Stabilized plethysms for the classical Lie groups

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Abstract

The plethysms of the Weyl characters associated to a classical Lie group by the symmetric functions stabilize in large rank. In the case of a power sum plethysm, we prove that the coefficients of the decomposition of this stabilized form on the basis of Weyl characters are branching coefficients which can be determined by a simple algorithm. This generalizes in particular some classical results on the power sum plethysms of Schur functions. The stabilization of power sum plethysms also implies that the specializations at q equal to 1 in certain parabolic Kazhdan-Lusztig polynomials do not depend on the rank of the root system considered. We also establish explicit formulas for the outer multiplicities appearing in the decomposition of the tensor square of any irreducible finite dimensional module into its symmetric and antisymmetric parts. These multiplicities can notably be expressed in terms of the Littlewood-Richardson coefficients.

1 Introduction

This paper is concerned with the plethysms of the Weyl characters associated to classical Lie groups by the symmetric functions. Let $\mathfrak{g}$ be a classical Lie group with rank n, and λ a partition. Recall that the Weyl character $s_\lambda^{\mathfrak{g}}$ is the character of the irreducible $\mathfrak{g}$-module $V^{\mathfrak{g}}(\lambda)$ of highest weight λ. Consider f a symmetric function of degree d and suppose $n \geq d\,|\lambda|$. We prove that the plethysm $f \circ s_\lambda^{\mathfrak{g}}$ of the Weyl character $s_\lambda^{\mathfrak{g}}$ by f decomposes on the basis $\{s_\mu^{\mathfrak{g}} \mid \mu \in \mathcal{P}_n\}$ with coefficients which do not depend on n (Proposition **??**). When $f = p_\ell$ is the power sum of degree ℓ, we establish that the coefficients so obtained are branching coefficients corresponding to the restriction to certain Levi subgroups (Theorem **??**). Suppose $n \geq \ell\,|\lambda|$ and set

$$p_\ell \circ s_\lambda^{\mathfrak{g}} = \sum_\mu a_{\lambda,\mu}^{\mathfrak{g},\ell} s_\mu^{\mathfrak{g}}.$$

For $\mathfrak{g} = \mathfrak{gl}_n$, it is well known that the coefficients $a_{\lambda,\mu}^{\mathfrak{gl}_n,\ell}$ are, up to a sign, Littlewood-Richardson coefficients. They are then obtained from the ℓ-quotient of the partition μ. We give a similar algorithm for computing the coefficients $a_{\lambda,\mu}^{\mathfrak{g},\ell}$ when $\mathfrak{g} = \mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n}$ or $\mathfrak{so}_{2n}$. This algorithm was originally introduced in [**?**] to decompose the plethysms $p_\ell \circ s_\lambda^{\mathfrak{so}_{2n+1}}$ on the basis of Weyl characters for any integers $n \geq 2$ and $\ell \geq 1$ (that is, with no restrictive conditions on the rank n). Although similar procedures also exist for $\mathfrak{g} = \mathfrak{sp}_{2n}$ or $\mathfrak{so}_{2n}$ when ℓ is odd, our method failed for the even power sum plethysms on the Weyl characters of type C_n or D_n. In the present paper, we show that this difficulty can be overcome by considering stabilized power sum plethysms, i.e. by assuming that $n \geq \ell\,|\lambda|$. Under this hypothesis, one has indeed $a_{\lambda,\mu}^{\mathfrak{so}_{2n+1},\ell} = a_{\lambda,\mu}^{\mathfrak{so}_{2n},\ell}$ and $a_{\lambda,\mu}^{\mathfrak{sp}_{2n},\ell} = a_{\lambda',\mu'}^{\mathfrak{so}_{2n+1},\ell}$. So

it suffices to consider the coefficients $a_{\lambda,\mu}^{\mathfrak{so}_{2n+1},\ell}$ for which there exists an algorithm in both cases ℓ even and ℓ odd.
We give two applications of the stabilization property of plethysms. By results conjectured by Lusztig [**?**] and proved by Kashiwara-Tanisaki [**?**], the power sum plethysm $p_\ell \circ s_\lambda^{\mathfrak{g}}$ can be interpreted as the character of the indecomposable $U_\xi(\mathfrak{g})$-module $L(\ell\lambda)$ where $\xi \in \mathbb{C}$ is a primitive 2ℓ-root of unity. The coefficients of the expansion of $\mathrm{char}(L(\ell\lambda))$ on the basis of Weyl characters are up to a sign, parabolic Kazhdan-Lusztig polynomials specialized at $q = 1$. These polynomials, introduced by Deodhar [**?**], are analogues of the Kazhdan-Lusztig polynomials obtained by replacing the affine Hecke algebra $\widehat{H}$ by one of its parabolic module $\widehat{H}\nu$ (ν being a weight of the affine root system considered). In Proposition **??**, we establish that the previous specializations of K-L parabolic polynomials are branching coefficients and do not depend on the rank n considered providing $n \geq \ell\,|\lambda|$. We conjecture that the parabolic K-L polynomials appearing in the expansion of $\mathrm{char}(L(\ell\lambda))$ are also independent of n when this hypothesis is verified. In Proposition **??**, we use our expression of the coefficients $a_{\lambda,\mu}^{\mathfrak{g},2}$ as branching coefficients, to derive explicit formulas giving the decompositions of the symmetric and antisymmetric parts of $V^{\mathfrak{g}}(\lambda)^{\otimes 2}$ in their irreducible components when $n \geq 2\,|\lambda|$. The corresponding multiplicities can then be expressed in terms of the Littlewood-Richardson coefficients. Note that we have not found a combinatorial interpretation of this result similar to that given in [**?**] from the combinatorics of domino tableaux.

The paper is organized as follows. In Section 2, we recall some basics on the representation theory of the classical Lie groups. Section 3 is concerned with plethysms $f \circ s_\lambda^{\mathfrak{g}}$ and their stabilization in large rank. In Section 4, we describe the algorithm of [**?**] which permits to compute the plethysms $p_\ell \circ s_\lambda^{\mathfrak{so}_{2n+1}}$ for any positive integer ℓ. We state Theorem **??** and Proposition **??** in Section 5. Finally, in Section 6, we express the multiplicities $a_{\lambda,\mu}^{\mathfrak{g},2}$ in terms of the Littlewood-Richardson coefficients.

2 Background on classical Lie groups

2.1 Root systems and Weyl groups

In the sequel G is one of the complex Lie groups Sp_{2n}, SO_{2n+1} or SO_{2n} and $\mathfrak{g}$ is its Lie algebra. We follow the convention of [**?**] to realize G as a subgroup of GL_N and $\mathfrak{g}$ as a subalgebra of $\mathfrak{gl}_N$ where

$$N = \begin{cases} 2n \text{ when } G = Sp_{2n} \\ 2n+1 \text{ when } G = SO_{2n+1} \\ 2n \text{ when } G = SO_{2n}. \end{cases}.$$

Let d_N be the linear subspace of $\mathfrak{gl}_N$ consisting of the diagonal matrices. For any $i \in I_n = \{1, ..., n\}$, write ε_i for the linear map $\varepsilon_i : d_N \to \mathbb{C}$ such that $\varepsilon_i(D) = \delta_{n-i+1}$ for any diagonal matrix D whose (i,i)-coefficient is δ_i. Then $(\varepsilon_1, ..., \varepsilon_n)$ is an orthonormal basis of the Euclidean space $\mathfrak{h}_{\mathbb{R}}^*$ (the real part of $\mathfrak{h}^*$). Let $(\cdot,\cdot)$ be the corresponding nondegenerate symmetric bilinear form defined on $\mathfrak{h}_{\mathbb{R}}^*$. Write R for the root system associated to G. For any $\alpha \in R$ we set $\alpha^\vee = \frac{\alpha}{(\alpha,\alpha)}$. The Lie algebra $\mathfrak{g}$ admits the diagonal decomposition $\mathfrak{g} = \mathfrak{h} \oplus_{\alpha \in R} \mathfrak{g}_\alpha$. We take for the set of positive roots:

$$\begin{cases} R^+ = \{\varepsilon_j - \varepsilon_i, \varepsilon_j + \varepsilon_i \text{ with } 1 \leq i < j \leq n\} \text{ for the root system } A_{n-1} \\ R^+ = \{\varepsilon_j - \varepsilon_i, \varepsilon_j + \varepsilon_i \text{ with } 1 \leq i < j \leq n\} \cup \{\varepsilon_i \text{ with } 1 \leq i \leq n\} \text{ for the root system } B_n \\ R^+ = \{\varepsilon_j - \varepsilon_i, \varepsilon_j + \varepsilon_i \text{ with } 1 \leq i < j \leq n\} \cup \{2\varepsilon_i \text{ with } 1 \leq i \leq n\} \text{ for the root system } C_n \\ R^+ = \{\varepsilon_j - \varepsilon_i, \varepsilon_j + \varepsilon_i \text{ with } 1 \leq i < j \leq n\} \text{ for the root system } D_n \end{cases}.$$

The Weyl group W of the Lie group G is the subgroup of the permutation group of the set $J_n = \{\overline{n}, ..., \overline{2}, \overline{1}, 1, 2, ..., n\}$ generated by the permutations

$$\begin{cases} s_i = (i, i+1)(\overline{i}, \overline{i+1}),\ i = 1, ..., n-1 \text{ and } s_n = (n, \overline{n}) \text{ for the root systems } B_n \text{ and } C_n \\ s_i = (i, i+1)(\overline{i}, \overline{i+1}),\ i = 1, ..., n-1 \text{ and } s'_n = (n, \overline{n-1})(n-1, \overline{n}) \text{ for the root system } D_n \end{cases}$$

where for $a \neq b$, (a, b) is the simple transposition which switches a and b. We identify the subgroup of W generated by $s_i = (i, i+1)(\overline{i}, \overline{i+1})$, $i = 1, ..., n-1$ with the symmetric group S_n. We denote by l the length function corresponding to the above set of generators. For any $w \in W$, we set $\varepsilon(w) = (-1)^{l(w)}$. The action of $w \in W$ on $\beta = (\beta_1, ..., \beta_n) \in \mathfrak{h}^*_{\mathbb{R}}$ is defined by

$$w \cdot (\beta_1, ..., \beta_n) = (\beta_1^{w^{-1}}, ..., \beta_n^{w^{-1}})$$

where $\beta_i^w = \beta_{w(i)}$ if $w(i) \in \{1, ..., n\}$ and $\beta_i^w = -\beta_{w(\overline{i})}$ otherwise. We denote by ρ the half sum of the positive roots of R^+. For any $x \in J_n$, we set $\overline{\overline{x}} = x$ and $|x| = x$ if x is unbarred, $|x| = \overline{x}$ otherwise.

Write P and P_+ respectively for the sets of weight and dominants weights associated to R. A partition of length m is a weakly *increasing* sequence of m nonnegative integers. Denote by $\mathcal{P}_m$ the set of partitions with at most m parts. Set $\mathcal{P} = \cup_{m \geq 0} \mathcal{P}_m$. Each partition $\lambda = (\lambda_1, ..., \lambda_m) \in \mathcal{P}_m$ will be identified with the dominant weight $\sum_{i=1}^m \lambda_i \varepsilon_i$. Then the irreducible finite dimensional polynomial representations of G are parametrized by the partitions of $\mathcal{P}_n$. For any $\lambda \in \mathcal{P}_n$, denote by $V^{\mathfrak{g}}(\lambda)$ the irreducible finite dimensional representation of G of highest weight λ. We will also need the irreducible rational representations of GL_n. They are indexed by the n-tuples

$$(\gamma^-, \gamma^+) = (-\gamma_q^-, ..., -\gamma_1^-, \gamma_1^+, \gamma_2^+, ..., \gamma_p^+) \tag{1}$$

where $\gamma^+ = (\gamma_1^+, \gamma_2^+, ..., \gamma_p^+)$ and $\gamma^- = (\gamma_1^-, ..., \gamma_q^-)$ are partitions of length p and q such that $p + q = n$. Write $\widetilde{\mathcal{P}}_n$ for the set of such n-tuples and denote also by $V^{\mathfrak{gl}_n}(\gamma)$ the irreducible rational representation of $\mathfrak{gl}_n$ of highest weight $\gamma = (\gamma^-, \gamma^+) \in \widetilde{\mathcal{P}}_n$. For any $\gamma = (\gamma^-, \gamma^+) \in \widetilde{\mathcal{P}}_n$, we set $|\gamma| = \sum \gamma_i^- + \sum \gamma_i^+$.

Write $s_\lambda^{\mathfrak{gl}_n}$ for the Weyl character (Schur function) of the finite-dimensional $\mathfrak{gl}_n$-module $V^{\mathfrak{gl}_n}(\lambda)$ of highest weight λ. The character ring of GL_n is $\Lambda_n = \mathbb{Z}[x_1, ..., x_n]^{\text{sym}}$ the ring of symmetric functions in n variables.
For any $\lambda \in \mathcal{P}_n$, we denote by $s_\lambda^{\mathfrak{g}}$ the Weyl character of $V^{\mathfrak{g}}(\lambda)$. Let $\mathcal{R}^{\mathfrak{g}}$ be the character ring of G. Then

$$\mathcal{R}^{\mathfrak{g}} = \mathbb{Z}[x_1, ..., x_n, x_1^{-1}, ..., x_n^{-1}]^W$$

is the $\mathbb{Z}$-algebra with basis $\{s_\lambda^{\mathfrak{g}} \mid \lambda \in \mathcal{P}_n\}$.

2.2 Levi subgroups

In the sequel, our computations will also make appear root subsystems of the classical root systems R described in the previous paragraph. Suppose that G is of type X_n with $X_n = A_{n-1}, B_n, C_n$ or D_n. Let $I = (i_1, ..., i_r)$ be an increasing sequence of integers belonging to I_n, that is $i_k \in I_n$ for any $k = 1, .., r$ and $i_1 < \cdots < i_r$. Then

$$R_I = \{\alpha \in R \cap \oplus_{i \in I} \mathbb{Z}\varepsilon_i\}$$

is a root subsystem of R of type X_r. Write R_I^+ for the set of positive roots in R_I. Then we have $R_I^+ = R_I \cap R^+$. The dominant weights associated to R_I have the form $\lambda = (\lambda_1, ..., \lambda_n)$ where $\lambda_i \neq 0$ only if $i \in I$ and $\lambda^{(I)} = (\lambda_{i_1}, ..., \lambda_{i_r}) \in \mathcal{P}_r$. We slightly abuse the notation by identifying λ with $\lambda^{(I)}$.
Consider an increasing sequence $X = (x_1, ..., x_r)$ of integers belonging to J_n such that $|x_k| = |x_{k'}|$ if and only if $k = k'$. For any integer $i = 1, ..., n$, set $\varepsilon_{\overline{i}} = -\varepsilon_i$. Then

$$R_{A,X} = \{\pm(\varepsilon_{x_j} - \varepsilon_{x_i}) \mid 1 \leq i < j \leq r\}$$

is a root subsystem of R of type A_{r-1}. To see this, consider the linear map $\theta_X : \mathbb{Z}^r \to \mathbb{Z}^n$ such that $\theta_X(\varepsilon_i) = \varepsilon_{x_i}$. The map θ is injective and preserves the scalar product in $\mathbb{Z}^r$ and $\mathbb{Z}^n$. Moreover the root system $\{\pm(\varepsilon_j - \varepsilon_i) \mid 1 \leq i < j \leq r\} \subset \mathbb{Z}^r$ of type A_{r-1} is sent on $R_{A,X}$ by θ_X. The set of positive roots in $R_{A,X}$ is equal to $R_{A,X}^+ = R_{A,X} \cap R^+$. Denote by $s \in \{1, ..., r\}$ the maximal integer such that $x_s < 0$. We associate to X, the increasing sequence of indices $I \subset I_n$ defined by

$$I = (\overline{x}_s, ..., \overline{x}_1, x_{s+1}, ..., x_r). \tag{2}$$

It will be useful to consider the weights corresponding to $R_{A,X}$ as the r-tuples $\beta = (\beta_{x_1}, ..., \beta_{x_r})$ with coordinates indexed by X. The coordinates $(\beta'_1, ..., \beta'_n)$ of β on the initial basis $(\varepsilon_1, ..., \varepsilon_n)$ are such that $\beta'_i = \beta_{x_a}$ if $i = x_a \in X$, $\beta'_i = -\beta_{x_a}$ if $\overline{i} = x_a \in X$ and $\beta'_i = 0$ otherwise. With this convention the dominant weights for $R_{A,X}$ have the form

$$\lambda^{(X)} = (\lambda_{x_1}, ..., \lambda_{x_r}) \in \widetilde{\mathcal{P}}_r. \tag{3}$$

This simply means that we have chosen to expand the weights of $R_{A,X}$ on the basis $\{\varepsilon_x \mid x \in X\}$ rather than on the basis $\{\varepsilon_i \mid i \in I\}$ to preserve the identification of the dominant weights with the nondecreasing r-tuples of integers.

Example 2.2.1 *Take $G = Sp_{10}$.*

- *For $I = (2, 4, 5)$ we have*

  $$R_I^+ = \{\varepsilon_5 \pm \varepsilon_4, \varepsilon_5 \pm \varepsilon_2, \varepsilon_4 \pm \varepsilon_2, 2\varepsilon_2, 2\varepsilon_4, 2\varepsilon_5\}$$

  which is the set of positive roots of a root system of type C_3. The weight $\lambda = (1, 2, 2)$ is dominant for G_I. Considered as a weight of Sp_{10}, we have $\lambda = (0, 1, 0, 2, 2)$.

- *For $X = (\overline{5}, \overline{2}, 1, 4)$ we have*

  $$R_{A,X}^+ = \{\varepsilon_4 - \varepsilon_1, \varepsilon_5 - \varepsilon_2, \varepsilon_1 + \varepsilon_2, \varepsilon_1 + \varepsilon_5, \varepsilon_4 + \varepsilon_2, \varepsilon_4 + \varepsilon_5\}$$

  which is the set of positive roots of a root system of type A_3. The weight $\gamma = (-3, -1, 4, 5)$ is dominant for G_X. Considered as a weight of Sp_{10}, we have $\gamma = (4, 1, 0, 5, 3)$.

Consider $p \geq 1$ an integer. Let $I^{(0)} = (i_1^{(0)}, ..., i_{r_0}^{(0)})$ be an increasing sequence of integers in I_n. For $k = 1, ..., p$, consider increasing sequences $X^{(k)} = (x_1^{(k)}, ..., x_{r_k}^{(k)}) \subset J_n$ of length r_k. Let s_k be maximal in $\{1, ..., r_k\}$ such that $x_{s_k}^{(k)} < 0$. Set

$$I^{(k)} = (\overline{x}_{s_k}^{(k)}, ..., \overline{x}_1^{(k)}, x_{s_k+1}^{(k)}, ..., x_{r_k}^{(k)}) \subset I_n. \tag{4}$$

We suppose that the sets $I^{(k)}$, $k=0,...,p$ are pairwise disjoint and verify $\cup_{k=0}^{p} I^{(k)} = I_n$. Set $\mathcal{I} = \{I^{(0)}, X^{(1)}, ..., X^{(p)}\}$ and

$$R_{\mathcal{I}} = R_{I^{(0)}} \cup \bigcup_{k=1}^{p} R_{A,X^{(k)}}$$

Then $\mathfrak{g}_{\mathcal{I}} = \mathfrak{h} \oplus_{\alpha \in R_{\mathcal{I}}} \mathfrak{g}_\alpha$ is a Lie subalgebra of $\mathfrak{g}$. Its corresponding Lie group $G_{\mathcal{I}}$ is a Levi subgroup. More precisely we have

$$G_{\mathcal{I}} \simeq \begin{cases} GL_{r_0} \times GL_{r_1} \times \cdots \times GL_{r_p} \text{ for } G = GL_n \\ SO_{2r_{0+1}} \times GL_{r_1} \times \cdots \times GL_{r_p} \text{ for } G = SO_{2n+1} \\ Sp_{2r_0} \times GL_{r_1} \times \cdots \times GL_{r_p} \text{ for } G = Sp_{2n} \\ SO_{2r_0} \times GL_{r_1} \times \cdots \times GL_{r_p} \text{ for } G = SO_{2n} \end{cases}.$$

The root system associated to $\mathfrak{g}_{\mathcal{I}}$ is $R_{\mathcal{I}}$. Denote by $P_{\mathcal{I}}^+$ its cone of dominant weights. The weight lattice of $G_{\mathcal{I}}$ coincides with that of G since the Lie algebras $\mathfrak{g}_{\mathcal{I}}$ and $\mathfrak{g}$ have the same Cartan subalgebra. The elements of $P_{\mathcal{I}}^+$ are the $(p+1)$-tuples $\lambda = (\lambda^{(0)}, \lambda^{(1)}, ..., \lambda^{(p)})$ where $\lambda^{(0)} = (\lambda_i \mid i \in I^{(0)})$ is a dominant weight of $R_{I^{(0)}}$ and for any $k = 1, ..., p$, $\lambda^{(k)} = (\lambda_i \mid i \in X^{(k)})$ is a dominant weight of $R_{A,X^{(k)}}$. For any $\lambda \in P_{\mathcal{I}}^+$, we denote by $V_{\mathcal{I}}(\lambda)$ the irreducible finite dimensional $G_{\mathcal{I}}$-module of highest weight λ. We set $|\lambda| = \left|\lambda^{(0)}\right| + \cdots + \left|\lambda^{(p)}\right|$. Each weight $\beta = (\beta^{(0)}, \beta^{(1)}, ..., \beta^{(p)}) \in P_{\mathcal{I}}$ can be considered as a weight $\beta = (\beta'_1, ..., \beta'_n)$ of P. With the convention (3) we have then $\beta'_i = \beta_{i_a^{(0)}}$ if $i = i_a^{(0)} \in I^{(0)}$ and for any $k = 1, ..., p$, $\beta'_i = \beta_{i_a^{(k)}}$ if $i = i_a^{(k)} \in X^{(k)}$, $\beta'_i = -\beta_{i_a^{(k)}}$ if $\bar{i} = i_a^{(k)} \in X^{(k)}$. In the sequel we identify the two expressions

$$\beta = (\beta^{(0)}, \beta^{(1)}, ..., \beta^{(p)}) \text{ and } \beta = (\beta'_1, ..., \beta'_n) \tag{5}$$

of the weights of $P_{\mathcal{I}}$.

2.3 Universal characters

For each Lie algebra $\mathfrak{g} = \mathfrak{so}_N$ or $\mathfrak{sp}_N$ and any partition $\nu \in \mathcal{P}_N$, we denote by $V^{\mathfrak{gl}_N}(\nu) \downarrow_{\mathfrak{g}}^{\mathfrak{gl}_N}$ the restriction of $V^{\mathfrak{gl}_N}(\nu)$ to $\mathfrak{g}$. Set

$$V^{\mathfrak{gl}_N}(\nu) \downarrow_{\mathfrak{so}_N}^{\mathfrak{gl}_N} = \bigoplus_{\lambda \in \mathcal{P}_n} V^{\mathfrak{so}_N}(\lambda)^{\oplus b_{\nu,\lambda}^{\mathfrak{so}_N}} \quad \text{and} \quad V^{\mathfrak{gl}_{2n}}(\nu) \downarrow_{\mathfrak{sp}_{2n}}^{\mathfrak{gl}_{2n}} = \bigoplus_{\lambda \in \mathcal{P}_n} V^{\mathfrak{sp}_{2n}}(\lambda)^{\oplus b_{\nu,\lambda}^{\mathfrak{sp}_{2n}}}.$$

This makes in particular appear the branching coefficients $b_{\nu,\lambda}^{\mathfrak{so}_N}$ and $b_{\nu,\lambda}^{\mathfrak{sp}_{2n}}$. The restriction map $r^{\mathfrak{g}}$ is defined by setting

$$r^{\mathfrak{g}} : \begin{cases} \mathbb{Z}[x_1, ..., x_N]^{\text{sym}} \to \mathcal{R}^{\mathfrak{g}} \\ s_\nu^{\mathfrak{gl}_N} \longmapsto \text{char}(V^{\mathfrak{gl}_N}(\nu) \downarrow_{\mathfrak{g}}^{\mathfrak{gl}_N}) \end{cases}.$$

We have then

$$r^{\mathfrak{g}}(s_\nu^{\mathfrak{gl}_N}) = \begin{cases} s_\nu^{\mathfrak{gl}_N}(x_1, ..., x_n, x_n^{-1}, ..., x_1^{-1}) \text{ when } N = 2n \\ s_\nu^{\mathfrak{gl}_N}(x_1, ..., x_n, 0, x_n^{-1}, ..., x_1^{-1}) \text{ when } N = 2n+1 \end{cases}.$$

Let $\mathcal{P}_n^{(2)}$ and $\mathcal{P}_n^{(1,1)}$ be the subsets of $\mathcal{P}_n$ containing the partitions with even length rows and the partitions with even length columns, respectively. When $\nu \in \mathcal{P}_n$ we have the following formulas for the branching coefficients $b_{\nu,\lambda}^{\mathfrak{so}_N}$ and $b_{\nu,\lambda}^{\mathfrak{sp}_{2n}}$:

Proposition 2.3.1 *(see [?] appendix p 295)*
Consider $\nu \in \mathcal{P}_n$. Then:

1. $b^{\mathfrak{so}_{2n+1}}_{\nu,\lambda} = b^{\mathfrak{so}_{2n}}_{\nu,\lambda} = \sum_{\gamma\in\mathcal{P}_n^{(2)}} c^{\nu}_{\lambda,\gamma}$
2. $b^{\mathfrak{sp}_{2n}}_{\nu,\lambda} = \sum_{\gamma\in\mathcal{P}_n^{(1,1)}} c^{\nu}_{\lambda,\gamma}$

where $c^{\nu}_{\gamma,\lambda}$ is the usual Littlewood-Richardson coefficient corresponding to the partitions γ, λ and ν.

Remarks:
(i) : Note that the equality $b^{\mathfrak{so}_{2n+1}}_{\nu,\lambda} = b^{\mathfrak{so}_{2n}}_{\nu,\lambda}$ becomes false in general when $\nu \notin \mathcal{P}_n$.
(ii) : By the above proposition we have for any $\nu \in \mathcal{P}_m$ with $m \leq n$

$$r^{\mathfrak{sp}_{2n}}(s^{\mathfrak{gl}_{2n}}_{\nu}) = \sum_{\lambda\in\mathcal{P}_m}\sum_{\gamma\in\mathcal{P}_m^{(1,1)}} c^{\nu}_{\lambda,\gamma} s^{\mathfrak{sp}_{2n}}_{\lambda} \text{ and } r^{\mathfrak{so}_N}(s^{\mathfrak{gl}_N}_{\nu}) = \sum_{\lambda\in\mathcal{P}_m}\sum_{\gamma\in\mathcal{P}_m^{(2)}} c^{\nu}_{\lambda,\gamma} s^{\mathfrak{so}_N}_{\lambda}. \tag{6}$$

By Proposition 1.5.3 in [?], one has also for any $\lambda \in \mathcal{P}_m$

$$\begin{aligned} s^{\mathfrak{sp}_{2n}}_{\lambda} &= \sum_{\nu\in\mathcal{P}_m, |\nu|\leq|\lambda|} (-1)^{\frac{|\nu|-|\lambda|}{2}} \sum_{\alpha=(\alpha_1>\cdots>\alpha_s>0)} c^{\lambda}_{\nu,\Gamma(\alpha)}\ r^{\mathfrak{sp}_{2n}}(s^{\mathfrak{gl}_{2n}}_{\nu}) \\ s^{\mathfrak{so}_N}_{\lambda} &= \sum_{\nu\in\mathcal{P}_m, |\nu|\leq|\lambda|} (-1)^{\frac{|\nu|-|\lambda|}{2}} \sum_{\alpha=(\alpha_1>\cdots>\alpha_s>0)} c^{\lambda}_{\nu,\Gamma'(\alpha)}\ r^{\mathfrak{so}_N}(s^{\mathfrak{gl}_N}_{\nu}) \end{aligned} \tag{7}$$

where $\Gamma(\alpha) = (\alpha_1 - 1, ..., \alpha_s - 1 \mid \alpha_1, ..., \alpha_s)$ in the Frobenius notation for the partitions. Observe that the coefficients appearing in the decompositions (6) and (7) do not depend on the rank n considered. Moreover they coincide for the orthogonal types B_n and D_n.

As suggested by the above decompositions, the manipulation of the Weyl characters is simplified by working with infinitely many variables. In [?], Koike and Terada have introduced a universal character ring for the classical Lie groups. This ring can be regarded as the ring $\Lambda = \mathbb{Z}[x_1, ..., x_n, ...]^{\text{sym}}$ of symmetric functions in countably many variables. It is equipped with three natural $\mathbb{Z}$-bases indexed by partitions, namely

$$\mathcal{B}^{\mathfrak{gl}}=\{\mathtt{s}^{\mathfrak{gl}}_{\lambda} \mid \lambda \in \mathcal{P}\},\ \mathcal{B}^{\mathfrak{sp}}=\{\mathtt{s}^{\mathfrak{sp}}_{\lambda} \mid \lambda \in \mathcal{P}\} \text{ and } \mathcal{B}^{\mathfrak{so}}=\{\mathtt{s}^{\mathfrak{so}}_{\lambda} \mid \lambda \in \mathcal{P}\}. \tag{8}$$

We have then

$$\mathtt{s}^{\mathfrak{gl}}_{\nu} = \sum_{\lambda\in\mathcal{P}}\sum_{\gamma\in\mathcal{P}^{(2)}} c^{\nu}_{\lambda,\gamma}\mathtt{s}^{\mathfrak{so}}_{\lambda},\ \mathtt{s}^{\mathfrak{gl}}_{\nu} = \sum_{\lambda\in\mathcal{P}}\sum_{\gamma\in\mathcal{P}^{(1,1)}} c^{\nu}_{\lambda,\gamma}\mathtt{s}^{\mathfrak{sp}}_{\lambda} \tag{9}$$

$$\mathtt{s}^{\mathfrak{sp}}_{\lambda} = \sum_{\nu\in\mathcal{P}, |\nu|\leq|\lambda|} (-1)^{\frac{|\nu|-|\lambda|}{2}} \sum_{\alpha=(\alpha_1>\cdots>\alpha_s>0)} c^{\lambda}_{\nu,\Gamma(\alpha)}\ \mathtt{s}^{\mathfrak{gl}}_{\nu} \tag{10}$$

$$\mathtt{s}^{\mathfrak{so}}_{\lambda} = \sum_{\nu\in\mathcal{P}, |\nu|\leq|\lambda|} (-1)^{\frac{|\nu|-|\lambda|}{2}} \sum_{\alpha=(\alpha_1>\cdots>\alpha_s>0)} c^{\lambda}_{\nu,\Gamma'(\alpha)}\ \mathtt{s}^{\mathfrak{gl}}_{\nu} \tag{11}$$

In the sequel we will write for short

$$b^{\mathfrak{so}}_{\nu,\lambda} = \sum_{\gamma\in\mathcal{P}^{(2)}} c^{\nu}_{\lambda,\gamma},\quad b^{\mathfrak{sp}}_{\nu,\lambda} = \sum_{\gamma\in\mathcal{P}^{(1,1)}} c^{\nu}_{\lambda,\gamma},\quad r^{\mathfrak{so}}_{\lambda,\nu} = \sum_{\alpha} c^{\lambda}_{\nu,\Gamma'(\alpha)} \text{ and } \quad r^{\mathfrak{sp}}_{\lambda,\nu} = \sum_{\alpha} c^{\lambda}_{\nu,\Gamma(\alpha)} \tag{12}$$

We denote by ω the linear involution defined on Λ by $\omega(\mathtt{s}_\lambda^{\mathfrak{gl}}) = \mathtt{s}_{\lambda'}^{\mathfrak{gl}}$. Then we have by Theorem 2.3.2 of [**?**]

$$\omega(\mathtt{s}_\lambda^{\mathfrak{so}}) = \mathtt{s}_{\lambda'}^{\mathfrak{sp}}. \tag{13}$$

Write $\pi_n : \mathbb{Z}[x_1, ..., x_n, ...]^{\mathrm{sym}} \to \mathbb{Z}[x_1, ..., x_n]^{\mathrm{sym}}$ for the ring homomorphism obtained by specializing each variable $x_i, i > n$ at 0. Then $\pi_n(\mathtt{s}_\lambda^{\mathfrak{gl}}) = s_\lambda^{\mathfrak{gl}_n}$. Let $\pi^{\mathfrak{sp}_{2n}}$ and $\pi^{\mathfrak{so}_N}$ be the specialization homomorphisms defined by setting $\pi^{\mathfrak{sp}_{2n}} = r^{\mathfrak{sp}_{2n}} \circ \pi_{2n}$ and $\pi^{\mathfrak{so}_N} = r^{\mathfrak{so}_N} \circ \pi_N$. For any partition $\lambda \in \mathcal{P}_n$ one has $s_\lambda^{\mathfrak{sp}_{2n}} = \pi^{\mathfrak{sp}_{2n}}(\mathtt{s}_\lambda^{\mathfrak{sp}})$ and $s_\lambda^{\mathfrak{so}_N} = \pi^{\mathfrak{so}_N}(\mathtt{s}_\lambda^{\mathfrak{so}})$. We shall also need the following proposition (see [**?**] Corollary 2.5.3).

Proposition 2.3.2 *Consider a Lie algebra $\mathfrak{g}$ of type $X_n \in \{B_n, C_n, D_n\}$. Let $\lambda \in \mathcal{P}_r$ and $\mu \in \mathcal{P}_s$. Suppose $n \geq r + s$ and set*

$$V^{\mathfrak{g}}(\lambda) \otimes V^{\mathfrak{g}}(\mu) = \bigoplus_{\nu \in \mathcal{P}_n} V^{\mathfrak{g}}(\nu)^{\oplus d_{\lambda,\mu}^\nu}.$$

Then the coefficients $d_{\lambda,\mu}^\nu$ neither depend on the rank n of $\mathfrak{g}$ nor on its type B, C or D.

Remarks:
(i) : The previous proposition implies the decompositions $\mathtt{s}_\lambda^{\mathfrak{sp}} \times \mathtt{s}_\mu^{\mathfrak{sp}} = \sum_{\nu \in \mathcal{P}} d_{\lambda,\mu}^\nu \mathtt{s}_\nu^{\mathfrak{sp}}$ and $\mathtt{s}_\lambda^{\mathfrak{so}} \times \mathtt{s}_\mu^{\mathfrak{so}} = \sum_{\nu \in \mathcal{P}} d_{\lambda,\mu}^\nu \mathtt{s}_\nu^{\mathfrak{so}}$ for any $\lambda, \mu \in \mathcal{P}$, in the ring Λ.
(ii) : The analogue result for $\mathfrak{g} = \mathfrak{gl}_n$ is well-known: the outer multiplicities $c_{\lambda,\mu}^\nu$ appearing in the decomposition of $V^{\mathfrak{gl}_n}(\lambda) \otimes V^{\mathfrak{gl}_n}(\mu)$ do not depend on n providing $n \geq r + s$.

3 Plethysms and stabilized plethysms

3.1 Plethysms on the Weyl characters

Consider $f \in \Lambda$ and $s_\lambda^{\mathfrak{g}}$ the Weyl character for $\mathfrak{g}$ associated to $\lambda \in \mathcal{P}_n$. Set $s_\lambda^{\mathfrak{g}} = \sum_{\beta \in \mathbb{Z}^n} a_\beta x^\beta$. As in the case of ordinary plethysms on symmetric functions (see [**?**] p 135), one defines the set of fictitious variables y_i such that

$$\prod_i (1 + ty_i) = \prod_\beta (1 + tx^\beta)^{a_\beta}.$$

Then the plethysm of the Weyl character $s_\lambda^{\mathfrak{g}}$ by the symmetric function f is defined by $f \circ s_\lambda^{\mathfrak{g}} = f(y_1, y_2, ...)$. In the sequel, we will focus on the power sum plethysms ψ_ℓ where ℓ is a positive integer. They are defined from the identity $\psi_\ell(s_\lambda^{\mathfrak{g}}) = p_\ell \circ s_\lambda^{\mathfrak{g}} = s_\lambda^{\mathfrak{g}}(x_1^\ell, ..., x_n^\ell)$. In particular, the map ψ_ℓ is linear on $\mathcal{R}^{\mathfrak{g}}$. The characters of the symmetric and antisymmetric parts of $V^{\mathfrak{g}}(\lambda)^{\otimes 2}$ can be expressed thanks to the plethysms by the complete and elementary symmetric functions h_2 and e_2. More precisely we have

$$h_2 \circ s_\lambda^{\mathfrak{g}} = \mathrm{char}(S^2(V^{\mathfrak{g}}(\lambda)) \text{ and } e_2 \circ s_\lambda^{\mathfrak{g}} = \mathrm{char}(\Lambda^2(V^{\mathfrak{g}}(\lambda)).$$

From the decomposition $V^{\mathfrak{g}}(\lambda)^{\otimes 2} = S^2(V^{\mathfrak{g}}(\lambda)) \oplus \Lambda^2(V^{\mathfrak{g}}(\lambda))$ we derive the relations

$$h_2 \circ s_\lambda^{\mathfrak{g}} = \frac{1}{2}((s_\lambda^{\mathfrak{g}})^2 + \psi_2(s_\lambda^{\mathfrak{g}})) \text{ and } e_2 \circ s_\lambda^{\mathfrak{g}} = \frac{1}{2}((s_\lambda^{\mathfrak{g}})^2 - \psi_2(s_\lambda^{\mathfrak{g}})). \tag{14}$$

3.2 Stabilized plethysms on the Schur functions

For any partition $\lambda \in \mathcal{P}_n$, the plethysm $\psi_\ell(s_\lambda^{\mathfrak{gl}_n})$ decomposes on the basis of Schur functions on the form

$$\psi_\ell(s_\lambda^{\mathfrak{gl}_n}) = \sum_{|\mu|=\ell|\lambda|} \varepsilon(\mu) c^\lambda_{(\mu^{(0)},...,\mu^{(\ell-1)})} s_\nu^{\mathfrak{gl}_n}. \tag{15}$$

Here $\varepsilon(\mu) \in \{-1,0,1\}$ and $\mu/\ell = (\mu^{(0)},...,\mu^{(\ell-1)})$ are respectively the ℓ-sign and the ℓ-quotient of the partition μ. We know briefly recall the algorithm which permits to obtain the sign $\varepsilon(\mu)$ and the ℓ-tuple of partitions μ/ℓ. Our description slightly differs from that which can be usually found in the literature (see [**?**] Example 8 p 12). This is because we have made our notation homogeneous with Section **??**.
Set $\rho_n = (1,2,...,n)$ and $I_n = \{1,2,...,n\}$. For any $k \in \{0,...,\ell-1\}$ consider the ordering sequences

$$I^{(k)} = (i \in I_n \mid \mu_i + i \equiv k mod \ell) \text{ and } J^{(k)} = (i \in I_n \mid i \equiv k mod \ell).$$

Set $r_k = \text{card}(I^{(k)})$ and write $I^{(k)} = (i_1^{(k)},...,i_{r_k}^{(k)})$.

1. If there exists $k \in \{0,...,\ell-1\}$ such that $\text{card}(I^{(k)}) \neq \text{card}(J^{(k)})$ then $\varepsilon(\mu) = 0$.
2. Otherwise let $\sigma_0 \in S_n$ be the permutation such that, for any $k = 0,...,\ell-1$, σ_0 increases from $I^{(k)}$ to $J^{(k)}$. Then we have $\varepsilon(\mu) = \varepsilon(\sigma_0)$ and $\mu/\ell = (\mu^{(0)},...,\mu^{(\ell-1)})$ where for any $k \in \{0,...,\ell-1\}$

$$\mu^{(k)} = \left(\frac{\mu_i + i + \ell - k}{\ell} \mid i \in I^{(k)} \right) - (1,2,...,r_k) \in \mathbb{Z}^{r_k}.$$

Example 3.2.1
Consider $\mu = (1,2,3,4,4,4,6,6)$ *and take* $\ell = 3$. *We have* $\mu + \rho_8 = (2,4,6,8,9,10,13,14)$. *Thus* $I^{(0)} = \{3,5\}, I^{(1)} = \{2,6,7\}, I^{(2)} = \{1,4,8\}$ *and* $J^{(0)} = \{3,6\}, J^{(1)} = \{1,4,7\}, J^{(2)} = \{2,5,8\}$. *Then* $\mu^{(0)} = (1,1), \mu^{(1)} = (1,2,2)$ *and* $\mu^{(2)} = (0,1,2)$. *Moreover*

$$\sigma_0 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 3 & 5 & 6 & 4 & 7 & 8 \end{pmatrix}.$$

Hence $\varepsilon(\mu) = -1$.

Remarks:
(i) : The coefficients appearing in the decomposition (**??**) do not depend on the rank n considered providing $n \geq \ell\,|\lambda|$. This is because $\varepsilon(\mu)$ and the nonempty parts appearing in the partitions $\mu^{(k)}$ of the above algorithm are not modified when empty parts are added to μ.
(ii) : When $n \geq \ell\,|\lambda|$, we write for short $a^{\ell,\mathfrak{gl}}_{\lambda,\mu} = \varepsilon(\mu) c^\lambda_{(\mu^{(0)},...,\mu^{(\ell-1)})}$. Then $a^{\ell,\mathfrak{gl}}_{\lambda,\mu} \neq 0$ only if $|\mu| = \ell\,|\lambda|$.

Proposition 3.2.2 *Consider* $f \in \Lambda$ *with degree* d *and* $\lambda \in \mathcal{P}_n$. *Then the coefficients of the expansion of* $f \circ s_\lambda^{\mathfrak{gl}_n}$ *on the basis of Schur functions do not depend on* n *providing* $n \geq d\,|\lambda|$.

Proof. By (**??**) and the previous remark, the proposition is true for the power sum plethysms $p_\ell \circ s_\lambda^{\mathfrak{gl}_n}$. The map $g \mapsto g \circ s_\lambda^{\mathfrak{gl}_n}$ is a ring homomorphism of Λ_n. The subspace Λ_n^d of polynomials in Λ_n with degree d is generated by the Newton polynomials $p_\beta = p_{\beta_1} \cdots p_{\beta_k}$, such that $\beta_i \in \mathbb{N}$

and $\beta_1 + \cdots + \beta_k = d$. So it suffices to prove the proposition for $f = p_\beta$. We have $p_\beta \circ s_\lambda^{\mathfrak{gl}_n} = p_{\beta_1} \circ s_\lambda^{\mathfrak{gl}_n} \times \cdots \times p_{\beta_k} \circ s_\lambda^{\mathfrak{gl}_n}$. Suppose $n \geq d\,|\lambda|$. For any $i = 1, ..., k$, we have $n \geq \beta_i\,|\lambda|$. Thus we can write $p_{\beta_i} \circ s_\lambda^{\mathfrak{gl}_n} = \sum_{\mu^{(i)} \in \mathcal{P}_n} a_{\lambda,\mu^{(i)}}^{\beta_i,\mathfrak{gl}}\, s_{\mu^{(i)}}^{\mathfrak{gl}_n}$. Moreover $a_{\lambda,\mu^{(i)}}^{\beta_i,\mathfrak{gl}} \neq 0$ only if $\left|\mu^{(i)}\right| = \beta_i\,|\lambda|$. By Remark (ii) following Proposition 2.3.2, we obtain that the coefficients of the decomposition of $p_\beta \circ s_\lambda^{\mathfrak{gl}_n}$ on the basis of Schur functions do not depend on n when $n \geq |\beta|\,|\lambda|$. ■

3.3 Stabilized plethysms on the Weyl characters

Lemma 3.3.1 *Consider $\lambda \in \mathcal{P}_m$, ℓ a positive integer and $\mathfrak{g}$ an orthogonal or symplectic Lie algebra with rank $n \geq m$.*

- *The coefficients of the expansion of the plethysm $p_\ell \circ s_\lambda^{\mathfrak{g}}$ on the basis of Weyl characters do not depend on n providing $n \geq \ell\,|\lambda|$.*
- *In this case, these coefficients coincide for $\mathfrak{g} = \mathfrak{so}_{2n+1}$ and $\mathfrak{g} = \mathfrak{so}_{2n}$.*
- *For any $n \geq \ell\,|\lambda|$, set*

$$p_\ell \circ s_\lambda^{\mathfrak{so}_N} = \sum_{\mu \in \mathcal{P}_n} a_{\lambda,\mu}^{\ell,\mathfrak{so}} s_\mu^{\mathfrak{so}_N} \text{ and } p_\ell \circ s_\lambda^{\mathfrak{sp}_{2n}} = \sum_{\mu \in \mathcal{P}_n} a_{\lambda,\mu}^{\ell,\mathfrak{sp}} s_\mu^{\mathfrak{sp}_{2n}}.$$

We have

$$\begin{aligned} a_{\lambda,\mu}^{\ell,\mathfrak{so}} &= \sum_{\nu \in \mathcal{P}_m, |\nu| \leq |\lambda|} (-1)^{\frac{|\lambda|-|\nu|}{2}} \sum_{\delta \in \mathcal{P}, |\delta| = \ell|\nu|} r_{\lambda,\nu}^{\mathfrak{so}}\; a_{\nu,\delta}^{\ell,\mathfrak{gl}}\; b_{\delta,\mu}^{\mathfrak{so}}, \\ a_{\lambda,\mu}^{\ell,\mathfrak{sp}} &= \sum_{\nu \in \mathcal{P}_m, |\nu| \leq |\lambda|} (-1)^{\frac{|\lambda|-|\nu|}{2}} \sum_{\delta \in \mathcal{P}, |\delta| = \ell|\nu|} r_{\lambda,\nu}^{\mathfrak{sp}}\; a_{\nu,\delta}^{\ell,\mathfrak{gl}}\; b_{\delta,\mu}^{\mathfrak{sp}}. \end{aligned}$$

Proof. We have $n \geq \ell\,|\lambda|$. Hence, the decomposition $s_\lambda^{\mathfrak{so}_N} = \sum_{\nu \in \mathcal{P}_m, |\nu| \leq |\lambda|} (-1)^{\frac{|\lambda|-|\nu|}{2}} r_{\lambda,\nu}^{\mathfrak{so}} r^{\mathfrak{so}_N}(s_\nu^{\mathfrak{gl}_N})$ holds. Since ψ_ℓ and $r^{\mathfrak{so}_N}$ commute, this gives

$$\begin{aligned} p_\ell \circ s_\lambda^{\mathfrak{so}_N} = \sum_{\nu \in \mathcal{P}_m, |\nu| \leq |\lambda|} (-1)^{\frac{|\lambda|-|\nu|}{2}} \sum_{\delta \in \mathcal{P}, |\delta| = \ell|\nu|} r_{\lambda,\nu}^{\mathfrak{so}}\; a_{\nu,\delta}^{\ell,\mathfrak{gl}}\, r^{\mathfrak{so}_N}(s_\delta^{\mathfrak{gl}_N}) = \\ \sum_{\mu \in \mathcal{P}_n} \sum_{\nu \in \mathcal{P}_m, |\nu| \leq |\lambda|} (-1)^{\frac{|\lambda|-|\nu|}{2}} \sum_{\delta \in \mathcal{P}, |\delta| = \ell|\nu|} r_{\lambda,\nu}^{\mathfrak{so}}\; a_{\nu,\delta}^{\ell,\mathfrak{gl}}\; b_{\delta,\mu}^{\mathfrak{so}}\, s_\mu^{\mathfrak{so}_N}. \end{aligned}$$

This yields the desired expression for the coefficients $a_{\lambda,\mu}^{\ell,\mathfrak{so}}$. In particular they do not depends on n and coincide for $\mathfrak{g} = \mathfrak{so}_{2n+1}$ and $\mathfrak{g} = \mathfrak{so}_{2n}$. The proof is similar for $\mathfrak{g} = \mathfrak{sp}_{2n}$. ■

Proposition 3.3.2 *Consider $f \in \Lambda$ with degree d and $\lambda \in \mathcal{P}_n$. Then the coefficients of the expansion of $f \circ s_\lambda^{\mathfrak{g}}$ on the basis of Schur functions do not depend on n providing $n \geq d\,|\lambda|$. In this case, these coefficients coincide for $\mathfrak{g} = \mathfrak{so}_{2n+1}$ and $\mathfrak{g} = \mathfrak{so}_{2n}$.*

Proof. The proposition follows from Lemma **??** by similar arguments to those of Proposition **??**. ■

According to the previous Lemma, we have the decompositions

$$p_\ell \circ \mathtt{s}_\lambda^{\mathfrak{gl}} = \sum_\mu a_{\lambda,\mu}^{\ell,\mathfrak{gl}} \mathtt{s}_\mu^{\mathfrak{gl}}, \quad p_\ell \circ \mathtt{s}_\lambda^{\mathfrak{sp}} = \sum_\mu a_{\lambda,\mu}^{\ell,\mathfrak{sp}} \mathtt{s}_\mu^{\mathfrak{sp}} \text{ and } \quad p_\ell \circ \mathtt{s}_\lambda^{\mathfrak{so}} = \sum_\mu a_{\lambda,\mu}^{\ell,\mathfrak{so}} \mathtt{s}_\mu^{\mathfrak{so}}$$

We shall need in Section **??** the following Lemma :

Lemma 3.3.3 *Consider $f \in \Lambda$ and $\lambda \in \mathcal{P}$. Then*

- $\omega(f \circ \mathtt{s}_\lambda^{\mathfrak{g}}) = f \circ \omega(\mathtt{s}_\lambda^{\mathfrak{g}})$ *if $|\lambda|$ is even,*
- $\omega(f \circ \mathtt{s}_\lambda^{\mathfrak{g}}) = \omega(f) \circ \omega(\mathtt{s}_\lambda^{\mathfrak{g}})$ *if $|\lambda|$ is odd.*

Proof. From Example 1 page 136 of [**?**] we have for any positive integer ℓ, $\omega(p_\ell \circ g) = p_\ell \circ \omega(g)$ if g is homogeneous of even degree and $\omega(p_\ell \circ g) = \omega(p_\ell) \circ \omega(g)$ if g is homogeneous of odd degree. Since ψ_ℓ is linear, this shows that $\omega(p_\ell \circ \mathtt{s}_\lambda^{\mathfrak{g}}) = p_\ell \circ \omega(\mathtt{s}_\lambda^{\mathfrak{g}})$ if $|\lambda|$ is even and $\omega(p_\ell \circ \mathtt{s}_\lambda^{\mathfrak{g}}) = \omega(p_\ell) \circ \omega(\mathtt{s}_\lambda^{\mathfrak{g}})$ if $|\lambda|$ is odd. Indeed, according to (10), $\mathtt{s}_\lambda^{\mathfrak{g}}$ is a sum of homogeneous functions of degrees equal to $|\lambda|$ modulo 2. The Lemma then follows since the maps ω and $f \mapsto f \circ \mathtt{s}_\lambda^{\mathfrak{g}}$ are ring homomorphisms of Λ. ■

Remark: Since $\omega(p_\ell) = (-1)^{\ell-1} p_\ell$, one has by the previous lemma $a_{\lambda,\mu}^{\ell,\mathfrak{sp}} = a_{\lambda',\mu'}^{\ell,\mathfrak{so}}$ if $|\lambda|$ is even and $a_{\lambda,\mu}^{\ell,\mathfrak{sp}} = (-1)^{\ell-1} a_{\lambda',\mu'}^{\ell,\mathfrak{so}}$ otherwise. This can also be verified by using the explicit formulas of Lemma **??**.

4 Power sum plethysms for Weyl characters of type B_n

4.1 Statement of the theorem

In Theorem 3.2.8 of [**?**], we have described an algorithm for computing the plethysms $p_\ell \circ s_\lambda^{\mathfrak{so}_{2n+1}}$ for any positive integer ℓ and any rank n. It notably permits to show that the decomposition of $p_\ell \circ s_\lambda^{\mathfrak{so}_{2n+1}}$ on the basis of Weyl characters makes appear branching coefficients corresponding to the restriction to a Levi subgroup of $\mathfrak{so}_{2n+1}$. Surprisingly, similar algorithms for $\mathfrak{sp}_{2n}$ and $\mathfrak{so}_{2n}$ only exists when ℓ is odd. In particular, the coefficients of the decomposition of $p_\ell \circ s_\lambda^{\mathfrak{sp}_{2n}}$ and $p_\ell \circ s_\lambda^{\mathfrak{so}_{2n}}$ on the basis of Weyl characters are not branching coefficients in general when ℓ is even. As we are going to see, this is nevertheless the case for the stabilized forms of these plethysms.

Theorems 3.2.8 and 3.2.10 of [**?**] can be reformulated as follows :

Theorem 4.1.1 *For any partition $\lambda \in \mathcal{P}_n$ and any positive integer ℓ we have*

$$p_\ell \circ s_\lambda^{\mathfrak{so}_{2n+1}} = \sum_{\mu \in \mathcal{P}_n} \varepsilon(\mu) [V^{\mathfrak{so}_{2n+1}}(\lambda) : V^{\mathfrak{L}_{\ell,\mu}}(\gamma_{\ell,\mu})] s_\mu^{\mathfrak{so}_{2n+1}} \tag{16}$$

where $\varepsilon(\mu) \in \{-1, 0, 1\}$, $\mathfrak{L}_{\ell,\mu}$ is a Levi-subgroup of $\mathfrak{so}_{2n+1}$ and γ_μ a dominant weight for $\mathfrak{L}_{\ell,\mu}$. Moreover, $\varepsilon(\mu)$, $\mathfrak{L}_{\ell,\mu}$ and $\gamma_{\ell,\mu}$ are determined by an algorithm which can be regarded as an analogue in type B_n of the computation of the ℓ-quotient (μ/ℓ).

We now recall the algorithm which permits to determinate $\varepsilon(\mu)$, $\mathfrak{L}_{\ell,\mu}$ and $\gamma_{\ell,\mu}$ in the above theorem. Set

$$J_n = \{\overline{n}, ..., \overline{1}, 1, ..., n\} \text{ and } L_n = \{\overline{n-1}, ..., \overline{1}, 0, 1, ..., n\}.$$

Let η be the bijection from J_n to L_n defined by $\eta(x) = x + 1$ if $x < 0$ and $\eta(x) = x$ otherwise. For each element $w \in W$ (the Weyl group of $\mathfrak{so}_{2n+1}$), denote by $\widetilde{w}$ the bijection from J_n to L_n defined by $\widetilde{w} = \eta \circ w$. This means that $\widetilde{w}(x) = w(x)$ if $w(x) > 0$ and $\widetilde{w}(x) = w(x) + 1$ if $w(x) < 0$. In particular w is determined by $\widetilde{w}$. For any $x \in J_n$, set $x^* = \overline{x} + 1$. The map $x \longmapsto x^*$ is involutive from L_n to itself. Since $w(\overline{x}) = \overline{w(x)}$, we have also

$$\widetilde{w}(\overline{x}) = \widetilde{w}(x)^*. \tag{17}$$

Hence, $\widetilde{w}$ is determined by the images of any subset $U_n \subset J_n$ such that $\text{card}(U_n) = n$ and $x \in U_n$ implies $\overline{x} \notin U_n$.
For any $k = 1, ..., \ell$ set

$$I^{(k)} = (i \in I_n \mid \mu_i + i \equiv k mod \ell) \text{ and } J^{(k)} = (x \in L_n \mid x \equiv k mod \ell). \tag{18}$$

Note that $(J^{(k)})^* = J^{(l-k+1)}$.

4.2 The even case $\ell = 2p$

For any $k = 1, ..., p$, set $s_k = \text{card}(I^{(k)})$, $r_k = \text{card}(I^{(k)}) + \text{card}(I^{(\ell-k+1)})$ and define $X^{(k)}$ as the increasing reordering of $\overline{I}^{(k)} \cup I^{(\ell-k+1)}$. Set

$$X^{(k)} = (i_1^{(k)}, ..., i_{r_k}^{(k)}). \tag{19}$$

1. If there exists $k \in \{1, ..., p\}$ such that $\text{card}(X^{(k)}) \neq \text{card}(J^{(k)})$ then $\varepsilon(\mu) = 0$.
2. Otherwise we have $\text{card}(J^{(\ell-k+1)}) = \text{card}(J^{(k)}) = r_k$ since $(J^{(k)})^* = J^{(l-k+1)}$. Let w_0 be the unique element of W such that, for any $k = 1, ..., p$, $\widetilde{w}_0$ increases from $X^{(k)}$ to $J^{(l-k+1)}$. Define $\alpha_k = \frac{1}{\ell}(\max J^{(k)} - k)$. For any $k = 1, ..., p$, consider $\widehat{\mu}^{(k)} \in \widetilde{\mathcal{P}}_{r_k}$ defined by

$$\widehat{\mu}^{(k)} = \left(\text{sign}(i) \frac{\mu_{|i|} + |i| + \text{sign}(i)k - \frac{1+\text{sign}(i)}{2}}{\ell} \mid i \in X^{(k)} \right) - (1, ..., r_k) + (\alpha_k + 1, ..., \alpha_k + 1). \tag{20}$$

   Set $\mathcal{I} = \{X^{(1)} ..., X^{(p)}\}$. We have then with the notation of Section 2.2

$$\varepsilon(\mu) = \varepsilon(w_0), \quad \mathfrak{L}_{\ell,\mu} = G_{\mathcal{I}} \text{ and } \quad \gamma_{\ell,\mu} = (\mu^{(1)}, ..., \mu^{(p)}) \in P_{\mathcal{I}}^+.$$

Example 4.2.1 *Consider* $\mu = (2, 5, 5, 6, 7, 9)$. *Then* $\mu + \rho_6 = (3, 7, 8, 10, 12, 15)$. *Hence* $I_2 = \{3, 4, 5\}$ *and* $I_1 = \{1, 2, 6\}$. *Moreover* $J_2 = \{\overline{4}, \overline{2}, 0, 2, 4, 6\}$ *and* $J_1 = \{\overline{5}, \overline{3}, \overline{1}, 1, 3, 5\}$. *Then* $\widetilde{w}_0$ *sends* $X_1 = \{\overline{6}, \overline{2}, \overline{1}, 3, 4, 5\}$ *on* J_2. *This gives*

$$\widetilde{w}_0 = \begin{pmatrix} \overline{6} & \overline{5} & \overline{4} & \overline{3} & \overline{2} & \overline{1} & 1 & 2 & 3 & 4 & 5 & 6 \\ \overline{4} & \overline{5} & \overline{3} & \overline{1} & \overline{2} & 0 & 1 & 3 & 2 & 4 & 6 & 5 \end{pmatrix}$$

*by using (**??**). Hence*

$$w_0 = \begin{pmatrix} \overline{6} & \overline{5} & \overline{4} & \overline{3} & \overline{2} & \overline{1} & 1 & 2 & 3 & 4 & 5 & 6 \\ \overline{5} & \overline{6} & \overline{4} & \overline{2} & \overline{3} & \overline{1} & 1 & 3 & 2 & 4 & 6 & 5 \end{pmatrix}.$$

We have $\varepsilon(\mu)=1$, $\alpha_1=2$ *and*

$$\gamma=(-7,-3,-1,4,5,6)-(1,2,3,4,5,6)+(3,3,3,3,3,3)=(-5,-2,-1,3,3,3).$$

Observe that $\mathfrak{L}_{\ell,\mu}\simeq GL_6$.

4.3 The odd case $\ell=2p+1$

In addition to the sets $X^{(k)}, k=1,...,p$ defined in (**??**), we have also to consider $I^{(p+1)}=\{i_1^{(p+1)},...,i_{r_{p+1}}^{(p+1)}\}$. Observe that $(J^{(p+1)})^*=J^{(p+1)}$. Let $X^{(p+1)}$ be the increasing reordering of $\overline{I}^{(p+1)}\cup I^{(p+1)}$.

- If $\mathrm{card}(I^{(p+1)})\neq\frac{1}{2}\mathrm{card}(J^{(p+1)})$ or if there exists $k\in\{1,...,p\}$ such that $\mathrm{card}(X^{(k)})\neq\mathrm{card}(J^{(k)})$ then $\varepsilon(\mu)=0$.
- Otherwise, we have $\mathrm{card}(J^{(p+1)})=2\mathrm{card}(I^{(p+1)})=2r_{p+1}$. Let w_0 be the unique element of W such that, for any $k=1,...,p$, $\widetilde{w}_0$ increases from $X^{(k)}$ to $J^{(l-k+1)}$ and increases from $X^{(p+1)}$ to $J^{(p+1)}$. Define

  $$\mu^{(p+1)}=\left(\frac{\mu_i+i+p}{\ell}\mid i\in I^{(p+1)}\right)-(1,...,r_{p+1})\in\mathcal{P}_{r_{p+1}}$$

  and for any $k=1,...,p$, $\mu^{(k)}$ as in the even case. Set $\mathcal{I}=\{I^{(p+1)},X^{(1)},...,X^{(p)}\}$. We have then with the notation of Section 2.2

  $$\varepsilon(\mu)=\varepsilon(w_0),\quad \mathfrak{L}_{\ell,\mu}=G_{\mathcal{I}}\text{ and }\quad \gamma_{\ell,\mu}=(\mu^{(p+1)},\mu^{(1)},...,\mu^{(p)})\in P_{\mathcal{I}}^+.$$

Example 4.3.1 *Consider* $\mu=(1,5,5,6,7,9)$ *and take* $\ell=3$. *We have* $\mu+\rho_6=(2,7,8,10,12,15)$. *Thus* $X^{(1)}=\{\overline{4},\overline{2},5,6\}, I^{(2)}=\{1,3\}, X^{(2)}=\{\overline{3},\overline{1},1,3\}$ *and* $J^{(1)}=\{\overline{5},\overline{2},1,4\}, J^{(2)}=\{\overline{4},\overline{1},2,5\}$. *In particular* $\alpha_2=1$. *Then*

$$\mu^{(1)}=\left(-\frac{10-1}{3}-1+1,-\frac{7-1}{3}-2+1,\frac{12}{3}-3+1,\frac{15}{3}-4+1\right)=(-2,-2,3,3)$$

and $\mu^{(2)}=(\frac{2+1}{3}-1,\frac{8+1}{3}-2)=(0,1)$. *Moreover, one has by using (**??**)*

$$\widetilde{w}_0=\begin{pmatrix}\overline{6}&\overline{5}&\overline{4}&\overline{3}&\overline{2}&\overline{1}&1&2&3&4&5&6\\\overline{3}&0&\overline{5}&\overline{4}&\overline{2}&\overline{1}&2&3&5&6&1&4\end{pmatrix}.$$

Hence

$$w_0=\begin{pmatrix}\overline{6}&\overline{5}&\overline{4}&\overline{3}&\overline{2}&\overline{1}&1&2&3&4&5&6\\\overline{4}&\overline{1}&\overline{6}&\overline{5}&\overline{3}&\overline{2}&2&3&5&6&1&4\end{pmatrix}$$

and $\varepsilon(\mu)=1$. *We have* $\mathfrak{L}_{\ell,\mu}\simeq SO_5\times GL_4$.

4.4 Further remarks

(i) : Suppose $\varepsilon(\mu) \neq 0$. In the even case, we have $\mathfrak{L}_{\ell,\mu} \simeq GL_{r_1} \times \cdots \times GL_{r_p}$. In the odd case, $\mathfrak{L}_{\ell,\mu}$ is not a direct product of linear groups since $\mathfrak{L}_{\ell,\mu} \simeq SO_{2r_p+1} \times GL_{r_1} \times \cdots \times GL_{r_p}$.

(ii) : When $\ell = 2$, we have always $\mathrm{card}(X^{(2)}) = 2n = \mathrm{card}(J_n)$. Hence $\varepsilon(\mu) \neq 0$ for all partition μ. Observe that it does not mean that the expansion (**??**) is infinite. In fact most of the branching coefficients $[V^{\mathfrak{so}_{2n+1}}(\lambda) : V^{\mathfrak{L}_{\ell,\mu}}(\gamma_{\ell,\mu})]$ cancels in this situation. Note also that we have always $\mathfrak{L}_{\ell,\mu} \simeq GL_n$ in this case.

(iii) : We have seen that the nonzero parts of the ℓ-quotient μ/ℓ does not depend on the number of zero parts in μ (see Remark after Example **??**). This notably implies the stability of the coefficients $a^{\ell,\mathfrak{gl}}_{\lambda,\mu}$. Nevertheless, one cannot deduce the stability of the coefficients $a^{\ell,\mathfrak{so}}_{\lambda,\mu}$ from the previous algorithms. Indeed, the dominant weights $\gamma_{\ell,\mu}$ given by the previous algorithm *do* not stabilize when the number of zero parts in μ increases. To illustrate this unpleasant situation, let us add parts equal to 0 in the partition μ of Example **??**. The dominant weights $\gamma_{\ell,\mu}$ then obtained are

$$(-3,-3,-3,-1,-1,2,5), \quad (-5,-2,1,1,1,3,3,3), \quad (-3,-3,-3,-1,-1,-1,-1,2,5) \text{ etc.}$$

To understand this phenomenon, we have to say a few words on the way Theorem **??** has been obtained in [**?**]. Fix a rank $n \geq 2$ and write $< \cdot, \cdot >_{\mathfrak{so}_{2n+1}}$ for the scalar product on $\mathcal{R}^{\mathfrak{so}_{2n+1}}$ which makes orthonormal the basis of Weyl characters $\{s^{\mathfrak{so}_{2n+1}}_\lambda \mid \lambda \in \mathcal{P}_n\}$. Since ψ_ℓ is linear on $\mathcal{R}^{\mathfrak{so}_{2n+1}}$, one can consider its adjoint map φ_ℓ for $< \cdot, \cdot >_{\mathfrak{so}_{2n+1}}$. Given $\lambda, \mu \in \mathcal{P}_n$ we have

$$< \psi_\ell(s^{\mathfrak{so}_{2n+1}}_\lambda), s^{\mathfrak{so}_{2n+1}}_\mu >_{\mathfrak{so}_{2n+1}} = < s^{\mathfrak{so}_{2n+1}}_\lambda, \varphi_\ell(s^{\mathfrak{so}_{2n+1}}_\mu) >_{\mathfrak{so}_{2n+1}} .$$

In [**?**], we have obtained the explicit decomposition of $\varphi_\ell(s^{\mathfrak{so}_{2n+1}}_\mu)$ on the basis of Weyl characters. Namely

$$\varphi_\ell(s^{\mathfrak{so}_{2n+1}}_\mu) = \varepsilon(\mu) \sum_{\lambda \in \mathcal{P}_n} [V^{\mathfrak{so}_{2n+1}}(\lambda) : V^{\mathfrak{L}_{\ell,\mu}}(\gamma_{\ell,\mu})] s^{\mathfrak{so}_{2n+1}}_\lambda$$

The previous considerations implies that $\varphi_\ell(s^{\mathfrak{so}_{2n+1}}_\mu)$ does not stabilize in general with the rank n. This is not incompatible with Proposition **??** which assets that $\psi_\ell(s^{\mathfrak{so}_{2n+1}}_\mu)$ stabilizes in large rank. Indeed we have $[V^{\mathfrak{so}_{2n+1}}(\lambda) : V^{\mathfrak{L}_{\ell,\mu}}(\gamma_{\ell,\mu})] \neq 0$ only if $|\gamma_{\ell,\mu}| \leq |\lambda|$. When $|\gamma_{\ell,\mu}|$ increases with n as in our example and λ is fixed, we will thus obtain $[V^{\mathfrak{so}_{2n+1}}(\lambda) : V^{\mathfrak{L}_{\ell,\mu}}(\gamma_{\ell,\mu})] = 0$ for n sufficiently large.

5 Coefficients $a^\ell_{\lambda,\mu}$, Levi subgroups and parabolic K-L polynomials

5.1 Coefficients $a^\ell_{\lambda,\mu}$ and restriction to Levi subgroups

By combining the results of Sections 3 and **??** we derive the following theorem which expresses $a^{\ell,\mathfrak{so}}_{\lambda,\mu}$ and $a^{\ell,\mathfrak{sp}}_{\lambda,\mu}$ as branching coefficients corresponding to restrictions to Levi subgroups.

Theorem 5.1.1 *Consider $\lambda \in \mathcal{P}_m$ and ℓ a positive integer. Let $\mathfrak{g}$ be a symplectic or orthogonal Lie group with rank $n \geq \ell\,|\lambda|$. Then we have :*

1. $a^{\ell,\mathfrak{so}}_{\lambda,\mu} = \varepsilon(\mu)[V^{\mathfrak{so}_{2n+1}}(\lambda) : V^{\mathfrak{L}_{\ell,\mu}}(\gamma_{\ell,\mu})]$ *where* $\varepsilon(\mu)$, $\mathfrak{L}_{\ell,\mu}$ *and* $\gamma_{\ell,\mu}$ *are determined by the algorithms of Section* **??**,

2. $a^{\ell,\mathfrak{sp}}_{\lambda,\mu} = \varepsilon(\mu')[V^{\mathfrak{so}_{2n+1}}(\lambda') : V^{\mathfrak{L}_{\ell,\mu'}}(\gamma_{\ell,\mu'})]$ *if* $|\lambda|$ *is even and*

3. $a^{\ell,\mathfrak{sp}}_{\lambda,\mu} = (-1)^{\ell-1}\varepsilon(\mu')[V^{\mathfrak{so}_{2n+1}}(\lambda') : V^{\mathfrak{L}_{\ell,\mu'}}(\gamma_{\ell,\mu'})]$ *if* $|\lambda|$ *is odd.*

Proof. Assertion 1 follows from Proposition **??** and Theorem **??**. By remark following Lemma **??**, one has $a^{\ell,\mathfrak{sp}}_{\lambda,\mu} = a^{\ell,\mathfrak{so}}_{\lambda',\mu'}$ if $|\lambda|$ is even and $a^{\ell,\mathfrak{sp}}_{\lambda,\mu} = (-1)^{\ell-1}a^{\ell,\mathfrak{so}}_{\lambda',\mu'}$ otherwise which proves assertion 2. Note that the assumption $n \geq \ell\,|\lambda|$ suffices to guarantee that λ' belongs to $\mathcal{P}_n$. ■

5.2 Coefficients $a^{\ell}_{\lambda,\mu}$ and parabolic K-L polynomials

Consider ℓ a positive integer and $\zeta \in \mathbb{C}$ such that ζ^2 is a primitive ℓ-th root of 1. We briefly recall in this paragraph the arguments which permits to establish that the coefficients of the plethysm $\psi_\ell(s^{\mathfrak{g}}_\lambda)$ on the basis of Weyl characters are, up to a sign, parabolic Kazhdan-Lusztig polynomials specialized at $q = 1$ (see [**?**] for more details).
For any $\lambda \in P_+$, denote by $V^{\mathfrak{g}}_q(\lambda)$ the finite dimensional $U_q(\mathfrak{g})$-module of highest weight λ. Its character is also the Weyl character $s^{\mathfrak{g}}_\lambda$. Let $U_{q,\mathbb{Z}}(\mathfrak{g})$ be the $\mathbb{Z}[q, q^{-1}]$-subalgebra of $U_q(\mathfrak{g})$ generated by the elements

$$E^{(k)}_i = \frac{E^{(k)}_i}{[k]_i!}, F^{(k)}_i = \frac{F^{(k)}_i}{[k]_i!} \text{ and } K^{\pm 1}_i$$

where $E_i, F_i, K^{\pm 1}_i$, $i \in I_n$ are the generators of $U_q(\mathfrak{g})$. The indeterminate q can be specialized at ζ in $U_{q,\mathbb{Z}}(\mathfrak{g})$. Thus it makes sense to define $U_\zeta(\mathfrak{g}) = U_{q,\mathbb{Z}}(\mathfrak{g}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}$ where $\mathbb{Z}[q, q^{-1}]$ acts on $\mathbb{C}$ by $q \mapsto \zeta$. Fix a highest weight vector v_λ in $V^{\mathfrak{g}}_q(\lambda)$. We have $V^{\mathfrak{g}}_q(\lambda) = U_q(\mathfrak{g}) \cdot v_\lambda$. Similarly $V^{\mathfrak{g}}_\zeta(\lambda) = U_\zeta(\mathfrak{g}) \cdot v_\lambda$ is a $U_\zeta(\mathfrak{g})$ module called a Weyl module and one has $\mathrm{char}(V^{\mathfrak{g}}_\zeta(\lambda)) = s^{\mathfrak{g}}_\lambda$. The module $V^{\mathfrak{g}}_\zeta(\lambda)$ is not simple but admits a unique simple quotient denoted by $L^{\mathfrak{g}}(\lambda)$.
For any $\beta \in P^{\mathfrak{g}}$, we denote by t_β the translation defined in $\mathfrak{h}^*_{\mathbb{R}}$ by $\gamma \longmapsto \gamma + \beta$. The extended affine Weyl group $\widehat{W}$ is the group

$$\widehat{W} = \{wt_\beta \mid w \in W, \beta \in P^{\mathfrak{g}}\}$$

with multiplication determined by the relations $t_\beta t_\gamma = t_{\beta+\gamma}$ and $wt_\beta = t_{w\cdot\beta}w$. The group $\widehat{W}$ is not a Coxeter group but contains the affine Weyl group $\widetilde{W}$. It makes sense to define a length function l on $\widehat{W}$ which extends the length function defined on W. For any integer $m \in \mathbb{Z}^*$ we obtain a faithful representation π_m of $\widehat{W}$ on P by setting for any $\beta, \gamma \in P, w \in W$

$$\pi_{-\ell}(w) \cdot \gamma = w \cdot \gamma \text{ and } \pi_{-\ell}(t_\beta) \cdot \gamma = \gamma - \ell\beta.$$

We write for simplicity $wt_\beta \cdot \gamma$ rather that $\pi_{-\ell}(wt_\beta) \cdot \gamma$. Hence for any $w \in W$ and any $\beta \in P$, we have $wt_\beta \cdot \gamma = w \cdot \gamma - \ell w \cdot \beta$. The parabolic Kazhdan-Lusztig polynomials $\{P^{-,\mathfrak{g}}_{\lambda,\mu}, (\lambda, \mu) \in \mathcal{P}_n \times \mathcal{P}_n\}$ are defined as the entries of the transition matrix between the natural basis of the parabolic modules $P_\nu = \widehat{H}\nu$ and a canonical one defined by Deodhar. This construction is similar to that of the Kazhdan-Lusztig basis in the affine Hecke algebras. In the context of this article, Kashiwara and Tanisaki have proved that the polynomials $P^-_{\lambda,\mu}$ have nonnegative integer coefficients.
From results due to Kazhdan-Lusztig and Kashiwara-Tanisaki one obtains the following decomposition of $\mathrm{char}(L^{\mathfrak{g}}(\lambda))$ on the basis of Weyl characters:

Theorem 5.2.1 *Consider $\lambda \in P_+$. The character of $L^{\mathfrak{g}}(\lambda)$ decomposes on the form*

$$\mathrm{char}(L^{\mathfrak{g}}(\lambda)) = \sum_{\mu} (-1)^{l(w(\lambda+\rho))-l(w(\mu+\rho))} P^{-,\mathfrak{g}}_{\mu+\rho,\lambda+\rho}(1) s^{\mathfrak{g}}_{\mu} \tag{21}$$

where the sum runs over the dominant weights $\mu \in P_+$ such that $\mu + \rho \in \widehat{W} \cdot (\lambda + \rho)$.

The Frobenius map Fr_ℓ is the algebra homomorphism defined from $U_\zeta(\mathfrak{g})$ to $U(\mathfrak{g})$ by $\mathrm{Fr}_\ell(K_i) = 1$ and

$$\mathrm{Fr}_\ell(E_i^{(k)}) = \begin{cases} e_i^{(k/\ell)} \text{ if } \ell \text{ divides } k \\ 0 \text{ otherwise} \end{cases} \quad \text{and } \mathrm{Fr}_\ell(F_i^{(k)}) = \begin{cases} f_i^{(k/\ell)} \text{ if } \ell \text{ divides } k \\ 0 \text{ otherwise} \end{cases}$$

where e_i, f_i, $i \in I_n$ are the Chevalley generators of the enveloping algebra $U(\mathfrak{g})$. This permits to endow each $U(\mathfrak{g})$-module M with the structure of a $U_\zeta(\mathfrak{g})$-module M^{Fr_ℓ}. Then we have

$$\mathrm{char}(M^{\mathrm{Fr}_\ell}) = \psi_\ell(\mathrm{char}(M))$$

in particular for any $\lambda \in P_+$, $\mathrm{char}(V^{\mathfrak{g}}(\lambda)^{\mathrm{Fr}_\ell}) = \psi_\ell(s^{\mathfrak{g}}_\lambda)$.
Each dominant weight $\lambda \in P_+$, can be uniquely decomposed on the form $\lambda = \overset{r}{\lambda} + \ell \overset{q}{\lambda}$ where $\overset{r}{\lambda}, \overset{q}{\lambda} \in P_+$ and $\overset{r}{\lambda} = (\overset{r}{\lambda}_1, ..., \overset{r}{\lambda}_n)$ verifies $0 \leq \overset{r}{\lambda}_{i+1} - \overset{r}{\lambda}_i < \ell$ for any $i \in I_n$.

Theorem 5.2.2 *(Lusztig) The simple $U_\zeta(\mathfrak{g})$-module $L^{\mathfrak{g}}(\lambda)$ is isomorphic to the tensor product*

$$L^{\mathfrak{g}}(\lambda) \simeq L^{\mathfrak{g}}(\overset{r}{\lambda}) \otimes V^{\mathfrak{g}}(\overset{q}{\lambda})^{\mathrm{Fr}_\ell}.$$

By replacing λ by $\ell\lambda$ in the previous theorem, we have $\overset{r}{\lambda} = 0$ and $\overset{q}{\lambda} = \lambda$. Thus $L^{\mathfrak{g}}(\ell\lambda) \simeq V^{\mathfrak{g}}(\lambda)^{\mathrm{Fr}_\ell}$. Then one deduces from (**??**) the following corollary.

Corollary 5.2.3 *(of Theorem **??**) For any nonnegative integer ℓ*

$$\psi_\ell(s^{\mathfrak{g}}_\lambda) = \sum_{\mu+\rho \in \widehat{W}\cdot\rho} (-1)^{l(w(\lambda+\rho))-l(w(\mu+\rho))} P^{-,\mathfrak{g}}_{\mu+\rho,\ell\lambda+\rho}(1) s^{\mathfrak{g}}_{\mu}.$$

The following proposition is a consequence of Theorem **??** and Corollary **??**.

Proposition 5.2.4 *Let ℓ be a positive integer and λ a partition of $\mathcal{P}_m$. Suppose $\mathfrak{g}$ is a symplectic or orthogonal Lie algebra with rank $n \geq \ell\,|\lambda|$. Then for any $\mu \in \mathcal{P}$ we have*

1. $P^{-,\mathfrak{so}_N}_{\mu+\rho,\ell\lambda+\rho}(1) = \left| a^{\ell,\mathfrak{so}}_{\lambda,\mu} \right| = [V^{\mathfrak{so}_{2n+1}}(\lambda) : V^{\mathfrak{L}_{\ell,\mu}}(\gamma_{\ell,\mu})]$ *where $\mathfrak{L}_{\ell,\mu}$ and $\gamma_{\ell,\mu}$ are determined by the algorithms of Section **??**,*
2. $P^{-,\mathfrak{sp}_{2n}}_{\mu+\rho,\ell\lambda+\rho}(1) = \left| a^{\ell,\mathfrak{sp}}_{\lambda,\mu} \right| = [V^{\mathfrak{so}_{2n+1}}(\lambda') : V^{\mathfrak{L}_{\ell,\mu}}(\gamma_{\ell,\mu'})]$.

In particular, $P^{-,\mathfrak{so}_N}_{\mu+\rho,\ell\lambda+\rho}(1)$ and $P^{-,\mathfrak{sp}_{2n}}_{\mu+\rho,\ell\lambda+\rho}(1)$ do not depend on the rank $n \geq \ell\,|\lambda|$ considered.

Conjecture 5.2.5 *The parabolic Kazhdan-Lusztig polynomials $P^{-,\mathfrak{so}_N}_{\mu+\rho,\ell\lambda+\rho}(q)$ and $P^{-,\mathfrak{sp}_{2n}}_{\mu+\rho,\ell\lambda+\rho}(q)$ do not depend on the rank n considered providing $n \geq \ell\,|\lambda|$*

Remark: When $\mathfrak{g} = \mathfrak{gl}_n$ and $n \geq \ell\,|\lambda|$, we have similarly to Proposition **??**, $P^{-,\mathfrak{gl}_n}_{\mu+\rho,\ell\lambda+\rho}(1) = \left|a^{\ell,\mathfrak{gl}}_{\lambda,\mu}\right| = c^{\lambda}_{(\mu^{(0)},\ldots,\mu^{(\ell-1)})}$ where $(\mu^{(0)},\ldots,\mu^{(\ell-1)}) = \mu/\ell$ is the ℓ-quotient of the partition μ. The polynomials $P^{-,\mathfrak{gl}_n}_{\mu+\rho,\ell\lambda+\rho}(q)$ are then independent of n. This can be proved combinatorially by considering the generalized Hall-Littlewood functions $G^{\ell,\mathfrak{gl}_n}_{\mu} = \sum_\lambda P^{-,\mathfrak{gl}_n}_{\mu+\rho,\ell\lambda+\rho}(q) s^{\mathfrak{gl}_n}_{\lambda}$. Indeed, it has been shown in [**?**] that the function $G^{\ell,\mathfrak{gl}_n}_{\mu}$ admits the decomposition

$$G^{\ell,\mathfrak{gl}_n}_{\mu} = \sum_{\nu\in\mathcal{P}} \sum_{T\in \mathrm{Tab}_\ell(\mu,\nu)} q^{s(T)} m_\nu.$$

Here $\mathrm{Tab}_\ell(\mu,\nu)$ is the set of ℓ-ribbon tableaux of shape μ and weight ν on I_n, s the spin statistic and m_ν the monomial function associated to ν. The set $S_{\mu,\ell}$ of partitions ν such that $\mathrm{Tab}_\ell(\mu,\nu) \neq \emptyset$ is finite and independent on n. Thus, for any $\nu \in S_{\mu,\ell}$, $\mathrm{Tab}_\ell(\mu,\nu)$ is also finite and independent on n. This shows that the coefficients of the expansion of the functions $G^{\ell,\mathfrak{gl}_n}_{\mu}$ on the basis of monomials m_ν do not depend on n. Since the coefficients of the transition matrix between the basis $\{m_\nu, \nu \in S_{\mu,\ell}\}$ and $\{s^{\mathfrak{gl}_n}_{\nu}, \nu \in S_{\mu,\ell}\}$ do not depend on n, the polynomials $P^{-,\mathfrak{gl}_n}_{\mu+\rho,\ell\lambda+\rho}(q)$ are independent of n.

6 Splitting $V^{\mathfrak{g}}(\lambda)^{\otimes 2}$ into its symmetric and antisymmetric parts

6.1 Decomposition of the plethysms $p_2 \circ s^{\mathfrak{g}}_{\lambda}$

Consider a partition $\lambda \in \mathcal{P}_m$. According to Theorem **??**, we have with the notation of Sections **??** and **??**

$$\begin{aligned} p_2 \circ s^{\mathfrak{gl}_n}_{\lambda} &= \sum_{\mu\in\mathcal{P}_n} \varepsilon(\mu) c^{\lambda}_{(\mu^{(0)},\mu^{(1)})} s^{\mathfrak{gl}_n}_{\mu}, \qquad (22)\\ p_2 \circ s^{\mathfrak{so}}_{\lambda} &= \sum_{\mu\in\mathcal{P}_n} \varepsilon(\mu) [V^{\mathfrak{so}_{2n+1}}(\lambda) : V^{\mathfrak{gl}_n}(\gamma_\mu)] s^{\mathfrak{so}}_{\mu} \text{ and } p_2 \circ s^{\mathfrak{sp}}_{\lambda} = (-1)^{|\lambda|} p_2 \circ s^{\mathfrak{so}}_{\lambda'} \end{aligned}$$

for any $n \geq 2\,|\lambda|$. Here we have written for short γ_μ for $\gamma_{2,\mu}$ and $V^{\mathfrak{gl}_n}(\gamma_\mu)$ instead of $V^{\mathfrak{L}_{2,\mu}}(\gamma_\mu)$ (see Remark (ii) of Section **??**). Since $n \geq m$ and $\gamma_\mu = (\gamma^-,\gamma^+)$ belongs to $\widetilde{\mathcal{P}}_n$ we have the following decomposition due to Littlewood :

$$[V^{\mathfrak{so}_{2n+1}}(\lambda) : V^{\mathfrak{gl}_n}(\gamma_\mu)] = \sum_{\delta,\xi\in\mathcal{P}_n} c^{\lambda}_{\delta,\xi} c^{\xi}_{\gamma^-,\gamma^+}. \qquad (23)$$

6.2 Symmetric and antisymmetric parts of $V^{\mathfrak{g}}(\lambda)^{\otimes 2}$

Consider $\lambda \in \mathcal{P}_m$. By Propositions **??** and **??**, for any rank $n \geq 2\,|\lambda|$ the plethysms $h_2 \circ s^{\mathfrak{g}}_{\lambda}$ and $e_2 \circ s^{\mathfrak{g}}_{\lambda}$ stabilize. Set

$$h_2 \circ s^{\mathfrak{g}}_{\lambda} = \sum_{\mu\in\mathcal{P}_n} m^{\mathfrak{G},+}_{\lambda,\mu} s^{\mathfrak{g}}_{\mu} \qquad \text{and} \qquad e_2 \circ s^{\mathfrak{g}}_{\lambda} = \sum_{\mu\in\mathcal{P}_n} m^{\mathfrak{G},-}_{\lambda,\mu} s^{\mathfrak{g}}_{\mu}$$

where $\mathfrak{G} = \mathfrak{gl}, \mathfrak{so}$ or $\mathfrak{sp}$ respectively when $\mathfrak{g} = \mathfrak{gl}_n, \mathfrak{so}_N$ or $\mathfrak{sp}_{2n}$. Recall that $h_2 \circ s^{\mathfrak{g}}_{\lambda}$ and $e_2 \circ s^{\mathfrak{g}}_{\lambda}$ are the characters of $S^2(V^{\mathfrak{g}}(\lambda)$ and $\Lambda^2(V^{\mathfrak{g}}(\lambda)$. By using (**??**) and Theorem **??**, we obtain for any rank

$n \geq 2\,|\lambda|$

$$\begin{aligned}
m^{\mathfrak{gl},\pm}_{\lambda,\mu} &= \frac{1}{2}(c^{\mu}_{\lambda,\lambda}, \pm\varepsilon(\mu)c^{\lambda}_{(\mu^{(0)},\mu^{(1)})}),\\
m^{\mathfrak{so},\pm}_{\lambda,\mu} &= \frac{1}{2}(d^{\mu}_{\lambda,\lambda}, \pm\varepsilon(\mu)[V^{\mathfrak{so}_{2n+1}}(\lambda) : V^{\mathfrak{gl}_n}(\gamma_\mu)],\\
m^{\mathfrak{sp},\pm}_{\lambda,\mu} &= \frac{1}{2}(d^{\mu'}_{\lambda',\lambda'}, \pm(-1)^{|\lambda|}\varepsilon(\mu')[V^{\mathfrak{so}_{2n+1}}(\lambda') : V^{\mathfrak{gl}_n}(\gamma_{\mu'})]
\end{aligned}$$

where the coefficients $d^{\mu}_{\lambda,\lambda}$ are the multiplicities appearing in Proposition 2.3.2. Now these multiplicities can be expressed in terms of the Littlewood coefficients [**?**]. Namely we have $d^{\mu}_{\lambda,\lambda} = \sum_{\delta,\xi,\eta} c^{\mu}_{\delta,\xi}c^{\lambda}_{\delta,\eta}c^{\lambda}_{\xi,\eta}$. In particular we recover the equality $d^{\mu'}_{\lambda',\lambda'} = d^{\mu}_{\lambda,\lambda}$ since $c^{\gamma}_{\delta,\eta} = c^{\gamma'}_{\delta',\eta'}$ for any partitions δ, η and γ. By using (**??**), this thus permits to express the multiplicities appearing in the symmetric and antisymmetric parts of $V^{\mathfrak{g}}(\lambda)^{\otimes 2}$ in terms of the Littlewood-Richardson coefficients.

Proposition 6.2.1 *With the above notation we have for any rank $n \geq 2\,|\lambda|$*

$$\begin{aligned}
m^{\mathfrak{gl},\pm}_{\lambda,\mu} &= \frac{1}{2}(c^{\mu}_{\lambda,\lambda} \pm \varepsilon(\mu)c^{\lambda}_{(\mu^{(0)},\mu^{(1)})}),\\
m^{\mathfrak{so},\pm}_{\lambda,\mu} &= \frac{1}{2}\left(\sum_{\delta,\xi,\eta\in\mathcal{P}_n} c^{\mu}_{\delta,\xi}c^{\lambda}_{\delta,\eta}c^{\lambda}_{\xi,\eta} \pm \varepsilon(\mu)\sum_{\delta,\xi\in\mathcal{P}_n} c^{\lambda}_{\delta,\xi}c^{\xi}_{\gamma^-,\gamma^+}\right)\\
m^{\mathfrak{sp},\pm}_{\lambda,\mu} &= \frac{1}{2}\left(\sum_{\delta,\xi,\eta\in\mathcal{P}_n} c^{\mu}_{\delta,\xi}c^{\lambda}_{\delta,\eta}c^{\lambda}_{\xi,\eta} \pm (-1)^{|\lambda|}\varepsilon(\mu')\sum_{\delta,\xi\in\mathcal{P}_n} c^{\lambda}_{\delta,\xi}c^{\xi}_{\kappa^-,\kappa^+}\right)
\end{aligned}$$

where $\gamma_\mu = (\gamma^-, \gamma^+)$ and $\gamma_{\mu'} = (\kappa^-, \kappa^+)$.

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