Extremal binary matrices without constant 2-squares

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Abstract

In this paper we solve, by computational means, an open problem of Erickson: denoting $[n] = \{1, \ldots, n\}$, what is the smallest integer n_0 such that, for every $n \geq n_0$ and every 2-coloring of the grid $[n] \times [n]$, there is a constant 2-square, i.e. a 2×2 subgrid $S = \{i, i+t\} \times \{j, j+t\}$ whose four points are colored the same? It has been shown recently that $13 \leq n_0 \leq \min(W(2,8), 5 \cdot 2^{2^{40}})$, where $W(2,8)$ is the still unknown 8th classical van der Waerden number. We obtain here the exact value $n_0 = 15$. In the process we display 2-colorings of $[13] \times \mathbb{Z}$ and $[14] \times [14]$ without constant 2-squares, and show that this is best possible.

1 Introduction

For a positive integer n, denote $[n] = \{1, \ldots, n\}$. In his lovely book, Martin J. Erickson posed the following problem [8, p.36].

Open Problem 4. *Find the minimum n such that if the n^2 lattice points of $[n] \times [n]$ are two-colored, there exist four points of one color lying on the vertices of a square with sides parallel to the coordinate axes.*

The author further hints in Exercise 4.24 of [8, p.70] that the least such n satisfies $n \leq 9(2^{81}+1)(2^{(2^{81}+1)^2}+1)$.

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Let us slightly rephrase the problem. A *2-square* in the grid $[n] \times [n]$ is a 2×2 subgrid of the form

$$S = \{i, i+t\} \times \{j, j+t\}$$

for some index $t \geq 1$. It follows from a theorem of Gallai (see Section 5.2) that if n is large enough, then for any 2-coloring

$$c : [n] \times [n] \to \{0, 1\},$$

the grid $[n] \times [n]$ must contain a *monochromatic* 2-square, i.e. whose four points are all colored 0 or 1 under c. The smallest integer n for which this occurs is denoted $R_2(S)$ in [2]. In that paper, it is shown that

$$13 \leq R_2(S) \leq \min(W(2,8), 5 \cdot 2^{2^{40}}),$$

where $W(2,l)$ is the classical lth van der Waerden number. Recall that $W(k,l)$ is the smallest $m \geq 1$ for which any k-coloring of $[m]$ contains a monochromatic subsequence forming an arithmetic progression of length l.

Perhaps surprisingly in view of such bounds, in this paper we solve Erickson's problem by computer and obtain the exact value

$$R_2(S) = 15.$$

The situation may be compared to that of van der Waerden numbers. The best known general upper bound for $W(2,l)$, due to Gowers [9], is

$$W(2,l) \;\leq\; 2^{2^{2^{2^{2^{l+9}}}}}.$$

However, the actual values of $W(2,l)$ for $3 \leq l \leq 6$, namely 9, 35, 178 and 1132, respectively, are much smaller than that. The value $W(2,6) = 1132$ has been obtained recently by clever reductions and massive parallel computations with a specially designed SAT solver [12]. At the time of writing, the numbers $W(2,l)$ are unknown for $l \geq 7$. See [10] for many more papers on Ramsey theory.

1.1 Erickson matrices

For convenience, we shall rather adopt the language of binary matrices, i.e. matrices with coefficients in the 2-element field $\mathbb{F}_2$. As for grids, a *2-square* in a matrix A is a 2×2 submatrix S with row indices $\{i, i+t\}$ and column indices $\{j, j+t\}$ for some $t \geq 1$. In this case, we say that S is of *span* $t+1$.

Definition 1.1 *An Erickson matrix is a binary matrix containing no constant 2-squares.*

We shall actually refine our claimed solution $R_2(S) = 15$ as follows.

Theorem 1.2 *There exist Erickson matrices of sizes* $13 \times \infty$ *and* 14×14. *There exist no Erickson matrices of size* 14×15.

In other words, every 2-coloring of the grid $[14]\times[15]$ contains a monochromatic 2-square. Conversely, there do exist 2-colorings of the grids $[13] \times \mathbb{Z}$ and $[14] \times [14]$ without monochromatic 2-squares.

The paper is organized as follows. In Section 2, we investigate Erickson matrices with special properties, such as antisymmetry, symmetry, and with constant first row, and obtain the maximal possible size in each case. In Section 3, we explain the backtracking algorithm we have used in the general case. In Section 4, we exhibit an extremal Erickson matrix of size $13 \times \infty$, and give a parametric description of all square Erickson matrices of size 14 up to automorphisms. This allows us to show that there are no Erickson matrices of size 14×15. In the last section, we discuss related structures such as Erickson tori, Erickson triangles and higher dimensional analogues, and conclude with some historical remarks about a theorem of Gallai which guarantees the existence of these objects.

Throughout the paper, we shall interchangeably display binary matrices as plain matrices or as black-and-white boards, with white for 0 and black for 1.

2 Three special cases

This work started with the objective of constructing a large Erickson matrix by computer. In order to cut down the volume of computations, we chose to focus on skew matrices with zero diagonal (see below). The first surprise occurred: even though millions of such matrices were found in size $n \times n$ with $n \leq 10$, only very few remained for $n = 11$ already, and none for $n \geq 12$. This gave a strong hint that in the general case, the maximum size of an Erickson matrix would not be too far from 11, and be actually reachable by exhaustive computer search. This turned out to be true: the maximum size of a square Erickson matrix is found here to be $n = 14$.

In this section, we study extremal square Erickson matrices in three special cases: skew, symmetric, and with constant first row. The maximum admissible size in each case turns out to be 11, 8 and 9, respectively (see below).

The construction method by computer in these instances does not need to be very sophisticated. In its most basic version, for each $n \geq 1$, one stores the set of Erickson matrices of size n with the given property, up to some automorphisms to reduce the load. A somewhat simple extension process then allows to pass from n to $n+1$. One useful trick consists in *completing constant elbows*. This means that, whenever an unassigned entry a_{ij} lies in a 2-square whose three other entries are equal to $\epsilon \in \mathbb{F}_2$, then a_{ij} must be set to $1-\epsilon$.

The reader may sense the importance of this trick by trying it out on this partially filled matrix:

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & \square \\ 0 & 0 & 0 & 0 & \square \\ 0 & 1 & 1 & 0 & \square \\ 0 & \square & \square & \square & 0 \end{pmatrix}.$$

By completing constant elbows, a few seconds suffice to check by hand that *there is a unique* 5×5 *Erickson matrix coinciding with the partial matrix* A.

2.1 Skew Erickson matrices

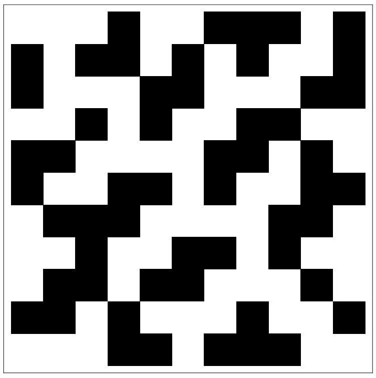
We start with *skew* binary matrices, i.e. square matrices $A = (a_{ij})$ over $\mathbb{F}_2$ satisfying $a_{ji} = 1 - a_{ij}$ for all $i \neq j$. No hypothesis is made on the diagonal entries a_{ii}. Those entries are in fact irrelevant in the present case for the Erickson property, as shown now.

Proposition 2.1 *Let A be a skew Erickson matrix. Denote by B any matrix obtained from A by negating some diagonal entries. Then B is also a skew Erickson matrix.*

Proof. Set $A = (a_{ij})$. We may assume that in B, a single diagonal entry of A is negated, say a_{ii} is replaced by $1 - a_{ii}$. Of course, B is skew since A is. Assume that B contains a constant 2-square S. Then one of the four positions of S is (i,i). This position cannot be the NW or SE corner of S, for otherwise S would also be skew, a contradiction. Therefore (i,i) must either be the NE or SW corner of S, and hence S is entirely contained in either the upper or the lower triangular part of B. Denote S' the symmetric image of S under transposition. Then one of the entries of S' is still $1 - a_{ii}$ at position (i,i) as in S, but the other three entries of S', being off-diagonal, are negated with respect to the corresponding entries in S. This implies that S' is in fact a constant 2-square in the original matrix A, a contradiction. ■

Consequently, when investigating skew Erickson matrices, we may freely assume that the diagonal is zero. This further reduces the volume of computations needed. Our computational result is that the largest possible size of a skew Erickson matrix is $n = 11$.

Theorem 2.2 *There exist no skew Erickson matrices of size* $n \geq 12$*. In size* $n = 11$*, there exist exactly 8 skew Erickson matrices with zero diagonal; up to automorphisms, they are all equivalent to the following one:*



Note that the automorphism group of the set of skew Erickson matrices is of order 8. It is generated by transposition, half-turn rotation, and negation of all the entries.

2.2 Symmetric Erickson matrices

The largest possible size for a symmetric Erickson matrix turns out to be 8. An example is given in Figure 1. The automorphism group of the set of symmetric Erickson matrices is the same as in the skew case.

Theorem 2.3 *There are no symmetric Erickson matrices of size $n \geq 9$. There are exactly 152 symmetric Erickson matrices of size 8, partitioned into 38 orbits under the automorphism group.*

More precisely, the 152 symmetric Erickson matrices of size 8 can be partitioned into five families $S_1, \ldots, S_5$, as follows.

The family S_1 is given by

$$S_1 = \begin{pmatrix} 0 & 0 & x_1 & 1 & 1 & 1 & 1 & x_2 \\ 0 & 1 & 0 & 0 & 1 & 0 & x_3 & 0 \\ x_1 & 0 & 1 & 0 & 0 & x_4 & 1 & 0 \\ 1 & 0 & 0 & 1 & x_5 & 1 & 0 & 0 \\ 1 & 1 & 0 & x_5 & 0 & 1 & 1 & 0 \\ 1 & 0 & x_4 & 1 & 1 & 0 & 1 & 1 \\ 1 & x_3 & 1 & 0 & 1 & 1 & 0 & 1 \\ x_2 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

subject to $(x_1, x_2, x_4) \neq (1, 1, 1)$. This family gives rise to 20 orbits. Eight orbits correspond to $x_1 = 0$ and come in pairs $(x_1 = 0, x_2, x_3, x_4, x_5), (x_1 = 0, 1 - x_2, 1 - x_3, 1 - x_4, 1 - x_5)$. The twelve solutions arising from $x_1 = 1$ give 12 more orbits which are all distinct.

The family

$$S_2 = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 & x_1 \\ 0 & 1 & 0 & 0 & 1 & 0 & x_2 & 0 \\ 1 & 0 & 1 & 0 & 0 & x_3 & 1 & 0 \\ 1 & 0 & 0 & 1 & x_4 & 1 & 0 & 0 \\ 1 & 1 & 0 & x_4 & 0 & 1 & 1 & 0 \\ 1 & 0 & x_3 & 1 & 1 & 0 & 1 & 0 \\ 1 & x_2 & 1 & 0 & 1 & 1 & 0 & 1 \\ x_1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

yields 8 orbits corresponding to the pairs $(x_1, x_2, x_3, x_4), (1 - x_1, 1 - x_2, 1 - x_3, 1 - x_4)$.

The family S_3 is given by

$$S_3 = \begin{pmatrix} x_1 & 1 & x_2 & 0 & 1 & 0 & 0 & x_3 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ x_2 & 1 & 0 & 1 & 1 & x_4 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & x_5 \\ 0 & 0 & x_4 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ x_3 & 1 & 0 & 0 & x_5 & 0 & 0 & 1 \end{pmatrix},$$

subject to the inequalities

$$(x_1, x_2) \neq (0,0),\ (x_1, x_3) \neq (1,1),\ (x_3, x_4) \neq (0,0),\ (x_3, x_5) \neq (0,0)\ .$$

This leaves 7 possibilites for the parameters giving rise to 7 orbits.

The next family

$$S_4 = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & x_1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & x_1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

gives rise to 2 orbits, and the last family represented by

$$S_5 = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

gives rise to a unique orbit.

Summarizing, the orbits of these families sum up to $38 = 20+8+7+2+1$, representing all 152 symmetric Erickson matrices of size 8×8.

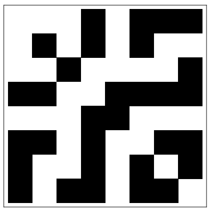


Figure 1: An extremal symmetric Erickson matrix of size 8

2.3 Erickson matrices with constant first row

We have found by computer that the largest size of a square Erickson matrix with constant first row is 9. It is probably possible to obtain this result by a case-by-case analysis carried out by hand. Indeed, this is how the authors of [2] have shown that a square Erickson matrix with constant *middle row* has maximal size 7, thereby proving that $R_2(S) \leq W(2,8)$.

Theorem 2.4 *There are square Erickson matrices with constant first row of size $n = 9$, but none of size $n \geq 10$. The following is an instance of size 9:*

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&0 \\ 0&1&0&1&0&1&0&1&0 \\ 0&0&1&1&0&0&1&1&0 \\ 1&0&0&0&1&1&1&0&0 \\ 0&1&1&0&1&0&0&1&0 \\ 1&1&0&0&0&0&1&1&1 \\ 0&1&0&1&1&0&1&0&1 \\ 1&0&1&1&0&0&0&0&1 \\ 0&0&1&0&0&1&0&1&1 \end{pmatrix}$$

3 The general case

We start here our study of general square Erickson matrices. After a brief description of their automorphism group, we explain the algorithm we have used to construct them and to uncover their maximal possible size.

3.1 Automorphisms

The automorphism group G of the set of square Erickson matrices is of order 16. It is isomorphic to the direct product

$$G \cong D_4 \times C_2,$$

where D_4 is the dihedral group of order 8 preserving the square, and C_2 is the group of order 2 negating the entries, i.e. exchanging 0 and 1.

3.2 The algorithm

Consider binary variables $x_1, x_2, x_3, \ldots$ organized in an array X as follows:

$$X \;=\; \begin{pmatrix} x_1 & x_3 & x_6 & x_{11} & x_{18} & \cdots \\ x_2 & x_4 & x_8 & x_{13} & \cdots & \cdots \\ x_5 & x_7 & x_9 & x_{15} & \cdots & \cdots \\ x_{10} & x_{12} & x_{14} & x_{16} & \cdots & \cdots \\ x_{17} & x_{19} & \cdots & & \ddots & \\ \vdots & \vdots & & & & \ddots \end{pmatrix}$$

Thus, for each $n \geq 1$, the first n^2 variables lie in an $n \times n$ square sub-array of X. Denote by $\bar{X}$ the set of all partial binary specialisations of X, where a finite number of variables x_i have been assigned a value in $\mathbb{F}_2$. We now describe a simple forcing procedure

$$\varphi : \bar{X} \to \bar{X},$$

meant to avoid constant 2-squares whenever possible. It is of course analogous to the process of "completing constant elbows" discussed in Section 2. Let $A \in \bar{X}$. If some unassigned variable x_t in A lies in a 2-square S whose three other entries are all equal in A to some $\varepsilon \in \mathbb{F}_2$, then set $x_t = 1 - \varepsilon$ in A and define $\varphi(A) \in \bar{X}$ to be the resulting partially specialized array. In other words, we set

$$\varphi(A) \;=\; A_{|x_t = 1-\varepsilon}.$$

In order for this to be well-defined, we may choose t to be the least index, if any, with the required property, and at the same time choose S to be the lexicographically smallest corresponding 2-square. If there is no such index t, then we set

$$\varphi(A) = A.$$

Note that, even if A does not contain constant 2-squares, it may well happen that $\varphi(A)$ does. This will happen if x_t lies in two 2-squares S, S', one of which having three 0's in A, and the other one having three 1's in A.

Of course, the simple forcing procedure φ may be iterated. Of special interest to us are the fixed points of φ. They are reached in a finite number of steps, depending of course on the initial argument A.

Let now $m \geq 0$, and denote by E_m the set of all partial binary specialisations $A \in \bar{X}$ with the following properties:

(1) The first m variables $x_1, \ldots, x_m$ have been specialized to 0 or 1 in A.

(2) A is the final fixed point of the forcing procedure φ applied iteratively on the element A' with all variables x_k unassigned for $k \geq m+1$ and coinciding with A for its first m coefficients $x_1, \ldots, x_m$.

(3) A contains no constant 2-squares on its assigned entries.

Note that, for each $A \in E_m$, most variables x_i with $i \geq m+1$ remain unassigned. But some of those variables may have been forced to a value 0 or 1 in A in order to satisfy the required properties. Indeed, even if only $x_1, \ldots, x_m$ have been specialized at first, further variables might get specialized by successive applications of φ.

Observe that the set E_{n^2} is in one-to-one correspondence with the set of all Erickson matrices of size $n \times n$.

The union $\bigcup_{m \geq 0} E_m$ has the structure of a rooted plane tree, where the root is the unique element of E_0 having no assigned variables. Given an element $A \in E_m$, its immediate left (respectively right) successor, if it exists, is the unique element $\tilde{A}$ of E_{m+1} which coincides with A for $x_1, \ldots, x_m$ and satisfies $x_{m+1} = 0$ (respectively $x_{m+1} = 1$).

Note that the tree $\bigcup_{m \geq 0} E_m$ is finite. This follows from the fact that square Erickson matrices are uniformly bounded in size.

Our algorithm is now the classical depth-first algorithm for visiting all vertices of a finite rooted plane tree: start at the root in the direction of its leftmost son and continue walking turning always to the left at each bifurcation, respectively turning around and backtracking when hitting a leaf.

An important point is that this algorithm has only very small memory requirements: we do not need to store the tree $\bigcup_{m \geq 0} E_m$, we only store the currently used element in E_m and use it to compute immediate successors or the unique immediate predecessor. This is crucial, since there happen to be more than 10^{12} Erickson matrices of size 9×9. (See below.)

Using the fact that the two subtrees issued from the two elements of E_1 (corresponding to $x_1 = 0$ and $x_1 = 1$ with all other entries unassigned) are mirrors of each other, we can halve the total amount of work by visiting only the subtree issued from the leftmost vertex (corresponding to $x_1 = 0$) of E_1. A further improvement is obtained by removing subtrees issued from solutions $A \in E_{n^2}$ such that A is lexicographically after the transposed solution A^t. The resulting subset of leaves in E_{n^2} then contains all Erickson matrices of size $n \times n$, up to transposition and up to negation $0 \leftrightarrow 1$ of all coefficients.

3.3 Outcome

Denote by $\mathrm{Er}(n)$ the number of square Erickson matrices of size n and with NW corner equal to 0. Thus, the total number of square Erickson matrices of size n is equal to $2\,\mathrm{Er}(n)$. Running the above algorithm yields the following values of $\mathrm{Er}(n)$, for $n = 2, 3, \ldots, 15$. The square Erickson matrices of maximal size 14 will be described in the next section.

n	$\mathrm{Er}(n)$
2	7
3	138
4	5490
5	390856
6	29169574
7	1533415720
8	29085496072
9	156515895928
10	54978562276
11	2510360996
12	1990028
13	570132
14	116114
15	0

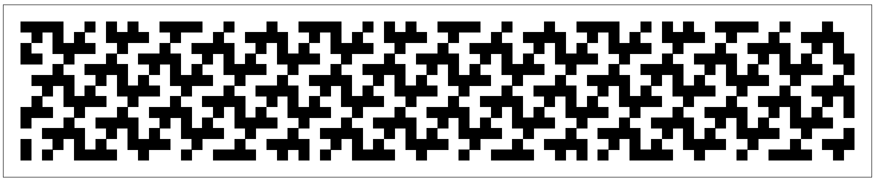


Figure 2: A $13 \times \infty$ Erickson matrix

4 Extremal Erickson matrices

The main computational result of this paper, obtained with the backtracking algorithm of Section 3.2, is the following.

Theorem 4.1 *There exist no Erickson matrices of size* $m \times n$ *with* $m \geq 14$ *and* $n \geq 15$. *These bounds are sharp.*

The next subsections exhibit extremal Erickson matrices, i.e. of size $13 \times \infty$ and 14×14. In the latter case, we give a complete parametric description of the possible matrices up to automorphisms. This parametrization is then used in Section 4.3 to show that there are no Erickson matrices of size 14×15.

4.1 An Erickson matrix of size $13 \times \infty$

We now exhibit an Erickson matrix of size $13 \times \infty$. The first row is periodic with period s of length 26, where

$$s = (1, 1, 1, 1, 0, 0, 1, 0, 1, 0, 1, 0, 0, 1, 1, 1, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0).$$

Theorem 4.2 *Let* A *be the binary matrix indexed over* $[13] \times \mathbb{Z}$ *defined as follows: the first row of* A *is 26-periodic with period* s, *and each subsequent row is obtained by rotating the preceding one 5 units to the left. Then* A *is a* $13 \times \infty$ *Erickson matrix.*

This construction can be enlarged to get a doubly-infinite binary matrix containing no constant 2-squares of span $t + 1 \leq 13$. The reader might appreciate the Escher-like structure of the corresponding grid coloring. (See Figure 3.)

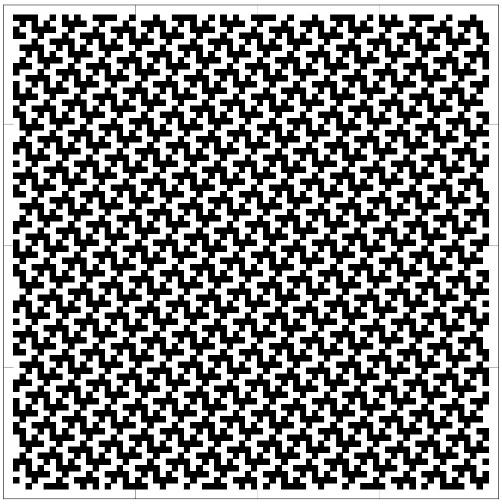


Figure 3: A binary coloring of $\mathbb{Z} \times \mathbb{Z}$ without constant 2-squares of span $t + 1 \leq 13$

Theorem 4.3 *Let B be the doubly-infinite, doubly-periodic binary matrix of biperiod 26×26, indexed over $\mathbb{Z} \times \mathbb{Z}$, whose first row is 26-periodic with period s, and where each row is obtained by translating the preceding one 5 units to the left. Then B contains no constant 2-squares of span less than or equal to 13.*

More Erickson matrices of size $13 \times \infty$ can be obtained from square Erickson *tori* of size 13. See Section 5.1.

4.2 Erickson matrices of size 14×14

We describe here the set of all extremal square Erickson matrices of size 14×14, harvested with the backtracking algorithm of Section 3.2.

For $n = 14$, the total number of binary Erickson matrices of size 14×14 is equal to 232228. These matrices are partitioned into exactly 14557 pairwise disjoint orbits under the automorphism group G. More precisely, these 14557 orbits divide up into exactly

- 14481 orbits of full length 16,
- 57 orbits of length 8, and

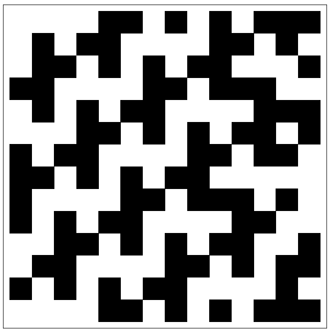


Figure 4: The lexicographically first square Erickson matrix of size 14

- 19 orbits of length 4.

The $8 \cdot 57$ Erickson matrices with a G-orbit of length 8 are all invariant under a half-turn rotation. The same is true for the $4 \cdot 19$ Erickson matrices with a G-orbit of length 4, which are further invariant under negation $0 \leftrightarrow 1$ of all entries followed by a 1/4-turn or a 3/4-turn rotation.

For the record, Figure 4 displays the lexicographically smallest Erickson matrix of size 14×14.

We now list the complete set of 232228 square Erickson matrices of size $n = 14$ up to automorphisms. They are partitioned into four families A_1, A_2, A_3, A_4.

4.2.1 The family A_1

$$A_1 = \begin{pmatrix} x_1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & x_2 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & x_3 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & x_4 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & x_5 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & x_6 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & x_7 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & x_8 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & x_9 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & x_{10} & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & x_{11} & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & x_{12} & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & x_{13} & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & x_{14} & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ x_{15} & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & x_{16} \end{pmatrix}$$

Any binary assignment of the 16 variables in A_1 yields an Erickson matrix, provided

$$(x_1, x_2, x_{15}, x_{16}) \neq (0,0,0,0), \;\; (x_1, x_2, x_{15}, x_{16}) \neq (1,1,1,1). \tag{1}$$

Thus, the binary specializations of A_1 yield $14 \cdot 2^{12} = 57344$ Erickson matrices. Under the action of the automorphism group G, they yield a total of 229376 Erickson matrices.

4.2.2 The family A_2

$$A_2 = \begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & x_1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & x_2 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & x_3 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & x_4 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & x_5 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & x_6 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
x_7 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}$$

Here the situation is simpler. Any binary assignment of the 7 variables in A_2 yields an Erickson matrix. This amounts to 128 Erickson matrices. As these matrices all have a G-orbit of length 16, they yield a total of $2^{11} = 2048$ Erickson matrices under the action of G.

4.2.3 The family A_3

$$A_3 = \begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & x_1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & x_2 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & x_3 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & x_4 & 1 & 1 & 0 & 1 & x_5 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}$$

Here again the situation is simple. Any binary assignment of the 5 variables in A_3 yields an Erickson matrix, with a G-orbit of length 16. Thus, this family yields a total of $2^9 = 512$ Erickson matrices under automorphisms.

4.2.4 The family A_4

$$
A_4 = \begin{pmatrix}
x_1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & x_2 & 1 & 1 & 0 & 1 & x_3 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
x_4 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & x_5 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
x_6 & 1 & 0 & 1 & 1 & x_7 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & x_8
\end{pmatrix}
$$

In the present case, a binary assignment of the 8 variables in A_4 yields an Erickson matrix if and only if

$$(x_1, x_3, x_6, x_8) \neq (0, 0, 0, 0),\ (x_1, x_3, x_6, x_8) \neq (1, 1, 1, 1), \tag{2}$$

$$(x_1, x_2) \neq (1, 1),\ (x_7, x_8) \neq (1, 1),\ (x_3, x_5) \neq (0, 0),\ (x_4, x_6) \neq (0, 0). \tag{3}$$

Because of symmetries in A_4, the orbits of its various specializations attain lengths 4, 8 and 16. This family, under the above affine constraints and the action of G, yields a total of 292 Erickson matrices.

Note that, up to a permutation of indices and up to negation of all entries, the matrices A_1 and A_4 are invariant by a rotation of order 4.

4.2.5 Summary

Summarizing the above, we obtain the following complete description of square Erickson matrices of size $n = 14$.

Theorem 4.4 *A binary square matrix of size 14 is an Erickson matrix if and only if it is equivalent, up to automorphisms, to a binary specialization of either A_1 under constraints (1), or A_2, or A_3, or A_4 under constraints (2,3). There are exactly* 232228 *square Erickson matrices of size 14, where*

$$232228 = 229376 + 2048 + 512 + 292$$

according to the partition into types A_1, A_2, A_3 and A_4.

4.2.6 Some linear algebra remarks

The matrices A_1, A_2, A_3, A_4 and closely related ones, viewed as real polynomial matrices, have the following selected linear algebraic properties.

• In A_1, the assignment $x_{10} = x_{13} = x_{14} = x_{15} = 1$ and $x_i = 0$ for all other indices $i \in [16]$ yields an Erickson matrix with determinant equal to 3^8. Similarly, in A_2, the assignment $(x_1, x_2, \ldots, x_7) = (0, 1, 0, 0, 0, 1, 1)$ yields an Erickson matrix with determinant equal to 5^5.

• The assignment $x_1 = 1$ in A_2 leads to a singular matrix with identical first and last rows. The determinant of A_2 is thus divisible by $x_1 - 1$.

• The determinant of A_3 is equal to

$$3(x_1-1)(990+35x_2-90x_3-495x_5-80x_2x_3-123x_2x_5+45x_3x_5+23x_2x_3x_5) \; .$$

Replacing every occurrence of 0 by -1 in A_3 and keeping all other entries unchanged, the resulting matrix A_3' with entries in $\{-1, 1, x_1, x_2, x_3, x_4, x_5\}$ has an even simpler determinant, equal to

$$-12800(x_1 - 1)(x_2 + 1)(x_3 + 5)(x_5 - 1) \; .$$

• Here again, the matrix A_4' with entries in $\{-1, 1, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$, obtained by replacing every occurrence of 0 by -1 in A_4 and keeping all other coefficients unchanged, has a remarkably simple determinant, equal to

$$460800(x_2 - x_6)(x_8 - x_4) \; .$$

• The characteristic polynomial of A_4 is divisible by $(X^2 - 5)(X^2 + 3)$.

• Every 14×14 Erickson matrix has rank equal to 13 or 14.

4.3 The case of size 14×15

For the proof of Theorem 4.1 to be complete, it remains to show that *there are no Erickson matrices of size* 14×15. This can be easily checked as follows, by using our parametric description of Erickson matrices of size 14×14. Up to reversion and negation, every first or last row or column of a 14×14 Erickson matrix is of one of the forms

$$\begin{aligned}&(0,0,0,0,1,1,0,1,*,1,0,1,1,1),\\&(0,1,0,0,1,1,1,1,0,0,1,0,*,0),\\&(x_1,0,1,0,0,1,1,1,x_2,0,0,1,0,*),\end{aligned}$$

with $(x_1, x_2) \neq (0,0)$ and with $*$ an arbitrary element of $\{0,1\}$. The first two rows arise in the families A_2 and A_3, and the last row occurs in A_4 and with x_2 specialized to 1 in A_1.

The presumed existence of a 14×15 Erickson matrix would imply that one of these rows arises as the second or second-to-last row or column of a 14×14 Erickson matrix. However, a direct inspection on A_1, A_2, A_3, A_4 shows that this is not the case, thereby settling our claim.

5 Related structures

We discuss here some variants of the notion of Erickson matrices.

5.1 Erickson tori of square size

Beside 2-colorings of the grids $[m] \times [n]$, it is also interesting to consider 2-colorings of toroidal grids. Matricially, this amounts to binary matrices with indices in $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. We define an *Erickson torus* of size $m \times n$ to be an array indexed by $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$, containing no constant 2-squares with indices of the form $\{(i,j),(i+t,j),(i,j+t),(i+t,j+t)\} \subset \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ for $t = 1, \ldots, \min(n,m) - 1$.

For each $n \geq 2$, the number $\tau(n)$ of Erickson tori of square size $n \times n$ is given by the following computer-constructed table.

n	$\tau(n)$
2	14
3	156
4	6112
5	114040
6	878448
7	360360
8	160416
$9, 10$	0
11	180224
12	0
13	427336
≥ 14	0

Of course, Erickson tori give rise to Erickson matrices. However, they behave somewhat differently. In particular, subarrays of Erickson tori are not generally Erickson tori of smaller size. This is a partial explanation of the gaps for $n = 9, 10, 12$ in the table above. Moreover, the group of automorphisms acting on Erickson tori of square size is larger than the group of automorphisms acting on Erickson matrices. It contains of course all 16 elements preserving Erickson matrices, but it also contains all shifts of indices corresponding to cyclic permutations of rows and columns. Moreover, in the case of Erickson tori of square size $n \times n$, we get all dilatations

$$a_{i,j} \longmapsto a_{\lambda i, \lambda j}$$

with $\lambda \in (\mathbb{Z}/n\mathbb{Z})^*$.

Erickson tori of square size $n \times n$ give rise to doubly periodic infinite binary arrays containing no constant 2-squares of size at most n. Every row and column of such an array defines an n-periodic binary sequence.

A particularly nice class of examples is given by Erickson tori associated to n-periodic row-sequences differing only by shifts of their indices. Such an Erickson torus of size $n \times n$ is thus completely described (up to a translation of its indices) by a row-period $(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \in \{0, 1\}^n$ and by the shift κ relating two adjacent rows. Reading indices modulo n, the associated infinite array is thus

$$\begin{array}{cccccccc} & \vdots & \vdots & \vdots & & & & \\ \cdots & \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & \alpha_0 & \cdots \\ \cdots & \alpha_\kappa & \alpha_{1+\kappa} & \alpha_{2+\kappa} & \cdots & \alpha_{n-1+\kappa} & \alpha_\kappa & \cdots \\ \cdots & \alpha_{2\kappa} & \alpha_{1+2\kappa} & \cdots & & & & \\ & \vdots & \vdots & & & & & \end{array}$$

Note that replacing the shift value κ by $n-\kappa$ gives rise to an isomorphic solution.

Examples of such "sequential" Erickson tori of square size exist for all relevant dimensions. The following table gives for each size n a full period $(\alpha_0, \ldots, \alpha_{n-1}) \in \{0,1\}^n$ together with the value κ indicating the shift of indices between adjacent rows.

n	$(\alpha_0, \alpha_1, \ldots, \alpha_{n-1})$	κ
2	$(0,1)$	1
3	$(0,1,1)$	1
4	$(0,0,1,1)$	$1,2$
5	$(0,0,1,0,1)$	2
6	$(0,0,0,1,1,1)$	2
7	$(0,0,0,1,0,1,1)$	$2,3$
8	$(0,0,1,0,1,1,0,1)$	3
11	$(0,0,0,1,0,0,1,0,1,1,1)$	$2,5$
13	$(0,0,0,1,0,0,1,1,1,1,0,0,1)$	5

5.2 Gallai's theorem

The presence of a constant 2-square in any binary square matrix of sufficiently large size follows from a d-dimensional generalization of van der Waerden's theorem due to Tibor Gallai in the 1930's. See e.g. [1] or [11, page 40]. Our attempt to locate the first written mention of this theorem revealed a somewhat confused situation, which we have tried to clarify and put in historical context in the last section. Here is Gallai's theorem as stated in [1], where a nice proof is included.

Theorem 5.1 *Let S be a finite subset of $\mathbb{N}^d$. For any k-coloring of $\mathbb{N}^d$, there is a positive integer $a \geq 1$ and a point $v \in \mathbb{N}^d$ such that the set $aS+v$ — i.e.,* *some translation of some dilation of S — is monochromatic. Furthermore, the dilation factor a and the coordinates of the translation point v are bounded by a function that depends only upon the set S and the number k (and not upon the particular coloring used).*

Erickson matrices correspond to the case $d = k = 2$ and

$$S = \{(0,0), (0,1), (1,0), (1,1)\}.$$

A simpler case, still with $d = k = 2$ but with $S = \{(0,0), (1,0), (0,1)\}$, corresponds to what might be called *Erickson triangles*. The maximal size of such a triangle without constant sub-triangle can be determined by hand, and is equal to 4. Thus, every binary triangular array

$$\begin{array}{llll} x_{11} & & & \\ x_{21} & x_{22} & & \\ x_{31} & x_{32} & x_{33} & \\ \vdots & \vdots & & \ddots \end{array}$$

of size at least 5 contains a constant subtriangle of the form $x_{ij} = x_{(i+t)j} = x_{(i+t)(j+t)}$. See e.g. [2, Lemma 1].

More generally, for $s \geq 2$, we define an *s-square* in $[n] \times [n]$ to be any $s \times s$ square subgrid of the form

$$\{i, i+t, \dots, i+(s-1)t\} \times \{j, j+t, \dots, j+(s-1)t\}$$

with s^2 points. For any positive integer $k \geq 2$, let us denote by $n(k,s)$ the smallest integer n for which any k-coloring of the grid $[n]^2$ contains a monochromatic s-square. Gallai's theorem ensures that $n(k,s)$ is finite. It would be interesting to understand the behavior of $n(k,s)$ beyond the case $k = s = 2$, for which $n(2,2) = 15$ by Theorem 1.2. The next interesting numbers to determine are

- $n(3,2)$, forcing constant 2-squares for any 3-coloring of $[n(3,2)]^2$,
- $n(2,3)$, forcing constant 3-squares for any 2-coloring of $[n(2,3)]^2$.

This may of course be generalized to any dimension $d \geq 3$. Besides constant "hypercubes", the simplest non-trivial case for $d = 3$ is given by

$$S = \{(0,0,0), (1,0,0), (0,1,0), (0,0,1)\}$$

and gives rise to what might be called "3−dimensional Erickson simplices." Here again, it would be most interesting to determine the smallest n forcing the presence of a monochromatic Erickson simplex in a binary cube of size $n \times n \times n$.

5.3 Historical remarks

[Warning: the information reported here still requires confirmation and approval from various sources.]

We take this opportunity to try and clarify some bibliographical and historical details, resulting from our attempt to locate the first written source of Gallai's theorem. Tibor Gallai himself apparently never published it. Rather, he told it to Richard Rado, who seemingly first stated it in [15]. (But not in his earlier paper [14], as mistakenly stated[1] in [7].) A prior and weaker version of Gallai's theorem appears in [13]. Quoting from Rado in [15, p. 123]:

> "This extension *[of van der Waerden's theorem]* was first proved by Dr. G. Grünwald, who kindly communicated it to me."

Yes, Grünwald is the original name of Gallai. The change from German-sounding to more Hungarian-sounding names was frequent for Hungarian Jews up to the 1930's, in a context of social pressure and harsh antisemitism [3, 16]. Yet the initial "G" in the above quotation is quite mysterious and confusing, as it should have been "T" for Tibor. Indeed, there is another Hungarian-Jewish mathematician called Géza Grünwald, not a relative but a close friend of Tibor Grünwald, and who died at age 31 in tragic circumstances during WWII [3, 16]. Another source of confusion comes from the fact that both Tibor and Géza Grünwald wrote joint papers with their common friend Paul Erdős. However, there is no controversy at all, and Tibor Grünwald is the author of Gallai's theorem. We will probably never know why Rado wrote "G. Grünwald" rather than "T. Grünwald" in [15]. See [4, 6] for moving obituaries of Gallai. See also [5].

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[1]Ron Graham kindly confirmed this to us

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