Abelian groupoids and non pointed additive categories

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Abstract

We show that, in any Mal’cev (and a fortiori protomodular) category $E$, the fibre $GrdXE$ of internal groupoids above the object $X$ is naturally Mal’cev and moreover shares with the category $Ab$ of abelian groups the property following which the domain of any split epimorphism is isomorphic with the direct sum of its codomain with its kernel. This allows us to point at a new class of "non pointed additive" categories. When furthermore the ground category $E$ is efficiently regular, we get a new way to produce Bear sums in the fibres $GrdXE$ and more generally in the fibres $n-GrdXE$.

Introduction

The main project of this work was to gather some properties (related to cohomological algebra) of the category $GrdC$ of internal groupoids inside a protomodular category $C$. In a way, the existence of the semi-direct product in the category $Gp$ of groups and the associated possible reduction of internal groupoids to crossed modules made that the systematic investigation of the category $GrdGp$ was not done, and no guiding example of such a protomodular context was existing. Actually it appears that our main results concerning $GrdC$ do hold when $C$ is only Mal’cev in the sense of [12] (see also [13] and [14]).

We show that, in the Mal’cev context, any groupoid is abelian in the sense of [7], which implies that any fibre $GrdX C$ of internal groupoids having $X$ as “object of objects” is naturally Mal’cev in the sense of [16]. Moreover we show that, given any split internal functor in the fibre $GrdX C$, the downward pullback:

$$
\begin{array}{c}
K_1[f_1] \ar[r]^{K_1} \ar@{=}[d] & W_1 \ar@{->>}[d]^{Z_1} \\
\Delta X \ar[r]_{\alpha_1, Z_1} & Z_1
\end{array}
$$

produces an upward pushout. In other word, when $C$ is Mal’cev, the fibre $GrdX C$ shares with the category $Ab$ of abelian groups (and more generally the fibre $GrdC = AbC$) the property following which the domain of any split epimorphism is isomorphic with the direct sum of its codomain with its kernel. It is all the more interesting since this is absolutely not the case in the fibres $AbGrdX Set$, $X \neq 1$ of abelian groupoids in $Set$. 

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This kind of results gives rise to new classes of “non pointed additive” categories which take place between the class of naturally Mal’cev categories \([16]\) and the one of essentially affine categories \([5]\). Subtle distinctions between different kinds of cohesion, which compensate the disorganisation determined by the absence of 0 (and consequently by the ordered set of subobjects of 1), uncomfortably demand to introduce a bit of terminology. We give a synthetic classification table in section 2.3. The most interesting intermediate class is the one of *penessentially affine categories* (see section 2.1 for the precise definition): it is a class of non pointed additive categories which are necessarily protomodular and such that any monomorphism is normal, and which, precisely, contains any fibre \(\text{Grd}_X \mathcal{C}\) in the Mal’cev context. This new structural approach of internal groupoids allows us to get a new way to produce Baer sums in these fibres, and more generally a new way to produce the cohomology groups \(H^n_\mathcal{E}(A)\). All this leads also to a more technical last section which is devoted to the fibre \(\text{Grd}_X \mathcal{E}\) when \(\mathcal{E}\) is only finitely complete.

## 1 Internal groupoids

Let \(\mathcal{E}\) be a finitely complete category, and \(\text{Grd}\mathcal{E}\) denote the category of internal groupoids in \(\mathcal{E}\). An internal groupoid \(Z_1\) in \(\mathcal{E}\) will be presented (see \([2]\)) as a reflexive graph \(Z_1 \rightrightarrows Z_0\) endowed with an operation \(\zeta_2:\)

\[
\begin{array}{c}
R^2[z_0] \\
\downarrow z_2 \\
\downarrow z_0 \\
R[z_0] \\
\downarrow z_1 \\
\downarrow z_0 \\
Z_1 \\
\downarrow \zeta_2 \\
\downarrow z_0 \\
Z_0 \\
\end{array}
\]

making the previous diagram satisfy all the simplicial identities (including the ones involving the degeneracies), where \(R[z_0]\) is the kernel equivalence relation of the map \(z_0\). In the set theoretical context, this operation \(\zeta_2\) associates the composite \(g.f^{-1}\) with any pair \((f,g)\) of arrows with same domain. We denote by \(\zeta_0 : \text{Grd}\mathcal{E} \to \mathcal{E}\) the forgetful functor which is a fibration. Any fibre \(\text{Grd}_X \mathcal{E}\) above an object \(X\) has an initial object \(\Delta X\), namely the discrete equivalence relation on \(X\), and a final object \(\nabla X\), namely the indiscrete equivalence relation on \(X\). This fibre is *quasi-pointed* in the sense that the unique map

\[
\omega : 0 \to 1 = \Delta X \to \nabla X
\]

is a monomorphism; this implies that any initial map is a monomorphism, and we can define the kernel of any map as its pullback along the initial map to the codomain. The fibre \(\text{Grd}_X \mathcal{E}\) is nothing but the category \(\text{Gp}\mathcal{E}\) of internal groups in \(\mathcal{E}\) which is necessarily pointed protomodular. It was shown in \([4]\) that any fibre \(\text{Grd}_X \mathcal{E}\) is still protomodular although non pointed. This involves an intrinsic notion of normal subobject and abelian object. They both have been characterized in \([7]\).
1.1 Abelian groupoids

Let us begin by the abelian groupoids. Consider the following pullback in $\text{Grd}_E$ which only retains the "endomorphisms" of $Z_1$:

$$
\begin{array}{c}
E_{n_1}Z_1 \rightarrow Z_1 \\
\downarrow \gamma_{Z_1} \quad \downarrow \gamma_{Z_1} \\
\Delta Z \rightarrow \nabla Z_0
\end{array}
$$

Let us recall [7] that:

**Proposition 1.1.** The groupoid $Z_1$ is abelian if and only if the group $\epsilon_1 : E_{n_1}Z_1 \rightarrow Z_0$ in the slice category $E/Z_0$ is abelian.

In the set theoretical context, this means that any group of endomaps in $Z_1$ is abelian. We shall denote by $\text{AbGrd}^{X}_E$ the full subcategory of $\text{Grd}^{X}_E$ whose objects are the abelian groupoids.

Now consider any internal functor $f_1 : W_1 \rightarrow Z_1$ in $\text{AbGrd}^{X}_E$. Suppose it is split by a functor $s_1$, and consider the following pullback determining the kernel of $f_1$:

$$
\begin{array}{c}
K_1[f_1] \xrightarrow{\overline{k}_1} W_1 \\
\downarrow \quad \downarrow \overline{\Delta} \\
\Delta X \rightarrow \nabla Z_1
\end{array}
$$

In the case $X = 1$, the upward square is actually a pushout in $\text{AbGrd}^{1}_E = \text{Ab}E$ the category of abelian groups in $E$. Does it still hold in any case? Suppose given a pair $h : K_1[f_1] \rightarrow V_1$, $t_1 : Z_1 \rightarrow V_1$ of internal functors in $\text{AbGrd}^{X}_E$.

**Lemma 1.1.** When $E = \text{Set}$, there is a factorization $g_1 : W_1 \rightarrow V_1$ such that $g_1 \overline{k}_1 = h_1$ and $g_1 \overline{s}_1 = t_1$ if and only if, for all pair $x \rightarrow y$ of maps in $K_1[f_1] \times Z_1$ with same domain, we have:

$$h_1(s_1 \phi \cdot \gamma \cdot s_1 \phi^{-1}) = t_1 \phi \cdot h_1 \gamma \cdot t_1 \phi^{-1}$$

**Proof.** For any $\delta : y \rightarrow y$ in $K_1[f_1]$, we must have $g_1 \delta = h_1 \delta$, and any $\phi : x \rightarrow y$ in $Z_1$, we must have $g_1 \cdot s_1 \phi = t_1 \phi$. Accordingly, for any $\psi : x \rightarrow y$ in $W_1$, we must have $g_1 \psi = g_1(\psi \cdot s_1 f_1 \psi^{-1}) \cdot g_1(s_1 f_1 \psi) = h_1(\psi \cdot s_1 f_1 \psi^{-1}) \cdot t_1(f_1 \psi)$. It remains to show that this definition is functorial, which is easily stated to be equivalent to our condition.

Accordingly, the category $\text{AbGrd}^{X}_E$ of abelian groupoids in the fibre above $X \neq 1$ does not share the classical property of $\text{Ab}E = \text{AbGrd}^{1}_E$ concerning the split epimorphisms.
1.2 Groupoids in Mal’cev and naturally Mal’cev categories

However we are going to show that this is the case as soon as the ground category \( E \) is Mal’cev. Recall that a category \( E \) is Mal’cev ([12], [13]) when it is finitely complete and such that any reflexive relation is actually an equivalence relation. When the category \( E \) is Mal’cev, we can truncate at level 2 (i.e. at the level of \( R[z_0] \)) the diagram defining a groupoid, see [13]. A category is naturally Mal’cev [16] when its is finitely complete and such that any object \( X \) is equipped with a natural Mal’cev operation. Any naturally Mal’cev category is Mal’cev.

**Theorem 1.1.** Suppose \( E \) is Mal’cev. Then any internal groupoid is abelian. Accordingly any fibre \( \text{Grd}_X E \) is naturally Mal’cev. Moreover, for any split epimorphism in \( \text{Grd}_X E \), the previous upward square is necessarily a pushout.

**Proof.** When \( E \) is Mal’cev, this is still the case for the slice category \( E/Z_0 \). On the other hand, any group in a Mal’cev category is abelian, see [13]. So, by Proposition 1.1, any groupoid is abelian. Any fibre \( \text{Grd}_X E \), being necessarily protomodular [4] and thus Mal’cev, is naturally Mal’cev, since any object in \( \text{Grd}_X E \) is abelian and produces a natural Mal’cev operation.

We are now going to show the next point by a classical method in Mal’cev categories. Consider the relation \( R \rightarrow K_1[f_1] \times Z_1 \) defined by \( \gamma R \phi \) if

\[
dom \gamma = \dom \phi \land h_1(s_1 \phi. \gamma.s_1 \phi^{-1}) = t_1 \phi.h_1 \gamma.t_1 \phi^{-1}
\]

Suppose \( \dom \gamma = \dom \phi = x \), then obviously we have \( 1_x R \phi, \gamma R 1_x \) and \( 1_x R 1_x \). Accordingly we can conclude that \( \gamma R \phi \) for all \( (\gamma, \phi) \) with same domain, whence, according to Lemma 1.1, the desired unique factorization \( g_1 : W_1 \rightarrow V_1 \).

This result holds a fortiori in any protomodular category \( E \). We have now an important structural property:

**Corollary 1.1.** Suppose \( E \) is Mal’cev. Then for any groupoid \( Z_1 \) the following upward left hand side square is a pushout in \( \text{Grd}_E \).

**Proof.** Let us consider the following diagram:

\[
\begin{array}{c}
\text{En}_1 Z_1 \\
\downarrow \quad \text{En}_1 Z_1 \\
\text{En}_1 Z_1 \\
\end{array}
\]

The whole rectangle and the right hand side squares being pullbacks, there is a unique dotted arrow which makes the downward square a pullback, and consequently the upward left hand side square a pushout in \( \text{Grd}_E \). According to the previous theorem. But, the functor \( (\_)_0 : \text{Grd}_E \rightarrow E \) being a fibration, it is still a pushout in \( \text{Grd}_E \).
It was shown in [5] that a finitely complete category \( \mathbb{E} \) is Mal’cev if and only if any reflexive graph \( Z_1' \) which is a subobject of a groupoid \( Z_1 \) is itself a groupoid. This property allows to strengthen the Theorem 1.1:

**Theorem 1.2.** Suppose \( \mathbb{E} \) is Mal’cev. Given any split epimorphism \( (f_1, s_1) : W_1 \rightarrowtail Z_1 \) in \( \text{Grd}_X \mathbb{E} \), there is a bijection between the pointed subobjects of the kernel \( K_1[f_1] \) and the pointed subobjects of \( (f_1, s_1) \).

**Proof.** Any pointed subobject \( j_1 \) of \( (f_1, s_1) \) produces a pointed subobject of \( K_1[f_1] \) by pullback along \( k_1 \):

\[
\begin{array}{ccc}
A_1 & \rightarrow & W'_1 \\
\downarrow & & \downarrow \bar{j}_1 \\
K_1[f_1] & \rightarrow & W_1 \\
\downarrow \Delta_X & & \downarrow \lambda \\
\Delta X & \rightarrow & Z_1
\end{array}
\]

Conversely suppose given a pointed subobject \( i_1 : A_1 \rightarrow K_1[f_1] \). Define \( W'_1 \) as the subobject of \( W_1 \) whose elements are those maps \( \tau : x \rightarrow y \in W_1 \) which satisfy \( \tau.s_1 f_1(\tau^{-1}) \in A_1 \). This subobject is given by the following right hand side pullback in \( \mathbb{E} \) where \( l = (w_1, \nu) \) (with \( \nu \) the map which internally corresponds to the mapping: \( \tau \mapsto \tau.s_1 f_1(\tau^{-1}) \)) is a natural retraction of \( k_1 : K_1[f_1] \rightarrow W_1 \):

\[
\begin{array}{ccc}
A_1 & \rightarrow & W'_1 \\
\downarrow i_1 & & \downarrow j_1 \\
K_1[f_1] & \rightarrow & W_1 \\
\downarrow k_1 & & \downarrow \lambda \\
\Delta X & \rightarrow & Z_1
\end{array}
\]

This produces a natural section \( k^1_1 \) of \( \lambda \). The object \( W'_1 \) clearly determines a subgraph \( W'_1 \) of the groupoid \( W_1 \). Since \( \mathbb{E} \) is Mal’cev, \( W'_1 \) is actually a subgroupoid such that the following square is a pullback in \( \text{Grd}_X \mathbb{E} \):

\[
\begin{array}{ccc}
A_1 & \rightarrow & W'_1 \\
\downarrow i_1 & & \downarrow j_1 \\
K_1[f_1] & \rightarrow & W_1 \\
\downarrow k_1 & & \downarrow \lambda \\
\Delta X & \rightarrow & Z_1
\end{array}
\]

\( \square \)

### 1.3 Connected equivalence relations

Let us now point out some properties related to commutator theory. First consider \( R \) and \( S \) two equivalence relations on an object \( X \) in any finitely complete category \( \mathbb{E} \). Let us recall the following definition from [9]:

\[
\begin{array}{ccc}
A_1 & \rightarrow & W'_1 \\
\downarrow i_1 & & \downarrow j_1 \\
K_1[f_1] & \rightarrow & W_1 \\
\downarrow k_1 & & \downarrow \lambda \\
\Delta X & \rightarrow & Z_1
\end{array}
\]

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Definition 1.1. A connector on the pair \((R, S)\) is a morphism
\[
p : R \times_X S \to X, \ (xRyS) \mapsto p(x, y, z)
\]
which satisfies the identities :
\[
1) \ xS p(x, y, z) \quad 1') \ zR p(x, y, z) \\
2) \ p(x, y, y) = x \quad 2') \ p(y, y, z) = z \\
3) \ p(x, y, p(y, u, v)) = p(x, u, v) \quad 3') \ p(p(x, y, u), u, v) = p(x, y, v)
\]
In set theoretical terms, Condition 1 means that with any triple \(xRyS\) we can associate a square:
\[
\begin{array}{c}
x \quad S \\
R \\
y \quad S \\
\end{array} \quad \begin{array}{c}
p(x, y, z) \\
\hat{R} \\
\end{array}
\]
More acutely, any connected pair produces a double equivalence relation in \(E\):
\[
\begin{array}{c}
R \times_X S \\
p_1 \\
p_0 \quad (d_0, p_0, p) \\
\end{array} \quad \begin{array}{c}
S \\
p_0 \quad (p_0, d_1) \\
R \quad d_0 \\
\end{array} \quad \begin{array}{c}
d_1 \\
X \\
\end{array}
\]
Example 1) An emblematical example is produced by a given discrete fibration \(f_1 : R \to Z_1\) with \(R\) an equivalence relation. For that consider the following diagram:
\[
\begin{array}{c}
R[f_1] \xrightarrow{p_1} R \quad f_1 \quad Z_1 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
R(d_0) \quad R[d_1] \quad d_0 \quad d_1 \quad z_1 \quad z_0 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
R[f_0] \xrightarrow{p_1} X \quad f_0 \quad Z_0
\end{array}
\]
It is clear that \(R[f_1]\) is isomorphic to \(R[f_0] \times_X R\) and that the map
\[
p : R[f_1] \xrightarrow{p_0} R \xrightarrow{d_1} X
\]
determines a connector.
2) Given any groupoid \(Z_1\), we have such a discrete fibration \(R[z_0] \to Z_1\):
\[
\begin{array}{c}
R[z_0] \xrightarrow{\zeta} Z_1 \\
z_0 \quad z_1 \quad z_0 \\
\end{array}
\]
\[
\begin{array}{c}
Z_1 \quad z_1 \quad Z_0
\end{array}
\]
which implies a connector on the pair \((R[z_0], R[z_1])\) explicited by the following diagram:

\[
\begin{array}{ccc}
x & \xrightarrow{\phi} & t \\
y & \xleftarrow{\psi} & z \\
\end{array}
\quad \xrightarrow{\sim} \quad
\begin{array}{ccc}
x & \xrightarrow{\sim} & t \\
y & \xleftarrow{\sim} & z \\
\end{array}
\]

The converse is true as well, see [13] and [9]; given a reflexive graph :

\[
\begin{array}{ccc}
z_1 & \xrightarrow{z} & z_0 \\
\end{array}
\]

any connector on the pair \((R[z_0], R[z_1])\) determines a groupoid structure.

Now let us observe that:

**Proposition 1.2.** Suppose \(p\) is a connector for the pair \((R, S)\). Then the following reflexive graph is underlying a groupoid we shall denote by \(R♯S\):

\[
\begin{array}{ccc}
R \times_X S & \xrightarrow{d, p_1} & X \\
d_0, p_0 & \xleftarrow{d_0, p_0} & d_0 \\
\end{array}
\]

**Proof.** Thank to the Yoneda embedding, it is enough to prove it in \(Set\). This is straightforward just setting:

\[(zRuSv)(xRySz) = xRp(u, z, y)Sv\]

The inverse of the arrow \(xRySz\) is \(zRp(x, y, z)Sx\).

**Remark**
1. When \(R \cap S = \Delta X\), the groupoid \(R♯S\) is actually an equivalence relation.
2. Let \(Z_1\) be any reflexive graph. We noticed it is a groupoid if and only if \([R[z_0], R[z_1]] = 0\). It is easy to check that:

\[R[z_0]♯R[z_1] \simeq Z_1^2\]

where \(Z_1^2\) is the groupoid whose objects are the maps and morphisms the commutative squares, in other words the groupoid which represents the natural transformations between functors with codomain \(Z_1\). Next we have:

**Proposition 1.3.** Given a discrete fibration \(f_1 : R \rightarrow Z_1\), the associated internal functor \(R[f_0]♯R \rightarrow R \rightarrow Z_1\) is fully faithful.

**Proof.** This functor \(\phi_1\) is given by the following diagram:

\[
\begin{array}{ccc}
R[f_1] & \xrightarrow{f_1, p_1} & Z_1 \\
\xrightarrow{d_1, p_1} & \xleftarrow{d_0, R(d_0)} & \xrightarrow{z_1} & \xleftarrow{f_0} & Z_0 \\
X & \xrightarrow{f_0} & Z_0 \\
\end{array}
\]
Thank to the Yoneda embedding, it is enough to prove it is fully faithful in $\text{Set}$. Suppose you have a map $\alpha : f(x) \to f(x')$. Since $\bar{\alpha}$ is a discrete fibration, there is an object $z \in X$ such that $zRx'$ and $f(z, x') = \alpha$. This implies that $f(z) = f(x)$. Accordingly $xR[f_0]zRx'$ is a map in $R[f_0]zR$ above $\alpha$. Suppose now that $\phi_1(xR[f_0]zRx') = \phi_1(xR[f_0]z'Rx')$. This means $f(z, x') = f(z', x')$.

In a Mal’cev category, the conditions 2) imply the other ones, and moreover a connector is necessarily unique when it exists, and thus the existence of a connector becomes a property.

**Example 1.1.** By Proposition 3.6, Proposition 2.12 and definition 3.1 in [17], two relations $R$ and $S$ in a Mal’cev variety $\mathcal{V}$ are connected if and only if $[R, S] = 0$ in the sense of Smith [19]. Accordingly we shall denote a connected pair of equivalence relations by the formula $[R, S] = 0$.

**Proposition 1.4.** Suppose $E$ is Mal’cev and $[R, S] = 0$. Then the following diagram (which is a pullback) is a pushout in $\text{GrdE}$:

$$
\begin{array}{ccc}
\Delta X & \longrightarrow & S \\
\downarrow & & \downarrow i_S \\
R & \longrightarrow & R\sharp S \\
\end{array}
$$

Proof. Let $f_1$ and $g_1$ be two functors making the following diagram commute:

$$
\begin{array}{ccc}
\Delta X & \longrightarrow & S \\
\downarrow & & \downarrow \zeta_1 \\
R & \longrightarrow & Z_1 \\
\end{array}
$$

We have $f_0 = g_0(= h_0) : X \to Z_0$. Wanting $h_1 : R \hookrightarrow S \to Z_1$ is given by the formula $h_1(xRySz) = g(y, z).f(x, y)$. This defines a functor $h_1 : R\sharp S \to Z_1$ if and only if, for all $xRySz$, we have $g(y, z).f(x, y) = f(p(x, y, z), g(x, p(x, y, z)))$. This is necessarily the case when $E$ is Mal’cev. For that, let us introduce the following relation $T$ on $R \times X$ defined by

$$(xRy)Tz \iff gSz \land g(y, z).f(x, y) = f(p(x, y, z), z).g(x, p(x, y, z))$$

For all $xRySz$, we have necessarily $(xRy)Ty$, $(yRy)Tz$ and $(yR)Ty$. Accordingly, for all $xRySz$, we have necessarily $(xR)Tz$. □

According to Remark 1 above, when we have $R \cap S = \Delta X$, the groupoid $R\sharp S$ being an equivalence relation, we have also $R\sharp S = R \vee S$.
1.4 The regular context

We shall end this section with a useful remark concerning pullbacks of split epimorphisms and discrete fibrations in the regular context:

**Proposition 1.5.** Suppose $E$ Mal’cev and regular [1]. Then any (downward) pullback of split epimorphism along a regular epimorphism produces an upward pushout:

$$
\begin{array}{c}
X & \xrightarrow{f} & Z \\
\downarrow{r} & & \downarrow{t} \\
X' & \xrightarrow{f'} & Z
\end{array}
$$

Any discrete fibration $f_1 : X_1 \to Z_1$ with $f_0$ regular epimorphic is cocartesian with respect to the functor $(f_0) : \text{Grd}E \to E$.

**Proof.** Consider the following diagram:

$$
\begin{array}{c}
\cdots \xrightarrow{h} & W & \cdots \\
X & \xrightarrow{f} & Z \\
\downarrow{r} & & \downarrow{t} \\
X' & \xrightarrow{f'} & Z
\end{array}
$$

with $g.f' = h.r$. We must find a map $\phi$ which makes the triangles commute. Since $f$ is a regular epimorphism, this is the case if and only if $R[f] \subset R[h]$.

Now the left hand side squares are still pullbacks. Since $E$ is Mal’cev, the pair $(R(r) : R[f'] \to R[f], s_0 : X \to R[f])$ is jointly strongly epic. So that the inclusion in question can be checked by composition with this pair. Checking by $s_0$ is straightforward. Checking by $R(r)$ is guaranteed by the existence of the map $g$. Let $f_1 : X_1 \to Z_1$ be any discrete fibration with $f_0$ regular epimorphic

$$
\begin{array}{c}
\cdots \xrightarrow{h_1} & W_1 & \cdots \\
X_1 & \xrightarrow{f_1} & Z_1 \\
\downarrow{z_1} & & \downarrow{w_1} \\
X & \xrightarrow{f_0} & Z_0
\end{array}
$$

where the pair $(h_0 = g_0.f_0, h_1)$ is underlying an internal functor $X_1 \to W_1$. By the previous part of this proof we have a map $g_1$ such that $g_1.s_0 = s_0.g_0$ and $g_1.f_1 = h_1$. The end of the proof (checking the commutation with the legs of the groupoids) is straightforward.

\[\square\]
2 Non pointed additive categories

The result asserted by Theorem 1.2 is actually underlying a stronger property which allows us to enrich the classification of non pointed additive categories. The weaker notion is the one of naturally Mal’cev category [16]. A naturally Mal’cev category is a Mal’cev category in which any pair \((R, S)\) of equivalence relations on an object \(X\) is connected. The stronger one is the notion of essentially affine category [4], namely finitely complete category with existence of pushouts of split monomorphisms along any map and such that, given any commutative square of split epimorphisms, the downward square is a pullback if and only if the upward square is a pushout:

\[ X' \xrightarrow{g} X \]
\[ f' \downarrow \quad \quad \downarrow f \]
\[ Y' \xrightarrow{h} Y \]

This is equivalent to saying that any change of base functor \(h^* : Pt_Y E \to Pt_Y E\) with respect with the fibration of points [4] is an equivalence of categories. Recall that the category \(E\) is naturally Mal’cev if and only if any fibre \(Pt_Y E\) is additive [5], and this last point is implied by the fact that the change of base functors \(h^*\) are equivalence of categories. The slice and coslice categories of a finitely complete additive category \(A\) are essentially affine. Notice then that, thanks to the Moore normalization, \(A/X\) is isomorphic to the fibre \(Grd_X A\). When the category \(E\) is pointed, the notions of naturally Mal’cev and essentially affine categories coincide with the notion of finitely complete additive category.

There is a well known intermediate notion, namely protomodular naturally Mal’cev categories (recall that a category is protomodular when any change of base functor \(h^*\) is conservative). This is the case, for instance, for the full subcategory \(Ab(Gp/Y)\) of the slice category \(Gp/Y\) whose object are group homomorphisms with abelian kernel. It is easy to check that the naturally Mal’cev protomodular category \(Ab(Gp/Y)\) is not essentially affine, since this would imply, considering the following diagram in \(Ab(Gp/Y)\), that any split epimorphism \(f : X \to Y\) with abelian kernel \(A\) is such that \(X = A \oplus Y\):

\[ A \xrightarrow{1} X \]
\[ \downarrow 1 \quad \quad \downarrow f \]
\[ Y \xrightarrow{1_Y} Y \]

The fibres \(AbGrd_X E\) of section 1.1 are other examples of naturally Mal’cev protomodular categories which are not essentially affine.

2.1 Penessentially affine categories

Let us introduce now two intermediate notions. Here is the first one:
Definition 2.1. A finitely complete category $\mathcal{E}$ is said to be antepenesentially affine when, for any square of split epimorphisms as above, the upward square is a pushout as soon as the downward square is a pullback.

The antepenesentially affine categories are stable by slice and coslice categories. According to Proposition 4 in [4], the previous definition is equivalent to saying that any change of base functor $h^*: \text{Pt}_Y\mathcal{E} \to \text{Pt}_Y\mathcal{E}$ is fully faithful. So, any essentially affine category is antepenesentially affine. On the other hand any fully faithful functor being conservative, any antepenesentially affine category is necessarily protomodular. Moreover any antepenesentially affine category is naturally Mal’cev for the same reasons as the essentially affine categories. On the other hand, again for the same reason as above, the protomodular naturally Mal’cev category $\text{Ab}(Gp/Y)$ and $\text{AbGrd}_{X}\mathcal{E}$ ($\mathcal{E}$ finitely complete) are not antepenesentially affine.

Definition 2.2. A finitely complete category $\mathcal{E}$ is said to be penesentially affine when it is antepenesentially affine and such that any (fully faithful) change of base functor $h^*$ is saturated on subobjects.

Recall that a left exact conservative functor $U: \mathcal{C} \to \mathcal{D}$ is saturated on subobjects when any subobject $j: d \hookrightarrow U(c)$ is isomorphic to the image by $U$ of some (unique up to isomorphism) subobject $i: c' \hookrightarrow c$. So, being penesentially affine implies that, given any downward parallelistic pullback as below and any pointed subobject $j': U' \hookrightarrow X'$ (with the retraction $\sigma'$ of $\sigma$ such that $f'j' = \sigma'$):

there is a (dotted) pushout of $\sigma'$ along $h$ which makes the upper upward diagram a pullback. The penesentially affine categories are stable by slice and coslice categories. Here is our first major structural point:

Theorem 2.1. Suppose $\mathcal{E}$ is Mal’cev. Then any fibre $\text{Grd}_{X}\mathcal{E}$ is penesentially affine.

Proof. Let us show first it is antepenesentially affine. Consider the following right hand side downward pullback in $\text{Grd}_{X}\mathcal{E}$:
Complete the diagram by the left hand side downward pullback, then the whole downward rectangle is a pullback. Now the upward left hand side square is a pushout as well as the whole upward rectangle. Accordingly the right hand side upward square is a pushout. The fact that the change of base functor along $h_1$ is saturated on subobjects is checked in the same way, thanks to Theorem 1.2.

2.2 Normal subobjects

Any penessential affine category is protomodular, and consequently yields an intrinsic notion of normal subobject. The aim of this subsection is to show that any penessential affine category is similar to an additive category insofar as any monomorphism is normal. Let us begin by the following more general observation:

**Proposition 2.1.** Let $\mathbb{E}$ be a naturally Mal’cev category. Then, given any monomorphism $s : Y \rightarrow X$ split by $f$, there is a unique equivalence relation $R$ on $X$ such that $s$ is normal to $R$ and $R \cap R[f] = \Delta X$. In any protomodular naturally Mal’cev category, and a fortiori in any antepenessentialy affine category, a split monomorphism is normal.

**Proof.** Consider the following diagram:

\[
\begin{array}{ccccccccc}
Y \times Y & \xrightarrow{s \times 1} & X \times Y & \xrightarrow{f \times 1} & Y \times Y \\
p_0 \downarrow & & p_x \downarrow & & p_0 \downarrow \\
X & \xrightarrow{f} & Y & \xrightarrow{s} & Y \\
\end{array}
\]

The right hand side downward square is a pullback of split epimorphisms in $\mathbb{E}$, and consequently a product in the additive fibre $Pt_Y \mathbb{E}$. Accordingly the left hand side upward square is a pushout. So the map $p_1 : Y \times Y \rightarrow Y$ produces a factorization $\psi : X \times Y \rightarrow X$ such that $\psi(1, f) = 1_X$ and $\psi(s \times 1) = s.p_1$. Whence a reflexive graph $(p_X, \psi) : X \times Y \Rightarrow X$ and thus a groupoid $X_1$ since $\mathbb{E}$ is naturally Mal’cev and thus satisfies the Lawvere condition following which any reflexive graph is a groupoid, see [16]. We can check $f.\psi = p_1.(f \times 1)$, thus we have a discrete fibration $f_1 : X_1 \rightarrow \nabla Y$. The codomain $\nabla Y$ being an equivalence relation, the domain $X_1$ is an equivalence relation we shall denote by $R$. The monomorphism $s$ is normal to $R$ since the left hand side downward square above is also a pullback. Moreover, by commutation of limits, the following square is a pullback in $Grd\mathbb{E}$:

\[
\begin{array}{ccc}
R \cap R[f] & \xrightarrow{} & \Delta Y \\
\downarrow & & \downarrow \\
R & \xrightarrow{X_1} & \nabla Y \\
\end{array}
\]
Since \( f \) is discrete fibration, the upper horizontal map is a discrete fibration and necessarily we have \( R \cap R[f] = \Delta X \).

Now suppose we have another equivalence \( S \) on \( X \) which is normal to \( s \) and such that \( S \cap R[f] = \Delta X \). By the first part of the assumption, there is a map \( \tilde{s} \) which makes the following downward left hand side square a pullback:

\[
\begin{array}{ccc}
Y \times Y & \overset{\tilde{s}}{\to} & S \\
\downarrow & & \downarrow \left( f \times f \right) \circ \left( d_0, d_1 \right) \\
Y & \overset{s}{\to} & Y \\
\end{array}
\]

and produces a splitting \( \tilde{s} \) of \( (f \times f).\left( d_0, d_1 \right) \). Accordingly we have a split epimorphism in the fibre \( P_{Y}E \):

\[
\begin{array}{ccc}
S & \overset{\tilde{s}}{\to} & Y \times Y \\
\downarrow & & \downarrow \left( f \times f \right) \circ \left( d_0, d_1 \right) \\
Y & \overset{s}{\to} & Y \\
\end{array}
\]

So, in this additive fibre, the domain of this split epimorphism is isomorphic to the product of its codomain by its kernel. But its kernel is \( S \cap R[f] = \Delta X \) by assumption, and thus \( S \simeq X \times Y \).

Now, when \( E \) is penessentially affine, we have more:

**Theorem 2.2.** Let \( E \) be a penessentially affine category. Then any monomorphism in \( E \) is normal.

**Proof.** Let \( m : X' \rightarrow X \) be any subobject. The change of base functor \( m^* : P_{X}E \rightarrow P_{X'}E \) is saturated on subobjects. Then consider the following diagram:

\[
\begin{array}{ccc}
1 \times m & m \times 1 & \to \ X \times X \\
\downarrow & & \downarrow \left( p_X, (1, m) \right) \\
X' \times X' & \overset{\tilde{m}}{\to} & R \\
\downarrow & & \downarrow \left( p_X, s_0 \right) \\
X' & \overset{m}{\to} & X \\
\end{array}
\]

The map \( 1 \times m \) determines a pointed subobject of \( (p_X, (1, m)) : X' \times X \rightrightarrows X' \). This produces a pointed subobject \( j : R \rightrightarrows X \times X \), and thus an equivalence relation on \( X \). Moreover the following vertical square is a pullback, which means
that $m$ is normal to $R$:

$$
\begin{array}{ccc}
X' \times X' & \stackrel{m}{\longrightarrow} & R \\
p_0 \downarrow & & \downarrow d_0 \\
X' & \stackrel{m}{\longrightarrow} & X 
\end{array}
$$

According to Theorem 2.1 we have the following:

**Corollary 2.1.** Let $E$ be any Mal’cev category. Then, in a fibre $\text{Grd}_X E$, any monomorphism is normal.

### 2.3 Classification table

We give, here, the classification table of our "non pointed additive" categories by decreasing order of generality:

<table>
<thead>
<tr>
<th>Category $\mathcal{C}$</th>
<th>Fibration: $\pi : Pt(\mathcal{C}) \rightarrow \mathcal{C}$</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>naturally Mal’cev</td>
<td>additive fibres</td>
<td>$\text{AutMal}$</td>
</tr>
<tr>
<td>protomodular and naturally Mal’cev</td>
<td>additive fibres + conservative change of base functors</td>
<td>$\text{AbGrd}_X E$ when $E$ finitely complete</td>
</tr>
<tr>
<td>antepenessentially affine</td>
<td>fully faithful change of base functors</td>
<td>$\text{Grd}_X E$ when $E$ Gumm</td>
</tr>
<tr>
<td>essentially affine</td>
<td>fully faithful saturated on subobj. change of base functors</td>
<td>$\text{Grd}_X E$ when $E$ Mal’cev</td>
</tr>
<tr>
<td></td>
<td>change of base functors are equivalences</td>
<td>$\text{Grd}_X \mathcal{A}$ when $\mathcal{A}$ fin., complete + additive</td>
</tr>
</tbody>
</table>

All the given examples do not belong to the next class. The category $\text{AutMal}$ is the variety of autonomous Mal’cev operations. A category is Gumm when it is finitely complete and satisfies the **Shifting Lemma** [10]. This means that, given any triple $R, S, T$ of equivalence relations on an object $X$ with $R \cap S \leq T$ the situation given by the continuous lines

$$
\begin{array}{ccc}
T & \xleftarrow{x} & S \\
\downarrow & & \downarrow \\
R & \xleftarrow{y} & T \\
\downarrow & & \downarrow \\
S & \xleftarrow{z} & T
\end{array}
$$

implies that $tTz$. A variety of universal algebras is Gumm if and only if it is congruence modular [15]. The Gumm categories are stable under slicing. Any regular Mal’cev category is Gumm. The table will be complete with the following:

**Proposition 2.2.** Suppose $E$ is a Gumm category. Then any internal groupoid is abelian. Accordingly any fibre $\text{Grd}_X E$ is naturally Mal’cev. Furthermore, any fibre $\text{Grd}_X E$ is antepenessentially affine.
Proof. Any internal Mal’cev operation on an object $X$ in a Gumm category is unique when it exists and necessarily associative and commutative, see Corollary 3.4 in [10]. This implies immediately that any internal group is abelian. The Gumm categories being stable under slicing, any internal groupoid is abelian by Proposition 1.1. In order to show that any fibre $\text{Grd}_X E$ is antepenentially affine, in the same way as in the proof of Theorem 2.1, it is sufficient to show that the square below Proposition 1.1 is a pushout, and that consequently the conditions of Lemma 1.1 are satisfied. For that, using the notations of this lemma, let us introduce the following mapping:

$$\tau : K_1[f_1 \times_X Z_1] \rightarrow V_1 \quad (\gamma, \phi) \mapsto t_1 \phi.h_1 \gamma.t_1 \phi^{-1}.h_1(s_1 \phi.\gamma.s_1 \phi^{-1})^{-1}$$

where $K_1[f_1 \times_X Z_1] = \{(\gamma, \phi)/\text{dom}\gamma = \text{dom}\phi\}$. The following diagram will complete the proof:

$$\begin{array}{ccc}
(\gamma, 1_x) & \xrightarrow{p_{K_1}} & (\gamma, \phi) \\
| & \searrow & | \\
(1_x, 1_x) & \xrightarrow{|p_{Z_1}|} & (1_x, \phi) \\
\tau & \downarrow & \\
& \tau & \\
\end{array}$$

where a kernel equivalence relation is denoted by the same symbol as the map itself. Clearly $R[p_{Z_1}] \cap R[p_{K_1}] \leq R[\tau]$. Moreover $\tau(1_x, 1_x) = 1_x = \tau(\gamma, 1_x)$ implies $\tau(1_x, \phi) = 1_y = \tau(\gamma, \phi)$, and $1_y = \tau(\gamma, \phi)$ is our condition.

The fibres $\text{Grd}_X E$ are not penessentialy in general, since the proof of Theorem 1.2 cannot apply to here.

2.4 Quasi-pointed penessentialy affine categories

We noticed that the fibres $\text{Grd}_X E$ are quasi-pointed. This particularity leads to further interesting observations. We recalled that a category is quasi-pointed when it has an initial object $0$ such that the unique map $\varpi : 0 \rightarrow 1$ is a monomorphism. The category $E/0 = \text{Pt}_0 E$ is then a full subcategory of $E$ stable under products and pullbacks. The inclusion $\text{Pt}_0 E \rightarrow E$ is a discrete fibration.

So it is stable by subobject, and by equivalence relation. Consequently, when moreover $E$ is regular, $\text{Pt}_0 E$ is stable under regular epimorphisms, which means that, when the domain of a regular epimorphism belongs to this subcategory, the codomain belongs to it as well. The quasi-pointed categories are stable by slice categories.

Definition 2.3. In a finitely complete quasi-pointed category, we shall call endosome of an object $X$ the object $\text{En}_X X$ defined by the following pullback:

$$\begin{array}{ccc}
\text{En}_X X & \xrightarrow{\epsilon_X} & X \\
\downarrow & & \downarrow \\
0 & \xrightarrow{\varpi} & 1
\end{array}$$
This construction determines a left exact functor \( En : E \to E/0 = Pt_0E \) which is a right adjoint to the inclusion. When \( E \) is regular, this functor preserves the regular epimorphisms. It is clear that when \( E \) is pointed this functor disappears, since it is nothing but the identity functor. Thanks to the following upper pullback, where the map \( \bar{\epsilon}_X \) is the unique map making the left hand side square commute and such that \( p_1 \bar{\epsilon}_X = \epsilon_X \), the functor \( En \) allows us to associate with any equivalence relation \( R \) on \( X \) a subobject \( I \) of \( EnX \) which we call the \textit{endonormalization} of the equivalence relation \( R \):

\[
\begin{array}{c}
I \\
i
\end{array} \longleftarrow \begin{array}{c}
\text{En}X \\
\epsilon_X
\end{array} \begin{array}{c}
\rightarrow \\
\times X
\end{array} \begin{array}{c}
\rightarrow \\
\text{En}_R
\end{array} \begin{array}{c}
R
\end{array} \begin{array}{c}
\rightarrow \\
\times X
\end{array} \begin{array}{c}
\rightarrow \\
\epsilon_X
\end{array} \begin{array}{c}
\rightarrow \\
X
\end{array} \begin{array}{c}
\rightarrow \\
\text{En}_X
\end{array} \begin{array}{c}
\rightarrow \\
\epsilon_X
\end{array} \begin{array}{c}
\rightarrow \\
X
\end{array} \begin{array}{c}
\rightarrow \\
0
\end{array}
\]

\textbf{Remark} The upper left hand side pullback, in the following diagram whose any square is a pullback, shows \( i \) is nothing but the classical normalization of the equivalence relation \( EnR \) (on the object \( EnX \)) in the pointed category \( Pt_0E \) since we have obviously \( \epsilon_X = \epsilon X \times \epsilon X.(0, 1) \):

\[
\begin{array}{c}
I \\
i
\end{array} \longleftarrow \begin{array}{c}
\text{En}X \\
\epsilon_X
\end{array} \begin{array}{c}
\rightarrow \\
\times X
\end{array} \begin{array}{c}
\rightarrow \\
\text{En}_R
\end{array} \begin{array}{c}
R
\end{array} \begin{array}{c}
\rightarrow \\
\times X
\end{array} \begin{array}{c}
\rightarrow \\
\epsilon_X
\end{array} \begin{array}{c}
\rightarrow \\
X
\end{array} \begin{array}{c}
\rightarrow \\
\text{En}_X
\end{array} \begin{array}{c}
\rightarrow \\
\epsilon_X
\end{array} \begin{array}{c}
\rightarrow \\
X
\end{array} \begin{array}{c}
\rightarrow \\
0
\end{array}
\]

Next we have:

\textbf{Proposition 2.3.} Suppose \( E \) penessentially affine and quasi-pointed. Then the endonormalization construction is bijective.

\textbf{Proof.} This is an immediate consequence of the fact that the change of base functor \( \alpha_X^* \) is saturated on subobjects. \( \square \)

\section{Baer sums and Baer categories}

When the naturally Mal’cev category \( E \) is moreover efficiently regular, there is a direction functor \( d : \mathbb{E}_g \to Ab(E) \) where \( \mathbb{E}_g \) is the full subcategory of objects with global support and \( AB(E) = Pt_1E \) is the category of global elements of \( E \) (which necessarily determine an internal abelian group structure in \( E \)). This functor \( d \) is a cofibration whose fibres are canonically endowed with a tensor
product, the so-called Baer sum, see [6]. Our aim will be to show there is, in the stronger context of penessentially affine categories, an alternative and simpler description of this Baer sum which mimics closely the classical Baer sum construction on exact sequences in abelian categories.

Recall the following [8]:

**Definition 3.1.** A category $\mathcal{C}$ is said to be efficiently regular when it is regular and such that any equivalence relation $T$ on an object $X$ which is a subobject $j : T \to R$ of an effective equivalence relation on $X$ by an effective monomorphism (which means that $j$ is the equalizer of some pair of maps in $\mathcal{C}$), is itself effective.

The efficiently regular categories are stable under slice and coslice categories. The category $GpTop$ (resp. $AbTop$) of (resp. abelian) topological groups is efficiently regular, but not Barr exact. A finitely complete regular additive category $\mathcal{A}$ is efficiently regular if and only if the kernel maps are stable under composition. In this context we can add some interesting piece of information:

**Proposition 3.1.** Suppose $\mathcal{E}$ naturally Mal’cev and efficiently regular. Then, given any monomorphism $s : Y \to X$ split by $f$, the equivalence relation $R$ on $X$ asserted by Proposition 2.1, to which $s$ is normal, is effective and produces a direct product decomposition $X \simeq Q \times Y$ where $Q$ is the quotient of $R$.

**Proof.** According to Proposition 2.1, we have a discrete fibration $f : R \to \nabla Y$. Certainly $\nabla Y$ is effective, and thus $R$ is effective, see [8]. Now consider the following diagram where $Q$ is the quotient of $R$:

$$
\begin{array}{ccc}
R & \xrightarrow{d_1} & X \\
\downarrow{f} & & \downarrow{f} \\
Y \times Y & \xrightarrow{p_0} & Y
\end{array}
\xrightarrow{q} Q
$$

Since the left hand side squares are pullbacks, then, according to the Barr-Kock theorem in regular categories, the right hand side square is a pullback, which gives us the direct product decomposition.

In the same order of idea, recall that, in a category $\mathcal{E}$ which is naturally Mal’cev and efficiently regular, the direction of an object $X$ with global support is given by the following diagram where, $\pi : X \times X \times X \to X$, written for $\pi_X$, is the value at $X$ of the natural Mal’cev operation:

$$
\begin{array}{ccc}
X \times X \times X & \xrightarrow{p_2} & X \times X \\
\downarrow{p_0} & & \downarrow{p_0} \\
X \times X & \xrightarrow{p_1} & X
\end{array}
\xrightarrow{q_X} dX
$$

The quotient $q_X$ of the upper equivalence relation does exist in the efficiently regular category $\mathcal{E}$ since the vertical diagram is a discrete fibration between
equivalence relations, see [8]. Actually the downward right hand side square is necessarily a pullback (\(E\) being regular) and the upward square a pushout (in a naturally Mal’cev category \(E\), the pair \((s_0, s_1) : X \times X \rightrightarrows X \times X \times X\), composing the edge of a pushout, is jointly strongly epic).

**Proposition 3.2.** Suppose \(D\) efficiently regular. Then any fibre \(Grd_X D\) is efficiently regular.

*Proof.* The regular epimorphisms in \(Grd_X D\) are the internal functors \(f_1 : X_1 \to Z_1\) such that the map \(f_1 : X_1 \to Z_1\) is a regular epimorphisms in \(D\). They are consequently stable under pullbacks. On the other hand, suppose the equivalence relation \(R_1 \rightrightarrows Z_1\) is effective. Then the underlying equivalence relation in \(D\) is still effective. Let \(q_1 : Z_1 \to Q_1\) be its quotient in \(D\). Then clearly the induced reflexive graph \(Q_1 \rightrightarrows X\) is underlying a groupoid \(Q_1\) and \(R_1 \rightrightarrows Z_1\) is the kernel relation of the internal functor \(q_1 : Z_1 \to Q_1\) in \(Grd_X D\). Accordingly \(Grd_X D\) is regular when \(D\) is regular. Suppose \(j_1 : S_1 \rightrightarrows R_1\) is an effective monomorphism in \(Grd_X D\). Then the underlying monomorphism \(j_1 : S_1 \rightrightarrows R_1\) is effective in \(D\) and the underlying equivalence relation \(S_1 \rightrightarrows Z_1\) is effective in \(D\). With the same arguments as above \(S_1 \rightrightarrows Z_1\) is an effective equivalence relation in \(Grd_X D\). \(\Box\)

### 3.1 Baer categories

Let us introduce the following:

**Definition 3.2.** We shall call Baer category any category \(E\) which is penessentially affine, efficiently regular, quasi-pointed and such that the endonormalization process reflects the effective monomorphism.

This implies that when the endonormalization (see Proposition 2.3) of an equivalence \(R\) is a kernel map, then \(R\) is effective, i.e. the kernel equivalence relation of some map. As a penessentially affine category, a Baer category is necessarily protomodular. Given a Baer category \(E\), the pointed subcategory \(E/0 = Pt_0 E\) is additive and efficiently regular, and consequently such that the kernel maps are stable under composition. The pertinence of this further definition comes from the following theorem which is our main structural point concerning internal groupoids:

**Theorem 3.1.** Let \(E\) be a Mal’cev efficiently regular category. Then any fibre \(Grd_X E\) is a Baer category.

*Proof.* Let \(R_1\) be an equivalence relation on \(Z_1\) in \(Grd_X E\). Its endonormaliza-
tion is given by the following pullback:

\[
\begin{array}{c}
\text{En}_1 Z_1 \\
\downarrow \\
Z_1 \\
\downarrow \\
Z_1 \times_0 Z_1
\end{array}
\xleftarrow{\Delta Z_1} \xrightarrow{\rho_0} \xleftarrow{\rho_1} Z_1
\]

Suppose that \(i_1\) is an effective monomorphism in \(\text{Grd}_X E\). Then the morphism \(i_1 : I_1 \to \text{En}_1 Z_1\) is an effective monomorphism in \(E\). By Theorem 1.2, we know that \(j_1 : R_1 \to Z_1 \times_0 Z_1\) is itself an effective monomorphism in \(E\). Since \(Z_1 \times_0 Z_1 \Rightarrow Z_1\) is an effective equivalence relation in \(E\), the equivalence relation \(R_1 \Rightarrow Z_1\) is effective in \(E\). Let \(q_1 : Z_1 \Rightarrow Q_1\) be its quotient in \(E\). Then clearly the induced reflexive graph \(Q_1 \Rightarrow X\) is an effective equivalence relation in \(E\). Since \(E\) is additive and efficiently regular, then \(j_1, i_1\) is still a kernel map. Accordingly, \(E\) being a Baer category, \(R\) is an effective equivalence relation in \(E\). Let \(q : X \Rightarrow Q\) be its quotient in \(E\). Since we have \(R \subseteq R[f]\), there is a factorization \(g : Q \Rightarrow Y\) which makes \(R\) effective in \(E/Y\).

We are in such a situation, for instance, with the categories \(E = \text{GpTop}\) and \(E = \text{GpHaus}\) of topological and Hausdorff groups. On the other hand we have:

**Proposition 3.3.** The Baer categories are stable under slice categories.

**Proof.** The only point which remains to check concerns the endonormalization process. So let \(f : X \to Y\) be an object in \(E/Y\) and \(R\) an equivalence relation on this object, which means that \(R \subseteq R[f]\). The endosome in \(E/Y\) of this object \(f\) is nothing but its kernel. Let us consider the following diagram in \(E\):

\[
\begin{array}{c}
I_R \\
\downarrow i \\
K[f] \\
\downarrow s_X \xrightarrow{(0,k)} X \\
\downarrow p_0 \\
X \times X \\
\downarrow p_1 \\
R[f] \\
\downarrow j \\
\xrightarrow{j_f} X \\
\downarrow f \\
Y
\end{array}
\]

Our assumption is that \(i\) is an effective monomorphism. This means it is a kernel map since the category \(E/0\) is additive. This is also the case for \(i_f\) since \(R[f]\) is an effective equivalence relation. Since \(E/0\) is additive and efficiently regular, then \(i_f, i\) is still a kernel map. Accordingly, \(E\) being a Baer category, \(R\) is an effective equivalence relation in \(E\). Let \(q : X \Rightarrow Q\) be its quotient in \(E\). Since we have \(R \subseteq R[f]\), there is a factorization \(g : Q \Rightarrow Y\) which makes \(R\) effective in \(E/Y\).
Proposition 3.4. In any Baer category $\mathbb{E}$ the following downward whole rectangle is a pullback and the following upward whole rectangle is a pushout:

\[
\begin{array}{ccc}
\text{En}X & \xrightarrow{\epsilon X} & X \times X \xrightarrow{q_X} dX \\
\downarrow p_0 & & \downarrow s_0 \\
0 & \xrightarrow{} & X \\
\end{array}
\]

Accordingly two objects with global support have same direction if and only if they have same endosome.

Proof. The downward left hand side square is a pullback and, $\mathbb{E}$ being penesionally affine, the associated upward square is a pushout. We just recall that the right hand part of the diagram fulfils the same property. Consequently $\text{En}X$ and $dX$ mutually determine each other.

Our second observation will be:

Proposition 3.5. Let $\mathbb{E}$ be any Baer category. Then the functor $\text{En} : \mathbb{E} \to \text{Pt}_0\mathbb{E}$ is cofibrant on regular epimorphisms. The associated cocartesian maps are regular epimorphisms.

Proof. This means that any regular epimorphism $g : \text{En}X \to C$ in $\text{Pt}_0\mathbb{E}$ determines a cocartesian map in $\mathbb{E}$. Clearly the condition on $g$ is equivalent to saying that $g$ is a regular epimorphism in $\mathbb{E}$. Now take $k : K \to \text{En}X$ the kernel of $g$ in the additive category $\text{Pt}_0\mathbb{E}$, and $R$ the associated equivalence relation on $X$ given by Proposition 2.3. It is an effective relation since its endonormalization $k$ is a kernel. Let $q : X \to Q$ be its quotient. Since the category $\mathbb{E}$ is regular, the functor $\text{En}$ preserves the quotients. So $\text{En}q$ is the quotient of the equivalence relation $\text{En}R$ and consequently the cokernel of its normalisation which is $k$, according to the remark following Definition 2.3. Thus the map $\text{En}q : \text{En}X \to \text{En}Q$ is nothing but (up to an isomorphism $\gamma$) our initial map $g : \text{En}X \to C$ which consequently appears to be the quotient of the equivalence relation $\text{En}R$. Whence a map $\epsilon$ given by the following diagram:

\[
\begin{array}{ccc}
\text{En}Q & \xrightarrow{\text{En}p_1} & \text{En}X \\
\downarrow & & \downarrow \epsilon R \downarrow \\
\text{En}p_0 & \xrightarrow{\text{En}q} & C \\
\downarrow & & \downarrow \epsilon \downarrow \\
R & \xrightarrow{p_1} & X \\
\end{array}
\]

A Baer category being necessarily protomodular, the middle square is a pushout since it is a pullback along a regular epimorphism. This implies the universal property of $q$ as a cocartesian map with respect to the functor $\text{En}$. By construction this map $q$ is a regular epimorphism.
The last observation of this proof gives the following corollary which is actually equivalent to the proposition itself:

**Corollary 3.1.** Let $\mathbb{E}$ be any Baer category. Then, along the map $\epsilon_X : EnX \twoheadrightarrow X$, there exists the pushout of any regular epimorphism in $Pt_0\mathbb{E}$ and the involved square is also a pullback:

\[
\begin{array}{c}
EnX \\
\downarrow \epsilon_X \\
X
\end{array} \quad \begin{array}{c}
\downarrow g \\
\rightarrow C
\end{array} \quad \begin{array}{c}
\downarrow \epsilon \\
\rightarrow Q
\end{array}
\]

Another immediate consequence of the previous proposition is the following:

**Corollary 3.2.** Let $\mathbb{E}$ be any Baer category. Then the fully faithful functor $\varpi^* : Pt_1\mathbb{E} \rightarrow Pt_0\mathbb{E}$ is cofibrant on regular epimorphisms.

### 3.2 An alternative description of Baer sums

We recalled that, when $\mathbb{E}$ is Mal’cev and efficiently regular, the functor $d$ is a cofibration whose fibres are actually groupoids (i.e. any map is cocartesian). When $\mathbb{E}$ is moreover a Baer category, we have, thanks to Proposition 3.4, a commutative triangle of functors:

\[
E_g \quad \begin{array}{c}
d \quad P_{t1}\mathbb{E} \quad = \quad Ab\mathbb{E}
\end{array}
\]

\[
\begin{array}{c}
E_n \\
\downarrow \\
P_{t0}\mathbb{E}
\end{array} \quad \begin{array}{c}
\downarrow \varpi^* \\
\rightarrow \varpi^*
\end{array}
\]

The functor $En$ takes its values in a larger category than $d$ and is only cofibrant on the regular epimorphisms. The fact that $\varpi^*$ is fully faithful implies that a cocartesian map $q$ with respect to $En$ which is associated with a map $g = \varpi(\gamma)$ (with $\gamma : dM = A \twoheadrightarrow C$ any regular epimorphism in $Pt_1\mathbb{E}$) is also a cocartesian map associated with $\gamma$ with respect to $d$.

#### 3.2.1 Level 1

Let us recall [6] that, when $\mathbb{E}$ is naturally Mal’cev and efficiently regular, any fibre of the direction functor $d$ is canonically endowed with a tensor product, namely the Baer sum. Let $a : 1 \rightarrow A$ be an element of $Pt_1\mathbb{E} = Ab\mathbb{E}$, $M$ and $N$ two elements in the fibre above $A$. Then $M \otimes N$ is defined as the codomain of the cocartesian map $\theta_+ : M \times N \twoheadrightarrow M \otimes N$ above $+: A \times A \rightarrow A$.

The previous remarks about the triangle of functors above allows us, in the context of Baer categories, the following easier construction of the Baer sum, very similar to the classical abelian case:

**Theorem 3.2.** Let $\mathbb{E}$ be any Baer category. Then the Baer sum of two objects $M$ and $N$ with global support and same direction $A$ is given by the following
pushout which is also a pullback:

\[
\begin{array}{ccc}
\omega^* A \times \omega^* A & \xrightarrow{+} & M \times N \\
\downarrow & & \downarrow \\
\omega^* A & \rightarrow & M \otimes N
\end{array}
\]

We classically denote by \( H^1_\omega(A) \) the abelian group structure determined by this Baer sum on the set \( \pi_0 d^{-1}(A) \) of the connected components of the fibre \( d^{-1}(A) \).

### 3.2.2 Level 2

Starting from a Baer category \( E \), any fibre \( Grd_X E \) is a Baer category, by Theorem 3.1. So the previous observations and constructions are the beginning of an iterative process. A groupoid \( Z_1 \) has a global support in the fibre \( Grd_X E \) if and only if it is connected, i.e. such that \( (z_0, z_1) : Z_1 \rightarrow Z_0 \times Z_0 = X \times X \) is a regular epimorphism. The direction of a connected abelian groupoid was defined in [7] as the pushout of \( s_0 : Z_1 \rightarrow Z_1 \times_0 Z_1 \) along \( (z_0, z_1) : Z_1 \times Z_1 \rightarrow dZ_1 \):

\[
\begin{array}{ccc}
Z_1 \times_0 Z_1 & \xrightarrow{s_0} & dZ_1 \\
\downarrow & & \downarrow \\
Z_1 & \xrightarrow{(z_0, z_1)} & Z_0 \times Z_0 \\
\downarrow & & \downarrow \\
Z_0 \times Z_0 & &
\end{array}
\]

For the same reasons as above this implies that the downward square is a pullback. In the Baer context, two connected groupoids in \( Grd_X E \) have same direction if and only if they have same endosome. The Baer sum of a pair \( (U_1, V_1) \) of connected groupoids having same endosome \( E_1 \) has thus its object of morphisms given by the following pushout in \( E \) which is also a pullback:

\[
\begin{array}{ccc}
E_1 \times_0 E_1 & \xrightarrow{+} & U_1 \times_0 V_1 \\
\downarrow & & \downarrow \\
E_1 & \rightarrow & (U_1 \otimes V_1)_1
\end{array}
\]

This is a much easier way than to go through the direction.

When the object \( X \) has a global support, the change of base along the terminal map \( \tau_X : X \rightarrow 1 \) produces an equivalence of categories [7]:

\[
\Gamma : Pt_1 E = AbE \rightarrow Pt_1 Grd_X E = AbGrd_X E ; A \mapsto K(A, 1) \times \nabla X
\]

where \( K(A, 1) \) denotes the groupoid structure (with object of objects 1) associated with the group structure \( A \).
Recall that an internal groupoid $\mathcal{Z}_1$ is said to be aspherical when it is connected and moreover $\mathcal{Z}_0$ has a global support (i.e. such that $\mathcal{Z}_0 \to 1$ is a regular epimorphism). The inverse of the functor $\Gamma$ allows to define the \textit{global direction} functor $\delta_1 : \text{Asp}\mathcal{E} \to P\text{t}_{\mathcal{E}} = \text{Ab}\mathcal{E}$, where $\text{Asp}\mathcal{E}$ is the full subcategory of aspherical groupoids (see [7]). It is given, this time, by the pushout of $s_0$ along the terminal map:

\[
\begin{array}{ccc}
Z_1 \times_0 Z_1 & \longrightarrow & d_1 Z_1 \\
p_0 \downarrow & & \downarrow \\
Z_1 & \longrightarrow & 1 \\
\langle z_0, z_1 \rangle \downarrow & & \downarrow \\
Z_0 \times Z_0 & \longrightarrow & \\
\end{array}
\]

This functor $d_1$ is a cofibration whose fibres are categories (and no longer groupoids as at level 1) canonically equipped with a tensor product called the \textit{global Baer sum}. Here again, the Baer categorical context will allow us to give an easier description of the global Baer sums of aspherical groupoids having same global direction.

\textbf{Definition 3.3.} Let $\mathcal{E}$ be a quasi-pointed category. The \textit{global endosome} of a groupoid $\mathcal{Z}_1$ is defined as the kernel of $\langle z_0, z_1 \rangle : Z_1 \to Z_0 \times Z_0$.

For exactly the same reasons as above, in a Baer category, two aspherical groupoids have same global direction if and only if they have same global endosome. The global Baer sum of a pair $(U_1, V_1)$ of aspherical groupoids with same global endosome $A$ is organized by the following pushout in $\mathcal{E}$ which is also a pullback:

\[
\begin{array}{ccc}
A \times A & \xrightarrow{\kappa_{U_1} \times \kappa_{V_1}} & U_1 \times V_1 \\
\downarrow & & \downarrow \\
A & \longrightarrow & (U_1 \otimes V_1)_1 \\
\end{array}
\]

Once again this is a much easier way than to go through the global direction. We denote by $H_2^\mathcal{E}(A)$ the abelian group structure determined by this global Baer sum on the set $\pi_0 d_1^{-1}(A)$ of the connected components of the fibre $d_1^{-1}(A)$ see [3] and [7].

3.2.3 The higher levels

The \textit{global direction} of an asperical n-groupoid $\mathcal{Z}_n$:

\[
\begin{array}{cccccccccccc}
\mathcal{Z}_n : Z_n & \xrightarrow{\partial_1} & Z_{n-1} & \xrightarrow{\partial_2} & \cdots & \xrightarrow{\partial_2} & Z_{n-2} & \xrightarrow{\partial_1} & Z_1 & \xrightarrow{\partial_0} & Z_0 \\
& \xrightarrow{\partial_0} & & \xrightarrow{\partial_0} & & \xrightarrow{\partial_0} & & \xrightarrow{\partial_0} & & \\
& & & & & & & & & & \\
\end{array}
\]
is defined in [18] and in [11] as a functor $d_n : n\text{-AspE} \to Pt_1E = AbE$ which is produced by the following pushout in $E$:

\[
\begin{array}{c}
Z_n \times_{n-1} Z_n \\
p_0 \downarrow \quad \downarrow p_1 \\
Z_n & \longrightarrow & d_nZ_n \\
\end{array}
\]

It is still a cofibration whose fibres are categories canonically endowed with a tensor product, still called the *global Baer sum* of $n$-groupoids. In a quasi-pointed category the *global endosome* of a $n$-groupoid $Z_n$ will be the kernel of the map $(z_0, z_1) : Z_n \to Z_{n-1} \times_{n-2} Z_{n-1}$. When $E$ is a Baer category, still two aspherical $n$-groupoids have same global direction if and only if they have same global endosome. The global Baer sum of of a pair $(U_n, V_n)$ of aspherical $n$-groupoids with same global endosome $A$ is organized by the following pushout in $E$ which is also a pullback:

\[
\begin{array}{c}
A \times A \\
\downarrow \quad \downarrow \\
A & \longrightarrow & (U_n \otimes V_n)_n \\
\end{array}
\]

We denote by $H^n_E(A)$ the abelian group structure determined by this global Baer sum on the set $\pi_0d_{n-1}^{-1}(A)$ of the connected components of the fibre $d_{n-1}^{-1}(A)$. These abelian groups $H^n_E(\cdot)$ were shown to have a Yoneda’s Ext style long cohomology sequence in [18] and [11].

### 4 The endosome of a groupoid

Now we have emphasized the role of the endosome of a groupoid, we shall end this work with some general observations about the endosome, when the ground category $E$ is only finitely complete. Let $Z_1$ be a groupoid in $E$. Let us begin by a useful property:

**Lemma 4.1.** Suppose we are given an internal functor $h_1 : R \to Z_1$ where $R$ is an equivalence relation on an object $Y$, then the following square is a pullback in the category $GrdE$:

\[
\begin{array}{c}
R[h_0] \cap R \\
\downarrow \quad \downarrow \\
R & \longrightarrow & Z_1 \\
\end{array}
\]
Proof. Straightforward by commutation of limits, considering the following rectangles, where the two squares on the left and the right hand side part on the right are pullbacks in $\text{Grd}E$: $$\begin{array}{ccc}
abla Y & \rightarrow & \nabla Z_0 \\
\downarrow & & \downarrow \\
R[h_0] \cap R & \rightarrow & R[h_0] \cap R \rightarrow \Delta Z_0 \\
\downarrow & & \downarrow \\
R & \rightarrow & \nabla h_0 \\
\downarrow & & \downarrow \\
\nabla h_0 & \rightarrow & \nabla Z_0
\end{array}$$
$$\begin{array}{ccc}
\nabla Y & \rightarrow & \nabla Z_0 \\
\downarrow & & \downarrow \\
R[h_0] \cap R & \rightarrow & R[h_0] \cap R \rightarrow \Delta Z_0 \\
\downarrow & & \downarrow \\
\nabla h_0 & \rightarrow & \nabla Z_0 \\
\downarrow & & \downarrow \\
Z_1 & \rightarrow & \Delta Z_0
\end{array}$$

We are now in position to specify the internal action of the endosome $E_{n_1}Z_1$ on the object $Z_1$ of morphisms:

**Proposition 4.1.** The endosome $E_{n_1}Z_1$ has a canonical action on $Z_1$ which is described by the following diagram where the vertical left hand side part is a kernel equivalence relation and the upper square a pullback:

$$\begin{array}{ccc}
Z_1 \times_0 Z_1 & \rightarrow & E_{n_1}Z_1 \\
\downarrow & & \downarrow \\
Z_1 & \rightarrow & E_{n_1}Z_1 \\
\downarrow & & \downarrow \\
(z_0, z_1) & \rightarrow & Z_0 \times Z_0
\end{array}$$

Proof. The previous lemma gives rise to the following upper left hand side pullback in $E$:

$$\begin{array}{ccc}
R[z_0] \cap R[z_1] & \rightarrow & E_{n_1}Z_1 \\
\downarrow & & \downarrow \\
Z_1 & \rightarrow & E_{n_1}Z_1 \\
\downarrow & & \downarrow \\
Z_0 & \rightarrow & Z_0
\end{array}$$

Furthermore the two lower left hand side squares are pullbacks since $Z_1$ is a groupoid. Now $R[z_0] \cap R[z_1]$ is nothing but the kernel equivalence relation of the map $(z_0, z_1) : Z_1 \rightarrow Z_0 \times Z_0$ whose underlying object is the object $Z_1 \times_0 Z_1$ of "parallel morphisms" of the groupoid $Z_1$ which, itself, is nothing but the object of morphisms of the groupoid $Z_1 \times_0 Z_1$. We have also $z_0.\epsilon_1 Z_1 = z_1.\epsilon_1 Z_1 = \epsilon_1 Z_1$. All this gives rise to the right hand side pullback with the $p_0$. The map $p_1 = z_1.\iota$ represents the canonical action of the endosome. In the set theoretical context the map $z_1 : Z_1 \times_0 Z_1 \rightarrow E_{n_1}Z_1$ associates with the pair $(\alpha, \beta) : x \Rightarrow y$ of parallel arrows in $Z_1$ the endomap $\beta.\alpha^{-1} : y \rightarrow y$. Notice that the section
induced by $s_0 : Z_0 \to Z_1$ of this map $\bar{z}_1$ is (by the equations defining it) nothing but the map $\bar{\epsilon}_1 Z_1 : \text{En}_1 Z_1 \to Z_1 \times_0 Z_1$ which associates with any endomap $\gamma : x \mapsto x$ the pair $(1_x, \gamma) : x \mapsto x$. 

We then get the following:

**Corollary 4.1.** Suppose $\mathcal{E}$ is naturally Mal’cev. Then the following square is a pushout in $\mathcal{E}$:

$$
\[
\begin{array}{ccc}
Z_1 & \xrightarrow{\bar{z}_1} & 
\text{En}_1 Z_1 \\
\downarrow{s_0} & & \downarrow{\alpha_1 \text{En}_1 Z_1} \\
Z_1 & \xleftarrow{s_0} & 
0
\end{array}
\]

Proof. These are the splittings of a split pullback, i.e. a product in the fibre $\text{Pt}_{Z_0}$, and, the category $\mathcal{E}$ being naturally Mal’cev, this fibre is additive. 

**References**


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