Abelian groupoids and non pointed additive categories

Dominique Bourn

Abstract

We show that, in any Mal'cev (and a fortiori protomodular) category $\mathbb{E}$, the fibre $Grd_X\mathbb{E}$ of internal groupoids above the object X is naturally Mal'cev and moreover shares with the category Ab of abelian groups the property following which the domain of any split epimorphism is isomorphic with the direct sum of its codomain with its kernel. This allows us to point at a new class of "non pointed additive" categories. When furthermore the ground category $\mathbb{E}$ is efficiently regular, we get a new way to produce Bear sums in the fibres $Grd_X\mathbb{E}$ and more generally in the fibres n-$Grd_X\mathbb{E}$.

Introduction

The main project of this work was to gather some properties (related to cohomological algebra) of the category $Grd\mathbb{C}$ of internal groupoids inside a protomodular [4] category $\mathbb{C}$. In a way, the existence of the semi-direct product in the category Gp of groups and the associated possible reduction of internal groupoids to crossed modules made that the systematic investigation of the category $GrdGp$ was not done, and no guiding example of such a protomodular context was existing. Actually it appears that our main results concerning $Grd\mathbb{C}$ do hold when $\mathbb{C}$ is only Mal'cev in the sense of [12] (see also [13] and [14]).

We show that, in the Mal'cev context, any groupoid is abelian in the sense of [7], which implies that any fibre $Grd_X\mathbb{C}$ of internal groupoids having X as "object of objects" is naturally Mal'cev in the sense of [16]. Moreover we show that, given any split internal functor in the fibre $Grd_X\mathbb{C}$, the downward pullback:

$$\begin{tikzcd} \underline{K}_1[f_\bullet] \arrow[r, tail, "\underline{k}_1"] \arrow[d, shift right] & \underline{W}_1 \arrow[d, shift right, "\underline{f}_1"'] \\ \Delta X \arrow[u, shift right] \arrow[r, tail, "\underline{\alpha}_1\underline{Z}_1"'] & \underline{Z}_1 \arrow[u, shift right, "\underline{s}_1"'] \end{tikzcd}$$

produces an upward pushout. In other word, when $\mathbb{C}$ is Mal'cev, the fibre $Grd_X\mathbb{C}$ shares with the category Ab of abelian groups (and more generally the fibre $Grd_1\mathbb{C} = Ab\mathbb{C}$) the property following which the domain of any split epimorphism is isomorphic with the direct sum of its codomain with its kernel. It is all the more interesting since this is absolutely not the case in the fibres $AbGrd_XSet$, $X \neq 1$ of abelian groupoids in Set.

This kind of results gives rise to new classes of "non pointed additive" categories which take place between the class of naturally Mal'cev categories [16] and the one of essentially affine categories [5]. Subtle distinctions between different kinds of cohesion, which compensate the disorganisation determined by the absence of 0 (and consequently by the ordered set of subobjects of 1), uncumfortably demand to introduce a bit of terminology. We give a synthetic classification table in section 2.3. The most interesting intermediate class is the one of *penessentially affine categories* (see section 2.1 for the precise definition): it is a class of non pointed additive categories which are necessarily protomodular and such that any monomorphism is normal, and which, precisely, contains any fibre $Grd_X\mathbb{C}$ in the Mal'cev context. This new structural approach of internal groupoids allows us to get a new way to produce Baer sums in these fibres, and more generally a new way to produce the cohomology groups $H^n_{\mathbb{C}}(A)$. All this leads also to a more technical last section which is devoted to the fibre $Grd_X\mathbb{E}$ when $\mathbb{E}$ is only finitely complete.

1 Internal groupoids

Let $\mathbb{E}$ be a finitely complete category, and $Grd\mathbb{E}$ denote the category of internal groupoids in $\mathbb{E}$. An internal groupoid $\underline{Z}_1$ in $\mathbb{E}$ will be presented (see [2]) as a reflexive graph $Z_1 \rightrightarrows Z_0$ endowed with an operation ζ_2:

$$\begin{tikzcd}
R^2[z_0] \arrow[r, shift left=2, "z_2"] \arrow[r, "z_1"] \arrow[r, shift right=2, "z_0"'] \arrow[r, bend left=60, "R(\zeta_2)"] & R[z_0] \arrow[r, shift left, "z_1"] \arrow[r, shift right, "z_0"'] \arrow[r, bend left=60, "\zeta_2"] & Z_1 \arrow[r, shift left=2, "z_1"] \arrow[r, shift right=2, "z_0"'] & Z_0 \arrow[l, "s_0"']
\end{tikzcd}$$

making the previous diagram satisfy all the simplicial identities (including the ones involving the degeneracies), where $R[z_0]$ is the kernel equivalence relation of the map z_0. In the set theoretical context, this operation ζ_2 associates the composite $g.f^{-1}$ with any pair (f, g) of arrows with same domain. We denote by $()_0 : Grd\mathbb{E} \to \mathbb{E}$ the forgetful functor which is a fibration. Any fibre $Grd_X\mathbb{E}$ above an object X has an initial object ΔX, namely the discrete equivalence relation on X, and a final object ∇X, namely the indiscrete equivalence relation on X. This fibre is *quasi-pointed* in the sense that the unique map

$$\varpi : 0 \to 1 \quad = \quad \Delta X \to \nabla X$$

is a monomorphism; this implies that any initial map is a monomorphism, and we can define the kernel of any map as its pullback along the initial map to the codomain. The fibre $Grd_1\mathbb{E}$ is nothing but the category $Gp\mathbb{E}$ of internal groups in $\mathbb{E}$ which is necessarily pointed protomodular. It was shown in [4] that any fibre $Grd_X\mathbb{E}$ is still protomodular although non pointed. This involves an intrinsic notion of normal subobject and abelian object. They both have been characterized in [7].

1.1 Abelian groupoids

Let us begin by the abelian groupoids. Consider the following pullback in $Grd\mathbb{E}$ which only retains the "endomorphisms" of $\underline{Z}_1$:

$$\begin{tikzcd}
\underline{En}_1\underline{Z}_1 \arrow[r, tail, "\underline{\epsilon}_1\underline{Z}_1"] \arrow[d, "\underline{e}_1\underline{Z}_1"'] & \underline{Z}_1 \arrow[d, "\underline{\omega}_1\underline{Z}_1"] \\
\Delta Z_0 \arrow[r, tail] & \nabla Z_0
\end{tikzcd}$$

Let us recall [7] that:

Proposition 1.1. *The groupoid $\underline{Z}_1$ is abelian if and only if the group $\epsilon_1 : En_1\underline{Z}_1 \to Z_0$ in the slice category $\mathbb{E}/Z_0$ is abelian.*

In the set theoretical context, this means that any group of endomaps in $\underline{Z}_1$ is abelian. We shall denote by $AbGrd_X\mathbb{E}$ the full subcategory of $Grd_X\mathbb{E}$ whose objects are the abelian groupoids.

Now consider any internal functor $\underline{f}_1 : \underline{W}_1 \to \underline{Z}_1$ in $AbGrd_X\mathbb{E}$. Suppose it is split by a functor $\underline{s}_1$, and consider the following pullback determining the kernel of $\underline{f}_1$:

$$\begin{tikzcd}
\underline{K}_1[\underline{f}_1] \arrow[r, tail, "\underline{k}_1"] \arrow[d, shift right] & \underline{W}_1 \arrow[d, shift right, "\underline{f}_1"'] \\
\Delta X \arrow[r, tail, "\underline{\alpha}_1\underline{Z}_1"'] \arrow[u, shift right] & \underline{Z}_1 \arrow[u, shift right, "\underline{s}_1"']
\end{tikzcd}$$

In the case $X = 1$, the upward square is actually a pushout in $AbGrd_1\mathbb{E} = Ab\mathbb{E}$ the category of abelian groups in $\mathbb{E}$. Does it still hold in any case? Suppose given a pair $\underline{h}_1 : \underline{K}_1[\underline{f}_1] \to \underline{V}_1, \underline{t}_1 : \underline{Z}_1 \to \underline{V}_1$ of internal functors in $AbGrd_X\mathbb{E}$.

Lemma 1.1. *When $\mathbb{E} = Set$, there is a factorization $\underline{g}_1 : \underline{W}_1 \to \underline{V}_1$ such that $\underline{g}_1.\underline{k}_1 = \underline{h}_1$ and $\underline{g}_1.\underline{s}_1 = \underline{t}_1$ if and only if, for all pair $x \xrightarrow{\gamma} x \xrightarrow{\phi} y$ of maps in $K[\underline{f}_1] \times \underline{Z}_1$ with same domain, we have:*

$$h_1(s_1\phi.\gamma.s_1\phi^{-1}) = t_1\phi.h_1\gamma.t_1\phi^{-1}$$

Proof. For any $\delta : y \to y$ in $\underline{K}_1[\underline{f}_1]$, we must have $g_1\delta = h_1\delta$, and any $\phi : x \to y$ in $\underline{Z}_1$, we must have $g_1.s_1\phi = t_1\phi$. Accordingly, for any $\psi : x \to y$ in $\underline{W}_1$, we must have $g_1\psi = g_1(\psi.s_1f_1\psi^{-1}).g_1(s_1f_1\psi) = h_1(\psi.s_1f_1\psi^{-1}).t_1(f_1\psi)$. It remains to show that this definition is functorial, which is easily stated to be equivalent to our condition. □

Accordingly, the category $AbGrd_X\mathbb{E}$ of abelian groupoids in the fibre above $X \neq 1$ does not share the classical property of $Ab\mathbb{E} = AbGrd_1\mathbb{E}$ concerning the split epimorphisms.

1.2 Groupoids in Mal'cev and naturally Mal'cev categories

However we are going to show that this is the case as soon as the ground category $\mathbb{E}$ is Mal'cev. Recall that a category $\mathbb{E}$ is Mal'cev ([12], [13]) when it is finitely complete and such that any reflexive relation is actually an equivalence relation. When the category $\mathbb{E}$ is Mal'cev, we can truncate at level 2 (i.e. at the level of $R[z_0]$) the diagram defining a groupoid, see [13]. A category is naturally Mal'cev [16] when its is finitely complete and such that any object X is equipped with a natural Mal'cev operation. Any naturally Mal'cev category is Mal'cev.

Theorem 1.1. *Suppose $\mathbb{E}$ is Mal'cev. Then any internal groupoid is abelian. Accordingly any fibre $Grd_X\mathbb{E}$ is naturally Mal'cev. Moreover, for any split epimorphism in $Grd_X\mathbb{E}$, the previous upward square is necessarily a pushout.*

Proof. When $\mathbb{E}$ is Mal'cev, this is still the case for the slice category $\mathbb{E}/Z_0$. On the other hand, any group in a Mal'cev category is abelian, see [13]. So, by Proposition 1.1, any groupoid is abelian. Any fibre $Grd_X\mathbb{E}$, being necessarily protomodular [4] and thus Mal'cev, is naturally Mal'cev, since any object in $Grd_X\mathbb{E}$ is abelian and produces a natural Mal'cev operation.

We are now going to show the next point by a classical method in Mal'cev categories. Consider the relation $R \rightarrowtail K_1[\underline{f}_1] \times Z_1$ defined by $\gamma R \phi$ if

$$dom\gamma = dom\phi \quad \wedge \quad h_1(s_1\phi.\gamma.s_1\phi^{-1}) = t_1\phi.h_1\gamma.t_1\phi^{-1}$$

Suppose $dom\gamma = dom\phi = x$, then obviously we have $1_xR\phi, \gamma R1_x$ and 1_xR1_x. Accordingly we can conclude that $\gamma R\phi$ for all (γ, ϕ) with same domain, whence, according to Lemma 1.1, the desired unique factorization $\underline{g}_1 : \underline{W}_1 \to \underline{V}_1$. $\square$

This result holds *a fortiori* in any protomodular category $\mathbb{E}$. We have now an important structural property:

Corollary 1.1. *Suppose $\mathbb{E}$ is Mal'cev. Then for any groupoid $\underline{Z}_1$ the following upward left hand side square is a pushout in $Grd\mathbb{E}$.*

Proof. Let us consider the following diagram:

$$\begin{tikzcd}
\underline{En}_1\underline{Z}_1 \arrow[r, dotted, tail, "\underline{\bar{\epsilon}}_1\underline{Z}_1"'] \arrow[rr, tail, bend left, "\underline{\epsilon}_1\underline{Z}_1"] \arrow[d, shift right, "\underline{e}_1\underline{Z}_1"'] & \underline{Z}_1 \times_0 \underline{Z}_1 \arrow[r, "p_1"] \arrow[d, shift right, "p_0"'] & \underline{Z}_1 \arrow[d, "\underline{\omega}_1\underline{Z}_1"] \\
\Delta Z_0 \arrow[r, tail] \arrow[u, shift right, "\underline{\alpha}_1\underline{En}_1\underline{Z}_1"'] & \underline{Z}_1 \arrow[r, "\underline{\omega}_1\underline{Z}_1"'] \arrow[u, shift right, "s_0"'] & \nabla Z_0
\end{tikzcd}$$

The whole rectangle and the right hand side squares being pullbacks, there is a unique dotted arrow which makes the downward square a pullback, and consequently the upward left hand side square a pushout in $Grd_{Z_0}\mathbb{E}$ according to the previous theorem. But, the functor $()_0 : Grd\mathbb{E} \to \mathbb{E}$ being a fibration, it is still a pushout in $Grd\mathbb{E}$. $\square$

It was shown in [5] that a finitely complete category $\mathbb{E}$ is Mal'cev if and only if any reflexive graph $\underline{Z}'_1$ which is a subobject of a groupoid $\underline{Z}_1$ is itself a groupoid. This property allows to strengthen the Theorem 1.1:

Theorem 1.2. *Suppose $\mathbb{E}$ is Mal'cev. Given any split epimorphism $(\underline{f}_1, \underline{s}_1) : \underline{W}_1 \rightleftarrows \underline{Z}_1$ in $Grd_X\mathbb{E}$, there is a bijection between the pointed subobjects of the kernel $\underline{K}_1[\underline{f}_1]$ and the pointed subobjects of $(\underline{f}_1, \underline{s}_1)$.*

Proof. Any pointed subobject $\underline{j}_1$ of $(\underline{f}_1, \underline{s}_1)$ produces a pointed subobject of $\underline{K}_1[\underline{f}_1]$ by pullback along $\underline{k}_1$:

$$\begin{tikzcd}
\underline{A}_1 \arrow[r, tail] \arrow[d, "\underline{i}_1"'] & \underline{W}'_1 \arrow[d, "\underline{j}_1"] \\
\underline{K}_1[\underline{f}_1] \arrow[r, tail, "\underline{k}_1"] \arrow[d, shift right] & \underline{W}_1 \arrow[d, shift right, "\underline{f}_1"'] \\
\Delta X \arrow[r, tail, "\underline{\alpha}_1\underline{Z}_1"'] \arrow[u, shift right] & \underline{Z}_1 \arrow[u, shift right, "\underline{s}_1"']
\end{tikzcd}$$

Conversely suppose given a pointed subobject $\underline{i}_1 : \underline{A}_1 \rightarrowtail \underline{K}_1$. Define W'_1 as the subobject of W_1 whose elements are those maps $\tau : x \to y \in W_1$ which satisfy $\tau.s_1 f_1(\tau^{-1}) \in A_1$. This subobject is given by the following right hand side pullback in $\mathbb{E}$ where $l = (w_1, \nu)$ (with ν the map which internally corresponds to the mapping:$\tau \mapsto \tau.s_1 f_1(\tau^{-1})$) is a natural retraction of $k_1 : K_1[\underline{f}_1] \rightarrowtail W_1$:

$$\begin{tikzcd}
A_1 \arrow[r, dotted, "k_1^A"] \arrow[d, "i_1"'] & W'_1 \arrow[r, "\lambda"] \arrow[d, "j_1"'] & A_1 \arrow[d, "i_1"] \\
K_1[\underline{f}_1] \arrow[r, "k_1"'] & W_1 \arrow[r, "l"'] & K_1[\underline{f}_1]
\end{tikzcd}$$

This produces a natural section k_1^A of λ. The object W'_1 clearly determines a subgraph $\underline{W}'_1$ of the groupoid $\underline{W}_1$. Since $\mathbb{E}$ is Mal'cev, $\underline{W}'_1$ is actually a subgroupoid such that the following square is a pullback in $Grd_X\mathbb{E}$:

$$\begin{tikzcd}
\underline{A}_1 \arrow[r, tail, "k_1^A"] \arrow[d, "\underline{i}_1"'] & \underline{W}'_1 \arrow[d, "\underline{j}_1"] \\
\underline{K}_1[\underline{f}_1] \arrow[r, tail, "\underline{k}_1"'] & \underline{W}_1
\end{tikzcd}$$

□

1.3 Connected equivalence relations

Let us now point out some properties related to commutator theory. First consider R and S two equivalence relations on an object X in any finitely complete category $\mathbb{E}$. Let us recall the following definition from [9]:

Definition 1.1. *A connector on the pair (R, S) is a morphism*

$$p : R \times_X S \to X,\ (xRySz) \mapsto p(x, y, z)$$

which satisfies the identities :

$$1)\ xSp(x, y, z) \quad 1')\ zRp(x, y, z)$$

$$2)\ p(x, y, y) = x \quad 2')\ p(y, y, z) = z$$

$$3)\ p(x, y, p(y, u, v)) = p(x, u, v) \quad 3')\ p(p(x, y, u), u, v) = p(x, y, v)$$

In set theoretical terms, Condition 1 means that with any triple $xRySz$ we can associate a square:

$$\begin{tikzcd}
x \arrow[r, dotted, dash, "S"] \arrow[d, dash, "R"'] & p(x,y,z) \arrow[d, dotted, dash, "R"] \\
y \arrow[r, dash, "S"'] & z.
\end{tikzcd}$$

More acutely, any connected pair produces a double equivalence relation in $\mathbb{E}$:

$$\begin{tikzcd}
R \times_X S \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "(d_0.p_0,p)"'] \arrow[d, shift right=2, "p_0"'] \arrow[d, shift left=2, "(p,d_1.p_0)"] & S \arrow[l] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] \\
R \arrow[u] \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & X \arrow[l] \arrow[u]
\end{tikzcd}$$

Example 1) An emblematical example is produced by a given discrete fibration $\underline{f}_1 : R \to \underline{Z}_1$ with R an equivalence relation. For that consider the following diagram:

$$\begin{tikzcd}
R[f_1] \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "p_0"'] \arrow[d, shift right=2, "R(d_0)"'] \arrow[d, shift left=2, "R(d_1)"] & R \arrow[l] \arrow[r, "f_1"] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & Z_1 \arrow[d, shift right=2, "z_1"'] \arrow[d, shift left=2, "z_0"] \\
R[f_0] \arrow[u] \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "p_0"'] & X \arrow[l] \arrow[u] \arrow[r, "f_0"'] & Z_0 \arrow[u]
\end{tikzcd}$$

It is clear that $R[f_1]$ is isomorphic to $R[f_0] \times_X R$ and that the map

$$p : R[f_1] \overset{p_0}{\to} R \overset{d_1}{\to} X$$

determines a connector.
2) Given any groupoid $\underline{Z}_1$, we have such a discrete fibration $R[z_0] \to \underline{Z}_1$:

$$\begin{tikzcd}
R[z_0] \arrow[r, "\zeta_2"] \arrow[d, shift right=2, "z_1"'] \arrow[d, shift left=2, "z_0"] & Z_1 \arrow[d, shift right=2, "z_1"'] \arrow[d, shift left=2, "z_0"] \\
Z_1 \arrow[u] \arrow[r, "z_1"'] & Z_0 \arrow[u]
\end{tikzcd}$$

which implies a connector on the pair $(R[z_0], R[z_1])$ explicited by the following diagram:

$$\begin{tikzcd} x \arrow[r, "\phi"] \arrow[dr, "\chi"] & t \\ y \arrow[r, "\psi"'] & z \end{tikzcd} \longmapsto \begin{tikzcd} x \arrow[r, dotted, no head] \arrow[dr, dotted, no head] & t \\ y \arrow[ur, "\phi.\chi^{-1}.\psi"'] \arrow[r, dotted, no head] & z \end{tikzcd}$$

The converse is true as well, see [13] and [9]; given a reflexive graph :

$$\begin{tikzcd} Z_1 \arrow[r, shift left=2, "z_1"] \arrow[r, shift right=2, "z_0"'] & Z_0 \arrow[l, "s_0"'] \end{tikzcd}$$

any connector on the pair $(R[z_0], R[z_1])$ determines a groupoid structure.

Now let us observe that:

Proposition 1.2. *Suppose p is a connector for the pair (R, S). Then the following reflexive graph is underlying a groupoid we shall denote by $R\sharp S$:*

$$\begin{tikzcd} R \times_X S \arrow[r, shift left=2, "d_1.p_1"] \arrow[r, shift right=2, "d_0.p_0"'] & X \arrow[l] \end{tikzcd}$$

Proof. Thank to the Yoneda embedding, it is enough to prove it in *Set*. This is straightforward just setting:

$$(zRuSv).(xRySz) = xRp(u, z, y)Sv$$

The inverse of the arrow $xRySz$ is $zRp(x, y, z)Sx$. □

Remark 1) When $R \cap S = \Delta X$, the groupoid $R\sharp S$ is actually an equivalence relation.
2) Let $\underline{Z}_1$ be any reflexive graph. We noticed it is a groupoid if and only if $[R[z_0], R[z_1]] = 0$. It is easy to check that:

$$R[z_0]\sharp R[z_1] \simeq \underline{Z}_1^2$$

where $\underline{Z}_1^2$ is the groupoid whose objects are the maps and morphisms the commutative squares, in other words the groupoid which represents the natural transformations between functors with codomain $\underline{Z}_1$. Next we have:

Proposition 1.3. *Given a discrete fibration $\underline{f}_1 : R \to \underline{Z}_1$, the associated internal functor $R[f_0]\sharp R \to R \to \underline{Z}_1$ is fully faithful.*

Proof. This functor $\underline{\phi}_1$ is given by the following diagram:

$$\begin{tikzcd} R[f_1] \arrow[rr, "f_1.p_i"] \arrow[d, shift right=3, "d_1.p_1"'] \arrow[d, shift left=3, "d_0.R(d_0)"] & & Z_1 \arrow[d, shift right=3, "z_1"'] \arrow[d, shift left=3, "z_0"] \\ X \arrow[u] \arrow[rr, "f_0"'] & & Z_0 \arrow[u] \end{tikzcd}$$

Thank to the Yoneda embedding, it is enough to prove it is fully faithful in Set. Suppose you have a map $\alpha : f(x) \to f(x')$. Since $\underline{f}_1$ is a discrete fibration, there is an object $z \in X$ such that zRx' and $f(z, x') = \alpha$. This implies that $f(z) = f(x)$. Accordingly $xR[f_0]zRx'$ is a map in $R[f_0]\sharp R$ above α. Suppose now that $\phi_1(xR[f_0]zRx') = \phi_1(xR[f_0]z'Rx')$. This means $f(z, x') = f(z', x')$. Since $\underline{f}_1$ is a discrete fibration, we have necessarily $z = z'$. $\square$

In a Mal'cev category, the conditions 2) imply the other ones, and moreover a connector is necessarily unique when it exists, and thus the existence of a connector becomes a property.

Example 1.1. *By Proposition 3.6, Proposition 2.12 and definition 3.1 in [17], two relations R and S in a Mal'cev variety $\mathcal{V}$ are connected if and only if $[R, S] = 0$ in the sense of Smith [19]. Accordingly we shall denote a connected pair of equivalence relations by the formula $[R, S] = 0$.*

Proposition 1.4. *Suppose $\mathbb{E}$ is Mal'cev and $[R, S] = 0$. Then the following diagram (which is a pullback) is a pushout in $Grd\mathbb{E}$:*

$$\begin{tikzcd} \Delta X \arrow[r, tail] \arrow[d] & S \arrow[d, "i_S"] \\ R \arrow[r, tail, "i_R"'] & R\sharp S \end{tikzcd}$$

Proof. Let $\underline{f}_1$ and $\underline{g}_1$ be two functors making the following diagram commute:

$$\begin{tikzcd} \Delta X \arrow[r, tail] \arrow[d] & S \arrow[d, "\underline{g}_1"] \\ R \arrow[r, "\underline{f}_1"'] & \underline{Z}_1 \end{tikzcd}$$

We have $f_0 = g_0 (= h_0) : X \to Z_0$. Wanting $\underline{h}_1.i_R = \underline{f}_1$ and $\underline{h}_1.i_S = \underline{g}_1$ implies that $h_1 : R \times_X S \to Z_1$ is given by the formula $h_1(xRySz) = g(y, z).f(x, y)$. This defines a functor $\underline{h}_1 : R\sharp S \to \underline{Z}_1$ if and only if, for all $xRySz$, we have $g(y, z).f(x, y) = f(p(x, y, z), z).g(x, p(x, y, z))$. This is necessarily the case when $\mathbb{E}$ is Mal'cev. For that, let us introduce the following relation T on $R \times X$ defined by

$$(xRy)Tz \quad \Leftrightarrow \quad ySz \quad \wedge \quad g(y, z).f(x, y) = f(p(x, y, z), z).g(x, p(x, y, z))$$

For all $xRySz$, we have necessarily $(xRy)Ty$, $(yRy)Ty$ and $(yRy)Tz$. Accordingly, for all $xRySz$, we have necessarily $(xRy)Tz$. $\square$

According to Remark 1 above, when we have $R \cap S = \Delta X$, the groupoid $R\sharp S$ being an equivalence relation, we have also $R\sharp S = R \vee S$

1.4 The regular context

We shall end this section with a useful remark concerning pullbacks of split epimorphisms and discrete fibrations in the regular context:

Proposition 1.5. *Suppose $\mathbb{E}$ Mal'cev and regular [1]. Then any (downward) pullback of split epimorphism along a regular epimorphism produces an upward pushout:*

$$\begin{tikzcd} X \arrow[r, two heads, "f"] \arrow[d, shift right, "x"'] & Z \arrow[d, shift right, "z"'] \\ X' \arrow[r, two heads, "f'"'] \arrow[u, shift right, "r"'] & Z \arrow[u, shift right, "t"'] \end{tikzcd}$$

Any disrete fibration $\underline{f}_1 : \underline{X}_1 \to \underline{Z}_1$ with f_0 regular epimorphic is cocartesian with respect to the functor $()_0 : Grd\mathbb{E} \to \mathbb{E}$.

Proof. Consider the following diagram:

$$\begin{tikzcd} R[f] \arrow[r, shift left=1.5, "p_1"] \arrow[r, shift right=1.5, "p_0"'] \arrow[d, shift right, "R(x)"'] & X \arrow[l] \arrow[r, two heads, "f"] \arrow[d, shift right, "x"'] \arrow[rr, bend left, "h"] & Z \arrow[r, dotted, "\phi"] \arrow[d, shift right, "z"'] & W \\ R[f'] \arrow[r, shift left=1.5, "p_1"] \arrow[r, shift right=1.5, "p_0"'] \arrow[u, shift right, "R(r)"'] & X' \arrow[l] \arrow[r, two heads, "f'"'] \arrow[u, shift right, "r"'] & Z \arrow[u, shift right, "t"'] \arrow[ur, "g"'] & \end{tikzcd}$$

with $g.f' = h.r$. We must find a map ϕ which makes the triangles commute. Since f is a regular epimorphism, this is the case if and only if $R[f] \subset R[h]$. Now the left hand side squares are still pullbacks. Since $\mathbb{E}$ is Mal'cev, the pair $(R(r) : R[f'] \to R[f], s_0 : X \to R[f])$ is jointly strongly epic. So that the inclusion in question can be checked by composition with this pair. Checking by s_0 is straightforward. Checking by $R(r)$ is guaranteed by the existence of the map g. Let $\underline{f}_1 : \underline{X}_1 \to \underline{Z}_1$ be any disrete fibration with f_0 regular epimorphic

$$\begin{tikzcd} R[f_1] \arrow[r, shift left=1.5, "p_1"] \arrow[r, shift right=1.5, "p_0"'] \arrow[d, shift right=2, "R(x_0)"'] \arrow[d, shift left=2, "R(x_1)"] & X_1 \arrow[l] \arrow[r, two heads, "f_1"] \arrow[d, shift right=2, "x_0"'] \arrow[d, shift left=2, "x_1"] \arrow[rr, bend left, "h_1"] & Z_1 \arrow[r, dotted, "g_1"] \arrow[d, shift right=2, "z_0"'] \arrow[d, shift left=2, "z_1"] & W_1 \arrow[d, shift right=2, "w_0"'] \arrow[d, shift left=2, "w_1"] \\ R[f_0] \arrow[r, shift left=1.5, "p_1"] \arrow[r, shift right=1.5, "p_0"'] \arrow[u] & X \arrow[l] \arrow[r, two heads, "f_0"'] \arrow[u] & Z_0 \arrow[r, "g_0"'] \arrow[u] & W_0 \arrow[u] \end{tikzcd}$$

where the pair $(h_0 = g_0.f_0, h_1)$ is underlying an internal functor $\underline{X}_1 \to \underline{W}_1$. By the previous part of this proof we have a map g_1 such that $g_1.s_0 = s_0.g_0$ and $g_1.f_1 = h_1$. The end of the proof (checking the commutation with the legs of the groupoids) is straightforward. $\square$

2 Non pointed additive categories

The result asserted by Theorem 1.2 is actually underlying a stronger property which allows us to enrich the classification of non pointed additive categories. The weaker notion is the one of naturally Mal'cev category [16]. A naturally Mal'cev category is a Mal'cev category in which any pair (R, S) of equivalence relations on an object X is connected. The stronger one is the notion of essentially affine category [4], namely finitely complete category with existence of pushouts of split monomorphisms along any map and such that, given any commutative square of split epimorphisms, the downward square is a pullback if and only if the upward square is a pushout:

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, shift right, "f'"'] & X \arrow[d, shift right, "f"'] \\ Y' \arrow[r, "h"'] \arrow[u, shift right, "s'"'] & Y \arrow[u, shift right, "s"'] \end{tikzcd}$$

This is equivalent to saying that any change of base functor $h^* : Pt_Y\mathbb{E} \to Pt_{Y'}\mathbb{E}$ with respect with the fibration of points [4] is an equivalence of categories. Recall that the category $\mathbb{E}$ is naturally Mal'cev if and only if any fibre $Pt_Y\mathbb{E}$ is additive [5], and this last point is implied by the fact that the change of base functors h^* are equivalence of categories. The slice and coslice categories of a finitely complete additive category $\mathbb{A}$ are essentially affine. Notice then that, thanks to the Moore normalization, $\mathbb{A}/X$ is isomorphic to the fibre $Grd_X\mathbb{A}$. When the category $\mathbb{E}$ is pointed, the notions of naturally Mal'cev and essentially affine categories coincide with the notion of finitely complete additive category.

There is a well known intermediate notion, namely protomodular naturally Mal'cev categories (recall that a category is protomodular when any change of base functor h^* is conservative). This is the case, for instance, for the full subcategory $Ab(Gp/Y)$ of the slice category Gp/Y whose object are group homomorphisms with abelian kernel. It is easy to check that the naturally Mal'cev protomodular category $Ab(Gp/Y)$ is not essentially affine, since this would imply, considering the following diagram in $Ab(Gp/Y)$, that any split epimorphism $f : X \to Y$ with abelian kernel A is such that $X = A \oplus Y$:

$$\begin{tikzcd} A \arrow[r] \arrow[d, shift left] & X \arrow[d, shift left, "f"] \\ 1 \arrow[r] \arrow[u, shift left] \arrow[dr] & Y \arrow[u, shift left, "s"] \arrow[d, "1_Y"] \\ & Y \end{tikzcd}$$

The fibres $AbGrd_X\mathbb{E}$ of section 1.1 are other examples of naturally Mal'cev protomodular categories which are not essentially affine.

2.1 Penessentially affine categories

Let us introduce now two intermediate notions. Here is the first one:

Definition 2.1. *A finitely complete category $\mathbb{E}$ is said to be antepenessentially affine when, for any square of split epimorphisms as above, the upward square is a pushout as soon as the downward square is a pullback.*

The antepenessensially affine categories are stable by slice and coslice categories. According to Proposition 4 in [4], the previous definition is equivalent to saying that any change of base functor $h^* : Pt_Y\mathbb{E} \to Pt_{Y'}\mathbb{E}$ is fully faithful. So, any essentially affine category is antepenessentially affine. On the other hand any fully faithful functor being conservative, any antepenessentially affine category is necessarily protomodular. Moreover any antepenessentially affine category is naturally Mal'cev for the same reasons as the essentially affine categories. On the other hand, again for the same reason as above, the protomodular naturally Mal'cev category $Ab(Gp/Y)$ and $AbGrd_X\mathbb{E}$ ($\mathbb{E}$ finitely complete) are not antepenessentially affine.

Definition 2.2. *A finitely complete category $\mathbb{E}$ is said to be penessentially affine when it is antepenessentially affine and such that any (fully faithful) change of base functor h^* is saturated on subobjects.*

Recall that a left exact conservative functor $U : \mathbb{C} \to \mathbb{D}$ is saturated on subobjects when any subobject $j : d \rightarrowtail U(c)$ is isomorphic to the image by U of some (unique up to isomorphism) subobject $i : c' \rightarrowtail c$. So, being penessentially affine implies that, given any downward parallelistic pullback as below and any pointed subobject $j' : U' \rightarrowtail X'$ (with the retraction ϕ' of σ' such that $f'.j' = \phi'$):

$$\begin{tikzcd}
& X' \arrow[rr, "g"] \arrow[ddl, shift left] & & X \arrow[ddl, shift left, "f"] \\
U' \arrow[ur, bend left, "j'"] \arrow[rr, dotted, "\gamma"] \arrow[d, shift left] & & U \arrow[ur, bend left, "j"] & \\
Y' \arrow[u, shift left, "\sigma'"] \arrow[uur, shift left] \arrow[rr, "h"'] & & Y \arrow[u, dotted, "\sigma"] \arrow[uur, shift left, "s"'] &
\end{tikzcd}$$

there is a (dotted) pushout of σ' along h which makes the upper upward diagram a pullback. The penessensially affine categories are stable by slice and coslice categories. Here is our first major structural point:

Theorem 2.1. *Suppose $\mathbb{E}$ is Mal'cev. Then any fibre $Grd_X\mathbb{E}$ is penessentially affine.*

Proof. Let us show first it is antepenesentially affine. Consider the following right hand side downward pullback in $Grd_X\mathbb{E}$:

$$\begin{tikzcd}
\underline{K}_1[\underline{f}'_1] \arrow[r, tail, "\underline{k}_1"] \arrow[d, shift right] & \underline{W}'_1 \arrow[r, "\underline{g}_1"] \arrow[d, shift right, "\underline{f}'_1"'] & \underline{W}_1 \arrow[d, shift right, "\underline{f}_1"'] \\
\Delta X \arrow[r, tail, "\underline{\alpha}_1\underline{Z}'_1"'] \arrow[u, shift right] & \underline{Z}'_1 \arrow[r, "\underline{h}_1"'] \arrow[u, shift right, "\underline{s}'_1"'] & \underline{Z}_1 \arrow[u, shift right, "\underline{s}_1"']
\end{tikzcd}$$

Complete the diagram by the left hand side downward pullback, then the whole downward rectangle is a pullback. Now the upward left hand side square is a pushout as well as the whole upward rectangle. Accordingly the right hand side upward square is a pushout. The fact that the change of base functor along $\underline{h}_1$ is saturated on subobjects is checked in the same way, thanks to Theorem 1.2. □

2.2 Normal subobjects

Any penessentially affine category is protomodular, and consequently yields an intrinsic notion of normal subobject. The aim of this subsection is to show that any penessentially affine category is similar to an additive category insofar as any monomorphism is normal. Let us begin by the following more general observation:

Proposition 2.1. *Let $\mathbb{E}$ be a naturally Mal'cev category. Then, given any monomorphism $s : Y \rightarrowtail X$ split by f, there is a unique equivalence relation R on X such that s is normal to R and $R \cap R[f] = \Delta X$. In any protomodular naturally Mal'cev category, and a fortiori in any antepenessentially affine category, a split monomorphism is normal.*

Proof. Consider the following diagram:

$$\begin{tikzcd}
Y \times Y \arrow[r, tail, "s \times 1"] \arrow[d, shift right, "p_0"'] & X \times Y \arrow[r, "f \times 1"] \arrow[d, shift right, "p_X"'] & Y \times Y \arrow[d, shift right, "p_0"'] \\
Y \arrow[r, tail, "s"'] \arrow[u, shift right, "s_0"'] & X \arrow[r, shift right, "f"'] \arrow[u, shift right, "(1,f)"'] & Y \arrow[l, shift right, "s"'] \arrow[u, shift right, "s_0"']
\end{tikzcd}$$

The right hand side downward square is a pullback of split epimorphisms in $\mathbb{E}$, and consequetly a product in the additive fibre $Pt_Y\mathbb{E}$. Accordingly the left hand side upward square is a pushout. So the map $p_1 : Y \times Y \to Y$ produces a factorization $\psi : X \times Y \to X$ such that $\psi.(1, f) = 1_X$ and $\psi.(s \times 1) = s.p_1$. Whence a reflexive graph $(p_X, \psi) : X \times Y \rightrightarrows X$ and thus a groupoid $\underline{X}_1$ since $\mathbb{E}$ is naturally Mal'cev and thus satisfies the *Lawvere condition* following which any reflexive graph is a groupoid, see [16]. We can check $f.\psi = p_1.(f \times 1)$, thus we have a discrete fibration $\underline{f}_1 : \underline{X}_1 \to \nabla Y$. The codomain ∇Y being an equivalence relation, the domain $\underline{X}_1$ is an equivalence relation we shall denote by R. The monomorphism s is normal to R since the left hand side downward square above is also a pullback. Moreover, by commutation of limits, the following square is a pullback in $Grd\mathbb{E}$:

$$\begin{tikzcd}
R \cap R[f] \arrow[r] \arrow[d] & \Delta Y \arrow[d] \\
R = \underline{X}_1 \arrow[r, "\underline{f}_1"'] & \nabla Y
\end{tikzcd}$$

Since $\underline{f}_1$ is discrete fibration, the upper horizontal map is a discrete fibration and necessarily we have $R \cap R[f] = \Delta X$.

Now suppose we have another equivalence S on X which is normal to s and such that $S \cap R[f] = \Delta X$. By the first part of the assumption, there is a map $\tilde{s}$ which makes the following downward left hand side square a pullback:

$$\begin{tikzcd}
Y \times Y \arrow[r, tail, "\tilde{s}"] \arrow[d, shift right, "p_0"'] & S \arrow[r, "(f\times f).(d_0,d_1)"] \arrow[d, shift right, "d_0"'] & Y \times Y \arrow[d, shift right, "p_0"'] \\
Y \arrow[r, tail, "s"'] \arrow[u, shift right, "s_0"'] & X \arrow[u, shift right, "s_0"'] \arrow[r, shift right, "f"'] & Y \arrow[u, shift right, "s_0"'] \arrow[l, shift right, "s"']
\end{tikzcd}$$

and produces a splitting $\tilde{s}$ of $(f \times f).(d_0, d_1)$. Accordingly we have a split epimorphism in the fibre $Pt_Y\mathbb{E}$:

$$\begin{tikzcd}
S \arrow[rr, shift left, "(f\times f).(d_0,d_1)"] \arrow[dr, shift left] & & Y \times Y \arrow[ll, shift left, "\tilde{s}"] \arrow[dl, shift left, "p_0"] \\
& Y \arrow[ul, shift left] \arrow[ur, shift left, "s_0"] &
\end{tikzcd}$$

So, in this additive fibre, the domain of this split epimorphism is isomorphic to the product of its codomain by its kernel. But its kernel is $S \cap R[f] \overset{d_0}{\to} X \overset{f}{\to} Y$. We have $S \cap R[f] = \Delta X$ by assumption, and thus $S \simeq X \times Y$. $\square$

Now, when $\mathbb{E}$ is penessentially affine, we have more:

Theorem 2.2. *Let $\mathbb{E}$ be a penessentially affine category. Then any monomorphism in $\mathbb{E}$ is normal.*

Proof. Let $m : X' \rightarrowtail X$ be any subobject. The change of base functor $m^* : Pt_X\mathbb{E} \to Pt_{X'}\mathbb{E}$ is saturated on subobjects. Then consider the following diagram:

$$\begin{tikzcd}
& X' \times X \arrow[rr, tail, "m\times 1"] \arrow[dl, shift left] & & X \times X \arrow[dl, shift left, "p_0"] \\
X' \times X' \arrow[ur, bend left, "1\times m"] \arrow[rr, dotted, "\tilde{m}"] \arrow[d, shift right, "p_0"'] & & R \arrow[ur, bend left, "j"] & \\
X' \arrow[u, shift right] \arrow[uur, shift left] \arrow[rr, tail, "m"'] & & X \arrow[u, dotted, "s_0"] \arrow[uur, shift left, "s_0"] &
\end{tikzcd}$$

The map $1 \times m$ determines a pointed subobject of $(p_{X'}, (1, m)) : X' \times X \leftrightarrows X'$. This produces a pointed subobject $j : R \rightarrowtail X \times X$, and thus an equivalence relation on X. Moreover the following vertical square is a pullback, which means that m is normal to R:

$$\begin{tikzcd}
X' \times X' \arrow[r, tail, "\tilde{m}"] \arrow[d, shift right, "p_0"'] & R \arrow[d, shift right, "d_0"'] \\
X' \arrow[r, tail, "m"'] \arrow[u, shift right, "s_0"'] & X \arrow[u, shift right, "s_0"']
\end{tikzcd}$$

□

According to Theorem 2.1 we have the following:

Corollary 2.1. *Let $\mathbb{E}$ be any Mal'cev category. Then, in a fibre $Grd_X\mathbb{E}$, any monomorphism is normal.*

2.3 Classification table

We give, here, the classification table of our "non pointed additive" categories by decreasing order of generality:

Category $\mathbb{C}$	**Fibration:** $\pi : Pt(\mathbb{C}) \to \mathbb{C}$	**Example**
naturally Mal'cev	additive fibres	$AutMal$
protomodular and naturally Mal'cev	additive fibres + conservative change of base functors	$AbGrd_X\mathbb{E}$ when $\mathbb{E}$ finitely complete
antepenessent. aff.	fully faithful change of base functors	$Grd_X\mathbb{E}$ when $\mathbb{E}$ Gumm
penesentially affine	fully faithful saturated on subobj. change of base functors	$Grd_X\mathbb{E}$ when $\mathbb{E}$ Mal'cev
essentially affine	change of base functors are equivalences	$Grd_X\mathbb{A}$ when $\mathbb{A}$ fin. complete + additive

All the given examples do not belong to the next class. The category $AutMal$ is the variety of autonomous Mal'cev operations. A category is Gumm when it is finitely complete and satisfies the *Shifting Lemma* [10]. This means that, given any triple R, S, T of equivalence relations on an object X with $R \cap S \leq T$ the situation given by the continuous lines

$$\begin{tikzcd}
x \arrow[r, dash, "S"] \arrow[d, dash, "R"'] \arrow[d, dash, bend right=70, "T"'] & t \arrow[d, dash, "R"] \arrow[d, dash, dotted, bend left=70, "T"] \\
y \arrow[r, dash, "S"'] & z
\end{tikzcd}$$

implies that tTz. A variety of universal algebras is Gumm if and only if it is congruence modular [15]. The Gumm categories are stable under slicing. Any regular Mal'cev category is Gumm. The table will be complete with the following:

Proposition 2.2. *Suppose $\mathbb{E}$ is a Gumm category. Then any internal groupoid is abelian. Accordingly any fibre $Grd_X\mathbb{E}$ is naturally Mal'cev. Furthermore, any fibre $Grd_X\mathbb{E}$ is antepenessentially affine.*

Proof. Any internal Mal'cev operation on an object X in a Gumm category is unique when it exists and necessarily associative and *commutative*, see Corollary 3.4 in [10]. This implies immediately that any internal group is abelian. The Gumm categories being stable under slicing, any internal groupoid is abelian by Proposition 1.1. In order to show that any fibre $Grd_X\mathbb{E}$ is antepenessentially affine, in the same way as in the proof of Theorem 2.1, it is sufficient to show that the square below Proposition 1.1 is a pushout, and that consequently the conditions of Lemma 1.1 are satisfied. For that, using the notations of this lemma, let us introduce the following mapping:

$$\tau : K_1[\underline{f}_1] \times_X Z_1 \to V_1 \quad (\gamma, \phi) \mapsto t_1\phi.h_1\gamma.t_1\phi^{-1}.h_1(s_1\phi.\gamma.s_1\phi^{-1})^{-1}$$

where $K_1[\underline{f}_1] \times_X Z_1 = \{(\gamma, \phi)/dom\gamma = dom\phi\}$. The following diagram will complete the proof:

$$\begin{tikzcd}
(\gamma, 1_x) \arrow[r, dash, "p_{K_1}"] \arrow[d, dash, "p_{Z_1}"'] \arrow[d, dash, bend right=60, "\tau"'] & (\gamma, \phi) \arrow[d, dash, "p_{Z_1}"] \arrow[d, dotted, dash, bend left=60, "\tau"] \\
(1_x, 1_x) \arrow[r, dash, "p_{K_1}"'] & (1_x, \phi)
\end{tikzcd}$$

where a kernel equivalence relation is denoted by the same symbol as the map itself. Clearly $R[p_{Z_1}] \cap R[p_{K_1}] \leq R[\tau]$. Moreover $\tau(1_x, 1_x) = 1_x = \tau(\gamma, 1_x)$ implies $\tau(1_x, \phi) = 1_y = \tau(\gamma, \phi)$, and $1_y = \tau(\gamma, \phi)$ is our condition. $\square$

The fibres $Grd_X\mathbb{E}$ are not penessentialy in general, since the proof of Theorem 1.2 cannot apply to here.

2.4 Quasi-pointed penessentially affine categories

We noticed that the fibres $Grd_X\mathbb{E}$ are quasi-pointed. This particularity leads to further interesting observations. We recalled that a category is quasi-pointed when it has an initial object 0 such that the unique map $\varpi : 0 \rightarrowtail 1$ is a monomorphism. The category $\mathbb{E}/0 = Pt_0\mathbb{E}$ is then a full subcategory of $\mathbb{E}$ stable under products and pullbacks. The inclusion $Pt_0\mathbb{E} \rightarrowtail \mathbb{E}$ is a discrete fibration. So it is stable by subobject, and by equivalence relation. Consequently, when moreover $\mathbb{E}$ is regular, $Pt_0\mathbb{E}$ is stable under regular epimorphisms, which means that, when the domain of a regular epimorphism belongs to this subcategory, the codomain belongs to it as well. The quasi-pointed categories are stable by slice categories.

Definition 2.3. *In a finitely complete quasi-pointed category, we shall call endosome of an object X the object EnX defined by the following pullback:*

$$\begin{tikzcd}
EnX \arrow[r, "\epsilon X"] \arrow[d] & X \arrow[d] \\
0 \arrow[r, "\varpi"'] & 1
\end{tikzcd}$$

This construction determines a left exact functor $En : \mathbb{E} \to \mathbb{E}/0 = Pt_0\mathbb{E}$ which is a right adjoint to the inclusion. When $\mathbb{E}$ is regular, this functor preserves the regular epimorphisms. It is clear that when $\mathbb{E}$ is pointed this functor disappears, since it is nothing but the identity functor. Thanks to the following upper pullback, where the map $\bar{\epsilon}X$ is the unique map making the left hand side square commute and such that $p_1.\bar{\epsilon}X = \epsilon X$, the functor En allows us to associate with any equivalence relation R on X a subobject I of EnX which we call the *endonormalization* of the equivalence relation R:

$$\begin{tikzcd}
I \arrow[r] \arrow[d, "i"'] & R \arrow[d, "{(d_0,d_1)}"] \\
EnX \arrow[r, "\bar{\epsilon}X"] \arrow[d, shift right] & X \times X \arrow[d, shift right, "p_0"'] \\
0 \arrow[r, tail, "\alpha_X"'] \arrow[u, shift right] & X \arrow[u, shift right, "s_0"']
\end{tikzcd}$$

Remark The upper left hand side pullback, in the following diagram whose any square is a pullback, shows i is nothing but the classical normalization of the equivalence relation EnR (on the object EnX) in the pointed category $Pt_0\mathbb{E}$ since we have obviously $\bar{\epsilon}X = \epsilon X \times \epsilon X.(0,1)$:

$$\begin{tikzcd}
I \arrow[r] \arrow[d, "i"'] & EnR \arrow[r] \arrow[d, "{En(d_0,d_1)}"'] & R \arrow[d, "{(d_0,d_1)}"] \\
EnX \arrow[r, "{(0,1)}"'] \arrow[rr, bend left, "\bar{\epsilon}X"] \arrow[d] & EnX \times EnX \arrow[r, "\epsilon X \times \epsilon X"'] \arrow[d] & X \times X \arrow[d, "p_0"'] \\
0 \arrow[r, tail] & EnX \arrow[r, tail, "\epsilon X"'] \arrow[d] & X \arrow[d] \\
 & 0 \arrow[r] & 1
\end{tikzcd}$$

Next we have:

Proposition 2.3. *Suppose $\mathbb{E}$ penessentially affine and quasi-pointed. Then the endonormalization construction is bijective.*

Proof. This is an immediate consequence of the fact that the change of base functor α_X^* is saturated on subobjects. □

3 Baer sums and Baer categories

When the naturally Mal'cev category $\mathbb{E}$ is moreover efficiently regular, there is a direction functor $d : \mathbb{E}_g \to Ab(\mathbb{E})$ where $\mathbb{E}_g$ is the full subcategory of objects with global support and $Ab(\mathbb{E}) = Pt_1\mathbb{E}$ is the category of global elements of $\mathbb{E}$ (which necessarily determine an internal abelian group structure in $\mathbb{E}$). This functor d is a cofibration whose fibres are canonically endowed with a tensor product, the so-called Baer sum, see [6]. Our aim will be to show there is, in the stronger context of penessensially affine categories, an alternative and simpler description of this Baer sum which mimics closely the classical Baer sum construction on exact sequences in abelian categories.

Recall the following [8]:

Definition 3.1. *A category $\mathbb{C}$ is said to be efficiently regular when it is regular and such that any equivalence relation T on an object X which is a subobject $j : T \rightarrowtail R$ of an effective equivalence relation on X by an effective monomorphism (which means that j is the equalizer of some pair of maps in $\mathbb{C}$), is itself effective.*

The efficiently regular categories are stable under slice and coslice categories. The category *GpTop* (resp. *AbTop*) of (resp. abelian) topological groups is efficiently regular, but not Barr exact. A finitely complete regular additive category $\mathbb{A}$ is efficiently regular if and only if the kernel maps are stable under composition. In this context we can add some interesting piece of information:

Proposition 3.1. *Suppose $\mathbb{E}$ naturally Mal'cev and efficiently regular. Then, given any monomorphism $s : Y \rightarrowtail X$ split by f, the equivalence relation R on X asserted by Proposition 2.1, to which s is normal, is effective and produces a direct product decomposition $X \simeq Q \times Y$ where Q is the quotient of R.*

Proof. According to Proposition 2.1, we have a discrete fibration $\underline{f}_1 : R \to \nabla Y$. Certainly ∇Y is effective, and thus R is effective, see [8]. Now consider the following diagram where Q is the quotient of R:

$$\begin{tikzcd} R \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] \arrow[d, shift right, "f_1"'] & X \arrow[l] \arrow[r, two heads, "q"] \arrow[d, shift right, "f"'] & Q \arrow[d] \\ Y \times Y \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "p_0"'] \arrow[u, shift right, "s_1"'] & Y \arrow[l] \arrow[r] \arrow[u, shift right, "s"'] & 1 \end{tikzcd}$$

Since the left hand side squares are pullbacks, then, according to the Barr-Kock theorem in regular categories, the right hand side square is a pullback, which gives us the direct product decomposition $\square$

In the same order of idea, recall that, in a category $\mathbb{E}$ which is naturally Mal'cev and efficiently regular, the *direction* of an object X with global support is given by the following diagram where, $\pi : X \times X \times X \to X$, written for π_X, is the value at X of the natural Mal'cev operation:

$$\begin{tikzcd} X \times X \times X \arrow[r, shift left=2, "p_2"] \arrow[r, shift right=2, "(p_0.p_0,\pi)"'] \arrow[d, shift right, "p_0"'] & X \times X \arrow[l] \arrow[r, two heads, "q_X"] \arrow[d, shift right, "p_0"'] & dX \arrow[d, shift right] \\ X \times X \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "p_0"'] \arrow[u, shift right, "s_0"'] & X \arrow[l] \arrow[r, two heads] \arrow[u, shift right, "s_0"'] & 1 \arrow[u, shift right, "\eta_X"'] \end{tikzcd}$$

The quotient q_X of the upper equivalence relation does exist in the efficiently regular category $\mathbb{E}$ since the vertical diagram is a discrete fibration between equivalence relations, see [8]. Actually the downward right hand side square is necessarily a pullback ($\mathbb{E}$ being regular) and the upward square a pushout (in a naturally Mal'cev category $\mathbb{E}$, the pair $(s_0, s_1) : X \times X \rightrightarrows X \times X \times X$, composing the edge of a pushout, is jointly strongly epic).

Proposition 3.2. *Suppose $\mathbb{D}$ efficiently regular. Then any fibre $Grd_X\mathbb{D}$ is efficiently regular.*

Proof. The regular epimorphisms in $Grd_X\mathbb{D}$ are the internal functors $\underline{f}_1 : \underline{X}_1 \to \underline{Z}_1$ such that the map $f_1 : X_1 \to Z_1$ is a regular epimorphisms in $\mathbb{D}$. They are consequently stable under pullbacks. On the other hand, suppose the equivalence relation $\underline{R}_1 \rightrightarrows \underline{X}_1$ is effective. Then the underlying equivalence relation in $\mathbb{D}$ is still effective. Let $q_1 : Z_1 \twoheadrightarrow Q_1$ be its quotient in $\mathbb{D}$. Then clearly the induced reflexive graph $Q_1 \rightrightarrows X$ is underlying a groupoid $\underline{Q}_1$ and $\underline{R}_1 \rightrightarrows \underline{Z}_1$ is the kernel relation of the internal functor $\underline{q}_1 : \underline{Z}_1 \twoheadrightarrow \underline{Q}_1$ in $Grd_X\mathbb{D}$. Accordingly $Grd_X\mathbb{D}$ is regular when $\mathbb{D}$ is regular. Suppose $\underline{j}_1 : \underline{S}_1 \rightarrowtail \underline{R}_1$ is an effective monomorphism in $Grd_X\mathbb{D}$. Then the underlying monomorphism $j_1 : S_1 \rightarrowtail R_1$ is effective in $\mathbb{D}$ and the underlying equivalence relation $S_1 \rightrightarrows Z_1$ is effective in $\mathbb{D}$. With the same arguments as above $\underline{S}_1 \rightrightarrows \underline{Z}_1$ is an effective equivalence relation in $Grd_X\mathbb{D}$. $\square$

3.1 Baer categories

Let us introduce the following:

Definition 3.2. *We shall call Baer category any category $\mathbb{E}$ which is penessentially affine, efficiently regular, quasi-pointed and such that the endonormalization process reflects the effective monomorphism.*

This implies that when the endonormalization (see Proposition 2.3) of an equivalence R is a kernel map, then R is effective, i.e. the kernel equivalence relation of some map. As a penessentially affine category, a Baer category is necessarily protomodular. Given a Baer category $\mathbb{E}$, the pointed subcategory $\mathbb{E}/0 = Pt_0\mathbb{E}$ is additive and efficiently regular, and consequently such that the kernel maps are stable under composition. The pertinence of this further definition comes from the following theorem which is our main structural point concerning internal groupoids:

Theorem 3.1. *Let $\mathbb{E}$ be a Mal'cev efficiently regular category. Then any fibre $Grd_X\mathbb{E}$ is a Baer category.*

Proof. Let $\underline{R}_1$ be an equivalence relation on $\underline{Z}_1$ in $Grd_X\mathbb{E}$. Its endonormalization is given by the following pullback:

$$
\begin{tikzcd}
\underline{I}_1 \arrow[r] \arrow[d, "\underline{i}_1"'] & \underline{R}_1 \arrow[d, "\underline{j}_1"] \\
\underline{En_1 Z_1} \arrow[r, "\underline{\bar{\epsilon}_1 Z_1}"'] \arrow[d, shift right, "\underline{e_1 Z_1}"'] & \underline{Z}_1 \times_0 \underline{Z}_1 \arrow[d, shift right, "p_0"'] \\
\Delta Z_0 \arrow[r, tail] \arrow[u, shift right] & \underline{Z}_1 \arrow[u, shift right, "s_0"']
\end{tikzcd}
$$

Suppose that $\underline{i}_1$ is an effective monomorphism in $Grd_X\mathbb{E}$. Then the morphism $i_1 : I_1 \to En_1\underline{Z}_1$ is an effective monomorphism in $\mathbb{E}$. By Theorem 1.2, we know that j_1 is a pullback of i_1 in $\mathbb{E}$. So that $j_1 : R_1 \rightarrowtail Z_1 \times_0 Z_1$ is itself an effective monomorphism in $\mathbb{E}$. Since $Z_1 \times_0 Z_1 \rightrightarrows Z_1 \overset{(z_0,z_1)}{\rightarrow} X \times X$ provides an effective relation in $\mathbb{E}$, the equivalence relation $R_1 \rightrightarrows Z_1$ is effective in $\mathbb{E}$. Let $q_1 : Z_1 \twoheadrightarrow Q_1$ be its quotient in $\mathbb{E}$. Then clearly the induced reflexive graph $Q_1 \rightrightarrows X$ is underlying a groupoid $\underline{Q}_1$ and $\underline{R}_1 \rightrightarrows \underline{Z}_1$ is the kernel relation of the internal functor $\underline{q}_1 : \underline{Z}_1 \twoheadrightarrow \underline{Q}_1$ in $Grd_X\mathbb{E}$. $\square$

We are in such a situation, for instance, with the categories $\mathbb{E} = GpTop$ and $\mathbb{E} = GpHaus$ of topological and Hausdorff groups. On the other hand we have:

Proposition 3.3. *The Baer categories are stable under slice categories.*

Proof. The only point which remains to check concerns the endonormalization process. So let $f : X \to Y$ be an object in $\mathbb{E}/Y$ and R an equivalence relation on this object, which means that $R \subset R[f]$. The endosome in $\mathbb{E}/Y$ of this object f is nothing but its kernel. Let us consider the following diagram in $\mathbb{E}$:

$$
\begin{tikzcd}
I_R \arrow[rr] \arrow[dr, "i"'] & & R \arrow[dr, "j"] & & & \\
& K[f] \arrow[rr, "{(0,k)}"] \arrow[dd] \arrow[dr, "i_f"'] & & R[f] \arrow[rr, "p_1"] \arrow[dd, "p_0"'] \arrow[dr, "j_f"] & & X \arrow[dd, "f"] \\
& & EnX \arrow[rr, "\epsilon X"'] \arrow[dl] & & X \times X \arrow[dl, "p_0"] & \\
& 0 \arrow[rr] & & X \arrow[rr, "f"'] & & Y
\end{tikzcd}
$$

Our assumption is that i is an effective monomorphism. This means it is a kernel map since the category $\mathbb{E}/0$ is additive. This is also the case for i_f since $R[f]$ is an effective equivalence relation. Since $\mathbb{E}/0$ is additive and efficiently regular, then $i_f.i$ is still a kernel map. Accordingly, $\mathbb{E}$ being a Baer category, R is an effective equivalence relation in $\mathbb{E}$. Let $q : X \twoheadrightarrow Q$ be its quotient in $\mathbb{E}$. Since we have $R \subset R[f]$, there is a factorization $g : Q \to Y$ which makes R effective in $\mathbb{E}/Y$. $\square$

Now our starting point to the way to Baer sums will be the following observation:

Proposition 3.4. *In any Baer category $\mathbb{E}$ the following downward whole rectangle is a pullback and the following upward whole rectangle is a pushout:*

$$\begin{tikzcd}
EnX \arrow[r, "\bar{\epsilon}X"] \arrow[d, shift right] & X \times X \arrow[r, two heads, "qx"] \arrow[d, shift right, "p_0"'] & dX \arrow[d, shift right] \\
0 \arrow[r, tail] \arrow[u, shift right] & X \arrow[r, two heads] \arrow[u, shift right, "s_0"'] & 1 \arrow[u, shift right, "\eta X"']
\end{tikzcd}$$

Accordingly two objects with global support have same direction if and only if they have same endosome.

Proof. The downward left hand side square is a pullback and, $\mathbb{E}$ being penessentially affine, the associated upward square is a pushout. We just recall that the right hand part of the diagram fulfils the same property. Consequently EnX and dX mutually determine each other. □

Our second observation will be:

Proposition 3.5. *Let $\mathbb{E}$ be any Baer category. Then the functor $En : \mathbb{E} \to Pt_0\mathbb{E}$ is cofibrant on regular epimorphisms. The associated cocartesian maps are regular epimorphisms.*

Proof. This means that any regular epimorphism $g : EnX \twoheadrightarrow C$ in $Pt_0\mathbb{E}$ determines a cocartesian map in $\mathbb{E}$. Clearly the condition on g is equivalent to saying that g is a regular epimorphism in $\mathbb{E}$. Now take $k : K \rightarrowtail EnX$ the kernel of g in the additive category $Pt_0\mathbb{E}$, and R the associated equivalence relation on X given by Proposition 2.3. It is an effective relation since its endonormalization k is a kernel. Let $q : X \twoheadrightarrow Q$ be its quotient. Since the category $\mathbb{E}$ is regular, the functor En preserves the quotients. So Enq is the quotient of the equivalence relation EnR and consequently the cokernel of its normalisation which is k, according to the remark following Definition 2.3. Thus the map $Enq : EnX \twoheadrightarrow EnQ$ is nothing but (up to an isomorphism γ) our initial map $g : EnX \twoheadrightarrow C$ which consequently appears to be the quotient of the equivalence relation EnR. Whence a map ϵ given by the following diagram:

$$\begin{tikzcd}
 & & EnQ \arrow[d, "\gamma"'] \arrow[dr] & \\
EnR \arrow[r, shift left=2, "Enp_1"] \arrow[r, shift right=2, "Enp_0"'] \arrow[d, "\epsilon R"'] & EnX \arrow[l] \arrow[ur, "Enq"] \arrow[r, two heads, "g"] \arrow[d, "\epsilon X"] & C \arrow[r] \arrow[d, dotted, "\epsilon"] & 0 \arrow[d] \\
R \arrow[r, shift left=2, "p_1"] \arrow[r, shift right=2, "p_0"'] & X \arrow[l] \arrow[r, two heads, "q"'] & Q \arrow[r] & 1
\end{tikzcd}$$

A Baer category being necessarily protomodular, the middle square is a pushout since it is a pullback along a regular epimorphism. This implies the universal property of q as a cocartesian map with respect to the functor En. By construction this map q is a regular epimorphism. □

The last observation of this proof gives the following corollary which is actually equivalent to the proposition itself:

Corollary 3.1. *Let $\mathbb{E}$ be any Baer category. Then, along the map $\epsilon X : EnX \rightarrowtail X$, there exists the pushout of any regular epimorphism in $Pt_0\mathbb{E}$ and the involved square is also a pullback:*

$$\begin{tikzcd} EnX \arrow[r, two heads, "g"] \arrow[d, "\epsilon X"'] & C \arrow[d, "\epsilon"] \\ X \arrow[r, two heads, "q"'] & Q \end{tikzcd}$$

Another immediate consequence of the previous proposition is the following:

Corollary 3.2. *Let $\mathbb{E}$ be any Baer category. Then the fully faithful functor $\varpi^* : Pt_1\mathbb{E} \to Pt_0\mathbb{E}$ is cofibrant on regular epimorphisms.*

3.2 An alternative description of Baer sums

We recalled that, when $\mathbb{E}$ is Mal'cev and efficiently regular, the functor d is a cofibration whose fibres are actually groupoids (i.e. any map is cocartesian). When $\mathbb{E}$ is moreover a Baer category, we have, thanks to Proposition 3.4, a commutative triangle of functors:

$$\begin{tikzcd} \mathbb{E}_g \arrow[rr, "d"] \arrow[dr, "En"'] & & Pt_1\mathbb{E} = Ab\mathbb{E} \arrow[dl, "\varpi^*"] \\ & Pt_0\mathbb{E} & \end{tikzcd}$$

The functor En takes its values in a larger category than d and is only cofibrant on the regular epimorphisms. The fact that ϖ^* is fully faithful implies that a cocartesian map q with respect to En which is associated with a map $g = \varpi(\gamma)$ (with $\gamma : dM = A \twoheadrightarrow C$ any regular epimorphism in $Pt_1\mathbb{E}$) is also a cocartesian map associated with γ with respect to d.

3.2.1 Level 1

Let us recall [6] that, when $\mathbb{E}$ is naturally Mal'cev and efficiently regular, any fibre of the direction functor d is canonically endowed with a tensor product, namely the Baer sum. Let $a : 1 \to A$ be an element of $Pt_1\mathbb{E} = Ab\mathbb{E}$, M and N two elements in the fibre above A. Then $M \otimes N$ is defined as the codomain of the cocartesian map $\theta_+ : M \times N \twoheadrightarrow M \otimes N$ above $+ : A \times A \twoheadrightarrow A$.

The previous remarks about the triangle of functors above allows us, in the context of Baer categories, the following easier construction of the Baer sum, very similar to the classical abelian case:

Theorem 3.2. *Let $\mathbb{E}$ be any Baer category. Then the Baer sum of two objects M and N with global support and same direction A is given by the following* *pushout which is also a pullback:*

$$\begin{array}{ccc} \varpi^*A \times \varpi^*A & \xrightarrow{\epsilon M \times \epsilon N} & M \times N \\ {\scriptstyle +}\downarrow & & \downarrow \\ \varpi^*A & \longrightarrow & M \otimes N \end{array}$$

We classically denote by $H^1_{\mathbb{E}}(A)$ the abelian group structure determined by this Baer sum on the set $\pi_0 d^{-1}(A)$ of the connected components of the fibre $d^{-1}(A)$.

3.2.2 Level 2

Starting from a Baer category $\mathbb{E}$, any fibre $Grd_X\mathbb{E}$ is a Baer category, by Theorem 3.1. So the previous observations and constructions are the beginning of an iterative process. A groupoid $\underline{Z}_1$ has a global support in the fibre $Grd_X\mathbb{E}$ if and only if it is connected, i.e. such that $(z_0, z_1) : Z_1 \to Z_0 \times Z_0 = X \times X$ is a regular epimorphism. The direction of a connected abelian groupoid was defined in [7] as the pushout of $s_0 : Z_1 \rightarrowtail Z_1 \times_0 Z_1$ along (z_0, z_1):

$$\begin{tikzcd} Z_1 \times_0 Z_1 \arrow[r] \arrow[d, shift right=2, "p_0"'] \arrow[d, dotted, shift left=2, "p_1"] & d\underline{Z}_1 \arrow[d, shift left] \\ Z_1 \arrow[r, "(z_0,z_1)"'] \arrow[u] \arrow[d, "(z_0,z_1)"'] & Z_0 \times Z_0 \arrow[u, shift left] \\ Z_0 \times Z_0 & \end{tikzcd}$$

For the same reasons as above this implies that the downward square is a pullback. In the Baer context, two connected groupoids in $Grd_X\mathbb{E}$ have same direction if and only if they have same endosome. The Baer sum of a pair $(\underline{U}_1, \underline{V}_1)$ of connected groupoids having same endosome $\underline{E}_1$ has thus its object of morphisms given by the following pushout in $\mathbb{E}$ which is also a pullback:

$$\begin{array}{ccc} E_1 \times_0 E_1 & \xrightarrow{\epsilon_1\underline{U}_1 \times_0 \epsilon_1\underline{V}_1} & U_1 \times_0 V_1 \\ {\scriptstyle +}\downarrow & & \downarrow \\ E_1 & \longrightarrow & (\underline{U}_1 \otimes \underline{V}_1)_1 \end{array}$$

This is a much easier way than to go through the direction.

When the object X has a global support, the change of base along the terminal map $\tau_X : X \to 1$ produces an equivalence of categories [7]:

$$\Gamma : Pt_1\mathbb{E} = Ab\mathbb{E} \to Pt_1 Grd_X\mathbb{E} = AbGrd_X\mathbb{E} \;\; ; \; A \mapsto K(A,1) \times \nabla X$$

where $K(A,1)$ denotes the groupoid structure (with object of objects 1) associated with the group stucture A.

Recall that an internal groupoid $\underline{Z}_1$ is said to be aspherical when it is connected and moreover Z_0 has a global support (i.e. such that $Z_0 \to 1$ is a regular epimorphism). The inverse of the functor Γ allows to define the *global direction* functor $d_1 : Asp\mathbb{E} \to Pt_1\mathbb{E} = Ab\mathbb{E}$, where $Asp\mathbb{E}$ is the full subcategory of aspherical groupoids (see [7]). It is given, this time, by the pushout of s_0 along the terminal map:

$$\begin{tikzcd}
Z_1 \times_0 Z_1 \arrow[r] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, dotted, "p_1"] & d_1\underline{Z}_1 \arrow[d, shift left] \\
Z_1 \arrow[u] \arrow[r] \arrow[d, "{(z_0,z_1)}"'] & 1 \arrow[u, shift left] \\
Z_0 \times Z_0 &
\end{tikzcd}$$

This functor d_1 is a cofibration whose fibres are categories (and no longer groupoids as at level 1) canonically equipped with a tensor product called the *global Baer sum*. Here again, the Baer categorical context will allow us to give an easier description of the global Baer sums of aspherical groupoids having same global direction.

Definition 3.3. *Let $\mathbb{E}$ be a quasi-pointed category. The global endosome of a groupoid $\underline{Z}_1$ is defined as the kernel of $(z_0, z_1) : Z_1 \to Z_0 \times Z_0$.*

For exactly the same reasons as above, in a Baer category, two aspherical groupoids have same global direction if and only if they have same global endosome. The global Baer sum of a pair $(\underline{U}_1, \underline{V}_1)$ of aspherical groupoids with same global endosome A is organized by the following pushout in $\mathbb{E}$ which is also a pullback:

$$\begin{tikzcd}
A \times A \arrow[r, "\kappa\underline{U}_1 \times \kappa\underline{V}_1"] \arrow[d, "+"'] & U_1 \times V_1 \arrow[d] \\
A \arrow[r] & (\underline{U}_1 \otimes \underline{V}_1)_1
\end{tikzcd}$$

Once again this is a much easier way than to go through the global direction. We denote by $H^2_{\mathbb{E}}(A)$ the abelian group structure determined by this global Baer sum on the set $\pi_0 d_1^{-1}(A)$ of the connected components of the fibre $d_1^{-1}(A)$ see [3] and [7].

3.2.3 The higher levels

The *global direction* of an asperical n-groupoid $\underline{Z}_n$:

$$\underline{Z}_n : \begin{tikzcd}
Z_n \arrow[r, shift left=2, "z_1"] \arrow[r, shift right=2, "z_0"'] & Z_{n-1} \arrow[l, "s_0"'] \arrow[r, shift left=2, "z_1"] \arrow[r, shift right=2, "z_0"'] & Z_{n-2} \ldots Z_1 \arrow[l, "s_0"'] \arrow[r, shift left=2, "z_1"] \arrow[r, shift right=2, "z_0"'] & Z_0 \arrow[l, "s_0"']
\end{tikzcd}$$

is defined in [18] and in [11] as a functor $d_n : n\text{-}Asp\mathbb{E} \to Pt_1\mathbb{E} = Ab\mathbb{E}$ which is produced by the following pushout in $\mathbb{E}$:

$$\begin{tikzcd}
Z_n \times_{n-1} Z_n \arrow[r] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, dotted, "p_1"] & d_n\underline{Z}_n \arrow[d, shift left] \\
Z_n \arrow[u] \arrow[r] \arrow[d, "(z_0,z_1)"'] & 1 \arrow[u, shift left] \\
Z_{n-1} \times_{n-2} Z_{n-1} &
\end{tikzcd}$$

It is still a cofibration whose fibres are categories canonically endowed with a tensor product, still called the *global Baer sum* of n-groupoids. In a quasi-pointed category the *global endosome* of a n-groupoid $\underline{Z}_n$ will be the kernel of the map $(z_0, z_1) : Z_n \to Z_{n-1} \times_{n-2} Z_{n-1}$. When $\mathbb{E}$ is a Baer category, still two aspherical n-groupoids have same global direction if and only if they have same global endosome. The global Baer sum of of a pair $(\underline{U}_n, \underline{V}_n)$ of aspherical n-groupoids with same global endosome A is organized by the following pushout in $\mathbb{E}$ which is also a pullback:

$$\begin{tikzcd}
A \times A \arrow[r, "\kappa\underline{U}_n \times \kappa\underline{V}_n"] \arrow[d, "+"'] & U_n \times V_n \arrow[d] \\
A \arrow[r] & (\underline{U}_n \otimes \underline{V}_n)_n
\end{tikzcd}$$

We denote by $H^n_{\mathbb{E}}(A)$ the abelian group structure determined by this global Baer sum on the set $\pi_0 d_n^{-1}(A)$ of the connected components of the fibre $d_n^{-1}(A)$. These abelian groups $H^n_{\mathbb{E}}(-)$ were shown to have a Yoneda's Ext style long cohomology sequence in [18] and [11].

4 The endosome of a groupoid

Now we have emphasized the role of the endosome of a groupoid, we shall end this work with some general observations about the endosome, when the ground category $\mathbb{E}$ is only finitely complete. Let $\underline{Z}_1$ be a groupoid in $\mathbb{E}$. Let us begin by a useful property:

Lemma 4.1. *Suppose we are given an internal functor $\underline{h}_1 : R \to \underline{Z}_1$ where R is an equivalence relation on an object Y, then the following square is a pullback in the category $Grd\mathbb{E}$:*

$$\begin{tikzcd}
R[h_0] \cap R \arrow[r] \arrow[d] & \underline{En}_1\underline{Z}_1 \arrow[d, "\epsilon_1\underline{Z}_1"] \\
R \arrow[r, "\underline{h}_1"'] & \underline{Z}_1
\end{tikzcd}$$

Proof. Straightforward by commutation of limits, considering the following rectangles, where the two squares on the left and the right hand side part on the right are pullbacks in $Grd\mathbb{E}$:

$$\begin{tikzcd}
R[h_0] \cap R \arrow[r] \arrow[d] & R[h_0] \arrow[r] \arrow[d] & \Delta Z_0 \arrow[d] \\
R \arrow[r] & \nabla Y \arrow[r, "\nabla h_0"'] & \nabla Z_0
\end{tikzcd}
\qquad
\begin{tikzcd}
R[h_0] \cap R \arrow[r] \arrow[d] & \underline{En_1 Z_1} \arrow[r, "e_1 \underline{Z}_1"] \arrow[d, "\epsilon_1 \underline{Z}_1"] & \Delta Z_0 \arrow[d] \\
R \arrow[r, "\underline{h}_1"'] & \underline{Z}_1 \arrow[r] & \nabla Z_0
\end{tikzcd}$$

□

We are now in position to specify the internal action of the endosome $\underline{En_1 Z_1}$ on the object Z_1 of morphisms:

Proposition 4.1. *The endosome $\underline{En_1 Z_1}$ has a canonical action on Z_1 which is described by the following diagram where the vertical left hand side part is a kernel equivalence relation and the upper square a pullback :*

$$\begin{tikzcd}
Z_1 \times_0 Z_1 \arrow[r, shift left, "\bar{z}_1"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, dotted, "p_1"] & En_1 \underline{Z}_1 \arrow[l, shift left, "\bar{\epsilon}_1 \underline{Z}_1"] \arrow[d, "e_1 \underline{Z}_1"] \\
Z_1 \arrow[r, shift left, "z_1"] \arrow[d, "(z_0, z_1)"'] & Z_0 \arrow[l, shift left, "s_0"] \\
Z_0 \times Z_0 &
\end{tikzcd}$$

Proof. The previous lemma gives rise to the following upper left hand side pullback in $\mathbb{E}$:

$$\begin{tikzcd}
R[z_0] \cap R[z_1] \arrow[r, "\bar{z}_1"] \arrow[d, "\iota"'] & En_1 \underline{Z}_1 \arrow[d, "\epsilon_1 \underline{Z}_1"] \\
R[z_0] \arrow[r, "\zeta_2"] \arrow[d, shift right, "z_0"'] \arrow[d, shift left, dotted, "z_1"] & Z_1 \arrow[d, shift right, "z_0"'] \arrow[d, shift left, "z_1"] \\
Z_1 \arrow[r, "z_1"'] & Z_0
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
Z_1 \times_0 Z_1 \arrow[r, "\bar{z}_1"] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, dotted, "p_1"] & En_1 \underline{Z}_1 \arrow[d, "e_1 \underline{Z}_1"] \\
Z_1 \arrow[r, "z_1"'] & Z_0
\end{tikzcd}$$

Furthermore the two lower left hand side squares are pullbacks since $\underline{Z}_1$ is a groupoid. Now $R[z_0] \cap R[z_1]$ is nothing but the kernel equivalence relation of the map $(z_0, z_1) : Z_1 \to Z_0 \times Z_0$ whose underlying object is the object $Z_1 \times_0 Z_1$ of "parallel morphisms" of the groupoid $\underline{Z}_1$ which, itself, is nothing but the object of morphisms of the groupoid $\underline{Z}_1 \times_0 \underline{Z}_1$. We have also $z_0.\epsilon_1 \underline{Z}_1 = z_1.\epsilon_1 \underline{Z}_1 = e_1 \underline{Z}_1$. All this gives rise to the right hand side pullback with the p_0. The map $p_1 = z_1.\iota$ represents the canonical action of the endosome. In the set theoretical context the map $\bar{z}_1 : Z_1 \times_0 Z_1 \to En_1 \underline{Z}_1$ associates with the pair $(\alpha, \beta) : x \rightrightarrows y$ of parallel arrows in $\underline{Z}_1$ the endomap $\beta.\alpha^{-1} : y \to y$. Notice that the section induced by $s_0 : Z_0 \to Z_1$ of this map $\bar{z}_1$ is (by the equations defining it) nothing but the map $\bar{\epsilon}_1\underline{Z}_1 : En_1\underline{Z}_1 \to Z_1 \times_0 Z_1$ which associates with any endomap $\gamma : x \to x$ the pair $(1_x, \gamma) : x \rightrightarrows x$. □

We then get the following:

Corollary 4.1. *Suppose $\mathbb{E}$ is naturally Mal'cev. Then the following square is a pushout in $\mathbb{E}$:*

$$\begin{tikzcd} Z_1 \times_0 Z_1 & En_1\underline{Z}_1 \arrow[l, "\bar{\epsilon}_1\underline{Z}_1"'] \\ Z_1 \arrow[u, "s_0"] & Z_0 \arrow[l, "s_0"] \arrow[u, "\alpha_1\underline{En_1\underline{Z}_1}"'] \end{tikzcd}$$

Proof. These are the splittings of a split pullback, i.e. a product in the fibre Pt_{Z_0}, and, the category $\mathbb{E}$ being naturally Mal'cev, this fibre is additive. □

References

[1] M. Barr, *Exact categories*, Springer L.N. in Math., **236**, 1971, 1-120.

[2] D. Bourn, *The shift functor and the comprehensive factorization for internal groupoids*, Cahiers de Top. et Géom. Diff. Catégoriques, **28**, 1987, 197-226.

[3] D. Bourn, *The tower of n-groupoids and the long cohomology sequence*, J. Pure Appl. Algebra, **62**, 1989, 137-183.

[4] D. Bourn, *Normalization equivalence, kernel equivalence, and affine categories*, Lecture Notes in Math., **1488**, Springer, 1991, 43-62.

[5] D. Bourn, *Mal'cev categories and fibration of pointed objects*, Applied Categorical Structures, **4**, 1996, 307-327.

[6] D. Bourn, *Baer sums and fibered aspects of Mal'cev operations*, Cahiers de Top. et Géom. Diff. Catégoriques, **40**, 1999, 297-316.

[7] D. Bourn, *Aspherical abelian groupoids and their directions*, J. Pure Appl. Algebra, **168**, 2002, 133-146.

[8] D. Bourn, *Baer sums in homological categories*, Journal of Algebra, **308**, 2007, 414-443.

[9] D. Bourn and M. Gran, *Centrality and connectors in Maltsev categories*, Algebra Universalis, **48**, 2002, 309-331.

[10] D. Bourn and M. Gran, *Categorical aspects of Modularity*, Fields Institute Communications, vol. **43**, 2004, 77-100.

[11] D. Bourn and D. Rodelo, *Cohomology without projectives*, Preprint Univ. Littoral, 2006.

[12] A. Carboni, J. Lambek and M.C. Pedicchio, *Diagram chasing in Mal'cev categories*, J. Pure Appl. Algebra, **69**, 1991, 271-284.

[13] A. Carboni, M.C. Pedicchio and N. Pirovano, *Internal graphs and internal groupoids in Mal'cev categories*, CMS Conference Proceedings, **13**, 1992, 97-109.

[14] M. Gran, *Internal categories in Mal'cev categories*, J. Pure Appl. Algebra, **143**, 1999, 281-291.

[15] H.P. Gumm, *Geometrical methods in Congruence Modular Algebras*, Mem. Amer. Math. Soc.,**45**, 1983.

[16] P.T. Johnstone, *Affine categories and naturally Mal'cev categories*, J. Pure Appl. Algebra, **61**, 1989, 251-256.

[17] M.C. Pedicchio, *A categorical approach to commutator theory*, Journal of Algebra, **177**, 1995, 647-657.

[18] D. Rodelo, *Direcções para a sucessão longa de co-homologia*, Thesis, Coimbra, 2005.

[19] J.D.H. Smith, *Mal'cev varieties*, Springer L.N. in Math., **554**, 1976.

Université du Littoral
Laboratoire de Mathématiques Pures et Appliquées
Bat. H. Poincaré
50 Rue F. Buisson BP 699
62228 Calais - France.

e-mail: bourn@lmpa.univ-littoral.fr

Key words : Mal'cev, protomodular, naturally Mal'cev categories; internal groupoids; Baer sum; long cohomology sequence.
subjclass :