APPROXIMATION AND CONVEX PROGRAMMING APPROACHES FOR SOLVING A MIN-MAX PROBLEM

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Abstract. Krylov subspace methods are among the best-known and most widely used numerical iterations for solving linear systems of equations and for computing eigenvalues of large matrices. These methods are polynomial methods whose convergence analysis is related to the behavior of polynomials on the spectrum of the matrix. This leads to discrete min-max approximation problems which enable us to obtain the upper bound of the relative Euclidean residual norm. More recently, to analyze the convergence rate of the GMRES method or the Arnoldi iteration method, we have presented in [1, 2] a new approach, based on an expression for the residual norm in terms of determinants of Krylov matrices. An interesting feature of our treatment is that it allows us to provide, among others, a thorough analysis of the convergence for normal matrices in the context of convex constrained optimization.
The goal of this paper is to reveal the relationship between these two approaches. More precisely, we prove that, for normal matrices, the Karush-Kuhn-Tucker (KKT) optimal conditions deduced from the convex maximization problem and the characterization properties of polynomial of best approximation on a finite set of points are identical. So the two approaches are mathematically equivalent.

Key words. Krylov subspaces, polynomials of best approximation, min-max problem, interpolation, convex optimization, KKT optimality conditions.

1. Introduction.

This paper is concerned with the question of convergence properties of Krylov subspace methods whose main tasks are the computation of the solution of linear systems of equations or of eigenvalue problems. They are projection methods for solving large sparse problems

$$A\,x = b, \quad \text{or} \tag{1.1}$$

$$A\,u = \lambda\,u, \tag{1.2}$$

using $\mathcal{K}_m(A, v) = span\left\{v,\ A\,v, \cdots, A^{m-1}v\right\}$, the Krylov subspace generated by v and A, where $v \in \mathbb{C}^N$ is a given initial approximate solution and A is a given non-singular complex matrix of size $N \times N$.
A wide variety of iterative methods falls within the Krylov subspace framework. We shall concentrate on two methods for non-Hermitian matrix problems : Arnoldi's method [11], which computes eigenvalues and eigenvectors of A and the generalized minimal residual method (GMRES) [13], which solves systems of equations. This latter constructs an approximate solution $x^{(m)}$ from the affine subspace $x^{(0)} + \mathcal{K}_m(A, r^{(0)})$ ($r^{(0)} = b - A\ x^{(0)}$ is the initial residual and $x^{(0)} \in \mathbb{C}^N$ is a given initial approximate solution of (1.1)) by imposing the Petrov-Galerkin condition $r^{(m)} \perp A\mathcal{K}_m(A, r^{(0)})$, where $r^{(m)} = b - A\,x^{(m)}$. Although it is guaranteed to produce the exact solution in at most N iterations, the interesting question here is how fast the residuals decrease below a tolerance sufficiently small for a specific application. The need to explore and to estimate the convergence rate of the iterative methods is

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motivated by providing, for large systems of linear equations, a good approximate solution computed quite early, after very few iterations. Understanding and quantifying the behaviour of Krylov subspace methods is an active area of research.

Numerous papers have dealt with this question by deriving upper bounds for the residual norm, which reveal some intrinsic links between the convergence properties and the spectral information available for A. The standard technique in most of these works [15, 12, 5, 4, 7], based on a representation of the method by using matrix-value polynomials, includes a similar min-max problem about a matrix polynomial of A. More precisely, the link between residuals and polynomials has inspired the search for bounds on the residual norm which are derived from analytic properties of some normalized associated polynomials as functions defined on the complex plane.
In recent years, a different approach from a purely algebraic point of view (divorced from approximation theory) was advocated by Sadok [16, 17] and Ipsen [6], followed by Zavorin, O'Leary and Elman [18], Liesen and Tichy [8], to study the convergence of the GMRES method. Related theoretical residual bounds are established, by exploring certain classes of matrices, concerning the obscure behaviour of this method, in particular the stagnation phenomenon. Nevertheless a great number of open questions remain.
More recently, to analyze the convergence rate of the GMRES and Arnoldi methods, we have presented in the papers [1, 2] a new tool, based on an expression for the residual norm in terms of determinants of Krylov matrices proposed in [17] instead of the pseudo-inverse of the next Krylov matrix as in [6]. An interesting feature of our treatment is that it allows us to provide, among others, a thorough analysis of the convergence for normal matrices in the context of convex constrained optimization. In this case, we have derived, at any step, an upper bound for the residual norm, which is proportional to a product of relative eigenvalue differences.

The purpose of the present work is thus to show the connection between these two approaches : the min-max polynomial approximation and the convex constrained optimization. More precisely, we establish that the Karush-Kuhn-Tucker (KKT) optimality conditions deduced from the convex maximization problem and the characterization properties of polynomial of best approximation on a finite set of points are equivalent.

1.1. Overview. The paper is organized as follows : in Section 2 and Section 3 we review briefly the GMRES and Arnoldi methods. We present the convergence analysis of these methods. We then distinguish between the approach based on the approximation theory and the other one based on the optimization theory. Section 4 contains necessary mathematical background in the approximation theory. We then provide results concerning the characterization of the polynomial of best approximation on finite point sets. We then apply these results to our situation. This section also gives a new bound concerning the Arnoldi method by bypassing the normalization constraint imposed on the polynomial of best approximation. Section 5 is devoted to stating the main results and to examining the KKT optimality conditions deduced from the convex maximization problem for both methods. We show the equivalence of the two formulations. Finally, short concluding remarks are made in Section 6.

Throughout the paper we assume exact arithmetic. So our analysis in this contribution does not depend on a specific implementation of these methods.

1.2. Preliminaries and notations. Let $\mathbb{P}_m$ denote the set of all polynomials of degree not exceeding m and let $\mathbb{P}_m^{(\lambda)}$ denote the set of polynomials of $\mathbb{P}_m$ with

value one at λ. We denote by $l_j(z) = \prod_{\nu=1, \upsilon\neq j}^{k} \dfrac{z-\mu_\upsilon}{\mu_j-\mu_\upsilon}$ the jth Lagrange interpolating polynomial of degree $k-1$ associated to the complex numbers μ_j $(j=1,\ldots,k)$.
Consider the complex number $z = \rho e^{\imath\,\theta}$ $(\rho > 0,\ \imath^2 = -1)$. Then $\overline{z} = Conj(z) = \rho e^{-\imath\,\theta}$ is the complex conjugate of z and $|z| = \rho$ is the modulus of z. We define $sgn(z)$ by $sgn(z) = \dfrac{z}{|z|} = e^{\imath\,\theta}$.
On the other hand, let the matrix $A \in \mathbb{C}^{N\times N}$ be a complex matrix with spectrum $\sigma(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$, i.e., the set of all eigenvalues of A. We assume A is diagonalizable matrix and the eigenvalues λ_i are distinct.
Let K_m be the Krylov matrix whose columns are $r^{(0)}, A\,r^{(0)}, \ldots, A^{m-1}\,r^{(0)}$. Let $\|\,.\,\|$ denote the Euclidean two-norm on $\mathbb{C}^N$ and the matrix norm induced by it. The superscripts "T" and "H" are the transpose and the conjugate transpose. The identity matrix is $I = (e_1, e_2, \ldots, e_N)$ with column e_i. According to the above assumptions, there exists an inversible matrix U such that

$$A = U\,\Lambda\,U^{-1}, \text{ where } \Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_N). \tag{1.3}$$

In this case, for any $p \in \mathbb{P}_m$ we know that

$$p(A) = p(U\,\Lambda\,U^{-1}) = U\,p(\Lambda)\,U^{-1} = U\,diag(p(\lambda_1), p(\lambda_2), \ldots, p(\lambda_N))\,U^{-1}. \tag{1.4}$$

Let $diag(\beta_1, \ldots, \beta_N)$ denote a diagonal matrix (the entries outside the main diagonal are all zero and the diagonal entries are β_i). Let V be the Vandermonde matrix defined as the $N \times N$ matrix whose generic element $v\ (i,j)$ is given by :

$$v\ (i,j) = \lambda_j^{i-1}\ ,\ i,j = 1, \ldots, N.$$

Let V_m denote the matrix with the first m columns of V, i.e.

$$V_m = \begin{pmatrix} 1 & \lambda_1 & \ldots & \lambda_1^{m-1} \\ 1 & \lambda_2 & \ldots & \lambda_2^{m-1} \\ \vdots & \vdots & \ldots & \vdots \\ 1 & \lambda_N & \ldots & \lambda_N^{m-1} \end{pmatrix}$$

and the matrix W_{m+1} is such that $W_{m+1} = [e_1\ ,\ V_m]$.

2. Outline of the convergence analysis for the GMRES method.

2.1. Review. In order to give the convergence problem and the notation to be used in the convergence analysis, we briefly review the GMRES for computing the solution of a linear system of equations . The basic idea behind solving this (large and sparse) problem is to project the problem onto the Krylov subspace of dimension $m \leq N$, solve the m-dimensional problem using a standard approach, and then recover the solution of the original problem from the solution of the projected problem.
As pointed out in the Introduction, the GMRES method starts with an initial guess $x^{(0)}$ for the solution of (1.1) and seeks the m-th approximate solution $x^{(m)}$ in the affine subspace

$$x^{(m)} \in x^{(0)} + \mathcal{K}_m(A, r^{(0)}),$$

satisfying the residual norm minimizing property

$$\left\|b - A\,x^{(m)}\right\| = \min_{u\in x^{(0)}+\mathcal{K}_m(A,r^{(0)})} \|b - A\,u\| = \min_{z\in\mathcal{K}_m(A,r^{(0)})} \left\|r^{(0)} - A\,z\right\|. \tag{2.1}$$

On the other hand, this approximation is of the form $x^{(m)} = x^{(0)} + p^*_{m-1}(A)\ r^{(0)}$, where $p^*_{m-1} \in \mathbb{P}_{m-1}$. So that, the residual $r^{(m)}$ has the polynomial representation $r^{(m)} = p^*_m(A)\ r^{(0)}$, where $p^*_m \in \mathbb{P}^{(0)}_m$, and therefore the problem (2.1) becomes

$$\left\| r^{(m)} \right\| = \min_{p \in \mathbb{P}^{(0)}_m} \left\| p(A) r^{(0)} \right\|. \tag{2.2}$$

Fundamental properties of GMRES are that the norm of $r^{(m)}$ is a non increasing function of m and it terminates in m steps if $r^{(m)} = 0$ and $r^{(m-1)} \neq 0$. Moreover, we have $r^{(m)} \neq 0$ if and only if $\dim(\mathcal{K}_{m+1}(A, r^{(0)})) = m + 1$. Therefore, while analyzing GMRES convergence, we will assume that the Krylov matrix K_{m+1} is of rank $m + 1$.

Before we focus on giving bound for the convergence quantities, we will explore two different ways of analyzing the convergence rate. It will be given in terms of so-called optimal polynomials for one and in terms of spectral decomposition of Krylov matrices for the other.

2.2. Analysis based on (uniform) approximation theory. The influence of the initial residual in (2.2) is usually suppressed in view of $\left\| p(A) r^{(0)} \right\| \leq \| p(A) \| \ \left\| r^{(0)} \right\|$, in which case the issue simplifies to bounding $\| p(A) \|$ for all $p \in \mathbb{P}^{(0)}_m$.

It follows that an estimate of the relative Euclidean residual norm $\dfrac{\left\| r^{(m)} \right\|}{\left\| r^{(0)} \right\|}$ is associated with the so-called Chebyshev polynomial of a matrix problem : $\min\limits_{p \in \mathbb{P}^{(0)}_m} \| p(A) \|$.

It is easy to see that, if $r^{(m)} = p(A)\ r^{(0)}$, then

$$\| r^{(m)} \| = \| U\ p(\Lambda)\, U^{-1} r^{(0)} \| \leq \kappa(U)\, \| r^{(0)} \|\, \| p(\Lambda) \| \leq \| r^{(0)} \| \kappa(U) \min_{p \in \mathbb{P}^{(0)}_m} \max_{\lambda \in \sigma(A)} |p(\lambda)|\,.$$

where $\kappa(U) = \| U \|\, \| U^{-1} \|$ is the condition number of U. However since the matrix U is not uniquely defined by the equation $A = U\, \Lambda\, U^{-1}$. We can write

$$\frac{\| r^{(m)} \|}{\| r^{(0)} \|} \leq \mu(A) \min_{p \in \mathbb{P}^{(0)}_m} \max_{\lambda \in \sigma(A)} |p(\lambda)|\,,$$

where

$$\mu(A) = \inf_U\ \kappa(U),$$

the infinimum being taken with respect to all matrices U for which $U^{-1}\, A\, U$ is diagonal.

We will assume now that the columns of the eigenvectors matrix U are of norm one. Then we give a new bound for the absolute residual norm of GMRES.

THEOREM 2.1. *Let us assume that the columns of U are normalized i.e. $\| u_i \| = 1$ where $U = [u_1, \ldots, u_N]$. If we expand $r^{(0)}$ in the eigen-basis $r^{(0)} = U\, \alpha$, then*

$$\| r^{(m)} \| \leq \left(\sum_{i=1}^{N} |\alpha_i| \right) \min_{p \in \mathbb{P}^{(0)}_m} \max_{\lambda \in \sigma(A)} |p(\lambda)|\,.$$

If the matrix A is normal ($U^H\, U = I$), then we have

$$\| r^{(m)} \| \leq \left(\sqrt{\sum_{i=1}^{N} |\alpha_i|^2} \right) \min_{p \in \mathbb{P}^{(0)}_m} \max_{\lambda \in \sigma(A)} |p(\lambda)|\,.$$

Proof. By noting that $\alpha = U^{-1} r^{(0)}$, then we obtain

$$\|r^{(m)}\| = \|U\, p(\Lambda)\, U^{-1} r^{(0)}\| = \|\sum_{i=1}^{N} \alpha_i\, p(\lambda_i)\, u_i\,\| \le \sum_{i=1}^{N} |\alpha_i|\, |p(\lambda_i)|$$

Hence $\|r^{(m)}\| \le (\sum_{i=1}^{N} |\alpha_i|) \min_{p\in\mathbb{P}_m^{(0)}} \max_{\lambda\in\sigma(A)} |p(\lambda)|$.

In normal case, it suffices to observe that $\|r_0\|^2 = \sum_{i=1}^{N} |\alpha_i|^2$ and the result follows since it is well known that

$$\|r^{(m)}\| \le \|r^{(0)}\| \min_{p\in\mathbb{P}_m^{(0)}} \max_{\lambda\in\sigma(A)} |p(\lambda)|\,. \tag{2.3}$$

□

2.3. Analysis based on optimization theory. This part of the section is devoted to formulating a different topic for residual analysis. An alternative, developed in [17, 1, 2], uses as a primary indicator the $(1,1)$ entry of the inverse of $K_l^H\, K_l$ where K_l is the Krylov matrix with l columns in GMRES method.

By using the Schur complement, we can express the residual norm $r^{(m)}$ ($\text{rank}(K_{m+1}) = m+1$) as follows

$$\|r^{(m)}\|^2 = \frac{1}{e_1^T\, (K_{m+1}^H K_{m+1})^{-1}\, e_1}.$$

Consequently, for normal matrices, with eigenvector decomposition of $r^{(0)}$ ($r^{(0)} = U\,\alpha$), we have

$$\frac{\|r^{(m)}\|^2}{\|r^{(0)}\|^2} = \frac{1}{e_1^T\, Z_{m+1}^{(V)}(\beta)^{-1}\, e_1} = F_m^{(V)}(\beta),$$

with $Z_{m+1}^{(V)}(\beta) = V_{m+1}^H D_\beta V_{m+1} \in \mathbb{C}^{(m+1)\times(m+1)}$, $D_\beta = diag(\beta_1, \ldots, \beta_N)$, and $\beta_i = \frac{|\alpha_i|^2}{\|\alpha\|^2} = \frac{|\alpha_i|^2}{\|r^{(0)}\|^2}$ ($\sum_{i=1}^{N} \beta_i = 1$).

Let $\Im(\beta)$ be the set of indices k such that $\beta_k \neq 0$ for $k = 1, \ldots, N$. If $\text{rank}(K_{m+1}) = m+1$ ($\|r^{(m)}\| \neq 0$) then clearly the cardinality of $\Im(\beta)$ is at least $m+1$.

Now an optimal bound of $\dfrac{\|r^{(m)}\|^2}{\|r^{(0)}\|^2}$ can be obtained by solving the following convex maximization problem

$$\max_{\substack{g(\beta)=0,\, g_i(\beta)\le 0\\ i=1,\ldots,N}} F_m^{(V)}(\beta),$$

where $g(\beta) = \sum_{i=1}^{N} \beta_i - 1$ et $g_i(\beta) = -\beta_i$ for $i = 1, \ldots, N$.

3. Outline of the convergence analysis for Arnoldi's method.

3.1. Review. The Arnoldi method consists in giving approximations to the solutions of the following eigenvalue problem : find u belonging to $\mathbb{C}^N$ and λ belonging to $\mathbb{C}$ such that

$$A\, u = \lambda\, u.$$

For a given vector $v_1 \in \mathbb{C}^N$, the Arnoldi method computes approximate eigenpair $\widetilde{\lambda}^{(m)}, \widetilde{u}^{(m)}$ from the standard Petrov-Galerkin condition

$$\widetilde{u}^{(m)} \in \mathcal{K}_m(A, v_1),$$
$$\text{and} \quad (A\, \widetilde{u}^{(m)} - \widetilde{\lambda}^{(m)} \widetilde{u}^{(m)}\ ,\ A^i\, v_1) = 0 \ \text{ for } \ i = 0, \ldots, m-1.$$

In terms of the distance $\|u - \mathcal{P}_m u\|$ of a given eigenvector u from the Krylov subspace $\mathcal{K}_m(A, v_1)$, it was shown in [14] the following result.

THEOREM 3.1. *Let $A_m = \mathcal{P}_m A \mathcal{P}_m$ and let $\gamma = \|\mathcal{P}_m(A - \lambda\, I)(I - \mathcal{P}_m)\|$. Then the residual norms of the pairs $\lambda, \mathcal{P}_m u$ and λ, u for the linear operator A_m satisfy, respectively*

$$\|(A_m - \lambda\, I)\ \mathcal{P}_m u)\| \le \gamma\ \|(I - \mathcal{P}_m)u\|\,,$$
$$\|(A_m - \lambda\, I)\ u\| \le \sqrt{|\lambda|^2 + \gamma^2}\ \|(I - \mathcal{P}_m)u\|\,.$$

Theorem 3.1 shows the convergence problem of the Arnoldi method can be attributed to the estimate $\|(I - \mathcal{P}_m)u\|$, provided the constants γ and $\sqrt{|\lambda|^2 + \gamma^2}$ are not influenced by any ill-conditioning consideration.

3.2. Analysis based on (uniform) approximation theory. As discussed in the previous section, the convergence analysis of the Arnoldi method is stated in terms of the distance $\|(I - \mathcal{P}_m)u\|$ of the exact eigenvector u from the Krylov subspace. The usual technique to estimate this distance assumes that A is diagonalizable and expands the initial vector v_1 in the eigen-basis as $v_1 = \sum_{j=1}^{N} \alpha_j\, u_j$.

We examine the convergence of a given eigenvalue which is indexed by 1, i.e., we consider u_1, the 1-st column column of U. In[14], when A is diagonalizable, it has given from the residual analysis in terms of eigenvectors the following

THEOREM 3.2. *Assuming $\alpha_1 \neq 0$, $\|u_j\| = 1$ for all j, then*

$$\|(I - \mathcal{P}_m)u_1\| \le \xi_1\, \epsilon_1^{(m)} \tag{3.1}$$

where $\xi_1 = \sum_{j \neq 1} \left| \frac{\alpha_j}{\alpha_1} \right|$ *and* $\epsilon_1^{(m)} = \min_{p \in \mathbb{P}_{m-1}^{(\lambda_1)}} \max_{j \neq 1} |p(\lambda_j)|\,.$

In the normal case, the same result holds but ξ_1 can be sharpened to

$$\xi_{1,normal} = \frac{\sqrt{\sum_{j=2}^{n} |\alpha_j|^2}}{|\alpha_1|},$$

We now establish a slightly different setting. Let us call ψ a function which has values

$$\psi(\lambda_j) = \delta_{1j} \tag{3.2}$$

To prove (3.1) we selected special polynomials which have value equal to one at λ_1. A less restrictive form of (3.1) can be established.

THEOREM 3.3. *Assume that A is diagonalizable and that the initial vector v_1 in Arnoldi's method has the expansion $v_1 = \sum_{k=1}^{N} \alpha_k u_k$ with respect to the eigenbasis $\{u_k\}_{k=1,\ldots,N}$ in which $\|u_k\| = 1, k = 1, 2, \ldots, N$ and $\alpha_1 \neq 0$. Then the following inequality holds:*

$$\|(I - \mathcal{P}_m)u_1\| \leq \hat{\xi}_1 \eta_1^{(m)} \tag{3.3}$$

where

$$\hat{\xi}_1 = \sum_{j=1}^{N} \left| \frac{\alpha_j}{\alpha_1} \right| \quad and \quad \eta_1^{(m)} = \min_{p \,\in\, \mathbb{P}_{m-1}} \max_{j} |\psi(\lambda_j) \; - \; p(\lambda_j)| \; . \tag{3.4}$$

Proof. Let x be any member of $\mathcal{K}_m(A, v_1)$. So $x = q(A)v_1$ where $q \in \mathbb{P}_{m-1}$, and

$$\|\alpha_1 u_1 - x\| = \|\alpha_1(1 - q(\lambda_1))u_1 - \sum_{j=2}^{N} \alpha_j q(\lambda_j) u_j\| = \left\| \sum_{j=1}^{N} [\psi(\lambda_j) - q(\lambda_j)] \alpha_j u_j \right\|$$

$$\leq \sum_{j=1}^{N} |\psi(\lambda_j) - q(\lambda_j)| \; |\alpha_j| \max_{j=1,\cdots,N} |\psi(\lambda_j) - q(\lambda_j)|$$

Therefore,

$$\|(I - \mathcal{P}_m)\alpha_1 u_1\| = \min_{x \,\in \mathcal{K}_m} \|\alpha_1 u_1 - x\| \leq |\alpha_1| \; |\hat{\xi}_1| \min_{q \,\in \mathbb{P}_{m-1}} \max_{j=1,\cdots,N} |\psi(\lambda_j) - q(\lambda_j)|$$

The result follows by dividing both sides of the above inequality by $|\alpha_1|$. ▯

Needless to say, a better expression can be obtained in the normal case, whereby the constant $\hat{\xi}_1$ is replaced by $\hat{\xi}_1 = \|\alpha\|/\alpha_1$. Here $\|\alpha\|$ is the 2-norm of the vector of $\alpha_i' s$. The main distinction between this result and the one in (3.1) is that the candidate polynomial q is no longer constrained to have value one at λ_1, so this result will generally be slightly sharper than (3.1).

3.3. Analysis based on optimization theory. Firstly, we are interested in the study of $\|(I - \mathcal{P}_m) \; \alpha_1 \; u_1\|$. In [2], it was shown that

$$\|(I - \mathcal{P}_m) \; \alpha_1 \; u_1\|^2 = \frac{1}{e_1^T \; (L_{m+1}^H L_{m+1})^{-1} \; e_1},$$

where L_{m+1} is such that $L_{m+1}^H L_{m+1}$ is nonsingular. A further computation with $L_{m+1} = U \, D_\alpha \, W_{m+1}$ and $U^H U = I$ gives

$$\frac{\|(I - \mathcal{P}_m) \; \alpha_1 \; u_1\|^2}{\| \; \alpha\|^2} = \frac{1}{e_1^T \; (Z_{m+1}^{(W)}(\beta))^{-1} \; e_1},$$

where $Z_{m+1}^{(W)}(\beta) \in \mathbb{C}^{(m+1)\times(m+1)}$ is the matrix defined by $Z_{m+1}^{(W)}(\beta) = W_{m+1}^H D_\beta W_{m+1}$ assumed regular. We define the function $F_m^{(W)}$ by $F_m^{(W)}(\beta) = e_1^T \; Z_{m+1}^{-1}(\beta) \; e_1$.

Again, since $\beta_i \geq 0$ and $\sum_{i=1}^{N} \beta_i = 1$, in order to get an optimal bound of $\dfrac{\|(I - \mathcal{P}_m)\ \alpha_1 u_1\|^2}{\|\alpha\|^2}$, we must solve the following maximization problem

$$\max_{\substack{g(\beta)=0,\ g_i(\beta)\leq 0\\ i=1,\ldots,N}} F_m^{(W)}(\beta).$$

Accordingly, for both methods we have the same maximization problem with slightly different objective functions. Note that the functions g, g_i are convex (affine functions) and the objective function $-F_m^{(\bullet)}$ is also convex (see appendix). It follows that this maximization problem can be viewed as convex constrained optimization problem. So in this case, it is known [3] that the local optimum is the global optimum and the Karush-Kuhn-Tucker (KKT) optimality conditions are necessary and sufficient.

4. Characterization of the Polynomial of Best Approximation. In order to characterize polynomial of best approximation of the finite min-max approximation problem, we follow the treatment of Lorentz [9, Chap. 2] to providing the essential details and tools. Our main concern is to derive two linear systems which characterize the optimal polynomial. They are fundamental to show the link with the other approach in the next section.

4.1. General Context. We begin this section by mentioning some details concerning additional notations used in the sequel.
Let $\mathcal{C}(S)$ denote the space of complex or real continuous functions on a compact subset S of $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), with the uniform norm $\|f\|_\infty = \max_{z\in S} |f(z)|$.
A set $\Phi = \{\varphi_1, \varphi_2, \ldots, \varphi_m\}$ from $\mathcal{C}(S)$ is a Chebyshev system, if it satisfies the Haar condition : each polynomial

$$p = a_1\ \varphi_1 + a_2\ \varphi_2 + \ldots + a_m\ \varphi_m\ ,$$

with not all coefficients equal to zero, has at most $(m-1)$ distinct zeros on S. The m-dimensional space E spanned by a such Φ is called a Chebyshev space. The reader may check that Φ is a Chebyshev system if and only if for any m distinct points $z_i \in S$ the following determinant is not zero :

$$\det(\varphi_j(z_i)) := \begin{vmatrix} \varphi_1(z_1) & \cdots & \varphi_1(z_m) \\ \vdots & \ldots & \vdots \\ \varphi_m(z_1) & \cdots & \varphi_m(z_m) \end{vmatrix}.$$

Let $f \in \mathcal{C}(S)$, $f \notin$ E. We say p^* is a best approximation to f from E if $\|f - p^*\|_\infty \leq \|f - p\|_\infty$, $\forall p \in$ E. i.e. $\|f - p^*\|_\infty = \min_{p\in \mathrm{E}} \max_{z\in S} |f(z) - p(z)|$.
For the proof of our first result, the following well-known characterization of best approximation will be useful. An elegant characterization of best approximations is also available in [10].

THEOREM 4.1.

A polynomial q^* is a polynomial of best approximation to $f \in \mathcal{C}(S)$ from E *if and only if there exist r extremal points $z_1, z_2, \ldots, z_r \in \mathcal{E}(f - q^*, S)$, and r numbers*

$\beta_1 > 0, \beta_2 > 0, \ldots, \beta_r > 0$ such that $\sum_{l=1}^{r} \beta_l = 1$ ($r \leq 2m+1$ in the complex case and $r \leq m+1$ in the real case), for which

$$\sum_{l=1}^{r} \beta_l \; [f(z_l) - q^*(z_l)] \; \overline{\varphi_j(z_l)} = 0, \;\; j = 1, \ldots, m. \tag{4.1}$$

Here $\mathcal{E}(f - q^*, S) := \{z : |f(z) - q^*(z)| = \|f - q^*\|_\infty \,, \; z \in S\}$ is the extremal set.

REMARK 4.2.

1. *The previous result can be applied to the best approximation on any finite set $\sigma \subset S$ having at least $(m+1)$ points.*
2. *The uniqueness of the polynomial of best approximation is guaranteed if* E *is a Chebyshev space[9, Chap. 2, p.26]. Moreover, we have $r = m+1$ if $S \subset \mathbb{R}$ and $m+1 \leq r \leq 2m+1$ if $S \subset \mathbb{C}$. This will be the case because we will deal with polynomials of $\mathbb{P}_m$.*

We now give an equivalent formulation of the relation (4.1). We then deduce auxiliary results from which the maximum and the polynomial of best approximation can be characterized.

COROLLARY 4.3. *A polynomial q^* such that $q^*(z) = a_1^* \, \varphi_1(z) + a_2^* \, \varphi_2(z) + \ldots + a_m^* \, \varphi_m(z)$ is a polynomial of best approximation to $f \in \mathcal{C}(S)$ from* E *if and only if there exist r points $z_1, z_2, \ldots, z_r$ and r numbers $\beta_1 > 0, \beta_2 > 0, \ldots, \beta_r > 0$ such that $\sum_{l=1}^{r} \beta_l = 1$ ($r \leq 2m+1$ in the complex case and $r \leq m+1$ in the real case), for which*

$$t_1^* \; f(z_l) + \sum_{j=2}^{m+1} t_j^* \; \varphi_{j-1}(z_l) = \varepsilon_l \frac{1}{\sqrt{\delta^*}}, \;\; l = 1, \ldots, r \;\; ; \tag{4.2}$$

with $t_1^* = \dfrac{1}{\delta^*}$ and $t_{j+1}^* = -\dfrac{a_j^*}{\delta^*}$, $j = 1, \ldots, m$ and

$$\sum_{l=1}^{r} \beta_l \; \varepsilon_l \; \overline{\varphi_j(z_l)} = 0, \;\; j = 1, \ldots, m \;, \tag{4.3}$$

$$\sum_{l=1}^{r} \beta_l \; \varepsilon_l \; \overline{f(z_l)} = \sqrt{\delta^*}, \tag{4.4}$$

where $\varepsilon_l = sgn(f(z_l) - q^*(z_l))$ and $\delta^* = \|f - q^*\|_\infty^2$.

Proof. let $\delta^* = \|f - q^*\|_\infty^2$. Then $z_l \in \mathcal{E}(f - q^*, S)$ is equivalent to

$$f(z_l) - q^*(z_l) = \sqrt{\delta^*} \varepsilon_l \;, \tag{4.5}$$

where $\varepsilon_l = sgn(f(z_l) - q^*(z_l))$. So by setting $t_1^* = \dfrac{1}{\delta^*}$ and $t_{j+1}^* = \dfrac{-a_j^*}{\delta^*}$, $j = 1, \ldots, m$; we obtain (4.2).

By using (4.5), $\sum_{l=1}^{r} \beta_l \; \varepsilon_l \; \overline{\varphi_j(z_l)} = 0$, $j = 1, \ldots, m$ is equivalent to (4.1). Then we have

$$\sum_{l=1}^{r} \beta_l \; \varepsilon_l \; \overline{q(z_l)} = 0 \;\; \text{for all polynomials } q \in \mathrm{E}. \tag{4.6}$$

On the other hand, again from (4.5) we have the relation $\sqrt{\delta^*}\beta_l = \beta_l\ \overline{\varepsilon_l}\ f(z_l) - \beta_l\ \overline{\varepsilon_l}\ q^*(z_l),\ l = 1, \dots, r$. To see that $\sum_{l=1}^{r} \beta_l = 1$ is equivalent to (4.4), it suffices to summing the terms in this relation and applying the conjugate of (4.6). □

We will now translate the above results to our finite min-max approximation problems.

4.2. Application to GMRES min-max problem. Let $\mathbb{Q}_m^{(0)}$ denote the set of all polynomials of degree not exceeding m with value zero at the origin. Our problem corresponds to $f \equiv \mathbf{1}$ and $\mathrm{E} = \mathbb{Q}_m^{(0)}$. So, we have $\|p_m^*\| = \min_{p\in\mathbb{P}_m^{(0)}} \max_{\lambda\in\sigma(A)} |p(\lambda)| = \min_{q\in\mathbb{Q}_m^{(0)}} \max_{\lambda\in\sigma(A)} |1 - q(\lambda)| = \|f - q_m^*\|_\infty$.

Hence the polynomial of best approximation q_m^* for $f \equiv \mathbf{1}$ (with respect to $\sigma(A)$) exists and is unique and $p_m^*(z) = 1 - q_m^*(z)$.

To simplify the notation, consider the extremal set labeled $\{\lambda_1, \dots, \lambda_r\} \subset \sigma(A)$. The extremal points satisfy $|1 - q_m^*(\lambda_j)| = \|\mathbf{1} - q_m^*\|_\infty = \sqrt{\delta^*}$ for $j = 1, \dots, r$.

So, according to the above definition of ε_j, the polynomial q_m^* satisfies the following interpolation conditions given in (4.2) :

$$q_m^*(\lambda_j) = 1 - \sqrt{\delta^*}\ \varepsilon_j \quad \text{for all } j = 1, \dots, r. \tag{4.7}$$

Moreover, Eq.(4.3) becomes

$$\sum_{l=1}^{r} \beta_l\ \varepsilon_l\ = \sqrt{\delta^*}.$$

The following theorem gives

THEOREM 4.4. *Let q^* a polynomial of best approximation for the function f such that $f(z) = 1$ on $\sigma(A)$ from $\mathrm{E} = \mathbb{Q}_m^{(0)}$. Then it exists $m+1$ eigenvalues such that*

$$\sum_{j=1}^{m+1} \varepsilon_j\ l_j^{(m+1)}(0) = \frac{1}{\sqrt{\delta^*}}, \tag{4.8}$$

with $l_j(z)$ is the jth Lagrange interpolating polynomial of degree m associated to $\{\lambda_1, \dots, \lambda_{m+1}\}$.

Proof. We may transform the interpolation conditions (4.7) to $\pi_m(\lambda_k) = \varepsilon_k$ for all $k = 1, \dots, r$; with $\pi_m(z) = \frac{1}{\sqrt{\delta^*}}(1 - q_m^*(z))$. Using the interpolation conditions corresponding to $\{\lambda_1, \dots, \lambda_{m+1}\}$, π_m can be written in the form of the Lagrange interpolation formula $\pi_m(z) = \sum_{k=1}^{m+1} \varepsilon_k\ l_k^{(m+1)}(z)$. Finally, taking into account that $\pi_m(0) = \frac{1}{\sqrt{\delta^*}}$, we obtain (4.8). □

4.3. Application to the Arnoldi min-max problems. In this part, we are concerned with the problem of best (uniform) approximation of the function ψ (3.2) by elements from $\mathbb{P}_{m-1}$:

$$\min_{q\in\mathbb{P}_{m-1}} \max_{j=1,\cdots,N} |\psi(\lambda_j) - q(\lambda_j)|$$

Our goal is to characterize the maximum and the polynomial of best approximation. We can now show the following result.

THEOREM 4.5. *There are r eigenvalues in $\sigma(A)$, where $m+1 \le r \le 2m+1$, at which the difference $\psi(z) - p^*(z)$ reaches its maximum modulus. In particular λ_1 is an extremal point verifying $\psi(\lambda_1) - p^*(\lambda_1) = \|\psi - p^*\| = \sqrt{\delta^*}$.*
In addition, consider any selection of m of the extremal points ($\neq \lambda_1$) and label these as $\lambda_2, \lambda_3, \ldots, \lambda_{m+1}$. Then the optimal polynomial p^* can be represented by

$$p^*(z) = -\sqrt{\delta^*} \sum_{k=2}^{m+1} e^{i\theta_k} \; l_k(z) \tag{4.9}$$

where $l_k(z) = \prod_{j=2 j\neq k}^{m+1} \left(\dfrac{z-\lambda_j}{\lambda_k - \lambda_j} \right)$, *$\theta_k$ is a real positive scalar for $k > 1$. Moreover, the following expression for $\eta_1^{(m)}$ follows,*

$$\eta_1^{(m)} = \frac{1}{1 - \sum\limits_{k=2}^{m+1} e^{i\theta_k} \prod\limits_{j=2 j\neq k}^{m+1} \frac{\lambda_1 - \lambda_j}{\lambda_k - \lambda_j}} \tag{4.10}$$

Proof. We begin by proving that λ_1 is an extremal point. Indeed, the characterization theorem (Theorem 3.1) gives by adapting (4.1) :

$$\sum_{l=1}^{r} \beta_l \; \overline{[\psi(z_l) - p^*(z_l)]} \; q(z_l) = 0, \quad j = 1, \ldots, m \; ; \qquad \text{for any } q \in \mathbb{P}_{m-1}. \tag{4.11}$$

For $l = 1, \ldots, r$, we have $\beta_l > 0$ and $\psi(z_l) - p^*(z_l) = \sqrt{\delta^*} e^{i\theta_l}$ ($\theta_l \in \mathbb{R}$). Suppose that all z_l are different from λ_1. Then, by remarking that $\psi(z_l) = 0$ and $p^*(z_l) = -\sqrt{\delta^*} e^{\iota\theta_l}$ ($l = 1, \ldots, r$) and by choosing $q = p^*$ in (4.11) , we obtain

$$\sum_{l=1}^{r} \beta_l \; \left[\overline{-p^*(z_l)}\right] \; p^*(z_l) = -\sum_{l=1}^{r} \beta_l \; |p^*(z_l)|^2 = 0.$$

That is, $\delta^* \sum\limits_{l=1}^{r} \beta_l \; = \delta^* = 0$, a contradiction.
Let's write $z_\nu = \lambda_1$. We have $1 - p^*(z_\nu) = \sqrt{\delta^*} e^{\iota\theta_1}$. Again by applying (4.11) with $q = p^*$, we obtain

$$\beta_\nu \; \left[\overline{1 - p^*(z_\nu)}\right] \; p^*(z_\nu) = \sum_{l=1, l\neq\nu}^{r} \beta_l \; |p^*(z_l)|^2 = \delta^* \sum_{l=1, l\neq\nu}^{r} \beta_l.$$

So we have

$$\beta_\nu \; \sqrt{\delta^*} e^{-\iota\theta_1} (1 - \sqrt{\delta^*} e^{\iota\theta_1}) \; = \beta_\nu \sqrt{\delta^*} e^{-\iota\theta_1} - \beta_\nu \delta^* = \delta^* \sum_{l=1, l\neq\nu}^{r} \beta_l.$$

Namely,

$$\beta_\nu \sqrt{\delta^*} e^{-\iota\theta_1} = \beta_\nu \delta^* + \delta^* \sum_{l=1, l\neq\nu}^{r} \beta_l = \delta^*.$$

Hence $\beta_\nu = \sqrt{\delta^*},\ e^{-\iota\theta_1} = 1$ and $1 - p^*(z_\nu) = \sqrt{\delta^*}$.
At the m extremal points labeled $\lambda_j, j = 2, \ldots, m+1$, we have $p^*(\lambda_j) = -\sqrt{\delta^*} e^{i\theta_j}$. Since p^* is a polynomial of degree $m-1$, then we can use Lagrange interpolation at the points $\lambda_2, \cdots, \lambda_{m+1}$ and this immediately leads to the expression (4.9). We now evaluate p^* at λ_1:

$$p^*(\lambda_1) = -\sqrt{\delta^*} \sum_{j=2}^{m+1} e^{i\theta_j} l_j(\lambda_1) = 1 - \sqrt{\delta^*} \to \sqrt{\delta^*}(1 - \sum_{j=2}^{m+1} e^{i\theta_j} l_j(\lambda_1)) = 1.$$

This leads to $\sqrt{\delta^*} = \dfrac{1}{1 - \sum_{j=2}^{m+1} e^{i\theta_j} l_j(\lambda_1)}$ from where follows (4.10). □

Let us return to the constrained finite min-max problem

$$\epsilon_1^{(m)} = \min_{p \in \mathbb{P}_{m-1}^{(\lambda_1)}} \max_{j \neq 1} |p(\lambda_j)| .$$

In [2], it was established the following,

Theorem 4.6. *There are r eigenvalues in $C = \Lambda(A) \backslash \{\lambda_1\}$, where $m+1 \leq r \leq 2m+1$, at which the optimal polynomial $1 - p^*(z) = 1 - (z - \lambda_1) q^*(z)$ reaches its maximum value. In addition, given any subset of m among these r eigenvalues, which can be labeled $\lambda_2, \lambda_3, \ldots, \lambda_{m+1}$, the polynomial p^* can be represented by*

$$1 - p^*(z) = \sum_{k=2}^{m+1} \rho e^{i\theta_k} l_k(z).$$

In particular,

$$\epsilon_1^{(m)} = \frac{1}{\sum\limits_{k=2}^{m+1} e^{i\theta_k} \prod\limits_{\substack{j=2 \\ j \neq k}}^{m+1} \dfrac{\lambda_1 - \lambda_j}{\lambda_k - \lambda_j}}. \tag{4.12}$$

Hence we can conclude that

$$\eta_1^{(m)} = \frac{\epsilon_1^{(m)}}{\epsilon_1^{(m)} - 1}.$$

5. Main result and equivalence. The main objectives of this section are to examine the KKT optimality conditions deduced from our convex maximization problem, state the main result establishing the equivalence between the two approaches, and point out the importance of obtained KKT-equations of which only the non active part is needed to prove the equivalence.

5.1. Karush-Kuhn-Tucker (KKT) optimality conditions. In this section, we shall give the details of the characterization of the solution of the convex maximization problem $\max\limits_{\substack{g(\beta)=0, g_i(\beta) \leq 0 \\ i=1,\ldots,N}} F_m^{(\bullet)}(\beta)$ obtained previously.
We begin with the following definitions.
Let $T = \{\beta$ such that $g(\beta) = 0$ and $g_i(\beta) \leq 0$ for $i = 1, \ldots, N\}$. Form the Lagrangian function : $\mathcal{L}_m(\beta, \delta, \mu) = F_m(\beta) - \delta\, g(\beta) - \sum\limits_{i=1}^{N} \mu_i g_i(\beta)$, where $\delta \in \mathbb{R}$;

$\mu = (\mu_1, \cdots\cdots, \mu_N) \in \mathbb{R}^N$.

According to Karush-Kuhn-Tucker (KKT) conditions [3], if $F_m(\beta)$ has a local maximizer β^* in T then there exist Lagrange multipliers $\delta^*, \mu^* = (\mu_1^*, \ldots, \mu_N^*)$ such that $(\beta^*, \delta^*, \mu^*)$, satisfy the following conditions :

(i) $\dfrac{\partial \mathcal{L}m}{\partial \beta_j}(\beta^*, \delta^*, \mu^*) = 0$;

(ii) $g(\beta^*) = 0$ and $(g_j(\beta^*) \leq 0$ and $\mu_j^* \geq 0$ for all $j = 1, \ldots, N)$;

(iii) $\mu_j^* g_j(\beta^*) = 0$ for all $j = 1, \ldots, N$.

5.2. Main result. Now we can state and prove the following main result.

Theorem 5.1. *Let be the nonsingular matrix $A \in \mathbb{C}^{N\times N}$ be normal given by (1.3). and let $m < N$. The following assertions hold*

1. *GMRES case : Assume that $\|r^{(m)}\| \neq 0$. Then we have*

$$(\min_{p\in\mathbb{P}_m^{(0)}} \max_{\lambda\in\sigma(A)} |p(\lambda)|)^2 = \max_{\substack{g(\beta)=0, g_i(\beta)\leq 0\\ i=1,\ldots,N}} F_m^{(V)}(\beta). \tag{5.1}$$

2. *Arnoldi's case : If $\|(I - \mathcal{P}_k)\ u_1\| \neq 0$ for $k \in \{1, \ldots, m\}$. . Then*

$$(\min_{q\in\mathbb{P}_{m-1}} \max_{j=1,\cdots,N} |\psi(\lambda_j) - q(\lambda_j)|)^2 = \max_{\substack{g(\beta)=0, g_i(\beta)\leq 0\\ i=1,\ldots,N}} F_m^{(W)}(\beta). \tag{5.2}$$

Proof. We will settle for proving only the first assertion, the other are outlined in the same manner.

Without loss of generality we will denote by $F_m = F_m^{(V)}$. As already mentioned, we have at least $m+1$ components of β^* are non null when $\|r^{(m)}\| \neq 0$. Thus, there exists $m + s$ $(s \geq 1)$ components of β^* labeled $\beta_1^*,\ \beta_2^*, \ldots, \beta_{m+s}^*$, to simplify the notation, such that $\beta_j^* \neq 0$ for all $j = 1, \ldots, m + s$.

From the complementarity conditions (iii), we deduce that $\mu_j^* = 0$ for all $j = 1, \ldots, m + s$ and $\mu_j^* > 0$ for all $j = m + s + 1, \ldots, N$. Hence the condition (i) can be re-expressed as

$$\begin{cases} \dfrac{\partial F_m}{\partial \beta_j}(\beta^*) = \delta^*, & \text{for all} \quad j = 1, \ldots, m + s, \text{ and} \\ \dfrac{\partial F_m}{\partial \beta_j}(\beta^*) = \delta^* - \mu_j^*, & \text{for all } j = m + s + 1, \ldots, N. \end{cases} \tag{5.3}$$

In view of Lemma of the Appendix in [1], F_m is differentiable on T and we have for all $j = 1, \ldots, N$

$$\frac{\partial F_m}{\partial \beta_j}(\beta) = F_m(\beta)^2 \left| e_j^T\ V_{m+1}\ t^{(m+1)} \right|^2, \tag{5.4}$$

where $t^{(m+1)} = (t_1^{(m+1)}, \ldots,\ t_{m+1}^{(m+1)})^T$ is such that

$$Z_{m+1}(\beta)\ t^{(m+1)} = V_{m+1}^H D_\beta V_{m+1} = e_1. \tag{5.5}$$

The choice of eigenvalues $\lambda_1, \ldots, \lambda_{m+s}$ corresponds to $\beta_1^*,\ \beta_2^*, \ldots, \beta_{m+s}^*$. Accordingly by taking $\beta = \beta^*$ and $t^{(m+1)} = t^*$ such that $Z_{m+1}(\beta^*)\ t^* = e_1$, the relations in (5.3) become

$$F_m(\beta^*)^2 \left| e_j^T\ V_{m+1}\ t^* \right|^2 = \delta^* \text{ for } \quad j = 1, \ldots, m + s, \tag{5.6}$$

and

$$F_m(\beta^*)^2\ \left|e_j^T\ V_{m+1}\ t^*\right|^2 = \delta^* - \mu_j^* \ \text{ for } j = m+s+1,\dots,N. \tag{5.7}$$

Again from [1], the Euler identity provides the relation $F_m(\beta) = \sum_{j=1}^{N} \beta_j \dfrac{\partial F_m}{\partial \beta_j}(\beta)$, because F_m is homogeneous of degree one. By noting that $\beta_j^* = 0$ for all $j = m+s+1,\dots,N$, $\sum_{j=1}^{m+s} \beta_j^* = 1$ and by using (5.4) and (5.6) , we deduce that $F_m(\beta^*) = \delta^*$ from the Euler identity. Therefore, from (5.6) we can write

$$e_j^T\ V_{m+1}\ t^* = \varepsilon_j\ \frac{1}{\sqrt{\delta^*}} \quad \text{for all} \quad j = 1,\dots,m+s. \tag{5.8}$$

Here we have written ε_j for $e^{\iota\theta_j}$ in the complex case and ε_j for ±1 in the real case. Combining $\frac{1}{e_1^T\ Z_{m+1}(\beta^*)^{-1}\ e_1} = F_m(\beta^*)$ and (5.5) , we obtain $t_1^* = \dfrac{1}{\delta^*}$.
The system (5.8) has $m+1$ unknowns, $m+s$ equations and $m+s$ parameters $\varepsilon_1,\dots,\varepsilon_{m+s}$. It is important to remark that s is chosen such that (5.8) has an unique solution which is given by

$$\begin{pmatrix} t_1^* \\ t_2^* \\ \vdots \\ t_{m+1}^* \end{pmatrix} = \frac{1}{\sqrt{\delta^*}} \begin{pmatrix} 1 & \lambda_1 & \dots\dots\dots & \lambda_1^m \\ 1 & \lambda_2 & \dots\dots\dots & \lambda_2^m \\ \vdots & \vdots & \dots & \vdots \\ 1 & \lambda_{m+1} & \dots\dots\dots & \lambda_{m+1}^m \end{pmatrix}^{-1} \times \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_{m+1} \end{pmatrix}.$$

Hence by using Cramer rules we obtain

$$t_1^* = \frac{1}{\sqrt{\delta^*}} \frac{\begin{vmatrix} \varepsilon_1 & \lambda_1 & \dots\dots\dots & \lambda_1^m \\ \varepsilon_2 & \lambda_2 & \dots\dots\dots & \lambda_2^m \\ \vdots & \vdots & \dots & \vdots \\ \varepsilon_{m+1} & \lambda_{m+1} & \dots\dots\dots & \lambda_{m+1}^m \end{vmatrix}}{\begin{vmatrix} 1 & \lambda_1 & \dots\dots\dots & \lambda_1^m \\ 1 & \lambda_2 & \dots\dots\dots & \lambda_2^m \\ \vdots & \vdots & \dots & \vdots \\ 1 & \lambda_{m+1} & \dots\dots\dots & \lambda_{m+1}^m \end{vmatrix}}. \tag{5.9}$$

Since $t_1^* = \dfrac{1}{\delta^*}$ and by expanding the numerator of (5.9) with respect to its first column we obtain $\frac{1}{\sqrt{\delta^*}} = \sum_{k=1}^{m+1} \varepsilon_k\ l_k^{(m+1)}(0)$, with ε_k are given by (5.8) which is identical to (4.7).
On the other hand, the optimal solutions β_j^* and the numbers ε_j can be deduced from (5.5). Indeed, using the fact that $Z_{m+1}(\beta^*)\ t^* = V_{m+1}^H D_{\beta^*} V_{m+1} t^* =\ e_1$ and (5.8), we obtain $e_j^T\ D_{\beta^*} V_{m+1}\ t^* = \beta_j^*\ \varepsilon_j\ \frac{1}{\sqrt{\delta^*}}$ for all $j = 1,\dots,m+s$. So, by applying V_{m+1}^H we have $\sum_{j=1}^{m+s} \beta_j^* \varepsilon_j = \sqrt{\delta^*}$ and $\sum_{j=1}^{m+s} \beta_j^* \varepsilon_j \overline{\lambda}_j^k = 0$ for all $k = 1,\dots,m$.
The relations (4.2), (4.3) and (4.4) in the corollary 4.3 are all satisfied, with $\varphi_j(z) = z^j$ and $r = m+s$. It follows that the Lagrange multiplier $\delta^*$is the same in the previous section.
Consequently $\delta^* = F(\beta^*) = \|\mathbf{1} - q_m^*\|_\infty$ and (5.1) is established. □

6. Concluding remarks. In this paper, we have established for normal matrices, the equivalence between the approximation and the optimization approaches for solving a min-max problems raised from the bound on the residual norm of the GMRES and Arnoldi methods.
Because of their convenient properties, the KKT equations are the key which allows us, by only using the non active part of these equations, to give a complete characterization of the residual bound at each step in the both methods. It also permits to reveal the connection between the two areas and to transfer some results from one to the other one. Unlike the approximation viewpoint, the active part of KKT equations enables us to give some precision about the extremal points. Indeed for the GMRES method for example, from the quantity (5.7), we deduce that

$$\mu_j^* = \delta^* - \ F_m(\beta^*)^2 \ \left| e_j^T \ V_{m+1} \ t^* \right|^2 > 0 \text{ for } j = m+s+1, \dots, N. \tag{6.1}$$

So we have

$$\left| e_j^T \ V_{m+1} \ t^* \right| < \frac{1}{\sqrt{\delta^*}} \text{ for } j = m+s+1, \dots, N.$$

Hence we obtain the characterization of extremal points by

$$\frac{1}{\sqrt{\delta^*}} = \left| e_i^T \ V_{m+1} \ t^* \right| = \max_{1 \le j \le N} \left| e_j^T \ V_{m+1} \ t^* \right| \ \text{ for } i = 1, \dots, m+s.$$

Thus the connection we established opens new insights in each area. We expect that the material presented about the optimization approach would lead to further work beyond normal matrices.

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