An eigenvalue criterion for stability of Navier-Stokes flows in $\mathbb{R}^3$.

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Abstract

We consider a resolvent equation arising from a stability problem for Navier-Stokes flows in the whole space $\mathbb{R}^3$ with nonzero velocity at infinity.

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1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be an exterior domain in $\mathbb{R}^3$. Consider the Navier-Stokes system with Oseen term

$$\partial_t u - \Delta u + \tau \cdot \partial_1 u + \tau \cdot (u \cdot \nabla)u + \nabla p = F, \;\; div\, u = 0 \quad \text{in } \Omega \times (0, \infty), \tag{1.1}$$

with the boundary conditions

$$u \,|\, \partial\Omega \times (0, \infty) = -e_1, \quad u(x,t) \to 0 \;\; (|x| \to \infty) \;\text{ for }\; t \in (0, \infty), \tag{1.2}$$

where $e_1 := (1, 0, 0)$, and where the data F do not depend on the time variable t. Let (u, p) be a solution to problem (1.1), (1.2), and let (U, P) be a solution of the corresponding stationary boundary value problem

$$-\Delta U + \tau \cdot \partial_1 U + \tau \cdot (U \cdot \nabla)U + \nabla P = F, \;\; div\, U = 0 \quad \text{in } \Omega, \tag{1.3}$$

$$U \,|\, \partial\Omega = -e_1, \quad U(x) \to 0 \;\; (|x| \to \infty). \tag{1.4}$$

In this situation, the question arises as to whether $u(t) - U$ tends to zero in some sense for t tending to infinity, provided $u(0) - U$ is small in a suitable way. This “stability problem” attracted much attention for some time now; see [3], [12], [13], [16], [17], [22], for example. Most of the results in these references are based on smallness assumptions on U. However, as explained in [19], [20], one would also like to find a criterion related to the spectrum of a suitable linear operator, similar to the situation with ODE. Recently Neustupa [21] came rather close to such a criterion. His result may be stated as follows:

Write P_2 for the usual Helmholtz operator on $L^2(\Omega)^3$. Define the operator L by

$$L(v) := P_2\big(\Delta v - \tau \cdot (U \cdot \nabla)v - \tau \cdot (v \cdot \nabla)U\big),$$

with v from a suitable function space. Let $\mathfrak{B}_{sym}$ denote the symmetric part of an operator $\mathfrak{B}$ given by $\mathfrak{B}(v) := -(U \cdot \nabla)v - (v \cdot \nabla)U$, and let H_0' be the finite dimensional subspace of $L^2(\Omega)^3$ consisting of the eigenfunctions associated to the positive eigenvalues of the operator $P_2\big(\Delta + \widetilde{a} \cdot \tau \cdot \mathfrak{B}_{sym}\big)$, where $\widetilde{a}$ is some fixed real number. (For rigorous definitions see Section 2.) Suppose there is some $R > 0$ and some non-increasing, integrable and square-integrable function $\varphi : [0, \infty) \mapsto [0, \infty)$ such that

$$\|\nabla e^{Lt}(f) \,|\, B_R\|_2 \le \varphi(t) \cdot \|f\|_2 \quad \text{for } \; t \in (0, \infty), \; f \in H_0'. \tag{1.5}$$

Then Neustupa [21] could show that for a strong solution (u,p) of (1.1), (1.3), the relation $\|\nabla(u(t)-U)\|_2 \to 0$ holds for $t \to \infty$ if $\|u(0)-U\|_{1,2}$ is small. Neustupa considers (1.5) as a substitute of the assumption that all eigenvalues of L have negative real part.

In the work at hand, we show that this point of view is justified at least in the case $\Omega = \mathbb{R}^3$ (the case of the whole space). It turned out that for such Ω, inequality (1.5) is valid provided all the eigenvalues of L have negative real part and the point 0 is almost in the resolvent of L, in the same sense as the point 0 is almost in the resolvent of respectively the Stokes and the Oseen operator. A precise statement of these conditions may be found in assertion (A1) and (A2) in Section 5 and 6, respectively. Our main result is stated in Theorem 7.3 at the end of the paper. This result is by no means obvious since the spectrum of L touches the imaginary axis from the left, independently of the concrete form of the function U.

2. Some notation, auxiliary results and assumptions.

For $A \subset \mathbb{R}^3$, we put $A^c := \mathbb{R}^3 \backslash A$. We write B_R for the open ball with center at the origin and radius $R > 0$. It will be convenient to use the notation $B_0 := \emptyset$. The length $\alpha_1 + \alpha_2 + \alpha_3$ of a multiindex $\alpha \in \mathbb{N}_0^3$ will be denoted by $|\alpha|$.

All our function spaces are to be understood as spaces of complex-valued functions. Let $p \in [1,\infty]$ and $A \subset \mathbb{R}^3$ measurable. Then we designate by $\| \, \|_p$ the norm of $L^p(A)$. In addition, we will make the usual convention that $\|f\|_p = \infty$ for any measurable function f from A into $\mathbb{C}$ with $f \notin L^p(A)$. This means conversely that any measurable function $f : A \mapsto \mathbb{C}$ is in $L^p(A)$ if and only if $\|f\|_p < \infty$. For measurable functions $f, g : \mathbb{R}^3 \mapsto \mathbb{C}$ with $\int_{\mathbb{R}^3} |f(x-y) \cdot g(y)| \, dy < \infty$ for a. e. $x \in \mathbb{R}^3$, or for measurable functions $f, g : \mathbb{R}^3 \mapsto [0,\infty)$, we define the convolution $f * g$ in an obvious way. For functions $f : \mathbb{R}^3 \mapsto \mathbb{C}, \ g : \mathbb{R}^3 \mapsto \mathbb{C}^3$, under analogous assumptions, we put $f * g := (f * g_j)_{1 \le j \le 3}$.

The space $C_0^\infty(\mathbb{R}^3)$ is defined in the standard way. For $p \in [1,\infty), \ m \in \mathbb{N}, \ B \subset \mathbb{R}^3$ open, the term $W^{m,p}(B)$ stands for the usual Sobolev space of exponent p and order m. We write $\| \, \|_{m,p}$ for the standard norm of this space. If $p = 2$, we use the symbol $H^m(B)$ instead of $W^{m,p}(B)$. The space $W^{m,p}_{loc}(B)$ is defined in an obvious way. We further put

$$\mathfrak{D}_0^{1,2}(\mathbb{R}^3) := \{ \, u \in W^{1,1}_{loc}(\mathbb{R}^3) \ : \ u \in L^6(\mathbb{R}^3), \ \partial_l u \in L^2(\mathbb{R}^3) \ \text{ for } \ 1 \le l \le 3 \, \}.$$

The mapping which associates the value $\| \, |\nabla u| \, \|_2$ to each $u \in \mathfrak{D}_0^{1,2}(\mathbb{R}^3)$ is a norm on $\mathfrak{D}_0^{1,2}(\mathbb{R}^3)$ ([10, Theorem II.5.1; p. 36, footnote 5]). In the following we will always suppose that $\mathfrak{D}_0^{1,2}(\mathbb{R}^3)$ is equipped with this norm. If $\mathfrak{B}$ is a vector space consisting of functions from a subset A of $\mathbb{R}^3$ into $\mathbb{C}$, we define

$$\mathfrak{B}^\sigma := \{ \, f : A \mapsto \mathbb{C}^\sigma \ : \ f_j \in \mathfrak{B} \ \text{ for } \ 1 \le j \le \sigma \, \}, \quad \text{for } \ \sigma \in \mathbb{N}, \ \sigma > 1.$$

Suppose the norm of $\mathfrak{B}$ is denoted by $\| \, \|_{\mathfrak{B}}$. Then we equip $\mathfrak{B}^\sigma$ with the norm $\| \, \|_{\mathfrak{B}}^{(\sigma)}$ given by $\|f\|_{\mathfrak{B}}^{(\sigma)} := \left(\sum_{\nu=1}^{\sigma} \|f\|_{\mathfrak{B}}^2 \right)^{1/2}$ for $f \in \mathfrak{B}^\sigma$, but we will always write $\| \, \|_{\mathfrak{B}}$ instead of $\| \, \|_{\mathfrak{B}}^{(\sigma)}$. We will use the abbreviation $\mathfrak{D}$ for the dual space $[\mathfrak{D}_0^{1,2}(\mathbb{R}^3)^3]'$ of $\mathfrak{D}_0^{1,2}(\mathbb{R}^3)^3$. As norm of this dual space, we choose the mapping $\| \, \|_{-1,2}$ defined by

$$\|L\|_{-1,2} := \{ \, |L(v)| / \|\nabla v\|_2 \ : \ v \in \mathfrak{D}_{loc}^{1,2}(\mathbb{R}^3)^3, \ v \ne 0 \, \} \ \ (L \in \mathfrak{D}).$$

Then, by standard results of functional analysis, the pair $(\mathfrak{D}, \| \, \|_{-1,2})$ is a Banach space. We further note

Lemma 2.1 *There is $C > 0$ with $\|u\|_6 \le C \cdot \|\nabla u\|_2$ for $u \in \mathfrak{D}_0^{1,2}(\mathbb{R}^3)^3$.*

For any $u \in \mathfrak{D}_0^{1,2}(\mathbb{R}^3)^3$, there is a sequence (φ_n) in $C_0^\infty(\mathbb{R}^3)^3$ such that $\|\varphi_n - u\|_6 \to 0$ and $\|\nabla(\varphi_n - u)\|_2 \to 0$.

Proof: The first statement of the lemma holds according to [10, Theorem II.5.1]. The second one follows from the first one and from [10, Theorem II.6.1]. $\diamond$

Lemma 2.2 *The equation $\|L\|_{-1,2} = \sup\{\, |L(v)|/\|\nabla v\|_2 \;:\; v \in C_0^\infty(\mathbb{R}^3)^3,\ v \ne 0 \,\}$ holds for $L \in \mathfrak{D}$. Let $f \in L^1_{loc}(\mathbb{R}^3)^3$ with*

$$\gamma_f := \sup\Big\{ \Big| \int_{\mathbb{R}^3} f \cdot v \, dx \Big| / \|\nabla v\|_2 \;:\; v \in C_0^\infty(\mathbb{R}^3)^3,\ v \ne 0 \Big\} < \infty. \tag{2.6}$$

Then the mapping $L_f : \mathfrak{D}_0^{1,2}(\mathbb{R}^3)^3 \ni v \mapsto \int_{\mathbb{R}^3} f \cdot v \, dx \in \mathbb{C}$ belongs to $\mathfrak{D}$, and $\|L_f\|_{-1,2} = \gamma_f$.

The lemma is an obvious consequence of the second statement of Lemma 2.1. In what follows, we will always write f instead of L_f if $f \in L^1_{loc}(\mathbb{R}^3)^3$ verifies (2.6). In this sense, the intersection $\mathfrak{D} \cap L^2(\mathbb{R}^3)^3$ is meaningful. We define

$$\|u\|_* := \|u\|_{-1,2} + \|u\|_2 \quad \text{for } \ u \in \mathfrak{D} \cap L^2(\mathbb{R}^3)^3.$$

Then the pair $\big(\mathfrak{D} \cap L^2(\mathbb{R}^3)^3, \|\ \|_*\big)$ is a Banach space. For $p \in (1, \infty)$, let $H_p(\mathbb{R}^3)$ denote the closure of the set $\{\, \varphi \in C_0^\infty(\mathbb{R}^3)^3 \;:\; \operatorname{div} \varphi = 0 \,\}$ with respect to the norm $\|\ \|_p$ of $L^p(\mathbb{R}^3)^3$ (space of solenoidal functions in $L^p(\mathbb{R}^3)^3$). Let us now present some results on the function spaces introduced up to now. We begin by stating some well-known L^p-inequalities for functions with domain $\mathbb{R}^n$. Since only the case $n = 3$ will be of interest in what follows, we limit ourselves to this case.

Theorem 2.1 (Young's inequality for integrals) *Let $p, q, r \in [1, \infty]$ with $1/p = 1/q + 1/r - 1$. Let $f, g : \mathbb{R}^3 \mapsto \mathbb{C}$ be measurable functions. Then $\|\, |f| * |g| \,\|_p \le \|f\|_q \cdot \|g\|_r$.*

This means in particular that in the case $f \in L^q(\mathbb{R}^3)$, $g \in L^r(\mathbb{R}^3)$, the integral $\int_{\mathbb{R}^3} |f(x-y) \cdot g(y)| \, dy$ is finite for a. e. $x \in \mathbb{R}^3$, and $\|f * g\|_p \le \|f\|_q \cdot \|g\|_r$.

Proof: see [1, Corollary 2.25]. $\diamond$

Theorem 2.2 (Hardy-Littlewood-Sobolev inequality) *Let $p, q \in (1, \infty)$, $\alpha \in (0, 3)$ with $1/p = 1/q - \alpha/3$. Then there is $C = C(p, q) > 0$ such that*

$$\Big(\int_{\mathbb{R}^3} \Big(\int_{\mathbb{R}^3} |x-y|^{-3+\alpha} \cdot |f(y)| \, dy \Big)^p dx \Big)^{1/p} \le C \cdot \|f\|_q \quad \text{for } \ f : \mathbb{R}^3 \mapsto \mathbb{C} \ \text{ measurable.}$$

Proof: see [24, p. 118-121]. $\diamond$

Next we present some further properties of the space $\mathfrak{D}$.

Lemma 2.3 *The set $C_0^\infty(\mathbb{R}^3)^3$ is dense in $\mathfrak{D}$.*

For $w \in \mathfrak{D} \cap L^2(\mathbb{R}^3)^3$, there is a sequence (φ_n) in $C_0^\infty(\mathbb{R}^3)^3$ with $\|\varphi_n - w\|_* \to 0$. In other words: $C_0^\infty(\mathbb{R}^3)^3$ is dense in $\mathfrak{D} \cap L^2(\mathbb{R}^3)^3$.

Proof: see [10, p. 75/76; Lemma VII.4.3]. $\diamond$

As a consequence of the first statement of Lemma 2.1, we note

Lemma 2.4 *$L^{6/5}(\mathbb{R}^3)^3 \subset \mathfrak{D}$, and there is $C > 0$ with $\|f\|_{-1,2} \le C \cdot \|f\|_{6/5}$ for $f \in L^{6/5}(\mathbb{R}^3)^3$.*

Next we recall some properties of the Newton potential.

Theorem 2.3 *Put* $E(z) := (4 \cdot \pi \cdot |z|)^{-1}$ *for* $z \in \mathbb{R}^3 \backslash \{0\}$. *Then* $E * \Phi \in C^\infty(\mathbb{R}^3)$, $\partial^\alpha(E * \Phi) = E * \partial^\alpha \Phi$, $\partial_l(E * \Phi) = (\partial_l E) * \Phi$, $-\Delta(E * \Phi) = \Phi$ *for* $\Phi \in C_0^\infty(\mathbb{R}^3)$, $\alpha \in \mathbb{N}_0^3$, $l \in \{1, 2, 3\}$.

Let $p \in (1, 3/2)$, $q \in (1, 3)$, $r \in (1, \infty)$. *Then there are constants* $\mathfrak{C}_1(p)$, $\mathfrak{C}_2(q)$, $\mathfrak{C}_3(r) > 0$ *with*

$$\|E * \Phi\|_{(1/p-2/3)^{-1}} \le \mathfrak{C}_1(p) \cdot \|\Phi\|_p, \quad \|\partial_l(E * \Phi)\|_{(1/q-1/3)^{-1}} \le \mathfrak{C}_2(q) \cdot \|\Phi\|_q,$$
$$\|\partial_m \partial_l(E * \Phi)\|_r \le \mathfrak{C}_3(r) \cdot \|\Phi\|_r, \quad \textit{for } \Phi \in C_0^\infty(\mathbb{R}^3),\ 1 \le l, m \le 3.$$

The proof of this theorem is well known. In fact, the first part follows from Lebesgue's theorem on dominated convergence; the estimate of $\partial_l(E * \Phi)$ is a consequence of Theorem 2.2, and the estimate of the second derivatives of $E * \Phi$ may be deduced from Calderon-Zygmund's inequality.

Lemma 2.5 *There is* $C > 0$ *and for any function* $w \in C_0^\infty(\mathbb{R}^3)^3$ *a function* $G \in C^\infty(\mathbb{R}^3)$ *with* $G \in W^{1,q}(\mathbb{R}^3)^3$ *for* $q \in (3/2, \infty)$, $div\, G = w$, *and* $C^{-1} \cdot \|w\|_{-1,2} \le \|G\|_2 \le C \cdot \|w\|_{-1,2}$.

Proof: Following [10, p. 391/392], we put $G_l := -\partial_l(E * w)$ for $w \in C_0^\infty(\mathbb{R}^3)$, $1 \le l \le 3$. Then the lemma follows with Theorem 2.3. Note in particular that $E * w \in \mathfrak{D}_0^{1,2}(\mathbb{R}^3)$ by that reference. ◇

Now we turn to the space $H_q(\mathbb{R}^3)$.

Theorem 2.4 *Let* $q \in (1, \infty)$. *Then, for any* $f \in L^q(\mathbb{R}^3)^3$, *there is a unique function* $P_q f \in H_q(\mathbb{R}^3)$ *and a function* $G_q f \in W^{1,1}_{loc}(\mathbb{R}^3)$, *unique up to a constant, such that* $\nabla G_q f \in L^q(\mathbb{R}^3)^3$ *and* $P_q f + \nabla G_q f = f$.

This defines a linear mapping $P_q : L^q(\mathbb{R}^3)^3 \mapsto H_q(\mathbb{R}^3)$. *There is* $C(q) > 0$ *with* $\|P_q f\|_q \le C(q) \cdot \|f\|_q$ *for* $f \in L^q(\mathbb{R}^3)^3$.

Proof: see [10, Section III.1]. ◇

Theorem 2.5 *Let* $q \in (1, \infty)$. *Then* $P_q \,|\, L^q(\mathbb{R}^3)^3 \cap L^2(\mathbb{R}^3)^3 = P_2 \,|\, L^q(\mathbb{R}^3)^3 \cap L^2(\mathbb{R}^3)^3$.

In view of this theorem, we will always write P instead of P_q.

Indications on the proof of Theorem 2.5: Let $P_q' : L^{q'}(\mathbb{R}^3)^3 \mapsto L^{q'}(\mathbb{R}^3)^3$ denote the dual operator of P_q. Then $P_q' = P_{q'}$; compare [10, Exercise III.1.6], [9]. Let $u \in L^{q'}(\mathbb{R}^3)^3$. Then the previous equation implies

$$\int_{\mathbb{R}^3} (P_2 - P_q)(u) \cdot \varphi \, dx = 0 \quad \text{for } \varphi \in C_0^\infty(\mathbb{R}^3)^3 \text{ with } \operatorname{div} \varphi = 0.$$

It follows by [10, Lemma III.1.1] that there is $g \in W^{1,1}_{loc}(\mathbb{R}^3)$ with $\nabla g = (P_q - P_2)(u)$. This implies that g, and hence $\big((P_q - P_2)(u)\big)_l$ for $1 \le l \le 3$, are distributional solutions of Laplace's equation in $\mathbb{R}^3$. This observation and Liouville's theorem yield $P_2 u = P_q u$. ◇

Theorem 2.6 *Let* $f \in L^2(\mathbb{R}^3)^3$. *Then* $f \in H_2(\mathbb{R}^3)$ *if and only if* $\int_{\mathbb{R}^3} f \cdot \nabla \varphi \, dx = 0$ *for* $\varphi \in C_0^\infty(\mathbb{R}^3)$.

Let $g \in H^1(\mathbb{R}^3)^3$. *Then* $div\, g = 0$ *if and only if* $g \in H_2(\mathbb{R}^3)$.

Proof: For the first statement of the lemma, we refer to [23, Lemma II.2.5.4]. The second one follows from the first one. ◇

In the next two theorems, we state some well-known results on the Oseen system and the Stokes resolvent problem, respectively.

Theorem 2.7 *For $L \in \mathfrak{D}$, there is a unique function $u \in \mathfrak{D}_0^{1,2}(\mathbb{R}^3)^3$ such that $div\, u = 0$ and*

$$\int_{\mathbb{R}^3} \big(\nabla u \cdot \nabla \varphi + \tau \cdot \partial_1 u \cdot \varphi \big)\, dx = L(\varphi) \quad \text{for } \varphi \in C_0^\infty(\mathbb{R}^3)^3 \ \text{ with } \ div\,\varphi = 0.$$

Proof: [10, Theorem VII.1.2, VII.2.1, II.5.1, II.6.1]. ◇

We remark that only the uniqueness statement in Theorem 2.7 will be needed in the following. But Theorem 2.7 in its entirety motivated assumption (A1) in Section 5, pertaining to the resolution of the perturbed Oseen problem (5.2).

Theorem 2.8 *Let $\sigma \in (0, \infty)$, and define*

$$g_1(r) := e^{-r} + r^{-2} \cdot (e^{-r} + r \cdot e^{-r} - 1), \quad g_2(r) := e^{-r} + 3 \cdot r^{-2} \cdot (e^{-r} + r \cdot e^{-r} - 1),$$

for $r \in \mathbb{C}\backslash\{0\}$,

$$F_{jk}^{(\sigma)}(z) := (4 \cdot \pi \cdot |z|)^{-1} \cdot \big(\delta_{jk} \cdot g_1(\sigma^{1/2} \cdot |z|) - z_j \cdot z_k \cdot g_2(\sigma^{1/2} \cdot |z|) \big),$$

for $z \in \mathbb{R}^3\backslash\{0\}$, $1 \le j, k \le 3$. (This means that the functions $F_{jk}^{(\sigma)}$ are the velocity part of a fundamental solution of the Stokes resolvent problem (2.7).) Let $s \in (1, \infty)$, $g \in L^s(\mathbb{R}^3)^3$. Define $w(g) := \Big(\sum_{k=1}^3 F_{jk}^{(\sigma)} * g_k \Big)_{1 \le j \le 3}$. Then $w(g) \in W^{2,s}(\mathbb{R}^3)^3$, and there is $\varrho(g) \in W_{loc}^{1,1}(\mathbb{R}^3)$ such that $\nabla \varrho(g) \in L^s(\mathbb{R}^3)^3$ and such that the pair $\big(w(g), \varrho(g) \big)$ solves the Stokes resolvent system

$$-\Delta u + \sigma \cdot u + \nabla \pi = g, \quad div\, u = 0 \quad in \ \mathbb{R}^3. \tag{2.7}$$

If $(u, \pi) \in W^{2,s}(\mathbb{R}^3)^3 \times W_{loc}^{1,1}(\mathbb{R}^3)$ with $\nabla \pi \in L^s(\mathbb{R}^3)^3$ is another solution of (2.7), then $w(g) = u$. There is $C(s, \sigma) > 0$ such that

$$\|w(f)\|_{2,s} \le C(s, \sigma) \cdot \|f\|_s \quad for \ f \in L^s(\mathbb{R}^3)^3.$$

Proof: see [18], [4, Theorem 1.3, Lemma 1.1]. ◇

Let $\tau \in (0, \infty)$ and a real-valued function $U \in W_{loc}^{1,1}(\mathbb{R}^3)^3$ be fixed throughout. We require that

$$U \in L^s(\mathbb{R}^3)^3 \ \text{ for } \ s \in (2,\, 3 + \epsilon_0], \quad \nabla U \in L^s(\mathbb{R}^3)^9 \ \text{ for } \ s \in (4/3,\, 3 + \epsilon_0], \tag{2.8}$$

with some $\epsilon_0 > 0$. Of course, this function U should be considered as the velocity part of the solution of problem (1.3), (1.4), but in the following, we will only need the properties of U stated in (2.8). We remark that for a solution of (1.3), (1.4), these properties hold under the assumption $F \in L^t(\mathbb{R}^3)^3$ for $t \in (1, q_0]$, with some $q_0 > 3$; see [11, Section IX.7]. For $v \in W_{loc}^{1,1}(\mathbb{R}^3)$, we put

$$\mathfrak{B}(v) := \Big(-\sum_{k=1}^3 (v_k \cdot \partial_k U_j + U_k \cdot \partial_k v_j) \Big)_{1 \le j \le 3},$$

$$\mathfrak{B}_{sym}(v) := \Big(-(1/2) \cdot \sum_{k=1}^3 v_k \cdot (\partial_j U_k + \partial_k U_j) \Big)_{1 \le j \le 3}.$$

Let us note some consequences of (2.8).

Corollary 2.1 *$U \in L^s(\mathbb{R}^3)^3$ for $s \in (2, \infty]$, and $\|U \,|\, B_R^c\|_\infty \to 0$ $(R \to \infty)$.*

Proof: By (2.8) and standard Sobolev inequalities (see [1, Theorem 4.12] for example), we have $U \in L^s(\mathbb{R}^3)^3$ for $2 < s \le \infty$. Moreover, the domains $\overline{B_R}^c$, for $R \in [1, \infty)$, say, verify an interior cone condition specified by a single cone having fixed height and vertex angle. Therefore, by [1, Theorem 4.12], there is a single constant $C(\epsilon_0) > 0$ with $\|v\|_\infty \le C(\epsilon_0) \cdot \|v\|_{1,3+\epsilon_0}$ for

$v \in W^{1,3+\epsilon_0}(\overline{B_R}^c)^3$ and for any $R \in [1, \infty)$. It follows that

$$\|U \mid \overline{B_R}^c\|_\infty \leq C(\epsilon_0) \cdot \|U \mid \overline{B_R}^c\|_{1,3+\epsilon_0} \quad \text{for } R \in [1, \infty).$$

But $U \in W^{1,3+\epsilon_0}(\mathbb{R}^3)^3$, so $\|U \mid \overline{B_R}^c\|_{1,3+\epsilon_0} \to 0 \quad (R \to \infty)$. Thus we may conclude that $\|U \mid \overline{B_R}^c\|_\infty \to 0 \; (R \to \infty)$. $\diamond$

A remark is perhaps in order with respect to the notation of the constants appearing in our estimates. Our generic constants will be denoted by the letter C. They will implicitly depend on certain quantities which may vary from section to section, but will be indicated at the beginning of each section. If these constants depend on some additional quantities $\gamma_1, ..., \gamma_n$, for some $n \in \mathbb{N}$, they will be denoted by $C(\gamma_1, ..., \gamma_n)$.

3. The scalar Oseen equation in $\mathbb{R}^3$.

In this section, we consider the scalar Oseen equation with resolvent term

$$-\Delta v + \tau \cdot \partial_1 v + \lambda \cdot v = \Phi \quad \text{in } \mathbb{R}^3. \tag{3.1}$$

The results we are going to obtain will later be used in order to solve a perturbed Oseen *system* with resolvent term in the whole space $\mathbb{R}^3$ (Section 5), and to estimate the solutions of this system (Section 6).

The generic constants in this section may depend on τ. Dependence on any other quantity will be indicated explicitly, as mentioned at the end of Section 2.

Definition 3.1 *Put* $s(x) := \tau \cdot (|x| - x_1)$ *for* $x \in \mathbb{R}^3$, $E^{(0)}(z) := (4 \cdot \pi \cdot |z|)^{-1} \cdot e^{-s(z)/2}$, $E^{(\lambda)}(z) := (4 \cdot \pi \cdot |z|)^{-1} \cdot e^{-(\lambda + (\tau/2)^2)^{1/2} \cdot |z| + \tau \cdot z_1/2}$ *for* $z \in \mathbb{R}^3 \backslash \{0\}$, $\lambda \in \mathbb{C} \backslash \{0\}$ *with* $\Re\lambda \geq 0$.

Note that here and in the following, we use the letter λ in order to denote nonzero complex numbers. Whenever we admit the value zero, we will use the letter ϱ.

We will establish some estimates of convolutions of $E^{(\varrho)}$. To this end, we begin by stating an observation for which we refer to [7, Lemma 4.3].

Lemma 3.1 *Let* $\beta \in (1, \infty)$. *Then* $\int_{\partial B_r} \big(1 + s(x)\big)^{-\beta} \, do_x \leq C(\beta) \cdot r \quad$ *for* $r \in (0, \infty)$.

The next lemma was proved in [6] (see [6, Lemma 4.8]).

Lemma 3.2 $\big(1 + s(x - y)\big)^{-1} \leq C \cdot (1 + |y|) \cdot \big(1 + \tau \cdot s(x)\big)^{-1}$ *for* $x, y \in \mathbb{R}^3$.

Now we can give pointwise estimates of the fundamental solution $E^{(\lambda)}$.

Theorem 3.1 *Let* $\mu, \gamma \in (0, \infty)$. *Then*

$$|\partial_z^\alpha E^{(\lambda)}(z)| \leq C(\mu, \gamma) \cdot |\lambda|^{-2 \cdot \gamma} \cdot \big(|z|^{-\gamma - 1 - |\alpha|/2} + |z|^{-\gamma - 1 - |\alpha|}\big) \cdot \big(1 + s(z)\big)^{-\mu} \cdot e^{-A \cdot |\lambda|^2 \cdot |z|} \tag{3.2}$$

for $z \in \mathbb{R}^3 \backslash \{0\}$, $\alpha \in \mathbb{N}_0^3$ *with* $|\alpha| \leq 2$, $\lambda \in \mathbb{C} \backslash \{0\}$ *with* $\Re\lambda \geq 0$ *and* $|\lambda| \leq (\tau/2)^2$, *where the constant* A *only depends on* τ. *Moreover,*

$$|\partial^\alpha E^{(\varrho)}(z)| \leq C(\mu) \cdot \big(|z|^{-1 - |\alpha|/2} + |z|^{-1 - |\alpha|}\big) \cdot \big(1 + s(z)\big)^{-\mu} \tag{3.3}$$

for z, α *as in (3.2), and* $\varrho \in \mathbb{C}$ *with* $\Re\varrho \geq 0$, $|\varrho| \leq (\tau/2)^2$.

Proof: Take λ as in (3.2). Abbreviate $\kappa := \tau/2$, and note that $|\lambda \cdot \kappa^{-2}| \leq 1$. We find

$$\begin{aligned}
&\Re[\,(\lambda+\kappa)^{1/2}-\kappa\,] = \kappa \cdot \Re[\,\lambda \cdot \kappa^{-2} \cdot \big(\,(\lambda \cdot \kappa^{-2}+1)^{1/2}+1\,\big)^{-1}\,] \\
&= \kappa \cdot [\,\Re\lambda \cdot \kappa^{-2} \cdot \big(1+\Re(\lambda \cdot \kappa^{-2}+1)^{1/2}\big) \;-\; \Im(\lambda \cdot \kappa^{-2}) \cdot \Im\overline{(\lambda \cdot \kappa^{-2}+1)^{1/2}}\,] \\
&\qquad \cdot [\,|\lambda \cdot \kappa^{-2}+1| + 2 \cdot \Re(\lambda \cdot \kappa^{-2}+1)^{1/2}+1\,]^{-1}, \\
&|\lambda \cdot \kappa^{-2}+1| \leq 2, \quad \Im(\lambda \cdot \kappa^{-2}) \cdot \Im\overline{(\lambda \cdot \kappa^{-2}+1)^{1/2}} \leq 0,
\end{aligned}$$

hence

$$\Re[\,(\lambda+\kappa^2)^{1/2}-\kappa]\;\geq\;\kappa \cdot [\,\Re(\lambda \cdot \kappa^{-2}) + |\Im(\lambda \cdot \kappa^{-2})| \cdot |\Im(\lambda \cdot \kappa^{-2}+1)^{1/2}|\,]/7. \tag{3.4}$$

There is some $\varphi \in [-\pi/2, \pi/2]$ such that $\lambda \cdot \kappa^{-2}+1 = |\lambda \cdot \kappa^{-2}+1| \cdot e^{i \cdot \varphi}$. Thus

$$\begin{aligned}
&|\Im(\lambda \cdot \kappa^{-2}+1)^{1/2}| = |\sin(\varphi/2)| \cdot |\lambda \cdot \kappa^{-2}+1|^{1/2} \geq |\sin(\varphi/2)| = [\,(1-\cos\varphi)/2\,]^{1/2} \\
&= [\,|\lambda \cdot \kappa^{-2}+1| - \Re(\lambda \cdot \kappa^{-2}+1)\,]^{1/2} \cdot [\,2 \cdot |\lambda \cdot \kappa^{-2}+1|\,]^{-1/2} \\
&\geq C \cdot [\,|\lambda \cdot \kappa^{-2}+1| - \Re(\lambda \cdot \kappa^{-2}+1)\,]^{1/2} \\
&= C \cdot |\Im(\lambda \cdot \kappa^{-2})| \cdot [\,|\lambda \cdot \kappa^{-2}+1| + \Re(\lambda \cdot \kappa^{2}+1)\,]^{-1/2} \geq C \cdot |\Im(\lambda \cdot \kappa^{-2})|.
\end{aligned}$$

This estimate and (3.4) yield

$$\Re[(\lambda+\kappa^2)^{1/2}-\kappa] \geq C \cdot \kappa \cdot [\,\Re(\lambda \cdot \kappa^{-2}) + \big(\,\Im(\lambda \cdot \kappa^{-2})\,\big)^2\,] \geq C \cdot |\lambda|^2 \cdot \kappa^{-3}. \tag{3.5}$$

Obviously

$$|(\lambda+\kappa^2)^{1/2}-\kappa| \leq |\lambda| \cdot \kappa^{-1} \cdot |(\lambda \cdot \kappa^{-2}+1)^{1/2}+1|^{-1} \leq |\lambda| \cdot \kappa^{-1}, \tag{3.6}$$

in particular $|(\lambda+\kappa^2)^{1/2}| \leq 2 \cdot \kappa$.

Now let $z \in \mathbb{R}^3 \backslash \{0\}$. Abbreviate $B(\lambda, z) := -(\lambda+\kappa^2)^{1/2} \cdot |z| + \kappa \cdot z_1$. Then we get with (3.5),

$$|e^{B(\lambda,z)}| \leq e^{-A_1 \cdot |\lambda|^2 \cdot |z|} \cdot e^{-\kappa \cdot (|z|-z_1)}, \tag{3.7}$$

with some constant $A_1 > 0$ only depending on τ. Thus

$$|E^{(\lambda)}(z)| \leq C \cdot |z|^{-1} \cdot e^{-A_1 \cdot |\lambda|^2 \cdot |z|} \cdot e^{-\kappa \cdot (|z|-z_1)}. \tag{3.8}$$

Let $l, m \in \{1, 2, 3\}$. Then

$$\partial_l E^{(\lambda)}(z) = (4 \cdot \pi)^{-1} \cdot [\,-z_l \cdot |z|^{-3} + |z|^{-1} \cdot \partial_{z_l} B(\lambda, z)\,] \cdot e^{B(\lambda,z)}, \tag{3.9}$$

$$\begin{aligned}
\partial_m \partial_l E^{(\lambda)}(z) = (4 \cdot \pi)^{-1} \cdot [\,&-\delta_{lm} \cdot |z|^{-3} + 3 \cdot z_l \cdot z_m \cdot |z|^{-5} \\
&-z_m \cdot |z|^{-3} \cdot \partial_{z_l} B(\lambda, z) + |z|^{-1} \cdot \partial_{z_l} \partial_{z_m} B(\lambda, z) - z_l \cdot |z|^{-3} \cdot \partial_{z_m} B(\lambda, z) \\
&+|z|^{-1} \cdot \partial_{z_l} B(\lambda, z) \cdot \partial_{z_m} B(\lambda, z)\,] \cdot e^{B(\lambda,z)}.
\end{aligned} \tag{3.10}$$

But for $\nu \in \{1, 2, 3\}$

$$\begin{aligned}
&|\partial_{z_\nu} B(\lambda, z)| = |\,\big(-(\lambda+\kappa^2)^{1/2}+\kappa\big) \cdot z_\nu \cdot |z|^{-1} + \kappa \cdot \partial_{z_\nu}(-|z|+z_1)\,| \\
&\leq C \cdot \big(\,|\lambda| + (|z|-z_1)^{1/2} \cdot |z|^{-1/2}\,\big),
\end{aligned}$$

where we used (3.6). Now we may conclude with (3.7) and (3.9):

$$\begin{aligned}
&|\partial_l E^{(\lambda)}(z)| \leq C \cdot \big(\,|z|^{-2} + |z|^{-1} \cdot |\lambda| + |z|^{-3/2} \cdot (|z|-z_1)^{1/2}\,\big) \cdot e^{-A_1 \cdot |\lambda| \cdot |z|} \cdot e^{-\kappa \cdot (|z|-z_1)} \\
&\leq C \cdot (|z|^{-2} + |z|^{-3/2}) \cdot e^{-A_1 \cdot |\lambda|^2 \cdot |z|/2} \cdot e^{-\kappa \cdot (|z|-z_1)/2}.
\end{aligned} \tag{3.11}$$

Similarly, with (3.10), (3.7), and because $|\partial_{z_l}\partial_{z_m} B(\lambda, z)| \le C \cdot |z|^{-1}$,

$$|\partial_l \partial_m E^{(\lambda)}(z)| \le C \cdot [\, |z|^{-3} + \big(\, |\lambda| + (|z| - z_1)^{1/2} \cdot |z|^{-1/2}\,\big) \cdot |z|^{-2} \tag{3.12}$$
$$+|z|^{-2} + \big(\, |\lambda| + (|z| - z_1)^{1/2} \cdot |z|^{-1/2}\,\big)^2 \cdot |z|^{-1}\,] \cdot e^{-A_1 \cdot |\lambda|^2 \cdot |z|} \cdot e^{-\kappa \cdot (|z| - z_1)}$$
$$\le C \cdot (|z|^{-3} + |z|^{-2}) \cdot e^{-A_1 \cdot |\lambda|^2 \cdot |z|/2} \cdot e^{-\kappa \cdot (|z| - z_1)/2}.$$

Recalling the abbreviation $s(x)$ from Definition 3.1, we observe in the case $s(x) \ge 1$ that $s(x)^{-1} \le 2 \cdot \big(\, 1 + s(x)\,\big)^{-1}$. If $s(x) < 1$, we get $1 \le 2 \cdot \big(\, 1 + s(x)\,\big)^{-1}$. Thus we find in the first case,

$$e^{-\kappa \cdot (|z| - z_1)/2} \le C(\mu) \cdot s(x)^{-\mu} \le C(\mu) \cdot \big(\, 1 + s(x)\,\big)^{-\mu},$$

and in the second one, $1 \le C(\mu) \cdot \big(\, 1 + s(x)\,\big)^{-\mu}$, hence in any case

$$e^{-\kappa \cdot (|z| - z_1)/2} \le C(\mu) \cdot \big(\, 1 + s(x)\,\big)^{-\mu}. \tag{3.13}$$

Moreover,

$$e^{-A_1 \cdot |\lambda|^2 \cdot |z|/2} \le C(\gamma) \cdot |\lambda|^{-2 \cdot \gamma} \cdot |z|^{-\gamma} \cdot e^{-A_1 \cdot |\lambda|^2 \cdot |z|/4}. \tag{3.14}$$

Inequality (3.2) follows from (3.8), (3.11), (3.12), (3.13) and (3.14). The estimate in (3.3) may be shown by similar, but somewhat simpler arguments. ◇

We exploit the preceding theorem to obtain L^p-estimates of convolutions of $E^{(\varrho)}$.

Theorem 3.2 *Let $p, q \in [1, 2]$ with $p \ge q$. Further suppose that $p < 2$ or $q > 1$. Then*

$$\|\, |E^{(\lambda)}| * |f|\, \|_p \le C(p, q) \cdot |\lambda|^{2 - 4 \cdot (1 - 1/q + 1/p)} \cdot \|f\|_q \tag{3.15}$$

for $f \in L^q(\mathbb{R}^3)$, $\lambda \in \mathbb{C} \backslash \{0\}$ with $\Re\lambda \ge 0$, $|\lambda| \le (\tau/2)^2$.
Let $q \in [1, 2)$, and take $p \in \big(\, (1/q - 1/2)^{-1}, \infty]$ if $q \ge 3/2$, and $p \in \big(\, (1/q - 1/2)^{-1}, (1/q - 2/3)^{-1}\,\big)$ if $q < 3/2$. Then, for $f \in L^q(\mathbb{R}^3)$, $\varrho \in \mathbb{C}$ with $\Re\varrho \ge 0$, $|\varrho| \le (\tau/2)^2$,

$$\|\, |E^{(\varrho)}| * |f|\, \|_p \le C(p, q) \cdot \|f\|_q. \tag{3.16}$$

Let $q \in [1, 3]$, $p \in \big(\, (1/q - 1/4)^{-1}, (1/q - 1/3)^{-1}\,\big)$. Then

$$\|\, |\partial_l E^{(\varrho)}| * |f|\, \|_p \le C(p, q) \cdot \|f\|_q \tag{3.17}$$

for $l \in \{1, 2, 3\}$ and for f, ϱ as in (3.16). Finally,

$$\|\, |E^{(\varrho)}| * |f|\, \|_6 + \|\, |\partial_l E^{(\varrho)}| * |f|\, \|_2 \le C \cdot \|f\|_{6/5} \tag{3.18}$$

for $l \in \{1, 2, 3\}$, $f \in L^{6/5}(\mathbb{R}^3)$, $\varrho \in \mathbb{C}$ with $\Re\varrho \ge 0$, $|\varrho| \le (\tau/2)^2$.

Proof: In the situation of (3.15), put $r := (1 - 1/q + 1/p)^{-1}$. Note that since $p \ge q$ and $q > 1$ or $p < 2$, we have $r \in [1, 2)$. Using (3.2) with $\alpha = 0$, $\mu = 2/r$, $\gamma = 0$, and referring to Lemma 3.1, we get

$$\int_{\mathbb{R}^3} |E^{(\lambda)}(x)|^r \, dx \le C \cdot \int_{\mathbb{R}^3} |x|^{-r} \cdot \big(\, 1 + s(x)\,\big)^{-2} \cdot e^{-A_1 \cdot |\lambda|^2 \cdot |x|} \, dx \tag{3.19}$$
$$\le C \cdot \Big(\int_{B_1} |x|^{-r} \, dz + \int_1^\infty \alpha^{-r+1} \cdot e^{-A \cdot |\lambda|^2 \cdot \alpha} \, d\alpha \Big) \le C \cdot \Big(1 + |\lambda|^{-4 + 2 \cdot r} \cdot \int_{|\lambda|^2}^\infty t^{-r+1} \cdot e^{-A \cdot t} \, dt \Big)$$
$$\le C \cdot \Big(1 + |\lambda|^{-4 + 2 \cdot r} \cdot \int_0^\infty t^{-r+1} \cdot e^{-A \cdot t} \, dt \Big) \le C \cdot (1 + |\lambda|^{-4 + 2 \cdot r}) \le C \cdot |\lambda|^{-4 + 2 \cdot r},$$

where the last and last but one inequality holds because $r < 2$. Now inequality (3.15) follows with Theorem 2.1.

In the situation of (3.16) and (3.17), we also put $r := (1 - 1/q + 1/p)^{-1}$. The exponents p and q are chosen in such a way that $r \in (2, 3)$ under the assumptions of (3.16), and $r \in (4/3,\, 3/2)$ under those of (3.17). Further observe that for $z \in \mathbb{R}^3 \backslash \{0\}$,

$$|E^{(\varrho)}(z)| \le C \cdot |z|^{-1}, \quad |\partial_l E^{(\varrho)}(z)| \le C \cdot \big(|z|^{-2} + |z|^{-3/2} \cdot \big(1 + s(z) \big)^{-3/2} \big). \tag{3.20}$$

Via estimates similar to (3.19), inequality (3.3) and Lemma 3.1 imply that $\int_{\mathbb{R}^3} |E^{(\varrho)}(x)|^r \, dx \le C(p, q)$ in the case of (3.16), and $\int_{\mathbb{R}^3} |\partial_l E^{(\varrho)}(x)|^r \, dx \le C(p, q)$ in the situation of (3.17). Now we obtain (3.16) and (3.17) by again applying Theorem 2.1. Finally, as concerns (3.18), we refer to the estimates in (3.20) once more, which allow us to apply Theorem 2.2 to $E^{(\varrho)} * f$ and $(\chi_{B_1} \cdot \partial_l E^{(\varrho)}) * f$, and Theorem 2.1 with $p = 2$, $q = 6/5$, $r = 3/2$ as well as Lemma 3.1 to $(\chi_{B_1^c} \cdot \partial_l E^{(\varrho)}) * f$. Inequality (3.18) then follows. $\diamond$

Theorem 3.3 *Let $\varrho \in \mathbb{C}$ with $\Re\varrho \ge 0$, $|\varrho| \le (\tau/2)^2$. Take $\Phi \in C_0^\infty(\mathbb{R}^3)$, and put $u := E^{(\varrho)} * \Phi$. Then $u \in C^\infty(\mathbb{R}^3)$, u verifies (3.1), and*

$$\partial^\alpha u = E^{(\varrho)} * \partial^\alpha \Phi \ \ \text{for} \ \ \alpha \in \mathbb{N}_0^3, \quad \partial_l u = (\partial_l E^{(\varrho)}) * \Phi \ \ \text{for} \ \ 1 \le l \le 3. \tag{3.21}$$

Let $q \in (1, \infty)$, $R \in (0, \infty)$. Then

$$\|\partial_l \partial_m u \,|\, B_R\|_q \le C(q, R) \cdot \|\Phi\|_q. \tag{3.22}$$

Proof: Theorem 3.1 yields in particular that for any $S > 0$,

$$|\partial^\beta E^{(\varrho)}(z)| \le C(S) \cdot |z|^{-1-|\beta|} \quad \text{for} \ \ z \in B_S \backslash \{0\}, \ \beta \in \mathbb{N}_0^3 \ \ \text{with} \ \ |\beta| \le 1. \tag{3.23}$$

In particular, we have $E^{(\varrho)} \in L^1_{loc}(\mathbb{R}^3)$, and we may conclude that $u \in C^\infty(\mathbb{R}^3)$,

$$\partial^\alpha u(x) = \int_{\mathbb{R}^3} E^{(\varrho)}(y) \cdot \partial^\alpha \Phi(x - y) \, dy \quad \text{for} \ \ x \in \mathbb{R}^3, \ \alpha \in \mathbb{N}_0^3.$$

This proves the first equation in (3.21). Let $R_0 > 0$ with *supp*$(\Phi) \subset B_{R_0}$, and take $l \in \{1,\, 2,\, 3\}$, $x \in \mathbb{R}^3$. It follows from the first equation in (3.21) that

$$\partial_l u(x) = \lim_{\epsilon \downarrow 0} \int_{B_{R_0+|x|} \backslash B_\epsilon(x)} E^{(\varrho)}(x - y) \cdot \partial_l \Phi(y) \, dy. \tag{3.24}$$

But for $\epsilon > 0$ smaller than $(R_0 + |x|)/2$, say, we integrate by parts in (3.24). Since in view of (3.23), we have

$$\int_{\partial B_\epsilon(x)} E^{(\varrho)}(x - y) \cdot \Phi(y) \cdot (x - y)_l / \epsilon \, do_y \to 0 \quad \text{for} \ \ \epsilon \downarrow 0, \tag{3.25}$$

we thus obtain the second equation in (3.21). Turning to the proof of the claim that u verifies (3.1), we observe that by (3.9), (3.10), for any $S > 0$,

$$|\partial^\alpha (E^{(\varrho)} - E)(z)| \le C(S) \cdot |z|^{-|\alpha|} \quad \text{for} \ \ z \in B_S \backslash \{0\}, \ \alpha \in \mathbb{N}_0^3 \ \text{with} \ |\alpha| \le 2, \tag{3.26}$$

where the function E was introduced in Theorem 2.3. Estimate (3.26) with $\alpha \in \mathbb{N}_0^3$, $|\alpha| = 1$ yields that

$$\int_{\partial B_\epsilon(x)} \partial_l (E^{(\varrho)} - E)(x - y) \cdot \Phi(y) \cdot (x - y)_l / \epsilon \, do_y \to 0 \ \ (\epsilon \downarrow 0), \quad \text{for} \ \ 1 \le l \le 3. \tag{3.27}$$

Since the function Φ is in particular Lipschitz continuous, we obtain

$$\int_{\partial B_\epsilon(x)} \partial_l E(x-y) \cdot \big(\Phi(y) - \Phi(x) \big) \cdot (x-y)_l/\epsilon \, do_y \to 0 \;\; (\epsilon \downarrow 0), \;\; \text{for} \;\; 1 \le l \le 3. \tag{3.28}$$

We further note that for $1 \le l \le 3$,

$$\int_{\partial B_\epsilon(x)} \partial_l E(x-y) \cdot (x-y)_l/\epsilon \, do_y = -1/3, \quad -\Delta E^{(\varrho)} + \tau \cdot \partial_1 E^{(\varrho)} + \varrho \cdot E^{(\varrho)} = 0, \tag{3.29}$$

$$-\Delta u + \tau \cdot \partial_1 u + \varrho \cdot u = E^{(\varrho)} * (-\Delta \Phi + \tau \cdot \partial_1 \Phi + \varrho \cdot \Phi), \tag{3.30}$$

with the last equation following from (3.21). After expressing the right-hand side of (3.30) as a limit like in (3.24), we integrate by parts and then apply (3.25), (3.27) – (3.30). It follows that u satisfies (3.1).

This leaves us to establish (3.22). So let $l, m \in \{1, 2, 3\}$. Theorem 2.1 with $r = 1$ and (3.26) yield

$$\| \big(\chi_{(0,\, 2\cdot R)} \cdot \partial_l \partial_m (E^{(\varrho)} - E) \big) * \Phi \|_q \le C(q, R) \cdot \|\Phi\|_q. \tag{3.31}$$

By Theorem 3.1, we have

$$|\partial_l \partial_m (E^{(\varrho)} - E)(z)| \le C \cdot \big(|z|^{-3} + |z|^{-2} \cdot \big(1 + s(z)\big)^{-2} \big) \quad \text{for} \;\; z \in \mathbb{R}^3 \backslash \{0\}.$$

Thus we get by Hölder's inequality and Lemma 3.1,

$$\big| \big(\chi_{(2\cdot R,\, \infty)} \cdot \partial_l \partial_m \big(E^{(\varrho)} - E \big) \big) * \Phi \big|(x) \le C(q) \cdot \|\Phi\|_q \quad \text{for} \;\; x \in \mathbb{R}^3,$$

hence

$$\| \big[\big(\chi_{(2\cdot R,\, \infty)} \cdot \partial_l \partial_m \big(E^{(\varrho)} - E \big) \big) * \Phi \big] \,|\, B_R \|_2 \le C(q, R) \cdot \|\Phi\|_q. \tag{3.32}$$

Combining (3.31), (3.32) and the last inequality in Theorem 2.3, we obtain (3.22). $\diamond$

By a density argument, we may deduce from Theorem 3.2 and 3.3:

Corollary 3.1 *Let $q \in (1, 2)$, $\varrho \in \mathbb{C}$ with $\Re\varrho \ge 0$ and $|\varrho| \le (\tau/2)^2$. Let $f \in L^q(\mathbb{R}^3)$, and put $u := E^{(\varrho)} * f$. Then $u \in W^{2,q}_{loc}(\mathbb{R}^3)$, $\partial_l u = (\partial^l E^{(\varrho)}) * f$ $(1 \le l \le 3)$, u verifies (3.1), and*

$$\|\partial_l \partial_m u \,|\, B_R\|_q \le C(q, R) \cdot \|f\|_q \quad \textit{for} \;\; 1 \le l, m \le 3.$$

Moreover,

$$\partial_l (E^{(\varrho)} * h) = E^{(\varrho)} * \partial_l h \quad \textit{for} \;\; l \in \{1, 2, 3\}, \; h \in W^{1,q}(\mathbb{R}^3). \tag{3.33}$$

Due to Corollary 3.1, we need not distinguish between $\partial_l (E^{(\varrho)} * f)$ and $(\partial_l E^{(\varrho)}) * f$, for f as in that corollary. Therefore we may write $\partial_l E^{(\varrho)} * f$ instead of $\partial_l (E^{(\varrho)} * f)$ or $(\partial_l E^{(\varrho)}) * f$.

We exploit some of the preceding results in order to show a uniqueness result for the scalar Oseen equation.

Theorem 3.4 *Let $\varrho \in \mathbb{C}$ with $\Re\varrho \ge 0$, $|\varrho| \le (\tau/2)^2$. Suppose that $u \in W^{2,1}_{loc}(\mathbb{R}^3)$ verifies the equation $-\Delta u + \tau \cdot \partial_1 u + \varrho \cdot u = 0$, and that*

$$u \,|\, B_{R_0^c} \in L^p(B_{R_0^c}), \;\; \nabla u \,|\, B_{R_0^c} \in L^{\overline{p}}(B_{R_0^c})^3 \quad \textit{for some} \;\; R_0 \in (0, \infty) \; \textit{and some} \; p, \overline{p} \in (1, \infty).$$

Then $u = 0$.

Proof: Let $\Phi \in C_0^\infty(\mathbb{R}^3)$, and put $\overline{x} := (-x_1, x_2, x_3)$, $\overline{\Phi}(x) := \Phi(\overline{x})$ for $x \in \mathbb{R}^3$, and $w(x) :=$

$(E^{(\varrho)} * \overline{\Phi})(\overline{x})$ $(x \in \mathbb{R}^3)$. Then we know by Theorem 3.3 that $w \in C^\infty(\mathbb{R}^3)$ and that the equation $-\Delta w - \tau \cdot \partial_1 w + \varrho \cdot w = \Phi$ is satisfied.

Let $\overline{R} \in [R_0, \infty)$ with $supp(\overline{\Phi}) \subset B_{\overline{R}}$. Note that $|\overline{x} - y| \geq |x|/2$ for $x \in B^c_{2\cdot\overline{R}}$, $y \in B_{\overline{R}}$. Thus, by referring to Lemma 3.2 and Theorem 3.1, we get for $x \in B^c_{2\cdot R}$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$:

$$|\partial^\alpha w(x)| \leq C(\overline{R}) \cdot \|\Phi\|_1 \cdot |x|^{-1-|\alpha|/2} \cdot \big(1 + s(x)\big)^{-1-|\alpha|/2}. \tag{3.34}$$

Moreover, due to our assumptions on $u \,|\, B_{R_0^c}$ and $\nabla u \,|\, B_{R_0^c}$, we may choose a sequence (R_n) in $[\overline{R}, \infty)$ such that $R_n \to \infty$ and the sequences $(\|u \,|\, \partial B_{R_n}\|_p)$ and $(\|\nabla u \,|\, \partial B_{R_n}\|_{\overline{p}})$ are bounded. But by Hölder's inequality, (3.34) and Lemma 3.1, we obtain

$$\begin{aligned} &\int_{\partial B_{R_n}} \big(|u \cdot \partial_l w| + |u \cdot w| + |\partial_l u \cdot w|\big)\, do_x \\ &\leq C(\overline{R}) \cdot \|\Phi\|_1 \cdot \big(\|u \,|\, \partial B_{R_n}\|_p + \|\nabla u \,|\, \partial B_{R_n}\|_{\overline{p}} \big) \cdot R_n^{-\epsilon} \end{aligned} \tag{3.35}$$

for $n \in \mathbb{N}$, with some $\epsilon = \epsilon(p, \overline{p}) > 0$. Note that the right-hand side of (3.35) tends to zero for $n \to \infty$. We further find

$$\int_{\mathbb{R}^3} u \cdot \Phi \, dx = \int_{B_{\overline{R}}} u \cdot \Phi \, dx = \lim_{n\to\infty} \int_{B_{R_n}} u \cdot (-\Delta w - \tau \cdot \partial_1 w + \varrho \cdot w)\, dx. \tag{3.36}$$

Integrating by parts on the right-hand side of (3.36), we obtain an integral over B_{R_n} which vanishes because $-\Delta u + \tau \cdot \partial_1 u + \varrho \cdot u = 0$. Moreover, we obtain surface integrals on ∂B_{R_n} which tend to zero for $n \to \infty$ due to (3.35). Since Φ was chosen arbitrarily in $C_0^\infty(\mathbb{R}^3)^3$, we may conclude $u = 0$. ◇

We further note

Theorem 3.5 *Let $f \in L^2(\mathbb{R}^3)$, $\lambda \in \mathbb{C}\backslash\{0\}$ with $\Re\lambda \geq 0$, $|\lambda| \leq (\tau/2)^2$. Then $E^{(\lambda)} * f \in H^2(\mathbb{R}^3)$ and $\|\partial_l \partial_m (E^{(\lambda)} * f)\|_2 \leq \|f\|_2$ for $1 \leq l, m \leq 3$.*

Let $g \in L^2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ for some $p \in (1, 2)$ (so that $E^{(0)} * g \in W^{2,p}_{loc}(\mathbb{R}^3)$ by Corollary 3.1). Then $\|\partial_l \partial_m (E^{(\lambda)} * g)\|_2 \leq \|g\|_2$ for $1 \leq l, m \leq 3$, $\|\partial_1 E^{(\lambda)} * g)\|_2 \leq \|g\|_2$.

Proof: We know by inequality (3.15) that $E^{(\lambda)} * f \in L^2(\mathbb{R}^3)$. Choosing the definition $\hat{g}(\xi) := (2\cdot\pi)^{-3/2} \cdot \int_{\mathbb{R}^3} e^{-i\cdot\xi\cdot y} \cdot g(y)\, dy$ for the Fourier transform of a function $g \in L^1(\mathbb{R}^3)$, we get $\hat{E}^{(\varrho)}(\xi) = (2\cdot\pi)^{-3/2} \cdot (\varrho + |\xi|^2 + \tau \cdot i \cdot \xi_1)^{-1}$ for $\xi \in \mathbb{R}^3$; compare [15, p. 19/20]. Thus, if $\Phi \in C_0^\infty(\mathbb{R}^3)$, we get $\|\partial_l \partial_m (E^{(\lambda)} * \Phi)\|_2 \leq \|\Phi\|_2$ for $1 \leq l, m \leq 3$ by Plancherel's theorem. If $\Im\lambda = 0$, we further get $\|\partial_1 (E^{(\lambda)} * \Phi)\|_2 \leq \|\Phi\|_2$. Now the first part of the theorem may be shown by a density argument and, as concerns derivatives of order 1, by interpolation. The second part follows by a continuity argument with respect to λ. ◇

The following lemma is a consequence of Theorem 3.5.

Lemma 3.3 *The inequalities $\|\nabla E^{(\varrho)} * w\|_2 \leq C \cdot \|w\|_{-1,2}$ and $\|E^{(\varrho)} * w\|_p \leq C(p) \cdot \|w\|_{-1,2}$ hold for $\varrho \in \mathbb{C}$ with $\Re\varrho \geq 0$, $|\varrho| \leq (\tau/2)^2$, $p \in (4, 6)$, $w \in C_0^\infty(\mathbb{R}^3)$.*

In addition, if $\varrho = 0$, the estimate $\|\partial_1 E^{(\varrho)} * w\|_{-1,2} \leq C \cdot \|w\|_{-1,2}$ is valid for $w \in C_0^\infty(\mathbb{R}^3)$.

Proof: Let $w \in C_0^\infty(\mathbb{R}^3)$, and choose $G := G(w)$ as in Lemma 2.5. Then we find for $k \in \{1, 2, 3\}$, using (3.33):

$$\|\partial_k E^{(\varrho)} * w\|_2 = \Big\| \sum_{l=1}^{3} \partial_k \partial_l (E^{(\varrho)} * G_l) \Big\|_2 \leq C \cdot \|G\|_2 \leq C \cdot \|w\|_{-1,2}, \tag{3.37}$$

where we applied Theorem 3.5 in the last but one inequality, and Lemma 2.5 in the last one.

Moreover, referring to (3.33), inequality (3.17) with $q = 2$, and finally Lemma 2.5, we find for $p \in (4, 6)$:

$$\|E^{(\varrho)} * w\|_p = \Big\| \sum_{l=1}^{3} (\partial_l E^{(\varrho)} * G_l) \Big\|_p \leq C(p) \cdot \|G\|_2 \leq C(p) \cdot \|w\|_{-1,2}.$$

If $\varrho = 0$, we may show the last inequality in Lemma 3.3 by an estimate as in (3.37), again based on (3.33), Theorem 3.5 and Lemma 2.5. $\diamond$

By a density argument, we may now define convolutions of $E^{(\varrho)}$ with $w \in \mathfrak{D}$. (Recall that $\mathfrak{D}$ is an abbreviation for $[\mathfrak{D}_0^{1,2}(\mathbb{R}^3)^3]'$.) The details are stated in

Corollary 3.2 *Let $\varrho \in \mathbb{C}$ with $\Re\varrho \geq 0$ and $|\varrho| \leq (\tau/2)^2$. Then there is a unique linear mapping $\Gamma_\varrho : \mathfrak{D} \mapsto \mathfrak{D}_0^{1,2}(\mathbb{R}^3)^3$ such that*

$$\Gamma_\varrho(w) = E^{(\varrho)} * w \ \textit{ for } \ w \in C_0^\infty(\mathbb{R}^3)^3, \quad \|\nabla\Gamma_\varrho(w)\|_2 \leq C \cdot \|w\|_{-1,2} \ \textit{ for } \ w \in \mathfrak{D}. \quad (3.38)$$

In addition, this operator satisfies the inequality

$$\|\Gamma_\varrho(w)\|_p \leq C(p) \cdot \|w\|_{-1,2} \ \textit{ for } \ w \in \mathfrak{D}, \ \ p \in (4, 6], \quad (3.39)$$

and in the case $\varrho = 0$, the relation $\partial_1\Gamma_\varrho(w) \in \mathfrak{D}$ is valid. Moreover, for $w \in \mathfrak{D} \cap L^2(\mathbb{R}^3)^3$,

$$\partial_l\Gamma_\varrho(w) = (\partial_l E^{(\varrho)}) * w, \quad \Gamma_\varrho(w) \in W^{2,1}_{loc}(\mathbb{R}^3)^3, \quad \partial_l\partial_m\Gamma_\varrho(w) \in L^2(\mathbb{R}^3)^3 \ \ (1 \leq l, m \leq 3), \quad (3.40)$$

$$-\Delta\Gamma_\varrho(w) + \tau \cdot \partial_1\Gamma_\varrho(w) + \varrho \cdot \Gamma_\varrho(w) = w. \quad (3.41)$$

Furthermore, if $w \in \mathfrak{D} \cap H_2(\mathbb{R}^3)$, the equation $div\,\Gamma_\varrho(w) = 0$ holds.
If $w \in L^2(\mathbb{R}^3)^3 \cap L^{6/5}(\mathbb{R}^3)^3$, or if $\varrho \neq 0$ and $w \in \mathfrak{D} \cap L^2(\mathbb{R}^3)^3$, we have $\Gamma_\varrho(w) = E^{(\varrho)} * w$.

Proof: Let $w \in \mathfrak{D}$. By Lemma 2.3, there is a sequence (w_n) in $C_0^\infty(\mathbb{R}^3)^3$ with $\|w_n - w\|_{-1,2} \to 0$. Thus, by (3.18) and Lemma 2.4, the sequence $(E^{(\varrho)} * w_n)$ converges in $L^6(\mathbb{R}^3)^3$, and Lemma 3.3 yields that the sequence $(\nabla E^{(\varrho)} * w_n)$ converges in $L^2(\mathbb{R}^3)^3$. In the case $\varrho = 0$, Lemma 3.3 further yields convergence of $(\partial_1 E^{(\varrho)} * w_n)$ in $\mathfrak{D}$. These references additionally imply that the respective limit functions do not depend on the choice of the sequence (w_n) with $\|w_n - w\|_{-1,2} \to 0$. Thus a linear operator $\Gamma_\varrho : \mathfrak{D} \mapsto \mathfrak{D}_0^{1,2}(\mathbb{R}^3)^3$ satisfying the first relation in (3.38) may be defined in an obvious way, and this operator satisfies the second relation in (3.38) as well as (3.39) with $q = 6$. Furthermore, due to Lemma 3.3, this operator fulfills (3.39) with $q \in (4, 6)$, and satisfies the relation $\Gamma_\varrho(w) \in \mathfrak{D}$ if $\varrho = 0$.

Let $w \in \mathfrak{D} \cap L^2(\mathbb{R}^3)^3$. Then we know by Lemma 2.3 there is a sequence (w_n) in $C_0^\infty(\mathbb{R}^3)^3$ with $\|w_n - w\|_* \to 0$. Inequality (3.17) and the relation $\|w_n - w\|_2 \to 0$ imply that

$$\|\partial_l E^{(\varrho)} * w_n - (\partial_l E^{(\varrho)}) * w\|_p \to 0 \ \ (n \to \infty), \quad \text{for } \ l \in \{1, 2, 3\}, \ p \in (4, 6).$$

On the other hand, $\|w_n - w\|_{-1,2} \to 0$, so we may conclude with (3.38) that

$$\|\partial_l E^{(\varrho)} * w_n - \partial_l\Gamma_\varrho(w)\|_2 \to 0 \ \ (n \to \infty), \quad \text{for } \ l \in \{1, 2, 3\}.$$

Thus we have proved the first relation in (3.40). The other statements in (3.40) as well as equation (3.41) follow from Theorem 3.5, (3.38), (3.39) (convergence of $(E^{(\varrho)} * w_n)$ in $L^6(\mathbb{R}^3)^3$), and Theorem 3.3.

If $w \in \mathfrak{D} \cap H_2(\mathbb{R}^3)$, we may choose a sequence (φ_n) in $C_0^\infty(\mathbb{R}^3)^3$ with $\|\varphi_n - w\|_2 \to 0$ and $\operatorname{div}\varphi_n = 0$ for $n \in \mathbb{N}$. Then $\partial_l\Gamma_\varrho(w) = (\partial_l E^{(\varrho)}) * w$ by (3.40), $\|(\partial_l E^{(\varrho)}) * (\varphi_n - w)\|_p \ (n \to \infty)$ for $p \in (4, 6)$, $1 \leq l \leq 3$ by (3.17), and $(\partial_l E^{(\varrho)}) * \varphi_n = E^{(\varrho)} * \partial_l\varphi_n$ for $n \in \mathbb{N}$, $1 \leq l \leq 3$ by (3.21). In this way we obtain $\operatorname{div}\Gamma_\varrho(w) = 0$.

If $w \in L^2(\mathbb{R}^3)^3 \cap L^{6/5}(\mathbb{R}^3)^3$, we have $w \in L^2(\mathbb{R}^3)^3 \cap \mathfrak{D}$ by Lemma 2.4, hence $\partial_l \Gamma_\varrho(w) = \partial_l E^{(\varrho)} * w$ by (3.40). On the other hand, the functions $\Gamma_\varrho(w)$ and and $E^{(\varrho)} * w$ belong to $\mathfrak{D}_0^{1,2}(\mathbb{R}^3)^3$, as follows by the definition of Γ_ϱ and (3.18), respectively. Now the inequality in Lemma 2.1 implies $\Gamma_\varrho(w) = E^{(\varrho)} * w$. If $\varrho \neq 0$ and $w \in \mathfrak{D} \cap L^2(\mathbb{R}^3)^3$, the preceding equation follows with (3.15). $\diamond$

4. Estimates of $P\mathfrak{B}$ and $P\mathfrak{B}_{sym}$.

In this section, any generic constant may depend on τ and U. Other quantities entering into these constants will be indicated explicitly.

Lemma 4.1 *For $q \in [6/5, 2]$, $v \in \mathfrak{D}_0^{1,2}(\mathbb{R}^3)^3$, the relation $\mathfrak{B}(v) \in L^q(\mathbb{R}^3)^3$ holds, and*

$$\|P\mathfrak{B}(v)\|_q \le C(q) \cdot \|\mathfrak{B}(v)\|_q \le C(q) \cdot \|\nabla v\|_2.$$

In particular, $P\mathfrak{B}(v) \in \mathfrak{D} \cap H_2(\mathbb{R}^3)$ for $v \in \mathfrak{D}_0^{1,2}(\mathbb{R}^3)^3$.

Proof: Take q, v as in the lemma. Then $(1/q - 1/6)^{-1} \in [3/2, 3]$ and $(1/q - 1/2)^{-1} \in [3, \infty]$, so that $\|\nabla U\|_{(1/q-1/6)^{-1}} < \infty$ and $\|U\|_{(1/q-1/2)^{-1}} < \infty$ by (2.8) and Corollary 2.1. Thus, with Lemma 2.1,

$$\|\mathfrak{B}(v)\|_q \le C \cdot \big(\|\nabla U\|_{(1/q-1/6)^{-1}} \cdot \|v\|_6 + \|U\|_{(1/q-1/2)^{-1}} \cdot \|\nabla v\|_2 \big) \le C(q) \cdot \|\nabla v\|_2.$$

Now the first part of the lemma follows with Theorem 2.4. The last statement is a consequence of Lemma 2.4 and the fact that P maps $L^2(\mathbb{R}^3)^3$ into $H_2(\mathbb{R}^3)$ and $L^{6/5}(\mathbb{R}^3)^3$ into $H_{6/5}(\mathbb{R}^3)$ (Theorem 2.5). $\diamond$

Lemma 4.2 *Let $q \in (1, 2)$, $\varrho \in \mathbb{C}$ with $\Re\varrho \ge 0$, $|\varrho| \le (\tau/2)^2$. Then $\mathfrak{B}\big(\Gamma_\varrho(w)\big) \in L^2(\mathbb{R}^3)^3 \cap L^q(\mathbb{R}^3)^3$ for $w \in \mathfrak{D}$, and*

$$P\mathfrak{B}\big(\Gamma_\varrho(w)\big) \in H_2(\mathbb{R}^3) \cap \mathfrak{D} \cap L^q(\mathbb{R}^3)^3 \quad (w \in \mathfrak{D}). \tag{4.1}$$

Moreover $\mathfrak{B}(E^{(\varrho)} * \Phi) \in L^q(\mathbb{R}^3)^3$ for $\Phi \in L^q(\mathbb{R}^3)^3$, and the following estimates are valid for $w \in \mathfrak{D}$, $\Phi \in L^q(\mathbb{R}^3)^3$:

$$\|P\mathfrak{B}\big(\Gamma_\varrho(w)\big)\|_* \le C \cdot \|w\|_{-1,2}, \quad \|P\mathfrak{B}(E^{(\varrho)} * \Phi)\|_q \le C(q) \cdot \|\Phi\|_q. \tag{4.2}$$

In addition, there are non-increasing functions D_1, $D_2^{(q)} : [0, \infty) \mapsto (0, \infty)$ depending on τ, U, and in the case of $D_2^{(q)}$ also on q, such that $D_1(R) \to 0$, $D_2^{(q)}(R) \to 0$ for $R \to \infty$, and

$$\|P\big[\chi_{B_R^c} \cdot \mathfrak{B}\big(\Gamma_\varrho(w)\big)\big]\|_* \le D_1(R) \cdot \|w\|_{-1,2}, \tag{4.3}$$

$$\|P\big[\chi_{B_R^c} \cdot \mathfrak{B}(E^{(\varrho)} * \Phi)\big]\|_q \le D_2^{(q)}(R) \cdot \|\Phi\|_q \quad \textit{for } R \in (0, \infty),\ w \in \mathfrak{D},\ \Phi \in L^q(\mathbb{R}^3)^3. \tag{4.4}$$

Proof: Let $\widetilde{q} \in \{6/5, q, 2\}$. Then $\widetilde{q} \le 2$, so $(1/\widetilde{q} - 1/3)^{-1} \le 6$. Moreover we have $\widetilde{q} > 1$, hence $(1/\widetilde{q} - 3/4)^{-1} > 4$ in the case $\widetilde{q} < 4/3$. Obviously $(1/\widetilde{q} - 1/3)^{-1} < (1/\widetilde{q} - 3/4)^{-1}$ in that latter case. Thus we may choose $\overline{p} \in (4, 6]$ with $(1/\widetilde{q} - 1/3)^{-1} \le \overline{p}$, and with $\overline{p} < (1/\widetilde{q} - 3/4)^{-1}$ in the case $\widetilde{q} < 4/3$. As a consequence, $(1/\widetilde{q} - 1/\overline{p})^{-1} \in (4/3, 3]$, so $\|\nabla U\|_{(1/\widetilde{q}-1/\overline{p})^{-1}} < \infty$ by (2.8). Moreover $(1/\widetilde{q} - 1/2)^{-1} \in (2, \infty]$, hence $\|U\|_{(1/\widetilde{q}-1/2)^{-1}} < \infty$ by Corollary 2.1. In addition, we get for $w \in \mathfrak{D}$, $R \in [0, \infty)$,

$$\begin{aligned} &\|\chi_{B_R^c} \cdot \mathfrak{B}\big(\Gamma_\varrho(w)\big)\|_{\widetilde{q}} \\ &\le C \cdot \big(\|\nabla U \,|\, B_R^c\|_{(1/\widetilde{q}-1/\overline{p})^{-1}} \cdot \|\Gamma_\varrho(w)\|_{\overline{p}} + \|U \,|\, B_R^c\|_{(1/\widetilde{q}-1/2)^{-1}} \cdot \|\nabla \Gamma_\varrho(w)\|_2 \big) \\ &\le C \cdot \big(\|\nabla U \,|\, B_R^c\|_{(1/\widetilde{q}-1/\overline{p})^{-1}} + \|U \,|\, B_R^c\|_{(1/\widetilde{q}-1/2)^{-1}} \big) \cdot \|w\|_{-1,2}, \end{aligned} \tag{4.5}$$

where the last inequality follows from (3.38), (3.39) and the fact that $\overline{p} \in (4,6]$. Now we may conclude that $\mathfrak{B}\big(\Gamma_\varrho(w)\big) \in L^{\widetilde{q}}(\mathbb{R}^3)^3$ for $\widetilde{q} \in \{6/5,\, q,\, 2\}$, $w \in \mathfrak{D}$, so the relation in (4.1) follows with Theorem 2.4 and Lemma 2.4. The latter references, inequality (4.5) with $\widetilde{q} \in \{6/5,\, 2\}$, as well as the relations $\|\nabla U\|_{(1/\widetilde{q}-1/\overline{p})^{-1}} < \infty$, $\|U\|_{(1/\widetilde{q}-1/2)^{-1}} < \infty$ (see above) and $\|U \,|\, B_R^c\|_\infty \to 0\,(R \to \infty)$ (see Corollary 2.1) yield that the first inequality in (4.2) is valid ($R = 0$ in (4.5)), and that there is a function D_1 with the properties stated in the lemma. Next take

$$\overline{r} \in \big(\,(1/q - 1/2)^{-1},\, \infty\big) \text{ if } q \geq 3/2, \quad \overline{r} \in \big(\,(1/q - 1/2)^{-1},\, (1/q - 2/3)^{-1}\,\big) \text{ if } q < 3/2,$$

and choose $r_0 \in \big(\,(1/q-1/4)^{-1},\, (1/q-1/3)^{-1}\,\big)$. Then $(1/q-1/\overline{r})^{-1} \in (3/2,\, 3)$, $(1/q-1/r_0)^{-1} \in (3,\infty)$, and

$$\begin{aligned} &\|\chi_{B_R^c} \cdot \mathfrak{B}(E^{(\varrho)} * \Phi)\|_q \qquad (4.6)\\ &\leq C \cdot \big(\, \|\nabla U \,|\, B_R^c\|_{(1/q-1/\overline{r})^{-1}} \cdot \|E^{(\varrho)} * \Phi\|_{\overline{r}} + \|U \,|\, B_R^c\|_{(1/q-1/r_0)^{-1}} \cdot \|\nabla E^{(\varrho)} * \Phi\|_{r_0}\,\big) \end{aligned}$$

for $\Phi \in L^q(\mathbb{R}^3)^3$. Now the second estimate in (4.2) as well as inequality (4.4) follow from (4.6), (2.8), Corollary 2.1, (3.16) and (3.17). $\diamond$

The ensuing theorem is a key technical result of our theory. It will allow us to solve the resolvent problem (5.8) related to the perturbed Oseen system (5.2), under the assumption that the resolvent parameter λ is small (Theorem 5.4), and to establish resolvent estimates for small λ (Theorem 6.2).

Theorem 4.1 *Let $q \in (1,2)$. Then there are functions D_3, $D_4^{(q)} : (0,\infty) \mapsto (0,\infty)$ depending on τ, U, and in the case of $D_2^{(q)}$ also on q, such that*

$$\begin{aligned} &\|P\mathfrak{B}\big(\Gamma_\lambda(w)\big) - P\mathfrak{B}\big(\Gamma_0(w)\big)\|_* \qquad (4.7)\\ &\leq \big(\,2 \cdot D_1(R) \;+\; D_3(R) \cdot \big[\,\widetilde{R}^{-1/2} + \ln^{1/2}(1/(1-1/\widetilde{R}))\,\big] \;+\; D_3(\widetilde{R}) \cdot |\lambda|^{1/3}\,\big) \cdot \|w\|_* \end{aligned}$$

for $\lambda \in \mathbb{C}\backslash\{0\}$ with $\Re\lambda \geq 0$, $|\lambda| \leq (\tau/2)^2$, $R \in (0,\infty)$, $\widetilde{R} \in [2 \cdot R + 1,\, \infty)$, $w \in \mathfrak{D} \cap L^2(\mathbb{R}^3)^3$;

$$\begin{aligned} &\|P\mathfrak{B}(E^{(\lambda)} * \Phi) - P\mathfrak{B}(E^{(0)} * \Phi)\|_q \qquad (4.8)\\ &\leq \big(\,2 \cdot D_2^{(q)}(R) \;+\; D_4^{(q)}(R) \cdot \widetilde{R}^{-1+2/q'} \;+\; D_4^{(q)}(\widetilde{R}) \cdot |\lambda|^{1/3}\,\big) \cdot \|\Phi\|_q \end{aligned}$$

for λ, R, $\widetilde{R}$ as in (4.7), and for $\Phi \in L^q(\mathbb{R}^3)^3$. The functions D_1 and $D_2^{(q)}$ were introduced in Lemma 4.2.

Proof: Let $\psi \in C^\infty(\mathbb{R})$ with $\psi \,|\, (-\infty,-1] = 0$, $0 \leq \psi \leq 1$, $\psi \,|\, [0,\infty) = 1$. For $R \in (1,\infty)$, $x \in \mathbb{R}^3$, put $\psi_R(x) := \psi(|x| - R)$, so that $\psi_R \in C^\infty(\mathbb{R}^3)$ with $\psi_R \,|\, B_{R-1} = 0$, $\psi \,|\, B_R^c = 1$, and $|\nabla\psi_R(x)| \leq C$ for $x \in \mathbb{R}^3$. Note that the upper bound of $|\nabla\psi_R(x)|$ is independent of R.

Take λ, R, $\widetilde{R}$ as in the theorem, and let $g \in C_0^\infty(\mathbb{R}^3)^3$. Then

$$P\mathfrak{B}(E^{(\lambda)} * g) - P\mathfrak{B}(E^{(0)} * g) = \sum_{i=1}^{3} \mathfrak{N}_i, \qquad (4.9)$$

with $\mathfrak{N}_1 := P\big(\,\chi_{B_R^c} \cdot \mathfrak{B}(E^{(\lambda)} * g - E^{(0)} * g)\,\big)$,

$$\mathfrak{N}_2 := P\big[\,\chi_{B_R} \cdot \mathfrak{B}\big(\,E^{(\lambda)} * (\chi_{B_{\widetilde{R}}^c} \cdot g) - E^{(0)} * (\chi_{B_{\widetilde{R}}^c} \cdot g)\,\big)\,\big],$$

$$\mathfrak{N}_3 := P\big[\,\chi_{B_R} \cdot \mathfrak{B}\big(\,E^{(\lambda)} * (\chi_{B_{\widetilde{R}}} \cdot g) - E^{(0)} * (\chi_{B_{\widetilde{R}}} \cdot g)\,\big)\,\big].$$

Let us abbreviate $u^{(\lambda)} := E^{(\lambda)} * (\chi_{B_{\widetilde{R}}} \cdot g),\ u^{(0)} := E^{(0)} * (\chi_{B_{\widetilde{R}}} \cdot g)$. By (3.16), (3.17), Theorem 3.3 and 3.4, we have

$$u^{(\lambda)} - u^{(0)} = -E^{(0)} * \lambda \cdot u^{(\lambda)}. \tag{4.10}$$

Take $\gamma \in (3/2,\, 2)$, for example $\gamma = 7/4$, and set $s := \big((1/\gamma - 1/4)^{-1} + (1/\gamma - 1/3)^{-1} \big)/2$. Then we get with (4.10) and (3.15) – (3.17),

$$\begin{aligned}
&\|u^{(\lambda)} - u^{(0)}\|_\infty + \|\nabla(u^{(\lambda)} - u^{(0)})\|_s = \|E^{(0)} * \lambda \cdot u^{(\lambda)}\|_\infty + \|\nabla E^{(0)} * \lambda \cdot u^{(\lambda)}\|_s \\
&\le C \cdot |\lambda| \cdot \|u^{(\lambda)}\|_\gamma \le C \cdot |\lambda|^{3-4/\gamma} \cdot \|g \,|\, B_{\widetilde{R}}\|_1 \le C \cdot |\lambda|^{1/3} \cdot \|g \,|\, B_{\widetilde{R}}\|_1,
\end{aligned}$$

where the last inequality holds because $\gamma > 3/2$ and $|\lambda| \le (\tau/2)^2$. Thus, with Theorem 2.4, for $\widetilde{q} \in \{6/5,\, q,\, 2\}$,

$$\begin{aligned}
\|\mathfrak{N}_3\|_{\widetilde{q}} &\le C \cdot \big(\|\nabla U \,|\, B_R\|_{\widetilde{q}} \cdot \|u^{(\lambda)} - u^{(0)}\|_\infty + \|U \,|\, B_R\|_{(1/\widetilde{q} - 1/s)^{-1}} \cdot \|\nabla(u^{(\lambda)} - u^{(0)})\|_s \big) \\
&\le C \cdot \big(R^{3\cdot(1/\widetilde{q} - 1/3)} \cdot \|\nabla U\|_3 + R^{3\cdot(1/\widetilde{q} - 1/s)} \cdot \|U\|_\infty \big) \cdot |\lambda|^{1/3} \cdot \|g \,|\, B_{\widetilde{R}}\|_1.
\end{aligned}$$

We may conclude, using Lemma 2.4,

$$\|\mathfrak{N}_3\|_* \le C(\widetilde{R}) \cdot |\lambda|^{1/3} \cdot \|g\|_2, \quad \|\mathfrak{N}_3\|_q \le C(q, \widetilde{R}) \cdot |\lambda|^{1/3} \cdot \|g\|_q. \tag{4.11}$$

As an immediate consequence of (4.3), (4.4), we get

$$\|\mathfrak{N}_1\|_* \le 2 \cdot D_1(R) \cdot \|g\|_{-1,2}, \quad \|\mathfrak{N}_1\|_q \le 2 \cdot D_2^{(q)}(R) \cdot \|g\|_q. \tag{4.12}$$

Let us now turn to the estimate of $\mathfrak{N}_2$. To start with, we observe that for $\widetilde{q} \in \{q,\, 6/5,\, 2\}$,

$$\begin{aligned}
\|\mathfrak{N}_2\|_{\widetilde{q}} &\le C \cdot (\|\nabla U \,|\, B_R\|_{\widetilde{q}} + \|U \,|\, B_R\|_{\widetilde{q}}) \qquad (4.13)\\
&\quad \cdot \sum_{\varrho \in \{0,\, \lambda\}} \big(\|\big[E^{(\varrho)} * (\chi_{B^c_{\widetilde{R}}} \cdot g) \big] \,|\, B_R\|_\infty + \|\big[\nabla E^{(\varrho)} * (\chi_{B^c_{\widetilde{R}}} \cdot g) \big] \,|\, B_R\|_\infty \big) \\
&\le C(\widetilde{q}, R) \cdot (\|\nabla U\|_2 + \|U\|_3) \\
&\quad \cdot \sum_{\varrho \in \{0,\, \lambda\}} \sum_{j=1}^{3} \Big(\sup_{x \in B_R} \Big| \int_{B^c_{\widetilde{R}}} E^{(\varrho)}(x-y) \cdot g_j(y)\, dy \Big| + \sum_{l=1}^{3} \sup_{x \in B_R} \Big| \int_{B^c_{\widetilde{R}}} \partial_l E^{(\varrho)}(x-y) \cdot g_j(y)\, dy \Big| \Big).
\end{aligned}$$

Now let $x \in B_R,\ j \in \{1,\, 2,\, 3\}$. Then

$$\begin{aligned}
&\Big| \int_{B^c_{\widetilde{R}}} E^{(\varrho)}(x-y) \cdot g_j(y)\, dy \Big| \qquad (4.14)\\
&\le \Big| \int_{\mathbb{R}^3} E^{(\varrho)}(x-y) \cdot (\psi_R \cdot g_j)(y)\, dy \Big| + \Big| \int_{B_{\widetilde{R}} \backslash B_{\widetilde{R}-1}} E^{(\varrho)}(x-y) \cdot (\psi_R \cdot g_j)(y)\, dy \Big| \\
&\le \Big| \int_{\mathbb{R}^3} v_{\widetilde{R}} \cdot g_j\, dy \Big| + \Big(\int_{B_{\widetilde{R}} \backslash B_{\widetilde{R}-1}} |E^{(\varrho)}(x-y)|^2\, dy \Big)^{1/2} \cdot \|g \,|\, B_{\widetilde{R}} \backslash B_{\widetilde{R}-1}\|_2,
\end{aligned}$$

with $v_{\widetilde{R}}(y) := E^{(\varrho)}(x-y) \cdot \psi_{\widetilde{R}}(y)$ for $y \in \mathbb{R}^3$. Note that $\psi_{\widetilde{R}} \,|\, B_{\widetilde{R}-1} = 0$. This latter observation, (3.3), Lemma 3.1, 3.2 and the estimate

$$|x-y| \ge |y|/2 + |y|/2 - |x| \ge |y|/2 + (\widetilde{R} - 1)/2 - |x| \ge |y|/2 + R - |x| \ge |y|/2 \tag{4.15}$$

for $y \in B^c_{\widetilde{R}-1}$ imply that $v_{\widetilde{R}} \in C^\infty(\mathbb{R}^3) \cap \mathfrak{D}_0^{1,2}(\mathbb{R}^3)$. Thus

$$\Big| \int_{\mathbb{R}^3} v_{\widetilde{R}} \cdot g_j\, dy \Big| \le \|g\|_{-1,2} \cdot \Big(\int_{B^c_{\widetilde{R}}} |\nabla_y \big(E^{(\varrho)}(x-y) \cdot \psi_{\widetilde{R}}(y) \big)|^2\, dy \Big)^{1/2} \tag{4.16}$$

$$\leq C \cdot \|g\|_{-1,2} \cdot \Big(\int_{B^c_{\widetilde{R}-1}} |x-y|^{-3} \cdot \big(1+s(x-y)\big)^{-3}\, dy$$
$$+ \int_{B_{\widetilde{R}} \setminus B_{\widetilde{R}-1}} |x-y|^{-2} \cdot \big(1+s(x-y)\big)^{-2}\, dy \Big)^{1/2}$$
$$\leq C(R) \cdot \|g\|_{-1,2} \cdot \Big(\int_{B^c_{\widetilde{R}-1}} |y|^{-3} \cdot \big(1+s(y)\big)^{-3}\, dy \; + \int_{B_{\widetilde{R}} \setminus B_{\widetilde{R}-1}} |y|^{-2} \cdot \big(1+s(y)\big)^{-2}\, dy \Big)^{1/2},$$

where the last inequality follows from (4.15) and Lemma 3.2. Now we apply Lemma 3.1 to obtain

$$\Big| \int_{\mathbb{R}^3} v_{\widetilde{R}} \cdot g_j \, dy \Big| \leq C(R) \cdot \|g\|_{-1,2} \cdot \big(\widetilde{R}^{-1/2} + \ln^{1/2}(\widetilde{R}/(\widetilde{R}-1)) \big). \tag{4.17}$$

Again using (3.3), (4.15), Lemma 3.2 and 3.1, we find

$$\Big(\int_{B_{\widetilde{R}} \setminus B_{\widetilde{R}-1}} |E^{(\varrho)}(x-y)|^2 \, dy \Big)^{1/2} \leq C(R) \cdot \ln^{1/2}(\widetilde{R}/(\widetilde{R}-1)). \tag{4.18}$$

Combining (4.14), (4.17) and (4.18), we get

$$\Big| \int_{B^c_{\widetilde{R}}} E^{(\varrho)}(x-y) \cdot g_j(y) \, dy \Big| \leq C(R) \cdot \|g\|_* \cdot \big(\widetilde{R}^{-1/2} + \ln^{1/2}(\widetilde{R}/(\widetilde{R}-1)) \big). \tag{4.19}$$

A similar reasoning, albeit somewhat simpler because $\nabla E^{(\varrho)}$ decays more rapidly that $E^{(\varrho)}$, allows us to conclude that

$$\Big| \int_{B^c_{\widetilde{R}}} \partial_l E^{(\varrho)}(x-y) \cdot g_j(y) \, dy \Big| \leq C(R) \cdot \|g\|_* \cdot \widetilde{R}^{-1/2} \quad (1 \leq l \leq 3). \tag{4.20}$$

Here and in (4.19) , x was an arbitrary element from B_R. Thus we may conclude from (4.13), (4.19) and (4.20), for $\widetilde{q} \in \{6/5,\, 2\}$:

$$\|\mathfrak{N}_2\|_{\widetilde{q}} \leq C(\widetilde{q}, R) \cdot \|g\|_* \cdot \big(\widetilde{R}^{-1/2} + \ln^{1/2}(\widetilde{R}/(\widetilde{R}-1)) \big), \tag{4.21}$$

hence by Lemma 2.4,

$$\|\mathfrak{N}_2\|_{-1,2} \leq C(R) \cdot \|g\|_* \cdot \big(\widetilde{R}^{-1/2} + \ln^{1/2}(\widetilde{R}/(\widetilde{R}-1)) \big). \tag{4.22}$$

Since $q < 2$, hence $q' > 2$, we get with a much simpler computation, based on Hölder's inequality, (3.3), (4.15), Lemma 3.1 and 3.2,

$$\Big| \int_{B^c_{\widetilde{R}}} \partial^\alpha E^{(\varrho)}(x-y) \cdot g_j(y) \, dy \Big| \leq \Big(\int_{B^c_{\widetilde{R}}} |\partial^\alpha E^{(\varrho)}(x-y)|^{q'} \, dy \Big)^{1/q'} \cdot \|g\|_q$$
$$\leq C(q) \cdot \widetilde{R}^{-1-|\alpha|/2+2/q'} \cdot \|g\|_q,$$

for $x \in B_R$, $1 \leq j \leq 3$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$. Now we again refer to (4.13), to obtain

$$\|\mathfrak{N}_2\|_q \leq C(q, R) \cdot \|g\|_q \cdot \widetilde{R}^{-1+2/q'}. \tag{4.23}$$

Next, using a density argument based on Lemma 2.3 and the first inequality in (4.2), we may deduce (4.7) from (4.9), (4.11), (4.12), (4.21) and (4.22). Finally the estimate in (4.8) follows from the second estimate in (4.2) (density argument), (4.9), (4.11), (4.12) and (4.23). $\diamond$

Lemma 4.3 *Let $q \in [1,\, 6/5]$. Then $\|\mathfrak{B}_{sym}(\Phi)\|_q \leq C(q) \cdot \|\Phi\|_2$ for $\Phi \in L^2(\mathbb{R}^3)^3$. In particular, the term $P\mathfrak{B}_{sym}(\Phi)$ is well defined for $\Phi \in L^2(\mathbb{R}^3)^3$. Moreover $\mathfrak{B}_{sym}(\Phi) \in L^2(\mathbb{R}^3)^3$ for $\Phi \in$*

$H^2(\mathbb{R}^3)^3$.

Proof: Since $(1/q - 1/2)^{-1} \in (2,3]$, we have by (2.8) that $\|\nabla U\|_{(1/q-1/2)^{-1}} < \infty$. Thus, for $\Phi \in L^2(\mathbb{R}^3)^3$,

$$\|\mathfrak{B}_{sym}(\Phi)\|_q \leq C \cdot \|\nabla U\|_{(1/q-1/2)^{-1}} \cdot \|\Phi\|_2 \leq C(q) \cdot \|\Phi\|_2.$$

If $\Phi \in H^2(\mathbb{R}^3)^3$, a standard Sobolev inequality yields $\|\Phi\|_\infty \leq C \cdot \|\Phi\|_{2,2}$, so $\|\mathfrak{B}_{sym}(\Phi)\|_2 \leq C \cdot \|\nabla U\|_2 \cdot \|\Phi\|_\infty < \infty$. $\diamond$

Theorem 4.2 *Let* $\sigma \in (0,\infty)$, $\widetilde{a} \in \mathbb{R}$, $f \in H^2(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3)$ *with* $\Delta f + \widetilde{a} \cdot \tau \cdot P\mathfrak{B}_{sym}(f) = \sigma \cdot f$. *If* $s \in [1,\, 6,5]$, *then* $f \in W^{2,s}(\mathbb{R}^3)^3$ *and*

$$\|\nabla f\|_2 \leq C(\widetilde{a}) \cdot \|f\|_2, \quad \|f\|_{2,s} \leq C(\widetilde{a}, s, \sigma) \cdot \|f\|_2. \tag{4.24}$$

In particular $f \in \mathfrak{D}$.

Proof: A simple variational argument yields $\|\nabla f\|_2^2 + \sigma \cdot \|f\|_2^2 = \int_{\mathbb{R}^3} g \cdot \overline{f}\; dx$, where $g := \widetilde{a} \cdot \tau \cdot P\mathfrak{B}_{sym}(f)$. It follows with Lemma 4.3, 2.1 and Theorem 2.4,

$$\|\nabla f\|_2^2 + \sigma \cdot \|f\|_2^2 \leq \tau \cdot |\widetilde{a}| \cdot \|P\mathfrak{B}_{sym}(f)\|_{6/5} \cdot \|f\|_6 \leq C(\widetilde{a}) \cdot \|f\|_2 \cdot \|\nabla f\|_2.$$

This implies $\|\nabla f\|_2 \leq C(\widetilde{a}) \cdot \|f\|_2$.

We know by Lemma 4.3 that $g \in H_2(\mathbb{R}^3)$. Moreover $-\Delta f + \sigma \cdot f = g$, $\operatorname{div} f = 0$ in $\mathbb{R}^3$, where the equation $\operatorname{div} f = 0$ follows from Theorem 2.6. In this situation, we may conclude with Theorem 2.8 that $f = \left(\sum_{k=1}^3 F_{jk}^{(\sigma)} * g_k\right)_{1\leq j\leq 3}$, with $F_{jk}^{(\sigma)}$ introduced in that reference. But $g \in L^s(\mathbb{R}^3)^3$ for $s \in [1,\, 6/5]$ according to Lemma 4.3, so Theorem 2.8 implies that $f \in W^{2,s}(\mathbb{R}^3)^3$ and $\|f\|_{2,s} \leq C(s,\sigma) \cdot \|g\|_s$ for $s \in (1,\, 6/5]$. The second inequality in (4.24) now follows with Lemma 4.3. Since the case $s = 6/5$ is admitted, Lemma 2.4 yields $f \in \mathfrak{D}$. $\diamond$

5. Solving the perturbed Oseen system (5.2) and the related resolvent problem (5.8).

In this section, we use the same convention on generic constants as in Section 4.

Our starting point is a simple result from operator theory, which the reader may easily verify.

Lemma 5.1 *Let* X, Y *be a vector spaces,* $A : X \mapsto Y$ *a linear and bijective operator, and* $B : X \mapsto Y$ *a linear operator. Let* $\mathfrak{I}_Y$ *denote the identical mapping of* Y *onto itself. Then the operator* $\mathfrak{I}_Y + B \circ A^{-1} : Y \mapsto Y$ *is bijective if and only if* $A + B : X \mapsto Y$ *has the same property. If one, and hence both, of these statements is true, we have*

$$(A+B)^{-1} = A^{-1} \circ (\mathfrak{I}_Y + B \circ A^{-1})^{-1}, \quad (\mathfrak{I}_Y + B \circ A^{-1})^{-1} = A \circ (A+B)^{-1}.$$

In the following, the role of A will be played by the operator $-\Delta + \tau \cdot \partial_1$, set up in a suitable function space, whereas B will correspond to $-\tau \cdot P\mathfrak{B}$. A suitable function space is given by

Theorem 5.1 *Let* $\mathfrak{H}$ *denote the space of all functions* $v \in \mathfrak{D}_0^{1,2}(\mathbb{R}^3)^3 \cap W_{loc}^{2,1}(\mathbb{R}^3)^3$ *such that* $\partial_l \partial_m v \in L^2(\mathbb{R}^3)^3$ *for* $l, m \in \{1, 2, 3\}$, $\partial_1 v \in \mathfrak{D}$ *and* $div\, v = 0$.

Define $\mathfrak{A}(v) := -\Delta v + \tau \cdot \partial_1 v$ *for* $v \in \mathfrak{H}$. *Then* $\mathfrak{A} : \mathfrak{H} \mapsto \mathfrak{D} \cap H_2(\mathbb{R}^3)$ *is linear and bijective, with* $A^{-1} = \Gamma_0 \,|\, \mathfrak{D} \cap H_2(\mathbb{R}^3)$.

Proof: Obviously $\mathfrak{A}(v) \in L^2(\mathbb{R}^3)^3$ and $\int_{\mathbb{R}^3} \mathfrak{A}(v) \cdot \nabla \varphi \; dx = 0$ for $\varphi \in C_0^\infty(\mathbb{R}^3)^3$, $v \in \mathfrak{H}$. Thus

Theorem 2.6 yields $\mathfrak{A}(v) \in H_2(\mathbb{R}^3)$ for $v \in \mathfrak{H}$. It is obvious that $\Delta v \in \mathfrak{D}$, so $\mathfrak{A}(v) \in \mathfrak{D}$ $(v \in \mathfrak{H})$. Therefore $\mathfrak{A} : \mathfrak{H} \mapsto \mathfrak{D} \cap H_2(\mathbb{R}^3)$ is well defined. We know by Corollary 3.2 that $\Gamma_0(w) \in \mathfrak{H}$ and $\mathfrak{A}\big(\Gamma_0(w)\big) = w$ for $w \in \mathfrak{D} \cap H_2(\mathbb{R}^3)$. This shows that $\mathfrak{A}$ is onto. Theorem 3.4 implies that $\mathfrak{A}$ is one-to-one. ◇

We now suppose that the following assumption is satisfied:

(A1) For any $G \in \mathfrak{D}$, there is one and only one function $u \in \mathfrak{D}_0^{1,2}(\mathbb{R}^3)^3$ with div $= 0$ and

$$\int_{\mathbb{R}^3} \big(\nabla u \cdot \nabla v + \tau \cdot \partial_1 u \cdot v - \tau \cdot P\mathfrak{B}(u) \cdot v \big)\, dx = G(v) \text{ for } v \in C_0^\infty(\mathbb{R}^3)^3 \text{ with } \operatorname{div} v = 0. \quad (5.1)$$

This means we assume that the perturbed Oseen system

$$-\Delta u + \tau \cdot \partial_1 u - \tau \cdot \mathfrak{B}(u) + \nabla \pi = g, \;\; \operatorname{div} u = 0 \;\text{ in }\; \mathbb{R}^3 \quad (5.2)$$

admits weak solutions in the same way as the Oseen system does (compare Theorem 2.7). We will now solve a version of (5.2) in which the pressure is eliminated.

Theorem 5.2 *The relation $P\mathfrak{B}(v) \in \mathfrak{D} \cap H_2(\mathbb{R}^3)$ holds for $v \in \mathfrak{H}$.*
Define $\widetilde{\mathfrak{A}} : \mathfrak{H} \mapsto \mathfrak{D} \cap H_2(\mathbb{R}^3)$ by $\widetilde{\mathfrak{A}}(v) := \mathfrak{A}(v) - \tau \cdot P\mathfrak{B}(v)$ for $v \in \mathfrak{H}$. Then $\widetilde{\mathfrak{A}}$ is well defined, linear and bijective.

Proof: Since $\mathfrak{H} \subset \mathfrak{D}_0^{1,2}(\mathbb{R}^3)^3$, the first claim of the theorem holds according to Lemma 4.1. In view of Theorem 5.1, we may conclude that the operator $\widetilde{\mathfrak{A}} : \mathfrak{H} \mapsto \mathfrak{D} \cap H_2(\mathbb{R}^3)$ is well defined. As an easy consequence of the uniqueness statement in (A1), we obtain that $\widetilde{\mathfrak{A}}$ is one-to-one. This leaves us to show that $\widetilde{\mathfrak{A}}$ is onto. To that end, take $\Phi \in \mathfrak{D} \cap H_2(\mathbb{R}^3)$, and let $u \in \mathfrak{D}_0^{1,2}(\mathbb{R}^3)^3$ be the solution of (5.1) with G given by

$$G(\varphi) := \int_{\mathbb{R}^3} \Phi \cdot \varphi \, dx \quad \text{for } \; \varphi \in C_0^\infty(\mathbb{R}^3)^3. \quad (5.3)$$

Then

$$\int_{\mathbb{R}^3} \big(\nabla u \cdot \nabla \varphi + \tau \cdot \partial_1 u \cdot \varphi \big)\, dx = \int_{\mathbb{R}^3} f \cdot \varphi \, dx \quad \text{for } \; \varphi \in C_0^\infty(\mathbb{R}^3)^3, \quad (5.4)$$

with $f := \Phi + \tau \cdot P\mathfrak{B}(u)$. By the first statement of Theorem 5.2, we have $f \in \mathfrak{D} \cap H_2(\mathbb{R}^3)^3$. Due to this and Theorem 5.1, we know there is $\widetilde{v} \in \mathfrak{H}$ with $\mathfrak{A}(\widetilde{v}) = f$. Now (5.4) and the uniqueness result in Theorem 2.7 yield $u = \widetilde{v}$, hence $u \in \mathfrak{H}$. At this point we may deduce from (5.4) and the definition of f that $\widetilde{\mathfrak{A}}(u) = \Phi$. This proves that $\widetilde{\mathfrak{A}}$ is onto. ◇

Corollary 5.1 *The mapping $\widetilde{Z}_0 : \mathfrak{D} \cap H_2(\mathbb{R}^3) \mapsto \mathfrak{D} \cap H_2(\mathbb{R}^3)$, with $\widetilde{Z}_0(w) := w - \tau \cdot P\mathfrak{B}\big(\Gamma_0(w)\big)$ for $w \in \mathfrak{D} \cap H_2(\mathbb{R}^3)$, is well defined, linear, bijective and bounded.*

Proof: The operators $\mathfrak{A}$ and $\widetilde{\mathfrak{A}}$, from Theorem 5.1 and 5.2, respectively, are bijective, so we get with Lemma 5.1 and the first statement in Theorem 5.2 that the operator $\mathfrak{V}$ from the space $\mathfrak{D} \cap H_2(\mathbb{R}^3)$ into itself, with

$$\mathfrak{V}(w) := w - \tau \cdot P\mathfrak{B}\big(\mathfrak{A}^{-1}(w)\big) \;\; (w \in \mathfrak{D} \cap H_2(\mathbb{R}^3)),$$

is bijective. Since $\mathfrak{A}^{-1} = \Gamma_0 \,|\, \mathfrak{D} \cap H_2(\mathbb{R}^3)$ by Theorem 5.1, we see that $\mathfrak{V} = \widetilde{Z}_0$, hence $\widetilde{Z}_0$ is bijective. The boundedness of $\widetilde{Z}_0$ follows from (4.2). ◇

Theorem 5.3 *Let $q \in (1, 2)$, and define an operator $Z_0^{(q)} : L^q(\mathbb{R}^3)^3 \mapsto L^q(\mathbb{R}^3)^3$ by $Z_0^{(q)}(\Phi) := \Phi - \tau \cdot P\mathfrak{B}(E^{(0)} * \Phi)$ for $\Phi \in L^q(\mathbb{R}^3)^3$. Then $Z_0^{(q)}$ is well defined, linear, bounded and bijective.*

Proof: We know by Lemma 4.2 that $Z_0^{(q)} : L^q(\mathbb{R}^3)^3 \mapsto L^q(\mathbb{R}^3)^3$ is well defined and bounded. The claim that $Z_0^{(q)}$ is bijective will be proved by reducing it to the fact that $\widetilde{Z}_0$ is one-to-one (Corollary 5.1). To this end take $R \in (0,\infty)$ and define $\mathfrak{S}_R : L^q(\mathbb{R}^3)^3 \mapsto L^q(\mathbb{R}^3)^3$ by $\mathfrak{S}_R(\Phi) := P\big(\chi_{B_R} \cdot \mathfrak{B}(E^{(0)} * \Phi)\big)$ for $\Phi \in L^q(\mathbb{R}^3)^3$. In order to show that $\mathfrak{S}_R$ is compact, we take a bounded sequence (Φ_n) in $L^q(\mathbb{R}^3)^3$. Then we may deduce from (3.16), (3.17) and Corollary 3.1 that the sequence $\big((E^{(0)} * \Phi_n)\,|\,B_R\big)_{n\geq 1}$ is bounded in $W^{2,q}(B_R)^3$. On the other hand, let $\epsilon \in (0,1)$. Then, for $\Phi \in L^q(\mathbb{R}^3)^3$,

$$\begin{aligned}\|\mathfrak{S}_R(\Phi)\|_q \;&\leq\; C \cdot \big(\|\nabla U|B_R\|_{a(\epsilon)} \cdot \|(E^{(0)} * \Phi)|B_R\|_{b(\epsilon)} \\ &\quad+\|U|B_R\|_{3+\epsilon} \cdot \|(\nabla E^{(0)} * \Phi)|B_R\|_{(1/q-1/(3+\epsilon))^{-1}}\big),\end{aligned} \tag{5.5}$$

where $a(\epsilon) := q+\epsilon,\ b(\epsilon) := \big(1/q - 1/(q+\epsilon)\big)$ in the case $q \geq 3/2$, and $a(\epsilon) := 3/2+\epsilon,\ b(\epsilon) := \big(1/q - 1/(3/2+\epsilon)\big)^{-1}$ if $q < 3/2$. But

$$\big(1/q - 1/(3/2+\epsilon)\big)^{-1} < 3 \cdot q/(3 - 2\cdot q) \ \text{ if } \ q < 3/2, \quad \big(1/q - 1/(3+\epsilon)\big)^{-1} < 3 \cdot q/(3-q).$$

Since the sequence $\big((E^{(0)} * \Phi_n)|B_R\big)_{n\geq 1}$ is bounded in $W^{2,q}(B_R)^3$, as remarked above, we may now apply the standard theory on compact imbeddings in Sobolev spaces. This theory implies there is a subsequence $(\widetilde{\Phi})_n$ of (Φ_n) such that the sequence $\big((E^{(0)} * \widetilde{\Phi}_n)|B_R\big)_{n\geq 1}$ converges in $L^{\left(1/q-1/(q+\epsilon)\right)^{-1}}(B_R)^3$ (case $q \geq 3/2$), or in $L^{\left(1/q-1/(3/2+\epsilon)\right)^{-1}}(B_R)^3$ (case $q < 3/2$), respectively, and the sequence $\big((\nabla E^{(0)} * \widetilde{\Phi}_n)|B_R\big)_{n\geq 1}$ is convergent in $L^{\left(1/q-1/(3+\epsilon)\right)^{-1}}(B_R)^3$. In view of (5.5), (2.8) and Corollary 2.1, we may conclude the sequence $\big(\mathfrak{S}_R(\Phi_n)\big)_{n\geq 1}$ converges in $L^q(\mathbb{R}^3)^3$. Thus we have shown that the operator $\mathfrak{S}_R : L^q(\mathbb{R}^3)^3 \mapsto L^q(\mathbb{R}^3)^3$ is compact. This is true for any $R > 0$. We further note that by (4.4), we may choose $R \in (0,\infty)$ so large that

$$\|P\big(\chi_{B_R^c} \cdot \mathfrak{B}(E^{(0)} * \Phi)\big)\|_q \leq (2\cdot\tau)^{-1} \cdot \|\Phi\|_q \quad \text{for } \ \Phi \in L^q(\mathbb{R}^3)^3. \tag{5.6}$$

Let us now fix such a value R. Then the operator

$$\mathfrak{G}_R : L^q(\mathbb{R}^3)^3 \ni \Phi \mapsto \Phi - \tau \cdot P\big(\chi_{B_R^c} \cdot \mathfrak{B}(E^{(0)} * \Phi)\big) \in L^q(\mathbb{R}^3)^3$$

is one-to-one. Moreover a simple fixed point argument based on (5.6) yields that $\mathfrak{G}_R$ is onto. Thus $\mathfrak{G}_R$ is linear and bijective, in particular Fredholm with index zero. Since $\mathfrak{S}_R$ is compact and $Z_0^{(q)} = \mathfrak{G}_R + \mathfrak{S}_R$, we may conclude that $Z_q^{(0)}$ is Fredholm with index zero.

Let us now show that $Z_0^{(q)}$ is one-to-one. To this end, take $\Phi \in L^q(\mathbb{R}^3)^3$ with $Z_0^{(q)}(\Phi) = 0$. Let $\overline{p} \in \big((1/q-1/4)^{-1},\ (1/q-1/3)^{-1}\big)$ with $\overline{p} > 3/2$,

$$p \in \big((1/q-1/2)^{-1},\ \infty\big) \ \text{ if } \ q \geq 3/2, \quad p \in \big((1/q-1/2)^{-1},\ (1/q-2/3)^{-1}\big) \ \text{ else.}$$

Since $\overline{p} < (1/q-1/3)^{-1}$, we have $2/3 - 1/\overline{p} < 1/2$. Obviously $(2/3+1/p)^{-1} < 3/2$. Thus we may choose

$$\gamma_0 \in (1,\,3/2) \cap \big((2/3+1/p)^{-1},\ 3/2\big) \ \text{ with } \ 1/\gamma_0 - 1/\overline{p} < 1/2.$$

Then $\gamma_0 < 3/2 < \overline{p}$, so the last relation implies $(1/\gamma_0 - 1/\overline{p})^{-1} > 2$, hence $\|U\|_{(1/\gamma_0-1/\overline{p})^{-1}} < \infty$ by Corollary 2.1. Since $\gamma_0 > (2/3+1/p)^{-1}$, we further have $(1/\gamma_0 - 1/p)^{-1} > 3/2$.

Now suppose that $q \geq 3/2$. Then $p > (1/q-1/2)^{-1} > 3$. On the other hand, $1/\gamma_0 - 1/3 > 1/3$, so we may conclude $(1/\gamma_0 - 1/p)^{-1} < 3$. It follows from (2.8) that $\|\nabla U\|_{(1/\gamma_0-1/p)^{-1}} < \infty$. Now we get with (3.16), (3.17),

$$\|\mathfrak{B}(E^{(0)} * \Phi)\|_{\gamma_0} \leq C \cdot \big(\|\nabla U\|_{(1/\gamma_0 - 1/p)^{-1}} \cdot \|E^{(0)} * \Phi\|_p + \|U\|_{(1/\gamma_0 - 1/\overline{p})^{-1}} \cdot \|\nabla E^{(0)} * \Phi\|_{\overline{p}} \big) \quad (5.7)$$
$$\leq C(p, \overline{p}) \cdot \|\Phi\|_q.$$

Since $Z_0^{(q)}(\Phi) = 0$, we may conclude with Theorem 2.4 that $\Phi \in L^{\gamma_0}(\mathbb{R}^3)^3$. Thus there is always some $q_1 \in (1,\, 3/2)$ with $\Phi \in L^{q_1}(\mathbb{R}^3)^3$.

Let $\overline{p}_1 \in \big((1/q_1 - 1/4)^{-1},\, (1/q_1 - 1/3)^{-1} \big)$. Since $q_1 < 3/2$, we have $(1/q_1 - 1/2)^{-1} < 6$, so we may choose $p_1 \in \big((1/q_1 - 1/2)^{-1},\, (1/q_1 - 2/3)^{-1} \big)$ with $p_1 < 6$. Then $\big(5/6 - 1/\overline{p}_1\big)^{-1} > 2$, $\big(5/6 - 1/p_1\big)^{-1} \in (3/2,\, 3)$, so that by (2.8) and Corollary 2.1, $\|U\|_{(5/6 - 1/\overline{p}_1)^{-1}} < \infty$ and $\|\nabla U\|_{(5/6 - 1/p_1)^{-1}} < \infty$. As a consequence, by an estimate as in (5.7), and by referring again to (3.15), (3.16), we get $\|\mathfrak{B}(E^{(0)} * \Phi)\|_{6/5} \leq C(\overline{p}_1, p_1, q_1) \cdot \|\Phi\|_{q_1}$. Thus we have found that $\mathfrak{B}(E^{(0)} * \Phi) \in L^{6/5}(\mathbb{R}^3)^3$. In view of Theorem 2.4 and the assumption $Z_0^{(q)}(\Phi) = 0$, we thus arrive at the relation $\Phi \in L^{6/5}(\mathbb{R}^3)^3$. Now inequality (3.18) yields $E^{(0)} * \Phi \in \mathfrak{D}_0^{1,2}(\mathbb{R}^3)^3$, hence $P\mathfrak{B}(E^{(0)} * \Phi) \in H_2(\mathbb{R}^3) \cap \mathfrak{D}$ by Lemma 4.1. Since $Z_0^{(q)}(\Phi) = 0$, we thus obtain $\Phi \in H_2(\mathbb{R}^3) \cap \mathfrak{D} \cap L^{6/5}(\mathbb{R}^3)^3$, so Corollary 3.2 implies $Z_0^{(q)}(\Phi) = \widetilde{Z}_0(\Phi)$. Therefore $\widetilde{Z}_0(\Phi) = 0$, and we may conclude with Corollary 5.1 that $\Phi = 0$. This proves that $Z_0^{(q)}$ is one-to-one. But a Fredholm operator with index zero which in addition is one-to-one must be bijective, so the proof of Theorem 5.3 is completed. ◇

Corollary 5.2 *Let $q \in (1, 2)$, $\lambda \in \mathbb{C}\backslash\{0\}$ with $\Re\lambda \geq 0$, $|\lambda| \leq (\tau/2)^2$. Then the operators*

$$\widetilde{Z}_\lambda : \mathfrak{D} \cap H_2(\mathbb{R}^3) \ni \Phi \mapsto \Phi - \tau \cdot P\mathfrak{B}(E^{(\lambda)} * \Phi) \in \mathfrak{D} \cap H_2(\mathbb{R}^3),$$
$$Z^{(q)} : L^q(\mathbb{R}^3)^3 \ni \Phi \mapsto \Phi - \tau \cdot P\mathfrak{B}(E^{(\lambda)} * \Phi) \in L^q(\mathbb{R}^3)^3$$

are well defined, linear and bounded. If $\psi, g \in \mathfrak{D} \cap H_2(\mathbb{R}^3)$ with $\widetilde{Z}_\lambda(\psi) = g$, and if we set $u := E^{(\lambda)} * \psi$, then $u \in H^2(\mathbb{R}^3)^3 \cap L^6(\mathbb{R}^3)^3$, and

$$-\Delta u + \tau \cdot \partial_1 u + \lambda \cdot u - \tau \cdot P\mathfrak{B}(u) = g, \quad div\, u = 0. \tag{5.8}$$

If in addition $g \in L^q(\mathbb{R}^3)^3$, the relations $\psi \in L^q(\mathbb{R}^3)^3$, $Z_\lambda^{(q)}(\psi) = g$ hold.

Proof: The corollary follows from (4.1), (4.2), Theorem 3.5 and Corollary 3.2. In particular, the last statement is a consequence of (4.1). ◇

Theorem 5.4 *There is $\epsilon_1 \in (0,\, (\tau/2)^2]$, depending on τ and U, such that $\widetilde{Z}_\lambda : \mathfrak{D} \cap H_2(\mathbb{R}^3) \mapsto \mathfrak{D} \cap H_2(\mathbb{R}^3)$ is bijective and $\|\Phi\|_* \leq C \cdot \|\widetilde{Z}_\lambda(\Phi)\|_*$ for $\Phi \in \mathfrak{D} \cap H_2(\mathbb{R}^3)$, $\lambda \in \mathbb{C}\backslash\{0\}$ with $\Re\lambda \geq 0$, $|\lambda| \leq \epsilon_1$.*

Let $q \in (1, 2)$. Then there is $\epsilon_2(q) \in (0,\, (\tau/2)^2]$, depending on τ, U and q, such that the operator $Z_\lambda^{(q)} : L^q(\mathbb{R}^3)^3 \mapsto L^q(\mathbb{R}^3)^3$ is bijective and $\|\Phi\|_q \leq C(q) \cdot \|Z_\lambda^{(q)}(\Phi)\|_q$ for $\Phi \in L^q(\mathbb{R}^3)^3$, $\lambda \in \mathbb{C}\backslash\{0\}$ with $\Re\lambda \geq 0$, $|\lambda| \leq \epsilon_2(q)$.

Proof: By Corollary 5.1 and the open mapping theorem, there is $C_0 > 0$ only depending on τ such that

$$\|w\|_* \leq C_0 \cdot \|\widetilde{Z}_0(w)\|_* \quad \text{for } w \in \mathfrak{D} \cap H_2(\mathbb{R}^3). \tag{5.9}$$

Similarly, by Theorem 5.3, there is $\widetilde{C}_0(q) > 0$, depending on τ and q, with

$$\|\Phi\|_q \leq \widetilde{C}_0(q) \cdot \|Z_0^{(q)}(\Phi)\|_q \quad \text{for } \Phi \in L^q(\mathbb{R}^3)^3. \tag{5.10}$$

Now, in view of (4.7), we may choose $R > 0$ so large that $2 \cdot D_1(R) \leq (12 \cdot C_0 \cdot \tau)^{-1}$, with $D_1(R)$

from Lemma 4.2. Since $\ln(1/(1-1/\widetilde{R})) \to 0$ ($\widetilde{R} \to \infty$), we may fix some $\widetilde{R} \in [2 \cdot R+1, \infty)$ with

$$D_3(R) \cdot [\widetilde{R}^{-1/2} + \ln^{1/2}(1/(1-1/\widetilde{R}))] \leq (12 \cdot C_0 \cdot \tau)^{-1},$$

where the constant $D_3(R)$ was introduced in Theorem 4.1. Finally we choose $\epsilon_1 \in (0, (\tau/2)^2]$ so small that $D_3(\widetilde{R}) \cdot \epsilon_1^{1/3} \leq (12 \cdot C_0 \cdot \tau)^{-1}$. Then it follows from (4.7) and the last statement of Corollary 3.2 that

$$\|\widetilde{Z}_\lambda(w) - \widetilde{Z}_0(w)\|_* \leq (4 \cdot C_0)^{-1} \cdot \|w\|_* \quad \text{for } w \in \mathfrak{D} \cap H_2(\mathbb{R}^3),\ \lambda \in \mathbb{C}\backslash\{0\} \tag{5.11}$$

with $\Re\lambda \geq 0,\ |\lambda| \leq \epsilon_1$. Thus we may deduce with (5.9) and a simple shoestring argument that $\|w\|_* \leq 2 \cdot C_0 \cdot \|\widetilde{Z}_\lambda(w)\|_*$.

Let $g \in \mathfrak{D} \cap H_2(\mathbb{R}^3)$, and put $\Phi_0 := (\widetilde{Z}_0)^{-1}(g),\ \Phi_{n+1} := (\widetilde{Z}_0)^{-1}\big(g - (\widetilde{Z}_\lambda - \widetilde{Z}_0)(\Phi_n)\big)$ for $n \in \mathbb{N}_0$. Then (5.9) and (5.11) yield convergence of the sequence (Φ_n) with respect to the norm $\|\ \|_*$. The limit function Φ verifies the equation $\widetilde{Z}_\lambda(\Phi) = g$. This proves that $\widetilde{Z}_\lambda$ is bijective. An analogous argument based on (5.10) and (4.8) implies existence of some $\epsilon_2(q) \in (0, (\tau/2)^2]$ with the desired properties. $\diamond$

6. Resolvent estimates for the perturbed Oseen system (5.2).

In the rest of this article, we write $\mathfrak{I}$ for the identical mapping of $H_2(\mathbb{R}^3)$ onto itself. Put $D(L) := H_2(\mathbb{R}^3) \cap H^2(\mathbb{R}^3)^3$. Since $H^2(\mathbb{R}^3) \subset \mathfrak{D}_0^{1,2}(\mathbb{R}^3)^3$ and because of Lemma 4.1, the term $P\mathfrak{B}(v)$ is well defined and belongs to $H_2(\mathbb{R}^3)$ for $v \in D(L)$, so we may define

$$Lv := \Delta v - \tau \cdot \partial_1 v + \tau \cdot P\mathfrak{B}(v) \quad (v \in D(L)). \tag{6.1}$$

Then $L : D(L) \mapsto H_2(\mathbb{R}^3)$ is linear and densely defined in $H_2(\mathbb{R}^3)$. We will use the usual notation $\varrho(L)$ for the resolvent set of L.

Note that if $g \in L^2(\mathbb{R}^3)^3,\ u \in D(L)$ with $Lu = Pg$, Theorem 2.4 yields some $\pi \in W^{1,1}_{loc}(\mathbb{R}^3)$ such that the pair (u, π) solves the perturbed Oseen problem (5.2). Thus estimates of the operator $(\lambda \cdot \mathfrak{I} - L)^{-1}$, for $\lambda \in \varrho(L)$, correspond to estimates of solutions of the resolvent problem (5.8).

The ensuing theorem is due to [2], [8], [14, Theorem 1.3.2].

Theorem 6.1 *There is a countable set $\mathfrak{K} \subset \mathbb{C}$ with $\sigma(L)\backslash\mathfrak{K} \subset \{\lambda \in \mathbb{C} : \Re\lambda \leq -(\Im\lambda)^2/\tau^2\}$. There is $a \in (0, \infty),\ \vartheta \in (\pi/2, \pi)$ such that*

$$S_{\vartheta,a} := \{\lambda \in \mathbb{C}\backslash\{a\} : |\arg(\lambda - a)| \leq \vartheta\} \subset \varrho(L).$$

We now require that the eigenvalues of L verify the following condition:

(A2) $\Re\lambda < 0$ for $\lambda \in \mathfrak{K}$.

Note that by (A2) and Theorem 6.1, we have

$$\{\lambda \in \mathbb{C}\backslash\{0\} : \Re\lambda \geq 0\} \cup \big(\{\lambda \in \mathbb{C} : \Re\lambda < 0\} \cap S_{\vartheta,a}\big) \subset \varrho(L). \tag{6.2}$$

In this section and in Section 7, we write C for constants which may depend on τ, U, a or ϑ. As usual in this article, if such a constant depends on additional quantities $\gamma_1, ..., \gamma_n$, for some $n \in \mathbb{N}$, we denote it by $C(\gamma_1, ..., \gamma_n)$.

Lemma 6.1 *Let $\lambda \in \mathbb{C}\backslash\{0\}$ with $\Re\lambda > 0$ and $|\lambda| \leq \epsilon_1$, with ϵ_1 from Theorem 5.4. Take $g \in \mathfrak{D} \cap H_2(\mathbb{R}^3)$. Then $(\lambda \cdot \mathfrak{I} - L)^{-1}(g) = E^{(\lambda)} * (\widetilde{Z}_\lambda)^{-1}(g)$ and $(\lambda \cdot \mathfrak{I} - L)^{-1}(g) \in \mathfrak{D} \cap D(L)$.*

Proof: We have to compare $u := (\lambda \cdot \mathfrak{I} - L)^{-1}(g)$ and $\widetilde{u} := E^{(\lambda)} * (\widetilde{Z}_\lambda)^{-1}(g)$. Corollary 5.2 and Theorem 2.6 yield that $\widetilde{u} \in D(L)$ and $(\lambda \cdot \mathfrak{I} - L)(\widetilde{u}) = g$. Since $\lambda \in \varrho(L)$ by (6.2), it follows that $u = \widetilde{u}$. Observing that $D(L) \subset \mathfrak{D}_0^{1,2}(\mathbb{R}^3)^3$ and $u = (1/\lambda) \cdot (Lu + g)$, and recalling Lemma 4.1, we obtain $u \in \mathfrak{D}$. ◇

The next theorem is the crucial element of our theory; it states resolvent estimates for the perturbed Oseen system (5.2), under the assumption that the resolvent parameter λ has small absolute value and non-negative real part.

Theorem 6.2 *The inequality $\|\nabla(\lambda \cdot \mathfrak{I} - L)^{-1}(g)\|_2 \le C \cdot \|g\|_*$ holds for $g \in \mathfrak{D} \cap H_2(\mathbb{R}^3)$, $\lambda \in \mathbb{C}\backslash\{0\}$ with $\Re\lambda \ge 0$, $|\lambda| \le \epsilon_1$, where ϵ_1 was introduced in Theorem 5.4.*

Let $s \in (1, 6/5]$, $\delta \in (0, 1]$. Then there is $\epsilon_3(s, \delta) \in (0, \epsilon_1]$, depending on τ, U, s and δ such that for $f \in L^s(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3)$, $R \in (0, \infty)$, $\lambda \in \mathbb{C}\backslash\{0\}$ with $\Re\lambda \ge 0$, $|\lambda| \le \epsilon_3(s, \delta)$, the ensuing estimates hold:

$$\|\nabla(\lambda \cdot \mathfrak{I} - L)^{-2}(f) \,|\, B_R\|_2 + \|\nabla\big[\,(\overline{\lambda} \cdot \mathfrak{I} - L)^{-1} \circ (\lambda \cdot \mathfrak{I} - L)^{-1}(f)\,\big] \,|\, B_R\|_2 \tag{6.3}$$
$$\le C(s, \delta, R) \cdot |\lambda|^{-4\cdot(1-1/s)-\delta} \cdot \|f\|_s,$$

$$\|\nabla(\lambda \cdot \mathfrak{I} - L)^{-3}(f) \,|\, B_R\|_2 \le C(s, \delta, R) \cdot |\lambda|^{-2-4\cdot(1-1/s)-\delta} \cdot \|f\|_s. \tag{6.4}$$

Proof: Take $g \in \mathfrak{D} \cap H_2(\mathbb{R}^3)$ and $\lambda \in \mathbb{C}\backslash\{0\}$ with $\Re\lambda \ge 0$, $|\lambda| \le \epsilon_1$. Then we get by Lemma 6.1, Corollary 3.2 and Theorem 5.4,

$$\|\nabla(\lambda \cdot \mathfrak{I} - L)^{-1}(g)\|_2 = \|E^{(\lambda)} * (\widetilde{Z}_\lambda)^{-1}(g)\|_2 \le C \cdot \|(\widetilde{Z}_\lambda)^{-1}(g)\|_{-1,2} \le C \cdot \|g\|_*.$$

This proves the first claim of the theorem. Now let $f \in L^s(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3)$, and take λ as before. Then $f \in L^{6/5}(\mathbb{R}^3)^3$ by interpolation, hence $f \in \mathfrak{D} \cap H_2(\mathbb{R}^3)$. Thus we may define $u^{(1)} := E^{(\lambda)} * (\widetilde{Z}_\lambda)^{-1}(f)$, and obtain $u^{(1)} \in \mathfrak{D} \cap H_2(\mathbb{R}^3) \cap H^2(\mathbb{R}^3)^3$ by Lemma 6.1. Repeating this argument, we put $u^{(i+1)} := E^{(\lambda)} * (\widetilde{Z}_\lambda)^{-1}(u^{(i)})$ for $i \in \{1, 2\}$, $w := E^{(\overline{\lambda})} * (\widetilde{Z}_\lambda)^{-1}(u^{(1)})$, and obtain $u^{(2)}$, $u^{(3)}$, $w \in \mathfrak{D} \cap H^2(\mathbb{R}^3)^3 \cap H_2(\mathbb{R}^3)$. Moreover Lemma 6.1 yields

$$u^{(i)} = (\lambda \cdot \mathfrak{I} - L)^{-i}(f), \quad w = (\overline{\lambda} \cdot \mathfrak{I} - L)^{-1} \circ (\lambda \cdot \mathfrak{I} - L)^{-1}(f). \tag{6.5}$$

Take $q \in [4/3, 2)$, and suppose that $|\lambda| \le \min\{\,\epsilon_2(r) \,:\, r \in \{s, q\}\,\}$, with $\epsilon_2(q)$, $\epsilon_2(s)$ from Theorem 5.4. Put $p := \big((1/q - 1/4)^{-1} + (1/q - 1/3)^{-1}\big)$. Since $q \ge 4/3$, we have $p \ge 2$. (Actually only values of q close to 2 are of interest because it is them who lead to values of δ close to 0 in (6.3) and (6.4), as will be seen below.)

Since $f \in \mathfrak{D} \cap H_2(\mathbb{R}^3) \cap L^s(\mathbb{R}^3)^3$, as explained above, we have $(\widetilde{Z}_\lambda)^{-1}(f) = (Z_\lambda^{(s)})^{-1}(f)$ by Corollary 5.2. In addition, inequality (3.15) implies $u^{(1)} \in L^s(\mathbb{R}^3)^3 \cap L^q(\mathbb{R}^3)^3$. Again referring to Corollary 5.2 and (3.15), we may conclude that $u^{(2)} \in L^q(\mathbb{R}^3)^3$,

$$(\widetilde{Z}_\lambda)^{-1}(u^{(1)}) = (Z_\lambda^{(r)})^{-1}(u^{(1)}) \text{ for } r \in \{q, s\}, \quad (\widetilde{Z}_\lambda)^{-1}(u^{(2)}) = (Z_\lambda^{(q)})^{-1}(u^{(2)}). \tag{6.6}$$

Recalling (6.5), and applying (3.17), (6.6), Theorem 5.4 and (3.15), we find

$$\|\nabla(\lambda \cdot \mathfrak{I} - L)^{-2}(f)|B_R\|_2 \le \|\nabla u^{(2)}|B_R\|_2 \le C(R) \cdot \|\nabla u^{(2)}\|_p \tag{6.7}$$
$$\le C(R, p, q) \cdot \|(Z_\lambda^{(q)})^{-1}(u^{(1)})\|_q \le C(R, p, q) \cdot \|u^{(1)}\|_q$$
$$\le C(R, p, q, s) \cdot |\lambda|^{2-4\cdot(1-1/s+1/q)} \cdot \|(Z_\lambda^{(s)})^{-1}(f)\|_s$$
$$\le C(R, p, q, s) \cdot |\lambda|^{2-4\cdot(1-1/s+1/q)} \cdot \|f\|_s = C(R, p, q, s) \cdot |\lambda|^{-4\cdot(1-1/s)-\delta} \cdot \|f\|_s,$$

with $\delta := 4 \cdot (1/q - 1/2)$. An analogous argument, starting with (6.5), yields

$$\|(\overline{\lambda} \cdot \mathfrak{I} - L)^{-1} \circ (\lambda \cdot \mathfrak{I} - L)^{-1}(f)\|_2 \leq C(R, p, q, s) \cdot |\lambda|^{-4 \cdot (1-1/s)-\delta} \cdot \|f\|_s.$$

Thus, since q may be chosen arbitratily in $[4/3, 2)$, we have proved (6.3). In order to estimate $(\lambda \cdot \mathfrak{I} - L)^{-3}(f)$, we again proceed as in (6.7), but with f replaced by $u^{(1)}$. Note that f may in fact be substituted by $u^{(1)}$ since $u^{(1)} \in L^s(\mathbb{R}^3)^3 \cap \mathfrak{D} \cap H_2(\mathbb{R}^3)$, as explained above, and because of (6.6). In this way we arrive at the inequality

$$\|\nabla(\lambda \cdot \mathfrak{I} - L)^{-3}(f)|B_R\|_2 \leq C(R, p, q, s) \cdot |\lambda|^{-4 \cdot (1-1/s)-\delta} \cdot \|u^{(1)}\|_s. \tag{6.8}$$

But by (3.15) and Theorem 5.4,

$$\|u^{(1)}\|_s \leq C(s) \cdot |\lambda|^{-2} \cdot \|(Z_\lambda^{(s)})^{-1}(f)\|_s \leq C(s) \cdot |\lambda|^{-2} \cdot \|f\|_s. \tag{6.9}$$

By combining (6.8) and (6.9), we arrive at (6.4). ◇

Corollary 6.1 *The inequality*

$$\|\nabla(\lambda \cdot \mathfrak{I} - L)^{-1}(f)\|_2 \leq C(\widetilde{a}, \sigma) \cdot \|f\|_2 \tag{6.10}$$

holds for σ, $\widetilde{a}$, f as in Theorem 4.2, and for $\lambda \in \mathbb{C}\backslash\{0\}$ with $\Re\lambda \geq 0$, $|\lambda| \leq \epsilon_1$.

Let $\delta \in (0, 1]$. Then there is $\epsilon_4(\delta) \in (0, \epsilon_1]$, depending on τ, U and δ, such that

$$\|\nabla(\lambda \cdot \mathfrak{I} - L)^{-2}(f) \,|\, B_R\|_2 + \|\nabla\big[(\overline{\lambda} \cdot \mathfrak{I} - L)^{-1} \circ (\lambda \cdot \mathfrak{I} - L)^{-1}(f)\big] \,|\, B_R\|_2 \tag{6.11}$$
$$\leq C(\widetilde{a}, \sigma, \delta, R) \cdot |\lambda|^{-\delta} \cdot \|f\|_2,$$

$$\|\nabla(\lambda \cdot \mathfrak{I} - L)^{-3}(f) \,|\, B_R\|_2 \leq C(\widetilde{a}, \sigma, \delta, R) \cdot |\lambda|^{-2-\delta} \cdot \|f\|_2 \tag{6.12}$$

for $R \in (0, \infty)$, $\lambda \in \mathbb{C}\backslash\{0\}$ with $\Re\lambda \geq 0$, $|\lambda| \leq \epsilon_4(\delta)$, and for σ, $\widetilde{a}$, f as in Theorem 4.2.

Proof: Take σ, $\widetilde{a}$, f as in Theorem 4.2. Then $\|f\|_* \leq \|f\|_2 + C \cdot \|f\|_{6/5} \leq C(\widetilde{a}, \sigma) \cdot \|f\|_2$ by Lemma 2.4 and (4.24). Inequality (6.10) now follows with the first statement of Theorem 6.2. Let $\delta \in (0, 1]$, and put $s := 1/(1 - \delta/8)$. Then $s \in (1, 6/5)$, so $\|f\|_s \leq C(\delta, \widetilde{a}, \sigma) \cdot \|f\|_2$ by (4.24), and $-4 \cdot (1 - 1/s) - \delta/2 = -\delta$. From these observations and inequalities (6.3) and (6.4) with δ replaced by $\delta/2$, we obtain (6.11) and (6.12), respectively. ◇

Lemma 6.2 *Let $\lambda \in \mathbb{C}$ with $\Re\lambda \geq 0$, and $g \in H_2(\mathbb{R}^3)$. Then*

$$\|\nabla(\lambda \cdot \mathfrak{I} - L)^{-1}(g)\|_2 \leq C \cdot (\|g\|_2 + \|(\lambda \cdot \mathfrak{I} - L)^{-1}(g)\|_2).$$

Proof: Put $u := (\lambda \cdot \mathfrak{I} - L)^{-1}(g)$. Then $u \in D(L)$ and $-\Delta u + \lambda \cdot u = g - \tau \cdot \partial_1 u + \tau \cdot P\mathfrak{B}(u)$, so that

$$\Re \int_{\mathbb{R}^3} (-\Delta u \cdot \overline{u} + \lambda \cdot |u|^2)\, dx = \Re \int_{\mathbb{R}^3} \big(g - \tau \cdot \partial_1 u + \tau \cdot P\mathfrak{B}(u)\big) \cdot \overline{u}\, dx. \tag{6.13}$$

But $\int_{\mathbb{R}^3} -\Delta u \cdot \overline{u}\, dx = \|\nabla u\|_2^2$, $\Re \int_{\mathbb{R}^3} \partial_1 u \cdot \overline{u}\, dx = 0$, so we deduce from (6.13) and Lemma 4.1 that

$$\|\nabla u\|_2^2 + \Re\lambda \cdot \|u\|_2^2 \leq \|g\|_2 \cdot \|u\|_2 + C \cdot \|\nabla u\|_2 \cdot \|u\|_2.$$

Since $\Re\lambda \geq 0$, the lemma now follows by a simple shoestring argument. ◇

Lemma 6.3 *Let γ_1, $\gamma_2 \in (0, \infty)$ with $\gamma_1 < \gamma_2$. Put $\mathfrak{M}_{\gamma_1,\gamma_2} := \{\lambda \in \mathbb{C} : \Re\lambda \geq 0,\ \gamma_1 \leq |\lambda| \leq \gamma_2\}$. Then $\mathfrak{M}_{\gamma_1,\gamma_2} \subset \varrho(L)$ (see (6.2)), and*

$$\|(\lambda \cdot \mathfrak{I} - L)^{-1}(\Phi)\|_2 + \|\nabla(\lambda \cdot \mathfrak{I} - L)^{-1}(\Phi)\|_2 \leq C(\gamma_1, \gamma_2) \cdot \|\Phi\|_2 \quad \textit{for } \Phi \in H_2(\mathbb{R}^3).$$

Proof: Recall that $\varrho(L)$ is an open set in $\mathbb{C}$, and the mapping $\varrho(L) \ni \lambda \mapsto (\lambda \cdot \mathfrak{I} - L)^{-1}$ is holomorphic, in particular continuous, with respect to the operator norm of linear bounded operators from $H_2(\mathbb{R}^3)$ into $H_2(\mathbb{R}^3)$. Thus an elementary argument involving finite coverings of $\mathfrak{M}_{\gamma_1,\gamma_2}$ and Neumann series of operators yields that $\|(\lambda \cdot \mathfrak{I} - L)^{-1}(\Phi)\|_2 \le C(\gamma_1, \gamma_2) \cdot \|\Phi\|_2$. Now the lemma follows with Lemma 6.2. $\diamond$

Theorem 6.3 *There is a constant $C_1 > 0$ depending on τ, U, ϑ and a such that*

$$|\lambda| \cdot \|(\lambda \cdot \mathfrak{I} - L)^{-1}(g)\|_2 \le C \cdot \|g\|_2 \tag{6.14}$$

for $g \in H_2(\mathbb{R}^3)$, $\lambda \in \mathbb{C}\backslash\{0\}$ with $\Re\lambda \ge 0$, $|\lambda| \ge C_1$, and for $\lambda \in S_{\vartheta,a}$ with $\Re\lambda < 0$ and $|\lambda| \ge C_1$;

$$|\lambda| \cdot \|\nabla(\lambda \cdot \mathfrak{I} - L)^{-1}(g)\|_2 \le C \cdot \|\nabla g\|_2 \tag{6.15}$$

for $g \in H_2(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)^3$, and for λ as in (6.14).

Proof: Let $\lambda \in \mathbb{C}\backslash\{0\}$ with $\Re\lambda \ge 0$ or $\lambda \in S_{\vartheta,a}$. This means by (6.2) that $\lambda \in \varrho(L)$. Let $g \in H_2(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)^3$, and put $u := (\lambda \cdot \mathfrak{I} - L)^{-1}(g)$. Then

$$-\Delta u + \lambda \cdot u = g - \tau \cdot \partial_1 u + \tau \cdot P\mathfrak{B}(u). \tag{6.16}$$

Multiplying this equation by $-\Delta\overline{u}$, integrating over $\mathbb{R}^3$, seperating real and imaginary parts, and then applying Hölder's inequality and Lemma 4.1, we get

$$\|\Delta u\|_2^2 + \Re\lambda \cdot \|\nabla u\|_2^2 \le C \cdot (\|\nabla g\|_2 \cdot \|\nabla u\|_2 + \|\nabla u\|_2 \cdot \|\Delta u\|_2), \tag{6.17}$$

$$|\Im\lambda| \cdot \|\nabla u\|_2^2 \le C \cdot (\|\nabla g\|_2 \cdot \|\nabla u\|_2 + \|\nabla u\|_2 \cdot \|\Delta u\|_2). \tag{6.18}$$

Now we distinguish between the cases $\Re\lambda \ge 0$ and $\Re\lambda < 0$. First consider the (more difficult) case $\Re\lambda < 0$. Then $\lambda \in S_{\vartheta,a}$, hence

$$\Re(\lambda - a) > -\cos(\pi - \vartheta) \cdot |\lambda - a|, \quad |\Im\lambda| \ge |\lambda - a| \cdot \sin(\pi - \vartheta).$$

We may thus deduce from (6.17) and (6.18), respectively,

$$\|\Delta u\|_2^2 + (-\cos(\pi - \vartheta) \cdot |\lambda - a| + a) \cdot \|\nabla u\|_2^2 \le \|\Delta u\|_2^2 + \Re\lambda \cdot \|\nabla u\|_2^2$$
$$\le C \cdot (\|\nabla g\|_2 \cdot \|\nabla u\|_2 + \|\nabla u\|_2 \cdot \|\Delta u\|_2),$$

$$|\lambda - a| \cdot \sin(\pi - \vartheta) \cdot \|\nabla u\|_2^2 \le C \cdot (\|\nabla g\|_2 \cdot \|\nabla u\|_2 + \|\nabla u\|_2 \cdot \|\Delta u\|_2).$$

The second inequality is multiplied by $2 \cdot \cot(\pi - \vartheta)$ and then added to the first one. It follows

$$\|\Delta u\|_2^2 + (\cos(\pi - \vartheta) \cdot |\lambda - a| + a) \cdot \|\nabla u\|_2^2 \tag{6.19}$$
$$\le C \cdot (\|\nabla g\|_2 \cdot \|\nabla u\|_2 + \|\nabla u\|_2 \cdot \|\Delta u\|_2) \le \alpha_1 \cdot (\|\nabla g\|_2 \cdot \|\nabla u\|_2 + \|\nabla u\|_2^2) + \|\Delta u\|_2^2/2,$$

with a constant α_1 depending on τ, U and ϑ. Now suppose in addition that $|\lambda| \ge 2 \cdot \alpha_1 / \cos(\pi - \vartheta)$. Then $\cos(\pi - \vartheta) \cdot |\lambda - a| + a \ge \cos(\pi - \vartheta) \cdot |\lambda| \ge 2 \cdot \alpha_1$, hence from (6.19)

$$\|\Delta u\|_2^2 + \big(\cos(\pi - \vartheta) \cdot |\lambda - a|/2 + a/2 + \alpha_1\big) \cdot \|\nabla u\|_2^2$$
$$\le \alpha_1 \cdot (\|\nabla g\|_2 \cdot \|\nabla u\|_2 + \|\nabla u\|_2^2) + \|\Delta u\|_2^2/2,$$

so that

$$\|\Delta u\|_2^2/2 + \big(\cos(\pi - \vartheta) \cdot |\lambda - a|/2 + a/2\big) \cdot \|\nabla u\|_2^2 \le \alpha_1 \cdot (\|\nabla g\|_2 \cdot \|\nabla u\|_2),$$

Since $\cos(\pi - \vartheta) \cdot |\lambda - a| + a \ge |\lambda| \cdot \cos(\pi - \vartheta)$, we now get $|\lambda| \cdot \|\nabla u\|_2^2 \le C \cdot \|\nabla g\|_2 \cdot \|\nabla u\|_2$, hence $|\lambda| \cdot \|\nabla u\|_2 \le C \cdot \|\nabla g\|_2$. Recall this inequality was proved under the assumptions $\lambda \in$

$S_{\vartheta,a},\ \Re\lambda < 0,\ |\lambda| \ge 2 \cdot \alpha_1/\cos(\pi - \vartheta)$.

Now we consider the case $\Re\lambda \ge 0$. Adding (6.17) and (6.18), we obtain

$$\|\Delta u\|_2^2 + (\Re\lambda + |\Im\lambda|) \cdot \|\nabla u\|_2^2 \le C \cdot (\|\nabla g\|_2 \cdot \|\nabla u\|_2 + \|\Delta u\|_2 \cdot \|\nabla u\|_2).$$

But $\Re\lambda + |\Im\lambda| \ge |\lambda|$, so we may conclude

$$\|\Delta u\|_2^2 + |\lambda| \cdot \|\nabla u\|_2^2 \le C \cdot (\|\nabla g\|_2 \cdot \|\nabla u\|_2 + \|\nabla u\|_2 \cdot \|\Delta u\|_2)$$
$$\le \alpha_2 \cdot (\|\nabla g\|_2 \cdot \|\nabla u\|_2 + \|\nabla u\|_2^2) + \|\Delta u\|_2^2/2,$$

with a constant $\alpha_2 > 0$ depending on τ and U. Thus, if $|\lambda| \ge 2 \cdot \alpha_2$, we arrive at the inequality

$$\|\Delta u\|_2^2/2 + |\lambda| \cdot \|\nabla u\|_2^2 \le \alpha_2 \cdot \|\nabla g\|_2 \cdot \|\nabla u\|_2.$$

It follows $|\lambda| \cdot \|\nabla u\|_2 \le C \cdot \|\nabla g\|_2$. This completes the proof of (6.15). In order to show (6.14), we multiply (6.16) with $\overline{u}$ instead of $-\Delta\overline{u}$. Then we obtain (6.14) by repeating the previous arguments with obvious modifications. ◇

Note that Theorem 6.2 presents resolvent estimates related to the operator L for the case that $|\lambda|$ is small, whereas Theorem 6.3 deals with the case of large $|\lambda|$. Lemma 6.3 might be considered as an (obvious) result for intermediate values.

7. Estimates of the semigroup e^{Lt}.

We recall that our convention at the beginning of Section 6 with respect to generic constants remains valid in this section.

By Theorem 6.1, (6.14) and [14, Theorem 1.3.4], the operator L defined in (6.1) generates an analytic semigroup in $H_2(\mathbb{R}^3)$ ([14, Definition 1.3.3]), which we denote by e^{Lt}. In what follows, we will exploit the resolvent estimates from Section 6 in order to evaluate this semigroup. We begin by introducing the constant

$$C_2 := \max\{\, C_1,\ \epsilon_4(1/16),\ 2^{-1/2},\ 2 \cdot a \cdot \tan(\pi - \vartheta)\,\},$$

where C_1 was chosen in Theorem 6.3, and $\epsilon_4(1/16)$ in Corollary 6.1. For the quantities a and ϑ, we refer to Theorem 6.1. Since $C_2 \ge 2 \cdot a \cdot \tan(\pi - \vartheta)$, we may choose $\vartheta_0 \in (\pi/2,\, \vartheta)$ so close to $\pi/2$ that for any $s \in [C_2, \infty)$, the relation

$$\{\, s \cdot e^{i\cdot\varphi}\ :\ \varphi \in [-\vartheta_0,\, -\pi/2] \cup [\pi/2,\, \vartheta_0]\,\} \cup \{\, r \cdot e^{i\cdot\vartheta_0}\ :\ r \in [s, \infty)\,\} \subset S_{\vartheta,a} \tag{7.1}$$

holds. Let $\alpha, \beta \in (0, \infty)$ with $\alpha < \beta$, $\beta \ge C_2$. Then we define the curves $\Gamma_i^{(\alpha,\beta)} \subset \mathbb{C}$, with $i \in \{1, ..., 5\}$, by setting

$$\Gamma_1^{(\alpha,\beta)} := \{\, \alpha \cdot e^{i\cdot\varphi}\ :\ \varphi \in [-\pi/2,\, \pi/2]\,\},\quad \Gamma_2^{(\alpha,\beta)} := \{\, i \cdot r\ :\ r \in [\alpha, \beta]\,\},$$
$$\Gamma_3^{(\alpha,\beta)} := \{\, i \cdot \beta + r \cdot e^{i\cdot\vartheta}\ :\ r \in [0, \infty)\,\},\quad \Gamma_i^{(\alpha,\beta)} := \{\, \overline{y}\ :\ y \in \Gamma_{i-2}^{(\alpha,\beta)}\,\}\ \text{ for }\ i \in \{4,\, 5\}.$$

Let $s \in [C_2, \infty)$ and define

$$\Lambda_1^{(s)} := \{\, s \cdot e^{i\cdot\varphi}\ :\ \varphi \in [-\vartheta_0, \vartheta_0]\,\},\quad \Lambda_2^{(s)} := \{\, r \cdot e^{i\cdot\vartheta_0}\ :\ r \in [s, \infty)\,\},\quad \Lambda_3^{(s)} := \{\, \overline{y}\ :\ y \in \Lambda_2^{(s)}\,\}.$$

Then, in view of (6.2), (7.1) and Theorem 6.1, and since $\beta \ge C_2 > a \cdot \tan(\pi - \vartheta)$, we have $\Gamma_\nu^{(\alpha,\beta)},\ \Lambda_\mu^{(s)} \subset \varrho(L)$ $(1 \le \nu \le 5,\ 1 \le \mu \le 3)$. As a consequence of these relations and [14, Theorem 1.3.4], we obtain

$$e^{Lt}(w) = (2\cdot\pi\cdot i)^{-1}\cdot\sum_{\nu=1}^{5}\int_{\Gamma_\nu^{(\alpha,\beta)}} e^{\lambda\cdot t}\cdot(\lambda\cdot I - L)^{-1}(w)\,d\lambda \tag{7.2}$$
$$= (2\cdot\pi\cdot i)^{-1}\cdot\sum_{\mu=1}^{3}\int_{\Lambda_\mu^{(s)}} e^{\lambda\cdot t}\cdot(\lambda\cdot I - L)^{-1}(w)\,d\lambda \quad \text{for } t\in(0,\infty),\ w\in H_2(\mathbb{R}^3).$$

A remark is perhaps in order with respect to the arguments we present in this section. The main difficulty consists in showing that for large t and for $\widetilde{a}$, σ, f as in Theorem 4.2, the term $\|\nabla e^{Lt}(f)\,|\,B_R\|_2$ is bounded by $C(R,\widetilde{a},\sigma)\cdot t^{-1-\epsilon}\cdot\|f\|_2$, for some $\epsilon>0$. (Incidentally we will chose $\epsilon = 1/8$, but this will only be for definiteness.) We will obtain such an estimate by considering the first sum on the right-hand side of (7.2). This means in particular that we have to show that

$$\Big\|\int_{\Gamma_1^{(\alpha,\beta)}} e^{\lambda\cdot t}\cdot\nabla(\lambda\cdot I - L)^{-1}(f)\,|\,B_R\,d\lambda\Big\|_2 \le C(\widetilde{a},\sigma,R)\cdot\|f\|_2\cdot t^{-1-\epsilon}$$

for large t. In view of (6.10), this should require $\alpha\le t^{-1-\epsilon}$. On other hand, in order to produce a factor $t^{-\gamma}$ for some $\gamma>0$ in the estimate of the integral $\int_{\Gamma_\nu^{(\alpha,\beta)}} e^{\lambda\cdot t}\cdot\nabla(\lambda\cdot I - L)^{-1}(f)\,|\,B_R\,d\lambda$ for $\nu=2$ and $\nu=4$, we introduce the local parameter $\varphi(r) := i\cdot r$ ($r\in[\alpha,\beta]$), and then integrate by parts with respect to r, so that the factor $e^{i\cdot r\cdot t}$ is transformed into $e^{i\cdot r\cdot t}\cdot(i\cdot t)^{-1}$. But this means that a single partial integration does not suffice to generate a factor $t^{-1-\epsilon}$. On the other hand, after two such integrations, we obtain a term $\nabla(i\cdot r\cdot I - L)^{-3}(f)\,|\,B_R$, which gives rise to a factor $r^{-2-\delta}$ for some $\delta>0$ (see (6.12)). Integrating this term on the interval $[\alpha,\beta]$ leads to a factor $\alpha^{-1-\delta} = t^{(1+\epsilon)(1+\delta)}$ which cancels the effect of the second partial integration. Therefore, recalling that the term $\nabla(i\cdot r\cdot I - L)^{-2}(f)\,|\,B_R$ only produces a factor $r^{-\delta}$ (see (6.11)), we perform a kind of interpolation between one and two partial integrations. To this end, we use fractional derivatives, as introduced in the next lemma.

Lemma 7.1 *Let $\kappa, b\in\mathbb{R}$ with $\kappa<b$, $\mu\in(0,1)$, $h\in C^1([\kappa,b])$ with $h(b)=0$. Define $\overline{h}:[\kappa,b]\mapsto\mathbb{C}$ by*

$$\overline{h}(r) := \Gamma(1-\mu)^{-1}\cdot\int_r^b (s-r)^{-1+\mu}\cdot h(s)\,ds \quad \textit{for } r\in[\kappa,b].$$

Then $\overline{h}\in C^1([\kappa,b])$ with

$$\overline{h}'(r) = \Gamma(1-\mu)^{-1}\cdot\int_r^b (\alpha-r)^{-1+\mu}\cdot h'(\alpha)\,d\alpha \quad \textit{for } r\in[\kappa,b]. \tag{7.3}$$

Define $\gamma:[\kappa,b]\ni r\mapsto\Gamma(\mu)^{-1}\cdot\int_r^b(s-r)^{-\mu}\cdot\overline{h}'(s)\,ds\in\mathbb{C}$. Then $h=-\gamma$.

We leave the proof of this lemma to the reader, and only remark that the equation $\gamma=-h$ may be reduced to the relation $B(\mu,1-\mu)=\Gamma(\mu)\cdot\Gamma(1-\mu)$ for $\mu\in(0,1)$, where B denotes the usual beta function.

Now we may prove an inequality which will be the key element in our estimate of the integrals over $\Gamma_2^{(\alpha,\beta)}$ and $\Gamma_4^{(\alpha,\beta)}$.

Lemma 7.2 *Let $\delta\in(0,\,1/4)$ and abbreviate $b:=\min\{\,\epsilon_4(\delta),\,2^{-1/2}\,\}$, with $\epsilon_4(\delta)$ from Corollary 6.1. Then, for $\widetilde{a}$, σ, f as in Theorem 4.2, $R\in(0,\infty)$, $\kappa\in(0,b)$, $t\in(0,\infty)$,*

$$\Big\|\int_\kappa^b e^{i\cdot r\cdot t}\cdot\nabla(i\cdot r\cdot\mathfrak{I} - L)^{-2}(f)\,|\,B_R\,dr\Big\|_2 \le C(\widetilde{a},\sigma,\delta,R)\cdot t^{-1/4}\cdot\kappa^{-\delta}\cdot\|f\|_2.$$

Proof: Take $\widetilde{a}, \sigma, f, R, \kappa, t$ as in the lemma. Note that by (6.2), we have $\{i{\cdot}r \,:\, r \in [\kappa, b]\} \subset \varrho(L)$. Therefore the mapping $g : [\kappa, b] \ni r \mapsto \nabla(i \cdot r \cdot \mathfrak{I} - L)^{-1}(f) \,|\, B_R \in L^2(B_R)^9$ is in particular twice continuously differentiable, with

$$g^{(\nu)}(r) = (-i)^\nu \cdot \nu \cdot \nabla(i \cdot r \cdot \mathfrak{I} - L)^{-(\nu+1)}(f) \,|\, B_R \quad \text{for } \nu \in \{1,\, 2\},\ r \in [\kappa, b].$$

Thus, due to the assumption $b \le \epsilon_4(\delta)$, inequalities (6.11) and (6.12) yield

$$r^\delta \cdot \|g'\|_2 + r^{2+\delta} \cdot \|g''(r)\|_2 \le C(\widetilde{a}, \sigma, \delta, R) \cdot \|f\|_2 \quad \text{for } r \in [\kappa, b]. \tag{7.4}$$

Put $h(r) := (i \cdot t)^{-1} \cdot (e^{i \cdot r \cdot t} - e^{i \cdot b \cdot t})$ for $r \in [\kappa, b]$. Define $\overline{h}$ and γ as in Lemma 7.1, with $\mu = 1/4$. Then we get by performing a partial integration, using the equation $\gamma' = -h'$ (Lemma 7.1), and changing the order of integration,

$$\int_\kappa^b e^{i \cdot r \cdot t} \cdot \nabla(i \cdot r \cdot \mathfrak{I} - L)^{-2}(f) \,|\, B_R \; dr = (-i) \cdot \int_\kappa^b \gamma'(r) \cdot g'(r) \; dr \tag{7.5}$$

$$= i \cdot \Gamma(1/4)^{-1} \cdot \int_\kappa^b \overline{h}'(s) \cdot \Big(\int_\kappa^s (s-r)^{-1/4} \cdot g''(r) \; dr \Big) \; ds$$

$$+ i \cdot \Gamma(1/4)^{-1} \cdot \int_\kappa^b (s-\kappa)^{-1/4} \cdot \overline{h}'(s) \; ds \cdot g'(\kappa).$$

Note that $\overline{h}'$ is a fractional derivative of h (of order 3/4). Thus we have transformed an integral of the form $\int_\kappa^b h' \cdot g' \; dr$ involving the derivative h' of h, into an integral of the form $\int_\kappa^b \overline{h}' \cdot \psi \; dr$ (modulo boundary terms) involving a fractional derivative of h, in contrast to the function h itself, which would arise in a standard partial integration.

The lemma follows from (7.4), (7.5) and the inequalities $|h(r)| \le 2/t,\ |h'(r)| \le 2$ for $r \in [\kappa, b]$. We omit the details because they were already elaborated in [5, proof of Lemma 6.2]. $\diamond$

In the following theorem, we estimate $\nabla e^{Lt}(f)|B_R$ for large values of t, with f given as in Theorem 4.2.

Theorem 7.1 *Put* $b := \min\{\, \epsilon_4(1/16),\, 2^{-1/2}\,\}$*, with* $\epsilon_4(1/16)$ *from Corollary 6.1. Let* $R \in (0, \infty)$*,* $t \in [b^{-1}, \infty)$*, and take* $\widetilde{a}$*,* σ*,* f *as in Theorem 4.2. Then*

$$\|\nabla e^{Lt}(f)|B_R\|_2 \le C(\widetilde{a}, \sigma, R) \cdot \|f\|_2 \cdot t^{-9/8}.$$

Proof: We start from the first equation in (7.2), with $\alpha = t^{-2}$. The latter assumption means in particular that $\alpha = t^{-2} \le t^{-1} \le b$. We further take $\beta \in [C_2, \infty)$, where C_2 was introduced at the beginning of this section. We find

$$\Big\| \nabla \Big(\int_{\Gamma_1^{(\alpha,\beta)}} e^{\lambda \cdot t} \cdot (\lambda \cdot \mathfrak{I} - L)^{-1}(f) \; d\lambda \Big) \,|\, B_R \Big\|_2 \tag{7.6}$$

$$\le \alpha \cdot \int_{-\pi/2}^{\pi/2} |e^{t \cdot \alpha \cdot e^{i \cdot \varphi}}| \cdot \|\nabla(\alpha \cdot e^{i \cdot \varphi} \cdot \mathfrak{I} - L)^{-1}(f) \,|\, B_R\|_2 \; d\varphi$$

$$\le C(\widetilde{a}, \sigma) \cdot \|f\|_2 \cdot \alpha \cdot e^{\alpha \cdot t} \le C(\widetilde{a}, \sigma) \cdot \|f\|_2 \cdot t^{-2},$$

where the last but one inequality holds because of (6.10). The last one is a consequence of the choice $\alpha = t^{-2}$. If $\lambda \in \Gamma_3^{(\alpha,\beta)} \cup \Gamma_5^{(\alpha,\beta)}$, we have $|\lambda| \ge |\beta| \ge C_2 \ge C_1$ and $\lambda \in S_{\vartheta,a},\ \Re\lambda \ge 0$, so inequality (6.15) is valid for such λ. This allows us to conclude that

$$\Big\|\nabla\Big(\sum_{\nu\in\{3,5\}}\int_{\Gamma_\nu^{(\alpha,\beta)}} e^{\lambda\cdot t}\cdot(\lambda\cdot\mathfrak{I}-L)^{-1}(f)\,d\lambda\Big)\,|\,B_R\Big\|_2 \tag{7.7}$$
$$\leq C\cdot\|\nabla f\|_2\cdot\int_0^\infty |e^{i\cdot\beta+r\cdot e^{i\cdot\vartheta}}|\cdot|i\cdot\beta+r\cdot e^{i\cdot\vartheta}|^{-1}\,dr$$
$$\leq C\cdot\|\nabla f\|_2\cdot\beta^{-1}\cdot\int_0^\infty e^{r\cdot t\cdot\cos\vartheta}\,dr\leq C(\widetilde{a})\cdot\|f\|_2\cdot(\beta\cdot t)^{-1},$$

where the last estimate follows with the first inequality in (4.24). This leaves us to deal with the main difficulty of this proof, that is, the estimate of the integrals over $\Gamma_2^{(\alpha,\beta)}$ and $\Gamma_4^{(\alpha,\beta)}$. To this end, we perform a partial integration. Noting that $b\leq C_2\leq\beta$, we obtain

$$\nabla\Big(\int_{\Gamma_2^{(\alpha,\beta)}} e^{\lambda\cdot t}\cdot(\lambda\cdot\mathfrak{I}-L)^{-1}(f)\,d\lambda\Big)\,|\,B_R \tag{7.8}$$
$$= i\cdot\int_\alpha^\beta e^{i\cdot r\cdot t}\cdot\nabla(i\cdot r\cdot\mathfrak{I}-L)^{-1}(f)\,|\,B_R\,dr=\sum_{j=1}^4\mathfrak{N}_j,$$

with

$$\mathfrak{N}_1 := t^{-1}\cdot e^{i\cdot t\cdot\beta}\cdot\nabla(i\cdot\beta\cdot\mathfrak{I}-L)^{-1}(f)\,|\,B_R,\quad \mathfrak{N}_2 := -t^{-1}\cdot e^{i\cdot t\cdot\alpha}\cdot\nabla(i\cdot\alpha\cdot\mathfrak{I}-L)^{-1}(f)\,|\,B_R,$$
$$\mathfrak{N}_3 := (i/t)\cdot\int_\alpha^b e^{i\cdot t\cdot r}\cdot\nabla(i\cdot r\cdot\mathfrak{I}-L)^{-2}(f)\,|\,B_R\,dr,$$
$$\mathfrak{N}_4 := (i/t)\cdot\int_b^\beta e^{i\cdot t\cdot r}\cdot\nabla(i\cdot r\cdot\mathfrak{I}-L)^{-2}(f)\,|\,B_R\,dr.$$

The integral over $\Gamma_4^{(\alpha,\beta)}$ is split into a sum $\sum_{j=1}^4\overline{\mathfrak{N}_j}$, where $\overline{\mathfrak{N}_j}$ is defined in an analogous way as $\mathfrak{N}_j$, for $j\in\{1,\ldots,4\}$. Recalling that $C_1\leq C_2\leq\beta$, we get from (6.15) and (4.24),

$$\|\mathfrak{N}_1\|_1+\|\overline{\mathfrak{N}}_1\|_2\leq C(\widetilde{a})\cdot(t\cdot\beta)^{-1}\cdot\|f\|_2. \tag{7.9}$$

We further find, using the standard resolvent equation, (6.10) and (6.11) with $\delta=1/16$,

$$\|\mathfrak{N}_2+\overline{\mathfrak{N}}_2\|_2\leq t^{-1}\cdot|e^{-i\cdot\alpha\cdot t}-e^{i\cdot\alpha\cdot t}|\cdot\|\nabla(-i\cdot\alpha\cdot\mathfrak{I}-L)^{-1}(f)\,|\,B_R\|_2 \tag{7.10}$$
$$+\,t^{-1}|e^{i\cdot\alpha\cdot t}|\cdot\|2\cdot i\cdot\alpha\cdot\nabla\big[(-i\cdot\alpha\cdot\mathfrak{I}-L)^{-1}\circ(i\cdot\alpha\cdot\mathfrak{I}-L)^{-1}(f)\big]\,|\,B_R\|_2$$
$$\leq C(\widetilde{a},\sigma,R)\cdot\|f\|_2\cdot t^{-1}\cdot\big(|\sin(\alpha\cdot t)|+\alpha^{15/16}\big)\leq C(\widetilde{a},\sigma,R)\cdot\|f\|_2\cdot(\alpha+\alpha^{15/16}/t).$$

Lemma 7.2 with $\delta=1/16$ yields

$$\|\mathfrak{N}_3\|_2+\|\overline{\mathfrak{N}}_3\|_2\leq C(\widetilde{a},\sigma,R)\cdot\|f\|_2\cdot t^{-5/4}\cdot\alpha^{-1/16}. \tag{7.11}$$

Concerning $\mathfrak{N}_4$, we perform an additional partial integration, to obtain

$$\mathfrak{N}_4=2\cdot i\cdot t^{-2}\cdot\int_b^\beta e^{i\cdot r\cdot t}\cdot\nabla(i\cdot r\cdot\mathfrak{I}-L)^{-3}(f)\,|\,B_R\,dr+t^{-2}\cdot e^{i\cdot\beta\cdot t}\cdot\nabla(i\cdot\beta\cdot\mathfrak{I}-L)^{-2}(f)\,|\,B_R$$
$$-t^{-2}\cdot e^{i\cdot b\cdot t}\cdot\nabla(i\cdot b\cdot\mathfrak{I}-L)^{-2}(f)\,|\,B_R.$$

Now we apply (6.15) with $\lambda=i\cdot\beta$, (6.11) with $\lambda=i\cdot b$, the inequality $b\leq\epsilon_4(1/16)$ and (4.24), to obtain

$$\|\mathfrak{N}_4\|_2\leq C\cdot t^{-2}\cdot\int_b^\beta\|\nabla(i\cdot r\cdot\mathfrak{I}-L)^{-3}(f)\|_2\,dr+C(\widetilde{a},\sigma,R)\cdot t^{-2}\cdot\|f\|_2. \tag{7.12}$$

The remaining integral in (7.12) is split into an integral from b to C_2 and into another one from C_2 to β. (Recall that $b \le C_2 \le \beta$.) But $\int_b^{C_2} \|(i \cdot r \cdot \mathfrak{I} - L)^{-3}(f)\|_2 \, dr \le C \cdot \|f\|_2$ by Lemma 6.3 with $\gamma_1 = b$, $\gamma_2 = C_2$, whereas

$$\int_{C_2}^{b} \|\nabla(i \cdot r \cdot \mathfrak{I} - L)^{-3}(f)\|_2 \, dr \le C \cdot \|\nabla f\|_2 \cdot \int_{C_2}^{\beta} r^{-3} \, dr \le C \cdot \|\nabla f\|_2$$

by (6.15). Thus, referring to (4.24), we obtain from (7.12),

$$\|\mathfrak{N}_4\|_2 \le C(\widetilde{a}, \sigma, R) \cdot \|f\|_2 \cdot t^{-2}. \tag{7.13}$$

An analogous estimate for $\|\overline{\mathfrak{N}}_4\|_2$ may be derived by the same arguments. Now, combining (7.8), the analogue of (7.8) for the integral over $\Gamma_4^{(\alpha,\beta)}$, (7.9) – (7.11), (7.13) and the analogue of (7.13) for $\overline{\mathfrak{N}}_4$, we obtain

$$\begin{aligned} &\Big\|\nabla\Big(\sum_{\nu\in\{2,\,4\}} \int_{\Gamma_\nu^{(\alpha,\beta)}} e^{\lambda\cdot t} \cdot (\lambda \cdot \mathfrak{I} - L)^{-1}(f) \; d\lambda\Big) \,|\, B_R\Big\|_2 \\ &\le C(\widetilde{a}, \sigma, R) \cdot \|f\|_2 \cdot \big((t \cdot \beta)^{-1} + \alpha + t^{-1} \cdot \alpha^{15/16} + t^{-5/4} \cdot \alpha^{-1/16} + t^{-2} \big) \\ &\le C(\widetilde{a}, \sigma, R) \cdot \|f\|_2 \cdot \big((t \cdot \beta)^{-1} + t^{-9/8} \big), \end{aligned} \tag{7.14}$$

where the last inequality holds because we chose $\alpha = t^{-2}$. By referring to (7.2), (7.6), (7.7), (7.14), we may conclude that $\|\nabla e^{Lt}(f)|B_R\|_2 \le C(\widetilde{a}, \sigma, R) \cdot \big((t \cdot \beta)^{-1} + t^{-9/8} \big)$. Letting β tend to infinity, we obtain the theorem. $\diamond$

Theorem 7.2 *Choose b as in Theorem 7.1. Then, for $t \in (0, b^{-1}]$, $\Phi \in H_2(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)^3$, the inequality $\|\nabla e^{Lt}(\Phi)\|_2 \le C(\vartheta_0) \cdot \|\nabla\Phi\|_2$ holds.*

Proof: Take t, Φ as in the theorem. Put $s_0 := 1/t$ if $t \le 1/C_2$, and $s_0 := C_2$ if $t > 1/C_2$. Then we have $s_0 \ge C_2$ in any case, so we may represent $e^{Lt}(\Phi)$ by the second sum in (7.2). Moreover, for $\lambda \in \Lambda_i^{(s_0)}$, $1 \le i \le 3$, we have $|\lambda| \ge s_0 \ge C_2 \ge C_1$, and in the case $\Re\lambda \le 0$ in addition $\lambda \in S_{\vartheta,a}$ (see (7.1)), hence

$$\|\nabla(\lambda \cdot \mathfrak{I} - L)^{-1}(\Phi)\|_2 \le C \cdot |\lambda|^{-1} \cdot \|\nabla\Phi\|_2 \tag{7.15}$$

by (6.15). In addition, we observe that $s_0 \cdot t = 1$ if $t \le 1/C_2$, and $s_0 \cdot t \le C_2 \cdot b^{-1}$ else. As a consequence, $s_0 \cdot t \le C$ in any case. Choosing $\psi_1(\varphi) := s_0 \cdot e^{i\cdot\varphi}$ ($\varphi \in [-\vartheta_0, \vartheta_0]$) as a representation of $\Lambda_1^{(s_0)}$, we get with (7.15):

$$\|e^{\psi_1(\varphi)\cdot t} \cdot \psi_1'(\varphi) \cdot \nabla\big(\psi_1(\varphi) \cdot \mathfrak{I} - L\big)^{-1}(\Phi)\|_2 \le e^{s_0\cdot\cos\varphi\cdot t} \cdot s_0 \cdot C \cdot |s_0 \cdot e^{i\cdot\varphi}|^{-1} \cdot \|\nabla\Phi\|_2 \le C \cdot \|\nabla\Phi\|_2,$$

where we used that $s_0 \cdot t \le C$, as remarked above. Moreover, introducing the local representation $\psi_2(r) := r \cdot e^{i\cdot\vartheta_0}$ ($r \in [s_0, \infty)$) of $\Lambda_2^{(s_0)}$, we find with (7.15),

$$\|e^{\psi_2(r)\cdot t} \cdot \psi_2'(r) \cdot \nabla\big(\psi_2(r) \cdot \mathfrak{I} - L\big)^{-1}(\Phi)\|_2 \le C(\vartheta_0) \cdot e^{r\cdot t\cdot\cos\vartheta_0} \cdot r^{-1} \cdot \|\nabla\Phi\|_2.$$

Furthermore, observing that $s_0 \cdot t \ge 1$,

$$\int_{s_0}^{\infty} e^{r\cdot t\cdot\cos\vartheta_0} \cdot r^{-1} \, dr = \int_{s_0\cdot t}^{\infty} e^{\alpha\cdot\cos\vartheta_0} \cdot \alpha^{-1} \, d\alpha \le \int_1^{\infty} e^{\alpha\cdot\cos\vartheta_0} \cdot \alpha^{-1} \, d\alpha \le C(\vartheta_0).$$

The same argument works for an analogous representation of $\Lambda_3^{(s_0)}$. Combining the preceding results, we get

$$\Big\|\nabla\Big(\int_{\Lambda_i^{(s_0)}} e^{\lambda\cdot t} \cdot (\lambda \cdot \mathfrak{I} - L)^{-1}(\Phi) \; d\lambda\Big)\Big\|_2 \le C(\vartheta_0) \cdot \|\nabla\Phi\|_2 \quad \text{for } i \in \{1,\,2,\,3\}.$$

This proves the theorem. $\diamond$

Theorem 7.3 *Let $\widetilde{a} \in \mathbb{R}$, $\sigma, R \in (0,\infty)$. Then there is a non-increasing function φ belonging to $L^1\big((0,\infty)\big) \cap L^2\big((0,\infty)\big)$, depending on τ, U, ϑ, a, ϑ_0, $\widetilde{a}$, σ and R, and verifying the inequality*

$$\|\nabla e^{Lt}(f)|B_R\|_2 \le \varphi(t) \cdot \|f\|_2 \ \ for \ \ t \in (0,\infty), \ f \in D(L) \ \ with \ \ \Delta f + \widetilde{a} \cdot \tau \cdot P\mathfrak{B}_{sym}(f) = \sigma \cdot f.$$

Proof: Again we abbreviate $b := \min\{\epsilon_4(1/16),\, 2^{-1/2}\}$. By Theorem 7.1, there is $\gamma_1 > 0$ depending on $\tau, U, R, \vartheta, a, \widetilde{a}, \sigma$ such that $\|\nabla e^{Lt}(f)|B_R\|_2 \le \gamma_1 \cdot t^{-9/8} \cdot \|f\|_2$ for $t \in [b^{-1},\infty)$ and for $f \in D(L)$ verifying the differential equation stated at the end of Theorem 7.3. Moreover Theorem 7.2 and 4.2 yield existence of a constant $\gamma_2 > 0$ depending on the same quantities and also on ϑ_0 such that $\|\nabla e^{Lt}(f)\|_2 \le \gamma_2 \cdot \|f\|_2$ for $t \in (0, b^{-1}]$ and for f as before. Thus, the function φ defined by $\varphi(t) := \gamma_1 \cdot t^{-9/8}$ for $t \in [b^{-1},\infty)$, $\varphi(t) := \gamma_2$ for $t \in (0, b^{-1})$, has the desired properties. $\diamond$

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