Eigenvalue bounds for the Schur complement with a pressurre convection-diffusion preconditioner in incompressible flow computations.

Paul Deuring, Université du Littoral, Laboratoire de Mathématiques Pures et Appliquées, BP 699, 62228 Calais cédex, France

Abstract

The article starts from the saddle point system arising when the Navier-Stokes system is implicitly or semi-implicitly discretized in time, if necessary is linearized by Picard iteration, and is discretized in space by a mixed finite element method. If a Krylov subspace method or an Uzawa type approach is used to solve this system, the Schur complement associated to it requires preconditioning. In the work at hand, we present upper and lower bounds for the eigenvalues of the Schur complement preconditioned by a pressure convection-diffusion matrix.

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1. Introduction

Consider the time-dependent Navier-Stokes equations,

$$\partial_t u - \nu \cdot \Delta u + (u \cdot \nabla)u + \nabla \pi = f, \quad \operatorname{div} u = 0, \tag{1.1}$$

or its stationary counterpart,

$$-\nu \cdot \Delta u + (u \cdot \nabla)u + \nabla \pi = f, \quad \operatorname{div} u = 0, \tag{1.2}$$

supplemented by boundary conditions and, in the time-dependent case, by initial conditions. When this (initial-) boundary value problem is discretized implicitly or semi-implicitly in time (if there is a time-variable), if necessary is linearized by Picard iteration, and is discretized in space by a mixed finite element method, then a variational problem of the following type arises: Find $u_h \in V_h,\ \pi_h \in P_h$ such that

$$a(u_h, w) + b_1(w, \pi_h) = \mathfrak{F}(w) \ \text{ for } \ w \in V_h, \quad b_2(u_h, \sigma) - c(\pi_h, \sigma) = \mathfrak{G}(\sigma) \ \text{ for } \ \sigma \in P_h. \tag{1.3}$$

Here h is a grid parameter, V_h and P_h are finite dimensional spaces, $\mathfrak{F} : V_h \mapsto \mathbb{R}$ and $\mathfrak{G} : P_h \mapsto \mathbb{R}$ are linear operators, b_1 and b_2 are bilinear forms corresponding to respectively the gradient and the divergence operator, and a is a bilinear form representing an "advection-diffusion-reaction operator" of the form $-\nu \cdot \Delta u + (v_0 \cdot \nabla)u + \theta \cdot u$. The parameter θ corresponds to the inverse of the time step in the evolutionary case, and equals 0 else. The function v_0 is the velocity approximation from the preceding step of the nonlinear iteration or from the preceding time step. In the case of LBB-stable mixed finite element methods, the form c vanishes; otherwise it plays the role of a "stabilization term". Other such terms may appear in the definition of b_1, b_2 and a, or may be incorporated into $\mathfrak{F}$ and $\mathfrak{G}$. In the LBB case, the forms b_1 and b_2 usually coincide. Algebraically, problem (1.3) corresponds to a saddle point system of the form

$$K \cdot \begin{pmatrix} X \\ P \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}, \quad \text{with } K := \begin{pmatrix} N & B_1^T \\ B_2 & -C \end{pmatrix}, \tag{1.4}$$

where N may be considered as a "vector advection-diffusion-reaction operator". The matrices $B_1,\ B_2$ are discrete gradient and divergence operators, possibly including stabilization terms;

C is a stabilization matrix which is zero in the case of LBB-stable finite elements. The solution vector X corresponds to the unknown u_h in (1.3), and the vector P to π_h. Since usually the size of K is large, iterative methods frequently are the most efficient means for solving (1.4). Following [20], we may distinguish two major classes of such solvers, that is, multigrid methods and Krylow subspace methods like GMRES. In general, the latter methods are used in two different ways: either they are applied to the global matrix K, or first the velocity part X is eliminated, and then the pressure part P is computed by solving a system with the pressure Schur complement $S := C + B_2 \cdot N^{-1} \cdot B_1^T$ as system matrix. As explained in [20], in both cases (not only in the second), and also in the case of some multigrid methods, a crucial problem consists in finding a suitable preconditioner for S. Under the assumption that the discrete advection-diffusion-reaction operator N can be efficiently approximated, such a preconditioner was proposed in [15]; it will be denoted by $\hat{S}^{-1}$ in what follows, and is given by $\hat{S}^{-1} := M_p^{-1} \cdot N_p \cdot A_p^{-1}$, where M_p and A_p are projections of the identity and of a Neumann Laplacian onto the pressure finite element space, and N_p is the projection of the velocity operator $-\nu \cdot \Delta u + (v_0 \cdot \nabla)u + \theta \cdot u$ onto the same space.

This choice of preconditioner is motivated in [15], [19] and [9, p. 347-348] for example. As concerns numerical tests, a great number of them have been performed by now, with very satisfactory results. We refer to [6], [7], [8], [9], [15], [19], [20], [24], [26] in this respect. As concerns other aspects of preconditioning the matrix K from (1.4), like symmetric preconditioners, multigrid methods, or computation of exterior flows, we mention [3], [4], [16], [17], [18], [23], [25], [28]. This list is by no means exhaustive; many more references may be found in [9].

In the work at hand, we are interested in a theoretical aspect: we want to determine upper and lower bounds of the eigenvalues of $\hat{S}^{-1} \cdot S$. These bounds are crucial in attempts to evaluate the performance of iterative methods applied to (1.4); compare [9, Chapter 4]. Partial results on such bounds were presented in [7] (Newton's method) and [8]; a detailed theory was given in [19]. In the latter article, it was shown in particular how to treat a large class of stabilized methods in a unified way. The arguments in [19] are largely based on matrix algebra, but they also refer to H^2-estimates of solutions to elliptic partial differential equations. These estimates, besides requiring unnecessary restrictions on the domain of solutions to (1.1) and (1.2), present the additional inconvenience that the constants appearing in them are not very explicit as concerns their dependence on the parameters of the problem at hand. But it is precisely this dependence which is of interest in view of performance analysis of iterative methods.

In the present paper we will present a theory which is self-contained, does not use any regularity results for partial differential equations, and allows us to trace all relevant parameters in an explicit way. It turned out that our upper bounds on the eigenvalues of $\hat{S}^{-1} \cdot S$ depend on ν and θ in the way described in [19]; compare (3.21) with [19, Corollary 9A]. But as concerns the lower bounds (see (3.22)), we get a somewhat different, more pessimistic result. Perhaps this discrepancy is related to the argument used in [19]: whereas the upper bounds were derived solely by means of matrix algebra, the estimate of the lower bounds involved H^2-estimates of solutions of elliptic partial differential equations.

Our arguments are based on a variational approach which we already used in [5] in order to deal with preconditioning of the Schur complement by a pressure mass matrix. In the present context, this approach consists in writing the eigenvalue equation $\hat{S}^{-1} \cdot S \cdot P = \lambda \cdot P$ as a variational problem, estimate the solutions of this problem, and then deduce from these estimates the desired bounds of λ. This program will be developed in form of an abstract theory (Section 2), which is afterwards applied to the stabilized finite element methods considered in [19], and to LBB-stable methods (Section 3).

2. Abstract theory.

Let V and M be finite dimensional Hilbert spaces with scalar products denoted by respectively $(\ ,\ )_V$ and $(\ ,\ )_M$, and with associated norms denoted by $\|\ \|_V$ and $\|\ \|_M$. Since we want to deal with the two cases of enclosed and non-enclosed flow at the same time, we fix some $m_0 \in M$. Typically the case $m_0 = 0$ is related to models of non-enclosed flow, whereas the case $m_0 \neq 0$ pertains to enclosed flows. We put $M_0 := \{ p \in M \ : \ (p, m_0)_M = 0 \}$. Of course, if $m_0 = 0$, we have $M = M_0$.

Moreover, we introduce another norm on V, denoted by $\|\ \|_a$ and supposed to be induced by a scalar product, which will not appear explicitly. The norms $\|\ \|_V$ and $\|\ \|_a$ are assumed to be linked by the inequality

$$\|v\|_V \leq K_1 \cdot \|v\|_a \quad \text{for} \ \ v \in V, \tag{2.1}$$

with some constant $K_1 > 0$. Next consider bilinear forms $a : V \times V \mapsto \mathbb{R}$, $b_1, b_2 : V \times M \mapsto \mathbb{R}$, $c : M \times M \mapsto \mathbb{R}$ such that c is symmetric and $c(p,p) \geq 0$ for $p \in M$, and such that there are constants $\epsilon \in [0, \infty)$, K_2, ..., $K_5 \in (0, \infty)$ with

$$K_2 \cdot \|v\|_a^2 \leq a(v,v), \quad |a(v,w)| \leq K_3 \cdot \|v\|_a \cdot \|w\|_V \quad \text{for} \ \ v, w \in V; \tag{2.2}$$

$$|b_1(v,p)| \leq K_4 \cdot \|v\|_V \cdot \|p\|_M, \quad |b_1(v,p) - b_2(v,p)| \leq \epsilon \cdot \|v\|_a \cdot \|p\|_M \quad \text{for} \ \ v \in V, \ p \in M; \tag{2.3}$$

$$|c(p,q)| \leq K_5 \cdot \|p\|_M \cdot \|q\|_M \quad \text{for} \ \ p, q \in M; \tag{2.4}$$

$$b_2(v, m_0) = 0 \ \text{ for } \ v \in V, \quad c(m_0, p) = 0 \ \text{ for } \ p \in M. \tag{2.5}$$

We further require the following weak inf-sup condition: there are constants K_6, $\widetilde{K}_6 > 0$ with

$$\|p\|_M \leq \Big(K_6 \cdot \sup\{ b_1(v,p) \ : \ v \in V, \ \|v\|_V = 1 \} \ + \ \widetilde{K}_6 \cdot c(p,p)^{1/2} \Big) \quad \text{for} \ \ p \in M_0. \tag{2.6}$$

Finally let $\mathfrak{F} : V \mapsto \mathbb{R}$, $\mathfrak{G} : M \mapsto \mathbb{R}$ be given linear operators. Then we suppose that the variational problem in (1.3), that is, the discrete Navier-Stokes system, is a special case of the following abstract problem: Find $v \in V$, $\varrho \in M_0$ with

$$a(v,w) + b_1(w,\varrho) = \mathfrak{F}(w) \ \text{ for } \ w \in V, \quad b_2(v,\sigma) - c(\varrho,\sigma) = \mathfrak{G}(\sigma) \ \text{ for } \ \sigma \in M_0. \tag{2.7}$$

In order to transform problem (2.7) into a linear system of equations, we put $n := \dim V$, $m := \dim M$, and fix a basis $(\varphi_1, ..., \varphi_n)$ of V, and a basis $(\psi_1, ..., \psi_m)$ of M. We further set

$$N := \big(a(\varphi_j, \varphi_i) \big)_{1 \leq i,j \leq n}, \ B_\nu := \big(b_\nu(\varphi_j, \psi_i) \big)_{1 \leq i \leq m, \, 1 \leq j \leq n} \ \text{ for } \ \nu \in \{1, 2\},$$
$$C := \big(c(\psi_j, \psi_i) \big)_{1 \leq i,j \leq m}, \quad F := \big(\mathfrak{F}(\varphi_j) \big)_{1 \leq j \leq n}, \quad G := \big(\mathfrak{G}(\psi_j) \big)_{1 \leq j \leq m}.$$

Then variational problem (2.7) corresponds to system (1.4). In the case of an enclosed flow ($m_0 \neq 0$), the system matrix K of (1.4) is rank deficient by 1. As indicated in Section 1, we want to study how well the pressure Schur complement $S = C + B_2 \cdot N^{-1} \cdot B_1^T$ may be approximated by a pressure convection-diffusion preconditioner. In the present abstract framework, we introduce this preconditioner via bilinear forms $d, \widetilde{a} : M \times M \mapsto \mathbb{R}$. In order to specify suitable assumptions on these forms, we introduce two seminorms on M, denoted by respectively $\|\ \|_{\widetilde{a}}$ and $\|\ \|_d$, supposed to be induced by bilinear, symmetric, positive semi-definite forms on M. These seminorms are required to be a norm on M_0. In addition we assume there is $K_7 > 0$ with

$$\|p\|_M \leq K_7 \cdot \|p\|_{\widetilde{a}} \quad \text{for} \ \ p \in M_0. \tag{2.8}$$

We further assume that d is symmetric, and that there are constants K_8, ..., K_{11}, $\mu \in (0,\infty)$ with

$$K_8 \cdot \|p\|_d^2 \leq d(p,p), \quad K_9 \cdot \|p\|_{\widetilde{a}}^2 \leq \widetilde{a}(p,p) \quad \text{for } p \in M_0; \tag{2.9}$$

$$|\mu \cdot d(p,q) - \widetilde{a}(p,q)| \leq K_{10} \cdot \|p\|_{\widetilde{a}} \cdot \|q\|_M, \tag{2.10}$$
$$|\mu \cdot d(p,q) - \widetilde{a}(p,q)| \leq K_{11} \cdot \|p\|_d \cdot \|q\|_M \quad \text{for } p,q \in M;$$

$$\sup\big\{ |\widetilde{a}(p,q)| \cdot \|p\|_{\widetilde{a}}^{-1} \cdot \|q\|_{\widetilde{a}}^{-1} \ : \ p,q \in M_0\backslash\{0\} \big\} < \infty, \tag{2.11}$$
$$\sup\big\{ |d(p,q)| \cdot \|p\|_d^{-1} \cdot \|q\|_d^{-1} \ : \ p,q \in M\backslash\{0\} \big\} < \infty,$$
$$\sup\big\{ \|p\|_M / \|p\|_d \ : \ p \in M_0\backslash\{0\} \big\} < \infty;$$

$$\widetilde{a}(p,m_0) = 0, \ \ d(p,m_0) = 0 \ \text{ for } \ p \in M_0. \tag{2.12}$$

Then we put

$$A_p := \big(\, d(\psi_j,\psi_i) \,\big)_{1\leq i,j\leq m}, \quad N_p := \big(\, \widetilde{a}(\psi_j,\psi_i) \,\big)_{1\leq i,j\leq m}, \quad Q_p := \big(\, (\psi_j,\psi_i)_M \,\big)_{1\leq i,j\leq M}.$$

Obviously the matrix Q_p is invertible. The ensuing three lemmas are stated in view of the case $m_0 \neq 0$. If $m_0 = 0$ (hence $M = M_0$), they are obvious.

Lemma 2.1 *Let $\alpha_0 \in \mathbb{R}^m$ with $m_0 = \sum_{j=1}^m \alpha_j \cdot \psi_j$. Put $\mathfrak{M}_0 = \big\{ \beta \in \mathbb{R}^m \ : \ \alpha \cdot \beta = 0 \big\}$. Then, for $\beta \in \mathbb{R}^m$, we have $\beta \in \mathfrak{M}_0$ iff $\sum_{j=1}^m (Q_p^{-1} \cdot \beta)_j \cdot \psi_j \in M_0$.*

Moreover, for $\beta \in \mathfrak{M}_0$, there are unique vectors $\varrho, \eta \in \mathbb{R}^m$ such that $N_p \cdot \varrho = \beta$ and $Q_p \cdot \varrho \in \mathfrak{M}_0$, $A_p \cdot \eta = \beta$ and $Q_p \cdot \eta \in \mathfrak{M}_0$.

Inversely, if $\eta \in \mathbb{R}^m$ with $Q_p \cdot \eta \in \mathfrak{M}_0$, then $A_p \cdot \eta$ and $N_p \cdot \eta$ belong to $\mathfrak{M}_0$.

Proof: Let $\beta \in \mathbb{R}^m$, and put $w := \sum_{j=1}^m (Q_p^{-1} \cdot \beta)_j \cdot \psi_j$. Then

$$(w, m_0)_M = (Q_p^{-1} \cdot \beta)^T \cdot Q_p \cdot \alpha = \beta \cdot \alpha.$$

This proves the first claim of the lemma. Now suppose that $\beta \in \mathfrak{M}_0$, so that $w \in M_0$. The linear mapping $L : M_0 \ni v \mapsto (w,v)_M \in \mathbb{R}$ is bounded with respect to the norm $\| \ \|_{\widetilde{a}}$ (see (2.8)), so by (2.9), (2.11) and the Lax-Milgram lemma, there is a unique element $\widetilde{v} \in M_0$ such that $\widetilde{a}(\widetilde{v}, v) = (w,v)_M$ for $v \in M_0$. Since $\widetilde{v}, w \in M_0$, we may conclude by (2.12) that

$$\widetilde{a}(\widetilde{v}, v) = (w,v)_M \quad \text{for } \ v \in M. \tag{2.13}$$

Let $\varrho \in \mathbb{R}^m$ with $\widetilde{v} = \sum_{j=1}^m \varrho_j \cdot \psi_j$. Then equation (2.13) yields

$$\sigma^T \cdot N_p \cdot \varrho = \sigma^T \cdot Q_p \cdot (Q_p^{-1} \cdot \beta) \quad \text{for } \ \sigma \in \mathbb{R}^m,$$

hence $N_p \cdot \varrho = \beta$. Moreover, since $\widetilde{v} \in M_0$, we have $(m_0, \widetilde{v})_M = 0$, so that $\alpha^T \cdot Q_p \cdot \varrho = 0$, and thus $Q_p \cdot \varrho \in \mathfrak{M}_0$.

If $\sigma \in \mathbb{R}^m$ with $Q_p \cdot \sigma \in \mathfrak{M}_0$ and $N_p \cdot \sigma = 0$, we put $v := \sum_{j=1}^m \sigma_j \cdot \psi_j$. Then $\widetilde{a}(v,v) = \sigma^T \cdot N_p \cdot \sigma = 0$. Furthermore, $\alpha^T \cdot Q_p \cdot \sigma = 0$ because $Q_p \cdot \sigma \in \mathfrak{M}_0$. Thus $(m_0, v)_M = \alpha^T \cdot Q_p \cdot \sigma = 0$, so that $v \in M_0$. The latter relation and (2.9) yield $\widetilde{a}(v,v) \geq K_9 \cdot \|v\|_{\widetilde{a}}^2$, so $v = 0$, hence $\sigma = 0$. This shows uniqueness of a vector $\varrho \in \mathbb{R}^m$ with $N_p \cdot \varrho = \beta$ and $Q_p \cdot \varrho \in \mathfrak{M}_0$. Using the same argument, but this time referring to the second and third relation in (2.1) instead of the first and (2.8), we obtain existence and uniqueness of $\eta \in \mathbb{R}^m$ with $A_p \cdot \eta = \beta$ and $Q_p \cdot \varrho \in \mathfrak{M}_0$. Concerning the last claim of the lemma, let $\eta \in \mathbb{R}^m$ with $Q_p \cdot \eta \in \mathfrak{M}_0$, that is, $v := \sum_{j=1}^m \eta_j \cdot \psi_j \in M_0$. Then assumption (2.12) yields $\widetilde{a}(v, m_0) = 0 = d(v, m_0)$, that is, $\alpha^T \cdot N_p \cdot \eta = 0 = \alpha^T \cdot A_p \cdot \eta$, hence $N_p \cdot \eta,\ A_p \cdot \eta \in \mathfrak{M}_0$. $\diamond$

In view of Lemma 2.1, we may define $N_p^{-1} \in \mathbb{R}^{m\times m}$ by the following two conditions. First, $N_p^{-1} \cdot \alpha := 0$, and second, if $\beta \in \mathfrak{M}_0$, then $N_p^{-1} \cdot \beta$ is the unique vector $\varrho \in \mathbb{R}^m$ with $N_p \cdot \varrho = \beta$ and $Q_p \cdot \varrho \in \mathfrak{M}_0$. The matrix $A_p^{-1} \in \mathbb{R}^{m\times m}$ is introduced in an analogous way. Note that if $m_0 = 0$, then N_p^{-1} and A_p^{-1} are the usual inverse of N_p and A_p, respectively. As indicated in Section 1, we consider $\hat{S} := A_p \cdot N_p^{-1} \cdot M_p$ as an approximation of S, or in other words, we precondition S by $\hat{S}^{-1} := M_p^{-1} \cdot N_p \cdot A_p^{-1}$. Note that $\hat{S}$ is singular in the case $m_0 \neq 0$, so the notation $\hat{S}^{-1}$ should be considered as formal (and therefore was introduced as a definition here). Concerning the matrix S, the following observation will be useful.

Lemma 2.2 *$S \cdot \beta \in \mathfrak{M}_0$ for $\beta \in \mathbb{R}^m$.*

Proof: Let $\beta \in \mathbb{R}^m$, and put $g := \sum_{j=1}^m \beta_j \cdot \psi_j$. By (2.2), (2.3), (2.1) and the Lax-Milgram lemma, there is $u \in V$ with $a(u, w) = b_1(w, g)$ for $w \in V$. Then $\alpha^T \cdot S \cdot \beta = c(m_0, g) + b_2(u, m_0)$, hence by (2.5) $\alpha^T \cdot S \cdot \beta = 0$. $\diamond$

Lemma 2.3 *Let $\lambda \in \mathbb{C},\ P \in \mathbb{C}^m\backslash\{0\}$ with $\hat{S}^{-1} \cdot S \cdot P = \lambda \cdot P$. If $m_0 = 0$, then $\lambda \neq 0$. If $m_0 \neq 0$, suppose in addition that $\lambda \neq 0$. Then $\Re P,\ \Im P \in Q_p^{-1} \cdot \mathfrak{M}_0$.*

Proof: First suppose that $m_0 = 0$. Then N_p and A_p are invertible (Lemma 2.1), hence $\hat{S}^{-1}$ is regular. Moreover, in view of (2.6) and (2.2), it is well known in that case that S is invertible; see [9, p. 274-275], for example. Since $P \neq 0$, it now follows from the eigenvalue equation that $\lambda \neq 0$.

In order to show the last claim of the lemma, let $\varrho \in \{\Re P, \Im P\}$. We have $S \cdot \varrho \in \mathfrak{M}_0$ by Lemma 2.2, hence $Q_p \cdot A_p^{-1} \cdot S \cdot \varrho \in \mathfrak{M}_0$ by Lemma 2.1 and the definition of A_p. Again referring to Lemma 2.1, we now get $N_p \cdot A_p^{-1} \cdot S \cdot \varrho \in \mathfrak{M}_0$, hence $\hat{S}^{-1} \cdot S \cdot \varrho \in Q_p^{-1} \cdot \mathfrak{M}_0$. It follows that $\Re(\lambda^{-1} \cdot \hat{S}^{-1} \cdot S \cdot \varrho),\ \Im(\lambda^{-1} \cdot \hat{S}^{-1} \cdot S \cdot \varrho) \in Q_p^{-1} \cdot \mathfrak{M}_0$. Now we may conclude from the eigenvalue equation that $\Re P,\ \Im P \in Q_p^{-1} \cdot \mathfrak{M}_0$. $\diamond$

Now we are in a position to establish the eigenvalue bounds for $\hat{S}^{-1} \cdot S$ which are the main result of this article.

Theorem 2.1 *Let $\lambda \in \mathbb{C},\ P \in \mathbb{C}\backslash\{0\}$ with $\hat{S}^{-1} \cdot S \cdot P = \lambda \cdot P$. In the case $m_0 \neq 0$, suppose in addition that $\lambda \neq 0$. Then $S \cdot P = \lambda \cdot \hat{S} \cdot P$,*

$$|\lambda| \ \leq\ 8 \cdot \mu \cdot \Big(K_5 + (K_4 \cdot K_1 + \epsilon) \cdot K_2^{-1} \cdot K_4 \cdot K_1\Big) \qquad (2.14)$$
$$+\ 16 \cdot K_{11}^2 \cdot K_8^{-1} \cdot \Big(K_5 + (K_4 \cdot K_1 + \epsilon) \cdot K_2^{-1} \cdot K_4 \cdot K_1\Big) \cdot K_7^2 \cdot K_9^{-1}.$$

If in addition the constant ϵ from (2.3) satifies the condition

$$\epsilon \leq K_2/(8 \cdot K_3 \cdot K_6), \qquad (2.15)$$

we further have

$$|\lambda| \geq \mu/\Big(128 \cdot (K_6^2 \cdot K_3^2 \cdot K_2^{-1} + \widetilde{K}_6^2) \cdot (1 + K_{10} \cdot K_9^{-1} \cdot K_7)\Big). \qquad (2.16)$$

Proof: First we write the eigenvalue equation in the form $S \cdot P = \lambda \cdot \hat{S} \cdot P$. This transformation is of course obvious in the case $m_0 = 0$. In order to justify it in the case $m_0 \neq 0$, take $\varrho \in \{\Re P,\ \Im P\}$. Then $Q_p \cdot \varrho \in \mathfrak{M}_0$ and $S \cdot \varrho \in \mathfrak{M}_0$ by Lemma 2.3 and 2.2, respectively. The relation $S \cdot \varrho \in \mathfrak{M}_0$ implies $Q_p \cdot A_p^{-1} \cdot S \cdot \varrho \in \mathfrak{M}_0$ by Lemma 2.1. Moreover, the eigenvalue equation yields $N_p \cdot A_p^{-1} \cdot S \cdot P = \lambda \cdot Q_p \cdot P$. Now it follows from Lemma 2.1 and the definition of N_p^{-1} that $A_p^{-1} \cdot S \cdot P = \lambda \cdot N_p^{-1} \cdot Q_p \cdot P$. Thus we may conclude that the equation $S \cdot P = \lambda \cdot \hat{S} \cdot P$ also holds in the case $m_0 \neq 0$.

Next we transform this equation into a variational problem. To this end, put

$$\pi_1 := \sum_{j=1}^{m} \Re P_j \cdot \psi_j, \quad \pi_2 := \sum_{j=1}^{m} \Im P_j \cdot \psi_j.$$

Since $Q_p \cdot \varrho \in \mathfrak{M}_0$ for $\varrho \in \{\Re P, \Im P\}$ by Lemma 2.3, we get $\pi_1, \pi_2 \in M_0$ (Lemma 2.1). Moreover, by (2.2), (2.3), (2.1), (2.9), (2.11), (2.8) and the Lax-Milgram lemma, there is a unique element $u_i \in V$, for $i \in \{1, 2\}$, such that

$$a(u_i, w) = b_1(w, \pi_i) \quad \text{for } w \in V, \tag{2.17}$$

and a unique element $\widetilde{u}_i \in M_0$ with

$$\widetilde{a}(\widetilde{u}_i, p) = (p, \pi_i)_M \quad \text{for } p \in M_0. \tag{2.18}$$

Due to (2.12), we thus have

$$\widetilde{a}(\widetilde{u}_i, p) = (p, \pi_i)_M \quad \text{for } p \in M, \ i \in \{1, 2\}.$$

Now we may deduce from the equation $S \cdot P = \lambda \cdot \hat{S} \cdot P$ that

$$c(\pi_1, p) + b_2(u_1, p) = \Re\lambda \cdot d(\widetilde{u}_1, p) - \Im\lambda \cdot d(\widetilde{u}_2, p), \tag{2.19}$$

$$c(\pi_2, p) + b_2(u_2, p) = \Re\lambda \cdot d(\widetilde{u}_2, p) + \Im\lambda \cdot d(\widetilde{u}_1, p) \quad \text{for } p \in M. \tag{2.20}$$

Without loss of generality, we may suppose that $\|\pi_1\|_M^2 + \|\pi\|_M^2 = 1$. In a first step, we use the approach from [5] in order to deduce upper bounds for u_i and $\widetilde{u}_i$. In fact, we obtain by (2.2), (2.17), (2.3), (2.1),

$$\|u_i\|_a^2 \leq K_2^{-1} \cdot a(u_i, u_i) = K_2^{-1} \cdot b_1(u_i, \pi_i) \leq K_2^{-1} \cdot K_4 \cdot \|u_i\|_V \cdot \|\pi_i\|_M \leq K_2^{-1} \cdot K_4 \cdot K_1 \cdot \|u_i\|_a,$$

hence

$$\|u_i\|_a \leq K_2^{-1} \cdot K_4 \cdot K_1 \quad \text{for } i \in \{1, 2\}. \tag{2.21}$$

Similarly, by (2.9), (2.18), (2.8),

$$\|\widetilde{u}_i\|_{\widetilde{a}}^2 \leq K_9^{-1} \cdot \widetilde{a}(\widetilde{u}_i, \widetilde{u}_i) = K_9^{-1} \cdot (\widetilde{u}_i, \pi_i)_M \leq K_9^{-1} \cdot \|\widetilde{u}_i\|_M \cdot \|\pi_i\|_M \leq K_9^{-1} \cdot K_7 \cdot \|\widetilde{u}_i\|_{\widetilde{a}},$$

hence

$$\|\widetilde{u}_i\|_{\widetilde{a}} \leq K_9^{-1} \cdot K_7 \quad \text{for } i \in \{1, 2\}. \tag{2.22}$$

Now let us turn to the proof of (2.14). By referring to (2.18), (2.19), (2.20), we get

$$\begin{aligned}
\Re\lambda &= \Re\lambda \cdot \sum_{i=1}^{2} (\pi_i, \pi_i)_M = \Re\lambda \cdot \sum_{i=1}^{2} (\pi_i, \pi_i)_M \; - \; \Im\lambda \cdot (\pi_1, \pi_2)_M \; + \; \Im\lambda \cdot (\pi_2, \pi_1)_M \\
&= \Re\lambda \cdot \widetilde{a}(\widetilde{u}_1, \pi_1) - \Im\lambda \cdot \widetilde{a}(\widetilde{u}_2, \pi_1) + \Re\lambda \cdot \widetilde{a}(\widetilde{u}_2, \pi_2) + \Im\lambda \cdot \widetilde{a}(\widetilde{u}_1, \pi_2) \\
&= \mu \cdot \Big(\Re\lambda \cdot d(\widetilde{u}_1, \pi_1) - \Im\lambda \cdot d(\widetilde{u}_2, \pi_1) + \Re\lambda \cdot d(\widetilde{u}_2, \pi_2) + \Im\lambda \cdot d(\widetilde{u}_1, \pi_2) \Big) \\
&\quad + \Re\lambda \cdot \sum_{i=1}^{2} \big(\widetilde{a}(\widetilde{u}_i, \pi_i) - \mu \cdot d(\widetilde{u}_i, \pi_i) \big)
\end{aligned}$$

$$+\Im\lambda\cdot\Big(-\widetilde{a}(\widetilde{u}_2,\pi_1)+\mu\cdot d(\widetilde{u}_2,\pi_1)+\widetilde{a}(\widetilde{u}_1,\pi_2)-\mu\cdot d(\widetilde{u}_1,\pi_2)\Big)$$

$$\leq \mu\cdot\Big(c(\pi_1,\pi_1)+b_2(u_1,\pi_1)+c(\pi_2,\pi_2)+b_2(u_2,\pi_2)\Big)\;+\;\Re\lambda\cdot\sum_{i=1}^{2}\big(\,\widetilde{a}(\widetilde{u}_i,\pi_i)-\mu\cdot d(\widetilde{u}_i,\pi_i)\,\big)$$

$$+\Im\lambda\cdot\Big(-\widetilde{a}(\widetilde{u}_2,\pi_1)+\mu\cdot d(\widetilde{u}_2,\pi_1)+\widetilde{a}(\widetilde{u}_1,\pi_2)-\mu\cdot d(\widetilde{u}_1,\pi_2)\Big).$$

Using (2.4), (2.3), (2.10), (2.1), (2.21), we may conclude

$$|\Re\lambda|\leq \mu\cdot\sum_{i=1}^{2}\big(\,K_5\cdot\|\pi_i\|_M^2+K_4\cdot\|u_i\|_V\cdot\|\pi_i\|_M+\epsilon\cdot\|u_i\|_a\cdot\|\pi_i\|_M\,\big) \tag{2.23}$$

$$+|\Re\lambda|\cdot\sum_{i=1}^{2}K_{11}\cdot\|\widetilde{u}_i\|_d\cdot\|\pi_i\|_M+|\Im\lambda|\cdot K_{11}\cdot\big(\,\|\widetilde{u}_1\|_d\cdot\|\pi_2\|_M+\|\widetilde{u}_2\|_d\cdot\|\pi_1\|_M\,\big)$$

$$\leq 2\cdot\mu\cdot\big(\,K_5+(K_4\cdot K_1+\epsilon)\cdot K_2^{-1}\cdot K_4\cdot K_1\,\big)\;+\;(|\Re\lambda|+|\Im\lambda|)\cdot K_{11}\cdot\sum_{i=1}^{2}\|\widetilde{u}_i\|_d$$

$$\leq 2\cdot\mu\cdot\big(\,K_5+(K_4\cdot K_1+\epsilon)\cdot K_2^{-1}\cdot K_4\cdot K_1\,\big)\;+\;(|\Re\lambda|+|\Im\lambda|)/4$$

$$+\,2\cdot(|\Re\lambda|+|\Im\lambda|)\cdot K_{11}^2\cdot\sum_{i=1}^{2}\|\widetilde{u}_i\|_d^2.$$

Again referring to (2.18) - (2.20), in a similar way we find

$$\Im\lambda=\mu\cdot\big(\,-c(\pi_1,\pi_2)-b_2(u_1,\pi_2)+c(\pi_2,\pi_1)+b_2(u_2,\pi_1)\,\big)$$

$$+\Im\lambda\cdot\sum_{i=1}^{2}\big(\,\widetilde{a}(\widetilde{u}_i,\pi_i)-\mu\cdot d(\widetilde{u}_i,\pi_i)\,\big)$$

$$+\Re\lambda\cdot\Big(-\widetilde{a}(\widetilde{u}_1,\pi_2)+\mu\cdot d(\widetilde{u}_1,\pi_2)+\widetilde{a}(\widetilde{u}_2,\pi_1)-\mu\cdot d(\widetilde{u}_2,\pi_1)\Big),$$

hence with (2.4), (2.3), (2.10), (2.1), (2.21),

$$|\Im\lambda|\leq 2\cdot\mu\cdot\big(\,K_5+(K_4\cdot K_1+\epsilon)\cdot K_2^{-1}\cdot K_4\cdot K_1\,\big)\;+\;(|\Re\lambda|+|\Im\lambda|)/4 \tag{2.24}$$

$$+\,2\cdot(|\Re\lambda|+|\Im\lambda|)\cdot K_{11}^2\cdot\sum_{i=1}^{2}\|\widetilde{u}_i\|_d^2.$$

On the other hand, due to the relation $\widetilde{u}_1,\widetilde{u}_2\in M_0$, by (2.9), the symmetry of d, (2.19), (2.20), (2.3), (2.4), (2.1), (2.21), (2.8) and (2.22), we obtain

$$|\Re\lambda|\cdot\sum_{i=1}^{2}\|\widetilde{u}_i\|_d^2\;\leq\;K_8^{-1}\cdot\Big|\Re\lambda\cdot\sum_{i=1}^{2}d(\widetilde{u}_i,\widetilde{u}_i)\Big| \tag{2.25}$$

$$=K_8^{-1}\cdot\Big|\Re\lambda\cdot d(\widetilde{u}_1,\widetilde{u}_1)-\Im\lambda\cdot d(\widetilde{u}_2,\widetilde{u}_1)+\Im\lambda\cdot d(\widetilde{u}_1,\widetilde{u}_2)+\Re\lambda\cdot d(\widetilde{u}_2,\widetilde{u}_2)\Big|$$

$$=K_8^{-1}\cdot\Big|\sum_{i=1}^{2}\big(\,c(\pi_i,\widetilde{u}_i)+b_2(u_i,\widetilde{u}_i)\,\big)\Big|$$

$$\leq K_8^{-1}\cdot\sum_{i=1}^{2}\big(\,K_5\cdot\|\pi_i\|_M\cdot\|\widetilde{u}_i\|_M+(K_4\cdot K_1+\epsilon)\cdot\|u_i\|_a\cdot\|\widetilde{u}_i\|_M\,\big)$$

$$\leq K_8^{-1} \cdot \sum_{i=1}^{2} \big(K_5 + (K_4 \cdot K_1 + \epsilon) \cdot K_2^{-1} \cdot K_4 \cdot K_1 \big) \cdot K_7 \cdot \|\widetilde{u}_i\|_{\widetilde{a}}$$

$$\leq 2 \cdot K_8^{-1} \cdot \big(K_5 + (K_4 \cdot K_1 + \epsilon) \cdot K_2^{-1} \cdot K_4 \cdot K_1 \big) \cdot K_7^2 \cdot K_9^{-1},$$

and similarly,

$$|\Im\lambda| \cdot \sum_{i=1}^{2} \|\widetilde{u}_i\|_d^2 \;\leq\; K_8^{-1} \cdot \big| c(\pi_2, \widetilde{u}_1) + b_2(u_2, \widetilde{u}_1) - c(\pi_1, \widetilde{u}_2) - b_2(u_1, \widetilde{u}_2) \big| \tag{2.26}$$

$$\leq 2 \cdot K_8^{-1} \cdot \big(K_5 + (K_4 \cdot K_1 + \epsilon) \cdot K_2^{-1} \cdot K_4 \cdot K_1 \big) \cdot K_7^2 \cdot K_9^{-1}.$$

Estimating the right-hand side of (2.23) and (2.24) by respectively (2.25) and (2.26), we arrive at the inequality

$$|\Re\lambda| + |\Im\lambda| \;\leq\; 4 \cdot \mu \cdot \big(K_5 + (K_4 \cdot K_1 + \epsilon) \cdot K_2^{-1} \cdot K_4 \cdot K_1 \big)$$
$$+ (|\Re\lambda| + |\Im\lambda|)/2 \;+\; 8 \cdot K_{11}^2 \cdot K_8^{-1} \cdot \big(K_5 + (K_4 \cdot K_1 + \epsilon) \cdot K_2^{-1} \cdot K_4 \cdot K_1 \big) \cdot K_7^2 \cdot K_9^{-1}.$$

Now inequality (2.14) follows. In order to prove (2.16), we choose an element $\overline{v}_i \in V$, for $i \in \{1,\, 2\}$, such that $\|\overline{v}_i\|_V = 1$ and $b_1(\overline{v}_i, \pi_i) \geq (1/2) \cdot \sup\{b_1(v, \pi_i) \;:\; v \in V,\; \|v\|_V = 1\}$. Since $\pi_i \in M_0$, and in view of (2.6), (2.17), we get

$$1 = \sum_{i=1}^{2} \|\pi_i\|_M^2 \leq \sum_{i=1}^{2} \big(2 \cdot K_6^2 \cdot b_1(\overline{v}_i, \pi_i) + \widetilde{K}_6^2 \cdot c(\pi_i, \pi_i)^{1/2} \big)^2 \tag{2.27}$$

$$\leq 2 \cdot \sum_{i=1}^{2} \big(4 \cdot K_6^2 \cdot b_1(\overline{v}_i, \pi_i)^2 + \widetilde{K}_6^2 \cdot c(\pi_i, \pi_i) \big) \;=\; 2 \cdot \sum_{i=1}^{2} \big(4 \cdot K_6^2 \cdot a(u_i, \overline{v}_i)^2 + \widetilde{K}_6^2 \cdot c(\pi_i, \pi_i) \big).$$

But for $i \in \{1,\, 2\}$, by (2.2), (2.17), (2.3),

$$a(u_i, \overline{v}_i)^2 \leq K_3^2 \cdot \|u_i\|_a^2 \cdot \|\overline{v}_i\|_V^2 = K_3^2 \cdot \|u_i\|_a^2 \leq K_3^2 \cdot K_2^{-1} \cdot a(u_i, u_i) \tag{2.28}$$
$$= K_3^2 \cdot K_2^{-1} \cdot b_1(u_i, \pi_i) \leq K_3^2 \cdot K_2^{-1} \cdot \big(|b_1(u_i, \pi_i) - b_2(u_i, \pi_i)| + b_2(u_i, \pi_i) \big)$$
$$\leq K_3^2 \cdot K_2^{-1} \cdot \big(\epsilon \cdot \|u_i\|_a \cdot \|\pi_i\|_M + b_2(u_i, \pi_i) \big)$$
$$\leq K_3^2 \cdot \|u_i\|_a^2/2 + K_3^2 \cdot K_2^{-2} \cdot \epsilon^2 + K_3^2 \cdot K_2^{-1} \cdot b_2(u_i, \pi_i).$$

By reading estimate (2.28) from the first equation onwards, we may conclude that

$$K_3^2 \cdot \|u_i\|_a/2 \leq K_3^2 \cdot K_2^{-2} \cdot \epsilon^2 + K_3^2 \cdot K_2^{-1} \cdot b_2(u_i, \pi_i).$$

Again referring to (2.28), we now get

$$a(u_i, \overline{v}_i)^2 \leq 2 \cdot K_3^2 \cdot K_2^{-2} \cdot \epsilon^2 + 2 \cdot K_3^2 \cdot K_2^{-1} \cdot b_2(u_i, \pi_i) \quad (i \in \{1,\, 2\}).$$

This result is inserted into (2.27); it follows

$$1 \leq 2 \cdot \sum_{i=1}^{2} \big(8 \cdot K_6^2 \cdot K_3^2 \cdot K_2^{-2} \cdot \epsilon^2 + 8 \cdot K_6^2 \cdot K_3^2 \cdot K_2^{-1} \cdot b_2(u_i, \pi_i) + \widetilde{K}_6^2 \cdot c(\pi_i, \pi_i) \big) \tag{2.29}$$

$$\leq 32 \cdot K_6^2 \cdot K_3^2 \cdot K_2^{-2} \cdot \epsilon^2 \;+\; 16 \cdot (K_6^2 \cdot K_3^2 \cdot K_2^{-1} + \widetilde{K}_6^2) \cdot \sum_{i=1}^{2} \big(b_2(u_i, \pi_i) + c(\pi_i, \pi_i) \big)$$

$$\leq 32 \cdot K_6^2 \cdot K_3^2 \cdot K_2^{-2} \cdot \epsilon^2 \;+\; 16 \cdot (K_6^2 \cdot K_3^2 \cdot K_2^{-1} + \widetilde{K}_6^2) \cdot \big(\Re\lambda \cdot d(\widetilde{u}_1, \pi_1) - \Im\lambda \cdot d(\widetilde{u}_2, \pi_1)$$
$$+ \Re\lambda \cdot d(\widetilde{u}_2, \pi_2) + \Im\lambda \cdot d(\widetilde{u}_1, \pi_2) \big),$$

where the last equation follows from (2.19), (2.20). But for $i, j \in \{1, 2\}$, by (2.18), (2.10), (2.22),

$$|d(\widetilde{u}_i, \pi_j)| = \mu^{-1} \cdot \big(|\widetilde{a}(\widetilde{u}_i, \pi_j)| + |\mu \cdot d(\widetilde{u}_i, \pi_j) - \widetilde{a}(\widetilde{u}_i, \pi_j)| \big) \tag{2.30}$$
$$\le \mu^{-1} \cdot \big(|(\pi_i, \pi_j)_M| + K_{10} \cdot \|\widetilde{u}_i\|_{\widetilde{a}} \cdot \|\pi_j\|_M \big) \le \mu^{-1} \cdot (1 + K_{10} \cdot \|\widetilde{u}_i\|_{\widetilde{a}})$$
$$\le \mu^{-1} \cdot \Big(1 + K_{10} \cdot K_9^{-1} \cdot K_7).$$

Due to (2.30), we may majorize the right-hand side of (2.29), to obtain

$$1 \le 32 \cdot K_6^2 \cdot K_3^2 \cdot K_2^{-2} \cdot \epsilon^2 + 64 \cdot (K_6^2 \cdot K_3^2 \cdot K_2^{-1} + \widetilde{K}_6^2) \cdot |\lambda| \cdot \mu^{-1} \cdot (1 + K_{10} \cdot K_9^{-1} \cdot K_7). \tag{2.31}$$

Therefore, if ϵ verifies condition (2.16), we get

$$1/2 \le 64 \cdot (K_6^2 \cdot K_3^2 \cdot K_2^{-1} + \widetilde{K}_6^2 \cdot) \cdot |\lambda| \cdot \mu^{-1} \cdot (1 + K_{10} \cdot K_9^{-1} \cdot K_7),$$

and inequality (2.16) follows. ◇

3. Applications.

Let us apply our abstract theory to the stabilized schemes considered in [19] and [20]. In order to introduce these schemes, we fix a bounded domain Ω with Lipschitz boundary Γ. We denote the outward unit normal to Ω by n. Let $\Gamma_D, \Gamma_N \subset \Gamma$ with $\Gamma_D \cap \Gamma_N = \emptyset,\ \ \Gamma_D \cup \Gamma_N = \Gamma$, and such that the set Γ_D has positive measure in Γ. Put $H^1_E(\Omega) := \{v \in H^1(\Omega)^2 \ : \ v \,|\, \Gamma_D = 0\}$. Note that $H^1_E(\Omega) = H^1_0(\Omega)^2$ if Γ_N has measure zero in Γ. Since the set Γ_D has positive measure, Poincaré's inequality holds, that is, there is $C_1 = C_1(\Omega, \Gamma_D) > 0$ with

$$\|v\|_2 \le C_1 \cdot \|\nabla v\|_2 \quad \text{for } \ v \in H^1_E(\Omega). \tag{3.1}$$

By a standard Sobolev inequality, we have $\|v\|_4 \le C \cdot \|v\|_{1,2}$ for $v \in H^1(\Omega)$, with $C = C(\Omega) > 0$ only depending on Ω. Therefore we may choose $C_2 = C_2(\Omega, \Gamma_D) > 0$ with

$$\|v\|_4 \le C_2 \cdot \|\nabla v\|_2 \quad \text{for } \ v \in H^1_E(\Omega). \tag{3.2}$$

Let $\mathfrak{T}$ be a subdivision of $\overline{\Omega}$ into a finite number of closed sets T, each of which is the closure of an open connected set with Lipschitz boundary. Put $h_T := \mathit{diam}\, T$ for $T \in \mathfrak{T}$, and $h := \max\{h_T \ : \ T \in \mathfrak{T}\}$. Let δ_T, with $T \in \mathfrak{T}$, be a family in $(0, \infty)$ such that

$$C_3 \cdot h_T^2 \le \delta_T \le C_4 \cdot h_T^2 \quad \text{for } \ T \in \mathfrak{T}, \tag{3.3}$$

for some $C_3, C_4 > 0$. Define the piecewise constant function $\delta : \overline{\Omega} \mapsto (0, \infty)$ by $\delta|T = \delta_T$ for $T \in \mathfrak{T}$. (Since we are only interested in δ as an L^2-function, we do not pay attention to its definition on ∂T, for $T \in \mathfrak{T}$.)

Let V_h, P_h be finite dimensional spaces with $V_h \subset H^1_E(\Omega),\ \ P_h \subset H^1(\Omega)$ and

$$v|T \in C^2(T)^2, \ \ p|T \in C^2(T) \ \text{ for } \ T \in \mathfrak{T}, \ \ v \in V_h, \ \ p \in P_h.$$

For $v \in V_h$ and $p \in P_h$, we define the L^2-functions $\Delta v,\ \Delta p$ in an obvious way. We assume the following inverse inequalities: there is some $C_5 > 0$ with

$$\|\Delta w\|_{2,T} \le C_5 \cdot h_T^{-1} \cdot \|\nabla w\|_{2,T}, \quad \|\nabla p\|_{2,T} \le C_5 \cdot h_T^{-1} \cdot \|p\|_{2,T}, \tag{3.4}$$
$$\|\Delta p\|_{2,T} \le C_5 \cdot h_T^{-1} \cdot \|\nabla p\|_{2,T} \quad \text{for } \ v \in V_h, \ p \in P_h, \ T \in \mathfrak{T}.$$

Such relations are satisfied by standard finite element spaces; see [22, p. 195, 281]. As a consequence of (3.3) and (3.4), we obtain

$$\|\delta^{1/2}\cdot\Delta v\|_2 \le C_4^{1/2}\cdot C_5\cdot\|\nabla v\|_2 \text{ for } v\in V_h, \quad \|\delta^{1/2}\cdot\nabla p\|_2 \le C_4^{1/2}\cdot C_5\cdot\|p\|_2 \text{ for } p\in P_h. \quad (3.5)$$

For $g \in H^1(\Omega)^2$ or $g \in H^1(\Omega)$, we put $|g|_{1,2} := \left(\sum_{i=1}^2 \|\partial_i g\|_2^2\right)^{1/2} = \|\nabla g\|_2$. Of course, the mapping $|\ |_{1,2}$ is only a seminorm on $H^1(\Omega)^2$ and $H^1(\Omega)$, but due to (3.1), it is a norm on $H^1_E(\Omega)$, and hence on V_h. We further require that V_h satisfies the following standard approximation property of finite element spaces (compare [1, Theorem 4.4.4]): for $v \in H^1_0(\Omega)^2$, there is $v_h \in V_h \cap H^1_0(\Omega)^2$ with

$$\Big(\sum_{T\in\mathfrak{T}} h_T^{-2}\cdot\|v-v_h\|_{2,T}^2\Big)^{1/2} \le C_6\cdot|v|_{1,2}, \quad |v-v_h|_{1,2} \le C_6\cdot|v|_{1,2}, \qquad (3.6)$$

where the constant C_6 typically depends only on the "chunkiness parameter" ([1, 4.2.16]) of an underlying grid. We are going to apply the theory from Section 2 with $V = V_h$, $M = P_h$, $\|\ \|_V := |\ \ |_{1,2}\,|\,V_h$. The scalar product $(\ ,\ )_M$ is chosen as the usual L^2-scalar product on P_h, hence $\|\ \|_M$ is the usual L^2-norm $\|\ \|_2$ on P_h. We fix parameters $\nu \in (0,\infty)$, $\beta,\theta \in [0,\infty)$ and $\varrho \in \{-1,\,0,\,1\}$, as well as a function $v_0 \in V_h$ with $v_0\cdot n \ge 0$ on Γ_N. The forms a, b_1, b_2 and c are defined as follows:

$$\begin{aligned} a(v,w) &:= \int_\Omega \Big(\nu\cdot\nabla v\cdot\nabla w + (v_0\cdot\nabla)v\cdot w + (1/2)\cdot\operatorname{div} v_0\cdot(v\cdot w) \\ &\quad + \theta\cdot(v\cdot w) + \beta\cdot\operatorname{div} v\cdot\operatorname{div} w \\ &\quad + \delta\cdot\big(-\nu\cdot\Delta v + (v_0\cdot\nabla)v + \theta\cdot v\big)\cdot\big(\varrho\cdot\nu\cdot\Delta w + (v_0\cdot\nabla)w\big)\Big)\,dx, \end{aligned}$$

$$b_1(v,q) := \int_\Omega \Big(-\operatorname{div} v\cdot q \;+\; \delta\cdot\big(\nu\cdot\Delta v + (v_0\cdot\nabla)v\big)\cdot\nabla q\Big)\,dx,$$

$$b_2(v,q) := b_1(v,q) - \int_\Omega \delta\cdot\big(2\cdot(v_0\cdot\nabla)v + \theta\cdot v\big)\cdot\nabla q\,dx, \quad c(p,q) := \int_\Omega \delta\cdot(\nabla p\cdot\nabla q)\,dx$$

for $v,w \in V_h$, $p,q \in P_h$. Let $\mathfrak{F} : V_h \mapsto \mathbb{R}$ and $\mathfrak{G} : P_h \mapsto \mathbb{R}$ be linear operators, which we consider as given.

With this choice of function spaces and bilinear forms, we have implicitly assumed that the Navier-Stokes system (1.1) or (1.2) is supplemented by a homogeneous boundary Dirichlet condition on Γ_D, and by a homogeneous traction condition on Γ_N. Moreover, it is inherent to our definition of the form a that in the time-dependent case, a fully or semi-implicit time discretization was used, and/or that the problem was linearized by Picard iteration. The function v_0 corresponds to a velocity approximation obtained in a previous time-step or in a previous step of the Picard iteration. Therefore it is reasonable to assume that $v_0 \in V_h$. As mentioned in Section 1, the quantity θ corresponds to the inverse of the time step and should be taken as zero in the stationary case. Concerning the discretization with respect to the space variables, we followed [19] so that our definition of a, b_1, b_2 and c covers the stabilized schemes of Brooks and Hughes [2], Hansbo and Szepessy [14], Franca and Frey [11], Tobiska and Lube [27], and Zhou and Feng [29] (least squares); compare the remarks in [19, p. 8]. We further note that instead of assuming the relation div $v_0 = 0$, which does not hold for most finite element methods, we introduced the additional term $(1/2)\cdot\operatorname{div} v_0\cdot(v\cdot w)$. Thus the form a is positive definite even if $\operatorname{div} v_0 \ne 0$. But it will turn out that in some situations ($\Gamma_N = \emptyset$), we will need smallness of this term, an assumption which, however, should be realistic; see our remarks below.

Next we define the mapping $\| \ \|_a$ by setting

$$\|v\|_a := \big(\nu \cdot |v|_{1,2}^2 + \theta \cdot \|v\|_2^2 + \beta \cdot \|\operatorname{div} v\|_2^2 \big)^{1/2} \quad \text{for } \ v \in V_h.$$

This mapping is a norm on V_h since the same is true for $| \ |_{1,2}$. Note that inequality (2.1) holds with $K_1 := \nu^{-1/2}$.

Now we distinguish the case that the measure of Γ_N is positive from the case this measure is zero. First suppose that the measure of Γ_N is positive, which means that the flow under consideration is non-enclosed. Then we make the additional assumption that there is a function $w_0 \in V_h$ with $\int_\Omega \operatorname{div} w_0 \, dx \neq 0$. This is not a very restrictive condition. In fact, it is fulfilled by any function $v \in V_h$ with $v \cdot n \geq 0$ on Γ_N and $v \cdot n > 0$ on a subset of Γ_N with positive measure.

As we consider a non-enclosed flow, the element m_0 appearing in our abstract theory is taken as zero. This means that $M = M_0$ in Section 2, so here the role of M and M_0 is played by P_h. Let us now turn to the question of how to choose the constants $K_2 - K_6$ and ϵ in (2.2) – (2.4) and (2.6). We begin by

Lemma 3.1 *If*

$$C_4 \leq \Big(2 \cdot \big(C_5 \cdot \nu^{1/2} + h \cdot \nu^{-1/2} \cdot \|v_0\|_\infty + h \cdot \theta^{1/2} \big) \cdot \big(C_5 \cdot \nu^{1/2} + h \cdot \nu^{-1/2} \cdot \|v_0\|_\infty \big) \Big)^{-1}, \quad (3.7)$$

then the first inequality in (2.2) holds with $K_2 = 1/2$. The second estimate in (2.2) and the estimates in (2.3) and (2.4) are valid with

$$\begin{aligned} K_3 &= \nu^{1/2} + 4 \cdot C_2^2 \cdot |v_0|_{1,2} \cdot \nu^{-1/2} + 3 \cdot \beta^{1/2} + \theta^{1/2} \cdot C_1 \\ &\quad + C_4 \cdot \big(C_5 \cdot \nu^{1/2} + h \cdot \nu^{-1/2} \cdot \|v_0\|_\infty + h \cdot \theta^{1/2} \big) \cdot \big(C_5 \cdot \nu + h \cdot \|v_0\|_\infty \big), \\ K_4 &= 3 + C_4 \cdot C_5 \cdot (C_5 \cdot \nu + h \cdot \|v_0\|_\infty), \quad K_5 = C_4 \cdot C_5^2, \\ \epsilon &= C_4 \cdot C_5 \cdot (2 \cdot \nu^{-1/2} \cdot h \cdot \|v_0\|_\infty + h \cdot \theta^{1/2}). \end{aligned}$$

Proof: For $v \in V_h$, we have by (3.5), (3.3),

$$\begin{aligned} &\|\nu \cdot \delta^{1/2} \cdot \Delta v\|_2 \leq \nu \cdot C_4^{1/2} \cdot C_5 \cdot \|\nabla v\|_2 \leq \nu^{1/2} \cdot C_4^{1/2} \cdot C_5 \cdot \|v\|_a, \\ &\|\delta^{1/2} \cdot (v_0 \cdot \nabla) v\|_2 \leq \|\delta^{1/2} \cdot v_0\|_\infty \cdot \|\nabla v\|_2 \leq C_4^{1/2} \cdot h \cdot \|v_0\|_\infty \cdot \|\nabla v\|_2 \\ &\quad \leq C_4^{1/2} \cdot h \cdot \nu^{-1/2} \cdot \|v_0\|_\infty \cdot \|v\|_a, \\ &\|\delta^{1/2} \cdot \theta \cdot v\|_2 \leq C_4^{1/2} \cdot h \cdot \theta \cdot \|v\|_2 \leq C_4^{1/2} \cdot h \cdot \theta^{1/2} \cdot \|v\|_a. \end{aligned}$$

With these inequalities and the relation $v \cdot n \geq 0$ on Γ_N, we get

$$\begin{aligned} &\|v\|_a^2 \leq \int_\Omega \big(\nu \cdot |\nabla v|^2 + \theta \cdot |v|^2 + \beta \cdot |\operatorname{div} v|^2 \big) \, dx + (1/2) \cdot \int_{\Gamma_N} (v_0 \cdot n) \cdot |v|^2 \, d\Gamma \\ &= a(v,v) - \int_\Omega \delta \cdot \big(-\nu \cdot \Delta v + (v_0 \cdot \nabla) v + \theta \cdot v \big) \cdot \big(\varrho \cdot \nu \cdot \Delta v + (v_0 \cdot \nabla) v \big) \, dx \\ &\leq a(v,v) + \big(\|\nu \cdot \delta^{1/2} \cdot \Delta v\|_2 + \|\delta^{1/2} \cdot (v_0 \cdot \nabla) v\|_2 + \|\delta^{1/2} \cdot \theta \cdot v\|_2 \big) \\ &\qquad \cdot \big(\|\nu \cdot \delta^{1/2} \cdot \Delta v\|_2 + \|\delta^{1/2} \cdot (v_0 \cdot \nabla) v\|_2 \big) \\ &\leq a(v,v) + C_4 \cdot \big(C_5 \cdot \nu^{1/2} + h \cdot \nu^{-1/2} \cdot \|v_0\|_\infty + h \cdot \theta^{1/2} \big) \cdot \big(C_5 \cdot \nu^{1/2} + h \cdot \nu^{-1/2} \cdot \|v_0\|_\infty \big) \cdot \|v\|_a^2. \end{aligned}$$

Thus, if the condition in (3.7) is satisfied, we obtain $\|v\|_a \leq a(v,v) + \|v\|_a^2/2$, hence $\|v\|_a^2/2 \leq a(v,v)$. We further find for $v, w \in V_h$,

$$
\begin{aligned}
|a(v,w)| &\leq \nu \cdot |v|_{1,2} \cdot |w|_{1,2} + \|v_0\|_4 \cdot \|\nabla v\|_2 \cdot \|w\|_4 + 3 \cdot \|\nabla v_0\|_2 \cdot \|v\|_4 \cdot \|w\|_4 \\
&\quad + \beta \cdot \|\mathrm{div}\, v\|_2 \cdot 3 \cdot \|\nabla w\|_2 + \theta \cdot \|v\|_2 \cdot \|w\|_2 \\
&\quad + \big(\|\nu \cdot \delta^{1/2} \cdot \Delta v\|_2 + \|\delta^{1/2} \cdot (v_0 \cdot \nabla) v\|_2 + \|\delta^{1/2} \cdot \theta \cdot v\|_2 \big) \\
&\quad \cdot \big(\|\nu \cdot \delta^{1/2} \cdot \Delta w\|_2 + \|\delta^{1/2} \cdot (v_0 \cdot \nabla) w\|_2 \big) \\
&\leq \Big(\nu^{1/2} + C_2^2 \cdot |v_0|_{1,2} \cdot \nu^{-1/2} + 3 \cdot C_2^2 \cdot |v_0|_{1,2} \cdot \nu^{-1/2} + 3 \cdot \beta^{1/2} + \theta^{1/2} \cdot C_1 \\
&\quad + C_4 \cdot \big(C_5 \cdot \nu^{1/2} + h \cdot \nu^{-1/2} \cdot \|v_0\|_\infty + h \cdot \theta^{1/2} \big) \cdot \big(C_5 \cdot \nu + h \cdot \|v_0\|_\infty \big) \Big) \cdot \|v\|_a \cdot |w|_{1,2},
\end{aligned}
$$

where we used (3.2) and (3.1). The preceding inequality shows that K_3 may be chosen as indicated in the lemma. Recalling (3.5), (3.3), we get

$$
\begin{aligned}
|b_1(v,p)| &\leq \|\mathrm{div}\, v\|_2 \cdot \|p\|_2 + \big(\|\nu \cdot \delta^{1/2} \cdot \Delta v\|_2 + \|\delta^{1/2} \cdot (v_0 \cdot \nabla) v\|_2 \big) \cdot \|\delta^{1/2} \cdot \nabla p\|_2 \\
&\leq \big(3 + C_4 \cdot C_5 \cdot (C_5 \cdot \nu + h \cdot \|v_0\|_\infty) \big) \cdot |v|_{1,2} \cdot \|p\|_2, \\
|c(p,q)| &\leq \|\delta^{1/2} \cdot \nabla p\|_2 \cdot \|\delta^{1/2} \cdot \nabla q\|_2 \leq C_4 \cdot C_5^2 \cdot \|p\|_2 \cdot \|q\|_2, \\
|b_1(v,p) - b_2(v,p)| &\leq \big(2 \cdot \|\delta^{1/2} \cdot (v_0 \cdot \nabla) v\|_2 + \|\delta^{1/2} \cdot \theta \cdot v\|_2 \big) \cdot \|\delta^{1/2} \cdot \nabla q\|_2 \\
&\leq C_4 \cdot C_5 \cdot (2 \cdot \nu^{-1/2} \cdot h \cdot \|v_0\|_\infty + h \cdot \theta^{1/2}) \cdot \|v\|_a \cdot \|p\|_2
\end{aligned}
$$

for $v \in V_h,\ p,\ q \in P_h$. The preceding three estimates explain the choice of K_4, K_5 and ϵ stated in the lemma. ◇

Next we look for a suitable constant K_6 in (2.6). Such a constant will be obtained by means of the ensuing weak inf-sup condition.

Theorem 3.1 *There are constants $C_7, C_8 > 0$, only depending on Ω and the constant C_6 from (3.6), such that for $p \in P_h$*

$$
\sup\Big\{ \int_\Omega \mathit{div}\, v \cdot p \, dx \cdot |v|_{1,2}^{-1} \;:\; v \in V_h \backslash \{0\} \Big\} + C_7 \cdot \Big(\sum_{T \in \mathfrak{T}} h_T^2 \cdot \|\nabla p\|_{2,T}^2 \Big)^{1/2} \geq C_8 \cdot \|p\|_2.
$$

Proof: We adapt the arguments from [1, p. 317] to our situation. Put $L_0^2(\Omega) := \{p \in L^2(\Omega) : \int_\Omega p \, dx = 0\}$. By the properties of projections on Hilbert spaces, we have

$$
\Big\| p - |\Omega|^{-1} \cdot \int_\Omega p \, dx \Big\|_2 + |\Omega|^{-1/2} \cdot \Big| \int_\Omega p \, dx \Big| \leq 2 \cdot \|p\|_2 \quad \text{for } p \in L^2(\Omega). \tag{3.8}
$$

According to one of our assumptions on V_h, there is a function $w_0 \in V_h$ such that $\gamma_0 := \int_\Omega \mathrm{div}\, w_0 \, dx$ does not vanish. As $\int_\Omega (-\mathrm{div}\, w_0 + |\Omega|^{-1} \cdot \gamma_0) \, dx = 0$, we may refer to [13, Theorem III.3.1] to choose a function $w_1 \in H_0^1(\Omega)^2$ with $\mathrm{div}\, w_1 = -\mathrm{div}\, w_0 + |\Omega|^{-1} \cdot \gamma_0$. It follows that $\mathrm{div}\,(w_1 + w_0) = |\Omega|^{-1} \cdot \gamma_0$.

Let $p \in L^2(\Omega) \backslash \{0\}$, and put $\alpha := |\Omega|^{-1} \cdot \int_\Omega (-p) \, dx$, so that $\int_\Omega (-p - \alpha) \, dx = 0$. Thus we may again refer to [13, Theorem III.3.1], which yields a function $w_2 \in H_0^1(\Omega)^2$ with

$$
\mathrm{div}\, w_2 = -p - \alpha, \quad \|w_2\|_{1,2} \leq \mathfrak{C}_1 \cdot \| -p - \alpha \|_2, \tag{3.9}
$$

where the constant $\mathfrak{C}_1$ depends only on Ω. Put

$$v := w_2 + \alpha \cdot |\Omega| \cdot \gamma_0^{-1} \cdot (w_0 + w_1), \quad \overline{v} := w_2 + \alpha \cdot |\Omega| \cdot \gamma_0^{-1} \cdot w_1.$$

Then

$$v \in H^1(\Omega)^2, \quad \operatorname{div} v = -p, \quad \overline{v} \in H_0^1(\Omega)^2, \tag{3.10}$$

$$|\overline{v}|_{1,2} \leq |w_2|_{1,2} + |\alpha| \cdot |\Omega| \cdot |\gamma_0|^{-1} \cdot |w_1|_{1,2} \leq \mathfrak{C}_2 \cdot \|p\|_2, \tag{3.11}$$

with $\mathfrak{C}_2 := (\mathfrak{C}_1 + |\Omega|^{1/2} \cdot \gamma_0^{-1} \cdot |w_1|_{1,2}) \cdot 2$. The last inequality follows from (3.9) and (3.8). A similar estimate yields

$$|v|_{1,2} \leq |w|_{1,2} + |\alpha| \cdot |\Omega| \cdot |\gamma_0|^{-1} \cdot |w_0 + w_1|_{1,2} \leq \mathfrak{C}_3 \cdot \|p\|_2, \tag{3.12}$$

with $\mathfrak{C}_3 := (\mathfrak{C}_1 + |\Omega|^{1/2} \cdot |\gamma_0|^{-1} \cdot |w_0 + w_1|_{1,2}) \cdot 2$. Since $\overline{v} \in H_0^1(\Omega)^2$, assumption (3.6) implies there is $v_h \in V_h \cap H_0^1(\Omega)^2$ with

$$\Big(\sum_{T \in \mathfrak{T}} h_T^{-2} \cdot \|\overline{v} - v_h\|_{2,T}^2\Big)^{1/2} \leq C_6 \cdot |\overline{v}|_{1,2}, \quad |\overline{v} - v_h|_{1,2} \leq C_6 \cdot |\overline{v}|_{1,2}. \tag{3.13}$$

Set $w_h := v_h + \alpha \cdot |\Omega| \cdot \gamma_0^{-1} \cdot w_0$. Note that $w_h \in V_h$. Using the notation

$$b(w,q) := -\int_\Omega \operatorname{div} w \cdot q \, dx \quad \text{for } \ w \in H^1(\Omega)^2, \ q \in L^2(\Omega),$$

we get

$$b(w_h, p) = b(v,p) + b(w_h - v,\, p) = \|p\|_2^2 + b(v_h - \overline{v},\, p) = \|p\|_2^2 + \int_\Omega (v_h - \overline{v}) \cdot \nabla p \, dx, \tag{3.14}$$

where the last but one equation follows from (3.10), and the last one holds by a partial integration and the fact that $v_h - \overline{v} \in H_0^1(\Omega)^2$.

For brevity, we set $\mathfrak{L}(p) := \Big(\sum_{T \in \mathfrak{T}} h_T^2 \cdot \|\nabla p\|_{2,T}^2\Big)^{1/2}$. Then, from (3.14), (3.13), (3.11), we get

$$\begin{aligned} b(w_h, p) &\geq \|p\|_2^2 - \Big(\sum_{T \in \mathfrak{T}} h_T^{-2} \cdot \|v_h - \overline{v}\|_{2,T}^2\Big)^{1/2} \cdot \mathfrak{L}(p) \geq \|p\|_2^2 - C_6 \cdot |\overline{v}|_{1,2} \cdot \mathfrak{L}(p) \\ &\geq \|p\|_2^2 - C_6 \cdot \mathfrak{C}_2 \cdot \|p\|_2 \cdot \mathfrak{L}(p), \end{aligned}$$

hence

$$b(w_h, p) + C_6 \cdot \mathfrak{C}_2 \cdot \|p\|_2 \cdot \mathfrak{L}(p) \geq \|p\|_2^2. \tag{3.15}$$

But with (3.10), $\|p\|_2 = \|\operatorname{div} v\|_2 \leq 3 \cdot |v|_{1,2}$. Due to this observation and (3.12), we may deduce from (3.15) that $b(w_h, p) + 3 \cdot C_6 \cdot \mathfrak{C}_2 \cdot |v|_{1,2} \cdot \mathfrak{L}(p) \geq \|p\|_2 \cdot \mathfrak{C}_3^{-1} \cdot |v|_{1,2}$, hence

$$b(w_h, p) \cdot |v|_{1,2}^{-1} + 3 \cdot C_6 \cdot \mathfrak{C}_2 \cdot \mathfrak{L}(p) \geq \mathfrak{C}_3^{-1} \cdot \|p\|_2. \tag{3.16}$$

On the other hand, by (3.13) (second inequality), (3.11), (3.12) and (3.10),

$$\begin{aligned} |w_h|_{1,2} &\leq |w_h - v|_{1,2} + |v|_{1,2} = |v_h - \overline{v}|_{1,2} + |v|_{1,2} \leq C_6 \cdot |\overline{v}|_{1,2} + |v|_{1,2} \\ &\leq (C_6 \cdot \mathfrak{C}_2 + \mathfrak{C}_3) \cdot \|p\|_2 = (C_6 \cdot \mathfrak{C}_2 + \mathfrak{C}_3) \cdot \|\operatorname{div} v\|_2 \leq 3 \cdot (C_6 \cdot \mathfrak{C}_2 + \mathfrak{C}_3) \cdot |v|_{1,2}. \end{aligned} \tag{3.17}$$

Now we distinguish two cases:

1st case: $b(w_h, p) \geq 0$. Then inequality (3.16) and (3.17) imply

$$3 \cdot (C_6 \cdot \mathfrak{C}_2 + \mathfrak{C}_3) \cdot b(w_h, p) \cdot |w_h|_{1,2}^{-1} + 3 \cdot C_6 \cdot \mathfrak{C}_2 \cdot \mathfrak{L}(p) \geq \mathfrak{C}_3^{-1} \cdot \|p\|_2. \tag{3.18}$$

2nd case: $b(w_h, p) < 0$. Then inequality (3.16) yields $3 \cdot C_6 \cdot \mathfrak{C}_2 \cdot \mathfrak{L}(p) \geq \mathfrak{C}_3^{-1} \cdot \|p\|_2$. On the other hand, $b(-w_h, p) > 0$, so inequality (3.18) now follows with w_h replaced by $-w_h$.

Since $w_h \in V_h$, we arrive in both cases at an inequality as stated in the theorem. ◇

Corollary 3.1 *Inequality (2.6) holds with*

$$K_6 := 2/C_8, \quad \widetilde{K}_6 = (2/C_8) \cdot \Big((C_7/2) \cdot C_3^{-1/2} + C_4^{1/2} \cdot (C_5 \cdot \nu + h \cdot \|v_0\|_\infty) \Big).$$

Proof: Let $p \in P_h$. Choose a function $\overline{v} \in V_h$ with $|\overline{v}|_{1,2} = 1$ and

$$\sup\{ \int_\Omega \operatorname{div} v \cdot p \, dx \cdot |v|_{1,2}^{-1} \; : \; v \in V_h \backslash \{0\} \} \leq 2 \cdot \int_\Omega \operatorname{div} \overline{v} \cdot p \, dx.$$

Then, with Theorem 3.1, (3.5), (3.3), we get

$$b_1(\overline{v}, p) + (C_7/2) \cdot \Big(\sum_{T \in \mathfrak{T}} h_T^2 \cdot \|\nabla p\|_{2,T}^2 \Big)^{1/2} \geq C_8 \cdot \|p\|_2/2 + \int_\Omega \delta \cdot \big(\nu \cdot \Delta \overline{v} + (v_0 \cdot \nabla) \overline{v} \big) \cdot \nabla p \, dx$$

$$\geq C_8 \cdot \|p\|_2/2 - \big(\|\delta^{1/2} \cdot \nu \cdot \Delta \overline{v}\|_2 + \|\delta^{1/2} \cdot (v_0 \cdot \nabla) \overline{v}\|_2 \big) \cdot \|\delta^{1/2} \cdot \nabla p\|_2$$

$$\geq C_8 \cdot \|p\|_2/2 - C_4^{1/2} \cdot (C_5 \cdot \nu + h \cdot \|v_0\|_\infty) \cdot |\overline{v}|_{1,2} \cdot \|\delta^{1/2} \cdot \nabla p\|_2.$$

It follows that

$$\sup\{ b_1(v, p) \cdot |v|_{1,2}^{-1} \; : \; v \in V \backslash \{0\} \} \; + \; (C_7/2) \cdot \Big(\sum_{T \in \mathfrak{T}} h_T^2 \cdot \|\nabla p\|_{2,T}^2 \Big)^{1/2}$$

$$+ \; C_4^{1/2} \cdot (C_5 \cdot \nu + h \cdot \|v_0\|_\infty) \cdot \Big(\sum_{T \in \mathfrak{T}} \delta_T \cdot \|\nabla p\|_{2,T}^2 \Big)^{1/2} \; \geq \; (C_8/2) \cdot \|p\|_2.$$

This estimate and the first inequality in (3.3) imply the corollary. ◇

Next we choose the norms $\| \; \|_d$ and $\| \; \|_{\widetilde{a}}$, as well as the forms d and $\widetilde{a}$: for $p, q \in P_h$, we put

$$\|p\|_d := \Big(|p|_{1,2}^2 + \int_{\Gamma_D} p^2 \, d\Gamma \Big)^{1/2}, \quad \|p\|_{\widetilde{a}} := \Big(\nu \cdot |p|_{1,2}^2 + \theta \cdot \|p\|_2^2 + \nu \cdot \int_{\Gamma_D} p^2 \, d\Gamma \Big)^{1/2},$$

$$d(p, q) := \int_\Omega \nabla p \cdot \nabla q \, dx + \int_{\Gamma_D} p \cdot q \, d\Gamma,$$

$$\widetilde{a}(p, q) := \int_\Omega \Big(\nu \cdot \nabla p \cdot \nabla q + (v_0 \cdot \nabla) p \cdot q + (1/2) \cdot \operatorname{div} v_0 \cdot p \cdot q + \theta \cdot p \cdot q$$

$$+ \; \delta \cdot \big(-\nu \cdot \Delta p + (v_0 \cdot \nabla) p + \theta \cdot p \big) \cdot (v_0 \cdot \nabla) q \Big) \, dx \; + \; \nu \cdot \int_{\Gamma_D} p \cdot q \, d\Gamma.$$

By [12, Theorem 5.11.2], the mapping $\|p\|_* := \Big(|p|_{1,2}^2 + \int_{\Gamma_D} p^2 \, d\Gamma \Big)^{1/2}$ $(p \in H^1(\Omega))$ is a norm on $H^1(\Omega)$ which is equivalent to the usual norm $\| \; \|_{1,2}$ of that space. Thus there is $C_9 = C_9(\Omega, \Gamma_D) > 0$ with

$$\|p\|_{1,2} \leq C_9 \cdot \|p\|_d \quad \text{for} \;\; p \in P_h. \tag{3.19}$$

It further follows that the mappings $\| \ \|_d$ and $\| \ \|_{\widetilde{a}}$ are norms on P_h. Obviously inequality (2.8) holds with $K_7 = C_9 \cdot \nu^{-1/2}$, or alternatively with $K_7 = \theta^{-1/2}$ if $\theta > 0$. Note that without the boundary integrals in the definition of $\| \ \|_d$ and $\| \ \|_{\widetilde{a}}$, these mappings would not be norms on P_h, as is required by our choice of P_h for the space M_0 from Section 2; compare the remarks in [9, p. 348-349]. Further note that the quantity ϱ was taken as zero in the definition of $\widetilde{a}$. Otherwise we would need an a-priori bound for $\left(\sum_{T\in\mathfrak{T}} \delta_T \cdot \|\Delta\pi_i\|_{2,T}^2\right)^{1/2}$ for $i \in \{1, 2\}$, where the pair (π_1, π_2) verifies equations (2.19), (2.20) and the relation $\|\pi_1\|_2^2 + \|\pi_2\|_2^2 = 1$. A term $\beta \cdot \operatorname{div} p \cdot \operatorname{div} q$ in the definition of $\widetilde{a}$ would have led to a similar problem, but in any case it is not clear what meaning should be assigned to the expression $\operatorname{div} p$ for a scalar function $p \in P_h$.

Let us now turn to the choice of the constants $K_8 - K_{11}$ and μ in (2.9) – (2.10). Obviously the first inequality in (2.9) holds with $K_8 = 1$. As concerns the second, it is valid with $K_9 = 1/2$ if C_4 verifies (3.7). This follows by the same arguments as in the proof of Lemma 3.1. Note in particular that

$$\int_\Gamma (v_0 \cdot n) \cdot p^2 \, dx = \int_{\Gamma_N} (v_0 \cdot n) \cdot p^2 \, dx \geq 0 \quad \text{for } \ p \in P_h$$

because $v_0 \in V_h$, so that $v \,|\, \Gamma_D = 0$, and because of the assumption $v_0 \cdot n \geq 0$ on Γ_N.

For $s \in [2, \infty)$, let $C_{10}(s) > 0$ with

$$\|p\|_r \leq C_{10}(r) \cdot \|p\|_{1,2} \quad \text{for } \ p \in P_h \tag{3.20}$$

(Sobolev's inequality). This constant $C_{10}(s)$ enters into our choice of the quantities K_{10} and K_{11} from (2.10):

Lemma 3.2 *Take $r \in (2, \infty]$. Then the estimates in (2.10) hold with $\mu = \nu$,*

$$K_{10} = \|v_0\|_\infty \cdot \nu^{-1/2} + \|div\, v_0\|_r \cdot C_{10}\big((1/2 - 1/r)^{-1} \big) \cdot C_9 \cdot \nu^{-1/2} + \theta^{1/2}$$
$$+ C_4 \cdot C_5 \cdot (C_5 \cdot \nu^{1/2} + h \cdot \|v_0\|_\infty \cdot \nu^{-1/2} + h \cdot \theta^{1/2}) \cdot \|v_0\|_\infty,$$

$$K_{11} = \|v_0\|_\infty + \|div\, v_0\|_r \cdot C_{10}\big((1/2 - 1/r)^{-1} \big) \cdot C_9 + \theta \cdot C_9$$
$$+ C_4 \cdot C_5 \cdot (C_5 \cdot \nu + h \cdot \|v_0\|_\infty + h \cdot \theta \cdot C_9) \cdot \|v_0\|_\infty.$$

Proof: Let $p, q \in P_h$. Then

$$|\nu \cdot d(p,q) - \widetilde{a}(p,q)| \ \leq \ \int_\Omega \big(|(v_0 \cdot \nabla)p \cdot q| + (1/2) \cdot |\operatorname{div} v_0 \cdot p \cdot q| + |\theta \cdot p \cdot q| \big) \, dx$$
$$+ \big(\|\nu \cdot \delta^{1/2} \cdot \Delta p\|_2 + \|\delta^{1/2} \cdot (v_0 \cdot \nabla)p\|_2 + \|\delta^{1/2} \cdot \theta \cdot p\|_2 \big) \cdot \|\delta^{1/2} \cdot (v_0 \cdot \nabla)q\|_2$$
$$\leq \|v_0\|_\infty \cdot \|\nabla p\|_2 \cdot \|q\|_2 + \|\operatorname{div} v_0\|_r \cdot \|p\|_{(1/2-1/r)^{-1}} \cdot \|q\|_2 + \theta \cdot \|p\|_2 \cdot \|q\|_2$$
$$+ C_4 \cdot C_5 \cdot (C_5 \cdot \nu \cdot \|\nabla p\|_2 + h \cdot \|v_0\|_\infty \cdot \|\nabla p\|_2 + h \cdot \theta \cdot \|p\|_2 \cdot \|q\|_2) \cdot \|v_0\|_\infty \cdot \|q\|_2$$
$$\leq \Big(\|v_0\|_\infty \cdot \nu^{-1/2} + \|\operatorname{div} v_0\|_r \cdot C_{10}\big((1/2 - 1/r)^{-1} \big) \cdot C_9 \cdot \nu^{-1/2} + \theta^{1/2}$$
$$+ C_4 \cdot C_5 \cdot (C_5 \cdot \nu^{1/2} + h \cdot \|v_0\|_\infty \cdot \nu^{-1/2} + h \cdot \theta^{1/2}) \cdot \|v_0\|_\infty \Big) \cdot \|p\|_{\widetilde{a}} \cdot \|q\|_2,$$

where we used (3.5), (3.20), (3.19) and (3.3). By an obvious variation of the preceding estimate, we see that $|\nu \cdot d(p,q) - \widetilde{a}(p,q)|$ is bounded also by

$$\Big(\|v_0\|_\infty + \|\operatorname{div} v_0\|_r \cdot C_{10}\big((1/2 - 1/r)^{-1} \big) \cdot C_9 + \theta \cdot C_9 + C_4 \cdot C_5 \cdot (C_5 \cdot \nu + h \cdot \|v_0\|_\infty$$
$$+ h \cdot \theta \cdot C_9) \cdot \|v_0\|_\infty \Big) \cdot \|p\|_d \cdot \|q\|_2$$

This completes the proof of the lemma. $\diamond$

We remark that the term $\|\mathrm{div}\, v_0\|_r$ does not pollute our estimates even though we have to require that $r > 2$. In fact, the velocity part u of a solution to (1.1) or (1.2) verifies the equation $\mathrm{div}\, u = 0$. Thus the function v_0, as a velocity approximation from a previous iteration step, typically verifies an inequality of the type $\|\mathrm{div}\, v_0\|_2 \le \mathfrak{C} \cdot h^\gamma$ for some $\gamma > 0$, with $\mathfrak{C} > 0$ only depending on u and standard grid parameters. Thus it should be expected that an inverse estimate yields a bound of $\|\mathrm{div}\, v_0\|_r$ with the same type of dependencies as those of the preceding constant $\mathfrak{C}$.

Let us interprete the preceding results. To this end, we consider the constants C_1, C_2, C_5, C_6, C_7, C_8 and $r \in (2, \infty]$ as given, whereas C_3 and C_4 should be chosen in such a way that inequalities (3.7) and (2.15) are fulfilled. We will return to this point below. For simplicity, we assume $\beta = 0$ and $C_3 = C_4$. We further require that θ and ν^{-1} are large with respect to the constants C_1, C_2, C_5, C_6, C_7, C_8, and also with respect to 1, $|v_0|_{1,2}$, $\|v_0\|_\infty$ and $\|\mathrm{div}\, v_0\|_r$. Moreover, we assume h to be so small that $h \cdot \|v_0\|_\infty \le \mathfrak{C} \cdot \nu$, $h \le \mathfrak{C} \cdot \nu$, $h \cdot \theta \le \mathfrak{C}$. Here and in the following, the symbol $\mathfrak{C}$ is to denote constants which are independent of ν, θ and h. We further require that the product $h \cdot \theta$ stays away from zero: $h \cdot \theta \ge \mathfrak{C}$. This means that the space step should not become small with respect to the time step.

In this situation, the right-hand side of (3.7) is larger than $\mathfrak{C} \cdot (\nu + h^2 \cdot \theta)^{-1}$. Thus we may take $C_4 = \alpha \cdot (\nu + h^2 \cdot \theta)^{-1}$, with $\alpha \in (0, 1]$ still to be determined. This is the type of choice for C_4 used in practical computations; compare [10, (4.1.145)]. Taking $K_7 = \theta^{-1/2}$, and recalling our above choice of K_1, ..., K_6, $\widetilde{K}_6$, K_8, ..., K_{11} and μ (in particular, see Lemma 3.1, 3.2 and Corollary 3.1), we thus have

$$
\begin{aligned}
&K_1 = \nu^{-1/2}, \quad K_2 = 1/2, \quad K_3 = O(\nu^{-1/2} + \theta^{1/2}), \quad K_4 \le \mathfrak{C}, \quad K_5 \le \mathfrak{C} \cdot C_4,\\
&K_6 = 2/C_8, \quad K_7 = \theta^{-1/2}, \quad K_8 = 1, \quad K_9 = 1/2,\\
&K_{10} \le \mathfrak{C} \cdot (\nu^{-1/2} + \theta^{1/2}) \quad K_{11} \le \mathfrak{C} \cdot \theta, \quad \epsilon \le \mathfrak{C} \cdot \big(C_4 \cdot (\nu^{1/2} + \theta^{-1/2}) \big), \quad \mu = \nu.
\end{aligned}
$$

(Concerning K_{11}, note that $C_4 \cdot h \cdot \theta \le \mathfrak{C} \cdot C_4 \cdot \nu \cdot \theta \le \mathfrak{C} \cdot \alpha \cdot \theta \le \mathfrak{C} \cdot \theta$.) We now find that the right-hand side of (2.15) is of order $O\big((\nu^{-1/2} + \theta^{1/2})^{-1} \big)$. On the other hand, recalling our assumptions on h, we get

$$
C_4 \cdot (\nu^{1/2} + \theta^{-1/2}) \le \mathfrak{C} \cdot \alpha \cdot (\nu + \theta^{-1})^{-1} \cdot (\nu^{1/2} + \theta^{-1/2}) \le \mathfrak{C} \cdot \alpha \cdot (\nu^{1/2} + \theta^{-1/2})^{-1}.
$$

Here we used that $h \cdot \theta \ge \mathfrak{C}$. We see that we may choose $\alpha \in (0, 1]$ independently of h, θ and ν such that condition (2.15) is satisfied. Now, with C_4 fixed, we get

$$
\begin{aligned}
&K_5 \le \mathfrak{C} \cdot \nu^{-1}, \quad \epsilon \le \mathfrak{C} \cdot (\nu^{1/2} + \theta^{-1/2})^{-1},\\
&\widetilde{K}_6 \le \mathfrak{C} \cdot \big(C_4^{-1/2} + C_4^{1/2} \cdot \nu \big) \le \mathfrak{C} \cdot \big((\nu + h^2 \cdot \theta)^{1/2} + (\nu + h^2 \cdot \theta)^{-1/2} \cdot \nu \big) \le \mathfrak{C} \cdot \nu^{1/2}.
\end{aligned}
$$

Referring to (2.14) and (2.16), we thus obtain the following bounds for $|\lambda|$:

$$
|\lambda| \le \mathfrak{C} \cdot \theta/\nu, \tag{3.21}
$$

$$
\begin{aligned}
|\lambda| &\ge \mathfrak{C} \cdot \nu \cdot \Big(\big((\nu^{-1/2} + \theta^{1/2})^2 + \nu^{-1/2} \big) \cdot \big(1 + (\nu^{-1/2} + \theta^{1/2}) \cdot \theta^{-1/2} \big) \Big)^{-1} \qquad (3.22)\\
&\ge \mathfrak{C} \cdot \nu \cdot \Big((\nu^{-1/2} + \theta^{1/2})^2 \cdot \big(2 + (\nu \cdot \theta)^{-1/2} \big) \Big)^{-1} \ge \mathfrak{C} \cdot \nu^2 \cdot (\nu \cdot \theta)^{1/2} \cdot (1 + (\nu \cdot \theta)^{1/2})^{-3}.
\end{aligned}
$$

Next we turn to the case that the measure of Γ_N in Γ is zero, so that $H^1_E(\Omega) = H^1_0(\Omega)^2$, and $V_h \subset H^1_0(\Omega)^2$. This means that we consider the case of an enclosed flow. The role of the element

m_0 in our abstract theory is played here by the constant function 1, so we obtain $M_0 = P_{h,0}$, with $P_{h,0} := \{ p \in P_h \, : \, \int_\Omega p \, dx = 0 \}$. In the definition of $\| \, \|_d$, $\| \, \|_{\widetilde{a}}$ and d, we drop the integral over Γ_N. This means in particular that $\| \, \|_d = | \; |_{1,2} \, | \, P_h$. By a standard variant of Poincaré's inequality (see [13, Theorem II.4.3], for example), there is $C_{11} > 0$ with

$$\|v\|_2 \le C_{11} \cdot |v|_{1,2} \quad \text{for } v \in L^2(\Omega) \quad \text{with } \int_\Omega v \, dx = 0. \tag{3.23}$$

Therefore the mappings $\| \, \|_d$ and $\| \, \|_{\widetilde{a}}$ are norms on $P_{h,0}$, as required in Section 2, and inequality (2.8) holds with $K_7 = C_{11} \cdot \nu^{-1/2}$, or alternatively with $K_7 = \theta^{-1/2}$ if $\theta > 0$. Obviously the relations in (2.5) and the second equation in (2.12) are valid. The first equation in (2.12), however, creates difficulties as we did not suppose $\operatorname{div} v_0 = 0$. In fact, we have to modify the definition of $\widetilde{a}$, for example putting

$$\widetilde{a}(p,q) := \int_\Omega \big(\nu \cdot \nabla p \cdot \nabla q + (v_0 \cdot \nabla) p \cdot q + \operatorname{div} v_0 \cdot p \cdot q + \theta \cdot p \cdot q$$
$$+ \delta \cdot (-\nu \cdot \Delta p + (v_0 \cdot \nabla) p + \theta \cdot p) \cdot (v_0 \cdot \nabla) q \big) \, dx \quad \text{for } p, q \in P_h,$$

and requiring a smallness condition on $\operatorname{div} v_0$:

$$\|\operatorname{div} v_0\|_2 \le \nu \cdot \big(2 \cdot C_{10}(4) \cdot (1 + C_{11}) \big)^{-2}. \tag{3.24}$$

As we indicated above in the context of Lemma 3.2, such a condition should not be a severe restriction. With this definition of $\widetilde{a}$, we have $\widetilde{a}(p,1) = 0$ for $p \in P_{h,0}$, so also the first equation in (2.12) holds. Moreover, due to (3.20) and (3.23), we have

$$\|(v_0 \cdot \nabla) p \cdot p + \operatorname{div} v_0 \cdot p^2\|_2 = \|(1/2) \cdot \operatorname{div} v_0 \cdot p^2\|_2 \le \|\operatorname{div} v_0\|_2 \cdot \|p\|_4^2/2$$
$$\le \big(C_{10}(4) \cdot (1 + C_{11}) \big)^2/2 \cdot \|\operatorname{div} v_0\|_2 \cdot |p|_{1,2}^2 \quad \text{for } p \in P_h.$$

Thus condition (3.24) and a reasoning as in the proof of Lemma 3.1 ensures that the second inequality in (2.9) holds with $K_9 = 1/4$ if condition (3.7) is fulfilled. As concerns K_6, the proof of Theorem 3.1 is now much simpler since $V_h \subset H_0^1(\Omega)^2$, and because the inequality stated in that theorem needs to be shown only for functions p belonging to $P_{h,0}$ instead of P_h. We leave the details to the reader.

To end this section, let us consider the case of a stable finite element method. For simplicity, we suppose $\Gamma_N = \emptyset$, that is, homogeneous Dirichlet boundary conditions are imposed everywhere on Γ, hence $H_E^1(\Omega) = H_0^1(\Omega)^2$, $V_h \subset H_0^1(\Omega)^2$. For $\| \, \|_V$, we again take the restriction of $| \; |_{1,2}$ to V_h, and for $\| \, \|_M$ the restriction of the usual L^2-norm to P_h. We choose $m_0 = 1$, so the role of the space M_0 in Section 2 is again played by $P_{h,0}$. Then the assumption that we consider a stable method means that we put

$$b(v,p) := b_1(v,p) := b_2(v,p) := -\int_\Omega \operatorname{div} v \cdot p \, dx, \quad c(p,q) := 0 \quad \text{for } v \in V_h, \; p, q \in P_h,$$

and that there is $C_{13} > 0$ with

$$\sup\{ \, b(v,p) \cdot |v|_{1,2}^{-1} \, : \, v \in V_h \backslash \{0\} \, \} \ge C_{13} \cdot \|p\|_2 \quad \text{for } p \in P_{h,0}.$$

We further define

$$a(v,w) := \int_\Omega \big(\nu \cdot \nabla v \cdot \nabla w + (v_0 \cdot \nabla) v \cdot w + (1/2) \cdot \operatorname{div} v_0 \cdot (v \cdot w) + \theta \cdot (v \cdot w) \big) \, dx,$$

$$\|v\|_a := (\nu \cdot |v|_{1,2}^2 + \theta \cdot \|v\|_2^2)^{1/2} \quad \text{for } v, w \in V_h,$$

$$\|p\|_d := |p|_{1,2}, \quad d(p,q) := \int_\Omega \nabla p \cdot \nabla q \, dx, \quad \|p\|_{\widetilde{a}} := (\nu \cdot |p|_{1,2}^2 + \theta \cdot \|p\|_2^2)^{1/2},$$

$$\widetilde{a}(p,q) := \int_\Omega \big(\nu \cdot \nabla p \cdot \nabla q + (v_0 \cdot \nabla) p \cdot q + \operatorname{div} v_0 \cdot p \cdot q + \theta \cdot p \cdot q \big) \, dx \quad \text{for } p, q \in P_h.$$

Due to (3.23), the mappings $\| \ \|_d$ and $\| \ \|_{\widetilde{a}}$ are norms on $P_{h,0}$. The relations in (2.5) and (2.12) are fulfilled. Inequality (2.1) holds with $K_1 = \nu^{-1/2}$, and (2.8) is again valid with $K_7 = C_{11} \cdot \nu^{-1/2}$ or $K_7 = \theta^{-1/2}$ if $\theta > 0$. The argument of the proof of Lemma 3.1 yields that we may choose

$$K_3 = \nu^{1/2} + 4 \cdot C_2^2 \cdot |v_0|_{1,2} \cdot \nu^{-1/2} + \theta^{1/2} \cdot C_1 \quad \text{and} \quad K_4 = 3.$$

Obviously we may take $K_1 = 1,\ K_5 = 0,\ \epsilon = 0,\ K_6 = C_{13},\ \widetilde{K}_6 = 0,\ K_8 = 1,\ \mu = \nu$. Condition (3.24) implies that the second inequality in (2.9) holds with $K_9 = 3/4$. As may be seen by the proof of Lemma 3.2, we may put

$$K_{10} = \|v_0\|_\infty \cdot \nu^{-1/2} + \|\operatorname{div} v_0\|_2 \cdot C_{10}\big((1/2 - 1/r)^{-1} \big) \cdot C_9 \cdot \nu^{-1/2} + \theta^{1/2},$$
$$K_{11} = \|v_0\|_\infty + \|\operatorname{div} v_0\|_2 \cdot C_{10}\big((1/2 - 1/r)^{-1} \big) \cdot C_9 + \theta \cdot C_9.$$

When we interprete these choices in the same way as above, we obtain the same bounds for λ as in (3.21) and (3.22).

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