# EXPANSION OF EULER'S CONSTANT IN TERMS OF ZETA NUMBERS

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Abstract. Many formulas involving Euler's constant $\gamma$ have been proved by different authors. Some of these expressions concern the expansion of $\gamma$ in terms of the Zeta function evaluated at integers. In this paper, using Padé approximation we prove a general formula depending on three parameters which contains as particular cases Euler's, Euler-Stieltjes, Flajolet-Vardi and some new formulas.

## 1. Introduction and motivation

The Euler's constant $\gamma = 0.577216...$ is the limit, when $n$ tends to infinity, of the sequence

$$(H_n - \log(n+1))_n$$

where $H_s$ is defined by

$$\Re(s) > -1, H_s := \int_0^1 \frac{1-x^s}{1-x}\mathrm{d}x = \sum_{k=1}^n \frac{1}{k} \text{ if } s = n \in \mathbf{N}.$$

An integral representation for Euler's constant is

$$\gamma = \int_0^1 \left( \frac{1}{\ln u} + \frac{1}{1-u} \right) \mathrm{d}u. \tag{1.1}$$

In [6], Sondow considers the double integral (so-called Beukers' integral [2])

$$I_n = \int_0^1 \int_0^1 \frac{(x\,(1-x)\,y\,(1-y))^n}{(1-x\,y)\ln(\,x\,y)}\,\mathrm{d}x\,\mathrm{d}y. \tag{1.2}$$

Applying Taylor expansion of $1/(1-xy)$ around 0, he proved the following identity

$$I_n = \binom{2n}{n}\gamma + L_n - A_n = \mathcal{O}(2^{-4n}n^{-1/2}) \tag{1.3}$$

where

$$A_n = \sum_{i=0}^n \binom{n}{i}^2 H_{n+i}$$

and

$$L_n = 2\sum_{k=1}^n \sum_{i=0}^{k-1} \binom{n}{i}^2 (H_{n-i} - H_i)\ln(n+k)$$

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After multiplication by $d_{2n} := LCM(1, 2, \cdots, 2n)$, it arises

$$d_{2n}\begin{pmatrix} 2n \\ n \end{pmatrix}\gamma = d_{2n}(A_n - L_n) + o(1).$$

Since $d_{2n}A_n \in \mathbf{Z}$, he deduced the following condition for irrationality of $\gamma$:
*($\{x\}$ denotes the fractional part of the real number $x$)*

**Theorem 1.** [6] *If $(\{d_{2n}L_n\})_n$ does not converge to 0, then $\gamma$ is irrational.*

*Remark.* Other conditions of the same type are given in [10].

In [8], we proved that the sequence involved in the paper by Sondow can be recovered by means of Padé approximation.

Let us consider the function $(\ln u)/(u-1)$ and its Padé Approximant $[n-1/n]$ of degree $(n-1/n)$ at the point $u = 1$,

$$\frac{\ln u}{u-1} = \frac{N_n(u)}{D_n(u)} + R_n(u) \tag{1.4}$$

where $N_n$ and $D_n$ are polynomials of degree $n-1$ and $n$, respectively, normalized by

$$N_n(1) = D_n(1) = 1, R_n(u) = \mathcal{O}(u^{2n}).$$

From the theory of Padé approximation, it is well known that $D_n$ is related with the shifted Legendre Polynomial $P_n^*$ orthogonal on the interval [0,1] with respect to the Lebesgue weight function. Some of these expressions are

$$P_n^*(t) = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix}^2 t^{n-k}(t-1)^k \tag{1.5}$$

$$= \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix}\begin{pmatrix} n+k \\ k \end{pmatrix}(-1)^{n+k}t^k. \tag{1.6}$$

The denominator of the Padé approximant $D_n$ has the following expression in terms of $P_n^*$

$$D_n(u) = P_n^*\left(\frac{1}{1-u}\right)(1-u)^n \begin{pmatrix} 2n \\ n \end{pmatrix}^{-1}. \tag{1.7}$$

Replacing $P_n^*$ by its expressions (1.5,1.6), formula (1.7) becomes

$$\begin{aligned} D_n(u) &= \begin{pmatrix} 2n \\ n \end{pmatrix}^{-1} \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix}^2 u^k \\ &= \begin{pmatrix} 2n \\ n \end{pmatrix}^{-1} \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix}\begin{pmatrix} n+k \\ k \end{pmatrix}(1-u)^{n-k}u^k. \end{aligned}$$

The numerator $N_n(u)$ of $[n-1/n]$ is related with the associated polynomial of the denominator:

$$\begin{aligned} N_n(u) &= 2\binom{2n}{n}^{-1}\sum_{k=1}^{n}\sum_{i=0}^{k-1}\binom{n}{i}^2(H_{n-i}-H_i)u^{k-1} \\ &= \binom{2n}{n}^{-1}\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\sum_{i=0}^{k-1}(u-1)^{n-k+i}\frac{(-1)^i}{i+1}. \end{aligned}$$

Formula (1.4) can be rewritten as

$$\frac{N_n(u)}{\ln u}+\frac{D_n(u)}{1-u}=-\frac{R_n(u)D_n(u)}{\ln u}$$

and so, from (1.1)

$$\gamma=\int_0^1\frac{1-u^nN_n(u)}{\ln u}\,\mathrm{d}u+\int_0^1\frac{1-u^nD_n(u)}{1-u}\,\mathrm{d}u-\int_0^1\frac{u^nR_n(u)D_n(u)}{\ln u}\,\mathrm{d}u.$$

For the first integral one gets:

$$\int_0^1\frac{1-u^nN_n(u)}{\ln(u)}\mathrm{d}u=-\binom{2n}{n}^{-1}L_n. \tag{1.8}$$

The second term can be expanded as

$$\begin{aligned} \int_0^1\frac{1-u^nD_n(u)}{1-u}\,\mathrm{d}u &= \binom{2n}{n}^{-1}\sum_{k=0}^{n}\binom{n}{k}^2H_{n+k} && (1.9) \\ &= \binom{2n}{n}^{-1}A_n. && (1.10) \end{aligned}$$

From the theory of Padé approximation, in formula (1.4), the remainder term $R_n$ has an integral representation

$$R_n(u)=\frac{(1-u)^n}{D_n(u)}\int_0^1\frac{t^nD_n(1-1/t)}{1-(1-u)t}\,\mathrm{d}t.$$

Thanks to formulas (1.8,1.10), $\gamma$ satisfies

$$\gamma = \frac{A_n-L_n}{\binom{2n}{n}}-\int_0^1\frac{u^nR_n(u)D_n(u)}{\ln(u)}\,\mathrm{d}u.$$

Thus another expression of the remainder term $I_n$ of Sondow (1.2) is

$$\begin{aligned} I_n &= \binom{2n}{n}\gamma - A_n + L_n = -\int_0^1 \frac{u^n R_n(u) D_n(u)}{\ln(u)}\,\mathrm{d}u \\ &= \int_0^1\int_0^1 \frac{u^n(1-u)^n}{\ln(u)}\frac{P_n^*(t)}{1-(1-u)t}\,\mathrm{d}t\,\mathrm{d}u \\ &= \int_0^1\int_0^1 \frac{u^n(1-u)^{2n}}{\ln(u)}\frac{t^n(1-t)^n}{(1-(1-u)t)^{n+1}}\,\mathrm{d}t\,\mathrm{d}u \end{aligned}$$

thanks to integration by parts and Rodrigues formula for orthogonal polynomials:

$$P_n^*(t) = \frac{(-1)^n}{n!}\frac{d}{dt}(t^n(1-t)^n).$$

Thus the approximation for Euler's constant $\gamma$ given by Sondow (1.3) can be recovered by means of the Padé approximation to the function $(\ln u)/(1-u)$.

In the same manner, Pilehrood [7] found irrationality criteria for the generalized Euler's constant. They defined the following linear form in logarithms

$$\begin{aligned} L_{(n_1,n_2)}(\alpha) &= \sum_{m=1}^{n_1}\sum_{k=0}^{m-1}\binom{n_1}{k}\binom{n_2}{k}(H_{n_1-k}+H_{n_2-k}-2H_k)\ln(m+n_1+\alpha-1) + \\ &\sum_{m=n_1+1}^{n_2}\sum_{k=m}^{n_2}(-1)^{k-1-n_1}/k\binom{n_2}{k}/\binom{k-1}{n_1}(\ln(m+n_1+\alpha-1). \end{aligned}$$

Actually, following the same idea as for Sondow's formula, it is possible to prove that Pilehrood's criterion also comes from Padé approximations $[n_2-1,n_1] = \dfrac{R_{n_2-1}(u)}{S_{n_1}(u)}$ (normalized by $R_{n_2-1}(1) = S_{n_1}(1) = 1$) to the function $\ln(u)/(u-1)$ at the point $u=1$. The linear form $L_{(n_1,n_2)}(\alpha)$ satisfies

$$L_{(n_1,n_2)}(\alpha) = \int_0^1 \left(1 - u^{n_1+\alpha-1}R_{n_2-1}(u)\right)\frac{1}{\ln(u)}\mathrm{d}u.$$

In the present paper, we use the same method for another expression of $\gamma$. With a change of variable in (1.1), another formula is the following

$$\gamma = \int_0^\infty \left(\frac{1}{1-e^{-t}} - \frac{1}{t}\right)e^{-t}dt. \tag{1.11}$$

The idea is to replace the exponential function by some suitable Padé approximations.

## 2. Results

In this section, we prove a general formula for the Euler's constant which depends on three parameters: $n$ is the degree of the numerator and $m$ the degree of the denominator of the Padé approximant used to approximate the function $e^{-t}$ in (1.11) and $s$ is an arbitrary complex number. In the following sections we will consider particular values of the parameters $n, m$ and $s$ to recover most of known formulas.

Notation: $\zeta(j,q) := \sum_{k=0}^{\infty} \frac{1}{(k+q)^j} = \frac{1}{\Gamma(j)} \int_0^{\infty} \frac{e^{-q\,t}}{1-e^{-t}} t^{j-1} dt$ is the Hurwitz Zeta function.

**Theorem 2.**
*If $s$ is some complex number with $\Re(s) > -1$, then for all integers $m, n$ such that $m+n \neq 0$, the following expression*

$$\gamma = H_s - \log(s+1) + \sum_{j=1}^{m} \frac{(-m)_j}{(-m-n)_j} \frac{1}{j} \frac{1}{(s+1)^j} - \sum_{j=2}^{\max(m,n)} \frac{(-m)_j - (-1)^j(-n)_j}{(-m-n)_j} \frac{1}{j} \zeta(j, s+1) + \varepsilon_{n,m}^{(s)},$$

*holds, where*

$$\varepsilon_{n,m}^{(s)} = (-1)^{n+1} \int_0^1 (1-x)^m x^n \zeta(m+n+1, s+2-x) \mathrm{d}x, \tag{2.1}$$

*satisfies*

$$\left|\varepsilon_{n,m}^{(s)}\right| \leq \frac{1}{(m+n+1)\begin{pmatrix} m+n \\ n \end{pmatrix}} \zeta(m+n+1, \Re(s)+1). \tag{2.2}$$

*If $s$ is real, $s > -1$, then,*

$$0 < (-1)^{n+1} \varepsilon_{n,m}^{(s)} \leq \frac{1}{(m+n+1)\begin{pmatrix} m+n \\ n \end{pmatrix}} \zeta(m+n+1, s+1).$$

*Proof.* In formula (1.11) we replace the exponential function by its Padé approximation

$$[n/m]_{e^{-t}} := \frac{P_n(t)}{Q_m(t)}$$

defined by the unique rational fraction of degree $(n/m)$ satisfying

$$e^{-t} = \frac{P_n(t)}{Q_m(t)} + R_{n,m}(t) \tag{2.3}$$

with $R_{n,m}(t) = \mathcal{O}(t^{n+m+1}), t \to 0$.

The expression of this fraction is well known (see for example [1] p.13):

$$P_n(t) = {}_1F_1(-n, -n-m, -t) := \sum_{k=0}^{n} \frac{(-n)_k}{(-m-n)_k} \frac{(-t)^k}{k!} \tag{2.4}$$

and

$$Q_m(t) = {}_1F_1(-m,-n-m,t) := \sum_{k=0}^{m} \frac{(-m)_k}{(-m-n)_k}\frac{t^k}{k!} \tag{2.5}$$

are polynomials of degree resp. $n$ and $m$, where $(a)_k$ is the Pochammer symbol defined by $(a)_k := a(a+1)(a+2)\dots(a+k-1)$.

The formula (2.3) leads to

$$\frac{Q_m(t)}{t} = \frac{t^{-1}(Q_m(t)-P_n(t))}{1-e^{-t}} - \frac{R_{n,m}(t)Q_m(t)}{t(1-e^{-t})}. \tag{2.6}$$

Using (2.4) and (2.5),

$$\begin{aligned} t^{-1}(Q_m(t)-P_n(t)) &= t^{-1}\left(\sum_{j=0}^{m}\frac{(-m)_j}{(-m-n)_j}\frac{t^j}{j!} - \sum_{j=0}^{n}\frac{(-n)_j}{(-m-n)_j}\frac{(-t)^j}{j!}\right) \\ &= \sum_{j=1}^{\max(m,n)} \frac{(-m)_j-(-1)^j(-n)_j}{(-m-n)_j}\frac{t^{j-1}}{j!}. \end{aligned}$$

To work with a general situation, we can multiply the identity (2.6) by some arbitrary function $f$. In order to obtain some further simplification, we choose $f(t) := e^{-st}$, with $s$ satisfying $\Re(s) > -1$ :

$$\frac{e^{-st}Q_m(t)}{t} = \frac{t^{-1}(Q_m(t)-P_n(t))e^{-st}}{1-e^{-t}} - \frac{R_{n,m}(t)Q_m(t)e^{-st}}{t(1-e^{-t})}.$$

So,

$$\begin{aligned} \gamma &= \int_0^\infty \left(\frac{1}{1-e^{-t}} - \frac{1}{t}\right)e^{-t}dt \\ &= \int_0^\infty \left(\frac{1-t^{-1}(Q_m(t)-P_n(t))e^{-st}}{1-e^{-t}} - \frac{1-Q_m(t)e^{-st}}{t} + \frac{R_{n,m}(t)Q_m(t)e^{-st}}{t(1-e^{-t})}\right)e^{-t}dt. \end{aligned}$$

Now we have to evaluate each term of the above formula.

Using the following two identities,

$$\int_0^\infty \frac{t^{j-1}e^{-st}}{1-e^{-t}}e^{-t}dt = (j-1)!\zeta(j,s+1), j \geq 2, \Re(s) > -1$$

and

$$\int_0^\infty t^{j-1}e^{-st}e^{-t}dt = \frac{(j-1)!}{(s+1)^j}, j \geq 1,$$

we get

$$\begin{aligned}\int_0^\infty \frac{1-t^{-1}(Q_m(t)-P_n(t))e^{-s\,t}}{1-e^{-t}}e^{-t}dt &= \int_0^\infty \frac{1-\ e^{-s\,t}\sum_{j=1}^{\max(m,n)}\frac{(-m)_j-(-1)^j(-n)_j}{(-m-n)_j}\frac{t^{j-1}}{j!}}{1-e^{-t}}e^{-t}dt \\ &= \int_0^\infty \frac{1-\ e^{-s\,t}}{1-e^{-t}}e^{-t}dt - \\ &\quad \sum_{j=2}^{\max(m,n)} \frac{(-m)_j-(-1)^j(-n)_j}{j!(-m-n)_j}\int_0^\infty \frac{e^{-s\,t}t^{j-1}}{1-e^{-t}}e^{-t}dt \\ &= H_s - \sum_{j=2}^{\max(m,n)} \frac{(-m)_j-(-1)^j(-n)_j}{(-m-n)_j}\frac{1}{j}\zeta(j,s+1)\end{aligned}$$

and

$$\begin{aligned}\int_0^\infty \frac{1-Q_m(t)e^{-s\,t}}{t}e^{-t}dt &= \int_0^\infty \frac{1-e^{-s\,t}\sum_{j=0}^{m}\frac{(-m)_j}{(-m-n)_j}\frac{t^j}{j!}}{t}e^{-t}dt \\ &= \int_0^\infty \frac{1-e^{-s\,t}}{t}e^{-t}dt - \sum_{j=1}^{m}\frac{(-m)_j}{j!(-m-n)_j}\int_0^\infty e^{-s\,t}t^{j-1}e^{-t}dt \\ &= \log(s+1) - \sum_{j=1}^{m}\frac{(-m)_j}{(-m-n)_j}\frac{1}{j}\frac{1}{(s+1)^j}.\end{aligned}$$

So, for all integers $m$ and $n$, the following expression

$$\gamma = H_s - \log(s+1) + \sum_{j=1}^{m}\frac{(-m)_j}{(-m-n)_j}\frac{1}{j}\frac{1}{(s+1)^j} - \sum_{j=2}^{\max(m,n)}\frac{(-m)_j-(-1)^j(-n)_j}{(-m-n)_j}\frac{1}{j}\zeta(j,s+1) + \varepsilon_{n,m}^{(s)},$$

holds, where the error term $\varepsilon_{n,m}^{(s)}$ satisfies

$$\varepsilon_{n,m}^{(s)} = \int_0^\infty \frac{R_{n,m}(t)Q_m(t)e^{-s\,t}}{t(1-e^{-t})}e^{-t}dt.$$

In formula (2.3), the error term $R_{n,m}$ for the Padé approximant to $e^{-t}$ is

$$R_{n,m}(t) = \frac{(-t)^{m+n+1}}{(m+n)!}\frac{e^{-t}}{Q_m(t)}\int_0^1 (x-1)^m x^n e^{xt}dx.$$

Thus

$$\begin{aligned}
\varepsilon_{n,m}^{(s)} &= \int_0^\infty \frac{(-t)^{m+n+1}}{(m+n)!}\frac{e^{-t}}{Q_m(t)}\frac{Q_m(t)e^{-s\,t}e^{-t}}{t(1-e^{-t})}dt\int_0^1 (x-1)^m x^n e^{xt}dx \\
&= (-1)^{n+1}\int_0^\infty \frac{t^{m+n}}{(m+n)!}\frac{e^{-s\,t}e^{-2t}}{1-e^{-t}}dt\int_0^1 (1-x)^m x^n e^{xt}dx \\
&= (-1)^{n+1}\int_0^1 (1-x)^m x^n \zeta(m+n+1, s+2-x)dx
\end{aligned}$$

(the function of the two variables $(t,x)$ to be integrated is bounded by $g(t) := t^{m+n}e^{(-s-1)t}$ on $(t,x) \in [0,\infty[\times[0,1])$.

So

$$\begin{aligned}
\left|\varepsilon_{n,m}^{(s)}\right| &= \left|\int_0^1 (1-x)^m x^n \zeta(m+n+1, -x+s+2)dx\right| \\
&\leq \frac{1}{(m+n+1)\begin{pmatrix} m+n \\ n \end{pmatrix}}\zeta(m+n+1, \Re(s)+1).
\end{aligned}$$

Thus,

$$\forall s \in \mathbf{C}, \Re(s) > -1, \lim_n \varepsilon_{n,m}^{(s)} = \lim_m \varepsilon_{n,m}^{(s)} = 0.$$

□

## 3. Particular cases

In this section we will consider all the possible values for the three parameters $s, m, n$, $s \in \mathbf{C}, \Re(s) > -1, m \in \mathbf{N}, n \in \mathbf{N}$.

### 3.1. $s = 0, m \in \mathbf{N}, n \in \mathbf{N}, m+n \neq 0$.

In that particular case, Theorem 1 reduces to

$$\gamma = \sum_{j=1}^{m} \frac{(-m)_j}{(-m-n)_j}\frac{1}{j} - \sum_{j=2}^{\max(m,n)} \frac{(-m)_j - (-1)^j(-n)_j}{(-m-n)_j}\frac{\zeta(j)}{j} + \varepsilon_{n,m}^{(0)} \tag{3.1}$$

with

$$0 < (-1)^{n+1}\varepsilon_{n,m}^{(0)} \leq \frac{1}{(m+n+1)\begin{pmatrix} m+n \\ n \end{pmatrix}}\zeta(m+n+1),$$

where $\zeta(q) := \sum_{p=1}^{\infty} \frac{1}{p^q} = \zeta(q,1)$ is the $\zeta$-function.

3.2. $\Re(s) > -1, m \in \mathbf{N}^*, n = 0$**.**

After simplification, we get

$$\gamma = H_s - \log(s+1) - \sum_{j=2}^{m} \frac{1}{j}\zeta(j, s+2) + \varepsilon_{0,m}^{(s)},$$

with $\left|\varepsilon_{0,m}^{(s)}\right| \leq \frac{1}{(m+1)}\zeta(m+1, \Re(s)+1)$

**Limit case:** $m \to \infty$

Since $\lim_{m\to\infty} \frac{1}{(m+1)}\zeta(m+1, \Re(s)+1) = 0$, we get the following series for $\gamma$:

$$\gamma = H_s - \log(s+1) - \sum_{j=2}^{\infty} \frac{1}{j}\zeta(j, s+2) \tag{3.2}$$

and for $s = 0$,

$$\gamma = 1 - \sum_{j=2}^{\infty} \frac{\zeta(j) - 1}{j} \text{ (Euler)}. \tag{3.3}$$

This formula combined with $\sum_{j=2}^{\infty} (\zeta(j) - 1) = 1$ gives

$$\gamma = \sum_{j=2}^{\infty} \frac{(j-1)(\zeta(j) - 1)}{j} \text{ (Euler)}. \tag{3.4}$$

*Remark.* Formula (3.2) can be proved directly from the definition of Euler's constant.

3.3. $\Re(s) > -1, m = 0, n \in \mathbf{N}^*$**.**

$$\gamma = H_s - \log(s+1) + \sum_{j=2}^{n} \frac{(-1)^j}{j}\zeta(j, s+1) + \varepsilon_{n,0}^{(s)}, \tag{3.5}$$

where $\left|\varepsilon_{n,0}^{(s)}\right| \leq \frac{1}{n+1}\zeta(n+1, \Re(s)+1)$.

**Limit case:** $n \to \infty$

$$\gamma = H_s - \log(s+1) + \sum_{j=2}^{\infty} \frac{(-1)^j}{j}\zeta(j, s+1). \tag{3.6}$$

which for $s = 0$ is due to Euler.

This formula combined with $\sum_{j=2}^{\infty}(-1)^j\zeta(j,s+1)=\dfrac{1}{s+1}$ gives

$$\gamma = H_s - \log(s+1) - \sum_{j=2}^{\infty}\frac{(-1)^j\,(j-1)\,(\zeta(j)-1)}{j}$$

which for $s=1$ is due to Flajolet-Vardi.

## 4. Case $m=\alpha n, m,n\in\mathbf{N}^*, s=k\in\mathbf{N}$

In the case $m=\alpha n$, it is possible to prove an integral formula for the error (2.1) . This expression implies that the sequence of the error is totally oscillating and also gives its speed of convergence to zero.

**Definition 1.**
*1) A sequence $(u_n)$ is called totally monotonic (TM) if there exists a non negative measure $d\mu$ with infinitely many points of increase such that*

$$\forall n\in\mathbf{N},\ u_n=\int_0^{\infty}x^n\ \mathrm{d}\mu(x).$$

*If the support of the measure $d\mu$ is the interval $[0,1/R]$, then $\forall n, u_{n+1}/u_n\le R$ and $\lim_n \dfrac{u_{n+1}}{u_n}=R$.*

*If $R=1$, it is equivalent to*

$$\forall n\in\mathbf{N},\forall k\in\mathbf{N},(-1)^k\Delta^k(u_n)>0$$

*where $\Delta^0(u_n):=u_n$ and $\Delta^{k+1}u_n=\Delta^k u_{n+1}-\Delta^k u_n$. (see [12], p. 108).*

*2)If $(-1)^n u_n$ is TM, then $(u_n)_n$ is called totally oscillating.*

*Remark: If a sequence is TM or TO, it is very easy to accelerate its convergence ([3], [9]) with for example the epsilon-algorithm or the modified moment method.*

**Theorem 3.** *For each integer $k\ge 0$, the sequence $(u_n^{(k)})_n$, defined by*

$$u_n^{(k)}:=(-1)^n\varepsilon_{n+1,\alpha(n+1)}^{(k)}$$

*is a TM sequence with an integral representation on the interval $[0,M_{k+1,\alpha}]$ where $M_{k+1,\alpha}=\frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{1}{(k+1)(k+2)^\alpha}$ for $\alpha>0$ and $M_{k+1,0}=\frac{1}{k+1}$.*

*Proof.* With a change of variable, another expression of the error (2.1) is the following:

$$\begin{aligned}\varepsilon_{n,\alpha n}^{(k)} &= (-1)^{n+1}\sum_{j=k+1}^{\infty}\int_0^1\left(\frac{X}{j+X}\right)^{\alpha n}\left(\frac{1-X}{j+X}\right)^n\frac{1}{j+X}dX\\ &= (-1)^{n+1}\sum_{j=k+1}^{\infty}\int_0^1\left(\frac{X^\alpha}{(j+X)^\alpha}\frac{1-X}{j+X}\right)^n\frac{1}{j+X}dX.\end{aligned}$$

Let us define

$$\Psi_j(X) := \frac{X^\alpha}{(j+X)^\alpha}\frac{1-X}{j+X}$$

and

$$M_{j,\alpha} := \max_{0\le X\le 1} \Psi_j(X).$$

The maximum of $\Psi_j$ is attained at $X = \dfrac{j\alpha}{j(\alpha+1)+1}$.

So

$$\begin{aligned} \text{for } \alpha &> 0,\ M_{j,\alpha} = \Psi_j\left(\frac{j\alpha}{j(\alpha+1)+1}\right) = \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{1}{j(j+1)^\alpha}, \\ \text{for } \alpha &= 0,\ M_{j,0} = \Psi_j(0) = \frac{1}{j}. \end{aligned}$$

Thus

$$\begin{aligned} \text{for } \alpha &> 0,\ 0 < (-1)^{n+1}\varepsilon^{(k)}_{n,\alpha n} \le \left(\frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\right)^n \zeta(n+1+\alpha n, k+1), \\ \text{for } \alpha &= 0,\ 0 < (-1)^{n+1}\varepsilon^{(k)}_{n,0} \le \int_0^1 (1-X)^n \sum_{j=s+1}^{\infty} \frac{1}{(j+X)^{n+1}} dX \le \frac{1}{(n+1)}\zeta(n+1,k+1). \end{aligned}$$

The function $\Psi_j$ applies $[0,1]$ on $[0, M_{j,\alpha}]$.

$$u = \Psi_j(X) \in [0, M_{j,\alpha}] \longrightarrow \Phi^{(\alpha)}_{j,1}(u) \le X \le \Phi^{(\alpha)}_{j,2}(u).$$

*For example, if* $\alpha = 1, \Phi^{(\alpha)}_{j,1}(u) = \dfrac{1-2ju-\sqrt{1-4ju-4uj^2}}{2(1+u)}, \Phi^{(\alpha)}_{j,2}(u) = \dfrac{1-2ju+\sqrt{1-4ju-4uj^2}}{2(1+u)}$.

$$\begin{aligned} (-1)^{n+1}\varepsilon^{(k)}_{n,\alpha n} &= \sum_{j=k+1}^{\infty} \int_0^{M_{j,\alpha}} u^n \left(\frac{d\Phi^{(\alpha)}_{j,1}(u)}{j+\Phi^{(\alpha)}_{j,1}(u)} - \frac{d\Phi^{(\alpha)}_{j,2}(u)}{j+\Phi^{(\alpha)}_{j,2}(u)}\right) \\ &= \int_0^1 u^n \sum_{j=k+1}^{\infty} Id_{[0,M_{j,\alpha}]}\varphi_{j,\alpha}(u)du. \end{aligned}$$

The functions $\Phi_{j,1}$ and $\Phi_{j,2}$ are increasing and decreasing, respectively, so all the functions $\varphi_{j,\alpha}$ are positive on $[0,1]$, and thus the sequence $\left((-1)^{n+1}\varepsilon^{(k)}_{n,\alpha n}\right)_{n\ge 1}$ is a totally monotonic sequence with an integral representation on the interval $[0, M_{k+1,\alpha}]$ and thus converges to 0 as $\left(\frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{1}{(k+1)(k+2)^\alpha}\right)^n$ for $\alpha > 0$ and as $1/(k+1)^n$ if $\alpha = 0$.

□

## 5. Limit case $s \in \mathbf{C}, \Re(s) > -1, m = \alpha n, n \to \infty$

Now, we consider the case where $m = \alpha n$ when $n$ tends to infinity. New formulas are proved. Of course, the general expression of Theorem 1 for $\gamma$ is true and we can derive the following series

$$\begin{aligned}\gamma &= H_s - \log(s+1) + \sum_{j=1}^{\infty} \frac{\alpha^j}{(\alpha+1)^j} \frac{1}{j} \frac{1}{(s+1)^j} - \sum_{j=2}^{\infty} \frac{\alpha^j - (-1)^j}{(\alpha+1)^j} \frac{1}{j} \zeta(j, s+1) \quad (5.1)\\ &= H_s - \log\left(s + \frac{1}{1+\alpha}\right) - \sum_{j=2}^{\infty} \frac{\alpha^j - (-1)^j}{(\alpha+1)^j} \frac{1}{j} \zeta(j, s+1). \quad (5.2)\end{aligned}$$

In the following, we recover formulas of Euler and Euler-Stieltjes.

(1) If we set $s = 0$, then

$$\gamma = \log(1+\alpha) - \sum_{j=2}^{\infty} \frac{\alpha^j - (-1)^j}{(\alpha+1)^j} \frac{1}{j} \zeta(j) \quad (5.3)$$

which are the Euler's formulas for $\alpha = 0$ or 1.

$$\text{For } \alpha = 0, \gamma = \sum_{j=2}^{\infty} \frac{(-1)^{\mathrm{j}}}{\mathrm{j}} \zeta(\mathrm{j}) \text{ (Euler)}. \quad (5.4)$$

$$For\ \alpha = 1, \gamma = \log 2 - \sum_{j=2}^{\infty} \frac{1 - (-1)^j}{2^j} \frac{1}{j} \zeta(j) = \log 2 - \sum_{p=1}^{\infty} \frac{\zeta(2p+1)}{(2p+1)2^{2p}} \text{ (Euler)}. \quad (5.5)$$

(2) If we set $s = 1$, then

$$\gamma = 1 - \log\left(1 + \frac{1}{1+\alpha}\right) - \sum_{j=2}^{\infty} \frac{\alpha^j - (-1)^j}{(\alpha+1)^j} \frac{1}{j} \left(\zeta(j) - 1\right). \quad (5.6)$$

For $\alpha = 1$, formula (5.6) gives

$$\gamma = 1 - \log\left(\frac{3}{2}\right) - \sum_{p=1}^{\infty} \frac{\zeta(2p+1) - 1}{4^p(2p+1)} \text{ (Euler} - \text{Stieltjes)}. \quad (5.7)$$

(3) Now, let us consider $s = 1/2$ and $\alpha = 1$. Then,

$$\gamma = 2 - \log(4) - \sum_{p=1}^{\infty} \frac{\zeta(2p+1, 3/2)}{4^p(2p+1)}. \quad (5.8)$$

If we add this equality with (5.5) and divide by 2, then

$$\begin{aligned}\gamma &= 1-\frac{\log(2)}{2}-\frac{1}{2}\sum_{p=1}^{\infty}\frac{\zeta(2p+1,3/2)+\zeta(2p+1)}{4^p(2p+1)} &(5.9)\\ &= 1-\frac{\log(2)}{2}-\frac{1}{2}\sum_{p=1}^{\infty}\frac{2^{2p+1}\left(\zeta(2p+1)-1\right)}{4^p(2p+1)} &(5.10)\\ &= 1-\frac{\log(2)}{2}-\sum_{p=1}^{\infty}\frac{\zeta(2p+1)-1}{2p+1}. &(5.11)\end{aligned}$$

## 6. Case $s = k \in \mathbf{N}, m = n \in \mathbf{N}^*$

In that case, the expression of $\gamma$ given in Theorem 2 only contains values of the Zeta function at odd integers:

$$\gamma = H_k-\log(k+1)+\sum_{j=1}^{m}\frac{(-m)_j}{(-2m)_j}\frac{1}{j}\frac{1}{(k+1)^j}-2\sum_{p=1}^{m}\frac{(-m)_{2p+1}}{(-2m)_{2p+1}}\frac{1}{2p+1}\zeta(2p+1,k+1)+\varepsilon^{(k)}_{m,m}, \tag{6.1}$$

where

$$(-1)^{m+1}\varepsilon^{(k)}_{m,m} = \int_0^1 u^m\sum_{j=k+1}^{\infty}Id_{[0,\frac{1}{4j(j+1)}]}\varphi_\alpha(u)du = O\left(\frac{1}{4(k+1)(k+2)}\right)^m \tag{6.2}$$

(1) $k = 0, m \in \mathbf{N}^*$.

$$\begin{aligned}\gamma &= \sum_{j=1}^{m}\frac{(-m)_j}{(-2m)_j}\frac{1}{j}-2\sum_{p=1}^{m}\frac{(-m)_{2p+1}}{(-2m)_{2p+1}}\frac{1}{2p+1}\zeta(2p+1)+\varepsilon^{(0)}_{m,m},\\ \text{where}(-1)^{\mathrm{m}+1}\varepsilon^{(0)}_{\mathrm{m,m}} &= \int_0^1 u^m\sum_{j=1}^{\infty}Id_{[0,\frac{1}{4j(j+1)}]}\varphi_{j,\alpha}(u)du = O(8^{-m}).\end{aligned}$$

If $m$ tends to infinity,

$$\gamma = \log 2-\sum_{p=1}^{\infty}\frac{1}{4^p}\frac{1}{2p+1}\zeta(2p+1)(\text{Euler}).$$

(2) $k = 1, m \in \mathbf{N}^*$.

$$\gamma = 1-\log 2+\sum_{j=1}^{m}\frac{(-m)_j}{(-2m)_j}\frac{1}{j}\frac{1}{2^j}-2\sum_{p=1}^{m}\frac{(-m)_{2p+1}}{(-2m)_{2p+1}}\frac{1}{2p+1}\zeta(2p+1,2)+\varepsilon^{(1)}_{m,m},$$

where

$$(-1)^{m+1}\varepsilon^{(1)}_{m,m} = \int_0^1 u^m\sum_{j=2}^{\infty}Id_{[0,\frac{1}{4j(j+1)}]}\varphi_{j,\alpha}(u)du = O(24^{-m}).$$

If $m$ tends to infinity,

$$\begin{aligned}\gamma &= 1-\log 2+\sum_{j=1}^{\infty}\frac{1}{2^j}\frac{1}{j}\frac{1}{2^j}-2\sum_{p=1}^{\infty}\frac{1}{2^{2p+1}}\frac{1}{2p+1}\zeta(2p+1,2)\\ &= 1-\log\left(\frac{3}{2}\right)-\sum_{p=1}^{\infty}\frac{1}{4^p}\frac{\zeta(2p+1)-1}{2p+1}(\mathrm{Euler-Stieltjes}).\end{aligned}$$

6.1. **An arithmetical property.**
In this section, we restrict ourselves to the case $m=n$ and $s=k\in\mathbf{N}$.

**Theorem 4.** *For each integer $k$, there exist sequences of integers $(a_m^{(k)})_m, (b_m^{(k)})_m, (c_{p,m}^{(k)})_{p,m}$ such that*

$$\lim_{m\to\infty}\left(a_m^{(k)}\ \gamma-b_m^{(k)}-a_m^{(k)}\ \log(k+1)-\sum_{p=1}^{m}c_{p,m}^{(k)}\zeta(2p+1,k+1)\right)=0. \tag{6.3}$$

*If $k=0$, then*

$$\lim_{m\to\infty}\left(a_m^{(0)}\ \gamma-b_m^{(0)}-\sum_{p=1}^{m}c_{p,m}^{(0)}\zeta(2p+1)\right)=0. \tag{6.4}$$

*Proof.* Let us define

$$E_m:=\sum_{j=1}^{m}\frac{(-m)_j}{(-2m)_j}\frac{1}{j}.$$

Using partial fraction decomposition of a rational fraction, we get the identity

$$\frac{(-m)_j}{(-2m)_j}\frac{1}{j}=\sum_{p=0}^{j-1}\frac{(-1)^{p-1+j}}{2m-p}\left(\begin{array}{c}j-1\\p\end{array}\right)\left(\begin{array}{c}m\\j\end{array}\right).$$

Thus

$$LCM(m+1,\cdots,2m)*E_m\in\mathbf{Z}.$$

If we set

$$\begin{aligned}a_m^{(k)} &:= LCM(m+1,\cdots,2m)(k+1)^m*LCM(1,\cdots,k),\mathrm{then}\\ b_m^{(k)} &:= a_m^{(k)}\left(H_k\ +\sum_{j=1}^{m}\frac{(-m)_j}{(-2m)_j}\frac{1}{j}\frac{1}{(k+1)^j}\right)\in\mathbf{Z}\ \mathrm{and}\\ c_{p,m}^{(k)} &:= a_m^{(k)}\ \left(2\frac{(-m)_{2p+1}}{(-2m)_{2p+1}}\frac{1}{2p+1}\right)\in\mathbf{Z},\forall p,1\le p\le m.\end{aligned}$$

then, from (6.1),

$$\begin{aligned} a_m^{(k)}\,\varepsilon_{m,m}^{(k)} &= a_m^{(k)}\left(\gamma - H_k + \log(k+1) - \sum_{j=1}^{m}\frac{(-m)_j}{(-2m)_j}\frac{1}{j}\frac{1}{(k+1)^j} + \right. \\ &\qquad \left. 2\sum_{p=1}^{m}\frac{(-m)_{2p+1}}{(-2m)_{2p+1}}\frac{1}{2p+1}\zeta(2p+1,k+1)\right) \\ &= \left(a_m^{(k)}\,\gamma - b_m^{(k)} - a_m^{(k)}\,\log(k+1) - \sum_{p=1}^{m} c_{p,m}^{(k)}\zeta(2p+1,k+1)\right). \end{aligned}$$

From analytic number theory ([11]), the asymptotics for $a_m^{(k)}$ satisfies

$$\lim_m \left(a_m^{(k)}\right)^{1/m} = e^2\,(k+1).$$

From (6.2), the error $\varepsilon_{m,m}^{(k)}$ satisfies

$$\lim_m \left(\varepsilon_{m,m}^{(k)}\right)^{1/m} = \frac{1}{4(k+1)(k+2)}.$$

Thus for all integer $k$,

$$\lim_m \left(a_m^{(k)}\varepsilon_{m,m}^{(k)}\right)^{1/m} = \frac{e^2}{4(k+2)} < 1,$$

and the theorem is proved.

□

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