Padé Approximation and Apostol-Bernoulli and -Euler Polynomials

Marc Prévost [1,2,3]

[1] *Univ Lille Nord de France, F-59000 Lille, France*

[2] *ULCO, LMPA J. Liouville, B.P. 699, F-62228 Calais*

[3] *CNRS, FR 2956*

Abstract

Using the Padé approximation of the exponential function, we obtain recurrence relations between Apostol-Bernoulli and between Apostol-Euler polynomials. As applications, we derive some new lacunary recurrence relations for Bernoulli and Euler polynomials with gap of length 4 and lacunary relations for Bernoulli and Euler numbers with gap of length 6.

Key words: Apostol-Bernoulli polynomials, Apostol-Euler polynomials, Bernoulli numbers, Euler numbers, Padé approximants
2000 MSC: 11B68, 41A21

---

1. Introduction

Classical Bernoulli and Euler polynomials play a fundamental role in various branches of mathematics including combinatorics, number theory and special functions. They are usually defined by means of the following generating functions

$$\frac{t}{e^t-1}e^{x\,t} = \sum_{k=0}^{\infty} B_k(x)\,\frac{t^k}{k!}, \qquad (|t|<2\pi), \tag{1.1}$$

$$\frac{2}{e^t+1}e^{x\,t} = \sum_{k=0}^{\infty} E_k(x)\,\frac{t^k}{k!}, \qquad (|t|<\pi). \tag{1.2}$$

The first polynomials are

$$\begin{aligned} B_0(x) &= 1, & E_0(x) &= 1,\\ B_1(x) &= x-1/2, & E_1(x) &= x-1/2,\\ B_2(x) &= x^2-x+1/6, & E_2(x) &= x^2-x. \end{aligned}$$

From (1.1) and (1.2), it is easily proved that

$$\begin{aligned} &B_n(1-x) = (-1)^n\,B_n(x), && E_n(1-x) = (-1)^n\,E_n(x),\\ &B_n(x+1)-B_n(x) = n\,x^{n-1}, && E_n(x+1)+E_n(x) = 2\,x^n,\\ &B_n(x+y) = \sum_{k=0}^{n}\binom{n}{k}B_k(x)y^{n-k}, && E_n(x+y) = \sum_{k=0}^{n}\binom{n}{k}E_k(x)y^{n-k}. \end{aligned}$$

Email address: prevost@lmpa.univ-littoral.fr (Marc Prévost [1,2,3])

The classical Bernoulli numbers B_n and the classical Euler numbers E_n are given by $B_n := B_n(0)$ and $E_n := 2^n E_n(1/2)$, respectively.

In 2000, T. Agoh [1] proved a general linear recurrence relation between Bernoulli and Euler polynomials.

More recently, in [2], Chen and Sun made use of Zeilberger's algorithm to prove most of the existing recurrence relations for Bernoulli and Euler polynomials. They also derived two new identities which are particular cases of the main theorem of this paper.

Some analogues of the classical Bernoulli polynomials were introduced by Apostol in order to evaluate the Hurwitz-Lerch zeta function:

$$\phi(z,s,a) := \sum_{n=0}^{\infty} \frac{z^n}{(a+n)^s}.$$

See [3] and also the recent book [4].

We begin by recalling here Apostol's definition as follows:

Definition 1. *(Apostol, [3]) The Apostol-Bernoulli polynomials $\mathcal{B}_k(x;\lambda)$ in the variable x are defined by means of the following generating function:*

$$\frac{t}{\lambda\, e^t - 1} e^{x\,t} \quad = \quad \sum_{k=0}^{\infty} \mathcal{B}_k(x;\lambda) \frac{t^k}{k!} \tag{1.3}$$

$$(|t| < 2\pi \textit{ when } \lambda = 1; |t| < |\log \lambda| \textit{ when } \lambda \neq 1).$$

Using the Gaussian hypergeometric functions, Luo [5] found formulas for $\mathcal{B}_k(x;\lambda)$ and in [6], Boyadzhiev found the relations of the Apostol-Bernoulli functions with the Euler polynomials and the derivative polynomials for the cotangent function.

Recently, Luo and Srivastava introduced the Apostol-Bernoulli polynomials of higher order (also called generalized Apostol-Bernoulli polynomials):

Definition 2. *(Luo and Srivastava, [7]) The Apostol-Bernoulli polynomials $\mathcal{B}_k^{(\alpha)}(x;\lambda)$ of order α in the variable x are defined by means of the generating function:*

$$\left(\frac{t}{\lambda\, e^t - 1}\right)^{\alpha} e^{x\,t} \quad = \quad \sum_{k=0}^{\infty} \mathcal{B}_k^{(\alpha)}(x;\lambda) \frac{t^k}{k!}, \tag{1.4}$$

$$(|t| < 2\pi \textit{ when } \lambda = 1; |t| < |\log \lambda| \textit{ when } \lambda \neq 1).$$

According to the definition, by setting $\alpha = 1$, we obtain the Apostol-Bernoulli polynomials $\mathcal{B}_k(x;\lambda)$. Moreover, we call $\mathcal{B}_k(\lambda) := \mathcal{B}_k(0;\lambda)$ the Apostol-Bernoulli numbers.

Explicit representation of $\mathcal{B}_k^{(\alpha)}(x;\lambda)$ in terms of a generalization of the Hurwitz-Lerch zeta function can be found in [8].

In the paper [9] submitted in 2004, Luo introduced Apostol-Euler polynomials of higher order α:

Definition 3. *(Luo, [9]) The Apostol-Euler polynomials $\mathcal{E}_k^{(\alpha)}(x;\lambda)$ of order α in the variable x are defined by means of the following generating function:*

$$\left(\frac{2}{\lambda\, e^t + 1}\right)^{\alpha} e^{x\,t} \quad = \quad \sum_{k=0}^{\infty} \mathcal{E}_k^{(\alpha)}(x;\lambda) \frac{t^k}{k!}, (\lambda \neq -1, |t| < |\log(-\lambda)|). \tag{1.5}$$

The Apostol-Euler polynomials $\mathcal{E}_k(x;\lambda)$ are given by $\mathcal{E}_k(x;\lambda) := \mathcal{E}_k^{(1)}(x;\lambda)$. The Apostol-Euler numbers $\mathcal{E}_k(\lambda)$ are given by $\mathcal{E}_k(\lambda) := 2^k\mathcal{E}_k(\frac{1}{2};\lambda)$.

Some relations between Apostol-Bernoulli and Apostol-Euler polynomials of order α can be found in [10]. For more results on these polynomials, the readers are referred to [11, 12, 13].

In this paper, we only consider Apostol-type polynomials $\mathcal{B}_k(x;\lambda)$ and $\mathcal{E}_k(x;\lambda)$. For $\lambda = 1$, they reduce to the classical Bernoulli polynomials $B_k(x)$ and the classical Euler polynomials $E_k(x)$, respectively. For $\lambda \neq 1$, $\deg \mathcal{B}_k(x;\lambda) = k-1$. For $\lambda \neq -1, \deg \mathcal{E}_k(x;\lambda) = k$. The first expressions of these polynomials are

$$
\begin{aligned}
&\mathcal{B}_0(x;\lambda) = 0, && \mathcal{E}_0(x;\lambda) = \frac{2}{1+\lambda},\\
&\mathcal{B}_1(x;\lambda) = \frac{1}{\lambda-1}, && \mathcal{E}_1(x;\lambda) = \frac{2(x-\lambda+x\lambda)}{(1+\lambda)^2},\\
&\mathcal{B}_2(x;\lambda) = \frac{2(-x-\lambda+x\lambda)}{(\lambda-1)^2}, && \mathcal{E}_2(x;\lambda) = \frac{2(x^2-\lambda-2x\lambda+2x^2\lambda+\lambda^2-2x\lambda^2+x^2\lambda^2)}{(1+\lambda)^3}.
\end{aligned}
$$

The core of the paper is Theorem 1. Using Padé approximation to e^t in the generating functions (1.3) and (1.5), we get a new relation between Apostol-type polynomials depending on three parameters n, m, p where n and m are respectively the degree of the numerator and the degree of the denominator of the Padé approximant used to approximate the function e^t and p is some positive integer. We will show that many known recurrence relations are particular cases of this general formula.

The paper is organized as follows. In the next section, we recall the definition of Padé approximant to a general series and its expression for the case of the exponential function. In Section 3, we apply Padé approximation to prove the announced general recurrence relations (Theorem 1) for the Apostol-Bernoulli and Apostol-Euler polynomials. In Section 4 we show that many known linear recurrence relations are particular cases of the general recurrence relations. A generalization of Theorem 1 is given in Section 5. In the last two sections, as applications, we will give some lacunary recurrence relations with a gap of length 4 between Bernoulli and Euler polynomials and with a gap of length 6 between Bernoulli and Euler numbers.

2. Padé approximant

In this section, we recall the definition of Padé approximation to general series and their expression in the case of the exponential function. Given a function f with a Taylor expansion

$$f(t) = \sum_{i=0}^{\infty} c_i t^i \tag{2.1}$$

in a neighborhood of the origin, a Padé approximant denoted $[n,m]_f$ to f is a rational fraction of degree n (resp. m) for the numerator (resp. the denominator):

$$[n,m]_f(t) = \frac{\alpha_0+\alpha_1 t+\cdots+\alpha_n t^n}{\beta_0+\beta_1 t+\cdots+\beta_m t^m},$$

whose Taylor expansion agrees with (2.1) as far as possible:

$$\sum_{i=0}^{\infty} c_i t^i - \frac{\alpha_0+\alpha_1 t+\cdots+\alpha_n t^n}{\beta_0+\beta_1 t+\cdots+\beta_m t^m} = O(t^{m+n+1}).$$

In the general case, the resulting linear system has unique solutions α_i, β_i (see, e.g., [14]).

Padé approximation is related with convergence acceleration [15], continued fractions [16, 17], orthogonal polynomials and quadrature formulas [18]. Moreover the denominators of Padé approximants satisfy a three terms recurrence [19, 20] and this property allows finding another proof of the irrationality of $\zeta(2)$ and $\zeta(3)$ (see [21]).

If $f(t) = e^t$ then

$$
\begin{aligned}
[n,m]_f(t) :&= \frac{P^{(n,m)}(t)}{Q^{(n,m)}(t)} = \frac{{}_1F_1(-n,-m-n,t)}{{}_1F_1(-m,-m-n,-t)} \\
&= \frac{\sum_{j=0}^{n} \frac{(-n)_j}{(-m-n)_j} \frac{t^j}{j!}}{\sum_{j=0}^{m} \frac{(-m)_j}{(-m-n)_j} \frac{(-t)^j}{j!}} = \frac{\sum_{j=0}^{n} \binom{n}{j} (n+m-j)! t^j}{\sum_{j=0}^{m} \binom{m}{j} (n+m-j)! (-t)^j},
\end{aligned}
$$

where the Pochhammer symbol $(a)_j$ is defined as

$$
\begin{aligned}
(a)_j &= a(a+1)\cdots(a+j-1) \text{ if } j \geq 1, \\
&= 1 \text{ if } j = 0
\end{aligned}
$$

and the hypergeometric series ${}_1F_1(a,b,z)$ is defined as ${}_1F_1(a,b,z) := \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!}$. In the sequel we write $[n,m](t)$ for the Padé approximant to e^t. The remainder term is defined by

$$
R^{(n,m)}(t) := e^t - [n,m](t) = e^t - \frac{P^{(n,m)}(t)}{Q^{(n,m)}(t)}
$$

and satisfies

$$
\begin{aligned}
R^{(n,m)}(t) &= t^{m+n+1} \frac{e^t}{Q^{(n,m)}(t)} \int_0^1 (x-1)^m x^n e^{-x\,t} dx \\
&= \frac{(-1)^m}{Q^{(n,m)}(t)} \sum_{j=0}^{\infty} \frac{t^{j+m+n+1}}{(j+m+n+1) \binom{m+n+j}{n} j!} \\
&= \frac{(-1)^m\, n!}{Q^{(n,m)}(t)} \sum_{j=0}^{\infty} \frac{(m+j)!}{j!\,(m+n+j+1)!} t^{j+m+n+1} \\
&= O(t^{n+m+1}).
\end{aligned}
$$

In the sequel, let

$$
\begin{aligned}
\alpha_j^{(n,m)} &:= \binom{n}{j} (n+m-j)!, \quad 0 \leq j \leq n, \\
\beta_j^{(n,m)} &:= \binom{m}{j} (n+m-j)! (-1)^j, \quad 0 \leq j \leq m, \\
\gamma_j^{(n,m)} &:= (-1)^m \frac{n!\,(m+j)!}{(m+n+1+j)!\,j!}, \quad j \geq 0.
\end{aligned}
$$

The polynomials $P^{(n,m)}, Q^{(n,m)}$ and the product $R^{(n,m)}\,Q^{(n,m)}$ are then given by

$$
\begin{aligned}
P^{(n,m)}(t) &= \sum_{j=0}^{n} \alpha_j^{(n,m)} t^j, \\
Q^{(n,m)}(t) &= \sum_{j=0}^{m} \beta_j^{(n,m)} t^j, \\
R^{(n,m)}(t)\,Q^{(n,m)}(t) &= \sum_{j=0}^{\infty} \gamma_j^{(n,m)} t^{j+m+n+1}.
\end{aligned}
$$

3. Recurrence relations for Apostol-type polynomials

Let us recall one method to derive the basic formula for Bernoulli numbers. It consists in considering the Taylor expansion of e^t around $t=0$.

The generating function of the Bernoulli numbers $B_k := \mathcal{B}_k(0;1)$ can be written as

$$(e^t-1)\sum_{k=0}^{\infty} B_k \frac{t^k}{k!} = t \Longleftrightarrow \sum_{j=1}^{\infty} \frac{t^j}{j!} \sum_{k=0}^{\infty} B_k \frac{t^k}{k!} = t, \quad (|t|<2\pi).$$

So, using the Cauchy product for series,

$$\sum_{p=1}^{\infty} t^p \sum_{\substack{k+j=p\\ k\geq 0, j\geq 1}} \frac{B_k}{k!\, j!} = t.$$

This implies $B_0 = 1$ and $\sum_{j=0}^{p-1} \binom{p}{j} B_j = 0$ for $p \geq 2$.

In this section, the main idea is to replace in the generating functions of Apostol-type polynomials (1.3) and (1.5), the exponential function e^t not by its Taylor expansion around $t=0$ but by its Padé approximant $[n,m](t)$. This gives the following main result.

Theorem 1. *For all integers $m \geq 0, n \geq 0$, the linear combination of Apostol-Bernoulli polynomials defined by*

$$\begin{aligned} A^{(p)}_{n,m}(x;\lambda) :=& \sum_{j=0}^{\max(n,m)} \left(\lambda\binom{n}{j} - (-1)^j\binom{m}{j}\right) \frac{(n+m-j)!}{(p-j)!} \mathcal{B}_{p-j}(x;\lambda) \\ =& \lambda\sum_{j=0}^{n} \binom{n}{j} \frac{(n+m-j)!}{(p-j)!} \mathcal{B}_{p-j}(x;\lambda) - \sum_{j=0}^{m} (-1)^j \binom{m}{j} \frac{(n+m-j)!}{(p-j)!} \mathcal{B}_{p-j}(x;\lambda) \end{aligned}$$

satisfies for $0 \leq p \leq m+n$,

$$A^{(p)}_{n,m}(x;\lambda) = \sum_{j=0}^{p-1} \binom{m}{j} (n+m-j)!(-1)^j \frac{x^{p-1-j}}{(p-1-j)!}, \tag{3.1}$$

and for $p \geq m+n+1$,

$$\begin{aligned} A^{(p)}_{n,m}(x;\lambda) = & \sum_{j=0}^{m} \binom{m}{j} (n+m-j)!(-1)^j \frac{x^{p-1-j}}{(p-1-j)!} - \\ & \lambda\,(-1)^m \frac{n!m!}{p!} \sum_{k=0}^{p-m-n-1} \binom{p-1-n-k}{m} \binom{p}{k} \mathcal{B}_k(x;\lambda). \end{aligned} \tag{3.2}$$

We use the convention $\mathcal{B}_k(x;\lambda) = 0$ for $k \leq -1$.

Also, for all integers $m \geq 0, n \geq 0$, the linear combination of Apostol-Euler polynomials defined by

$$\begin{aligned} \widehat{A}^{(p)}_{n,m}(x;\lambda) :=& \sum_{j=0}^{\max(n,m)} \left(\lambda\binom{n}{j} + (-1)^j\binom{m}{j}\right) \frac{(n+m-j)!}{(p-j)!} \mathcal{E}_{p-j}(x;\lambda) \\ =& \lambda\sum_{j=0}^{n} \binom{n}{j} \frac{(n+m-j)!}{(p-j)!} \mathcal{E}_{p-j}(x;\lambda) + \sum_{j=0}^{m} (-1)^j \binom{m}{j} \frac{(n+m-j)!}{(p-j)!} \mathcal{E}_{p-j}(x;\lambda) \end{aligned}$$

satisfies for $0 \leq p \leq m+n$,

$$\widehat{A}^{(p)}_{n,m}(x;\lambda) = 2\ \sum_{j=0}^{p} \binom{m}{j}\ (n+m-j)!(-1)^j\, \frac{x^{p-j}}{(p-j)!}, \tag{3.3}$$

and for $p \geq m+n+1$,

$$\begin{aligned}\widehat{A}^{(p)}_{n,m}(x;\lambda) \quad &= \quad 2\ \sum_{j=0}^{m} \binom{m}{j}\ (n+m-j)!(-1)^j\, \frac{x^{p-j}}{(p-j)!} - \\ &\lambda\ (-1)^m\ \frac{n!\ m!}{p!} \sum_{k=0}^{p-m-n-1} \binom{p-1-n-k}{m}\ \binom{p}{k}\,\mathcal{E}_k(x;\lambda). \end{aligned}\tag{3.4}$$

We use the convention $\mathcal{E}_k(x;\lambda) = 0$ for $k \leq -1$.

Proof. We start from the generating function (1.3),

$$\big(\lambda\, e^t - 1\big) \sum_{k=0}^{\infty} \mathcal{B}_k(x;\lambda)\, \frac{t^k}{k!} = t\, e^{x\,t}, \qquad (\lambda = 1, |t| < 2\pi; \lambda \neq 1, |t| < |\log\lambda|),$$

and for some integers n, m, we replace e^t by its Padé approximant, previously defined,

$$e^t = [n,m](t) + R^{(n,m)}(t) = \frac{P^{(n,m)}(t)}{Q^{(n,m)}(t)} + R^{(n,m)}(t).$$

We get

$$\left(\lambda\frac{P^{(n,m)}(t)}{Q^{(n,m)}(t)} + \lambda R^{(n,m)}(t) - 1\right) \sum_{k=0}^{\infty} \mathcal{B}_k(x;\lambda)\frac{t^k}{k!} = t\, e^{x\,t}$$

and

$$\Big(\lambda P^{(n,m)}(t) - Q^{(n,m)}(t) + \lambda R^{(n,m)}(t)\ Q^{(n,m)}(t)\Big) \sum_{k=0}^{\infty} \mathcal{B}_k(x;\lambda)\frac{t^k}{k!} = t\, e^{x\,t} Q^{(n,m)}(t).$$

Thus

$$\Big(\lambda P^{(n,m)}(t) - Q^{(n,m)}(t)\Big) \sum_{k=0}^{\infty} \mathcal{B}_k(x;\lambda)\frac{t^k}{k!} = t\, e^{x\,t}\ Q^{(n,m)}(t) - \lambda R^{(n,m)}(t)\ Q^{(n,m)}(t) \sum_{k=0}^{\infty} \mathcal{B}_k(x;\lambda)\frac{t^k}{k!}.$$

From the expression of the Padé approximant, it arises the following relation

$$\begin{aligned}&\sum_{j=0}^{\max(n,m)} \Big(\lambda\alpha_j^{(n,m)}\ - \beta_j^{(n,m)}\Big)\ t^j \sum_{k=0}^{\infty} \mathcal{B}_k(x;\lambda)\frac{t^k}{k!} = \\ &\qquad t\sum_{k=0}^{\infty} \frac{x^k t^k}{k!} \sum_{j=0}^{m} \beta_j^{(n,m)}\ t^j - \lambda \sum_{j=0}^{\infty} \gamma_j^{(n,m)} t^{j+m+n+1} \sum_{k=0}^{\infty} \mathcal{B}_k(x;\lambda)\frac{t^k}{k!}\end{aligned}$$

which gives, applying the Cauchy product for series,

$$\begin{aligned}&\sum_{p=0}^{\infty} t^p \sum_{\substack{k+j=p\\ k\geq 0,\ j\geq 0}} \Big(\lambda\alpha_j^{(n,m)} -\ \beta_j^{(n,m)}\Big)\, \frac{\mathcal{B}_k(x;\lambda)}{k!} = \\ &\qquad \sum_{p=0}^{\infty} t^p \sum_{\substack{k+j=p-1\\ k\geq 0,\ j\geq 0}} \beta_j^{(n,m)}\, \frac{x^k}{k!} - \lambda \sum_{p=0}^{\infty} t^p \sum_{\substack{k+j=p-m-n-1\\ k\geq 0,\ j\geq 0}} \gamma_j^{(n,m)}\, \frac{\mathcal{B}_k(x;\lambda)}{k!}.\end{aligned}$$

Equating the coefficients of t^p gives for $0 \le p \le m+n$

$$\sum_{\substack{k+j=p\\ k\ge 0,\ j\ge 0}} \left(\lambda \alpha_j^{(n,m)} - \beta_j^{(n,m)}\right) \frac{\mathcal{B}_k(x;\lambda)}{k!} = \sum_{\substack{k+j=p-1\\ k\ge 0,\ j\ge 0}} \beta_j^{(n,m)} \frac{x^k}{k!} \ ,$$

and for $p = r+m+n,\ r \ge 1$

$$\sum_{\substack{k+j=p\\ k\ge 0,\ j\ge 0}} \left(\lambda \alpha_j^{(n,m)} - \beta_j^{(n,m)}\right) \frac{\mathcal{B}_k(x;\lambda)}{k!} = \sum_{\substack{k+j=p-1\\ k\ge 0,\ j\ge 0}} \beta_j^{(n,m)} \frac{x^k}{k!} - \lambda \sum_{\substack{k+j=r-1\\ k\ge 0,\ j\ge 0}} \gamma_j^{(n,m)} \frac{\mathcal{B}_k(x;\lambda)}{k!},$$

which proves the expression of $A_{n,m}^{(p)}(x;\lambda)$ in Theorem 1.

For $\widehat{A}_{n,m}^{(p)}(x;\lambda)$ the proof is similar. We start from the generating function (1.5)

$$\left(\lambda\, e^t + 1\right) \sum_{k=0}^{\infty} \mathcal{E}_k(x;\lambda)\, \frac{t^k}{k!} = 2\, e^{x\,t}, \qquad (|t| < |\log(-\lambda)|),$$

and as previously we replace e^t by its Padé approximant $[n,m](t)$. We get

$$\begin{aligned} \sum_{p=0}^{\infty} t^p \sum_{\substack{k+j=p\\ k\ge 0,\ j\ge 0}} \left(\lambda \alpha_j^{(n,m)} + \beta_j^{(n,m)}\right) \frac{\mathcal{E}_k(x;\lambda)}{k!} &= 2\sum_{p=0}^{\infty} t^p \sum_{\substack{k+j=p\\ k\ge 0,\ j\ge 0}} \beta_j^{(n,m)} \frac{x^k}{k!} - \\ &\quad \lambda \sum_{p=0}^{\infty} t^p \sum_{\substack{k+j=p-m-n-1\\ k\ge 0,\ j\ge 0}} \gamma_j^{(n,m)} \frac{\mathcal{E}_k(x;\lambda)}{k!}. \end{aligned}$$

So, for $0 \le p \le m+n$

$$\sum_{\substack{k+j=p\\ k\ge 0,\ j\ge 0}} \left(\lambda \alpha_j^{(n,m)} + \beta_j^{(n,m)}\right) \frac{\mathcal{E}_k(x;\lambda)}{k!} = 2 \sum_{\substack{k+j=p\\ k\ge 0,\ j\ge 0}} \beta_j^{(n,m)} \frac{x^k}{k!},$$

and for $p = m+n+r,\ r \ge 1$,

$$\sum_{\substack{k+j=p\\ k\ge 0,\ j\ge 0}} \left(\lambda \alpha_j^{(n,m)} + \beta_j^{(n,m)}\right) \frac{\mathcal{E}_k(x;\lambda)}{k!} = 2 \sum_{\substack{k+j=p\\ k\ge 0,\ j\ge 0}} \beta_j^{(n,m)} \frac{x^k}{k!} - \lambda \sum_{\substack{k+j=r-1\\ k\ge 0,\ j\ge 0}} \gamma_j^{(n,m)} \frac{\mathcal{E}_k(x;\lambda)}{k!},$$

which proves the expression of $\widehat{A}_{n,m}^{(p)}(x;\lambda)$. ■

Theorem 1 provides a recurrence relation of length $\max(n,m)$ from $\mathcal{B}_{p-\max(n,m)}(x;\lambda)$ (resp. $\mathcal{E}_{p-\max(n,m)}(x;\lambda)$) to $\mathcal{B}_p(x;\lambda)$ (resp. $\mathcal{E}_p(x;\lambda)$) for p less than $m+n$. On the other hand, if p is greater than $m+n$, we obtain also a recurrence relation for the same Apostol-type polynomials but with supplementary first terms from $\mathcal{B}_0(x;\lambda)$ (resp. $\mathcal{E}_0(x;\lambda)$) to $\mathcal{B}_{p-m-n-1}(x;\lambda)$ (resp. $\mathcal{E}_{p-m-n-1}(x;\lambda)$).

Of course, Theorem 1 is valid for Bernoulli and Euler polynomials and also for Bernoulli and Euler numbers which are particular Apostol-type polynomials. By this mean we recover known results, see [1, 2, 22, 23, 24, 25, 26], as shown in the next sections.

4. Applications

In this section, we will consider the value of the parameter p with respect to n, m.

Let us consider the particular value $p = m + n$. For this particular case, from formulas (3.1) and (3.3), we can prove the following corollary.

Corollary 1. *For $m \geq 0, n \geq 0$,*

$$\lambda \sum_{k=0}^{n} \binom{n}{k} \mathcal{B}_{m+k}(x;\lambda) - (-1)^m \sum_{k=0}^{m} (-1)^k \binom{m}{k} \mathcal{B}_{n+k}(x;\lambda) = ((m+n)x - n)x^{n-1}(x-1)^{m-1},$$

$$\lambda \sum_{k=0}^{n} \binom{n}{k} \mathcal{E}_{m+k}(x;\lambda) + (-1)^m \sum_{k=0}^{m} (-1)^k \binom{m}{k} \mathcal{E}_{n+k}(x;\lambda) = 2x^n(x-1)^m.$$

Proof. For $p = m+n$, the expression of $A_{n,m}^{(p)}(x;\lambda)$ and $\widehat{A}_{n,m}^{(p)}(x;\lambda)$ can be simplified as

$$A_{n,m}^{(m+n)}(x;\lambda) = \lambda \sum_{k=0}^{n} \binom{n}{k} \mathcal{B}_{m+k}(x;\lambda) - (-1)^m \sum_{k=0}^{m} (-1)^k \binom{m}{k} \mathcal{B}_{n+k}(x;\lambda)$$

and

$$\widehat{A}_{n,m}^{(m+n)}(x;\lambda) = \lambda \sum_{k=0}^{n} \binom{n}{k} \mathcal{E}_{m+k}(x;\lambda) + (-1)^m \sum_{k=0}^{m} (-1)^k \binom{m}{k} \mathcal{E}_{n+k}(x;\lambda).$$

Using the identities

$$\sum_{j=0}^{m} \binom{m}{j} (-1)^j \, j \, x^{j+1} = -m(1-x)^{m-1}x^2, \qquad \sum_{j=0}^{m} \binom{m}{j} (-1)^j \; x^{m-j} = (x-1)^m$$

and formulas (3.1) and (3.3), we get Corollary 1. ∎

Let $s, r \geq 1$. Setting $p = m+n-s$ and $p = m+n+r$ in Theorem 1, we will get Corollaries 2 and 3.

Corollary 2. *For $n \in \mathbb{N}$, $m \in \mathbb{N}$, $s \in \mathbb{N}$ such that $1 \leq s \leq n+m$,*

$$\lambda \sum_{k=0}^{n} \binom{n}{k} (m-s+k+1)_s \mathcal{B}_{m-s+k}(x;\lambda) - (-1)^m \sum_{k=0}^{m} (-1)^k \binom{m}{k} (n-s+k+1)_s \mathcal{B}_{n-s+k}(x;\lambda) =$$

$$\sum_{j=0}^{m+n-s-1} \binom{m}{j} (m+n-s-j)_{s+1} (-1)^j x^{m+n-s-1-j},$$

$$\lambda \sum_{k=0}^{n} \binom{n}{k} (m-s+k+1)_s \mathcal{E}_{m-s+k}(x;\lambda) + (-1)^m \sum_{k=0}^{m} (-1)^k \binom{m}{k} (n-s+k+1)_s \mathcal{E}_{n-s+k}(x;\lambda) =$$

$$2 \sum_{j=0}^{m+n-s} \binom{m}{j} (m+n-s+1-j)_s (-1)^j x^{m+n-s-j}.$$

Remark 1. *For $\lambda = 1$ and $s = 1$, Corollary 1 and Corollary 2 reduce to the following formulas*

$$\sum_{k=0}^{n} \binom{n}{k} B_{m+k}(x) - (-1)^m \sum_{k=0}^{m} (-1)^k \binom{m}{k} B_{n+k}(x) = ((m+n)x - n)x^{n-1}(x-1)^{m-1},$$
$$\sum_{k=0}^{n} \binom{n}{k} E_{m+k}(x) + (-1)^m \sum_{k=0}^{m} (-1)^k \binom{m}{k} E_{n+k}(x) = 2x^n(x-1)^m,$$

$$\sum_{k=0}^{n} \binom{n}{k} (m+k) B_{m+k-1}(x) - (-1)^m \sum_{k=0}^{m} (-1)^k \binom{m}{k} (n+k) B_{n+k-1}(x) = \sum_{j=0}^{m+n-2} \binom{m}{j} (m+n-1-j)(m+n-j)(-1)^j x^{m+n-2-j},$$

$$\sum_{k=0}^{n} \binom{n}{k} (m+k) E_{m+k-1}(x) + (-1)^m \sum_{k=0}^{m} (-1)^k \binom{m}{k} (n+k) E_{n+k-1}(x) = 2 \sum_{j=0}^{m+n-1} \binom{m}{j} (m+n-j)(-1)^j x^{m+n-1-j},$$

proved by Wu and all [26] (generalizing Kaneko's formula [22]) and by Momiyama in [25] for Bernoulli numbers but without reference to a previous more general result by Agoh in [1].

Remark 2. *For $\lambda = 1$, $s = 3$, $m = n$, $x = 0$ we find the formula proved by Chen and Sun in [2] for the particular case of Bernoulli numbers.*

Corollary 3. *For $n \in \mathbb{N}$, $m \in \mathbb{N}$, $r \in \mathbb{N}, r \geq 1$,*

$$\lambda \sum_{k=0}^{n} \binom{n}{k} \frac{\mathcal{B}_{m+r+k}(x;\lambda)}{(m+1+k)_r} - (-1)^m \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{\mathcal{B}_{n+r+k}(x;\lambda)}{(n+1+k)_r} =$$
$$\sum_{j=0}^{m} \binom{m}{j} (-1)^j \frac{x^{m+n+r-1-j}}{(m+n+1-j)_{r-1}} - \lambda(-1)^m \frac{n!\, m!}{(n+m+r)!} \sum_{k=0}^{r-1} \binom{m+r-1-k}{m} \binom{m+n+r}{k} \mathcal{B}_k(x;\lambda),$$

$$\lambda \sum_{k=0}^{n} \binom{n}{k} \frac{\mathcal{E}_{m+r+k}(x;\lambda)}{(m+1+k)_r} + (-1)^m \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{\mathcal{E}_{n+r+k}(x;\lambda)}{(n+1+k)_r} =$$
$$2 \sum_{j=0}^{m} \binom{m}{j} (-1)^j \frac{x^{n+m+r-j}}{(m+n+1-j)_r} - \lambda(-1)^m \frac{n!\, m!}{(n+m+r)!} \sum_{k=0}^{r-1} \binom{m+r-1-k}{m} \binom{m+n+r}{k} \mathcal{E}_k(x;\lambda).$$

Remark 3. *When $\lambda = 1, x = 0$ and $r = 1$, the first formula of Corollary 3 reduces to a result due to Gelfand [24].*

5. Extension of Theorem 1

The formulas in Theorem 1 involve the Apostol-Bernoulli polynomials $\mathcal{B}_k(x;\lambda)$ or Apostol-Euler polynomials $\mathcal{E}_k(x;\lambda)$. In the present section, using the same method of Padé approximants as in Section 3, we prove the same type of recurrence relations for the sequences $(\,l^{-k}\mathcal{B}_k(x;\lambda))_k$ and $(\,l^{-k}\mathcal{E}_k(x;\lambda))_k$ where l is some positive integer.

Theorem 2. *For Apostol-Bernoulli polynomials, for $n \in \mathbb{N}$, $m \in \mathbb{N}$, $p \in \mathbb{N}$, $l \in \mathbb{N}$, if $p \leq m+n$, then*

$$\sum_{j=0}^{\max(n,m)} \left(\lambda^l \binom{n}{j} - (-1)^j \binom{m}{j}\right) \frac{(n+m-j)!}{(p-j)!} \frac{\mathcal{B}_{p-j}(x;\lambda)}{l^{p-j}} = \sum_{k=0}^{l-1} \lambda^k \sum_{j=0}^{p-1} \binom{m}{j} \frac{(-1)^j (n+m-j)!}{l^{p-j}} \frac{(x+k)^{p-1-j}}{(p-1-j)!},$$

and if $p \geq m+n+1$, then

$$\sum_{j=0}^{\max(n,m)} \left(\lambda^l \binom{n}{j} - (-1)^j \binom{m}{j}\right) \tfrac{(n+m-j)!}{(p-j)!} \tfrac{\mathcal{B}_{p-j}}{l^{p-j}}(x;\lambda) = \sum_{k=0}^{l-1} \lambda^k \sum_{j=0}^{m} \binom{m}{j} (n+m-j)! \tfrac{(-1)^j}{l^{p-j}} \tfrac{(x+k)^{p-1-j}}{(p-1-j)!} - \lambda^l (-1)^m \tfrac{n!m!}{p!} \sum_{k=0}^{p-m-n-1} \binom{p-1-n-k}{m} \binom{p}{k} \tfrac{\mathcal{B}_k(x;\lambda)}{l^k}.$$

For Apostol-Euler polynomials, for $n \in \mathbb{N}$, $m \in \mathbb{N}$, $p \in \mathbb{N}$, $l \in \mathbb{N}$, if $p \leq m+n$, then

$$\sum_{j=0}^{\max(n,m)} \left(\lambda^l \binom{n}{j} - (-1)^{l+j} \binom{m}{j}\right) \tfrac{(n+m-j)!}{(p-j)!} \tfrac{\mathcal{E}_{p-j}(x;\lambda)}{l^{p-j}} = 2(-1)^{l-1} \sum_{k=0}^{l-1} (-\lambda)^k \sum_{j=0}^{p} \binom{m}{j} (n+m-j)! \tfrac{(-1)^j}{l^{p-j}} \tfrac{(x+k)^{p-j}}{(p-j)!},$$

and if $p \geq m+n+1$, then

$$\sum_{j=0}^{\max(n,m)} \left(\lambda^l \binom{n}{j} - (-1)^{l+j} \binom{m}{j}\right) \tfrac{(n+m-j)!}{(p-j)!} \tfrac{\mathcal{E}_{p-j}(x;\lambda)}{l^{p-j}} = 2(-1)^{l-1} \sum_{k=0}^{l-1} (-\lambda)^k \sum_{j=0}^{m} \binom{m}{j} (n+m-j)! \tfrac{(-1)^j}{l^{p-j}} \tfrac{(x+k)^{p-j}}{(p-j)!} - \lambda^l (-1)^m \tfrac{n!\,m!}{p!} \textstyle\sum_{k=0}^{p-m-n-1} \binom{p-1-n-k}{m} \binom{p}{k} \tfrac{\mathcal{E}_k(x;\lambda)}{l^k}.$$

Proof. The details of the proof will be omitted. For the Apostol-Bernoulli polynomials, multiply the generating function (1.3) by $(\lambda^l e^{l\,t} - 1)$ to obtain

$$t\, e^{x\,t}(\lambda^{l-1} e^{(l-1)\,t} + \cdots + \lambda\, e^t + 1) = (\lambda^l e^{l\,t} - 1) \sum_{k=0}^{\infty} \mathcal{B}_k(x;\lambda) \frac{t^k}{k!}, \tag{5.1}$$

and substitute $e^{l\,t}$ in the right hand-side member of (5.1) by its Padé approximant $[n,m](lt)$ as in the proof of Theorem 1. For the Apostol-Euler polynomials, we must multiply (1.5) by $(\lambda^l e^{l\,t} + (-1)^{l-1})$ to obtain

$$2\, e^{x\,t}(\lambda^{l-1} e^{(l-1)\,t} - \lambda^{l-2} e^{(l-2)\,t} + \cdots + (-1)^{l-1}) = (\lambda^l e^{l\,t} + (-1)^{l-1}) \sum_{k=0}^{\infty} \mathcal{E}_k(x;\lambda) \frac{t^k}{k!}.$$

Remark 4. *If $\lambda = 1$, $x = 0$ and $p = m+n-1$ the first formula of Theorem 2 reduces to a result due to Gessel [23]. If $\lambda = 1$, $x = 0$ and $p = m+n-3$ it reduces to a result proved in [2].*

6. Limit case: $m = \rho\, n$, $n \to \infty$.

Let us recall the expression for $A^{(p)}_{n,m}$:

$$A^{(p)}_{n,m}(x;\lambda) = \lambda \sum_{j=0}^{n} \binom{n}{j} \frac{(n+m-j)!}{(p-j)!} \mathcal{B}_{p-j}(x;\lambda) - \sum_{j=0}^{m} (-1)^j \binom{m}{j} \frac{(n+m-j)!}{(p-j)!} \mathcal{B}_{p-j}(x;\lambda). \tag{6.1}$$

If n, m tend to infinity, then of course, $p \leq m+n$ and so by Theorem 1

$$A^{(p)}_{n,m}(x;\lambda) = \sum_{j=0}^{p-1} \binom{m}{j} (n+m-j)! (-1)^j \frac{x^{p-1-j}}{(p-1-j)!}.$$

Let us assume that $m = \rho\ n$ with n going to infinity. After dividing the previous identity by $(n+m-p)!$, we get

$$\frac{A^{(p)}_{n,\rho\, n}(x;\lambda)}{(n+\rho\, n-p)!} = \sum_{j=0}^{p-1} \binom{\rho\, n}{j} \frac{(n+\rho\, n-j)!}{(n+\rho\, n-p)!} (-1)^j \frac{x^{p-1-j}}{(p-1-j)!}. \tag{6.2}$$

Using

$$\binom{n}{j} \underset{n\to\infty}{\sim} \frac{n^j}{j!}, \quad (j \geq 0)$$

and

$$\binom{n+\rho\, n-j}{n+\rho\, n-p} \underset{n\to\infty}{\sim} \frac{(1+\rho)^{p-j}}{(p-j)!} n^{p-j}, \quad (0 \leq j \leq p)$$

we get from (6.1)

$$\frac{A^{(p)}_{n,\rho\, n}(x;\lambda)}{(n+\rho\, n-p)!} \underset{n\to\infty}{\sim} \frac{n^p}{p!} \sum_{j=0}^{p} \binom{p}{j} (\lambda - (-\rho)^j)(1+\rho)^{p-j} \mathcal{B}_{p-j}(x;\lambda).$$

Similarly, the right hand-side member of Eq. (6.2) satisfies

$$\begin{aligned}\sum_{j=0}^{p-1} \binom{\rho\, n}{j} \frac{(n+\rho\, n-j)!(-1)^j x^{p-1-j}}{(n+\rho\, n-p)!(p-1-j)!} &\underset{n\to\infty}{\sim} \frac{n^p}{(p-1)!} \sum_{j=0}^{p-1} \rho^j\, (1+\rho)^{p-j}\, (-1)^j \binom{p-1}{j} x^{p-1-j} \\ &\underset{n\to\infty}{\sim} \frac{n^p}{(p-1)!} (1+\rho)(x\,(1+\rho)-\rho)^{p-1}.\end{aligned}$$

Then, we are led to the following theorem.

Theorem 3. *For all complex number ρ and for all integer p, the following formulas hold:*

$$\sum_{j=0}^{p} \binom{p}{j} (\lambda - (-\rho)^j)(1+\rho)^{p-j} \mathcal{B}_{p-j}(x;\lambda) = p\,(1+\rho)\,(x\,(1+\rho)-\rho)^{p-1}, \tag{6.3}$$

$$\sum_{j=0}^{p} \binom{p}{j} (\lambda + (-\rho)^j)(1+\rho)^{p-j} \mathcal{E}_{p-j}(x;\lambda) = 2\,(x\,(1+\rho)-\rho)^{p}. \tag{6.4}$$

Proof. It can be found that the generating functions of both sides of (6.3) are $(1+\rho)te^{(x(1+\rho)-\rho)t}$ and the generating functions of both sides of (6.4) are $2e^{(x(1+\rho)-\rho)t}$.

Remark 5. *Set $X = x - \rho/(1+\rho)$, the previous identities turn to*

$$\lambda\mathcal{B}_p(X+1;\lambda) - \mathcal{B}_p(X;\lambda) = p\, X^{p-1}, \quad (p \geq 0),$$

and

$$\lambda\mathcal{E}_p(X+1;\lambda) + \mathcal{E}_p(X;\lambda) = 2\, X^{p}, \quad (p \geq 0)$$

which can be found in [3, 9].

7. Lacunary recurrence relations

Typically, the calculation of Apostol-type polynomials requires the calculation of all the previous polynomials. For the particular case of Bernoulli numbers, in [27], Ramanujan proved a recurrence relation between the Bernoulli numbers with a gap of length 6. Lehmer [28] in 1935 extended these methods to Euler numbers, Genocchi numbers and Lucas numbers. In [29], using the "multisectioning" technique, Hare constructed lacunary recursion formulas of arbitrary size.

In this section, we use Theorem 3 to get recurrence relations for the sequences $(B_{4k+a}(x))_k$, $(E_{4k+a}(x))_k$, with $a \in \{0,1,2,3\}$. For the particular case $x = 0$, we will give relations for $(B_{6k+a})_k$, $(E_{6k+a})_k$, where $a \in \{0,2,4\}$.

The sequence of Bernoulli polynomials $(B_k(x) := \mathcal{B}_k(x;1))_k$ satisfies the property (see Theorem 3)

$\sum_{j=0}^{p} \binom{p}{j} (1-(-\rho)^j)(1+\rho)^{-j} B_{p-j}(x) = p \ \left(x \ - \frac{\rho}{1+\rho}\right)^{p-1}$, for each complex number ρ.

If we take $\rho = i, (i^2=-1)$, then

$\sum_{j=0}^{p} \binom{p}{j} (1-(-\rho)^j)(1+\rho)^{-j} B_{p-j}(x) = \sum_{k=0}^{p} \binom{p}{k} 2^{-k/2} \left(e^{-i\,k\,\pi/4} - (-1)^k e^{i\,k\,\pi/4}\right) B_{p-k}(x)$,

and

$p \ (x \ - \rho/(1+\rho))^{p-1} = p\, 2^{1-p} \ \ (2x \ -1-i)^{p-1} = p\, 2^{1-p} \sum_{k=0}^{p-1} \binom{p-1}{k} (2x-1)^{p-1-k} \, (-i)^k$.

Thus

$$\sum_{k=0}^{p} \binom{p}{k} 2^{-k/2} \left(e^{-i\,k\,\pi/4} - (-1)^k e^{i\,k\,\pi/4}\right) B_{p-k}(x) = p\, 2^{1-p} \sum_{k=0}^{p-1} \binom{p-1}{k} (2x-1)^{p-1-k} \, (-i)^k. \tag{7.1}$$

Let us calculate the coefficient $c_k = \left(e^{-i\,k\,\pi/4} - (-1)^k e^{i\,k\,\pi/4}\right)$:

$$\begin{array}{lcll} k & \equiv & 0(4) & c_k = 0, \\ k & \equiv & 1(4) & c_k = (-1)^{(k-1)/4}\sqrt{2}, \\ k & \equiv & 2(4) & c_k = -2(-1)^{(k-2)/4}\, i, \\ k & \equiv & 3(4) & c_k = -(-1)^{(k-3)/4}\sqrt{2}. \end{array}$$

So, if we consider only the imaginary part of formula (7.1), this will provide a recurrence relation with a gap of length 4. After simplification the following formula

$$\sum_{k\equiv 2(4)}^{p} \binom{p}{k} 2^{-k/2} (-1)^{(k-2)/4} B_{p-k}(x) = p\, 2^{-p} \sum_{\substack{k=1 \\ k \text{ odd}}}^{p-1} \binom{p-1}{k} (2x-1)^{p-1-k} \, (-1)^{(k-1)/2}$$

holds. So, according to the value of p modulo 4, for all integer $q \geq 1$

$$\begin{aligned}
\sum_{j=1}^{q} \binom{4q}{4j-2} (-4)^j B_{4j-2}(x) &= (-1)^q\, 4\, q\, 2^{1-2q} \sum_{\substack{k=1 \\ k \text{ odd}}}^{4q-1} \binom{4q-1}{k} (2x-1)^{4q-1-k} \, (-1)^{\frac{k-1}{2}}, \\
\sum_{j=1}^{q} \binom{4q+1}{4j-1} (-4)^j B_{4j-1}(x) &= (-1)^q\, (4q+1)\, 2^{-2q} \sum_{\substack{k=1 \\ k \text{ odd}}}^{4q} \binom{4q}{k} (2x-1)^{4q-k} \, (-1)^{\frac{k-1}{2}}, \\
\sum_{j=0}^{q} \binom{4q+2}{4j} (-4)^j B_{4j}(x) &= (-1)^q\, (4q+2)\, 2^{-1-2q} \sum_{\substack{k=1 \\ k \text{ odd}}}^{4q+1} \binom{4q+1}{k} (2x-1)^{4q+1-k} \, (-1)^{\frac{k-1}{2}}, \\
\sum_{j=0}^{q} \binom{4q+3}{4j+1} (-4)^j B_{4j+1}(x) &= (-1)^q\, (4q+3)\, 2^{-2-2q} \sum_{\substack{k=1 \\ k \text{ odd}}}^{4q+2} \binom{4q+2}{k} (2x-1)^{4q+2-k} \, (-1)^{\frac{k-1}{2}}.
\end{aligned}$$

The sequence of Euler polynomials $(E_k(x) := \mathcal{E}_k(x;1))_k$ satisfies the property (see Theorem 3)

$\sum_{j=0}^{p} \binom{p}{j} (1+(-\rho)^j)(1+\rho)^{-j} E_{p-j}(x) = 2 \ (x \ - \rho/(1+\rho))^{p}$, for each complex number ρ.

If we take $\rho = i, (i^2=-1)$, then

$\sum_{j=0}^{p} \binom{p}{j} (1+(-\rho)^j)(1+\rho)^{-j} E_{p-j}(x) = \sum_{k=0}^{p} \binom{p}{k} 2^{-k/2} \left(e^{-i\,k\,\pi/4} + (-1)^k e^{i\,k\,\pi/4}\right) E_{p-k}(x)$, and $2\,(x\ -\rho/(1+\rho))^p =\ 2^{1-p}\ \ (2x\ \ -1-i)^p = 2^{1-p}\sum_{k=0}^{p}\binom{p}{k}(2x-1)^{p-k}\,(-i)^k$.

Thus

$$\sum_{k=0}^{p}\binom{p}{k}2^{-k/2}\left(e^{-i\,k\,\pi/4}+(-1)^k e^{i\,k\,\pi/4}\right)E_{p-k}(x) = 2^{1-p}\sum_{k=0}^{p}\binom{p}{k}(2x-1)^{p-k}\,(-i)^k. \tag{7.2}$$

Let us calculate the coefficient $\hat{c}_k = \left(e^{-i\,k\,\pi/4}+(-1)^k e^{i\,k\,\pi/4}\right)$:

$$\begin{array}{lcll} k & \equiv & 0(4) & \hat{c}_k = 2\,(-1)^{k/4}, \\ k & \equiv & 1(4) & \hat{c}_k = i\sqrt{2}(-1)^{(k+3)/4}, \\ k & \equiv & 2(4) & \hat{c}_k = 0, \\ k & \equiv & 3(4) & \hat{c}_k = i\sqrt{2}(-1)^{(k+1)/4}. \end{array}$$

So, if we consider only the real part of formula (7.2), this will provide recurrence with a gap of length 4. After simplification the following formula

$$\sum_{k\equiv 0(4)}^{p}\binom{p}{k}2^{-k/2}(-1)^{k/4}E_{p-k}(x) = 2^{-p}\sum_{\substack{k=0\\ k\ \text{even}}}^{p}\binom{p}{k}(2x-1)^{p-k}\,(-1)^{k/2}$$

holds. Thus

$$\begin{aligned} \sum_{j=0}^{q}\binom{4q}{4j}(-4)^j E_{4j}(x) &= (-1)^q\,2^{-2q}\sum_{\substack{k=0\\ k\ \text{even}}}^{4q}\binom{4q}{k}(2x-1)^{4q-k}\,(-1)^{k/2}, \\ \sum_{j=0}^{q}\binom{4q+1}{4j+1}(-4)^j E_{4j+1}(x) &= (-1)^q\ 2^{-1-2q}\sum_{\substack{k=0\\ k\ \text{even}}}^{4q+1}\binom{4q+1}{k}(2x-1)^{4q+1-k}\,(-1)^{k/2}, \\ \sum_{j=0}^{q}\binom{4q+2}{4j+2}(-4)^j E_{4j+2}(x) &= (-1)^q\ 2^{-2-2q}\sum_{\substack{k=0\ ,\\ k\ \text{even}}}^{4q+2}\binom{4q+2}{k}(2x-1)^{4q+2-k}\,(-1)^{k/2}, \\ \sum_{j=0}^{q}\binom{4q+3}{4j+3}(-4)^j E_{4j+3}(x) &= (-1)^q\ 2^{-3-2q}\sum_{\substack{k=0\\ k\ \text{even}}}^{4q+3}\binom{4q+3}{k}(2x-1)^{4q+3-k}\,(-1)^{k/2}. \end{aligned}$$

■

In Theorem 3, if we set $\lambda = 1, x = 0$ in formula (6.3) and $\lambda = 1, x = 1/2$ in formula (6.4) then

$$\sum_{j=0}^{p}\binom{p}{j}(1-(-\rho)^j)(1+\rho)^{p-j}B_{p-j} = p\,(1+\rho)(-\rho)^{p-1}\,, \tag{7.3}$$

$$\sum_{j=0}^{p}\binom{p}{j}(1+(-\rho)^j)\left(\frac{1+\rho}{2}\right)^{p-j}E_{p-j} = 2\left(\frac{1-\rho}{2}\right)^p. \tag{7.4}$$

Setting $\rho = \mu = -1/2 + i\sqrt{3}/2$ in (7.3) and (7.4), we can derive some recurrence relations with gap of length 6, for Bernoulli and Euler numbers (the first three identities have been proved by Ramanujan in [27]). See also [28, 30].

Theorem 4. *For odd integer $p \geq 3$, we have*

$$\sum_{\substack{k=0\\k\equiv 4(6)}}^{p-1} \binom{p}{k} B_k = -\frac{p}{6} \quad \text{if } p \equiv 1(6), \tag{7.5}$$

$$\sum_{\substack{k=0\\k\equiv 0(6)}}^{p-1} \binom{p}{k} B_k = \frac{p}{3} \quad \text{if } p \equiv 3(6), \tag{7.6}$$

$$\sum_{\substack{k=0\\k\equiv 2(6)}}^{p-1} \binom{p}{k} B_k = \frac{p}{3} \quad \text{if } p \equiv 5(6). \tag{7.7}$$

For even integer p, we have

$$2^{-p}E_p + 3 \sum_{\substack{k=0\\k\equiv 0(6)}}^{p} \binom{p}{k} 2^{-k}E_k = 2^{1-p}\left(1+(-3)^{p/2}\right) \quad \text{if } p \equiv 0(6), \tag{7.8}$$

$$2^{-p}E_p + 3 \sum_{\substack{k=0\\k\equiv 2(6)}}^{p} \binom{p}{k} 2^{-k}E_k = 2^{1-p}\left(1+(-3)^{p/2}\right) \quad \text{if } p \equiv 2(6), \tag{7.9}$$

$$2^{-p}E_p + 3 \sum_{\substack{k=0\\k\equiv 4(6)}}^{p} \binom{p}{k} 2^{-k}E_k = 2^{1-p}\left(1+(-3)^{p/2}\right) \quad \text{if } p \equiv 4(6). \tag{7.10}$$

Proof. Let us assume that p is an odd integer greater than 3, then from (7.3), we obtain

$$\sum_{\substack{j=1\\p-j\equiv 0(2)}}^{p} \binom{p}{j} (1-(-\rho)^j)(1+\rho)^{-j} B_{p-j} = \frac{p}{2}(1+\rho)^{1-p}(1+(-\rho)^{p-1}). \tag{7.11}$$

Let us take $\rho = \mu$, with $\mu = -1/2 + i\sqrt{3}/2$. For odd integer $p \geq 3$, using $1/(1+\mu) = -\mu$, we get

$$\sum_{\substack{j=1\\j\equiv 1(2)}}^{p} \binom{p}{j} \left((1+\mu)^{-j} - \left(\frac{-\mu}{1+\mu}\right)^j\right) B_{p-j} = \frac{p}{2}\left((1+\mu)^{1-p} + \left(\frac{\mu}{1+\mu}\right)^{p-1}\right),$$

then

$$\sum_{\substack{j=1\\j\equiv 1(2)}}^{p} \binom{p}{j} \left((-\mu)^j - \mu^{2j}\right) B_{p-j} = \frac{p}{2}\left(\mu^{p-1} + \mu^{2p-2}\right). \tag{7.12}$$

The coefficient $d_j = (-\mu)^j - \mu^{2j}$ have some repeated values according to the value modulo 6 of the odd integer j:

$$\begin{aligned} j &\equiv \pm 1(6), & d_j &= 1 \\ j &\equiv 3(6), & d_j &= -2. \end{aligned}$$

Then (7.12) indicates

$$\sum_{\substack{j=1\\p-j\equiv \pm 1(6)}}^{p} \binom{p}{j} B_{p-j} - 2 \sum_{\substack{j=1\\p-j\equiv 3(6)}}^{p} \binom{p}{j} B_{p-j} = \frac{p}{2}\left(\mu^{p-1} + \mu^{2p-2}\right).$$

By difference with the relation

$$\sum_{\substack{j=1\\p-j\equiv 0(2)}}^{p} \binom{p}{j} B_{p-j} = \frac{p}{2}$$

obtained from (7.11) by setting $\rho = 0$, it arises

$$3 \sum_{\substack{j=0\\p-j\equiv 3(6)}}^{p-1} \binom{p}{j} B_j = -\frac{p}{2}\left(\mu^{p-1} + \mu^{2p-2}\right) + \frac{p}{2},$$

which proves formulas (7.5), (7.6) and (7.7).

In a similar way, by making use of (7.4), we obtain formulas (7.8) – (7.10). ■

Acknowledgment. I would like to thank the anonymous referees for pointing out several errors in earlier versions of this paper.

References

[1] T. Agoh, Recurrences for Bernoulli and Euler polynomials and numbers, Expo. Math. 18 (3) (2000) 197–214.

[2] W. Y. C. Chen, L. H. Sun, Extended Zeilberger's algorithm for identities on Bernoulli and Euler polynomials, J. Number Theory 129 (9) (2009) 2111–2132.

[3] T. M. Apostol, On the Lerch zeta function, Pacific J. Math. 1 (1951) 161–167.

[4] S. Kanemitsu, H. Tsukada, Vistas of special functions, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007.

[5] Q.-M. Luo, On the Apostol-Bernoulli polynomials, Cent. Eur. J. Math. 2 (4) (2004) 509–515.

[6] K. N. Boyadzhiev, Apostol-Bernoulli functions, derivative polynomials and Eulerian polynomials, Adv. Appl. Discrete Math. 1 (2) (2008) 109–122.

[7] Q.-M. Luo, H. M. Srivastava, Some generalizations of the Apostol-Bernoulli and Apostol-Euler polynomials, J. Math. Anal. Appl. 308 (1) (2005) 290–302.

[8] M. Garg, K. Jain, H. M. Srivastava, Some relationships between the generalized Apostol-Bernoulli polynomials and Hurwitz-Lerch zeta functions, Integral Transforms Spec. Funct. 17 (11) (2006) 803–815.

[9] Q.-M. Luo, Apostol-Euler polynomials of higher order and Gaussian hypergeometric functions, Taiwanese J. Math. 10 (4) (2006) 917–925.

[10] Q.-M. Luo, H. M. Srivastava, Some relationships between the Apostol-Bernoulli and Apostol-Euler polynomials, Comput. Math. Appl. 51 (3-4) (2006) 631–642.

[11] Q.-M. Luo, The multiplication formulas for the Apostol-Bernoulli and Apostol-Euler polynomials of higher order, Integral Transforms Spec. Funct. 20 (5-6) (2009) 377–391.

[12] J. Choi, P. J. Anderson, H. M. Srivastava, Some q-extensions of the Apostol-Bernoulli and the Apostol-Euler polynomials of order n, and the multiple Hurwitz zeta function, Appl. Math. Comput. 199 (2) (2008) 723–737.

[13] Q.-M. Luo, Fourier expansions and integral representations for the Apostol-Bernoulli and Apostol-Euler polynomials, Math. Comp. 78 (2009) 2193–2208.

[14] G. A. Baker, Jr., P. Graves-Morris, Padé approximants, 2nd Edition, Vol. 59 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 1996.

[15] C. Brezinski, Convergence acceleration during the 20th century, J. Comput. Appl. Math. 122 (1-2) (2000) 1–21, numerical analysis 2000, Vol. II: Interpolation and extrapolation.

[16] C. Brezinski, Computation of Padé approximants and continued fractions, J. Comput. Appl. Math. 2 (2) (1976) 113–123.

[17] W. B. Jones, W. J. Thron, Continued fractions, Vol. 11 of Encyclopedia of Mathematics and its Applications, Addison-Wesley Publishing Co., Reading, Mass., 1980.

[18] W. Gautschi, Orthogonal polynomials—constructive theory and applications, in: Proceedings of the international conference on computational and applied mathematics (Leuven, 1984), Vol. 12/13, 1985, pp. 61–76.

[19] C. Brezinski, Padé-type approximation and general orthogonal polynomials, Vol. 50 of International Series of Numerical Mathematics, Birkhäuser Verlag, Basel, 1980.

[20] C. Brezinski, A direct proof of the Christoffel-Darboux identity and its equivalence to the recurrence relationship, J. Comput. Appl. Math. 32 (1-2) (1990) 17–25, extrapolation and rational approximation (Luminy, 1989).

[21] M. Prévost, A new proof of the irrationality of $\zeta(2)$ and $\zeta(3)$ using Padé approximants, J. Comput. Appl. Math. 67 (2) (1996) 219–235.

[22] M. Kaneko, A recurrence formula for the Bernoulli numbers, Proc. Japan Acad. Ser. A Math. Sci. 71 (8) (1995) 192–193.

[23] I. M. Gessel, Applications of the classical umbral calculus, Algebra Universalis 49 (4) (2003) 397–434, dedicated to the memory of Gian-Carlo Rota.

[24] M. B. Gel′fand, A note on a certain relation among Bernoulli numbers, Baškir. Gos. Univ. Učen. Zap. Vyp. 31 (Ser. Mat. 3) (1968) 215–216.

[25] H. Momiyama, A new recurrence formula for Bernoulli numbers, Fibonacci Quart. 39 (3) (2001) 285–288.

[26] K.-J. Wu, Z.-W. Sun, H. Pan, Some identities for Bernoulli and Euler polynomials, Fibonacci Quart. 42 (4) (2004) 295–299.

[27] S. Ramanujan, Some properties of Bernoulli's numbers [J. Indian Math. Soc. **3** (1911), 219–234], in: Collected papers of Srinivasa Ramanujan, AMS Chelsea Publ., Providence, RI, 2000, pp. 1–14.

[28] D. H. Lehmer, Lacunary recurrence formulas for the numbers of Bernoulli and Euler, Ann. of Math. (2) 36 (3) (1935) 637–649.

[29] K. G. Hare, Multisectioning, rational poly-exponential functions and parallel computation, M.Sc. Thesis, Mathematics, Simon Fraser University, (1999) 219–234.

[30] Y.-H. Chen, Multiple zeta values and application to the lacunary recurrence formulas of Bernoulli's numbers, Journal of Physics Conference Series 96 (2008) 012212.