 2i\pi \alpha X_k(\omega) - \mathbb{E} \exp 2i\pi \alpha X_k| \right| < \infty$$

By construction, $\mathbb{P}$– almost surely:

$$\int_0^1 \left| \sum_{k=1}^{\infty} a_k |\exp 2i\pi \alpha X_k(\omega) - \mathbb{E} \exp 2i\pi \alpha X_k|^2\right| d\alpha$$

$$= \sum_{k=1}^{\infty} |a_k|^2 \int_0^1 |\exp (2i\pi \alpha X_k(\omega)) - \mathbb{E} \exp (2i\pi \alpha X_k)|^2 d\alpha$$

$$\geq \sum_{k=1}^{\infty} |a_k|^2 \int_0^1 |1 - |\mathbb{E} \exp (2i\pi \alpha X_k)||^2 d\alpha$$

Assume now that the law of $X_k$ is uniform on the $2k + 1$ integers of $[k^2, (k + 1)^2 - 1]$, for all $k \geq 1$.

$$|\mathbb{E} \exp (2i\pi \alpha X_k)| = \frac{1}{2k + 1} \left| \frac{\sin \pi \alpha(2k + 1)}{\sin \pi \alpha} \right|$$

Using Lebesgue convergence theorem, we get

$$\lim_{k \to +\infty} \int_0^1 |1 - |\mathbb{E} \exp (2i\pi \alpha X_k)||^2 d\alpha = 1$$

and we also get the divergence of the series with positive terms

$$\sum_{k=1}^{\infty} |a_k|^2 \int_0^1 |1 - |\mathbb{E} \exp (2i\pi \alpha X_k)||^2 d\alpha = \infty$$

A contradiction with uniform convergence of the centered part.

2.2. The expectation part in the non periodic case.

**Theorem 2.3.** Let $(X_k)_{k \geq 0}$ be a sequence of independent real valued random variables Assume we are in the non periodic case. Assume we are in the polynomial or in the subexponential case. Let $(a_k)_{k \geq 1}$ be a sequence of complex numbers such that, for any compact set $K$ which does not contain 0:

$$\forall \varepsilon > 0, \exists N > 0, \sup_{m > n \geq N} \sup_{\alpha \in K} \sup_{j \in \mathbb{Z} - \{0\}} \left| \sum_{k=n}^{m} a_k \varphi_{X_k}(j\alpha) \right| < \varepsilon.$$
Assume moreover that:

\[ \sum_{n \geq 1} \sqrt{\frac{\sum_{k \geq n} |a_k|^2}{nc_n}} < +\infty \]

then in both cases, there exists a measurable set \( \Omega_0 \) with \( \mathbb{P}(\Omega_0) = 1 \) such that for all \( \omega \in \Omega_0 \), for any \( f \in B(\mathbb{T}) \) such that \( \int_{\mathbb{T}} f(t) dt = 0 \) : for \( \alpha \in \mathbb{R} - \{0\} \), \( F(\alpha, \omega) = \sum_{k=0}^{\infty} a_k f(\alpha X_k(\omega)) \) is well defined, \( \alpha \mapsto F(\alpha, \omega) \) is continuous and the series defining \( F \) converges uniformly on every compact set which does not contain 0.

**Remark 2.3.**

1. Of course, if hypothesis \((H)\) is fulfilled only for one compact \( K \), \( F \) is continuous and uniformly convergent on this compact \( K \).
2. It is worth noticing that the set \( \Omega_0 \) does not depend on the function \( f \) in \( B(\mathbb{T}) \).
3. Conditions on the law of the process \( (X_k)_{k \geq 0} \) will be given later to fulfill hypothesis \((H)\).

Of course, condition \((H)\) may not be convenient to check. To make it more checkable, it is possible to split up the hypothesis on the sequence \((a_k)\) and the characteristic function \( \varphi_{X_k} \) either using Abel’s summation (corollary 2.4) method or using Cauchy Schwarz inequality (corollary 2.5).

**Corollary 2.4.** Let \( (X_k)_{k \geq 0} \) be a sequence of independent real valued random variables. Assume we are in the polynomial or in the subexponential case and in the non periodic case.

Assume that, for any compact set \( K \) which does not contain 0 :

\[ (H') \quad \sup_{N \geq 1} \sup_{\alpha \in K} \sup_{j \in \mathbb{Z} - \{0\}} \sum_{k=0}^{N} \varphi_{X_k}(j\alpha) < \infty. \]

Let \((a_k)_{k \geq 1}\) be a sequence of complex numbers enjoying the following properties

1. \( \sum_{n \geq 1} \sqrt{\frac{\sum_{k \geq n} |a_k|^2}{nc_n}} < +\infty \)
2. \( \sum_{k \geq 1} |a_k - a_{k+1}| \) converges

then there exists a measurable set \( \Omega_0 \) with \( \mathbb{P}(\Omega_0) = 1 \) such that for all \( \omega \in \Omega_0 \), for any \( f \in B(\mathbb{T}) \) such that \( \int_{\mathbb{T}} f(t) dt = 0 \) : for \( \alpha \in \mathbb{R} - \{0\} \), \( F(\alpha, \omega) \) is well defined, \( \alpha \mapsto F(\alpha, \omega) \) is continuous and the series defining \( F \) converges uniformly on every compact set which does not contain \( \{0\} \).

**Corollary 2.5.** Let \( (X_k)_{k \geq 0} \) be a sequence of independent real valued random variables. Assume we are in the polynomial or in the subexponential case and in the non periodic case.

Assume that, for any compact set \( K \) which does not contain 0 :

\[ (H'') \quad \forall \varepsilon > 0, \exists N > 0, \sup_{m \geq n \geq N} \sup_{\alpha \in K} \sup_{j \in \mathbb{Z} - \{0\}} \left( \sum_{k=n}^{m} |\varphi_{X_k}(j\alpha)|^2 \right) < \varepsilon. \]
Let \((a_k)_{k \geq 1}\) be a sequence of complex numbers enjoying

\[
\sum_{n \geq 1} \frac{\sqrt{\sum_{k \geq n} |a_k|^2}}{nc_n} < +\infty
\]

then there exists a measurable set \(\Omega_0\) with \(\mathbb{P}(\Omega_0) = 1\) such that for all \(\omega \in \Omega_0\), for any \(f \in B(\mathbb{T})\) such that \(\int_{\mathbb{T}} f(t)dt = 0\) : for \(\alpha \in \mathbb{R} - \{0\}\), \(F(\alpha, \omega)\) is well defined, \(\alpha \mapsto F(\alpha, \omega)\) is continuous and the series defining \(F\) converges uniformly on every compact set which does not contain \(\{0\}\).

**Remark 2.4.**
The previous corollaries will be useful for example when the law of \(X_k\) is obtained by convolution product (see corollary 5.2).

2.3. The expectation part in the periodic case. In the periodic situation, to obtain convergence properties for a large class of functions, we need to control a phenomenon of uniform distribution of the sequence \(\{\alpha j\}\) (fractionary part of the sequence \(\alpha j\)) (see example 5.3 and corollary 5.3), as the modulus of the characteristic function does not go to zero anymore when \(|j|\) goes to infinity. One way to do this is to link the regularity of the set of functions \(f\) with the arithmetical properties of \(\alpha\). Thus, we only obtain pointwise convergence in this case.

**Theorem 2.6.** Let \((X_k)_{k \geq 0}\) be a sequence of independent real valued random variables. Assume we are in the periodic case. Assume we are in the polynomial case or in the subexponential case. Let \((a_k)_{k \geq 1}\) be a sequence of complex numbers, let \(\eta > 1\) and:

\[
E_\eta = \{\alpha \in \mathbb{T} \setminus \{0\}, \forall \varepsilon > 0, \exists N > 0, \sup_{m > n \geq N} \sup_{j \in \mathbb{Z} - \{0\}} |\alpha j|^{-\eta} \sum_{k=0}^{m} a_k \varphi X_k(j) < \varepsilon\}
\]

Assume moreover that:

\[
\sum_{n \geq 1} \frac{\sqrt{\sum_{k \geq n} |a_k|^2}}{nc_n} < +\infty
\]

then there exists a measurable set \(\Omega_0\) with \(\mathbb{P}(\Omega_0) = 1\) such that for all \(\omega \in \Omega_0\), for any \(f \in C_\eta(\mathbb{T})\) such that \(\int_{\mathbb{T}} f(t)dt = 0\) : for \(\alpha \in E_\eta\), \(F(\alpha, \omega) = \sum_{k=0}^{\infty} a_k f(\alpha X_k(\omega))\) is well defined.

**Remark 2.5.** We shall give examples with \(\eta\) close to one and such that \(E_\eta\) has full Lebesgue measure (see section 5.2). We will also be able to give explicit \(\alpha \in E_\eta\).

Is it possible to give results on the continuity of \(\alpha \mapsto F(\alpha, \omega)\)? We give a positive answer in the particular case when \(f(t) = e^{2i\pi t}\).
Theorem 2.7. Let \((X_k)_{k \geq 0}\) be a sequence of independent real valued random variables. Assume we are in the polynomial case or in the subexponential case and in the periodic case. Let \((a_k)_{k \geq 1}\) be a sequence of complex numbers such that, for any compact set \(K\) which does not contain 0:

\[
(\mathcal{H}''') \quad \forall \varepsilon > 0, \exists N > 0, \sup_{m > n \geq N} \sup_{\alpha \in K} \left| \sum_{k=n}^{m} a_k \varphi X_k(\alpha) \right| < \varepsilon.
\]

Assume moreover that:

\[
\sum_{n \geq 1} \frac{\sqrt{\sum_{k \geq n} |a_k|^2}}{n c_n} < +\infty
\]

then there exists a measurable set \(\Omega_0\) with \(P(\Omega_0) = 1\) such that for all \(\omega \in \Omega_0\): for \(\alpha \in \mathbb{R} - \{0\}\), \(F(\alpha, \omega) = \sum_{k=0}^{\infty} a_k e^{2\pi i (\alpha X_k(\omega))}\) is well defined, \(\alpha \mapsto F(\alpha, \omega)\) is continuous and the series defining \(F\) converges uniformly on every compact set which does not contain 0.

3. Proof of theorems 2.1, 2.3, 2.6 and 2.7

First, we split formally \(F\) into two parts as follows:

\[
\sum_{k} a_k f(\alpha X_k(\omega)) = \sum_{k} a_k [f(\alpha X_k(\omega)) - \mathbb{E}(f(\alpha X_k))] + \sum_{k} a_k \mathbb{E}(f(\alpha X_k))
\]

**Step 1**: (first part of the sum (centered part) in the non periodic and periodic cases)

We begin by proving theorem 2.1 in the non periodic case.

We will prove a result of uniform convergence for the centered part.

Let \((N_k)_{k \geq 1}\) be a strictly increasing sequence of integers and define

\[
\forall k \geq 1, \quad P_k(\alpha) = \sum_{l=N_k+1}^{N_{k+1}} a_l [f(\alpha X_l(\omega)) - \mathbb{E}(f(\alpha X_l))]
\]

where \(f \in B(\mathbb{T})\). We want to study the following series, for all \(M \geq 1\):

\[
\sum_{k} \sup_{\alpha \in [-M, M]} |P_k(\alpha)|
\]

We have:

\[
|P_k(\alpha)| \leq \sum_{j \in \mathbb{Z}} |\tilde{f}(j)| \sum_{l=N_k+1}^{N_{k+1}} a_l \left| \exp(2\pi j \alpha X_l(\omega)) - \mathbb{E}\exp(2\pi j \alpha X_l) \right|
\]

Hence, using theorem 2.2, there exists a positive integrable random variable \(\xi\) such that

\[
(3) \quad \sup_{\alpha \in [-M, M]} |P_k(\alpha)| \leq \xi ||f|| \sqrt{\log(M \Phi_\beta(N_{k+1}) \sum_{j=N_{k+1}}^{N_k} |a_j|^2)}
\]
where
\[
\xi = \sup_{j \in \mathbb{Z}} \sup_{k \geq 1} \sup_{\alpha \in I_M} \left| \frac{1}{\sqrt{A_{k,M}^2 \log (|j| + 2)}} \sum_{l=N_{k+1}}^{N_{k+1}} a_l \left[ e^{2i\pi \alpha j X_l(\omega)} - E e^{2i\pi \alpha j X_l} \right] \right|
\]
with
\[
A_{k,M}^2 = \log (M \Phi_\beta(N_{k+1})) \sum_{l=N_{k+1}}^{N_{k+1}} |a_l|^2
\]
First, in the polynomial case, that is to say when there exists \( d > 0 \) with \( \Phi_\beta(N) = \mathcal{O}(N^d) \) then we choose \( N_k = 2^{2^k} \) and we need to prove that
\[
\sum_k 2^{k/2} \left( \sum_{l=2^{2^k}+1}^{2^{2^k+1}} |a_l|^2 \right)^{1/2} < +\infty
\]
now we use the following equivalent:
\[
\sum_{l=2^{2^k}+1}^{2^{2^k+1}} \frac{1}{l(\log(l))^{1/2}} \approx 2^{k/2}
\]
which may be computed by comparing series and integral, hence:
\[
2^{k/2} \left( \sum_{l=2^{2^k}+1}^{2^{2^k+1}} |a_l|^2 \right)^{1/2} \leq C \sum_{l=2^{2^k}+1}^{2^{2^k+1}} \frac{1}{l(\log(l))^{1/2}} \left( \sum_{l=2^{2^k}+1}^{\infty} |a_l|^2 \right)^{1/2}
\]
\[
\leq C \sum_{l=2^{2^k}+1}^{2^{2^k+1}} \frac{\left( \sum_{j=l}^{\infty} |a_j|^2 \right)^{1/2}}{l(\log(l))^{1/2}}
\]
and, using condition (2):
\[
\sum_k 2^{k/2} \left( \sum_{l=2^{2^k}+1}^{2^{2^k+1}} |a_l|^2 \right)^{1/2} \leq \sum_k C \sum_{l=2^{2^k}+1}^{2^{2^k+1}} \frac{\left( \sum_{j=l}^{\infty} |a_j|^2 \right)^{1/2}}{l(\log(l))^{1/2}}
\]
\[
\leq \sum_{n \geq 2} \frac{\sqrt{\sum_{k \geq n} |a_k|^2}}{n \sqrt{\log n}} < +\infty
\]
this implies:
\[
\sum_{k \geq 1} \sup_{\alpha \in [-M,M]} |P_k(\alpha)| < \infty
\]
almost everywhere on the measurable set \( \Omega_\alpha = \{ \omega \in \Omega \mid \xi(\omega) < \infty \} \). By construction, this set does not depend on the choice of \( f \).

Secondly, in the subexponential case, that is when there exists \( \gamma \in ]0, 1[ \) with 
\[
\Phi_\beta(N) = O(2^{N^\gamma})
\]
we choose \( N_k = 2^k \) and we need to prove that
\[
\sum_k 2^{\gamma k/2} \left( \sum_{l=2^k+1}^{2^{k+1}} |a_l|^2 \right)^{1/2} < +\infty
\]
Using the following equivalent:
\[
\sum_{l=2^k+1}^{2^{k+1}} \frac{1}{\lfloor l/2 \rfloor} \approx 2^{\gamma k/2}
\]
and doing the same kind of computation as before, using condition (2):
\[
\sum_{n \geq 2} \frac{\sqrt{\sum_{k \geq n} |a_k|^2}}{n^{1-\gamma}} < +\infty
\]
implies
\[
\sum_{k \geq 1} \sup_{\alpha \in [-M, M]} |P_k(\alpha)| < \infty
\]
We also get from (3), for all \( \alpha \in \mathbb{R} \):
\[
|P_k(\alpha)| \leq \xi(||f||) \sqrt{\log(|\alpha|+2)} \sqrt{\log(\Phi_\beta(N_{k+1}))} \sum_{j=N_{k+1}}^{N_{k+1+1}} |a_j|^2
\]
summing on \( k \), we get the inequality:
\[
|F(\alpha, \omega) - \mathbb{E}(F(\alpha, .))| \leq C \xi(\omega)||f|| \sqrt{\log(|\alpha|+2)}
\]
where \( C \) only depends on \( (a_k) \) and \( \mathbb{E}(\xi) < \infty \).
This ends the proof of theorem 2.1 which is also the first step of the proof of theorem 2.3.

In the periodic case, using remark 2.2, we can see that we only need to take \( f \in A(T) \) and in the previous inequality the \( \sqrt{\log(|\alpha|+2}) \) is replaced by a constant.

**Step 2:** (second part of the sum (expectation part) in the non periodic case, theorem 2.3)
Let \( K \) be a compact set which does not contain zero and \( \alpha \in K \). Let \( n < m \)
be two integers,

\[ \left| \sum_{k=n}^{m} a_k \mathbb{E}(f(\alpha X_k)) \right| = \left| \sum_{j \in \mathbb{Z}^*} \sum_{k=n}^{m} a_k \hat{f}(j) \mathbb{E} \left( \exp(2i\pi j \alpha X_k) \right) \right| \]

(5) \quad = \left| \sum_{j \in \mathbb{Z}^*} \sum_{k=n}^{m} a_k \hat{f}(j) \varphi_{X_k}(j \alpha) \right|

(6) \quad = \left| \sum_{j \in \mathbb{Z}^*} \hat{f}(j) \left( \sum_{k=n}^{m} a_k \varphi_{X_k}(j \alpha) \right) \right|

\[ \leq \left( \sum_{j \in \mathbb{Z}^*} |\hat{f}(j)| \right) \sup_{j \in \mathbb{Z}^*} \left| \sum_{k=n}^{m} a_k \varphi_{X_k}(j \alpha) \right| \]

At this point, to prove theorem 2.3, we can conclude directly by using hypothesis (H) to get

\[ \sup_{n<m} \sup_{\alpha \in K} \left| \sum_{k=n}^{m} a_k \mathbb{E}(f(\alpha X_k)) \right| < \varepsilon \]

as long as \( m \) and \( n \) are large enough.

To prove corollary 2.4, we use Abel’s summation. Let \( \phi_p = \sum_{k=0}^{p-1} \varphi_{X_k} \), we have :

(8) \quad \sum_{k=n}^{m} a_k \varphi_{X_k}(j \alpha) = \sum_{k=n}^{m} a_k (\phi_{k+1}(j \alpha) - \phi_k(j \alpha))

(9) \quad = \sum_{k=n+1}^{m+1} a_k \phi_{k}(j \alpha) - \sum_{k=n}^{m} a_k \phi_k(j \alpha)

(10) \quad = -a_n \phi_n(j \alpha) + a_m \phi_{m+1}(j \alpha)

(11) \quad + \sum_{k=n+1}^{m} (a_{k-1} - a_k) \phi_k(j \alpha)

and :

\[ (7) \leq \|f\| \sup_{N \geq 1} \sup_{\alpha \in K} \sup_{j \in \mathbb{Z}^*} |\phi_N(j \alpha)| \left[ |a_m| + |a_n| + \sum_{k=n+1}^{m} |a_k - a_{k-1}| \right] \]

we now conclude using hypothesis \( \mathcal{H}' \) and hypothesis (1) on the sequence \( (a_n) \) in corollary 2.4 :

\[ \sup_{n<m} \sup_{\alpha \in K} \left| \sum_{k=n}^{m} a_k \mathbb{E}(f(\alpha X_k)) \right| \leq \|f\| \left[ |a_m| + |a_n| + \sum_{k=n+1}^{m} |a_k - a_{k-1}| \right] \sup_{N \geq 1} \sup_{\alpha \in K} \sup_{j \in \mathbb{Z}^*} |\phi_N(j \alpha)| < \varepsilon \]
as long as \( m \) and \( n \) are large enough.

To prove corollary 2.5, we use Cauchy Schwarz inequality in the following way:

\[
\left| \sum_{k=n}^{m} a_k \varphi_{X_k}(j\alpha) \right| \leq \sqrt{\sum_{k=n}^{m} |a_k|^2} \sqrt{\sum_{k=n}^{m} \varphi_{X_k}(j\alpha)^2}
\]

and we conclude using condition \( \mathcal{H}'' \).

- **Step 3**: (second part of the sum (expectation part) in the periodic case, theorems 2.6 and 2.7)

We take \( f \in \mathcal{C}_\eta \) and \( \alpha \in E_\eta \), in the same way as in the beginning of step 2, we have

\[
\left| \sum_{k=n}^{m} a_k \mathbb{E}(f(\alpha X_k)) \right| = \left| \sum_{j \in \mathbb{Z}^*} \sum_{k=n}^{m} a_k \hat{f}(j) \mathbb{E}(\exp(2i\pi j\alpha X_k)) \right|
\leq \left( \sum_{j \in \mathbb{Z}^*} |\hat{f}(j)||j|^\eta \right) \sup_{j \in \mathbb{Z}^*} |j^{-\eta}| \sum_{k=n}^{m} a_k \varphi_{X_k}(j\alpha)
\]

and we conclude using the definition of \( E_\eta \).

As for theorem 2.7, that is to say for \( f(t) = e^{2i\pi t} \), if \( K \) is a compact which does not contain 0 and \( \alpha \in K \), we have

\[
\left| \sum_{k=n}^{m} a_k \mathbb{E}(f(\alpha X_k)) \right| = \left| \sum_{k=n}^{m} a_k \varphi_{X_k}(\alpha) \right|
\]

and the conclusion follows from hypothesis \( \mathcal{H}'' \).

**Remark 3.1.** The most general hypothesis we can put on the sequence \( (a_k)_{k \geq 1} \) is the following: there exists a strictly increasing sequence \( (N_k)_{k \geq 1} \) such that

\[
\sum_{k=1}^{\infty} \log \Phi_j(N_{k+1}) \sum_{l=N_k+1}^{N_{k+1}} |a_l|^2 < \infty
\]

### 4. Proof of theorem 2.2

Let us begin by restating some inequalities obtained by Fernique ([4]) which will be useful in the proof of theorem 2.2. For more information on gaussian techniques in this framework, see [3] and [2].

**Inequality 4.1.** Let \( (G_k)_{k \geq 1} \) be a sequence of Banach space valued gaussian random variables \((B, \| \cdot \|)\) defined on a probabilised space \((\Omega, \mathcal{A}, \mathbb{P})\). Then:

\[
\mathbb{E} \sup_{k \geq 1} \| G_k \| \leq K_1 \left\{ \sup_{k \geq 1} \mathbb{E} \| G_k \| + \mathbb{E} \sup_{k \geq 1} \| \lambda_k \sigma_k \| \right\}
\]
where \((\lambda_k)_{k \geq 1}\) is an isonormal sequence, \(K_1\) a universal constant and for all \(k \geq 1\),

\[\sigma_k = \sup_{f \in B' : \|f\| \leq 1} \|G_k \cdot f - G_f\|_{L^p} \]

**Inequality 4.2.** Let \(g\) be a real valued stationary gaussian random variable, separable and continuous in quadratic mean. Let \(m\) be its associated spectral measure on \(\mathbb{R}^+\) defined by

\[E[|g(s) - g(t)|^2] = 2 \int_0^\infty |1 - \cos 2\pi u(s - t)|m(du)\]

We have

\[E \sup_{\alpha \in [0,1]} g(\alpha) \leq K \left\{ \sqrt{\int_0^\infty \min(u^2, 1)m(du)} + \int_0^\infty \sqrt{m(\{e^{u^2}, \infty\})} dx \right\}\]

where \(K\) is a universal constant.

**Inequality 4.3.** (decoupling) Let \(X = \{X(t) : t \in T\}\) be a gaussian random function defined on a finite or countable set \(T\). Let \(\{T_k, k \in [1, n]\}\) be a covering of \(T\). Let \(S = T_1 \times \cdots \times T_n\). The following inequality holds:

\[E \left\{ \sup_{k \in [1, n]} \left[ \sup_{t \in T_k} X(t) - E \sup_{t \in T_k} X(t) \right] \right\} \leq \frac{\pi}{2} \cdot \sup_{s \in S} \left\{ E \left[ \sup_{k \in [1, n]} X(s_k) \right] \right\}\]

The following estimation generalizes inequality 4.2 and will be useful in the almost periodic case because it gives estimations on arbitrarily large intervals.

**Inequality 4.4.** Let \(g\) a real valued stationnary gaussian random function, separable and continuous in quadratic mean. Let \(m\) its associated spectral measure on \(\mathbb{R}^+\) defined as in inequality 4.2. There exists a universal constant \(K\) such that

\[E \sup_{\alpha \in [-M, M]} g(\alpha) \leq K \left\{ \sqrt{\int_0^\infty \min(2Mu^2, 1)m(du)} + \int_0^\infty \sqrt{m\left(\frac{e^{u^2}}{2M}, \infty\right)} dx \right\}\]

Let us now come to the proof.

**Step 1:** In this part, we replace our problem by a question of regularity of trajectories of random gaussian functions.

Let us consider an independent copy of \(X = (X_k)_{k \geq 1}\) denoted by \(X' = (X'_k)_{k \geq 1}\) defined on another probability space \((\Omega', \mathcal{A}', P')\). We call \(E_\alpha\) the integration symbol whose index refers to the space of integration.

Using classical convexity properties, to prove (1), it is enough to show

\[\sum_{k=1}^\Lambda a_k \left[ e^{2i\pi \alpha k} X_k - e^{2i\pi \alpha'} X'_k \right] \leq \left( A_{\alpha, \alpha'}^2 \log (\|j\| + 2) \right) \]

Let us now come to the proof.

**Step 1:** In this part, we replace our problem by a question of regularity of trajectories of random gaussian functions.

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Using classical convexity properties, to prove (1), it is enough to show

\[\sum_{k=1}^\Lambda a_k \left[ e^{2i\pi \alpha} X_k - e^{2i\pi \alpha'} X'_k \right] \leq \left( A_{\alpha, \alpha'}^2 \log (\|j\| + 2) \right) \]
Let us now symmetrize the problem: consider the following separable family of random functions, with continuous trajectories
\[(f_k)_{k \geq 1} = \{f_k(\alpha, j) = a_k(\exp 2i\pi \alpha j X_k - \exp 2i\pi \alpha j X'_k), \alpha \in I_M, j \in \mathbb{Z}\}_{k \geq 1}\]
By construction \(f\) is a symmetric family of random functions, that is to say their law is sign-invariant. More precisely, call \(f'_{k, j}\) a sequence of independent Rademacher random variables (taking the values +1 and -1 with probability 1/2), defined on a third space \((\Omega', \mathcal{A}', \mathbb{P}')\), independent of \(X\) and \(X'\). \(\{f_k, k \geq 1\}\) and \(\{\varepsilon_k f_k, k \geq 1\}\) have the same law. Thus for all integers \((\Lambda, \lambda)\) such that \(\Lambda \geq \lambda, \sum_{k=\lambda}^{\Lambda} f_k\) and \(\sum_{k=\lambda}^{\Lambda} \varepsilon_k f_k\) also have the same law.

That is why (12) can be written on a larger space of integration in the following way:
\[\mathbb{E}_{X, X', \varepsilon} \sup_{j \in \mathbb{Z}} \sup_{\lambda \geq 1} \sup_{\lambda \geq \lambda} \sup_{\alpha \in I_M} \left| \sum_{k=\lambda}^{\Lambda} \varepsilon_k f_k(\alpha, j) \right| \frac{\sum_{k=\lambda}^{\Lambda} \varepsilon_k f_k(\alpha, j)}{A_{\Lambda, \lambda, M}^2 \log (|j| + 2)} \]
We deduce a sufficient condition for (12) to be realised
\[\mathbb{E}_{X, \varepsilon} \sup_{j \in \mathbb{Z}} \sup_{\lambda \geq 1} \sup_{\lambda \geq \lambda} \sup_{\alpha \in I_M} \left| \sum_{k=\lambda}^{\Lambda} \varepsilon_k a_k \exp 2i\pi \alpha j X_k \right| \frac{\sum_{k=\lambda}^{\Lambda} \varepsilon_k a_k \exp 2i\pi \alpha j X_k}{A_{\Lambda, \lambda, M}^2 \log (|j| + 2)} \]
We then use a precious tool in the theory of gaussian random functions: the contraction principle. This tool is built on a quite simple idea: replace the choice of signs by a sequence of gaussian random variables with mean zero and variance one. This idea can be explained by the following property: given \(g\) a gaussian random variable with mean zero and variance 1 and \(\varepsilon\) a Rademacher random variable, if \(g\) and \(\varepsilon\) are independent, then \(g\) and \(\varepsilon |g|\) have the same law.

As a consequence, in order to prove (13), we show
\[\mathbb{E}_{X, g, g'} \sup_{j \in \mathbb{Z}} \sup_{\lambda \geq 1} \sup_{\lambda \geq \lambda} \sup_{\alpha \in I_M} \left| \sum_{k=\lambda}^{\Lambda} a_k (g_k \cos 2\pi \alpha j X_k + g'_k \sin 2\pi \alpha j X_k) \right| \frac{\sum_{k=\lambda}^{\Lambda} a_k (g_k \cos 2\pi \alpha j X_k + g'_k \sin 2\pi \alpha j X_k)}{A_{\Lambda, \lambda, M}^2 \log (|j| + 2)} < +\infty\]
where \(\{g_k, k \geq 1\}\) et \(\{g'_k, k \geq 1\}\) are two sequences of independent identically distributed random variables with law \(\mathcal{N}(0, 1)\), independent of \(X\) and \(\varepsilon\), defined on two other probability spaces.
Conditionally to \(X\), the problem is reduced to studying the regularity of the trajectories of stationary gaussian random variables. This concludes the first step of the proof.

-Step 2: In this part, we use the gaussian tools introduced in the beginning.
Conditionally to \( X \), call \( G(\lambda, \Lambda, j, \alpha) \) the following quantity

\[
\frac{1}{\sqrt{A_{\lambda, \Lambda}^2 \log (|j| + 3)}} \sum_{k=\lambda}^{\Lambda} a_k \left[ g_k \cos (2\pi \alpha j X_k) + g_k' \sin (2\pi \alpha j X_k) \right]
\]

If \( j, \lambda \) and \( \Lambda \) are fixed, \( G(\alpha) := G(\lambda, \Lambda, j, \alpha) \) is a random function with almost surely continuous trajectories (up to a modification of trajectories). That is why it is enough to show that \( G \) is bounded on \( I_M \cap \mathbb{Q} \). Moreover, we will assume that \( |j| \leq J \) where \( J \) is a large fixed integer.

Let us begin by finding an upper bound for

\[
E_{g, \tilde{g}} \sup_{|j| \leq J} \sup_{\lambda \geq 1} \sup_{\Lambda \geq \lambda} \sup_{\alpha \in I_M \cap \mathbb{Q}} |G(\lambda, \Lambda, j, \alpha)|
\]

First remark that if \( Y_t (t \in T) \) is a gaussian random function defined on \( T \), then for all \( t_0 \in T \) we have (see [4] (480))

\[
E \sup_{t \in T} |Y_t| \leq E|Y_{t_0}| + E \sup_{t \in T} Y_t
\]

In this way, we get rid of the absolute value. Apply this remark to \( G(\lambda, \Lambda, j, \alpha) \) with \( \alpha = 0, \lambda = 1 \) and \( j = 0 \) and let us find an upper bound for

\[
(14) \quad E_{g, \tilde{g}} \sup_{|j| \leq J} \sup_{\lambda \geq 1} \sup_{\Lambda \geq \lambda} \sup_{\alpha \in I_M \cap \mathbb{Q}} G(\lambda, \Lambda, j, \alpha)
\]

In order to apply the decoupling inequality 4.3, define

\[
T = \{-J, \ldots, J\} \times H \times (I_M \cap \mathbb{Q})
\]

where \( J \) is a large enough integer and \( H \) is the upper triangle of dimension 2 in \( \mathbb{N} \times \mathbb{N} \) (see figure 1 below) (\( \lambda \in \mathbb{N} \) and \( \Lambda \geq \lambda \)). A point in \( t \in T \) will be written \( t = (j, \lambda, \Lambda, \alpha) \).

This set \( T \) is at most countable and we will find an upper bound for \( E_{g, \tilde{g}} \sup_{t \in T} G(t) \) independently of \( J \) and then conclude by taking the supremum on \( j \in J \). Define

\[
T_j = \{j\} \times H \times (I_M \cap \mathbb{Q})
\]

It is obvious that \( \{T_j\}_{j=-J, \ldots, J} \) is a covering of \( T \). Define

\[
S = T_{-J} \times \cdots \times T_J
\]

Using inequality 4.3, we have

\[
E_{g, \tilde{g}} \sup_{t \in T} G(t) \leq \frac{\pi}{2} \sup_{s \in S} E_{g, \tilde{g}} \sup_{-J \leq j \leq J} G(s_j) + \sup_{-J \leq j \leq J} E_{g, \tilde{g}} \sup_{t \in T_j} G(t)
\]

where \( s_j \) is a point in \( T_j \).

**Step 3** : We study now

\[
\sup_{s \in S} \sup_{-J \leq j \leq J} G(s_j).
\]
We can rewrite this in the following way:

\[
\sup_{g;g_0} \sup_{-J \leq j \leq J} \sum_{k=\lambda_j}^{\Lambda_j} a_k \left[ g_k \cos(2\pi j \alpha_j X_k) + g'_k \sin(2\pi j \alpha_j X_k) \right] \sqrt{A_{\lambda_j,\Lambda_j,M}^2 \log(|j| + 2)}
\]

where the first supremum is taken on

\[
\{(\alpha_{-J}, \cdots, \alpha_J) \in (I_M \cap \mathbb{Q})^{2J+1}, (\lambda_{-J}, \cdots, \lambda_J) \in H^{2J+1} \}
\]

Fix \((\alpha_{-J}, \cdots, \alpha_J) \in (I_M \cap \mathbb{Q})^{2J+1}\) and \((\lambda_{-J}, \cdots, \lambda_J) \in H^{2J+1}\).

Define the gaussian process

\[
G_j := \sum_{k=\lambda_j}^{\Lambda_j} a_k \left[ g_k \cos(2\pi j \alpha_j X_k) + g'_k \sin(2\pi j \alpha_j X_k) \right] \sqrt{A_{\lambda_j,\Lambda_j,M}^2 \log(|j| + 2)}
\]

In order to get an upper bound for 15, we remark that

\[
\sup_{g;g_0} \sup_{-J \leq j \leq J} G_j
\]

is less than

\[
\sum_{k=\lambda_j}^{\Lambda_j} a_k \left[ g_k \cos(2\pi j \alpha_j X_k) + g'_k \sin(2\pi j \alpha_j X_k) \right] \sqrt{A_{\lambda_j,\Lambda_j,M}^2 \log(|j| + 2)}
\]

Applying inequality 4.1 to the finite sequence of random gaussian functions

\[
(G(j, \lambda_j, \Lambda_j, \alpha))_{-J \leq j \leq J}
\]

we prove that \(\mathbb{E}_{g,g'} \sup_{-J \leq j \leq J} \sup_{\alpha \in I_M} |G(j, \lambda_j, \Lambda_j, \alpha)|\) is less than

\[
C \left\{ \sup_{-J \leq j \leq J} \mathbb{E}_{g,g'} \sup_{\alpha \in I_M} |G(j, \lambda_j, \Lambda_j, \alpha)| + \mathbb{E}_\xi \sup_{-J \leq j \leq J} |\xi_j q_j| \right\}
\]

where \(C\) is a universal constant, \((\xi_j)_{-J \leq j \leq J}\) is an isonormal sequence and

\[
q_j \leq \sup_{\alpha \in I_M} ||G(j, \lambda_j \Lambda_j, \alpha)||_{2,g,g'}
\]

This gives us the following upper bound

\[
q_j \leq \frac{1}{\sqrt{\log(M \Psi_j(\Lambda_j)) \log(|j| + 2)}}
\]

As for all \(j\) we have \(\Lambda_j \geq 1\) and \(M \geq 1\) we easily get

\[
\mathbb{E}_\xi \sup_{-J \leq j \leq J} |\xi_j q_j| \leq \mathbb{E}_\xi \sup_{-J \leq j \leq J} \left| \frac{1}{\sqrt{\log(|j| + 2)}} \xi_j \right|
\]

\[
\leq C \mathbb{E}_\xi \sup_{j \in \mathbb{Z}} \left| \frac{\xi_j}{\sqrt{\log(|j| + 2)}} \right|
\]
The exponential integrability of gaussian vectors gives
\[
\mathbb{E}_\xi \sup_{j \in \mathbb{Z}} \frac{\xi_j}{\sqrt{\log |j| + 2}} = C < \infty
\]
and hence
\[
\sup_{(\lambda_{-J}, \Lambda_{-J}), \cdots, (\lambda_J, \Lambda_J)} \mathbb{E}_\xi \sup_{-J \leq j \leq J} |\xi_j q_j| = C < \infty
\]
indepenently of \( J \) and of the sequence \((\lambda_{-J}, \Lambda_{-J}), \cdots, (\lambda_J, \Lambda_J) \in H^{2J+1}\).

We then get independently of \( J \) on the whole integration space
\[
\mathbb{E}_X \sup_{(\lambda_{-J}, \Lambda_{-J}), \cdots, (\lambda_J, \Lambda_J)} \mathbb{E}_\xi \sup_{-J \leq j \leq J} |\xi_j q_j| \leq C < \infty
\]
where \( C \) is a constant.

Let us now try to find an upper bound for
\[
\sup_{((\lambda_{-J}, \Lambda_{-J}), \cdots, (\lambda_J, \Lambda_J))} \sup_{\alpha \in I_M} |G(j, \lambda_j, \Lambda_j, \alpha)|
\]
indipendently of \( J \).
We will use inequality 4.4. Let us choose a finite sequence
\[
((\lambda_{-J}, \Lambda_{-J}), \cdots, (\lambda_J, \Lambda_J)
\]
and an integer \(|j| \leq J\). The gaussian random function \( G_j(\alpha) := G(j, \lambda_j, \Lambda_j, \alpha) \)
is stationary. Its associated spectral measure on \( \mathbb{R}^+ \) is defined by
\[
m_j = \frac{1}{\log(|j| + 3) \log(M \Phi_{\beta}(\Lambda_j))} \sum_{k=\lambda_j}^{\Lambda_j} a_k^2 \delta_{j\lambda_k}
\]
where \( \delta_u \) is the Dirac measure in the point \( u \).
We get
\[
\mathbb{E}_{g,g'} \sup_{\alpha \in I_M} |G_j(\alpha)|
\leq C \left\{ \sqrt{\int_0^\infty \min\left(\frac{u^2}{2M}, 1\right) m_j(du) + \int_0^\infty \sqrt{m_j \left(\frac{e^{x^2}}{2M}, \infty\right)} \, dx} \right\}
\]
It is obvious that the first term is less than
\[
\sqrt{m_j(\mathbb{R}^+)} \leq C \frac{1}{\sqrt{\log(2M) \log(|j| + 2)}}
\]
where \( C \) is a universal constant because for all \( j \) we have \( \Phi_{\beta}(\Lambda_j) \geq 2 \).
For the second term, it can be rewritten in the following way

\[
\int_0^\infty \sqrt{m_j \left[ \frac{\exp[,2]}{2M}, \infty \right]} \, dx = \left[ \frac{1}{\log(|\lambda| + 3) \log(M \Phi_\beta(\Lambda_j)) \left( \sum_{k=\Lambda_j}^{\Lambda_j} |a_k|^2 \right)} \right]^{\frac{1}{2}}
\]

\[
\times \int_0^\infty \sqrt{\sum_{k=\Lambda_j}^{\Lambda_j} |a_k|^2 1_{\{2M|jX_k| > e^{x^2}\}}} \, dx
\]

Using

\[
\forall k \geq 1, \quad 1_{\{2M|jX_k| > e^{x^2}\}} \leq 1_{\{2M|j\sup_{l \leq k} |X_l| > e^{x^2}\}}
\]

we cut \( \mathbb{R}^+ \) in the integral according to the increasing subdivision

\[
\{0\} \cup \{ \sqrt{\log^+ (2M|j| \sup_{l \leq k} |X_l|)} : \lambda_j \leq k \leq \Lambda_j \}
\]

We get thus an upper bound for the previous integral

\[
\sum_{k=\Lambda_j}^{\Lambda_j} \left[ \sqrt{\log^+ (2M|j| \sup_{l \leq k} |X_l|)} - \sqrt{\log^+ (2M|j| \sup_{l \leq k-1} |X_l|)} \right] \left( \sum_{l=\lambda_j}^{\Lambda_j} |a_l|^2 \right)^{\frac{1}{2}}
\]

\[
\leq \left( \sum_{l=\lambda_j}^{\Lambda_j} |a_l|^2 \right)^{\frac{1}{2}} \sum_{k=\Lambda_j}^{\Lambda_j} \left[ \sqrt{\log^+ (2M|j| \sup_{l \leq k} |X_l|)} \right]
\]

\[
- \left[ \sqrt{\log^+ (2M|j| \sup_{l \leq k-1} |X_l|)} \right]
\]

\[
\leq 2 \left( \sum_{k=\Lambda_j}^{\Lambda_j} |a_k|^2 \right)^{\frac{1}{2}} \sqrt{\log^+ (2M|j| \sup_{l \leq \Lambda_j} |X_l|)}
\]
Consequently, $\forall |j| \leq J$,

$$
\int_{0}^{\infty} \sqrt{m_j \left( \left\lfloor \exp \frac{x^2}{M} \right\rfloor , \infty \right)} \, dx \leq 2 \sqrt{\frac{\log^+ (2M|j| \sup_{l \leq \Lambda_j} |X_l|)}{\log (|j| + 3) \log (M\Phi_\beta(\Lambda_j))}}
$$

$$
\leq 4 \sqrt{\frac{\log^+ (2M \sup_{l \leq \Lambda_j} |X_l|)}{\log (M\Phi_\beta(\Lambda_j))}}
$$

$$
\leq 4 \sup_{1 \leq l \leq \Lambda_j} \sqrt{\frac{\log^+ (2M |X_l|)}{\log (M\Phi_\beta(l))}}
$$

$$
\leq 4 \sup_{N \geq 1} \sqrt{\frac{\log^+ (2M |X_N|)}{\log (M\Phi_\beta(N))}}
$$

$$
\leq 4 \left( 1 + \sup_{N \geq 1} \sqrt{\frac{\log^+ (|X_N|)}{\log \Phi_\beta(N)}} \right)
$$

Finally, on the whole integration space, we get the following upper bound:

$$
E_X \sup_{(\lambda, \Lambda) \in \mathbb{H}^{2j+1}} \sup_{-J \leq j \leq J} \sup_{\alpha \in \Lambda_M} |G_j(\alpha)| \leq C \left( 1 + \sup_{N \geq 1} \sqrt{\frac{\log^+ (|X_N|)}{\log \Phi_\beta(N)}} \right) < \infty
$$
Let us fix $j$. In order to apply inequality 4.1, we need to replace the supremum on $(\lambda, \Lambda) \in H$ by a supremum on only one variable, in other words, we need to renumber $H$ using one variable $h \in \mathbb{N}^2$ given by the formula:

$$h = \lambda + \frac{\Lambda(\Lambda - 1)}{2}$$

(see figure 1).

Let us fix $j$. Inequality 4.1 gives us:

$$\mathbb{E}_{g,g'} \sup_{h \geq 1} \sup_{\alpha \in I_M \cap \mathbb{Q}} G(j, h, \alpha, \alpha) \leq K_1 \left( \sup_{h \geq 1} \mathbb{E}_{g,g'} \sup_{\alpha \in I_M \cap \mathbb{Q}} G(j, h, \alpha) + \mathbb{E}_{g,g'} \sup_{h \geq 1} |\lambda_h \sigma_h| \right)$$

where $(\lambda_h)_{h \geq 1}$ is a isonormal sequence. The inequality:

$$\frac{\Lambda(\Lambda - 1)}{2} \leq h \leq \frac{\Lambda(\Lambda + 1)}{2}$$

gives a polynomial dependence between $\Lambda$ and $h$, hence:

$$\sigma_h = O\left(\frac{1}{\sqrt{\log h}}\right)$$

and the first term in the right hand side of inequality (23) is dealt with in the same way as before. Finally, we get:

$$\mathbb{E}_{g,g'} \sup_{(\lambda, \Lambda) \in H} \sup_{\alpha \in I_M \cap \mathbb{Q}} G(j, \lambda, \Lambda, \alpha) \leq C \left( 1 + \sup_{N \geq 1} \sqrt{\frac{\log^+(|X_N|)}{\log \Phi_{\beta}(N)}} \right)$$

That is to say, by integrating on the whole space,

$$\mathbb{E}_X \sup_{1 \leq j \leq n} \mathbb{E}_{g,g'} \sup_{(\lambda, \Lambda) \in H} \sup_{\alpha \in I_M \cap \mathbb{Q}} G(j, \lambda, \Lambda, \alpha) \leq C \left( 1 + \mathbb{E}_X \sup_{N \geq 1} \sqrt{\frac{\log^+(|X_N|)}{\log \Phi_{\beta}(N)}} \right)$$

Let us prove that

$$\mathbb{E}_X \sup_{N \geq 1} \sqrt{\frac{\log^+(|X_N|)}{\log \Phi_{\beta}(N)}} < \infty$$

Using Jensen inequality, we can get rid of the square root. Let $\delta > 0$. For all $N \geq 1$, we have

$$\beta \log^+ |X_N| \leq \beta \log^+ \left[ \frac{|X_N|}{\Phi_{\beta}(N)} \right] + \beta \log^+ [\Phi_{\beta}(N)]$$

noticing that $\Phi_{\beta}(N)) \geq 2$ it is sufficient to show

$$\mathbb{E} \sup_{N \geq 1} \log^+ \left[ \frac{|X_N|^2}{\Phi_{\beta}(N)} \right] < \infty$$
Using now the inequality $\log^+(x) \leq x$ for any $x \geq 0$, it is sufficient to prove

$$\sum_{N \geq 1} \mathbb{E}|X_N|^\beta \Phi_\beta(N) < \infty$$

And as $\mathbb{E}|X_N|^\beta \leq \Phi_\beta(N)$ and $\Phi_\beta(N) \geq N$, if we chose $\delta = \frac{3}{2}$ we get the conclusion. Steps 3 (see 21), step 4 (see 24) lead us to the announced result of theorem 2.2.

5. Applications

5.1. Examples in the non periodic case. Let us begin by giving an example where the $(X_k)$ are uniformly distributed:

Example 5.1. Suppose that $\mathcal{L}(X_k) = \mathcal{U}([\mu_k - \sigma_k/2, \mu_k + \sigma_k/2])$ with $\sigma_k > 0$ and $\mathbb{E}(X_k) = \mu_k$ with $\mu_k = \mathcal{O}(k^d)$ for some $d > 0$. The characteristic function of $X_k$ can easily be computed:

$$\varphi_{X_k}(t) = \frac{e^{2\pi t \mu_k}}{\pi t \sigma_k} \sin(\pi t \sigma_k)$$

Using condition $\mathcal{H}$ of theorem 2.3, the following condition

$$(25) \quad \sum_{n \geq 1} \frac{|a_n|}{\sigma_n}$$

and

$$\sum_{n \geq 1} \sqrt{\frac{\sum_{k \geq n} |a_k|^2}{n \log n}} < +\infty$$

are sufficient to get the convergence of $\sum_{k=0}^{\infty} a_k f(\alpha X_k(\omega))$. Notice that using corollary 2.5 (Cauchy-Schwarz inequality), condition (25) is replaced by

$$\sum_{n \geq 1} \frac{1}{\sigma_n^2} < +\infty$$

The subexponential case could be dealt with in the same way. If we consider the border case $a_k = \mathcal{O}(k^{-1/2-\epsilon})$, it is sufficient that:

$$\exists \eta > 0, \sigma_k \geq k^{\frac{1}{2}+\eta}$$

in this case:

$$\mathbb{P}\{\forall k, X_k \in [\mu_k - \frac{k^{\frac{1}{2}+\eta}}{2}, \mu_k + \frac{k^{\frac{1}{2}+\eta}}{2}]\} = 1$$

which gives an information on the possible dispersion of the variables $X_k$.

Here are other examples where the conditions of our theorems can be quite easily verified.
Corollary 5.1. Let \((X_k)_{k \geq 1}\) be a sequence of real independent random variables whose law can be written in the following way for all \(k \geq 1\) : 
\[\mathcal{L}(X_k) = \mathcal{L}(\sigma_k \cdot X + \mu_k)\] 
where \(X\) verifies \(\mathbb{E}|X|^\beta < \infty\) for some \(\beta > 0\) and \(\sigma_k > 0\). Moreover, we assume that there exist \(d > 0\) and \(\delta > 0\) such that \(|\sigma_k| = O(k^d)\), \(|\mu_k| = O(k^d)\) and the function \(t \mapsto t^\delta \mathbb{E}\exp(2i\pi t X)\) is bounded on \(\mathbb{R}\), and let \((a_k)_{k \geq 1}\) be a sequence of real or complex numbers satisfying the following two conditions:

1. \(|a_k| = O(k^{-\gamma})\) with \(\gamma > 1/2\)
2. \(\sum_{k=1}^{\infty} \frac{1}{|a_k|^\delta} < \infty\)

Then there exists a measurable set \(\Omega_o\) with full measure \((\mathbb{P}(\Omega_o) = 1)\) such that for any \(\omega \in \Omega_o\) for all \(f \in B(\mathbb{T})\) such that \(\int_{\mathbb{T}} f(t)dt = 0\) : for any compact set \(K\) which does not contain \(0\), the application \(t \in K \mapsto F(t) = \sum_{k \geq 1} a_k f(tX_k(\omega))\) is continuous and the series defining \(F\) converges uniformly on \(K\).

The proof of corollary 5.1 relies on corollary 2.5.

Example 5.2. The random variable \(X\) may have a gaussian law with mean zero and variance one, a Cauchy law, the first Laplace law, an exponential law with parameter \(\.load{\gamma}\), a Poisson law with parameter \(\gamma\). Namely, for simulation purpose, it would be interesting to have “localised” variables \(X_k\), that is to say to choose the smallest \(\sigma_k\) (or the smallest \(d\)). In this case, it is possible as we only have to choose \(\delta\) big enough so that the series \(\sum_{k \geq 1} \frac{1}{|a_k|^\delta}\) converges.

We discuss now the case when the laws of \(X_k\) are generated by a convolution product of a given law \(\mu\):

Corollary 5.2. Let \((X_k)_{k \geq 1}\) be a sequence of real valued independent random variables such that for all integer \(k \geq 1\), \(\mathcal{L}(X_k) = \mu^{*k}\) where \(\mu\) is a probability measure on \(\mathbb{R}\) with \(\mathbb{E}|X_1|^\delta < \infty\) for some \(\delta\). Assume the following:

(a) \(\varphi_{X_1}(t) = 1 \iff t = 0\) \((X_1\ \text{strictly aperiodic})\)

(b) \(\exists \delta > 0, \sup_{t \in \mathbb{R}} |t^\delta \mathbb{E}\exp(2i\pi t X_1)| = q < \infty\)

Let \((a_k)_{k \geq 1}\) be a sequence of real or complex numbers such that the sequence \(|a_k|\) is decreasing and fulfills the two following conditions:

1. \(|a_k| = O(k^{-\beta})\) avec \(\beta > 1/2\)
2. \(\sum_{k=1}^{\infty} |a_k - a_{k+1}| < \infty\)

Then there exists a measurable set \(\Omega_o\) with full measure \((\mathbb{P}(\Omega_o) = 1)\) such that for any \(\omega \in \Omega_o\), for all \(f \in B(\mathbb{T})\) such that \(\int_{\mathbb{T}} f(t)dt = 0\), for any compact set \(K\) which does not contain \(0\), the function \(t \in K \mapsto F(t, \omega) = \sum_{k \geq 1} a_k f(tX_k(\omega))\) is continuous and the series defining \(F\) converges uniformly on \(K\).
\[ \sum_{k \geq 1} a_k f(t X_k(\omega)) \] is continuous and the series defining \( F \) converges uniformly on \( K \).

**Remark 5.1.** If \( X_1 \) is aperiodic (\(|\mathcal{F}(t)| = 1 \iff t = 0\)), then the condition on the differences \(|a_k - a_{k+1}|\) may be removed, using corollary 2.5 and the same kind of computation as in the following proof.

The random variable \( X_1 \) being real valued, its characteristic function is not periodic. Take for example a gaussian law with mean zero and variance one.

**Proof:** Let \( K \) be a compact set which does not contain 0. Using Abel’s summation, it is sufficient to prove

\[ \sup_{t \in K} \sup_{|j| \geq 1} \left| \sum_{k=1}^{N} (E \exp 2i\pi j t X_1)^k \right| < \infty \]

independently of \( N \). Let us split the supremum on \( j \) respectively into the supremum on the indexes \( J(q) \) and \( \bar{J}(q) \) where \( J(q) = \{ j \in \mathbb{Z}^*: |j|^q \leq \left[ \frac{2q}{\varepsilon} \right] \} \) and \( 2\varepsilon \) is the distance between 0 and the fixed compact \( K \).

On one hand, using (a), one can prove that:

\[ \forall \varepsilon > 0 \quad \inf_{|t| > \varepsilon} |t|^q |1 - E \exp(2i\pi t X_1)| > 0 \]

this implies

\[ \sup_{t \in K} \sup_{j \in J(q)} \left| \sum_{k=1}^{N} (E \exp 2i\pi j t X_1)^k \right| \leq \sup_{t \in K} \sup_{j \in \bar{J}(q)} C(\varepsilon) |jt|^q \leq C(K) \left[ \frac{2q}{\varepsilon^q} \right] \]

On the other hand, using (b),

\[ \sup_{t \in K} \sup_{j \in J(q)} \left| \sum_{k=1}^{N} (E \exp 2i\pi j t X_1)^k \right| \leq \sup_{t \in K} \sup_{j \in \bar{J}(q)} \sum_{k=1}^{N} \left( \frac{q}{|j|^q} \right)^k \]

\[ \leq C \sum_{k=1}^{N} \frac{1}{2^k} \leq 2C \]

where \( C \) is a universal constant.

\[ \square \]

5.2. **Examples in the periodic case.** Recall that

\[ C_\eta(T) = \{ f : T \to \mathbb{C}, \sum_{j \in \mathbb{Z}} |f(j)||j|^{\eta} < +\infty \} \]

For these examples, we need here an extra definition.

We say that the irrationality measure of \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) is \( \eta \geq 1 \) if for all \( \varepsilon > 0 \), there exists \( C(\alpha, \varepsilon) > 0 \) such that

\[ \forall j \in \mathbb{Z} \setminus \{0\}, |j\alpha|_{\mathbb{Z}} \geq \frac{C(\alpha, \varepsilon)}{|j|^{\eta + \varepsilon}} \]
where $|x|_\mathbb{Z}$ denotes the distance between $x$ and the nearest integer. It is well known that Lebesgue almost every irrational number has an irrationality measure 1.

Moreover, by the theorem of Thue-Siegel-Roth [8], the irrationality measure of every irrational real algebraic number is 1.

We present again the example used at the end of section 2.1 and we prove that it fulfills the hypothesis of theorem 2.6

**Example 5.3.** $X_k$ is a sequence of independent random variables with disjoint supports. For each $k \geq 1$, the support of $X_k$ is the set of integers belonging to $[k^2, (k + 1)^2 - 1]$. The law of $X_k$ is uniform on this set of integers. A computation shows that

$$|\varphi_{X_k}(j\alpha)| = \frac{1}{2k + 1} \left| \frac{\sin(\pi j \alpha (2k + 1))}{\sin(\pi j \alpha)} \right|$$

And using Cauchy-Schwarz inequality, we get

$$\left| \sum_{k=n}^{m} a_k \varphi_{X_k}(j\alpha) \right| \leq \frac{1}{|\sin(\pi j \alpha)|} \sqrt{\sum_{k=n}^{m} |a_k|^2} \sqrt{\sum_{k=n}^{m} |\varphi_{X_k}(j\alpha)|^2}$$

$$\leq \frac{1}{|\sin(\pi j \alpha)|} \sqrt{\sum_{k=n}^{m} |a_k|^2} \sqrt{\sum_{k=n}^{m} \frac{1}{(2k + 1)^2}}$$

As $|\sin|$ has period $\pi$, $|\sin(\pi j \alpha)| = |\sin(\pi |\alpha j|_\mathbb{Z})|$ and as $|\alpha j|_\mathbb{Z} \leq \frac{1}{2}$,

$$|\sin(\pi |\alpha j|_\mathbb{Z})| \geq |\alpha j|_\mathbb{Z}$$

therefore

$$\left| \sum_{k=n}^{m} a_k \varphi_{X_k}(j\alpha) \right| \leq \frac{1}{|\alpha j|_\mathbb{Z}} \sqrt{\sum_{k=n}^{m} |a_k|^2} \sqrt{\sum_{k=n}^{m} \frac{1}{(2k + 1)^2}}$$

At this point (for the expectation part), note that we do not need the convergence of the series $\sum |a_k|^2$. However, we still need the conditions necessary for the convergence of the centered part (see theorem 2.1). Anyway we have

$$\left| \sum_{k=n}^{m} a_k \varphi_{X_k}(j\alpha) \right| \leq \frac{\varepsilon}{|\alpha j|_\mathbb{Z}}$$

as long as $m$ and $n$ are large enough.

Take $\alpha$ with an irrationality measure 1 and fix $\varepsilon > 0$, there exists $C(\alpha, \varepsilon) > 0$ such that

$$\forall j \in \mathbb{Z} \setminus \{0\}, \quad |j\alpha|_\mathbb{Z} \geq C(\alpha, \varepsilon) |j|^{1+\varepsilon}$$

and

$$\forall j \in \mathbb{Z} \setminus \{0\}, \quad |j|^{-1+\varepsilon} \left| \sum_{k=n}^{m} a_k \varphi_{X_k}(j\alpha) \right| \leq \frac{C}{C(\alpha, \varepsilon)} \varepsilon$$
as long as \( n \) and \( m \) are large enough. This means that \( \alpha \in E_{1+\epsilon} \). So \( E_{1+\epsilon} \) contains all the non zero irrational of irrationality measure 1. This is a set of Lebesgue measure one and this contains all the real irrational algebraic numbers.

Finally, here is again an example where the laws of \( X_k \) are generated by a product of convolution.

**Corollary 5.3.** Let \( (X_k)_{k \geq 1} \) be an sequence of independent random variables such that for all integer \( k \geq 1 \), \( \mathcal{L}(X_k) = \mu^k \) where \( \mu \) is a probability measure on \( \mathbb{R} \) with \( \mathbb{E}|X_1|^{\beta} < \infty \) for some \( \beta > 0 \). Assume we are in the periodic case. Let \( (a_k)_{k \geq 1} \) be a sequence of complex numbers such that \( |a_k| = O(k^{-\gamma}) \) with \( \gamma > 1/2 \).

Suppose that \( X_1 \) gives a strictly positive mass to two consecutive integers. Then there exists a measurable set \( \Omega_o \) with full measure (\( \mathbb{P}(\Omega_o) = 1 \)) such that for any \( \omega \in \Omega_o \), for all \( t > 0 \) and for all \( f \in C_{1+\epsilon}(\mathbb{T}) \) such that \( f(t) = 0 \), the function \( \alpha \mapsto F(t, \alpha) = \sum_{k \geq 1} a_k f(\alpha X_k(\omega)) \) is Lebesgue-almost everywhere well defined in particular.

**Proof.** First note that as \( X_1 \) gives a strictly positive mass to two consecutive integers, \( X_1 \) is aperiodic meaning that \( |X_1(t)| = 1 \Leftrightarrow t = 0 \).

Using theorem 2.6, we only need to prove

\[
\forall t > 0, \lambda(E_{1+\epsilon}) = 1
\]

where \( \lambda \) is the Lebesgue measure on \( \mathbb{T} \).

We first use Cauchy-Schwarz inequality, the fact that the law of \( X_k \) is a convolution product and that \( \sum_{k} |a_k|^2 < +\infty \)

\[
\sum_{k=n}^{m} a_k \varphi_{X_k}(j\alpha) \leq \sqrt{\sum_{k=n}^{m} |a_k|^2} \sqrt{\sum_{k=n}^{m} |\varphi_{X_k}(j\alpha)|^{2k}} \leq \varepsilon \sqrt{1 - |\varphi_{X_1}(j\alpha)|^2}
\]

as long as \( n \) and \( m \) are large enough.

Remark that these inequalities are true for every \( \alpha \) non zero irrational number as \( X_1 \) is aperiodic.

Now let \( Y \) a random variable such that \( Y(\Omega) \subset \mathbb{Z} \). Let \( p_k = \mathbb{P}(Y = k) \), \( \varphi_Y(t) = \mathbb{E}(e^{2\pi itY}) = \sum_{k \in \mathbb{Z}} p_k e^{2\pi itk} \) and \( \Re(\varphi_Y(t)) = \sum_{k \in \mathbb{Z}} p_k \cos(2\pi tk) \) hence, if we suppose \( p_1 > 0 \):

\[
1 - \Re(\varphi_Y(t)) = \sum_{k \in \mathbb{Z}} p_k (1 - \cos(2\pi tk)) = \sum_{k \in \mathbb{Z}} 2p_k \sin^2(\pi tk) \geq 2p_1 \sin^2(\pi t)
\]

We take now \( Y = X_1 - X_1' \) where \( X_1' \) is an independent copy of \( X_1 \). \( \varphi_Y(t) = |\varphi_{X_1}(t)|^2 \) and as \( X_1 \) gives a strictly positive mass to two consecutive integers,
\[ p_1 = \mathbb{P}(X_1 - X'_1 = 1) > 0. \] We thus get
\[ 1 - |\phi_{X_1}(\alpha j)|^2 > 2p_1 \sin^2(\pi \alpha j) \]
As \( \sin^2 \) has period \( \pi \), \( \sin^2(\pi \alpha j) = \sin^2(\pi |\alpha j| \mathbb{Z}) \) and as \( |\alpha j| \mathbb{Z} \leq \frac{1}{7} \),
\[ \sin^2(\pi |\alpha j| \mathbb{Z}) \geq |\alpha j|^2 \]
Consequently:
\[ \frac{1}{\sqrt{1 - |\phi_{X_1}(j\alpha)|^2}} \leq \frac{C}{|\alpha j| \mathbb{Z}} \]
where \( C \) is a constant depending only on \( X_1 \).
Take \( \alpha \) with an irrationality measure 1 and fix \( \epsilon > 0 \), there exists \( C(\alpha, \epsilon) > 0 \) such that
\[ \forall j \in \mathbb{Z} \setminus \{0\}, \ |j\alpha| \mathbb{Z} \geq \frac{C(\alpha, \epsilon)}{|j|^{1+\epsilon}} \]
and
\[ \forall j \in \mathbb{Z} \setminus \{0\}, \ |j|^{-(1+\epsilon)} \left| \sum_{k=n}^{m} a_k \phi_{X_k}(j\alpha) \right| \leq \frac{C}{C(\alpha, \epsilon) \mathbb{Z}} \]
as long as \( n \) and \( m \) are large enough. This means that \( \alpha \in E_{1+\epsilon} \). So \( E_{1+\epsilon} \) contains all the non zero irrational of irrationality measure 1. This is a set of Lebesgue measure one and this contains all the real irrational algebraic numbers. \( \square \)

**Example 5.4.** If the law of \( X_1 \) is a Poisson law with parameter 1, we use:
\[ \forall t \in \mathbb{T}, \ |\phi_{X_1}(t)| \leq e^{\cos(2\pi t) - 1} \]
and of course \( X_1 \) gives a strictly positive mass to two consecutive integers.

**References**


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