# ASYMPTOTIC BEHAVIOUR OF LARGE EIGENVALUES OF JAYNES-CUMMINGS TYPE MODELS

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Abstract. We consider a class of unbounded self-adjoint operators including the Hamiltonian of the Jaynes-Cummings model without rotating-wave approximation (RWA). The corresponding operators are defined by infinite Jacobi matrices with discrete spectrum. Our purpose is to give an asymptotic description of large eigenvalues.

## 1. Introduction

1.1. **Formulation of the result.** We consider an infinite real Jacobi matrix

$$J = \begin{pmatrix} d(1) & a(1) & 0 & 0 & 0 & \dots \\ a(1) & d(2) & a(2) & 0 & 0 & \dots \\ 0 & a(2) & d(3) & a(3) & 0 & \dots \\ 0 & 0 & a(3) & d(4) & a(4) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{1.1}$$

where the form of entries $\{d(k)\}_{k=1}^{\infty}$, $\{a(k)\}_{k=1}^{\infty}$ is motivated by the structure of the Hamiltonian of the Jaynes-Cummings model without rotating-wave approximation (RWA). Following [9] this model can be described by using the entries of the form

$$\begin{cases} d(k) = k + (-1)^k \rho \\ a(k) = a_1 k^{1/2} \end{cases} \tag{1.2}$$

where $\rho \in \mathbb{R}$ and $a_1 > 0$ are some constants. We write $x \in l^2 = l^2(\mathbb{N}^*)$ ($\mathbb{N}^* = \{1, 2, \dots\}$) if and only if $x : \mathbb{N}^* \to \mathbb{C}$ satisfies

$$\|x\|_{l^2} := \Big( \sum_{k=1}^{\infty} |x(k)|^2 \Big)^{1/2} < \infty,$$

we denote

$$\mathcal{D} := \{x \in l^2 : \sum_{k=1}^{\infty} d(k)^2 |x(k)|^2 < \infty\} \tag{1.3}$$

and define $J : \mathcal{D} \to l^2$ by the formula

$$(Jx)(k) = d(k)x(k) + a(k)x(k+1) + a(k-1)x(k-1) \tag{1.4}$$

where we assume $x(0) = 0$ and $a(0) = 0$. It is well known that $J$ is bounded from below self-adjoint operator with compact resolvent in the Hilbert space $l^2$ equipped with the standard scalar product $\langle x, y\rangle_{l^2} = \sum_{k=1}^{\infty} \overline{x(k)} y(k)$. Therefore one can find an

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orthogonal basis $\{w_n\}_{n=1}^{\infty}$ such that $Jw_n = \lambda_n(J)w_n$ $(n \in \mathbb{N}^*)$ where $\{\lambda_n(J)\}_{n=1}^{\infty}$ is the non-decreasing sequence of real eigenvalues. The aim of this paper is to describe the asymptotic behaviour of $\lambda_n(J)$ when $n \to \infty$. We will prove

**Theorem 1.1.** *Assume that $J$ is defined as above and $\rho < \frac{1}{2}$. Then the asymptotic formula*

$$\lambda_n(J) = n - a_1^2 + O(n^{-1/4}\ln(n)) \tag{1.5}$$

*holds when $n \to \infty$.*

In Section 1.2 we compare our results with other known results, in Section 1.3 we formulate Theorem 1.2 which is a generalization of Theorem 1.1 motivated by a model of interactions with $N$-level atoms (cf. A. Boutet de Monvel, S. Naboko, L. O. Silva [1]). In particular Theorem 1.2 allows us to establish asymptotic formula for $\lambda_n(J)$ when the entries satisfy

$$\begin{cases} d(k) = k + v(k) \\ a(k) = a_1 k^{\gamma} + a_1' k^{\gamma-1} + O(k^{\gamma-2}) \end{cases} \tag{1.6}$$

where $0 < \gamma \le 1/2$, $a_1 > 0$, $a_1' \in \mathbb{R}$ are fixed and $v : \mathbb{N}^* \to \mathbb{R}$ is real valued and periodic of period $N$, i.e.

$$v(k+N) = v(k) \text{ for } k \in \mathbb{N}^*. \tag{1.7}$$

In Section 1.4 we present a plan of the paper.

1.2. **Discussion.** In this section we describe known results about the asymptotic behaviour of large eigenvalues for so called modified Jaynes-Cummings models, i.e. when the entries of $J$ satisfy

$$\begin{cases} d(k) = k^{\alpha} + v(k) \\ a(k) = a_1 k^{\gamma} \end{cases} \tag{1.8}$$

where $\alpha > \gamma > 0$, $a_1 > 0$ are fixed and $v$ is real valued periodic of period $N$.

It appears that the asymptotic behaviour of $\lambda_n(J)$ strongly depends on whether $\alpha - \gamma > 1$ or not. In fact the easy case $\alpha - \gamma > 1$ it is possible to apply approximation methods based on an idea of successive diagonalizations which was first applied to the problem of eigenvalue asymptotics of Jacobi matrices in the paper of J. Janas and S. Naboko [7]. The name of modified Jaynes-Cummings models was introduced in the paper of A. Boutet de Monvel, S. Naboko, L. O. Silva [1] treating the case of entries (1.8) with $\alpha = 2$ and $\gamma = \frac{1}{2}$. The result obtained in [1] in the case $\alpha = 2$ and $\gamma = \frac{1}{2}$ gives the asymptotic formula

$$\lambda_n(J) = n^2 + v(n) + O(n^{-1}) \quad (n \to \infty) \tag{1.9}$$

More general results of M. Malejki [8] and A. Boutet de Monvel, L. Zielinski [3] applied to the case of entries (1.8) give the asymptotic formula

$$\lambda_n(J) = n^{\alpha} + v(n) + O(n^{\gamma-2\kappa} + n^{2\gamma-\alpha}) \quad (n \to \infty) \tag{1.10}$$

where $\kappa := \alpha - 1 - \gamma > 0$. We observe that under the additional conditions $\alpha \le 2$ and $\gamma < \frac{2}{3}(\alpha - 1)$ we have $\alpha - 2\gamma > 0$ and $2\kappa - \gamma = 2(\alpha - 1) - 3\gamma > 0$, hence we obtain the asymptotic behaviour of the difference

$$\lambda_n(J) - n^{\alpha} = v(n) + o(1) \quad (n \to \infty) \tag{1.11}$$

reflecting the oscillations determined by the periodic nature of $v$.

The case $\alpha = 1$ and $0 < \gamma \leq 1/2$ investigatd in this paper exhibits a radical change of the asymptotic behaviour of $\lambda_n(J)$: the new phenomenon consists in the absence of periodic modulations of large eigenvalues. This phenomenon was already described in our earlier paper [5] treating the case $\alpha = 1$ and $0 < \gamma < 1/2$. In this paper we follow a general framework of [5] but in order to improve the remainder estimates we need to refine our approach constructing suitable approximations by means of truncated Fourier series.

1.3. **A generalization.** We consider the diagonal entries of the form

$$d(k) = k + v(k) \tag{1.12}$$

where $v : \mathbb{N}^* \to \mathbb{R}$ is real valued and periodic of period $N$ and we denote

$$\langle v \rangle := \frac{1}{N} \sum_{1 \leq k \leq N} v(k), \tag{1.13}$$

$$\rho_N := \max_{1 \leq k \leq N} |v(k) - \langle v \rangle| \tag{1.14}$$

We consider the following hypotheses

$$\begin{cases} \rho_N < \frac{1}{2} \text{ when } N = 2 \\ \rho_N < \frac{1}{\pi\sqrt{N}} \quad \text{when } N \geq 3 \end{cases} \tag{H1}$$

Concerning the off-diagonal entries we fix $0 < \gamma \leq \frac{1}{2}$ and assume that for some constants $C$, $C'$, $C''$, $c > 0$ one has

$$ck^\gamma \leq a(k) \leq Ck^\gamma \tag{H2$a$}$$

$$|\delta a(k)| \leq C' k^{\gamma-1} \tag{H2$b$}$$

$$|\delta^2 a(k)| \leq C'' k^{\gamma-2} \tag{H2$c$}$$

where we have denoted

$$\delta a(k) := a(k+1) - a(k) \tag{1.15}$$

$$\delta^2 a(k) := a(k+2) - 2a(k+1) + a(k). \tag{1.16}$$

Notice that (H2) is satisfied if large $k$ behaviour of $a(k)$ given by (1.6).

Our main result is the following

**Theorem 1.2.** *Assume that the entries* $\{d(k)\}_{k=1}^\infty$ *are of the form* (1.12) *with* $v$ *satisfying* (1.7), (H1) *and* $\{a(k)\}_{k=1}^\infty$ *satisfy* (H2). *If* $J$ *is defined by* (1.4), *then*

$$\lambda_n(J) = n + \langle v \rangle + a(n-1)^2 - a(n)^2 + O(n^{-\gamma/2} \ln(n)) \tag{1.17}$$

*holds as* $n \to \infty$.

Notice that in the case $a(k) = a_1 k^{1/2}$ we have $a(n-1)^2 - a(n)^2 = -a_1^2$, i.e. the asymptotic formula (1.17) becomes (1.5).

1.4. **Plan of the paper.** We follow the approach used in our earlier paper [5].

Step 1. We consider an approximation of the operator $J$ by a sequence of operators $J_n^+$ which are easier to investigate.

More precisely the asymptotic formula for the $n$-th eigenvalue of $J_n^+$ is given in Theorem 2.1 and in Proposition 2.2 we claim that the difference between the $n$-th eigenvalue of $J$ and $n$-th eigenvalue of $J_n^+$ is $O(n^{3\gamma-2})$ which allows us to reduce the proof of Theorem 1.2 to the proof of Theorem 2.1, i.e.

$$(\text{Theorem 2.1} + \text{Proposition 2.2}) \Rightarrow \text{Theorem 1.2} \Rightarrow \text{Theorem 1.1}$$

We prove Proposition 2.2 reasoning as in [5] (see the details of the proof in Section 7) except one difference: the definition of $J_n^+$ is modified. The key point to define $J_n^+$ is the fact that the value of the $n$-th eigenvalue is close to $n$ and its asymptotic behaviour depends only on the values of entries $d(k)$, $a(k)$ for $k$ satisfying $|n-k| \leq \tilde{C} n^\gamma$ where $\tilde{C}$ is fixed large enough. The simplest approximation of $a(k)$ by $a(n)$ used in [5] gives the error

$$\sup_{\{k\in\mathbb{N}^*:\, |n-k|\leq \tilde{C}n^\gamma\}} |a(k)-a(n)| \; = O(n^{2\gamma-1}).$$

In this paper we use the approximation of $a(k)$ by $a(n)+\delta a(n)(k-n)$ which gives the error

$$\sup_{\{k\in\mathbb{N}^*:\, |n-k|\leq \tilde{C}n^\gamma\}} |a(k)-a(n)-\delta a(n)(k-n)| \; = O(n^{3\gamma-2})$$

Step 2. We deduce Theorem 2.1 from the trace formula stated in Proposition 3.2.

More precisely reasoning similarly as in [5] (see the details of the proof in Section 7) we prove Proposition 3.3 allowing us to obtain the asymptotic formula from the trace formula and some rough eigenvalue estimates stated in Lemmas 2.3, 2.4, 3.1, i.e.

$$(\text{Proposition 3.2} + \text{Proposition 3.3} + \text{Lemmas 2.3, 2.4, 3.1}) \Rightarrow \text{Theorem 1.2}$$

Step 3. Reformulation of the trace formula of Proposition 3.2.

More precisely Lemma 3.4 allows us to introduce an auxiliary cut-off and Lemma 3.6 allows us to use the Fourier transform, reducing the proof of Proposition 3.2 to the proof of Proposition 3.5, i.e.

$$(\text{Proposition 3.5} + \text{Lemmas 3.4, 3.6 }) \Rightarrow \text{Proposition 3.2}$$

Finally in Lemma 3.8 we check that the analytic expansion formula reduces the proof of Proposition 3.5 to the proof of Proposition 3.7, i.e.

$$(\text{Proposition 3.7} + \text{Lemma 3.8 }) \Rightarrow \text{Proposition 3.5}$$

Step 4. It remains to prove Proposition 3.7.

The basic ingredient of the proof is a suitable unitary conjugation introduced in Section 2 and investigated in Section 4 (cf. Proposition 4.1) by means of the discrete Fourier transform. Throughout all the proof we consider phase functions of the same structure using three basic tools

a) norm estimates (cf. Lemma 4.2) allowing us to control the remainders,

b) the formula for the phase function of compositions (cf. Section 5.1 and Lemma 5.1)

c) the stationary phase estimate (cf. Lemma 5.2) allowing us to conclude estimates of Proposition 3.7.

The norm estimates of Lemma 4.2 are not optimal: the logarithmic factor can be dropped under additional conditions of regularity with respect to $j$. Since the presence of logarithmic factors makes no difference in considered remainder estimates, our choice is to use the simplest assumptions and non-optimal norm estimates.

In order to apply the stationary phase method we separate the principal part of phase functions and remainders. The principal part has non-degenerate critical points and coincides with the phase function considered in [5] (cf. Lemma 6.2). A suitable control of the reminders is ensured by Lemmas 5.3, 5.4 and 6.1.

## 2. Preliminary considerations

2.1. **Modified operators $J_n^+$.** If $L$ is a self-adjoint bounded from below operator with compact resolvent in $l^2$, then there is an orthogonal basis $\{w_n\}_{n=1}^{\infty}$ such that $Lw_n = \lambda_n(L)w_n$ $(n \in \mathbb{N}^*)$ and the sequence $\{\lambda_n(L)\}_{n=1}^{\infty}$ is non-decreasing. We say that $\lambda_n(L)$ is the $n$-th eigenvalue of $L$.

We assume that $J$ is as in Theorem 1.2. Since $\lambda_n(J+\mu) = \lambda_n(J)+\mu$ holds for every $\mu \in \mathbb{R}$, without loss of generality we assume further on

$$\langle v \rangle = 0 \tag{2.1}$$

and write

$$\rho_N := \max_{1\le j\le N} |v(j)|. \tag{2.2}$$

Our analysis is based on an approximation of the entries $\{d(k)\}_{k=1}^{\infty}$, $\{a(k)\}_{k=1}^{\infty}$ by $\{d_n(k)\}_{k=1}^{\infty}$, $\{a_n(k)\}_{k=1}^{\infty}$ so that the $n$-th eigenvalue of $J$ is close to the $n$-th eigenvalue of the operator $J_n^+$ defined by the matrix

$$\begin{pmatrix} d_n(1) & a_n(1) & 0 & 0 & 0 & \dots \\ a_n(1) & d_n(2) & a_n(2) & 0 & 0 & \dots \\ 0 & a_n(2) & d_n(3) & a_n(3) & 0 & \dots \\ 0 & 0 & a_n(3) & d_n(4) & a_n(4) & \\ \vdots & \vdots & & \ddots & \ddots & \ddots \end{pmatrix} \tag{2.3}$$

For this purpose we fix a cut-off function $\theta_0 \in C^{\infty}(\mathbb{R})$ satisfying the conditions

$$\begin{cases} \theta_0(t) = 1 \quad \text{if } |t| \le \frac{1}{6} \\ \theta_0(t) = 0 \text{ if } |t| \ge \frac{1}{5} \\ 0 \le \theta_0(t) \le 1 \text{ for } t \in \mathbb{R} \end{cases} \tag{2.4}$$

and for $\tau > 0$ we denote

$$\theta_{\tau,n}(s) := \theta_0\left(\frac{s-n}{\tau}\right). \tag{2.5}$$

Then we define $d_n$, $a_n : \mathbb{Z} \to \mathbb{R}$ by

$$d_n(k) = k + v(k)\theta_{n,n}(k)^2 \tag{2.6}$$

$$a_n(k) := (a(n) + \delta a(n)(k-n))\,\theta_{2n,n}(k) \tag{2.7}$$

and consider $J_n^+ : \mathcal{D} \to l^2$ acting according to the formula

$$(J_n^+x)(k) = d_n(k)x(k) + a_n(k)x(k+1) + a_n(k-1)x(k-1). \tag{2.8}$$

We observe that the assumptions made on $\{a(k)\}_{k=1}^{\infty}$ and the definition of $\{a_n(k)\}_{k=1}^{\infty}$ imply the fact that for every $m \in \mathbb{N}$ there is a constant $C_m > 0$ such that

$$\sup_{k\in\mathbb{Z}} |\delta^m a_n(k)| \leq C_m n^{\gamma-m} \tag{2.9}$$

where we have adopted the notations

$$\delta^0 b(k) := b(k), \qquad \delta^m b(k) := \delta^{m-1} b(k+1) - \delta^{m-1} b(k) \ \ (m \in \mathbb{N}^*) \tag{2.10}$$

Instead of Theorem 1.2 we will prove

**Theorem 2.1.** *If $\langle v \rangle = 0$ and $J_n^+$ is as above, then one has*

$$\lambda_n(J_n^+) \ = n + a_n(n-1)^2 - a_n(n)^2 + O(n^{-\gamma/2}\ln(n)) \tag{2.11}$$

Since $a_n(n-1)^2 - a_n(n)^2 = a(n-1)^2 - a(n)^2$ and

$$\gamma \leq \tfrac{1}{2} \ \ \Rightarrow \ \ 3\gamma - 2 \leq \gamma - 1 \leq -\tfrac{1}{2}\gamma \tag{2.12}$$

it is clear that Theorem 1.2 follows from Theorem 2.2 if

$$\lambda_n(J) = \lambda_n(J_n^+) + O(n^{3\gamma-2})\,. \tag{2.13}$$

However the above estimate can be obtained reasoning similarly as in Section 7 of [5]. More precisely we have

**Proposition 2.2.** *If $J$ is as in Theorem 1.2 and $J_n^+$ as above, then* (2.13) *holds.*

*Proof.* See Section 7.1. □

2.2. **The case $v = 0$.** In the case $v = 0$ we can use the ideas of the successive diagonalization to obtain the asymptotic behaviour of large eigenvalues. We give here the remainder estimate of the same type as in [2] but the presentation below is slightly simplified with respect to [2].

We denote by $\mathcal{B}(l^2)$ the algebra of bounded operators on $l^2$ and $||\cdot||_{\mathcal{B}(l^2)}$ denotes the norm of $\mathcal{B}(l^2)$. We introduce $\Lambda^+ : \mathcal{D} \to l^2$ by the formula

$$(\Lambda^+ x)(k) = kx(k) \tag{2.14}$$

and for any $b : \mathbb{N}^* \to \mathbb{C}$ the linear operator $b(\Lambda^+)$ is defined by the usual functional calculus, i.e. $b(\Lambda^+)$ is closed in $l^2$ and

$$(b(\Lambda^+)x)(k) = b(k)x(k) \tag{2.15}$$

holds for every $x \in l^2$ with finite $\operatorname{supp} x := \{k \in \mathbb{N}^* : x(k) \neq 0\}$.

We introduce $J_{0,n}^+ : \mathcal{D} \to l^2$ by the formula

$$J_{0,n}^+ := \Lambda^+ + A_n^+ \tag{2.16}$$

where $A_n^+$ is the finite rank operator

$$A_n^+ := \begin{pmatrix} 0 & a_n(1) & 0 & 0 & 0 & \dots \\ a_n(1) & 0 & a_n(2) & 0 & 0 & \dots \\ 0 & a_n(2) & 0 & a_n(3) & 0 & \dots \\ 0 & 0 & a_n(3) & 0 & a_n(4) & \\ \vdots & \vdots & & \ddots & \ddots & \ddots \end{pmatrix} \tag{2.17}$$

According to these notations we have

$$J_n^+ = J_{0,n}^+ + v_n(\Lambda^+) \tag{2.18}$$

with

$$v_n(k) := v(k)\theta_{n,n}(k)^2. \tag{2.19}$$

The asymptotic behaviour of eigenvalues of $J^+_{0,n}$ is given in the following

**Lemma 2.3.** *Let $a_n(k)$ be as before and let $l_n : \mathbb{Z} \to \mathbb{R}$ be given by*

$$l_n(k) := k + a_n(k-1)^2 - a_n(k)^2. \tag{2.20}$$

*Then there exists a constant $C > 0$ such that*

$$|\lambda_k(J^+_{0,n}) - \lambda_k(l_n(\Lambda^+))| \leq C n^{3\gamma-2} \tag{2.21}$$

*holds for all $k \in \mathbb{N}^*$.*

*Proof.* We introduce finite rank operators

$$B^+_n := \begin{pmatrix} 0 & \mathrm{i}a_n(1) & 0 & 0 & 0 & \dots \\ -\mathrm{i}a_n(1) & 0 & \mathrm{i}a_n(2) & 0 & 0 & \dots \\ 0 & -\mathrm{i}a_n(2) & 0 & \mathrm{i}a_n(3) & 0 & \dots \\ 0 & 0 & -\mathrm{i}a_n(3) & 0 & \mathrm{i}a_n(4) & \\ \vdots & \vdots & & \ddots & \ddots & \ddots \end{pmatrix} \tag{2.22}$$

which satisfy the equality

$$\mathrm{i}(\Lambda^+ B^+_n - B^+_n \Lambda^+) = [\mathrm{i}\Lambda^+,\, B^+_n] = A^+_n \tag{2.23}$$

(cf. [5] , Section 5) and we claim that

$$\mathrm{e}^{\mathrm{i}B^+_n} J^+_{0,n} \mathrm{e}^{-\mathrm{i}B^+_n} = l_n(\Lambda^+) + R^+_n \tag{2.24}$$

holds with

$$||R^+_n||_{\mathcal{B}(l^2)} = O(n^{3\gamma-2}). \tag{2.25}$$

Since $\lambda_k(J^+_{0,n}) = \lambda_k(l_n(\Lambda^+) + R^+_n)$, the min-max principle ensures

$$|\lambda_k(l_n(\Lambda^+) + R^+_n) - \lambda_k(l_n(\Lambda^+))| \leq ||R^+_n||_{\mathcal{B}(l^2)} \tag{2.26}$$

and it is clear that (2.21) follows from (2.25).

In order to prove (2.25) we denote $J_{0,n}(t) := \Lambda^+ + tA^+_n$ for $t \in \mathbb{R}$ and introduce

$$G_n(t) := \mathrm{e}^{\mathrm{i}tB^+_n} J_{0,n}(t) \mathrm{e}^{-\mathrm{i}tB^+_n} \tag{2.27}$$

Then the differentiation with respect to $t \in \mathbb{R}$ gives the expression

$$\frac{\mathrm{d}}{\mathrm{d}t} G_n(t) = \mathrm{e}^{\mathrm{i}tB^+_n} \Big( \mathbb{D}_{B^+_n} J_{0,n}(t) \Big) \mathrm{e}^{-\mathrm{i}tB^+_n}$$

where

$$\mathbb{D}_{B^+_n} J_{0,n}(t) := \frac{\mathrm{d}}{\mathrm{d}t} J_{0,n}(t) + [\mathrm{i}B^+_n, J_{0,n}(t)].$$

Using the calculations described in [5], Section 5.2, we find that

$$\mathbb{D}_{B^+_n} J_{0,n}(t) = t\,[\mathrm{i}B^+_n,\, A^+_n] = 2t\, a_{1,n}(\Lambda^+) \tag{2.28}$$

holds with $a_{1,n}(k) := a_n(k-1)^2 - a_n(k)^2$. Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t} G_n(t) = 2t\, \mathrm{e}^{\mathrm{i}tB^+_n} a_{1,n}(\Lambda^+) \mathrm{e}^{-\mathrm{i}tB^+_n} = 2t \big( a_{1,n}(\Lambda^+) + R_n(t) \big) \tag{2.29}$$

with

$$R_n(t) = \int_0^1 \mathrm{d}s\, \mathrm{e}^{\mathrm{i}stB^+_n} [\mathrm{i}B^+_n,\, a_{1,n}(\Lambda^+)] \mathrm{e}^{-\mathrm{i}stB^+_n}\,. \tag{2.30}$$

Since $||[S, a_{1,n}(\Lambda^+)]||_{\mathcal{B}(l^2)} = ||(\delta a_{1,n})(\Lambda^+)]||_{\mathcal{B}(l^2)} = O(n^{2\gamma-2})$, it is clear that

$$||R_n(t)||_{\mathcal{B}(l^2)} \le ||[B_n^+,\, a_{1,n}(\Lambda^+)]||_{\mathcal{B}(l^2)} = O(n^{3\gamma-2}). \tag{2.31}$$

To complete the proof we observe that

$$J_{0,n}^+ - \Lambda^+ = G_n(1) - G_n(0) = \int_0^1 \mathrm{d}t\, 2t\,\big(a_{1,n}(\Lambda^+) + R_n(t)\big) = a_{1,n}(\Lambda^+) + R_n^+$$

and (2.25) follows from (2.31). □

**Lemma 2.4.** *Let $l_n(k)$ be as in* (2.20)*. Then there exist $n_0$, $C_0 > 0$ such that*

$$|\lambda_k(J_{0,n}^+) - l_n(k)| \le C_0 n^{3\gamma-2} \tag{2.32}$$

*holds for $n \ge n_0$, $k \in \mathbb{N}^*$.*

*Proof.* Due to (2.21) it suffices to check that

$$n \ge n_1 \quad\Rightarrow\quad \lambda_k(l_n(\Lambda^+)) = l_n(k) \text{ for all } k \in \mathbb{N}^* \tag{2.33}$$

However $\sigma(l_n(\Lambda^+)) = \{l_n(k) :\, k \in \mathbb{N}^*\}$ and it remains to show that

$$n \ge n_1 \quad\Rightarrow\quad l_n(k) < l_n(k+1) \text{ for all } k \in \mathbb{N}^* \tag{2.34}$$

For this purpose we write

$$l_n(k) = k + a_{1,n}(k) \tag{2.35}$$

with $a_{1,n}(k) := a_n(k-1)^2 - a_n(k)^2$ and we observe that (2.9) ensure that

$$\sup_{k\ge1} |a_{1,n}(k)| \le C n^{2\gamma-1}, \tag{2.36}$$

$$\sup_{k\ge1} |\delta a_{1,n}(k)| \le C_0 n^{2\gamma-2} \tag{2.37}$$

hold for some constants $C$, $C_0 > 0$. Therefore

$$\sup_{k\ge1} |l_n(k+1) - l_n(k) - 1| \le C_0 n^{2\gamma-2} \tag{2.38}$$

and for any $\varepsilon > 0$ we can choose $n(\varepsilon)$ such that $C_0 n(\varepsilon)^{2\gamma-2} < \varepsilon$. Hence

$$n \ge n(\varepsilon) \quad\Rightarrow\quad l_n(k+1) - l_n(k) > 1 - \varepsilon \text{ for all } k \in \mathbb{N}^* \tag{2.39}$$

completes the proof. □

2.3. **Rough estimates of eigenvalues.** In this section we localize the eigenvalues of $J_n^+$ using the min-max principle to compare the eigenvalues of $J_n^+$ and $J_{0,n}^+$. Moreover we introduce more notations using the unitary conjugation from the proof of Lemma 2.3.

Due to the min-max principle we can estimate

$$|\lambda_k(J_n^+) - \lambda_k(J_{0,n}^+)| \le ||v_n(\Lambda^+)||_{\mathcal{B}(l^2)} = \rho_N \tag{2.40}$$

and using Lemma 2.4 we obtain

$$|\lambda_k(J_n^+) - l_n(k)| \le \rho_N + C_0 n^{3\gamma-2}. \tag{2.41}$$

For $C > 0$ we introduce the intervals

$$\Delta_{k,n}^C := [l_n(k) - \rho_N - C n^{3\gamma-2},\, l_n(k) + \rho_N + C n^{3\gamma-2}]. \tag{2.42}$$

Due to $\rho_N < \frac{1}{2}$ and (2.39) it is possible to choose $n_C$ sufficiently large to ensure

$$n \ge n_C \;\Rightarrow\; \Delta_{k,n}^C \cap \Delta_{k+1,n}^C = \emptyset\,, \tag{2.43}$$

hence

$$n \geq n_{C_0} \Rightarrow \{\lambda_k(J_n^+)\} = \sigma(J_n^+) \cap \Delta_{k,n}^{C_0}. \tag{2.44}$$

Further on instead of $J_n^+$ we will investigate the operators

$$L_n^+ = l_n(\Lambda^+) + \tilde{V}_n^+ \tag{2.45}$$

with

$$\tilde{V}_n^+ := \mathrm{e}^{-\mathrm{i}B_n^+} v_n(\Lambda^+) \mathrm{e}^{\mathrm{i}B_n^+}, \tag{2.46}$$

where $B_n^+$ are as in Section 2.2. We claim that there is a constant $C_0' > 0$ such that

$$|\lambda_k(J_n^+) - \lambda_k(L_n^+)| \leq C_0' n^{3\gamma-2}. \tag{2.47}$$

Indeed, $\lambda_k(J_n^+) = \lambda_k(\tilde{J}_n^+)$ holds with

$$\tilde{J}_n^+ := \mathrm{e}^{-\mathrm{i}B_n^+} J_n^+ \mathrm{e}^{\mathrm{i}B_n^+} = \mathrm{e}^{-\mathrm{i}B_n^+} J_{0,n}^+ \mathrm{e}^{\mathrm{i}B_n^+} + \tilde{V}_n^+ = L_n^+ + R_n^+$$

where $R_n^+$ is as in (2.24), hence the min-max principle gives

$$|\lambda_k(\tilde{J}_n^+) - \lambda_k(L_n^+)| \leq ||R_n^+||_{\mathcal{B}(l^2)} \tag{2.48}$$

and the right-hand side of (2.48) is $O(n^{3\gamma-2})$ due to (2.25).

At the end of this section we notice that combining (2.41) with (2.47) we find that

$$|\lambda_k(L_n^+) - l_n(k)| \leq \rho_N + C_1 n^{3\gamma-2} \tag{2.49}$$

holds with $C_1 = C_0 + C_0'$ and similarly as above we conclude

$$n \geq n_{C_1} \Rightarrow \{\lambda_k(L_n^+)\} = \sigma(L_n^+) \cap \Delta_{k,n}^{C_1}. \tag{2.50}$$

## 3. Recovery of the behaviour of large eigenvalues.

3.1. **Extension to $\mathcal{H} := l^2(\mathbb{Z})$.** Similarly as in [5] we replace the Hilbert space $l^2$ by $\mathcal{H} := l^2(\mathbb{Z})$. More precisely we write $x \in \mathcal{H}$ if and only if $x : \mathbb{Z} \to \mathbb{C}$ satisfies

$$\|x\|_{\mathcal{H}} := \Big( \sum_{k\in\mathbb{Z}} |x(k)|^2 \Big)^{1/2} < \infty \tag{3.1}$$

and $\langle x, y \rangle = \sum_{k\in\mathbb{Z}} \overline{x(k)} y(k)$ is the scalar product of $x, y \in \mathcal{H}$. We denote by $\mathcal{B}(\mathcal{H})$ the algebra of bounded operators on $\mathcal{H}$ and $||\cdot||$ denotes the norm of $\mathcal{B}(\mathcal{H})$. We denote by $\{\mathrm{e}_n\}_{n\in\mathbb{Z}}$ the canonical basis of $\mathcal{H}$ ( i.e. $\mathrm{e}_n(n) = 1$ and $\mathrm{e}_n(j) = 0$ if $j \neq n$) and we define $S \in \mathcal{B}(\mathcal{H})$ as the shift

$$S\mathrm{e}_n = \mathrm{e}_{n+1} \quad (n \in \mathbb{Z}). \tag{3.2}$$

We introduce

$$\mathcal{H}_1 := \{x \in \mathcal{H} : \sum_{k\in\mathbb{Z}} k^2 |x(k)|^2 < \infty\} \tag{3.3}$$

and we define $\Lambda : \mathcal{H}_1 \to \mathcal{H}$ by the formula

$$(\Lambda x)(k) = kx(k) \quad (k \in \mathbb{Z}) \tag{3.4}$$

For any $b : \mathbb{Z} \to \mathbb{C}$ we define the linear operator $b(\Lambda)$ by the usual functional calculus, i.e. $b(\Lambda)$ is closed in $\mathcal{H}$ and

$$(b(\Lambda)x)(k) = b(k)x(k) \tag{3.5}$$

holds for every $x \in \mathcal{H}$ with finite $\operatorname{supp} x := \{k \in \mathbb{Z} : x(k) \neq 0\}$.

Further on we investigate $J_n' : \mathcal{H}_1 \to \mathcal{H}$ defined by

$$J_n' = J_{0,n}' + v_n(\Lambda) \tag{3.6}$$

with $v_n : \mathbb{Z} \to \mathbb{R}$ given by (2.19) and $J'_{0,n} : \mathcal{H}_1 \to \mathcal{H}$ given by

$$J'_{0,n} := \Lambda + S\, a_n(\Lambda) + a_n(\Lambda)\, S^{-1}. \tag{3.7}$$

Then for $k \geq 1$ we have $J'_n \mathrm{e}_k = J^+_n \mathrm{e}_k$, while for $k \leq 0$ we have $J'_n \mathrm{e}_k = k\mathrm{e}_k$. Thus it is clear that the spectrum of $J'_n$ has the form

$$\sigma(J'_n) = \sigma(J^+_n) \cup \{k \in \mathbb{Z} :\ k \leq 0\} \tag{3.8}$$

Moreover we have

**Lemma 3.1.** *If $v$ is periodic of period $N$ then*

$$\sup_{k\in\mathbb{Z}} \ |\lambda_{k+N}(J'_n) - \lambda_k(J'_n) - N| \ = O(n^{\gamma-1}). \tag{3.9}$$

*Proof.* Due to $||v_n(\Lambda) - S^{-N} v_n(\Lambda) S^N|| = O(n^{\gamma-1})$ and the estimates of Section 2.3 we deduce (3.9) reasoning similarly as in Section 5.3 of [5]. $\square$

Next we define

$$\tilde{V}_n := \mathrm{e}^{-\mathrm{i}B_n} v_n(\Lambda) \mathrm{e}^{\mathrm{i}B_n} \tag{3.10}$$

with

$$B_n := \mathrm{i}\left(a_n(\Lambda)\, S^{-1} \ - \ S a_n(\Lambda)\right). \tag{3.11}$$

Then for $k \geq 1$ we have $B_n \mathrm{e}_k = B^+_n \mathrm{e}_k$ and $\tilde{V}_n \mathrm{e}_k = \tilde{V}^+_n \mathrm{e}_k$, while for $k \leq 0$ we have $B_n \mathrm{e}_k = \tilde{V}_n \mathrm{e}_k = 0$. Further on we define

$$L_n := l_n(\Lambda) + \tilde{V}_n. \tag{3.12}$$

Then for $k \geq 1$ we have $L_n \mathrm{e}_k = L^+_n \mathrm{e}_k$, while for $k \leq 0$ we have $L_n \mathrm{e}_k = k\mathrm{e}_k$ and it is clear that the spectrum of $L_n$ has the form

$$\sigma(L_n) = \sigma(L^+_n) \cup \{k \in \mathbb{Z} :\ k \leq 0\}. \tag{3.13}$$

Moreover using Lemma 3.1 and

$$\sup_{k\in\mathbb{Z}} |\lambda_k(J'_n) - \lambda_k(L_n)| \ = O(n^{3\gamma-2}) \tag{3.14}$$

we deduce

$$\sup_{k\in\mathbb{Z}} \ |\lambda_{k+N}(L_n) - \lambda_k(L_n) - N| \ = O(n^{\gamma-1}). \tag{3.15}$$

3.2. **A trace formula.** Further on $\hat{\chi}$ denotes the Fourier transform of an integrable function $\chi : \mathbb{R} \to \mathbb{C}$, i.e.

$$\hat{\chi}(t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda}{2\pi}\, \mathrm{e}^{-\mathrm{i}t\lambda} \chi(\lambda). \tag{3.16}$$

We recall that the assumption $\hat{\chi} \in C_0^\infty(\mathbb{R})$ ensures $\chi \in \mathcal{S}(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ denotes the Schwartz class of rapidly decreasing functions on $\mathbb{R}$. We denote by $\mathcal{B}_1(\mathcal{H})$ the ideal of trace class operators on $\mathcal{H}$ with the norm $\|Q\|_{\mathcal{B}_1(\mathcal{H})} = \mathrm{tr}(Q^*Q)^{1/2}$. We define

$$L_n = L_{0,n} + \tilde{V}_n \tag{3.17}$$

where

$$L_{0,n} := l_n(\Lambda)$$

with $l_n(k) = k + a_n(k-1)^2 - a_n(k)^2$ and

$$\tilde{V}_n := \mathrm{e}^{-\mathrm{i}B_n} v_n(\Lambda) \mathrm{e}^{\mathrm{i}B_n}$$

with $B_n = \mathrm{i}\left(a_n(\Lambda) S^{-1} \ - \ S\, a_n(\Lambda)\right)$. We will prove the following trace formula

**Proposition 3.2.** *Assume that* $\chi \in \mathcal{S}(\mathbb{R})$ *is such that* $\hat{\chi} \in C_0^\infty(\mathbb{R})$ *and define*

$$\mathcal{G}_n^0 := \operatorname{tr} \chi(L_n - l(n)) - \operatorname{tr} \chi(L_{0,n} - l(n)), \tag{3.18}$$

*where*

$$l(n) := n + a(n-1)^2 - a(n)^2 = l_n(n). \tag{3.19}$$

*Then one has*

$$\mathcal{G}_n^0 = O(n^{-\gamma/2} \ln(n)). \tag{3.20}$$

Then we deduce Theorem 2.1 using

**Proposition 3.3.** *Assume that*

*(i)* (2.49), (2.50) *and* (3.15) *hold,*
*(ii) the hypothesis* (H1) *holds,*
*(iii)* (3.20) *holds for every* $\chi \in \mathcal{S}(\mathbb{R})$ *such that* $\hat{\chi} \in C_0^\infty(\mathbb{R})$.
*Then one has* $\lambda_n(L_n) - l(n) = O(n^{-\gamma/2} \ln(n))$.

*Proof.* The proof of Proposition 3.3 can be easily reduced to the proof given in Section 6.1 of [5], where we considered $n$ instead of $l(n)$. We give details in Section 7.2. □

3.3. **An auxiliary cut-off.** In this section we show that the estimate

$$\mathcal{G}_n^0 = \mathcal{G}_n + O(n^{-\gamma}) \tag{3.21}$$

holds with $\mathcal{G}_n^0$ given by (3.18) and

$$\mathcal{G}_n := \operatorname{tr} \Big( \theta_{n^\gamma, n}(L_{0,n}) \big( \chi(L_n - l(n)) - \chi(L_{0,n} - l(n)) \big) \Big) \tag{3.22}$$

where $\theta_{\tau,n}$ is as in Section 2. It is clear that (3.21) follows from

**Lemma 3.4.** *Let* $\chi \in \mathcal{S}(\mathbb{R})$. *Then*

$$||(I - \theta_{n^\gamma,n}(L_{0,n}))\chi(L_n - l(n))||_{\mathcal{B}_1(\mathcal{H})} = O(n^{-\gamma}) \tag{3.23}$$

*and for every* $\mu > 0$ *one has*

$$||(I - \theta_{n^\gamma,n}(L_{0,n}))\chi(L_{0,n} - l(n))||_{\mathcal{B}_1(\mathcal{H})} = O(n^{-\mu}). \tag{3.24}$$

*Proof.* First of all we observe that there is a constant $C > 0$ such that

$$||(I + (L_{0,n} - l(n))^2)^{-1}||_{\mathcal{B}_1(\mathcal{H})} = \sum_{j \in \mathbb{Z}} \frac{1}{1 + (l_n(j) - l(n))^2} \le C.$$

and a similar estimate holds for $L_n$, i.e.

$$||(I + (L_n - l(n))^2)^{-1}||_{\mathcal{B}_1(\mathcal{H})} = \sum_{j \in \mathbb{Z}} \frac{1}{1 + (\lambda_j(L_n) - l(n))^2} \le C. \tag{3.25}$$

Next we claim that for every $\mu > 0$ we can estimate

$$||(I - \theta_{n^\gamma,n}(L_n))\chi(L_n - l(n))||_{\mathcal{B}_1(\mathcal{H})} = O(n^{-\mu}). \tag{3.26}$$

Indeed, if $\chi_0(s) := (1 + s^2)\chi(s)$ then for every $\mu > 0$ we have

$$\sup_{s \in \mathbb{R}} |(1 - \theta_{n^\gamma,n}(s))\chi_0(s - l(n))| = O(n^{-\mu}), \tag{3.27}$$

hence

$$||(I - \theta_{n^\gamma,n}(L_n))\chi_0(L_n - l(n))|| = O(n^{-\mu}). \tag{3.28}$$

Since the left-hand side of (3.26) can be estimated by

$$||(I-\theta_{n^\gamma,n}(L_n))\chi_0(L_n-l(n))|| \times ||(1+(L_n-l(n))^2)^{-1}||_{\mathcal{B}_1(\mathcal{H})}$$

we deduce (3.26) from (3.25) and (3.28). It is clear that reasoning similarly with $L_{0,n}$ we obtain (3.24).

If the operator $T$ is self-adjoint, the operaor $R$ is bounded and $\theta \in C_0^\infty(\mathbb{R})$, then there exists a constant $C = C(\theta)$ such that

$$||\theta(T+R)-\theta(T)|| \leq C||R||. \tag{3.29}$$

Using (3.29) with $T = n^{-\gamma}(L_{0,n}-n)$ and $R = n^{-\gamma}\tilde{V}_n$ we can estimate

$$||\theta_{n^\gamma,n}(L_n)-\theta_{n^\gamma,n}(L_{0,n})|| \leq C_0||n^{-\gamma}\tilde{V}_n|| = O(n^{-\gamma}). \tag{3.30}$$

Using (3.30) and (3.28) we obtain

$$||(I-\theta_{n^\gamma,n}(L_{0,n}))\chi_0(L_n-l(n))|| = O(n^{-\gamma}). \tag{3.31}$$

Combining the last estimate with (3.25) we obtain (3.23). □

3.4. **Use of the Fourier transform.** We will deduce the trace formula from the following

**Proposition 3.5.** *For $t \in \mathbb{R}$ we define $U_n(t) := \mathrm{e}^{-\mathrm{i}tL_{0,n}}\mathrm{e}^{\mathrm{i}tL_n}$ and*

$$u_{n,j}(t) := \langle \mathrm{e}_j,\, U_n(t)\mathrm{e}_j\rangle. \tag{3.32}$$

*Then for every $t_0 > 0$ one has the estimate*

$$\sup_{-t_0\leq t\leq t_0}\ \sup_{|j-n|\leq n^\gamma}\left|\frac{\mathrm{d}}{\mathrm{d}t}u_{n,j}(t)\right| = O(n^{-\gamma/2}). \tag{3.33}$$

Indeed, we have

**Lemma 3.6.** *Assume $0 < \mu \leq \gamma$. If for every $t_0 > 0$ the left-hand side of (3.33) is $O(n^{-\mu})$, then $\mathcal{G}_n = O(n^{-\mu}\ln(n))$.*

*Proof.* We assume that $\operatorname{supp}\hat{\chi} \subset [-t_0,\, t_0]$ and write the inverse Fourier formula

$$\chi(\lambda) = \int_{-\infty}^{\infty}\mathrm{d}t\,\mathrm{e}^{\mathrm{i}t\lambda}\hat{\chi}(t) = \int_{-t_0}^{t_0}\mathrm{d}t\,\mathrm{e}^{\mathrm{i}t\lambda}\hat{\chi}(t). \tag{3.34}$$

Using $L_n - l(n)$ and $L_{0,n}-l(n)$ in the place of $\lambda$ we find the expression

$$\chi(L_n-l(n)) - \chi(L_{0,n}-l(n)) = \int \mathrm{d}t\,\hat{\chi}(t)\,\mathrm{e}^{-\mathrm{i}tl(n)}\left(\mathrm{e}^{\mathrm{i}tL_n}-\mathrm{e}^{\mathrm{i}tL_{0,n}}\right), \tag{3.35}$$

hence

$$\mathcal{G}_n = \int \mathrm{d}t\,\hat{\chi}(t)\,\mathrm{e}^{-\mathrm{i}tl(n)}\operatorname{tr}\left(\theta_{n^\gamma,n}(L_{0,n})\mathrm{e}^{\mathrm{i}tL_{0,n}}(U_n(t)-I)\right) \tag{3.36}$$

However

$$\operatorname{tr}\left(\theta_{n^\gamma,n}(L_{0,n})\mathrm{e}^{\mathrm{i}tL_{0,n}}(U_n(t)-I)\right) = \sum_{j\in\mathbb{Z}}\langle \mathrm{e}^{-\mathrm{i}tL_{0,n}}\theta_{n^\gamma,n}(L_{0,n})\mathrm{e}_j, (U_n(t)-I)\mathrm{e}_j\rangle$$

and using $\mathrm{e}^{-\mathrm{i}tL_{0,n}}\theta_{n^\gamma,n}(L_{0,n})\mathrm{e}_j = \mathrm{e}^{-\mathrm{i}tl(j)}\theta_{n^\gamma,n}(l(j))\mathrm{e}_j$ we can express

$$\mathcal{G}_n = \sum_{j\in\mathbb{Z}}\mathcal{G}_n(j) \tag{3.37}$$

with

$$\mathcal{G}_n(j) = \int \mathrm{d}t\, \hat{\chi}(t)\, \mathrm{e}^{\mathrm{i}t/2}\, \mathrm{e}^{\mathrm{i}t(l(j)-l(n)-1/2)} \theta_{n^\gamma,n}(l(j))(u_{n,j}(t)-1). \tag{3.38}$$

Next we observe that choosing $n_0$ large enough and $c_0 > 0$ small enough we ensure

$$n \geq n_0 \;\Rightarrow\; |l(j) - l(n) - \tfrac{1}{2}| \geq c_0(1 + |j-n|)$$

and we can express

$$\mathrm{e}^{\mathrm{i}t(l(j)-l(n)-1/2)} = \frac{-\mathrm{i}}{l(j)-l(n)-\frac{1}{2}} \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{e}^{\mathrm{i}t(l(j)-l(n)-1/2)},$$

hence integrating by parts we obtain

$$\mathcal{G}_n(j) = \mathrm{i}\mathcal{G}_{1,n}(j) + \mathrm{i}\mathcal{G}_{2,n}(j)$$

with

$$\mathcal{G}_{1,n}(j) = \int \mathrm{d}t\, \hat{\chi}(t)\, \mathrm{e}^{\mathrm{i}t(l(j)-l(n))} \frac{\theta_{n^\gamma,n}(l(j))}{l(j)-l(n)-\frac{1}{2}} \frac{\mathrm{d}}{\mathrm{d}t} u_{n,j}(t)$$

and

$$\mathcal{G}_{2,n}(j) = \int \mathrm{d}t\, \frac{\mathrm{d}}{\mathrm{d}t}\big(\hat{\chi}(t)\, \mathrm{e}^{\mathrm{i}t/2}\big)\, \mathrm{e}^{\mathrm{i}t(l(j)-l(n)-1/2)} \frac{\theta_{n^\gamma,n}(l(j))}{l(j)-l(n)-\frac{1}{2}} \,(u_{n,j}(t)-1).$$

Since $\operatorname{supp}\hat{\chi} \subset [-t_0,\, t_0]$ we can estimate

$$|\mathcal{G}_{1,n}(j)| \leq C \frac{\theta_{n^\gamma,n}(l(j))}{1+|j-n|} \sup_{-t_0 \leq t \leq t_0} \left| \frac{\mathrm{d}}{\mathrm{d}t} u_{n,j}(t) \right|$$

and

$$|\mathcal{G}_{2,n}(j)| \leq C \frac{\theta_{n^\gamma,n}(l(j))}{1+|j-n|} \sup_{-t_0 \leq t \leq t_0} |u_{n,j}(t) - 1| \;.$$

Combining $\theta_{n^\gamma,n}(l(j)) \neq 0 \;\Rightarrow\; |j-n| \leq n^\gamma$ with

$$\sup_{-t_0 \leq t \leq t_0} |u_n(t) - 1| \leq \sup_{-t_0 \leq t \leq t_0} \left| \frac{\mathrm{d}}{\mathrm{d}t} u_{n,j}(t) \right|$$

we find that the estimate

$$|\mathcal{G}_n| \leq \sum_{|j-n| \leq n^\gamma} \frac{C_0}{1+|j-n|}\, n^{-\mu}$$

holds if the left hand side of (3.33) is $O(n^{-\mu})$. To complete the proof we observe that

$$\sum_{|k| \leq n^\gamma} \frac{1}{1+|k|} \quad \leq 1 + 2\ln(n).$$

□

3.5. **Expansion of $U_n(t)$.** We observe that

$$-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\,U_n(t)=\mathrm{e}^{-\mathrm{i}tL_{0,n}}(L_n-L_{0,n})\mathrm{e}^{\mathrm{i}tL_n}$$

ensures

$$-\mathrm{i}\,\frac{\mathrm{d}}{\mathrm{d}t}\,U_n(t)=H_n(t)U_n(t),\quad U_n(0)=I \tag{3.39}$$

with

$$H_n(t):=\mathrm{e}^{-\mathrm{i}tL_{0,n}}(L_n-L_{0,n})\mathrm{e}^{\mathrm{i}tL_{0,n}}. \tag{3.40}$$

Then we have the following expansion formula

$$U_n(t)-I=\mathrm{i}\int_0^t \mathrm{d}t_1\,H_n(t_1)\,+\sum_{\nu=2}^{\infty}\mathrm{i}^\nu\int_0^t \mathrm{d}t_1\ldots\int_0^{t_{\nu-1}}\mathrm{d}t_\nu\,H_n(t_1)\ldots H_n(t_\nu). \tag{3.41}$$

For $\nu\in\mathbb{N}^*$, $(t_1,\ldots,t_\nu)\in\mathbb{R}^\nu$ we denote

$$g_{1,n,j}(t_1)=\mathrm{i}\langle \mathrm{e}_j,\,H_n(t_1)\mathrm{e}_j\rangle \tag{3.42}$$

and

$$g_{\nu,n,j}(t_1,\ldots,t_\nu)=\mathrm{i}^\nu\langle \mathrm{e}_j,\,H_n(t_1)\ldots H_n(t_\nu)\mathrm{e}_j\rangle. \tag{3.43}$$

In Sections 5 and 6 we prove that

**Proposition 3.7.** *Let $t_0>0$ be fixed.*

*a) If $C=C(t_0)$ is large enough, then one has the estimate*

$$\sup_{-t_0\le t_1\le t_0}\ \sup_{|j-n|\le n^\gamma}|g_{1,n,j}(t_1)|\le Cn^{-\gamma/2} \tag{3.44}$$

*b) If $0<\varepsilon\le\gamma/24$ and $C=C(t_0,\varepsilon)$ is large enough, then the estimates*

$$\sup_{-t_0\le t_1,\ldots,t_{\nu-1}\le t_0}\ \sup_{|j-n|\le n^\gamma}\int_{-t_0}^{t_0}\mathrm{d}t_\nu\,|g_{\nu,n,j}(t_1,\ldots,t_\nu)|\ \le Cn^{-\gamma/2} \tag{3.45}$$

*hold for $\nu\le n^\varepsilon$.*

At the end of this section we check

**Lemma 3.8.** *If* (3.44) *holds and* (3.45) *hold for $\nu\le n^\varepsilon$, then* (3.33) *holds.*

*Proof.* Using the expansion (3.41) we find

$$\frac{\mathrm{d}}{\mathrm{d}t}u_{n,j}(t)=g_{1,n,j}(t)+\sum_{\nu=2}^{\infty}u_{\nu,n,j}(t), \tag{3.46}$$

where

$$u_{2,n,j}(t)=\int_0^t\mathrm{d}t_2\,g_{2,n,j}(t,t_2)$$

and

$$u_{\nu,n,j}(t)=\int_0^t\mathrm{d}t_2\ldots\int_0^{t_{\nu-1}}\mathrm{d}t_\nu\,g_{\nu,n,j}(t,t_2,\ldots,t_\nu)$$

for $\nu\ge 3$. Then using (3.45) we can estimate

$$\sum_{2\le\nu<n^\varepsilon}|u_{\nu,n,j}(t)|\le\sum_{2\le\nu<n^\varepsilon}\frac{Ct_0^{\nu-1}n^{-\gamma/2}}{(\nu-1)!}\ \le C\mathrm{e}^{t_0}n^{-\gamma/2}$$

for $t \in [-t_0, t_0]$ and $|j-n| \le n^\gamma$. To complete the proof it remains to observe that

$$\sum_{\nu > n^\varepsilon} |u_{\nu,n,j}(t)| \le \sum_{\nu > n^\varepsilon} \frac{C t_0^{\nu-1}}{(\nu-1)!} \le \frac{C \mathrm{e}^{t_0}}{([n^\varepsilon]-1)!}, \tag{3.47}$$

where $[s] := \max\{k \in \mathbb{Z} : \, k \le s\}$. Since $k! \sim (k/\mathrm{e})^k$ it is clear that the right-hand side of (3.47) is rapidly decreasing when $n \to \infty$. □

## 4. Approximation of $\mathrm{e}^{\mathrm{i}B_n}$

4.1. **Statement of the result.** Let $B_n = \mathrm{i}\big(a_n(\Lambda)\, S^{-1} - S\, a_n(\Lambda)\big)$ as before, denote

$$\Theta_n := \theta_{n,n}(\Lambda) \tag{4.1}$$

where $\theta_n$ is as in Section 2.1 and for $n \in \mathbb{N}^*$, $j \in \mathbb{Z}$, $\xi \in \mathbb{R}$ define

$$\tilde{\psi}_n(j, \mathrm{e}^{\mathrm{i}\xi}) := 2a_n(j) \sin(\xi) \left(1 - \delta a(n) \cos(\xi)\right) \tag{4.2}$$

The purpose of this section is to prove

**Proposition 4.1.** *For $n \in \mathbb{N}^*$ let $Q_n \in \mathcal{B}(\mathcal{H})$ be defined by the formula*

$$\left(Q_n \mathrm{e}_k\right)(j) = \int_0^{2\pi} \frac{\mathrm{d}\xi}{2\pi}\, \theta_{n,n}(j)\, \mathrm{e}^{\mathrm{i}\tilde{\psi}_n(j,\mathrm{e}^{\mathrm{i}\xi}) + \mathrm{i}(k-j)\xi}\,. \tag{4.3}$$

*Then*

$$\mathrm{e}^{\mathrm{i}B_n} \Theta_n = Q_n + R_n \tag{4.4}$$

*holds with*

$$||R_n|| = O(n^{\gamma-1} \sqrt{\ln(n)}\,). \tag{4.5}$$

*Remarks.* Since $\theta_{n,n}(j) \neq 0 \Rightarrow |j-n| < \frac{n}{5}$, the operators $Q_n$ are of finite rank. Further on we consider $\tilde{\psi}_n(j, \mathrm{e}^{\mathrm{i}\xi})$ only for $j \in \mathbb{Z}$ satisfying $|j-n| < \frac{n}{3}$ and using

$$|j-n| \le n/3 \quad \Rightarrow \quad a_n(j) = a(n) + (j-n)\delta a(n) \tag{4.6}$$

we find that for $|j-n| < \frac{n}{3}$ the expression

$$\tilde{\psi}_n(j, \mathrm{e}^{\mathrm{i}\xi}) := \psi_n(\mathrm{e}^{\mathrm{i}\xi}) + (j-n)\varphi_n(\mathrm{e}^{\mathrm{i}\xi}) \tag{4.7}$$

holds with

$$\begin{cases} \psi_n(\mathrm{e}^{\mathrm{i}\xi}) := 2a(n) \sin(\xi) \left(1 - \delta a(n) \cos(\xi)\right) \\ \varphi_n(\mathrm{e}^{\mathrm{i}\xi}) := 2\delta a(n) \sin(\xi) \big(1 - \delta a(n) \cos(\xi)\big) \end{cases} \tag{4.8}$$

Below we explain how the expression (4.3) is related with Fourier series of $2\pi$-periodic function of the variable $\xi$.

4.2. **Use of the Fourier series.** At the beginning we introduce some notations. We denote $\mathcal{C} := \{z \in \mathbb{C} : |z| = 1\}$ and define $\mathrm{L}^2(\mathcal{C})$ as the Hilbert space equipped with the orthonormal basis $\{f_n\}_{n\in\mathbb{Z}}$ where $f_n(\mathrm{e}^{\mathrm{i}\xi}) = \mathrm{e}^{in\xi}$ for $\xi \in \mathbb{R}$, i.e. the isometric isomorphism $\mathcal{F}_0 : \mathrm{L}^2(\mathcal{C}) \to \mathcal{H}$ is given by the formula $\mathcal{F}_0 f_n = \mathrm{e}_n$ and more generally

$$(\mathcal{F}_0 f)(j) = \langle f_j,\, f\rangle_{\mathrm{L}^2(\mathcal{C})} = \int_0^{2\pi} \frac{\mathrm{d}\xi}{2\pi}\, f(\mathrm{e}^{\mathrm{i}\xi})\, \mathrm{e}^{-\mathrm{i}j\xi} \tag{4.9}$$

For $m = 0, 1, 2, \ldots$ we write $p \in \mathrm{C}^m(\mathcal{C})$ when $\xi \to p(\mathrm{e}^{\mathrm{i}\xi})$ is a function $\mathbb{R} \to \mathbb{C}$ of class $\mathrm{C}^m$ and we use the notation

$$||p||_{\mathrm{C}^m(\mathcal{C})} := \max_{0\le i\le m} \sup_{\xi\in\mathbb{R}} \left|\left(\tfrac{\mathrm{d}}{\mathrm{d}\xi}\right)^i p(\mathrm{e}^{\mathrm{i}\xi})\right| \tag{4.10}$$

If $p \in \mathrm{C}^0(\mathcal{C})$ and $p(S) \in \mathcal{B}(\mathcal{H})$ is defined by the usual functional calculus, then one has $(\mathcal{F}_0^{-1} p(S)\mathcal{F}_0 f)(\mathrm{e}^{\mathrm{i}\xi}) = p(\mathrm{e}^{\mathrm{i}\xi}) f(\mathrm{e}^{\mathrm{i}\xi})$ and

$$\left(p(S)\mathrm{e}_k\right)(j) = \langle f_j,\, p f_k\rangle_{\mathrm{L}^2(\mathcal{C})} = \int_0^{2\pi} \frac{\mathrm{d}\xi}{2\pi}\, p(\mathrm{e}^{\mathrm{i}\xi})\, \mathrm{e}^{\mathrm{i}(k-j)\xi}\,. \tag{4.11}$$

More generally we consider $q(j,\cdot) \in \mathrm{C}^0(\mathcal{C})$ for every $j \in \mathbb{Z}$ and define

$$\left(q(\Lambda, S)\mathrm{e}_k\right)(j) = \int_0^{2\pi} \frac{\mathrm{d}\xi}{2\pi}\, q(j, \mathrm{e}^{\mathrm{i}\xi})\, \mathrm{e}^{\mathrm{i}(k-j)\xi}\,. \tag{4.12}$$

Assume that $b : \mathbb{Z} \to \mathbb{C}$ is bounded. Then by definition

$$b(\Lambda)p(S) = q(\Lambda, S) \text{ if } q(j, \mathrm{e}^{\mathrm{i}\xi}) = b(j)p(\mathrm{e}^{\mathrm{i}\xi}) \tag{4.13}$$

and in the case $\tilde{q}(j,\cdot) = b(j)q(j,\cdot)$ we can identify $\tilde{q}(\Lambda, S)$ with $b(\Lambda)q(\Lambda, S)$. Following this convention we can write the operator $Q_n$ defined by (4.3) in the form

$$Q_n = \theta_{n,n}(\Lambda)(\mathrm{e}^{\mathrm{i}\tilde{\psi}_n})(\Lambda, S) = \Theta_n(\mathrm{e}^{\mathrm{i}\tilde{\psi}_n})(\Lambda, S). \tag{4.14}$$

Similarly if $p \in \mathrm{C}^0(\mathcal{C})$ and $\tilde{q}(j,\cdot) = q(j,\cdot)p(\cdot)$, then we can identify $\tilde{q}(\Lambda, S)$ with $q(\Lambda, S)p(S)$. This property is a special case of a more general expression for compositions: we claim that the composition of the operator $q_1(\Lambda, S)$ and the adjoint of $q_2(\Lambda, S)$ can be expressed by the formula

$$\langle \mathrm{e}_j, q_1(\Lambda, S)\big(q_2(\Lambda, S)\big)^* \mathrm{e}_k\rangle = \int_0^{2\pi} \frac{\mathrm{d}\xi}{2\pi}\, q_1(j, \mathrm{e}^{\mathrm{i}\xi})\overline{q_2(k, \mathrm{e}^{\mathrm{i}\xi})}\, \mathrm{e}^{\mathrm{i}(k-j)\xi} \tag{4.15}$$

In order to check (4.15) we introduce $Q_i := q_i(\Lambda, S)\mathcal{F}_0 : \mathrm{L}^2(\mathcal{C}) \to \mathcal{H}$ and observe that

$$\langle Q_i^* \mathrm{e}_j, f_k\rangle_{\mathrm{L}^2(\mathcal{C})} = \langle \mathrm{e}_j, q_i(\Lambda, S)\mathrm{e}_k\rangle = \big(q_i(\Lambda, S)\mathrm{e}_k\big)(j) = \langle \overline{q_i(j,\cdot)} f_j, f_k\rangle_{\mathrm{L}^2(\mathcal{C})} \tag{4.16}$$

Clearly (4.16) ensures ${Q_i}^* \mathrm{e}_j = \overline{q_i(j,\cdot)} f_j$ for $i = 1$ and 2, hence

$$\langle \mathrm{e}_j, Q_1 {Q_2}^* \mathrm{e}_k\rangle = \langle {Q_1}^* \mathrm{e}_j, {Q_2}^* \mathrm{e}_k\rangle_{\mathrm{L}^2(\mathcal{C})} = \langle\, \overline{q_1(j,\cdot)} f_j, \overline{q_2(k,\cdot)} f_k\rangle_{\mathrm{L}^2(\mathcal{C})}. \tag{4.17}$$

However the right-hand side of (4.17) coincides with the right hand-side of (4.15) and due to $q_1(\Lambda, S)\big(q_2(\Lambda, S)\big)^* = Q_1 {Q_2}^*$ the left-hand side of (4.17) coincides with the left hand-side of (4.15).

We have the following norm estimates

**Lemma 4.2.** *For $j \in \mathbb{Z}$, $n \in \mathbb{Z}^*$ let $q_n(j,\cdot) \in \mathrm{C}^1(\mathcal{C})$ and*

$$\tilde{\psi}_n(j, \mathrm{e}^{\mathrm{i}\xi}) = \psi_n(\mathrm{e}^{\mathrm{i}\xi}) + (j-n)\varphi_n(\mathrm{e}^{\mathrm{i}\xi}) \text{ when } |j-n| \le n/3 \tag{4.18}$$

*holds with some real-valued* $\psi_n \in \mathrm{C}^0(\mathcal{C})$ *and* $\varphi_n \in \mathrm{C}^2(\mathcal{C})$*. If*

$$\sup_{n\in\mathbb{N}^*} ||\varphi_n||_{\mathrm{C}^2(\mathcal{C})} \leq \tfrac{1}{2} \tag{4.19}$$

*then*

$$||\theta_{3n/2,n}(\Lambda)(\mathrm{e}^{\mathrm{i}\tilde{\psi}_n} q_n)(\Lambda, S)|| \leq 4\sqrt{\ln(n)} \sup_{|j-n|\leq n/3} ||q_n(j,\cdot)||_{\mathrm{C}^1(\mathcal{C})} \tag{4.20}$$

*Proof.* See Section 7.3. □

4.3. **An auxiliary computation.** For $0 \leq t \leq 1$ we define

$$\tilde{\psi}_n(j, \mathrm{e}^{\mathrm{i}\xi}) := 2a_n(j)t\sin(\xi)\left(1 - t\delta a(n)\cos(\xi)\right). \tag{4.21}$$

In what follows we always assume that $j \in \mathbb{Z}$ satisfies $|j-n| \leq n/3$. Thus (4.6) allows us to express

$$\tilde{\psi}_n^t(j, \mathrm{e}^{\mathrm{i}\xi}) = \psi_n^t(\mathrm{e}^{\mathrm{i}\xi}) + (j-n)\varphi_n^t(\mathrm{e}^{\mathrm{i}\xi}) \tag{4.22}$$

with

$$\begin{cases} \psi_n^t(\mathrm{e}^{\mathrm{i}\xi}) := 2ta(n)\sin(\xi)\left(1 - t\delta a(n)\cos(\xi)\right) \\ \varphi_n^t(\mathrm{e}^{\mathrm{i}\xi}) := 2t\delta a(n)\sin(\xi)\left(1 - t\delta a(n)\cos(\xi)\right) \end{cases} \tag{4.23}$$

We observe that

$$\varphi_n^t(\mathrm{e}^{\mathrm{i}\xi}) = \tilde{\psi}_n^t(j+1, \mathrm{e}^{\mathrm{i}\xi}) - \tilde{\psi}_n^t(j, \mathrm{e}^{\mathrm{i}\xi}) \tag{4.24}$$

and for every $m \in \mathbb{N}$ we can find $C_m > 0$ such that

$$\sup_{0\leq t\leq 1} ||\psi_n^t||_{\mathrm{C}^m(\mathcal{C})} \leq C_m n^{\gamma}, \tag{4.25}$$

$$\sup_{0\leq t\leq 1} ||\varphi_n^t||_{\mathrm{C}^m(\mathcal{C})} \leq C_m n^{\gamma-1}. \tag{4.26}$$

We will show the following

**Lemma 4.3.** *Let* $\tilde{\psi}_n^t$, $\varphi_n^t$ *be given by* (4.21), (4.24) *and let* $j \in \mathbb{Z}$ *be such that* $|j-n| \leq n/3$*. Then for* $0 \leq t \leq 1$ *we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\psi}_n^t(j, \mathrm{e}^{\mathrm{i}\xi}) + 2a_n(j)\,\mathrm{Im}\Big(\mathrm{e}^{\mathrm{i}\varphi_n^t(\mathrm{e}^{\mathrm{i}\xi})-\mathrm{i}\xi}\Big) = a_n(j)r_n^t(\mathrm{e}^{\mathrm{i}\xi}) \tag{4.27}$$

*with* $r_n^t$ *satisfying*

$$\sup_{0\leq t\leq 1}\ \sup_{\xi\in\mathbb{R}} |r_n^t(\mathrm{e}^{\mathrm{i}\xi})| = O(n^{2(\gamma-1)}). \tag{4.28}$$

*Proof.* We assume that $0 \leq t \leq 1$ and $|j-n| \leq n/3$. Since

$$\left|\mathrm{e}^{\mathrm{i}\varphi_n^t} - 1 - \mathrm{i}\varphi_n^t\right| \leq |\varphi_n^t|^2 = O(n^{2(\gamma-1)}) \tag{4.29}$$

we have the expression

$$\mathrm{e}^{\mathrm{i}\varphi_n^t(\mathrm{e}^{\mathrm{i}\xi})}\mathrm{e}^{-\mathrm{i}\xi} = (1 + \mathrm{i}\varphi_n^t(\mathrm{e}^{\mathrm{i}\xi}))\mathrm{e}^{-\mathrm{i}\xi} + O(n^{2(\gamma-1)}).$$

However $\varphi_n^t(\mathrm{e}^{\mathrm{i}\xi}) = 2t\delta a(n)\sin(\xi) + O(n^{2(\gamma-1)})$ by definition of $\varphi_n^t$, hence

$$\mathrm{e}^{\mathrm{i}\varphi_n^t(\mathrm{e}^{\mathrm{i}\xi})}\mathrm{e}^{-\mathrm{i}\xi} = (1 + 2\mathrm{i}t\delta a(n)\sin(\xi))\mathrm{e}^{-\mathrm{i}\xi} + O(n^{2(\gamma-1)})$$

and

$$2a_n(j)\,\mathrm{Im}\Big(\mathrm{e}^{\mathrm{i}\varphi_n^t(\mathrm{e}^{\mathrm{i}\xi})-\mathrm{i}\xi}\Big) = 2a_n(j)\left(-\sin(\xi) + 2t\delta a(n)\sin(\xi)\cos(\xi) + O(n^{2(\gamma-1)})\right)$$

To complete the proof it remains to observe that the differentiation of (4.21) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\psi}_n^t(j, \mathrm{e}^{\mathrm{i}\xi}) = 2a_n(j)\left(\sin(\xi) - 2t\delta a(n)\sin(\xi)\cos(\xi)\right).$$

□

4.4. **Proof of Proposition 4.1.** We denote

$$Q_n^t := \Theta_n\big(\mathrm{e}^{\mathrm{i}\tilde{\psi}_n^t}\big)(\Lambda, S) \tag{4.30}$$

and observe that Lemma 4.2 ensures

$$\sup_{0\le t\le 1} ||Q_n^t|| = O(\sqrt{\ln(n)}\,)\,. \tag{4.31}$$

Since $Q_n^0 = \Theta_n$, we can express

$$Q_n^1 - \mathrm{e}^{\mathrm{i}B_n}\Theta_n = \int_0^1 \mathrm{d}t\, \frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathrm{e}^{\mathrm{i}(1-t)B_n} Q_n^t\Big) = \int_0^1 \mathrm{d}t\, \mathrm{e}^{\mathrm{i}(1-t)B_n}\tilde{Q}_n^t \tag{4.32}$$

with

$$\tilde{Q}_n^t := \left(\frac{\mathrm{d}}{\mathrm{d}t} - \mathrm{i}B_n\right) Q_n^t \tag{4.33}$$

and it remains to show

$$\sup_{0\le t\le 1} ||\tilde{Q}_n^t||\ = O(n^{\gamma-1}\sqrt{\ln(n)}) \tag{4.34}$$

To begin the proof of (4.34) we show that $B_n$ can be replaced by

$$B_n' := \mathrm{i}a_n(\Lambda)(S^{-1} - S). \tag{4.35}$$

For this purpose we first observe that the estiamtes

$$||[S,\, a_n(\Lambda)]|| = ||\delta a_n(\Lambda)|| = O(n^{\gamma-1})$$
$$||[S^{\pm 1},\, \Theta_n]a_n(\Lambda)|| = O(n^{\gamma-1})$$

imply

$$||B_n\Theta_n\ - \Theta_n B_n'||\ = O(n^{\gamma-1}) \tag{4.36}$$

Next we consider auxiliary operators

$$\hat{Q}_n^t := \theta_{3n/2,n}(\Lambda)\big(\mathrm{e}^{\mathrm{i}\tilde{\psi}_n^t}\big)(\Lambda, S).$$

Since $\theta_{3n/2,n}(s) \ne 1 \Longrightarrow |s-n| \le \frac{3n}{2\times 6} \Longrightarrow \theta_{n,n}(s) = 0$ ensures $B_n Q_n^t = B_n\Theta_n\hat{Q}_n^t$ we can estimate the norm of

$$B_n Q_n^t - \Theta_n B_n'\hat{Q}_n^t = (B_n\Theta_n\ - \Theta_n B_n')\hat{Q}_n^t \tag{4.37}$$

by $O(n^{\gamma-1}\sqrt{\ln(n)})$ due to (4.36) and $||\hat{Q}_n^t|| = O(\sqrt{\ln(n)})$. Thus instead of (4.34) it suffices to show that

$$\sup_{0\le t\le 1} ||\,\frac{\mathrm{d}}{\mathrm{d}t}Q_n^t - \mathrm{i}P_n^t\,||\ = O(n^{\gamma-1}\sqrt{\ln(n)}) \tag{4.38}$$

holds with

$$P_n^t := \Theta_n B_n'\hat{Q}_n^t.$$

However $\theta_{3n/2,n}(s) \ne 1 \Longrightarrow \theta_{n,n}(s\pm 1) = 0$ for $n \ge 20$ allows us to express

$$P_n^t = \Theta_n B_n'\big(\mathrm{e}^{\mathrm{i}\tilde{\psi}_n^t}\big)(\Lambda, S) \tag{4.39}$$

for $n \ge 20$. Further on we assume $n \ge 20$ and observe that

$$\Big(\big(\mathrm{e}^{\mathrm{i}\tilde{\psi}_n^t}\big)(\Lambda, S)\mathrm{e}_k\Big)\,(j) = \int_0^{2\pi} \frac{\mathrm{d}\xi}{2\pi}\;\; x_n^{t,\xi}(j)\,\mathrm{e}^{\mathrm{i}k\xi}$$

holds with

$$x_n^{t,\xi}(j) := \mathrm{e}^{\mathrm{i}\tilde{\psi}_n^t(j,\mathrm{e}^{\mathrm{i}\xi}) - \mathrm{i}j\xi},$$

hence using (4.39) we can express

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}Q_n^t\mathrm{e}_k - \mathrm{i}P_n^t\mathrm{e}_k\right)(j) = \theta_{n,n}(j)\int_0^{2\pi}\frac{\mathrm{d}\xi}{2\pi}\; y_n^{t,\xi}(j)\,\mathrm{e}^{\mathrm{i}k\xi} \tag{4.40}$$

with

$$y_n^{t,\xi} := \frac{\mathrm{d}}{\mathrm{d}t}x_n^{t,\xi} + a_n(j)(S^{-1}-S)x_n^{t,\xi}.$$

However

$$\frac{\mathrm{d}}{\mathrm{d}t}x_n^{t,\xi}(j) = x_n^{t,\xi}(j)\,\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\psi}_n^t(j,\mathrm{e}^{\mathrm{i}\xi})$$

and

$$\big(S^{-1}x_n^{t,\xi}\big)(j) = x_n^{t,\xi}(j+1) = x_n^{t,\xi}(j)\mathrm{e}^{\mathrm{i}\varphi_n^t(\mathrm{e}^{\mathrm{i}\xi})-\mathrm{i}\xi},$$

$$\big(Sx_n^{t,\xi}\big)(j) = x_n^{t,\xi}(j-1) = x_n^{t,\xi}(j)\mathrm{e}^{-\mathrm{i}\varphi_n^t(\mathrm{e}^{\mathrm{i}\xi})+\mathrm{i}\xi}.$$

Thus

$$(S^{-1}-S)x_n^{t,\xi}(j) = 2\mathrm{i}\;\mathrm{Im}\Big(\mathrm{e}^{\mathrm{i}\varphi_n^t(\mathrm{e}^{\mathrm{i}\xi})-\mathrm{i}\xi}\Big)$$

and (4.40) can be written in the form

$$\theta_{n,n}(j)\int_0^{2\pi}\frac{\mathrm{d}\xi}{2\pi}\,\mathrm{i}\,a_n(j)\,r_n^t(\mathrm{e}^{\mathrm{i}\xi})\,x_n^{t,\xi}(j)\mathrm{e}^{\mathrm{i}k\xi}$$

with $r_n^t$ as in (4.27), hence

$$\frac{\mathrm{d}}{\mathrm{d}t}Q_n^t + \mathrm{i}P_n^t = \mathrm{i}\,a_n(\Lambda)\,Q_n^t\,r_n^t(S). \tag{4.41}$$

Since Lemma 4.3 ensures $||r_n^t(S)|| = O(n^{2(\gamma-1)})$, using (4.34) and $||a_n(\Lambda)|| = O(n^{\gamma})$ we conclude that the norm of (4.41) is $O(n^{3\gamma-2})$. To complete the proof of Proposition 4.1 we observe that $\gamma \le \frac{1}{2}$ ensures $n^{3\gamma-2} \le n^{\gamma-1}$.

## 5. Proof of the assertion of Proposition 3.7 *a)*

5.1. **A composition formula.** We assume that $\psi_n$, $\varphi_n$, $\psi_n^0$, $\varphi_n^0$ are functions $\mathcal{C}\to\mathbb{R}$ of class $C^m$ with $m\ge 2$. For $j\in\mathbb{Z}$ let $\tilde{\psi}_n$ be defined by (4.22) and $\tilde{\psi}_n^0$ by

$$\tilde{\psi}_n^0(j,\mathrm{e}^{\mathrm{i}\xi}) = \psi_n^0(\mathrm{e}^{\mathrm{i}\xi}) + (j-n)\varphi_n^0(\mathrm{e}^{\mathrm{i}\xi}). \tag{5.1}$$

We want to express the kernel

$$K_n(j,k) := \langle \mathrm{e}_j,\,(\mathrm{e}^{\mathrm{i}\tilde{\psi}_n^0})(\Lambda,S)\big((\mathrm{e}^{\mathrm{i}\tilde{\psi}_n})(\Lambda,S)\big)^*\mathrm{e}_k\rangle \tag{5.2}$$

under the assumption

$$||\varphi_n||_{\mathrm{C}^m(\mathcal{C})} = O(n^{\gamma-1}). \tag{5.3}$$

First of all we observe that

$$K_n(j,k) = \int_0^{2\pi}\frac{\mathrm{d}\xi}{2\pi}\,\mathrm{e}^{\mathrm{i}\tilde{\psi}_n^0(j,\mathrm{e}^{\mathrm{i}\xi})-\mathrm{i}\tilde{\psi}_n(k,\mathrm{e}^{\mathrm{i}\xi})+\mathrm{i}(k-j)\xi}$$

and using $\tilde{\psi}_n(j,\mathrm{e}^{\mathrm{i}\xi}) - \tilde{\psi}_n(k,\mathrm{e}^{\mathrm{i}\xi}) = (j-k)\varphi_n(\mathrm{e}^{\mathrm{i}\xi})$ we find

$$K_n(j,k) = \int_0^{2\pi}\frac{\mathrm{d}\xi}{2\pi}\,\mathrm{e}^{\mathrm{i}(\tilde{\psi}_n^0-\tilde{\psi}_n)(j,\mathrm{e}^{\mathrm{i}\xi})+\mathrm{i}(k-j)(\xi-\varphi_n(\mathrm{e}^{\mathrm{i}\xi}))}. \tag{5.4}$$

We introduce the function $\eta_n:\mathbb{R}\to\mathbb{R}$ by the formula

$$\eta_n(\xi) := \xi - \varphi_n(\mathrm{e}^{\mathrm{i}\xi}). \tag{5.5}$$

Then $\eta_n(\xi + 2\pi) = \eta_n(\xi) + 2\pi$ and due to (5.3) the derivative

$$\eta_n'(\xi) := \frac{\mathrm{d}\eta_n}{\mathrm{d}\xi}(\xi) = 1 + O(n^{\gamma-1}).$$

Therefore choosing $n_0$ large enough we ensure the fact that $\eta_n$ is bijective $\mathbb{R} \to \mathbb{R}$ for $n \geq n_0$ and we can define $\xi_n : \mathbb{R} \to \mathbb{R}$ as the inverse of $\eta_n$. Thus we have

$$\xi_n(\eta) - \varphi_n(\mathrm{e}^{\mathrm{i}\xi_n(\eta)}) = \eta \tag{5.6}$$

and $\xi_n(\eta + 2\pi) = \xi_n(\eta) + 2\pi$. Next we observe that $K_n(j,k)$ is obtained by integrating $2\pi$-periodic function of $\xi$ on the interval $[0,\, 2\pi]$ and the same value can be obtained by integrating on the interval $[\xi_n(0),\, \xi_n(0) + 2\pi] = \xi_n([0,\, 2\pi])$. Thus the change of variable $\xi = \xi_n(\eta)$ for $n \geq n_0$ allows us to write

$$K_n(j,k) = \int_0^{2\pi} \frac{\mathrm{d}\eta}{2\pi}\, \mathrm{e}^{\mathrm{i}(\tilde{\psi}_n^0 - \tilde{\psi}_n)(j, \mathrm{e}^{\mathrm{i}\xi_n(\eta)}) + \mathrm{i}(k-j)\eta} \xi_n'(\eta) \tag{5.7}$$

However $\eta \to \xi_n(\eta) - \eta$ is $2\pi$-periodic, hence we can define $\tilde{\xi}_n : \mathcal{C} \to \mathbb{R}$ by the formula

$$\tilde{\xi}_n(\mathrm{e}^{\mathrm{i}\eta}) = \xi_n(\eta) - \eta. \tag{5.8}$$

Then for $n \geq n_0$ we define $\vartheta_n : \mathcal{C} \to \mathcal{C}$ by the formula

$$\vartheta_n(\mathrm{e}^{\mathrm{i}\eta}) := \mathrm{e}^{\mathrm{i}\eta}\mathrm{e}^{\mathrm{i}\tilde{\xi}_n(\mathrm{e}^{\mathrm{i}\eta})} = \mathrm{e}^{\mathrm{i}\xi_n(\eta)} \tag{5.9}$$

and introduce $\tilde{\vartheta}_n : \mathbb{Z} \times \mathcal{C} \to \mathbb{Z} \times \mathcal{C}$ by the formula

$$\tilde{\vartheta}_n(j, \mathrm{e}^{\mathrm{i}\eta}) := (j, \vartheta_n(\mathrm{e}^{\mathrm{i}\eta})). \tag{5.10}$$

The expression (5.7) means that

$$(\mathrm{e}^{\mathrm{i}\tilde{\psi}_n^0})(\Lambda, S)\big((\mathrm{e}^{\mathrm{i}\tilde{\psi}_n})(\Lambda, S)\big)^* = (\mathrm{e}^{\mathrm{i}(\tilde{\psi}_n^0 - \tilde{\psi}_n)\circ\tilde{\vartheta}_n})(\Lambda, S)\, q_n(S) \tag{5.11}$$

with

$$q_n(\mathrm{e}^{\mathrm{i}\eta}) := 1 + \frac{\mathrm{d}}{\mathrm{d}\eta}\big(\tilde{\xi}_n(\mathrm{e}^{\mathrm{i}\eta})\big) = \xi_n'(\eta). \tag{5.12}$$

We claim that $\xi_n'(\eta) = 1 + O(n^{\gamma-1})$. Indeed, denoting $\varphi_n^{(1)}(\mathrm{e}^{\mathrm{i}\xi}) := \frac{\mathrm{d}}{\mathrm{d}\xi}\big(\varphi_n(\mathrm{e}^{\mathrm{i}\xi})\big)$ and differentiating (5.6) we obtain

$$\xi_n'(\eta)\big(1 - \varphi_n^{(1)}(\mathrm{e}^{\mathrm{i}\xi_n(\eta)})\big) = 1 \tag{5.13}$$

hence

$$\xi_n'(\eta) - 1 = \frac{\varphi_n^{(1)}(\mathrm{e}^{\mathrm{i}\xi_n(\eta)})}{1 - \varphi_n^{(1)}(\mathrm{e}^{\mathrm{i}\xi_n(\eta)})} = O(n^{\gamma-1})\,. \tag{5.14}$$

Moreover successive differentiations of (5.14) allow us to estimate

$$\frac{\mathrm{d}^i\xi_n}{\mathrm{d}\eta^i}(\eta) = O(n^{\gamma-1})$$

for $i = 2, \ldots, m$ by induction with respect to $i$, hence the estimate

$$||\tilde{\xi}_n||_{C^m(\mathcal{C})} = O(n^{1-\gamma}) \tag{5.15}$$

holds for every $m \in \mathbb{N}$.

5.2. **Approximation of $H_n(t)$.** Since $v$ is of period $N$ we can express

$$v(k) = \sum_{\omega\in\Omega} c_\omega \mathrm{e}^{\mathrm{i}\omega k} \tag{5.16}$$

where $c_\omega \in \mathbb{C}$ are constants and $\Omega = \{2\pi k/N \,:\, k \in [0, N[\cap\mathbb{Z}\}$. Moreover our assumption $\langle v\rangle = 0$ ensures $c_0 = 0$ and we can express

$$v(\Lambda) = \sum_{w\in\Omega^*} c_\omega \mathrm{e}^{\mathrm{i}\omega\Lambda} \tag{5.17}$$

with $\Omega^* = \Omega\setminus\{0\}$. By definition $v_n(\Lambda) = (\theta_{n,n}^2 v)(\Lambda) = \Theta_n^2 v(\Lambda)$ and we can express

$$H_n(t) = \mathrm{e}^{-\mathrm{i}tL_{0,n}}\mathrm{e}^{\mathrm{i}B_n}\Theta_n^2 v(\Lambda)\mathrm{e}^{-\mathrm{i}B_n}\mathrm{e}^{\mathrm{i}tL_{0,n}} = \sum_{\omega\in\Omega^*} c_\omega H_n^{\omega,t} \tag{5.18}$$

with

$$H_n^{\omega,t} := \mathrm{e}^{-\mathrm{i}tL_{0,n}}\mathrm{e}^{\mathrm{i}B_n}\Theta_n^2\mathrm{e}^{\mathrm{i}\omega\Lambda}\mathrm{e}^{-\mathrm{i}B_n}\mathrm{e}^{-\mathrm{i}tL_{0,n}}. \tag{5.19}$$

For $s\in\mathbb{R}$ we define $\tau_s : \mathcal{C}\to\mathcal{C}$ by the formula

$$\tau_s(\mathrm{e}^{\mathrm{i}\xi}) := \mathrm{e}^{\mathrm{i}(\xi-s)} \tag{5.20}$$

and $\tilde{\tau}_s : \mathbb{Z}\times\mathcal{C}\to\mathbb{Z}\times\mathcal{C}$ by the formula

$$\tilde{\tau}_s(j,\mathrm{e}^{\mathrm{i}\xi}) := (j,\tau_s(\mathrm{e}^{\mathrm{i}\xi})) = (j,\mathrm{e}^{\mathrm{i}(\xi-s)}). \tag{5.21}$$

**Lemma 5.1.** *Let $\tilde{\psi}_n$ by given by* (4.22), (4.23), *$\tilde{\vartheta}_n$ by* (5.10) *and*

$$\tilde{\psi}_n^{\omega,t} =: (\tilde{\psi}_n\circ\tilde{\tau}_\omega - \tilde{\psi}_n)\circ\tilde{\vartheta}_n\circ\tilde{\tau}_t\,. \tag{5.22}$$

*Then*

$$H_n^{\omega,t} = \Theta_n^2\mathrm{e}^{\mathrm{i}\omega\Lambda}(\mathrm{e}^{\mathrm{i}\tilde{\psi}_n^{\omega,t}})(\Lambda,S) + R_n^{\omega,t} \tag{5.23}$$

*holds with*

$$||R_n^{\omega,t}|| = O(n^{\gamma-1}\ln(n)) \tag{5.24}$$

*Proof.* We first consider the case $t=0$.

Step 1. Let $Q_n = \Theta_n(\mathrm{e}^{\mathrm{i}\tilde{\psi}_n})(\Lambda,S)$ be as in Proposition 4.1. Then

$$H_n^{\omega,0} = \mathrm{e}^{\mathrm{i}B_n}\Theta_n\mathrm{e}^{\mathrm{i}\omega\Lambda}(Q_n^* + R_n^*) = (Q_n+R_n)\mathrm{e}^{\mathrm{i}\omega\Lambda}Q_n^* + \mathrm{e}^{\mathrm{i}B_n}\Theta_n\mathrm{e}^{\mathrm{i}\omega\Lambda}R_n^*$$

holds with $||R_n|| = O(n^{\gamma-1}\sqrt{\ln(n)})$. Thus

$$H_n^{\omega,0} = Q_n\mathrm{e}^{\mathrm{i}\omega\Lambda}Q_n^* + R_{n,1}^{\omega,0} \tag{5.25}$$

and $||Q_n|| = O(\sqrt{\ln(n)})$ holds due to Lemma 4.2, hence we can estimate

$$||R_{n,1}^{\omega,0}|| \le ||R_n||\,(||Q_n||+1) \,= O(n^{\gamma-1}\ln(n))\,. \tag{5.26}$$

Step 2. We observe that

$$\mathrm{e}^{-\mathrm{i}\omega\Lambda}Q_n\mathrm{e}^{\mathrm{i}\omega\Lambda} = \Theta_n(\mathrm{e}^{\mathrm{i}\tilde{\psi}_n\circ\tilde{\tau}_\omega})(\Lambda,S) \tag{5.27}$$

Indeed, using $\mathrm{e}^{\mathrm{i}\omega\Lambda}\mathrm{e}_m = \mathrm{e}^{\mathrm{i}\omega m}\mathrm{e}_m$ we find that

$$\begin{aligned}\langle \mathrm{e}_j, \mathrm{e}^{-\mathrm{i}\omega\Lambda}Q_n\mathrm{e}^{\mathrm{i}\omega\Lambda}\mathrm{e}_k\rangle &= \mathrm{e}^{\mathrm{i}\omega(k-j)}\langle \mathrm{e}_j, Q_n\mathrm{e}_k\rangle\\ &= \theta_{n,n}(j)\int_0^{2\pi}\frac{\mathrm{d}\xi}{2\pi}\,\mathrm{e}^{\mathrm{i}(k-j)(\xi+\omega)+\mathrm{i}\tilde{\psi}_n(j,\mathrm{e}^{\mathrm{i}\xi})}\\ &= \theta_{n,n}(j)\int_\omega^{\omega+2\pi}\frac{\mathrm{d}\eta}{2\pi}\,\mathrm{e}^{\mathrm{i}(k-j)\eta+\mathrm{i}\tilde{\psi}_n(j,\mathrm{e}^{\mathrm{i}(\eta-\omega)})}\end{aligned}$$

and the integral of $2\pi$-periodic functions on the interval $[\omega, \omega + 2\pi]$ is the same as on the interval $[0, 2\pi]$, i.e. (5.27) follows.

Step 3. To show Lemma 5.1 in the case $t = 0$ we transform

$$Q_n \mathrm{e}^{\mathrm{i}\omega\Lambda} Q_n^* = \Theta_n \mathrm{e}^{\mathrm{i}\omega\Lambda} (\mathrm{e}^{\mathrm{i}\tilde{\psi}_n \circ \tilde{\tau}_\omega})(\Lambda, S)\big((\mathrm{e}^{\mathrm{i}\tilde{\psi}_n})(\Lambda, S)\big)^* \Theta_n \tag{5.28}$$

and use the composition formula (5.11) to write

$$Q_n \mathrm{e}^{\mathrm{i}\omega\Lambda} Q_n^* = \Theta_n \mathrm{e}^{\mathrm{i}\omega\Lambda} (\mathrm{e}^{\mathrm{i}\tilde{\psi}_n^{\omega,0}})(\Lambda, S) q_n(S) \Theta_n \tag{5.29}$$

with $q_n = 1 + O(n^{\gamma-1})$ and

$$\tilde{\psi}_n^{\omega,0} = (\tilde{\psi}_n \circ \tilde{\tau}_\omega - \tilde{\psi}_n) \circ \tilde{\vartheta}_n \,.$$

Due to $||\Theta_n (\mathrm{e}^{\mathrm{i}\tilde{\psi}_n^{\omega,0}})(\Lambda, S)|| = O(\sqrt{\ln(n)})$ we have

$$Q_n \mathrm{e}^{\mathrm{i}\omega\Lambda} Q_n^* = Q_n^{\omega,0} + R_{n,2}^{\omega,0} \tag{5.30}$$

with

$$Q_n^{\omega,0} = \mathrm{e}^{\mathrm{i}\omega\Lambda} \Theta_n (\mathrm{e}^{\mathrm{i}\tilde{\psi}_n^{\omega,0}})(\Lambda, S) \Theta_n \tag{5.31}$$

and

$$||R_{n,2}^{\omega,0}|| = O(n^{\gamma-1} \sqrt{\ln(n)}). \tag{5.32}$$

To complete the proof of Lemma 5.1 when $t = 0$ it suffices to show that the norm of

$$P_n^\omega := Q_n^{\omega,0} - \Theta_n^2 (\mathrm{e}^{\mathrm{i}\tilde{\psi}_n^{\omega,0}})(\Lambda, S)$$

is $O(n^{\gamma-1}\sqrt{\ln(n)})$.

Step 4. We show $||P_n^\omega|| = O(n^{\gamma-1}\sqrt{\ln(n)})$. We observe that

$$P_n^\omega = [\Theta_n (\mathrm{e}^{\mathrm{i}\tilde{\psi}_n^{\omega,0}})(\Lambda, S),\, \Theta_n]$$

and $\Theta_n = \theta_0(\frac{1}{n}\Lambda - I)$, hence (3.29) allows us to estimate

$$||P_n^\omega|| \le C ||[\Theta_n\, \mathrm{e}^{\mathrm{i}\tilde{\psi}_n^{\omega,0}}(\Lambda, S),\, \tfrac{1}{n}\Lambda]|| = C n^{-1} ||\tilde{P}_n^\omega||$$

where $\tilde{P}_n^\omega := [\Theta_n (\mathrm{e}^{\mathrm{i}\tilde{\psi}_n^{\omega,0}})(\Lambda, S),\, \Lambda]$. However using $\Lambda \mathrm{e}_m = m\mathrm{e}_m$ we find

$$\begin{aligned}\langle \mathrm{e}_j, \tilde{P}_n^\omega \mathrm{e}_k\rangle &= \langle \mathrm{e}_j, \Theta_n (\mathrm{e}^{\mathrm{i}\tilde{\psi}_n^{\omega,0}})(\Lambda, S)\Lambda \mathrm{e}_k\rangle - \langle \Lambda \mathrm{e}_j, \Theta_n (\mathrm{e}^{\mathrm{i}\tilde{\psi}_n^{\omega,0}})(\Lambda, S)\mathrm{e}_k\rangle \\ &= (k-j)\langle \mathrm{e}_j, \Theta_n\, \mathrm{e}^{\mathrm{i}\tilde{\psi}_n^{\omega,0}}(\Lambda, S)\mathrm{e}_k\rangle\end{aligned}$$

and using $(k-j)\mathrm{e}^{\mathrm{i}(k-j)\xi} = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\xi}\big(\mathrm{e}^{\mathrm{i}(k-j)\xi}\big)$ we can write

$$\langle \mathrm{e}_j, \tilde{P}_n^\omega \mathrm{e}_k\rangle = -\theta_{n,n}(j) \int_0^{2\pi} \frac{\mathrm{d}\xi}{2\pi}\, \mathrm{e}^{\mathrm{i}\tilde{\psi}_n^{\omega,0}(j,\mathrm{e}^{\mathrm{i}\xi})}\, \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\xi}\big(\mathrm{e}^{\mathrm{i}(k-j)\xi}\big),$$

i.e. the integration by parts allows us to conclude that

$$\tilde{P}_n^\omega = \Theta_n \big(\mathrm{e}^{\mathrm{i}\tilde{\psi}_n^{\omega,0}} q_n^\omega\big)(\Lambda, S) \tag{5.33}$$

holds with

$$q_n^\omega(j, \mathrm{e}^{\mathrm{i}\xi}) := \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\xi}\big(\tilde{\psi}_n(j, \mathrm{e}^{\mathrm{i}\xi})\big).$$

It is clear that $||\tilde{P}_n^\omega|| = O(n^\gamma \sqrt{\ln(n)})$ follows from Lemma 4.2 and

$$\sup_{|j-n| \le n/3} ||q_n^\omega(j, \cdot)||_{\mathrm{C}^1(\mathcal{C})} = O(n^\gamma)$$

Step 5. We complete the proof of Lemma 5.1 in the general case. For this purpose we write $l_n(k) = k + a_{1,n}(k)$ and introduce

$$\tilde{H}_n^{\omega,t}(s) := \mathrm{e}^{-\mathrm{i}t\Lambda}\mathrm{e}^{-\mathrm{i}sa_{1,n}(\Lambda)}H_n^{\omega,0}\mathrm{e}^{\mathrm{i}sa_{1,n}(\Lambda)}\mathrm{e}^{\mathrm{i}t\Lambda} \tag{5.34}$$

for $s, t \in \mathbb{R}$. We claim that

$$H_n^{\omega,t} = \tilde{H}_n^{\omega,t}(1) = \tilde{H}_n^{\omega,t}(0) + \tilde{R}_n^{\omega,t} \tag{5.35}$$

holds with

$$\sup_{t\in\mathbb{R}} ||\tilde{R}_n^{\omega,t}|| = O(n^{3\gamma-2}). \tag{5.36}$$

Indeed, since

$$\frac{\mathrm{d}}{\mathrm{d}s}\tilde{H}_n^{\omega,t}(s) = \mathrm{e}^{-\mathrm{i}t\Lambda}\mathrm{e}^{-\mathrm{i}sa_{1,n}(\Lambda)}[\mathrm{i}H_n^{\omega,0},\, a_{1,n}(\Lambda)]\,\mathrm{e}^{\mathrm{i}sa_{1,n}(\Lambda)}\mathrm{e}^{\mathrm{i}t\Lambda} \tag{5.37}$$

it suffices to show

$$||[H_n^{\omega,0},\, a_{1,n}(\Lambda)]|| = O(n^{3\gamma-2}). \tag{5.38}$$

However $||[S,\, a_{1,n}(\Lambda)]|| = O(n^{2\gamma-2})$ implies $||[B_n,\, a_{1,n}(\Lambda)]|| = O(n^{3\gamma-2})$, hence the norm of

$$[\mathrm{e}^{\mathrm{i}B_n},\, a_{1,n}(\Lambda)] = \int_0^1 \mathrm{d}t\,\mathrm{e}^{\mathrm{i}tB_n}\,[\mathrm{i}B_n,\, a_{1,n}(\Lambda)]\,\mathrm{e}^{\mathrm{i}(1-t)B_n}.$$

is $O(n^{3\gamma-2})$ and (5.36) follows.

To complete the proof we express

$$\tilde{H}_n^{\omega,t}(0) = \mathrm{e}^{-\mathrm{i}t\Lambda}\Theta_n^2(\mathrm{e}^{\mathrm{i}\tilde{\psi}_n^{\omega,0}})(\Lambda,S)\mathrm{e}^{\mathrm{i}t\Lambda} + \tilde{R}_{0,n}^{\omega,t}$$

where the norm of $\tilde{R}_{0,n}^{\omega,t} := \mathrm{e}^{-\mathrm{i}t\Lambda}R_n^{\omega,0}\mathrm{e}^{\mathrm{i}t\Lambda}$ was estimated by $O(n^{\gamma-1}\sqrt{\ln(n)})$ in Steps 1-4. It remains to observe that we find

$$\mathrm{e}^{-\mathrm{i}t\Lambda}(\mathrm{e}^{\mathrm{i}\tilde{\psi}_n^{\omega,0}})(\Lambda,S)\mathrm{e}^{\mathrm{i}t\Lambda} = (\mathrm{e}^{\mathrm{i}\tilde{\psi}_n^{\omega,0}\circ\tilde{\tau}_t})(\Lambda,S)$$

reasoning as in Step 2 and by definition $\tilde{\psi}_n^{\omega,0}\circ\tilde{\tau}_t = \tilde{\psi}_n^{\omega,t}$. □

5.3. **A stationary phase estimate.** We show the following

**Lemma 5.2.** *For $b \in \mathrm{C}^2(\mathcal{C})$ and $\mu \in \mathbb{R}$ denote*

$$\mathcal{J}(b,\mu) := \int_0^{2\pi} \mathrm{d}\eta\,\mathrm{e}^{\mathrm{i}\mu\cos(\eta)}b(\mathrm{e}^{\mathrm{i}\eta})\,. \tag{5.39}$$

*Then there is a constant $C_0$ such that*

$$|\mathcal{J}(b,\mu)| \le \frac{C_0}{|\mu|^{1/2}}\Big(||b||_{\mathrm{C}^0(\mathcal{C})} + \frac{1}{|\mu|^{1/2}}||b||_{\mathrm{C}^2(\mathcal{C})}\Big). \tag{5.40}$$

*Proof.* Using a partition of unity on the circle we can use two cut-off functions $\chi_k \in \mathrm{C}^\infty(\mathbb{R})$ with $\operatorname{supp}\chi_k\ \subset ]-\frac{3}{4}\pi,\, \frac{3}{4}\pi[$ for $k = \pm 1$, reducing the problem to the estimate

$$\left|\int_{-\infty}^{\infty} b_k(\xi)\mathrm{e}^{\mathrm{i}k\mu\cos(\xi)}\chi_k(\xi)\mathrm{d}\xi\right| \le \frac{C}{|\mu|^{1/2}}||b_k||_{\mathrm{C}^0(\mathbb{R})} + \frac{C}{|\mu|}||b_k||_{\mathrm{C}^2(\mathbb{R})} \quad (k = \pm 1)$$

for $b_k \in \mathrm{C}^2(\mathbb{R})$. If $|\xi| \le \frac{3}{4}\pi$ then we can write

$$b_k(\xi) = b_k(0) + q_k(\xi)\xi = b_k(0) + \tilde{q}_k(\xi)\sin(\xi)$$

with $\tilde{q}_k(\xi) := q_k(\xi)\frac{\xi}{\sin(\xi)}$. However the standard stationary phase method ensures

$$\left|b_k(0)\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}k\mu\cos(\xi)}\chi_k(\xi)\mathrm{d}\xi\right| \le |b_k(0)|\,C_0|\mu|^{-1/2}$$

and writing $\mathrm{e}^{\mathrm{i}k\mu\cos(\xi)}\sin(\xi) = \frac{\mathrm{i}}{k\mu}\frac{\mathrm{d}}{\mathrm{d}\xi}\mathrm{e}^{\mathrm{i}k\mu\cos(\xi)}$ we observe that the integration by parts gives

$$\int_{-\infty}^{\infty} \tilde{q}_k(\xi)\sin(\xi)\mathrm{e}^{\mathrm{i}k\mu\cos(\xi)}\chi_k(\xi)\mathrm{d}\xi = \frac{\mathrm{i}k}{\mu}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}k\mu\cos(\xi)}\frac{\mathrm{d}}{\mathrm{d}\xi}(\tilde{q}_k\chi_k)(\xi)\mathrm{d}\xi. \tag{5.41}$$

Since the absolute value of the right hand side of (5.41) can be estimated by $\frac{C_1}{|\mu|}||b_k||_{\mathrm{C}^2(\mathbb{R})}$ the proof is complete. □

5.4. **End of the proof of Proposition 3.7** *a*)**.** Step 1. We begin by

**Lemma 5.3.** *For every $m \in \mathbb{N}$ one can find a constant $C_m$ such that*

$$||f_n \circ \vartheta_n - f_n||_{\mathrm{C}^{m-1}(\mathcal{C})} \le C_m n^{\gamma-1}||f_n||_{\mathrm{C}^m(\mathcal{C})} \tag{5.42}$$

*holds for $f_n \in \mathrm{C}^m(\mathcal{C})$.*

*Proof.* For $s \in \mathbb{R}$ we define $\vartheta_n^s(\mathrm{e}^{\mathrm{i}\eta}) := \mathrm{e}^{\mathrm{i}\eta}\mathrm{e}^{\mathrm{i}s\tilde{\xi}_n(\mathrm{e}^{\mathrm{i}\eta})}$. If $m \in \mathbb{N}^*$, then there exists a constant $C'_m$ such that

$$\sup_{0\le s\le 1} ||g_n \circ \vartheta_n^s||_{\mathrm{C}^{m-1}(\mathcal{C})} \le C'_m||g_n||_{\mathrm{C}^{m-1}(\mathcal{C})} \tag{5.43}$$

holds for every $g_n \in \mathrm{C}^{m-1}(\mathcal{C})$. Indeed, using the standard differentiation of composed functions we easily deduce (5.43) by induction with respect to $m$. Next we introduce $g_n(\mathrm{e}^{\mathrm{i}\eta}) := \frac{\mathrm{d}}{\mathrm{d}\eta}\big(f_n(\mathrm{e}^{\mathrm{i}\eta})\big)$ and observe that

$$\frac{\mathrm{d}}{\mathrm{d}s}f_n\big(\mathrm{e}^{\mathrm{i}\eta}\mathrm{e}^{\mathrm{i}s\tilde{\xi}_n(\mathrm{e}^{\mathrm{i}\eta})}\big) = \tilde{\xi}_n(\mathrm{e}^{\mathrm{i}\eta})g_n\big(\mathrm{e}^{\mathrm{i}\eta}\mathrm{e}^{\mathrm{i}s\tilde{\xi}_n(\mathrm{e}^{\mathrm{i}\eta})}\big) \tag{5.44}$$

But the $||\cdot||_{\mathrm{C}^{m-1}(\mathcal{C})}$-norm of (5.44) can be estimated by $C''_m||\tilde{\xi}_n||_{C^{m-1}(\mathcal{C})}||g_n||_{\mathrm{C}^{m-1}(\mathcal{C})}$ due to (5.43) and we complete the proof using $||\tilde{\xi}_n||_{\mathrm{C}^{m-1}(\mathcal{C})} = O(n^{\gamma-1})$. □

Step 2. In order to apply the stationary phase estimate of Lemma 5.2 we define a suitable principal part of the phase function $\tilde{\psi}_n^{\omega,t}$ introduced in Lemma 5.1. By definition

$$\tilde{\psi}_n^{\omega,t}(j,\mathrm{e}^{\mathrm{i}\eta}) = \psi_n^{\omega,t}(\mathrm{e}^{\mathrm{i}\eta}) + (j-n)\varphi_n^{\omega,t}(\mathrm{e}^{\mathrm{i}\eta}) \tag{5.45}$$

holds with

$$\begin{cases}\psi_n^{\omega,t} := (\psi_n \circ \tau_\omega - \psi_n)\circ\vartheta_n\circ\tau_t = \psi_n^{\omega,0}\circ\tau_t\\ \varphi_n^{\omega,t} := (\varphi_n \circ \tau_\omega - \varphi_n)\circ\vartheta_n\circ\tau_\tau = \varphi_n^{\omega,0}\circ\tau_t\end{cases} \tag{5.46}$$

and we can decompose

$$\psi_n = \psi_{n,1} + \psi_{n,2} \tag{5.47}$$

with

$$\begin{cases}\psi_{n,1}(\mathrm{e}^{\mathrm{i}\xi}) := 2a(n)\sin(\xi) = 2a(n)\,\mathrm{Im}\,(\mathrm{e}^{\mathrm{i}\xi}),\\ \psi_{n,2}(\mathrm{e}^{\mathrm{i}\xi}) := -2a(n)\delta a(n)\sin(\xi)\cos(\xi)\end{cases} \tag{5.48}$$

Next we observe that the decomposition (5.47) allows us to write

$$\psi_n^{\omega,0} = (\psi_{n,1}\circ\tau_\omega - \psi_{n,1})\circ\vartheta_n + (\psi_{n,2}\circ\tau_\omega - \psi_{n,2})\circ\vartheta_n$$

and we can express

$$\psi_n^{\omega,0} = \psi_{n,1}^{\omega,0} + \psi_{n,2}^{\omega,0} \tag{5.49}$$

with

$$\psi_{n,1}^{\omega,0} := \psi_{n,1}\circ\tau_\omega - \psi_{n,1},$$

i.e.

$$\psi_{n,1}^{\omega,0}(\mathrm{e}^{\mathrm{i}\xi}) = 2a(n)\,\mathrm{Im}\Big(\mathrm{e}^{\mathrm{i}\xi}(\mathrm{e}^{-\mathrm{i}\omega}-1)\Big)$$

and

$$\psi_{n,2}^{\omega,0} := \psi_{n,1}^{\omega,0}\circ\vartheta_n - \psi_{n,1}^{\omega,0} + (\psi_{n,2}\circ\tau_\omega - \psi_{n,2})\circ\vartheta_n.$$

Denoting $\psi_{n,k}^{\omega,t} := \psi_{n,k}^{\omega,0}\circ\tau_t$ for $k=1,2$ we can decompose

$$\psi_n^{\omega,t} = (\psi_{n,1}^{\omega,0} + \psi_{n,2}^{\omega,0})\circ\tau_t = \psi_{n,1}^{\omega,t} + \psi_{n,2}^{\omega,t} \tag{5.50}$$

and according to the above notations we find the expression

$$\psi_{n,1}^{\omega,t}(\mathrm{e}^{\mathrm{i}\xi}) = 2a(n)\,\mathrm{Im}\Big(\mathrm{e}^{\mathrm{i}(\xi-t)}(\mathrm{e}^{-\mathrm{i}\omega}-1)\Big) =$$

$$2a(n)\big(\sin(\xi-\omega-t)-\sin(\xi-t)\big) = -4a(n)\sin(\tfrac{\omega}{2})\cos(\xi-t-\tfrac{\omega}{2}).$$

We have the following

**Lemma 5.4.** *For every $m\in\mathbb{N}$ there is a constant $C_m$ such that*

$$||\psi_{n,1}^{\omega,t}||_{\mathrm{C}^m(\mathcal{C})} = ||\psi_{n,1}^{\omega,0}||_{\mathrm{C}^m(\mathcal{C})} \le C_m n^\gamma \tag{5.51}$$

$$||\psi_{n,2}^{\omega,t}||_{\mathrm{C}^m(\mathcal{C})} = ||\psi_{n,2}^{\omega,0}||_{\mathrm{C}^m(\mathcal{C})} \le C_m n^{2\gamma-1} \le C_m \tag{5.52}$$

$$||\varphi_n^{\omega,t}||_{\mathrm{C}^m(\mathcal{C})} = ||\varphi_n^{\omega,0}||_{\mathrm{C}^m(\mathcal{C})} \le C_m n^{\gamma-1} \tag{5.53}$$

*Proof.* It is clear that (5.51) directly follows from the definition of $\psi_{n,1}$ and (5.53) follows from (5.3) and (5.43). Then using $a(n)\delta a(n) = O(n^{2\gamma-1})$ and the definition of $\psi_{n,2}$ we obtain $||\psi_{n,2}||_{\mathrm{C}^m(\mathcal{C})} = O(n^{2\gamma-1})$ and (5.43) ensures

$$||(\psi_{n,2}\circ\tau_\omega - \psi_{n,2})\circ\vartheta_n||_{\mathrm{C}^m(\mathcal{C})} \le C'_m n^{2\gamma-1}. \tag{5.54}$$

Moreover Lemma 5.2 allows us to estimate

$$||\psi_{n,1}^{\omega,0}\circ\vartheta_n - \psi_{n,1}^{\omega,0}||_{\mathrm{C}^{m-1}(\mathcal{C})} \le C'_m n^{\gamma-1}||\psi_{n,1}^{\omega,0}||_{\mathrm{C}^m(\mathcal{C})}, \tag{5.55}$$

hence to complete the proof of (5.52) it remains to use (5.54) and to observe that the right-hand side of (5.55) can be estimated by $C'_m n^{2\gamma-1}$ due to (5.51). ☐

Step 3. We write

$$g_{1,n,j}(t) = \mathrm{i}\sum_{\omega\in\Omega^*} c_\omega\langle \mathrm{e}_j, H_n^{\omega,t}\mathrm{e}_j\rangle = \mathrm{i}\sum_{\omega\in\Omega^*} c_\omega \mathrm{e}^{\mathrm{i}j\omega} g_n^{\omega,t}(j) + O(n^{\gamma-1}\ln(n)) \tag{5.56}$$

with

$$g_n^{\omega,t}(j) := \langle \mathrm{e}_j, \Theta_n^2(\mathrm{e}^{\mathrm{i}\tilde{\psi}_n^{\omega,t}})(\Lambda,S)\mathrm{e}_j\rangle = \theta_{n,n}(j)^2\int_0^{2\pi}\frac{\mathrm{d}\eta}{2\pi}\,\mathrm{e}^{\mathrm{i}\tilde{\psi}_n^{\omega}(j,\mathrm{e}^{\mathrm{i}\eta})}. \tag{5.57}$$

It remains to show the estimate

$$\sup_{-t_0\le t\le t_0}\ \sup_{|j-n|\le n^\gamma} |g_n^{\omega,t}(j)| \le Cn^{-\gamma/2}. \tag{5.58}$$

For this purpose we write

$$\theta_{n,n}(j)^2\mathrm{e}^{\mathrm{i}\tilde{\psi}_n^{\omega}(j,\mathrm{e}^{\mathrm{i}\eta})} = \mathrm{e}^{\mathrm{i}\psi_{n,1}^{\omega,t}(\mathrm{e}^{\mathrm{i}\eta})} b_n^{\omega,t}(j,\mathrm{e}^{\mathrm{i}\eta}) \tag{5.59}$$

with

$$b_n^{\omega,t}(j,\mathrm{e}^{\mathrm{i}\eta}) := \theta_{n,n}(j)^2\mathrm{e}^{\mathrm{i}\psi_{n,2}^{\omega,t}(\mathrm{e}^{\mathrm{i}\eta})+\mathrm{i}(j-n)\varphi_n^{\omega,t}(\mathrm{e}^{\mathrm{i}\eta})} \tag{5.60}$$

and we observe that (5.53) ensures

$$\sup_{|j-n|\le n^\gamma} ||(j-n)\varphi_n^{\omega,t}||_{\mathrm{C}^2(\mathcal{C})} \le C_2 n^{2\gamma-1} \le C_2. \tag{5.61}$$

Combining (5.61) with (5.52) we obtain

$$\sup_{|j-n|\le n^\gamma} ||b_n^{\omega,t}(j,\cdot)||_{\mathrm{C}^2(\mathcal{C})} \le C' \tag{5.62}$$

and using the expression

$$\psi_{n,1}^{\omega,t}(\mathrm{e}^{\mathrm{i}\xi}) = -4a(n)\sin(\tfrac{\omega}{2})\cos(\xi - t - \tfrac{\omega}{2})$$

we observe that

$$g_n^{\omega,t}(j) = \int_0^{2\pi} \frac{\mathrm{d}\eta}{2\pi} \mathrm{e}^{\mathrm{i}\mu_n^\omega \cos(\eta - t - \omega/2)} b_n^{\omega,t}(j,\mathrm{e}^{\mathrm{i}\eta}), \tag{5.63}$$

holds with $\mu_n^\omega := -4a(n)\sin(\omega/2)$. Performing the change of variable $\xi = \eta - t - \omega/2$ and using the notation of Lemma 5.2 we find

$$g_n^{\omega,t}(j) = \mathcal{J}(b_n^{\omega,t} \circ \tilde{\tau}_{t+\omega/2}(j,\cdot),\, \mu_n^\omega) \,= O(|\mu_n^\omega|^{-1/2}) \tag{5.64}$$

uniformly with respect to $j \in [n - n^\gamma,\, n + n^\gamma]$. To complete the proof we observe that we can find $c_0 > 0$ such that

$$|\mu_n^\omega| \ge c_0\, n^\gamma \tag{5.65}$$

holds when $\omega \in \Omega^*$.

## 6. Proof of the assertion of Proposition 3.7 *b*)

**6.1. Approximation of $H_n^{\omega_1,t_1}\dots H_n^{\omega_\nu,t_\nu}$.** For $\nu \in \mathbb{N}^*$ we consider $\underline{t} = (t_1,\dots,t_\nu) \in \mathbb{R}^\nu$, $\underline{\omega} = (\omega_1,\dots,\omega_\nu) \in (\Omega^*)^\nu$ and denote

$$H_n^{\underline{\omega},\underline{t}} := H_n^{\omega_1,t_1}\dots H_n^{\omega_\nu,t_\nu}. \tag{6.1}$$

Our aim is to find an approximation of $H_n^{\underline{\omega},\underline{t}}$ by

$$Q_n^{\underline{\omega},\underline{t}} :=\ \mathrm{e}^{\mathrm{i}|\omega|_1\Lambda}\Theta_n^{2\nu}\big(\mathrm{e}^{\mathrm{i}\tilde{\psi}_n^{\underline{\omega},\underline{t}}}\big)(\Lambda,S), \tag{6.2}$$

where $|(\omega_1,\dots,\omega_\nu)|_1 := \omega_1 + \dots + \omega_\nu$ and we want to separate a principal part of $\tilde{\psi}_n^{\underline{\omega},\underline{t}}$ so that the stationary phase method could be applied similarly as in Section 5.

Our analysis uses induction with respect to $\nu$. We write $\underline{\omega} = (\underline{\omega}',\omega) \in (\Omega^*)^{\nu-1}\times\Omega^*$, $\underline{t} = (\underline{t}',t) \in \mathbb{R}^{\nu-1}\times\mathbb{R}$ and using

$$H_n^{\underline{\omega},\underline{t}} = H_n^{\underline{\omega}',\underline{t}'} H_n^{\omega,t} \tag{6.3}$$

we consider the decomposition

$$\tilde{\psi}_n^{\underline{\omega},\underline{t}}(j,\mathrm{e}^{\mathrm{i}\eta}) = (\psi_{n,1}^{\underline{\omega},\underline{t}} + \psi_{n,2}^{\underline{\omega},\underline{t}})(\mathrm{e}^{\mathrm{i}\eta}) + (j-n)(\varphi_{n,1}^{\underline{\omega},\underline{t}} + \varphi_{n,2}^{\underline{\omega},\underline{t}})(\mathrm{e}^{\mathrm{i}\eta}) \tag{6.4}$$

where the principal parts $\psi_{n,1}^{\underline{\omega},\underline{t}}$, $\varphi_{n,1}^{\underline{\omega},\underline{t}}$ are defined by induction as follows:

1) For $\nu = 1$ we use $\psi_{n,j}^{\omega,t}$ as in (5.49), $\varphi_{n,1}^{\omega,t} = \varphi_n^{\omega,t}$ given by (5.46), $\varphi_{n,2}^{\omega,t} = 0$

2) In the case $\nu \ge 2$ we use the induction formulas

$$\begin{cases} \psi_{n,1}^{\underline{\omega},\underline{t}} := (\psi_{n,1}^{\underline{\omega}',\underline{t}'} - \psi_{n,1}^{-\omega,t}) \circ \tau_\omega \\ \varphi_{n,1}^{\underline{\omega},\underline{t}} := (\varphi_{n,1}^{\underline{\omega}',\underline{t}'} - \varphi_{n,1}^{-\omega,t}) \circ \tau_\omega, \end{cases} \tag{6.5}$$

where $-\omega$ is identified with $2\pi - \omega \in \Omega^*$.

We observe that (6.5) ensures

$$||\psi_{n,1}^{\underline{\omega},\underline{t}}||_{\mathrm{C}^m(\mathcal{C})} \le ||\psi_{n,1}^{\underline{\omega}',\underline{t}'}||_{\mathrm{C}^m(\mathcal{C})} + ||\psi_{n,1}^{-\omega,t}||_{\mathrm{C}^m(\mathcal{C})}, \tag{6.6}$$

$$||\varphi_{n,1}^{\underline{\omega},\underline{t}}||_{\mathrm{C}^m(\mathcal{C})} \le ||\varphi_{n,1}^{\underline{\omega}',\underline{t}'}||_{\mathrm{C}^m(\mathcal{C})} + ||\varphi_{n,1}^{-\omega,t}||_{\mathrm{C}^m(\mathcal{C})} \tag{6.7}$$

and we claim that for every $m \in \mathbb{N}$ there exists a constant $C_m$ (independent of $\underline{\omega}$, $\underline{t}$, $n$) such that

$$||\psi_{n,1}^{\underline{\omega},\underline{t}}||_{\mathrm{C}^m(\mathcal{C})} \le C_m \nu n^{\gamma} \tag{6.8}$$

$$||\varphi_{n,1}^{\underline{\omega},\underline{t}}||_{\mathrm{C}^m(\mathcal{C})} \le C_m \nu n^{\gamma-1} \tag{6.9}$$

hold for $\underline{\omega} \in (\Omega^*)^\nu$, $\underline{t} \in [-t_0, t_0]^\nu$. Indeed, the estimates (6.8), (6.9) hold in the case $\nu = 1$ (cf. Section 5) and the induction assumption that

$$||\psi_{n,1}^{\underline{\omega}',\underline{t}'}||_{\mathrm{C}^m(\mathcal{C})} \le C_m (\nu-1) n^{\gamma} \tag{6.10}$$

$$||\varphi_{n,1}^{\underline{\omega}',\underline{t}'}||_{\mathrm{C}^m(\mathcal{C})} \le C_m (\nu-1) n^{\gamma-1} \tag{6.11}$$

hold for a certain $\nu \ge 2$ implies (6.8), (6.9) due to (6.6), (6.7). Moreover further on we assume that $C_m \le C_{m+1}$ for all $m \in \mathbb{N}$.

We will prove

**Lemma 6.1.** *We assume that $Q_n^{\underline{\omega},\underline{t}}$ is given by (6.2) and $\tilde{\psi}_n^{\underline{\omega},\underline{t}}$ is expressed by (6.4) with $\psi_{n,1}^{\underline{\omega},\underline{t}}$, $\varphi_{n,1}^{\underline{\omega},\underline{t}}$ defined by (6.5). Let $0 < \varepsilon \le 1/8$ be fixed. Then we can find $\psi_{n,2}^{\underline{\omega},\underline{t}}$, $\varphi_{n,2}^{\underline{\omega},\underline{t}}$ and constants $\hat{C}$, $\hat{n}$ (independent of $\underline{\omega}$, $\underline{t}, n$) such that the estimates*

$$||\psi_{n,2}^{\underline{\omega},\underline{t}}||_{\mathrm{C}^3(\mathcal{C})} \le \hat{C} \nu n^{\varepsilon}, \tag{6.12}$$

$$||\varphi_{n,2}^{\underline{\omega},\underline{t}}||_{\mathrm{C}^3(\mathcal{C})} \le \hat{C} \nu n^{2(\gamma-1)+\varepsilon} \tag{6.13}$$

*hold when $\nu \le n^{\varepsilon}$ and $n \ge \hat{n}$. Moreover*

$$H_n^{\underline{\omega},\underline{t}} = Q_n^{\underline{\omega},\underline{t}} + R_n^{\underline{\omega},\underline{t}} \tag{6.14}$$

*holds with $R_n^{\underline{\omega},\underline{t}}$ satisfying*

$$||R_n^{\underline{\omega},\underline{t}}|| \le \nu n^{\gamma-1+3\varepsilon} \tag{6.15}$$

*when $\nu \le n^{\varepsilon}$ and $n \ge \hat{n}$.*

The proof is given in Section 6.2 and uses induction with respect to $\nu$. We first observe that the composition formula from Section 5.1 can be applied to $\varphi_n^{-\omega,t}$ instead of $\varphi_n$. More precisely : we assume that $\hat{n}$ is large enough and for $n \ge \hat{n}$ we consider the bijection $\mathbb{R} \to \mathbb{R}$ given by the formula

$$\eta_n^{\omega,t}(\xi) := \xi - \varphi_n^{-\omega,t}(\mathrm{e}^{\mathrm{i}\xi}). \tag{6.16}$$

We denote by $\xi_n^{\omega,t}$ its inverse, i.e.

$$\xi_n^{\omega,t}(\eta) - \varphi_n^{-\omega,t}(\mathrm{e}^{\mathrm{i}\xi_n^{\omega,t}(\eta)}) = \eta \tag{6.17}$$

holds for $\eta \in \mathbb{R}$ and we define $\tilde{\xi}_n^{\omega,t} : \mathcal{C} \to \mathbb{C}$ by the formula

$$\tilde{\xi}_n^{\omega,t}(\mathrm{e}^{\mathrm{i}\eta}) = \xi_n^{\omega,t}(\eta) - \eta. \tag{6.18}$$

For $n \ge n_0$ we define $\vartheta_n^{\omega,t} : \mathcal{C} \to \mathcal{C}$ by the formula

$$\vartheta_n^{\omega,t}(\mathrm{e}^{\mathrm{i}\eta}) := \mathrm{e}^{\mathrm{i}\eta}\mathrm{e}^{\mathrm{i}\tilde{\xi}_n^{\omega,t}(\mathrm{e}^{\mathrm{i}\eta})} \tag{6.19}$$

and $\tilde{\vartheta}_n^{\omega,t} : \mathbb{Z} \times \mathcal{C} \to \mathbb{Z} \times \mathcal{C}$ by the formula

$$\tilde{\vartheta}_n^{\omega,t}(j, \mathrm{e}^{\mathrm{i}\eta}) := (j, \vartheta_n^{\omega,t}(\mathrm{e}^{\mathrm{i}\eta})). \tag{6.20}$$

We observe that reasoning as in Lemma 5.3 for every $m \in \mathbb{N}$ we can find a constant $C'_m$ such that

$$||f_n \circ \vartheta_n^{\omega,t} - f_n||_{\mathrm{C}^{m-1}(\mathcal{C})} \le C'_m n^{\gamma-1} ||f_n||_{\mathrm{C}^m(\mathcal{C})} \tag{6.21}$$

holds for $f_n \in \mathrm{C}^m(\mathcal{C})$ and we can assume $C'_m \le C'_{m+1}$ for all $m \in \mathbb{N}$.

6.2. **Proof of Lemma 6.1.** Let $\nu = 1$. Then Lemma 5.1 ensures

$$H_n^{\omega,t} = Q_n^{\omega,t} + R_n^{\omega,t} \tag{6.22}$$

with $||R_n^{\omega,t}|| = O(n^{\gamma-1}\ln(n))$ and Lemma 4.2 ensures

$$||Q_n^{\omega,t}|| \leq 4\sqrt{\ln(n)}\,. \tag{6.23}$$

Thus the assertion of Lemma 6.1 holds when $\nu = 1$.

Next we consider $\nu \geq 2$ and $\underline{\omega} = (\underline{\omega}', \omega) \in (\Omega^*)^{\nu-1} \times \Omega^*$, $\underline{t} = (\underline{t}', t) \in \mathbb{R}^{\nu-1} \times \mathbb{R}$. Reasoning by induction we assume that we have already defined $\psi_{n,2}^{\underline{\omega}',\underline{t}'}$, $\varphi_{n,2}^{\underline{\omega}',\underline{t}'}$ satisfying

$$||\psi_{n,2}^{\underline{\omega}',\underline{t}'}||_{\mathrm{C}^3(\mathcal{C})} \leq \hat{C}(\nu-1)n^{\varepsilon}, \tag{6.24}$$

$$||\varphi_{n,2}^{\underline{\omega}',\underline{t}'}||_{\mathrm{C}^3(\mathcal{C})} \leq \hat{C}(\nu-1)n^{2(\gamma-1)+\varepsilon} \tag{6.25}$$

and

$$H_n^{\underline{\omega}',\underline{t}'} = Q_n^{\underline{\omega}',\underline{t}'} + R_n^{\underline{\omega}',\underline{t}'} \tag{6.26}$$

holds with

$$Q_n^{\underline{\omega}',\underline{t}'} = \mathrm{e}^{\mathrm{i}|\underline{\omega}'|_1\Lambda}\Theta_n^{2(\nu-1)}(\mathrm{e}^{\mathrm{i}\tilde{\psi}_n^{\underline{\omega}',\underline{t}'}})(\Lambda, S), \tag{6.27}$$

where

$$\tilde{\psi}_n^{\underline{\omega}',\underline{t}'}(j, \mathrm{e}^{\mathrm{i}\eta}) = (\psi_{n,1}^{\underline{\omega}',\underline{t}'} + \psi_{n,2}^{\underline{\omega}',\underline{t}'})(\mathrm{e}^{\mathrm{i}\eta}) + (j-n)(\varphi_{n,1}^{\underline{\omega}',\underline{t}'} + \varphi_{n,2}^{\underline{\omega}',\underline{t}'})(\mathrm{e}^{\mathrm{i}\eta}) \tag{6.28}$$

and $R_n^{\underline{\omega}',\underline{t}'}$ satisfy

$$||R_n^{\underline{\omega}',\underline{t}'}|| \leq (\nu-1)n^{\gamma-1+3\varepsilon}. \tag{6.29}$$

Further on we assume $\hat{n} \geq 4\hat{C}^2$. Then using $\nu \leq n^{\varepsilon}$ and $\gamma - 1 + \varepsilon \leq -3/4$ we can estimate

$$||\varphi_{n,2}^{\underline{\omega}',\underline{t}'}||_{\mathrm{C}^3(\mathcal{C})} \leq \hat{C}n^{2(\gamma-1)+2\varepsilon} \leq \hat{C}n^{-3/4} \leq \frac{1}{2}$$

for $n \geq \hat{n}$ and Lemma 4.2 ensures

$$||Q_n^{\underline{\omega}',\underline{t}'}|| \leq 4\sqrt{\ln(n)}\,. \tag{6.30}$$

We will estimate the difference between $H_n^{\underline{\omega},\underline{t}}$ and $Q_n^{\underline{\omega},\underline{t}}$ in several steps assuming $\nu \leq n^{\varepsilon}$ and $n \geq \hat{n}$.

Step 1. We identify $-\omega$ with $2\pi - \omega$ and check that

$$H_n^{\omega,t} = Q_n^{\underline{\omega}',\underline{t}'}(Q_n^{-\omega,t})^* + R_{n,1}^{\omega,t} \tag{6.31}$$

holds with $R_{n,1}^{\omega,t}$ satisfying

$$||R_{n,1}^{\omega,t}|| \leq ||R_n^{\underline{\omega}',\underline{t}'}|| + \tilde{C}_1 n^{\gamma-1}\ln^{3/2}(n), \tag{6.32}$$

where $\tilde{C}_1$ is a constant independent of $\underline{\omega}$, $\underline{t}, n$. Indeed, we observe that $\mathrm{e}^{2\pi\mathrm{i}\Lambda} = I$ ensures $\mathrm{e}^{-\mathrm{i}\omega\Lambda} = \mathrm{e}^{\mathrm{i}(2\pi-\omega)\Lambda}$ and $H_n^{\omega,t} = (H_n^{-\omega,t})^* = (H_n^{2\pi-\omega,t})^*$, hence we can write

$$H_n^{\omega,t} = H_n^{\underline{\omega}',\underline{t}'}(H_n^{-\omega,t})^* = Q_n^{\underline{\omega}',\underline{t}'}(Q_n^{-\omega,t})^* + R_{n,1}^{\underline{\omega},\underline{t}}$$

with

$$R_{n,1}^{\underline{\omega},\underline{t}} := R_n^{\underline{\omega}',\underline{t}'}(H_n^{-\omega,t})^* + Q_n^{\underline{\omega}',\underline{t}'}(R_n^{-\omega,t})^*$$

and $R_n^{-\omega,t} := R_n^{2\pi-\omega,t}$. Using (6.30) and $||H_n^{-\omega,t}|| = 1$ we can estimate

$$||R_{n,1}^{\underline{\omega},\underline{t}}|| \leq ||R_n^{\underline{\omega}',\underline{t}'}|| + 4\sqrt{\ln(n)}||R_n^{-\omega,t}|| \tag{6.33}$$

and (6.32) follows from $||R_n^{-\omega,t}|| = O(n^{\gamma-1}\ln(n))$.

Step 2. We analyse

$$P^{\underline{\omega},\underline{t}}_{0,n} := \Theta^{2(\nu-1)}_n(\mathrm{e}^{\mathrm{i}\tilde{\psi}^{\underline{\omega}',\underline{t}'}_n})(\Lambda, S)\big((\mathrm{e}^{\mathrm{i}\tilde{\psi}^{-\omega,t}_n})(\Lambda, S)\big)^*, \tag{6.34}$$

where $\tilde{\psi}^{-\omega,t}_n := \tilde{\psi}^{2\pi-\omega,t}_n$. Reasoning as in Section 5.1 we find the expression

$$P^{\underline{\omega},\underline{t}}_{0,n} = \Theta^{2(\nu-1)}_n(\mathrm{e}^{\mathrm{i}\tilde{\psi}^{\underline{\omega},\underline{t}}_{0,n}})(\Lambda, S)q^{\omega,t}_{0,n}(S) \tag{6.35}$$

where

$$\tilde{\psi}^{\underline{\omega},\underline{t}}_{0,n} := (\tilde{\psi}^{\underline{\omega}',\underline{t}'}_n - \tilde{\psi}^{-\omega,t}_n) \circ \tilde{\vartheta}^{\,\omega,t}_n \tag{6.36}$$

and $q^{\omega,t}_{0,n} = 1 + r^{\omega,t}_{0,n}$ holds with

$$|r^{\omega,t}_{0,n}(\mathrm{e}^{\mathrm{i}\xi})| \leq \tilde{C}_0 n^{\gamma-1}. \tag{6.37}$$

Next we describe

$$\tilde{\psi}^{\underline{\omega},\underline{t}}_{0,n}(j, \mathrm{e}^{\mathrm{i}\eta}) = \psi^{\underline{\omega},\underline{t}}_{0,n}(\mathrm{e}^{\mathrm{i}\eta}) + (j-n)\varphi^{\underline{\omega},\underline{t}}_{0,n}(\mathrm{e}^{\mathrm{i}\eta}). \tag{6.38}$$

We first combine the decomposition formula (6.4) with (6.36) to express

$$\psi^{\underline{\omega},\underline{t}}_{0,n} = (\psi^{\underline{\omega}',\underline{t}'}_{n,1} - \psi^{-\omega,t}_{n,1} + \psi^{\underline{\omega}',\underline{t}'}_{n,2} - \psi^{-\omega,t}_{n,2}) \circ \vartheta^{\,\omega,t}_n. \tag{6.39}$$

Then we consider the decomposition

$$\psi^{\underline{\omega},\underline{t}}_{0,n} = \psi^{\underline{\omega},\underline{t}}_{0,n,1} + \psi^{\underline{\omega},\underline{t}}_{0,n,2} + \psi^{\underline{\omega},\underline{t}}_{0,n,3} \tag{6.40}$$

defined as follows

$$\psi^{\underline{\omega},\underline{t}}_{0,n,1} := \psi^{\underline{\omega}',\underline{t}'}_{n,1} - \psi^{-\omega,t}_{n,1}, \tag{6.41}$$

$$\psi^{\underline{\omega},\underline{t}}_{0,n,2} := \big(\psi^{\underline{\omega},\underline{t}}_{0,n,1} \circ \vartheta^{\,\omega,t}_n - \psi^{\underline{\omega},\underline{t}}_{0,n,1}\big) + (\psi^{\underline{\omega}',\underline{t}'}_{n,2} - \psi^{-\omega,t}_{n,2}), \tag{6.42}$$

$$\psi^{\underline{\omega},\underline{t}}_{0,n,3} := (\psi^{\underline{\omega}',\underline{t}'}_{n,2} - \psi^{-\omega,t}_{n,2}) \circ \vartheta^{\,\omega,t}_n - (\psi^{\underline{\omega}',\underline{t}'}_{n,2} - \psi^{-\omega,t}_{n,2}). \tag{6.43}$$

Due to (6.10) and (5.51) we can estimate

$$||\psi^{\underline{\omega},\underline{t}}_{0,n,1}||_{\mathrm{C}^m(\mathcal{C})} \leq ||\psi^{\underline{\omega}',\underline{t}'}_{n,1}||_{\mathrm{C}^m(\mathcal{C})} + ||\psi^{-\omega,t}_{n,1}||_{\mathrm{C}^m(\mathcal{C})} \leq C_m \nu n^{\gamma}. \tag{6.44}$$

and assuming $\hat{C} \geq 2C_3$ we ensure

$$||\psi^{-\omega,t}_{n,2}||_{\mathrm{C}^3(\mathcal{C})} \leq C_3 \leq \tfrac{1}{2}\hat{C}. \tag{6.45}$$

Combining the last estimate with induction assumption (6.24) we can estimate

$$||\psi^{\underline{\omega}',\underline{t}'}_{n,2} - \psi^{-\omega,t}_{n,2}||_{\mathrm{C}^3(\mathcal{C})} \leq \hat{C}(\nu - \tfrac{1}{2})n^{\varepsilon}. \tag{6.46}$$

However (6.21) and (6.44) ensure

$$\begin{aligned} ||\psi^{\underline{\omega},\underline{t}}_{0,n,1} \circ \vartheta^{\,\omega,t}_n - \psi^{\underline{\omega},\underline{t}}_{0,n,1}||_{\mathrm{C}^3(\mathcal{C})} &\leq C'_4 n^{\gamma-1} ||\psi^{\underline{\omega},\underline{t}}_{0,n,1}||_{\mathrm{C}^4(\mathcal{C})} \\ &\leq C'_4 C_4 \nu n^{2\gamma-1} \leq C'_4 C_4 n^{\varepsilon} \leq \tfrac{1}{2}\hat{C} n^{\varepsilon} \end{aligned}$$

if we choose $\hat{C} \geq 2C'_4 C_4$. Thus $\hat{C} \geq 2C'_4 C_4$ ensures

$$||\psi^{\underline{\omega},\underline{t}}_{0,n,2}||_{\mathrm{C}^3(\mathcal{C})} \leq \hat{C}\nu n^{\varepsilon}. \tag{6.47}$$

The last estimate and (6.21) give

$$||\psi^{\underline{\omega},\underline{t}}_{0,n,3}||_{\mathrm{C}^2(\mathcal{C})} \leq C'_3 \hat{C} \nu n^{\varepsilon+\gamma-1}. \tag{6.48}$$

Reasoning exactly as above we observe that

$$\varphi^{\underline{\omega},\underline{t}}_{0,n} = (\varphi^{\underline{\omega},\underline{t}'}_{n,1} - \varphi^{-\omega,t}_n + \varphi^{\underline{\omega}',\underline{t}'}_{n,2}) \circ \vartheta^{\,\omega,t}_n \tag{6.49}$$

and consider the decomposition

$$\varphi_{0,n}^{\underline{\omega},\underline{t}} = \varphi_{0,n,1}^{\underline{\omega},\underline{t}} + \varphi_{0,n,2}^{\underline{\omega},\underline{t}} + \varphi_{0,n,3}^{\underline{\omega},\underline{t}} \tag{6.50}$$

defined by

$$\varphi_{0,n,1}^{\underline{\omega},\underline{t}} := \varphi_{n,1}^{\underline{\omega}',\underline{t}'} - \varphi_n^{-\omega,t}, \tag{6.51}$$

$$\varphi_{0,n,2}^{\underline{\omega},\underline{t}} := \left(\varphi_{0,n,1}^{\underline{\omega},\underline{t}} \circ \vartheta_n^{\omega,t} - \varphi_{0,n,1}^{\underline{\omega},\underline{t}}\right) + \varphi_{n,2}^{\underline{\omega}',\underline{t}'}, \tag{6.52}$$

$$\varphi_{n,3}^{\underline{\omega},\underline{t}} := \varphi_{n,2}^{\underline{\omega}',\underline{t}'} \circ \vartheta_n^{\omega,t} - \varphi_{n,2}^{\underline{\omega}',\underline{t}'} \tag{6.53}$$

and obtain the estimates

$$||\varphi_{0,n,1}^{\underline{\omega},\underline{t}}||_{\mathrm{C}^m(\mathcal{C})} \leq C_m \nu n^{\gamma-1}. \tag{6.54}$$

$$||\varphi_{0,n,2}^{\underline{\omega},\underline{t}}||_{\mathrm{C}^3(\mathcal{C})} \leq \hat{C}\nu n^{\varepsilon+2(\gamma-1)} \tag{6.55}$$

$$||\varphi_{0,n,3}^{\underline{\omega},\underline{t}}||_{\mathrm{C}^2(\mathcal{C})} \leq C_3'\hat{C}\nu n^{\varepsilon+3(\gamma-1)}. \tag{6.56}$$

Step 3. We observe that

$$Q_n^{\underline{\omega}',\underline{t}'}(Q_n^{-\omega,t})^* = \mathrm{e}^{\mathrm{i}|\omega|_1\Lambda} P_n^{\underline{\omega},\underline{t}}\, \Theta_n^2 \tag{6.57}$$

holds with $P_n^{\underline{\omega},\underline{t}} := \mathrm{e}^{-\mathrm{i}\omega\Lambda} P_{0,n}^{\underline{\omega},\underline{t}}\, \mathrm{e}^{\mathrm{i}\omega\Lambda}$, where $P_{0,n}^{\underline{\omega},\underline{t}}$ is as in Step 2, i.e.

$$P_n^{\underline{\omega},\underline{t}} = \Theta_n^{2(\nu-1)} \mathrm{e}^{-\mathrm{i}\omega\Lambda}(\mathrm{e}^{\mathrm{i}\tilde{\psi}_{0,n}^{\underline{\omega},\underline{t}}})(\Lambda, S) q_{0,n}^{\omega,t}(S)\, \mathrm{e}^{\mathrm{i}\omega\Lambda}.$$

However reasoning as in Step 2 of the proof of Lemma 5.1 we find

$$P_n^{\underline{\omega},\underline{t}} = \Theta_n^{2(\nu-1)}(\mathrm{e}^{\mathrm{i}\tilde{\psi}_{0,n}^{\underline{\omega},\underline{t}} \circ \tilde{\tau}_\omega})(\Lambda, S) q_n^{\omega,t}(S) \tag{6.58}$$

with $q_n^{\omega,t} = q_{0,n}^{\underline{\omega},\underline{t}} \circ \tau_\omega$ and below we consider expressions for $\tilde{\psi}_{0,n}^{\underline{\omega},\underline{t}} \circ \tilde{\tau}_\omega$.

We use $\psi_{0,n,i}$, $\varphi_{0,n,i}$ ($i = 1$, 2, 3) defined in Step 2 and denote

$$\psi_{n,i}^{\underline{\omega},\underline{t}} := \psi_{0,n,i}^{\underline{\omega},\underline{t}} \circ \tau_\omega, \qquad \varphi_{n,i}^{\underline{\omega},\underline{t}} := \varphi_{0,n,i}^{\underline{\omega},\underline{t}} \circ \tau_\omega. \tag{6.59}$$

It is clear that the expressions for $\psi_{n,1}^{\underline{\omega},\underline{t}}$, $\varphi_{n,1}^{\underline{\omega},\underline{t}}$ used above are the same as in the definition given in Section 6.1 and

$$||\psi_{n,i}^{\underline{\omega},\underline{t}}||_{\mathrm{C}^m(\mathcal{C})} = ||\psi_{0,n,i}^{\underline{\omega},\underline{t}}||_{\mathrm{C}^m(\mathcal{C})}, \quad ||\varphi_{n,i}^{\underline{\omega},\underline{t}}||_{\mathrm{C}^m(\mathcal{C})} = ||\varphi_{0,n,i}^{\underline{\omega},\underline{t}}||_{\mathrm{C}^m(\mathcal{C})}. \tag{6.60}$$

Next we assume that $\tilde{\psi}_n^{\underline{\omega},\underline{t}}$ is given by (6.4) with $\psi_{n,i}^{\underline{\omega},\underline{t}}$, $\varphi_{n,i}^{\underline{\omega},\underline{t}}$ as in (6.59) and we are going to estimate the difference between $P_n^{\underline{\omega},\underline{t}}$ and

$$\tilde{P}_n^{\underline{\omega},\underline{t}} := \Theta_n^{2(\nu-1)}(\mathrm{e}^{\mathrm{i}\tilde{\psi}_n^{\underline{\omega},\underline{t}}})(\Lambda, S). \tag{6.61}$$

For this purpose we write

$$\tilde{\psi}_{0,n}^{\underline{\omega},\underline{t}} \circ \tilde{\tau}_\omega = \tilde{\psi}_{n,1}^{\underline{\omega},\underline{t}} + \tilde{\psi}_{n,2}^{\underline{\omega},\underline{t}} + \tilde{\psi}_{n,3}^{\underline{\omega},\underline{t}}$$

and we observe that introducing $r_{n,3}^{\underline{\omega},\underline{t}} := \mathrm{e}^{\mathrm{i}\tilde{\psi}_{n,3}^{\underline{\omega},\underline{t}}} - 1$, $r_n^{\omega,t} := 1 - q_n^{\omega,t}$ we can express

$$P_n^{\underline{\omega},\underline{t}} = (\tilde{P}_n^{\underline{\omega},\underline{t}} + \tilde{R}_n^{\underline{\omega},\underline{t}})(I + r_n^{\omega,t}(S)) \tag{6.62}$$

with

$$\tilde{R}_n^{\underline{\omega},\underline{t}} := \Theta_n^{2(\nu-1)}(\mathrm{e}^{\mathrm{i}\tilde{\psi}_n^{\underline{\omega},\underline{t}}} r_{n,3}^{\underline{\omega},\underline{t}})(\Lambda, S). \tag{6.63}$$

However applying the estimates obtained earlier to

$$\tilde{\psi}_{n,3}^{\underline{\omega},\underline{t}}(j, \mathrm{e}^{\mathrm{i}\eta}) = \psi_{n,3}^{\underline{\omega},\underline{t}}(\mathrm{e}^{\mathrm{i}\eta}) + (j-n)\varphi_{n,3}^{\underline{\omega},\underline{t}}(\mathrm{e}^{\mathrm{i}\eta})$$

we find that

$$||r_{n,3}^{\underline{\omega},\underline{t}}(j,\cdot)||_{C^1(\mathcal{C})} \leq ||\tilde{\psi}_{n,3}^{\underline{\omega},\underline{t}}(j,\cdot)||_{\mathrm{C}^1(\mathcal{C})} \leq C_3'\hat{C}\nu n^{\varepsilon+\gamma-1}$$

hold for $j \in \mathbb{Z}$ satisfying $|j-n| \le n/3$, hence

$$||\tilde{R}_n^{\omega,\underline{t}}|| \le 4\sqrt{\ln(n)}\, C_3'\hat{C}\nu n^{\varepsilon+\gamma-1}. \tag{6.64}$$

Using moreover $||\tilde{P}_n^{\omega,\underline{t}}|| \le 4\ln(n)$ and $|r_n^{\omega,t}(\mathrm{e}^{\mathrm{i}\xi})| \le \tilde{C}_0 n^{\gamma-1}$, we obtain

$$P_n^{\omega,\underline{t}} = \tilde{P}_n^{\omega,\underline{t}} + R_{n,3}^{\omega,\underline{t}} \tag{6.65}$$

with

$$||R_{n,3}^{\omega,\underline{t}}|| \le 4\ln(n)\,\tilde{C}_0 n^{\gamma-1} + 4\sqrt{\ln(n)}\, C_3'\hat{C} n^{\gamma-1+2\varepsilon}(1+\tilde{C}_0). \tag{6.66}$$

Step 4. Combining the results of Step 1, 2 and 3 we obtain

$$H_n^{\omega,\underline{t}} = \mathrm{e}^{\mathrm{i}|\underline{\omega}'|_1\Lambda}\tilde{P}_n^{\omega,\underline{t}}\Theta_n^2 + R_{n,4}^{\omega,\underline{t}} \tag{6.67}$$

with

$$||R_{n,4}^{\omega,\underline{t}}|| \le ||R_{n,1}^{\omega,\underline{t}}|| + ||R_{n,3}^{\omega,\underline{t}}|| \le ||R_{n,1}^{\omega',\underline{t}'}|| + \tilde{C}_2 n^{2\varepsilon+\gamma-1}\ln^2(n) \tag{6.68}$$

and the constant $\tilde{C}_2$ depends on $\tilde{C}_0$, $\tilde{C}_1$, $\hat{C}$, $C_4$ and $C_4'$ only (recall that $C_m \le C_{m+1}$ and $C_m' \le C_{m+1}'$ for all $m \in \mathbb{N}$). It remains to analyse

$$Q_n^{\omega,\underline{t}} - \mathrm{e}^{\mathrm{i}|\underline{\omega}'|_1\Lambda}\tilde{P}_n^{\omega,\underline{t}} = \mathrm{e}^{\mathrm{i}|\underline{\omega}'|_1\Lambda}[\tilde{P}_n^{\omega,\underline{t}}, \Theta_n^2].$$

However reasoning as in Step 4 of the proof of Lemma 5.1 we obtain

$$||[\tilde{P}_n^{\omega,\underline{t}}, \Theta_n^2]|| \le \tilde{C}_3 n^{\gamma-1}\sqrt{\ln(n)} \tag{6.69}$$

with a constant $\tilde{C}_3$ depending only on $C_2$. Finally we can choose $\hat{n}$ depending on $\tilde{C}_2$, $\tilde{C}_3$ so that

$$n \ge \tilde{n} \implies \tilde{C}_2 n^{2\varepsilon+\gamma-1}\ln^2(n) + \tilde{C}_3 n^{\gamma-1}\sqrt{\ln(n)} \le n^{3\varepsilon+\gamma-1},$$

hence $H_n^{\omega,\underline{t}} = Q_n^{\omega,\underline{t}} + R_n^{\omega,\underline{t}}$ with

$$||R_n^{\omega,\underline{t}}|| \le ||R_n^{\omega',\underline{t}'}|| + n^{3\varepsilon+\gamma-1} \tag{6.70}$$

for $n \ge \hat{n}$. We complete the proof of (6.15) using (6.70) and the induction assumption (6.29).

6.3. **Special properties of $\psi_{1,n}^{\omega,\underline{t}}$.** We have

**Lemma 6.2.** *If $\psi_{1,n}^{\omega,\underline{t}}$ are defined as before, then*

$$\psi_{1,n}^{\omega,\underline{t}}(\mathrm{e}^{\mathrm{i}\eta}) = \mu_n^{\omega,\underline{t}}\cos(\eta - \eta_{1,n}^{\omega,\underline{t}}) \tag{6.71}$$

*holds with some real valued Lebesgue measurable functions $\underline{t} \to \eta_{1,n}^{\omega,\underline{t}}$ and $\underline{t} \to \mu_n^{\omega,\underline{t}}$. Moreover we can find a constant $c_0 > 0$ and real valued Lebesgue measurable functions $\underline{t}' \to t_{0,n}^{\omega,\underline{t}'}$ such that for $\underline{t} = (\underline{t}', t) \in \mathbb{R}^{\nu-1} \times \mathbb{R}$ one has*

$$|\mu_n^{\omega,\underline{t}}| \ge c_0 n^{\gamma}\,|t - t_{0,n}^{\omega,\underline{t}'}|_{\mathrm{mod}\,\pi}, \tag{6.72}$$

*where $|s|_{\mathrm{mod}\,\pi} = \mathrm{dist}\,(s + \pi\mathbb{Z}, \pi\mathbb{Z})$.*

*Proof.* We observe that $\psi_{1,n}^{\omega,\underline{t}}(\mathrm{e}^{\mathrm{i}\xi}) = \mathrm{Im}\big(\Psi_n^{\omega}(\underline{t}, \mathrm{e}^{\mathrm{i}\xi})\big)$ holds with $\Psi_n^{\omega}(\underline{t}, \mathrm{e}^{\mathrm{i}\xi})$ defined in Section 4.3 of [5]. Moreover in Section 4.3 of [5] we checked that

$$\Psi_n^{\omega}(\underline{t}, \mathrm{e}^{\mathrm{i}\xi}) = \Psi_n^{\omega}(\underline{t}, 1)\mathrm{e}^{\mathrm{i}\xi}, \tag{6.73}$$

hence (6.71) holds and the estimate follows from Lemma 4.1 in [5]. □

6.4. **End of the proof.** To begin we express

$$g_{\nu,n,j}(\underline{t}) = \mathrm{i}^{\nu} \sum_{\omega\in\Omega^*} c_{\omega_1}\dots c_{\omega_\nu}\langle \mathrm{e}_j,\, H_n^{\underline{\omega},\underline{t}}\,\mathrm{e}_j\rangle \tag{6.74}$$

and introduce

$$g_{n,j}^{\underline{\omega},\underline{t}} := \langle \mathrm{e}_j,\, Q_n^{\underline{\omega},\underline{t}}\,\mathrm{e}_j\rangle = \mathrm{e}^{\mathrm{i}j|\underline{\omega}|_1}\theta_{n,n}(j)^{2\nu}\int_0^{2\pi}\frac{\mathrm{d}\eta}{2\pi}\,\mathrm{e}^{\mathrm{i}\tilde{\psi}_n^{\underline{\omega},\underline{t}}(j,\mathrm{e}^{\mathrm{i}\eta})}$$

Then due to Lemma 6.1 it suffices to show that the estimates

$$\sup_{-t_0\le t_1,\dots,t_{\nu-1}\le t_0}\ \sup_{|j-n|\le n^\gamma}\left|\int_{-t_0}^{t_0}\mathrm{d}t_\nu\, g_{n,j}^{\underline{\omega},\underline{t}}\right| \le Cn^{-\gamma/2} \tag{6.75}$$

hold for $\nu \le n^\varepsilon$. Using (6.9) and (6.13) we find that

$$\tilde{\varphi}_n^{\underline{\omega},\underline{t}}(j,\mathrm{e}^{\mathrm{i}\xi}) := (j-n)\big(\varphi_{n,1}^{\underline{\omega},\underline{t}} + \varphi_{n,2}^{\underline{\omega},\underline{t}}\big)(\mathrm{e}^{\mathrm{i}\xi})$$

satisfies

$$\sup_{|j-n|\le n^\gamma}||\tilde{\varphi}_n^{\underline{\omega},\underline{t}}(j,\cdot)||_{\mathrm{C}^2(\mathcal{C})} \le Cn^{3\varepsilon}. \tag{6.76}$$

Using the expression (6.71) and the change of variable $\xi = \eta - \eta_{1,n}^{\underline{\omega},\underline{t}}$ we can express

$$\int_0^{2\pi}\mathrm{d}\eta\,\mathrm{e}^{\mathrm{i}\tilde{\psi}_n^{\underline{\omega},\underline{t}}(j,\mathrm{e}^{\mathrm{i}\eta})} = \mathcal{J}(b_n^{\underline{\omega},\underline{t}}(j,\cdot),\mu_n^{\underline{\omega},\underline{t}}) \tag{6.77}$$

with $b_n^{\underline{\omega},\underline{t}} := \big(\mathrm{e}^{\mathrm{i}\psi_{n,2}^{\underline{\omega},\underline{t}}+\mathrm{i}\tilde{\varphi}_n^{\underline{\omega},\underline{t}}}\big)\circ\tilde{\tau}_{\eta_{1,n}^{\underline{\omega},\underline{t}}}$. Using (6.76) and (6.12) we obtain

$$\sup_{|j-n|\le n^\gamma}||b_n^{\omega,t}(j,\cdot)||_{\mathrm{C}^2(\mathcal{C})} \le C'n^{6\varepsilon} \tag{6.78}$$

Further on we abbreviate $\mathcal{J}_{n,j}^{\underline{\omega},\underline{t}} := \mathcal{J}(b_n^{\underline{\omega},\underline{t}}(j,\cdot),\mu_n^{\underline{\omega},\underline{t}})$. Since $|\mathcal{J}_{n,j}^{\underline{\omega},\underline{t}}| \le 2\pi$ it is clear that

$$\left|\int_{k\pi+t_{0,n}^{\underline{\omega},\underline{t}'}-n^{-\gamma/2}}^{k\pi+t_{0,n}^{\underline{\omega},\underline{t}'}+n^{-\gamma/2}}\mathrm{d}t\,\mathcal{J}_{n,j}^{\underline{\omega},\underline{t}}\right| \le 4\pi n^{-\gamma/2}$$

and it remains to integrate $\mathcal{J}_{n,j}^{\underline{\omega},\underline{t}}$ over

$$\Delta_n := [-t_0,t_0]\setminus\bigcup_{k\in\mathbb{Z}}[k\pi + t_{0,n}^{\underline{\omega},\underline{t}'} - n^{-\gamma/2},\, k\pi + t_{0,n}^{\underline{\omega},\underline{t}'} + n^{-\gamma/2}].$$

However combining Lemma 5.2 and 6.2 we find the estimate

$$\sup_{|j-n|\le n^\gamma}|\mathcal{J}_{n,j}^{\underline{\omega},\underline{t}}| \le \frac{C}{n^{\gamma/2}|t-t_{0,n}^{\underline{\omega},\underline{t}'}|_{\mathrm{mod}\,\pi}^{1/2}}\left(1+\frac{C'n^{6\varepsilon}}{n^{\gamma/2}|t-t_{0,n}^{\underline{\omega},\underline{t}'}|_{\mathrm{mod}\,\pi}^{1/2}}\right)$$

and due to $6\varepsilon \le \gamma/4$ we can estimate

$$t\in\Delta_n \Rightarrow \frac{n^{6\varepsilon}}{n^{\gamma/2}|t-t_{0,n}^{\underline{\omega},\underline{t}'}|_{\mathrm{mod}\,\pi}^{1/2}} \le n^{6\varepsilon-\gamma/4}\le 1.$$

Since $t\to|t|^{-1/2}$ is locally integrable on $\mathbb{R}$ we complete the proof writing

$$\sup_{|j-n|\le n^\gamma}\int_{\Delta_n}\mathrm{d}t\,|\mathcal{J}_{n,j}^{\underline{\omega},\underline{t}}| \le \frac{C(1+C')}{n^{\gamma/2}}\int_{-t_0}^{t_0}\frac{\mathrm{d}t}{|t-t_{0,n}^{\underline{\omega},\underline{t}'}|_{\mathrm{mod}\,\pi}^{1/2}} \le \frac{C''}{n^{\gamma/2}}\,.$$

## 7. Appendix

7.1. **Proof of Proposition 2.2.** We define

$$(\tilde{J}_n^+ x)(k) = d_n(k)x(k) + \tilde{a}_n(k)x(k+1) + \tilde{a}_n(k-1)x(k-1). \tag{7.1}$$

with

$$\tilde{a}_n(k) = \begin{cases} a(k) \text{ if } n - C_0(n+1)^\gamma \leq k \leq n + C_0(n+1)^\gamma \\ a_n(k) \text{ otherwise} \end{cases} \tag{7.2}$$

where $C_0$ is fixed large enough. We claim that

$$||\tilde{J}_n^+ - J_n^+||_{\mathcal{B}(l^2)} = O(n^{3\gamma-2}). \tag{7.3}$$

Indeed, $|j| \leq n/3$ ensures $a_n(n+j) = a(n) + \delta a(n)j$ and $|\delta^2 a(n+j)| \leq \tilde{C}n^{\gamma-2}$, hence we can estimate

$$\sup_{k\in\mathbb{N}^*} |\tilde{a}_n(k) - a_n(k)| = \sup_{|j|\leq C_0(n+1)^\gamma} |a(n+j) - a(n) - \delta a(n)j|$$

$$\leq \sup_{|j|\leq C_0(n+1)^\gamma} j^2\, \tilde{C}n^{\gamma-2} = O(n^{3\gamma-2})\,.$$

However (7.3) and the min-max principle give

$$\sup_{k\in\mathbb{N}^*} |\lambda_k(\tilde{J}_n^+) - \lambda_k(J_n^+)| = O(n^{3\gamma-2})\,.$$

To complete the proof it suffices to know that choosing $C_0$ large enough we ensure the estimates

$$\lambda_n(J) = \lambda_n(\tilde{J}_n) + O(n^{-\nu}) \tag{7.4}$$

for any $\nu > 0$. However reasoning exactly as in Section 7 of [5] we deduce (7.4) from

$$\lambda_n(\tilde{J}_n) \in ]\lambda_n(J) - 2\lambda_n(J)^{-\nu}, \lambda_n(J) + 2\lambda_n(J)^{-\nu}] \tag{7.5}$$

which holds for $n \geq n_\nu$ if $n_\nu$ is chosen large enough.

7.2. **Proof of Proposition 3.3.**

**Lemma 7.1.** *Let $\varepsilon > 0$ be fixed. If $\chi \in \mathcal{S}(\mathbb{R})$ then*

$$\operatorname{tr} \chi(L_{0,n} - l(n)) = \sum_{|k|\leq n^\varepsilon} \chi(k) + O(n^{2\gamma-3+2\varepsilon}) \tag{7.6}$$

*and for any $\mu > 0$ the estimate*

$$\operatorname{tr} \chi(L_n - l(n)) = \sum_{|k|\leq n^\varepsilon} \chi(k + r_n(k)) + O(n^{-\mu}) \tag{7.7}$$

*holds with*

$$r_n(k) := \lambda_{n+k}(L_n) - k - l(n) \tag{7.8}$$

*Proof.* Our assumptions ensure $l(n) = n + O(\ln(n))$ and rough estimates of Section 2.3 ensure

$$|k| \geq n^\varepsilon \implies |\lambda_{n+k}(L_n) - l(n)| \geq \tfrac{1}{2}\, n^\varepsilon$$

for $n \geq n_\varepsilon$, hence

$$\operatorname{tr} \chi(L_n - l(n)) = \sum_{|k|\leq n^\varepsilon} \chi(\lambda_{n+k}(L_n) - l(n)) + O(n^{-\mu})$$

follows from $\chi \in \mathcal{S}(\mathbb{R})$ and we find (7.7) writing $\lambda_{n+k}(L_n) - l(n) = k + r_n(k)$.

Next we observe that

$$r_{0,n}(k) := l(n+k) - k - l(n) = a_{1,n}(n+k) - a_{1,n}(n) = O(|k|n^{2\gamma-3}) \tag{7.9}$$

and writing $\lambda_{n+k}(L_{0,n}) - l(n) = l(n+k) - l(n) = k + r_{0,n}(k)$ we find that

$$\operatorname{tr}\chi(L_{0,n} - l(n)) = \sum_{|k|\le n^{\varepsilon}} \chi(k + r_{0,n}(k)) \quad + O(n^{-\mu})$$

holds for every $\mu > 0$ and to complete the proof of (7.6) it suffices to observe that

$$\sup_{|k|\le n^{\varepsilon}} |\chi(k + r_{0,n}(k)) - \chi(k)| \quad = O(n^{2\gamma-3+\varepsilon}).$$

□

*Proof of Proposition 3.3.* We first observe that $r_n(k) = \lambda_{n+k}(L_n) - l(n+k) + r_{0,n}(k)$ and using (7.9) with rough estimates of Section 2.3 we can estimate

$$\sup_{|k|\le n^{\varepsilon}} |r_n(k)| \le \rho_N + Cn^{2\gamma-3+\varepsilon}. \tag{7.10}$$

We fix $\varepsilon \le 1/8$ and $\rho'_N > \rho_N$. Due to (7.10) we can fix $n_0$ such that $|r_n(k)| \le \rho'_n$ holds for $n \ge n_0$ similarly as in Section 6.1 of [5]. All further steps of the proof follow exactly as in Section 6.1 of [5].

7.3. **Proof of Lemma 4.2.** We denote

$$\tilde{\varphi}_n(j, \mathrm{e}^{\mathrm{i}\xi}) := (j-n)\varphi_n(\mathrm{e}^{\mathrm{i}\xi}) \tag{7.11}$$

and observe that

$$(\mathrm{e}^{\mathrm{i}\tilde{\psi}_n} q_n)(\Lambda, S) = (\mathrm{e}^{\mathrm{i}\tilde{\varphi}_n} q_n)(\Lambda, S)\mathrm{e}^{\mathrm{i}\psi_n(S)}. \tag{7.12}$$

Since $||\mathrm{e}^{\mathrm{i}\psi_n(S)}|| = 1$ it remains to show

$$||\tilde{\Theta}_n(\mathrm{e}^{\mathrm{i}\tilde{\varphi}_n} q_n)(\Lambda, S)|| \le 4\sqrt{\ln(n)} \sup_{|j-n|\le n/3} ||q_n(j,\cdot)||_{\mathrm{C}^1(\mathcal{C})}, \tag{7.13}$$

where $\tilde{\Theta}_n := \theta_{3n/2,n}(\Lambda)$. Using the Schur test of boundedness in $\mathcal{H} = l^2(\mathbb{Z})$ we write

$$||\tilde{\Theta}_n(\mathrm{e}^{\mathrm{i}\tilde{\varphi}_n} q_n)(\Lambda, S)\big((\mathrm{e}^{\mathrm{i}\tilde{\varphi}_n} q_n)(\Lambda, S)\big)^*\tilde{\Theta}_n|| \le \sup_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |K_n(j,k)| \tag{7.14}$$

with

$$K_n(j,k) := \langle \mathrm{e}_j, \tilde{\Theta}_n(\mathrm{e}^{\mathrm{i}\tilde{\varphi}_n} q_n)(\Lambda, S)\big((\mathrm{e}^{\mathrm{i}\tilde{\varphi}_n} q_n)(\Lambda, S)\big)^*\tilde{\Theta}_n \mathrm{e}_k\rangle.$$

Since $\tilde{\varphi}_n(j, \mathrm{e}^{\mathrm{i}\xi}) - \tilde{\varphi}_n(k, \mathrm{e}^{\mathrm{i}\xi}) = (j-k)\varphi_n(\mathrm{e}^{\mathrm{i}\xi})$ we can express

$$K_n(j,k) = \int_0^{2\pi} \frac{\mathrm{d}\xi}{2\pi}\, \mathrm{e}^{\mathrm{i}(k-j)(\xi+\varphi_n(\mathrm{e}^{\mathrm{i}\xi}))}\theta_{3n/2,n}(j)q_n(j,\mathrm{e}^{\mathrm{i}\xi})\,\overline{q}_n(k,\mathrm{e}^{\mathrm{i}\xi})\,\theta_{3n/2,n}(k).$$

We introduce

$$b_n(j,\mathrm{e}^{\mathrm{i}\xi},k) := \frac{\theta_{3n/2,n}(j)q_n(j,\mathrm{e}^{\mathrm{i}\xi})\,\overline{q}_n(k,\mathrm{e}^{\mathrm{i}\xi})\,\theta_{3n/2,n}(k)}{1 + \frac{\mathrm{d}}{\mathrm{d}\xi}(\varphi_n(\mathrm{e}^{\mathrm{i}\xi}))}$$

and we observe that

$$\mathrm{i}(j-k)K_n(j,k) = \int_0^{2\pi} \frac{\mathrm{d}\xi}{2\pi}\frac{\mathrm{d}}{\mathrm{d}\xi}\big(\mathrm{e}^{\mathrm{i}(k-j)(\xi+\varphi_n(\mathrm{e}^{\mathrm{i}\xi}))}\big)b_n(j,\mathrm{e}^{\mathrm{i}\xi},k).$$

Thus the integration by parts gives

$$\mathrm{i}(j-k)K_n(j,k) = -\int_0^{2\pi} \frac{\mathrm{d}\xi}{2\pi}\mathrm{e}^{\mathrm{i}(k-j)(\xi+\varphi_n(\mathrm{e}^{\mathrm{i}\xi}))}\frac{\mathrm{d}}{\mathrm{d}\xi}\big(b_n(j,\mathrm{e}^{\mathrm{i}\xi},k)\big)$$

and for $j \neq k$ we can estimate

$$|K_n(j,k)| \leq \frac{||b_n(j,\cdot,k)||_{\mathrm{C}^1(\mathcal{C})}}{|j-k|}.$$

A direct computation using $||\varphi_n||_{\mathrm{C}^1(\mathcal{C})} \leq \frac{1}{2}$ allows us to estimate

$$\sup_{j,k\in\mathbb{Z}} ||b_n(j,\cdot,k)||_{\mathrm{C}^1(\mathcal{C})} \leq \sup_{|j-n|\leq n/3} 6||q_n(j,\cdot)||^2_{\mathrm{C}^1(\mathcal{C})}.$$

and

$$\sum_{k\in\mathbb{Z}} |K_n(j,k)| \leq ||q_n(j,\cdot)||^2_{\mathrm{C}^1(\mathcal{C})} \Big(1 + 12 \sum_{1\leq m\leq n/3} \frac{1}{m}\Big) \tag{7.15}$$

and it is clear that the right-hand side of (7.15) can be estimated by $16\ln(n)$ which completes the proof due to (7.14).

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