Legendre modified moments for Euler's constant

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Abstract

Polynomials moments are often used for the computation of Gauss quadrature to stabilize the numerical calculation of the orthogonal polynomials, see ([5–7] or numerical resolution of linear systems ([4]). These modified moments can also be used to accelerate the convergence of sequences to a real or complex numbers if the error satisfies some properties as done in ([3,12]). In this paper, we use Legendre modified moments to accelerate the convergence of the sequence $H_n - \log(n+1)$ to the Euler's constant γ. A formula for the error is given. It is proved that it is a totally monotonic sequence. At last, we give applications to the arithmetic property of γ.

Key words: Legendre moments, Euler's constant, Padé approximations
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1 Introduction

The Euler constant $\gamma = 0.577216...$ is the limit of the sequence

$$H_n - \log(n+1)$$

where H_n is the harmonic number defined by

$$\sum_{k=1}^{n} \frac{1}{k}.$$

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An integral representation for Euler's constant is

$$\gamma = \int_0^1 \left(\frac{1}{\ln u} + \frac{1}{1-u} \right) \, \mathrm{d}u. \tag{1}$$

The sequence (S_n)

$$\begin{aligned} S_n := & \int_0^1 \left(\frac{1-u^n}{\log u} + \frac{1-u^n}{1-u} \right) \mathrm{d}u \\ = & H_n - \log(n+1). \end{aligned}$$

converges to γ but very slowly, as $\mathcal{O}(1/n)$.

The error $\gamma - S_n$ satisfies:

$$\gamma - S_n = \int_0^1 u^n \left(\frac{1}{\log u} + \frac{1}{1-u} \right) \mathrm{d}u \tag{2}$$

and can be considered as moments with respect to the weight function

$w(u) = \left(\frac{1}{\log u} + \frac{1}{1-u} \right)$ on the interval $[0, 1]$.

2 Legendre modified moments

Suppose we are given a sequence of real or complex numbers $(x_n)_n$ converging to l and satisfying the property

$$x_n - l = \int_0^1 u^n \mathrm{d}\mu(u)$$

where $d\mu$ is a positive measure on the interval $[0, 1]$.

If the error of a sequence is of this form, then a way to decrease this error and so accelerate the convergence is to use modified moments, i.e. by replacing the monomial u^n by some suitable polynomials P_n normalized by $P_n(1) = 1$ In that case, the error between the limit l and the transformed sequence y_n

becomes

$$y_n - l = \int_0^1 P_n(u)\mathrm{d}\mu(u).$$

Of course, each y_n is a linear combination of $x_0, x_1, \ldots, x_n$.

To improve the convergence, the polynomial P_n can be chosen to be orthogonal with respect to some weight. In our case, because of the behavior of the weight function around 1 ($w(u) \sim \mathcal{O}(1), u = 1$), a good choice will be the shifted Legendre polynomials which are orthogonal on $[0, 1]$

$$\int_0^1 P_n^*(t)P_m^*(t)\mathrm{d}t = 0, n \neq m.$$

These polynomials can be expressed in different bases:

$$P_n^*(t) = \sum_{k=0}^{n} \binom{n}{k}^2 t^{n-k}(t-1)^k \tag{3}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n+k} t^k \tag{4}$$

Theorem 1 *For $0 \leq m \leq n$, let us define*

$$J_{n,m} = \int_0^1 u^{n-m} P_n^*(u) \left(\frac{1}{\ln u} + \frac{1}{1-u} \right) \mathrm{d}u \tag{5}$$

$$L_{n,m} = -\int_0^1 \left(\frac{1 - u^{n-m} P_n^*(u)}{\ln u} \right) \mathrm{d}u \tag{6}$$

$$A_{n,m} = \int_0^1 \left(\frac{1 - u^{n-m} P_n^*(u)}{1-u} \right) \mathrm{d}u \tag{7}$$

then

$$\gamma = A_{n,m} - L_{n,m} + J_{n,m} \tag{8}$$

$$L_{n,m} = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n+k} \ln(n - m + k + 1) \tag{9}$$

$$A_{n,m} = 2H_n \tag{10}$$

Proof

We first prove the identity (8) linking Euler's constant γ, the linear combination of logarithms numbers $L_{n,m}$, the rational numbers $A_{n,m}$ and the integrals $J_{n,m}$. From formula (1), one substitute the integrand $\left(\frac{1}{\ln u} + \frac{1}{1-u}\right)$ by an approximation involving Legendre Polynomials as follows:

$$\gamma = \int_0^1 \left(\frac{1}{\ln u} + \frac{1}{1-u}\right) \, \mathrm{d}u \tag{11}$$

$$= \int_0^1 \left(\frac{1 - u^{n-m} P_n^*(u)}{\ln u} + \frac{1 - u^{n-m} P_n^*(u)}{1-u}\right) \, \mathrm{d}u + \tag{12}$$

$$\int_0^1 u^{n-m} P_n^*(u) \left(\frac{1}{\ln u} + \frac{1}{1-u}\right) \, \mathrm{d}u. \tag{13}$$

The expression (4) of P_n^* leads to analogous expressions $L_{n,m}$.
By linearity

$$L_{n,m} = -\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n+k} \int_0^1 \left(\frac{1 - u^{k+n-m}}{\ln u}\right) \mathrm{d}u \tag{14}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n+k} \ln(n - m + k + 1) \tag{15}$$

$A_{n,m}$ is treated quite differently:
From the orthogonality relation between two polynomials P_n^*

$$\int_0^1 P_n^*(u) q(u) \, \mathrm{d}u = 0, \text{for all polynomial } q \text{ of degree less than } n. \tag{16}$$

by taking $q(u) = \frac{1 - u^{n-m}}{1-u}$, another expression for $A_{n,m}$ is

$$A_{n,m} = \int_0^1 P_n^*(u) q(u) \, \mathrm{d}u + \int_0^1 \frac{1 - P_n^*(u)}{1-u} \, \mathrm{d}u = \int_0^1 \frac{P_n^*(1) - P_n^*(u)}{1-u} \, \mathrm{d}u \tag{17}$$

and so $A_{n,m}$ is independent of $0 \le m \le n$.

Let us now compute the integral in (17).

Legendre polynomials satisfy a three term recurrence relation which is

$$(n+1)P^*_{n+1}(u) = (2n+1)(2u-1)P^*_n(u) - nP^*_{n-1}(u) \tag{18}$$
$$P^*_0(u) = 1 \quad P^*_1(u) = 2u - 1 \tag{19}$$

Thus, $A_{n,m}$'s also satisfy a similar recurrence relation

$$(n+1)A_{n+1,m} = (2n+1)A_{n,m} - nA_{n-1,m} \tag{20}$$
$$A_{0,m} = 0 \quad A_{1,m} = 2 \tag{21}$$

With (20) and (21), it is not difficult to prove that

$$A_{n,m} = 2H_n, 0 \leq m \leq n \tag{22}$$

□

In section 4 , we will prove the asymptotic formula

$$\gamma = A_{n,m} - L_{n,m} + \mathcal{O}(4^{-n}). \tag{23}$$

Actually, we will prove that the error term $J_{n,m}$ is a totally monotone sequence (i.e. a sequence of monomial moments with respect a positive measure), converging to 0 as 4^{-n}.

Before, we need to investigate the analytic property of the weight function w which provides a new integral formula for the Euler's constant.

3 New formula for Euler's constant

Euler constant can be written as sum of series or with integral representation. (see http://numbers.computation.free.fr/Constants/Gamma/gamma.html).

In this section, from an integral representation of the weight w proved in Lemma 2, we can write a new integral formula for γ.

Lemma 2 *The function* $\dfrac{1}{\ln(1-u)} + \dfrac{1}{u}$ *is a Markov-Stieltjes function. More* *precisely,*

$$\frac{1}{\ln(1-u)} + \frac{1}{u} = \int_0^1 \frac{1}{1-u\,t}\mu(t)\,\mathrm{d}t \tag{24}$$

where the weight function μ is

$$\mu(t) := \frac{1}{t\,(\ln^2(1/t-1)+\pi^2)}$$

Proof

After a change of variable ($u \to (1-u)$ and $x = 1/t - 1$), formula (24) is equivalent to

$$\frac{1}{\ln(u)} + \frac{1}{1-u} = \int_0^\infty \frac{1}{x+u}\frac{1}{\ln^2 x + \pi^2}\,\mathrm{d}x \tag{25}$$

The weight function μ can be found with the Stieltjes inversion formula (see [15]).

Another way to prove formula (25) is to apply residue theorem to the function

$$f(x) := \frac{1}{x+u}\frac{1}{\ln x + i\pi}.$$

Taking the determination of $\ln x$ on the complex plane cut along the positive real axis, the poles of f are $x = -u$ and $x = -1$.

Let us define γ_r a small semi-circle $z = re^{i\theta}, -\pi/2 \le \theta \le \pi/2, r > 0$. D_r^+ the line $z = x + ir, x$ running from 0 *to* R, Γ_R the circle $z = Re^{i\theta}, 0 \le \theta \le 2\pi$ and D_r^- the line $z = x - ir,$ for x from R *to* 0.

Now, we compute $\int_{\mathcal{C}} f(x)\,\mathrm{d}x$ where $\mathcal{C}$ is the union of D_r^+, Γ_R, D_r^- and γ_r, with the theorem of residue to obtain

$$\int_0^\infty \frac{1}{x+u}\left(\frac{1}{\ln x + i\pi} + \frac{-1}{\ln x - i\pi}\right)\,\mathrm{d}x = \int_0^\infty \frac{1}{x+u}\left(\frac{-2i\pi}{\ln^2 x + \pi^2}\right)\,\mathrm{d}x \tag{26}$$

$$= -2i\pi\left(\frac{1}{\ln u} + \frac{1}{1-u}\right) \tag{27}$$

□

Now, we are in position to prove a new formula for the Euler's constant γ.

Theorem 3 *The Euler's constant* γ *satisfies*

$$\gamma = \int_{-\infty}^{+\infty} \frac{\ln(1+e^{-z})e^{z}}{z^2+\pi^2}\,\mathrm{d}z \tag{28}$$

Proof

In the integral representation (1) of γ , let us substitute the integrand by the expression (24). This leads to

$$\gamma = \int_{0}^{1} -\frac{\ln(1-t)}{t}\frac{1}{t(\ln^2(1/t-1)+\pi^2)}\,\mathrm{d}t \tag{29}$$

$$= \int_{-\infty}^{\infty} \frac{\ln(1+e^{-z})e^{z}}{z^2+\pi^2}\,\mathrm{d}z \tag{30}$$

with the change of variable $t = (1+e^{z})^{-1}$.

□

With the expression of the weight function as a Markov Stieltjes function, it is possible to deduce some interesting properties for the error, for example to show that it is a sequence of moments.

4 Behavior of the error

Theorem 4 *For each fixed integer* m*, the sequence* $((-1)^m J_{n,m})_n$ *defined in Theorem 1 is totally monotonic. More precisely*

$$(-1)^m J_{n,m} = \int_{0}^{1/4} v^n \rho_m(v)\,\mathrm{d}v = \mathcal{O}(4^{-n}), \tag{31}$$

where the weight function is

$$\rho_m(v) = \int_{\frac{1-\sqrt{1-4\,v}}{2}}^{\frac{1+\sqrt{1-4\,v}}{2}} \left(\frac{u-u^2-v}{u\,v}\right)^m \frac{1}{(u-u^2-v)\left(\pi^2+\ln^2\left(\frac{-u\,v}{u^2-u+v}\right)\right)}\,\mathrm{d}u.$$

Proof

$J_{n,m} = \int_0^1 u^{n-m} P_n^*(u) \left(\frac{1}{\ln u} + \frac{1}{1-u}\right) \mathrm{d}u$ appears as Legendre modified moments of the weight function $\left(\frac{1}{\ln u} + \frac{1}{1-u}\right)$.
For some particular cases of weight function, a sequence of polynomial modified moments can be itself a sequence of monomial moments, with respect to a positive measure (see [13]). Using Rodrigues formula for orthogonal polynomials, Lemma 2, Fubini's theorem and after n integrations by parts, it arises

$$J_{n,m} = \int_0^1 u^{n-m} \frac{(-1)^n}{n!} \frac{\mathrm{d}^n}{\mathrm{d}u^n} \left(u^n (1-u)^n\right) \left(\frac{1}{\ln u} + \frac{1}{1-u}\right) \mathrm{d}u \tag{32}$$

$$= \int_0^1 u^{n-m} \frac{(-1)^n}{n!} \frac{\mathrm{d}^n}{\mathrm{d}u^n} \left(u^n (1-u)^n\right) \mathrm{d}u \int_0^1 \frac{1}{1-(1-u)\,t} \mu(t)\, \mathrm{d}t \tag{33}$$

$$= \int_0^1 \int_0^1 \frac{(-1)^n}{n!} u^n (1-u)^n \, \mathrm{d}u \frac{\mathrm{d}^n}{\mathrm{d}u^n} \left(\frac{u^{n-m}}{1-(1-u)\,t}\right) \mu(t)\, \mathrm{d}t. \tag{34}$$

The computation of $\frac{\mathrm{d}^n}{\mathrm{d}u^n} \left(\frac{u^{n-m}}{1-(1-u)\,t}\right)$ needs the partial decomposition of the rational function $\frac{u^{n-m}}{1-(1-u)\,t} = q(u) + \left(\frac{t-1}{t}\right)^{n-m} \frac{1}{1-(1-u)t}$, where q is polynomial of degree $n-m-1$.

Another expression of $J_{n,m}$ is then

$$J_{n,m} = \int_0^1 \int_0^1 u^n (1-u)^n \left(\frac{t-1}{t}\right)^{n-m} \frac{t^n}{(1-(1-u)t)^{n+1}} \mu(t)\, \mathrm{d}t\, \mathrm{d}u. \tag{35}$$

We do the following change of variable

$$v = \frac{u(1-u)(1-t)}{1-(1-u)t} \in [0, 1/4] \Leftrightarrow t = \phi(v) = \frac{u^2 - u + v}{(u-v)(u-1)} \in [0, 1].$$

Let $\phi_1(v)$ and $\phi_2(v)$ denote the two roots of the quadratic equation $v = u - u^2$, $\phi_1(v) = \frac{1+\sqrt{1-4\,v}}{2}$, $\phi_2(v) = \frac{1-\sqrt{1-4\,v}}{2}$.

$$J_{n,m} = \int_0^{1/4} v^n \mathrm{d}v \int_{\phi_1(v)}^{\phi_2(v)} \left(\frac{-\phi(v)}{\phi(v)-1} \right)^m \frac{u^2 \mu(\phi(v))}{(u-1)(u-v)^2} \frac{(-1)^m \mathrm{d}u}{1-(1-u)\phi(v)} \tag{36}$$

which can be simplified to give the result.

□

As quoted in [3,12,13], the sequence $(A_{n,m} - L_{n,m})_n$ converging to γ with an error totally monotonic on the interval $[0, R]$ with $R = 1/4$ can be accelerated by the ε-algorithm to obtain an error of order $\mathcal{O}\left(((2/R-1) - \sqrt{(2/R-1)^2 - 1})^n \right) = \mathcal{O}\left((7 - \sqrt{48})^n \right)$

Remark: it exists some sequence converging to γ with an error of order $\mathcal{O}(e^{-8n})$ ([2]), but they don't not provide arithmetic property for γ, as we can do in the last section.

5 Approximation of γ by rational numbers

In the numerical computation of formula

$$L_{n,m} = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n+k} \ln(n-m+k+1),$$

the problem is the evaluation of logarithmic functions.
A mean to avoid this drawback is the substitution of $\ln(n-m+k+1)$ by some suitable approximations. We will show now that Padé approximants are good enough to preserve the speed of convergence $\mathcal{O}(4^{-n})$.
Another expression of $L_{n,m}$ is

$$L_{n,m} = \ln(n-m+1) + \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n+k} \ln\left(1 + \frac{k}{n-m+1}\right). \tag{37}$$

By substituting in $L_{n,m}$, $\ln(n+1+k-m)$ by its Padé approximant $[n,n]$, a rational approximation is obtained as following:

for $n-m+1 = 2^p, p \in \mathbf{Z}$, let us define

$$\tilde{L}_{n,m} := p\,[n/n]_{t=1} + \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n+k} [n/n]_{t=k/(n-m+1)} \tag{38}$$

where $[n/n]$ is the Padé approximant of $\ln(1+t)$ at $t=0$:

$$[n/n]_t = \frac{t\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\left(\sum_{i=0}^{k-1}\frac{t^{i-k+n}(-1)^i}{i+1}\right)}{\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}t^{n-k}} \tag{39}$$

If we replace in (8), the quantities $L_{n,m}$ by $\tilde{L}_{n,m}$, we get an approximation of γ by rational numbers:

Theorem 5 *For integers n, m, p such that $n-m+1 = 2^p$*

$$\gamma = A_{n,m} - \tilde{L}_{n,m} + \mathcal{O}(4^{-n}) \tag{40}$$

Proof

The Padé error for the logarithmic function is

$$\ln(1+x) - [n/n]_x = \frac{(-1)^n x^{n+1}}{P_n^*(-1/x)}\int_0^1 \frac{t^n\,(1-t)^n}{(1+x\,t)^{n+1}}\,\mathrm{d}t \tag{41}$$

Let us set

$$L'_{n,m} := \ln(n-m+1) + \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n+k}[n/n]_{t=k/(n-m+1)} \tag{42}$$

We have to evaluate the difference $\delta_{n,m} := L_{n,m} - L'_{n,m}$. For sake of simplicity, we set $\zeta_k = \dfrac{k}{n-m+1}$.

$$\delta_{n,m} = \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n+k}\left(\ln(1+\zeta_k) - [n/n]_{t=\zeta_k}\right) \tag{43}$$

$$= \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^k\frac{\zeta_k^{n+1}}{P_n^*(-\zeta_k^{-1})}\int_0^1 \frac{t^n\,(1-t)^n}{(1+\zeta_k\,t)^{n+1}}\,\mathrm{d}t \tag{44}$$

Since $\zeta_k \in [0,1]$ and P_n^* has all its roots in $[0,1]$, $\left|\dfrac{\zeta_k^n}{P_n^*(-\zeta_k^{-1})}\right| \le \dfrac{1}{|P_n^*(-1)|}$. On the other hand, the integral

$$\int_0^1 \frac{t^n\,(1-t)^n}{(1+\zeta_k\,t)^{n+1}}\mathrm{d}t \le 4^{-n}\int_0^1 \frac{1}{(1+\zeta_k\,t)^{n+1}}\mathrm{d}t \le 4^{-n}\frac{1}{n\,\zeta_k}. \tag{45}$$

So,

$$\begin{aligned}
|\delta_{n,m}| &\le \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\left|\frac{\zeta_k^{n+1}}{P_n^*(-\zeta_k^{-1})}\right|\left|\int_0^1 \frac{t^n\,(1-t)^n}{(1+\zeta_k\,t)^{n+1}}\,\mathrm{d}t\right| \\
&\le \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\frac{\zeta_k}{|P_n^*(-1)|}4^{-n}\frac{1}{n\,\zeta_k} \\
&\le \frac{1}{|nP_n^*(-1)|}\,4^{-n}\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k} = \frac{1}{|nP_n^*(-1)|}\,4^{-n}\,|P_n^*(-1)| = (n\,4^n)^{-1}
\end{aligned}$$

The goal is partly reached since the error between $L_{n,m}$ and its approximation is less than $J_{n,m}$. Now, let us consider the approximation of $\ln(n-m+1)$. It is difficult to approximate this number (which tends to infinity) with an error less than 4^{-n}. So, we consider sequences of integers n, such that $n-m+1$ is a power of 2: $n-m+1 = 2^p$. With this hypothesis, $\ln(n-m+1) = p\,\ln 2$

In (41), if $x=1$, $\ln 2 - [n/n]_{x=1} = \dfrac{(-1)^n}{P_n^*(-1)}\displaystyle\int_0^1 \frac{t^n\,(1-t)^n}{(1+\,t)^{n+1}}\,\mathrm{d}t$. The asymptotic for Legendre polynomials are well known

$$P_n(\alpha) \sim (\alpha + \sqrt{\alpha^2-1})^n, \text{ for } \alpha \in \mathbf{R}\setminus[-1,1].$$

Thus the shifted Legendre Polynomials satisfy

$$P_n^*(t) \sim ((2t-1) + 2\sqrt{t^2-t})^n, \text{ for } t \in \mathbf{R}\setminus[0,1].$$

The maximum of the fraction $\left(\dfrac{t(1-t)}{1+t}\right)$ for $t \in [0,1]$ is obtained for $t = \sqrt{2}-1$, and its value is $(3-2\sqrt{2})$. Thus

$$|\ln 2 - [n/n]_{x=1}| \leq \frac{(3-2\sqrt{2})^n}{(3+2\sqrt{2})^n}\ \ln 2 \tag{46}$$

For $n-m+1 = 2^p$, $\ln(2^p) - p\,[n/n]_{x=1} \leq p\,(3-2\sqrt{2})^{2\,n}$ which is a $o(4^{-n}/n)$. At last, the error $\left|L_{n,m} - \tilde{L}_{n,m}\right|$ satisfies

$$\left|L_{n,m} - \tilde{L}_{n,m}\right| = \mathcal{O}(4^{-n}/n).$$

Now,

$$\gamma - A_{n,m} + \tilde{L}_{n,m} = J_{n,m} + \tilde{L}_{n,m} + L_{n,m} = \mathcal{O}(4^{-n}) + \mathcal{O}(4^{-n}/n) = \mathcal{O}(J_{n,m}) \tag{47}$$

and the theorem is proved.

□

6 Application to the arithmetic property of γ

The irrationality of γ remains an open question. In this section, we prove some sufficient conditions for the proof of the irrationality of Euler's constant. Corollary 6 is similar to the results found in [9–11], but it involves only the computation of rational numbers.

On the other hand, corollary 7 is new because it depends on the decrease of a numerical sequence involving logarithms numbers.

Corollary 6 *Let us denote by* $\{x\}$ *the fractional part of the real number* x*:*

$$\{x\} := x - \lfloor x \rfloor$$

and $d_n := \mathbf{LCM}(1,\ldots,n)$*. If for some integer* m*,* $\{d_{2^p+m-1}(-1)^m \tilde{L}_{2^p+m-1,m}\}$ *does not converge to* 0 *when* p *tends to infinity, then* γ *is irrational.*

Proof

Suppose that γ is rational, then it exists a pair of integers A, B such that $\gamma = \frac{A}{B}$. Then, for n greater than some integer N, $d_n\gamma$ is an integer. For integers n, m, p such that $n-m+1 = 2^p$,the relation

$$\gamma = A_{n,m} - \tilde{L}_{n,m} + \mathcal{O}(J_{n,m}),$$

leads to

$$d_n\gamma = d_nA_{n,m} - d_n\tilde{L}_{n,m} + d_n\mathcal{O}(J_{n,m}).$$

$A_{n,m} = 2H_n$, so $d_n A_{n,m}$ is an integer, and thus the fractional part of $(-1)^m d_n \tilde{L}_{n,m}$ is equal to the fractional part of the positive sequence $(-1)^m d_n \mathcal{O}(J_{n,m})$ which converges to zero since $\lim_n d_n^{1/n} = e$ ([8]).

So, if for some integer m, the fractional part $(\{d_{2^p+m-1}(-1)^m \tilde{L}_{2^p+m-1,m}\})_p$ does not converge to 0, then γ is irrational.

□

Another sufficient condition comes from the property of the error term in the asymptotic formula (23) and from the upper and lower bound of the $LCM(1, \ldots, n)$:

Corollary 7 *Let $\mathcal{P}$ the following property: A sequence $(x_n)_n$ satisfies $\mathcal{P}$ if*

$$\forall N \in \mathbf{N}, \exists n \geq N, x_n - x_{n+1} < 0.$$

If for some integer m, the sequence $\{d_{2^p}(-1)^m L_{2^p,m}\}$ satisfies $\mathcal{P}$ then γ is irrational.

Proof

For the proof, we exploit the property of totally monotonic sequences (TMS). A sequence u_n is called TMS if there exists a non negative measure $d\mu$ with infinitely many points of increase such that

$$\forall n \in \mathbf{N},\ u_n = \int_0^\infty x^n \, \mathrm{d}\mu(x).$$

If the support of the measure $d\mu$ is the interval $[0, 1/R]$, then $\forall n, u_{n+1}/u_n \leq R$ and $\lim_n \frac{u_{n+1}}{u_n} = R$. If $R = 1$, it is equivalent to

$$\forall n \in \mathbf{N}, \forall k \in \mathbf{N}, (-1)^k \Delta^k(u_n) > 0$$

where $\Delta^0(u_n) := u_n$ and $\Delta^{k+1} u_n = \Delta^k u_{n+1} - \Delta^k u_n$. (see [15], p. 108).

The previous properties can be applied to the sequence $J_{n,m}$ for which we prove some convergence properties. Since $\{d_n(-1)^m J_{n,m}\} = d_n(-1)^m J_{n,m} = \{d_n(-1)^m L_{n,m}\}$, if they are not satisfied by $\{d_n(-1)^m L_{n,m}\}$ then γ is irrational.

First we will prove that $J_{n,m}$ satisfies $d_{2n}(-1)^m J_{2n,m} < d_n(-1)^m J_{n,m}$: the numbers d_n and $J_{n,m}$ satisfy $2^n \leq d_n < e^{1.039\, n}$ (see [14], p.12-13 for the lower bound and [8] for the upper one) $\frac{J_{n+1,m}}{J_{n,m}} < 1/4$ (property of totally monotonic sequence [15], p.135).

$$\frac{d_n J_{n,m}}{d_{2n} J_{2n,m}} > \frac{2^n}{e^{1.039\times\, 2n}}\, 4^n > 1.0014$$

Thus, for all integer m, $\left(d_{2^p}(-1)^m J_{2^p,m}\right)_{p\in\mathbf{N}}$ is a positive decreasing sequence, converging to 0. So, if $\left(\{d_{2^p}(-1)^m L_{2^p,m}\}\right)_p$ is non decreasing for p greater than any integer, then γ is irrational.

$\square$

Consequence

If for some m, $\{d_{2^p}(-1)^m L_{2^p,m}\}$ satisfies $\mathcal{P}$, then γ is irrational.

Suppose that for some m, and some p $\{d_{2^p}(-1)^m L_{2^p,m}\} < \{d_{2^{p+1}}(-1)^m L_{2^{p+1},m}\}$, then it implies that if γ is rational then its denominator is greater than 2^p.

Numerical computation show that it is true for $m = 1, p = 15$. So, if γ is rational, then its denominator is greater than $2^{15} = 32768$.

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