ASYMPTOTICS OF LARGE EIGENVALUES FOR A CLASS OF BAND MATRICES

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Abstract. We investigate the asymptotic behaviour of large eigenvalues for a class of finite difference self-adjoint operators with compact resolvent in l^2.

1. Introduction

1.1. **General remarks.** Infinite tridiagonal tridiagonal matrices called Jacobi matrices have been investigated in many recent papers in relation with various questions of pure and applied mathematics (see [1,5,10–12]). In [1,2,5,6,13] the authors investigate Jacobi matrices acting in l^2 as unbounded self-adjoint operators with discrete spectrum and asymptotic formulas for large eigenvalues are given. This type of analysis is of particular interest in Quantum Physics when information about physical parameters can be deduced from the spectral asymptotics of a concrete model. Except for recent work [7] there are no corresponding work concerning the asymptotic analysis of large eigenvalues of higher order symmetric difference operators. This fact is not so surprising cause higher order difference operators had not been studied from the spectral point of view, up to the last few years. On the other hand there are works already, dealing with spectral properties of difference operators of higher order, for example see [3,4,9]. It is natural to ask if known results on tridiagonal matrices can be generalized to higher order difference operators and to look for applications. As an example of possible application let us mention here the problem of the behaviour of large singular values for a non-symmetric Jacobi matrix J with discrete spectrum. Indeed, these singular values are eigenvalues of J^*J which is a symmetric difference operator of order four. The aim of this paper is to obtain a simple remainder estimate in the asymptotics of eigenvalues for a large class of symmetric higher order difference operators. Applying this result to tridiagonal matrices we find that

(i) in this paper we obtain asymptotic estimates of eigenvalue for Jacobi matrices which cannot be treated in [1,2,5,6] (cf. Section 1.3),
(ii) the remainder estimates of [1,2,5,6] are more precise than ours.

Since our assumptions are weaker than in [1,2,5–7], we must overcome some additional difficulties, but the main idea of our approach remains the same as in [2]. Although the approach of this paper is used to obtain the simplest remainder estimate, it is possible to follow the idea of [2] in order to compute further terms of the asymptotics with smaller remainders under stronger "conditions of smoothness" imposed on the entries. It is an open question how to extend the methods of the present work to the case of

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non-smooth entries. Finally notice that we (and most the above authors) have not studied asymptotic properties at infinity of eigenvectors.

1.2. **Formulation.** We denote by l^2 the Hilbert space of square sommable complex valued sequences $x\colon \mathbb{N}^* \to \mathbb{C}$ with the norm

$$\|x\|_{l^2} = \left(\sum_{j=1}^{\infty} |x(j)|^2 \right)^{1/2}.$$

We fix $d\colon \mathbb{N}^* \to \mathbb{R}$ and introduce

$$\mathcal{D} := \{x \in l^2 \colon \sum_{j=1}^{\infty} |d(j)x(j)|^2 < \infty\} \tag{1.1}$$

Then we consider the self-adjoint operator $D\colon \mathcal{D} \to l^2$ given by the formula

$$(Dx)(j) = d(j)x(j) \text{ for } x \in \mathcal{D} \tag{1.2}$$

and a finite difference operator $A'\colon \mathcal{D} \to l^2$ of the form

$$(A'x)(j) = \sum_{1 \le l \le m} \left(a_l(j)x(j+l) + a_l(j-l)x(j-l) \right), \tag{1.3}$$

where the coefficients $a_l\colon \mathbb{N}^* \to \mathbb{R}$, $l = 1, \dots, m$ satisfy

$$\frac{|a_l(j-l)| + |a_l(j)|}{d(j)} \xrightarrow[j \to \infty]{} 0 \tag{1.4}$$

and we assume $a_l(j-l) = 0 = x(j-l)$ when $j \le l$ in (1.3). We investigate the operator

$$A = D + A' \tag{1.5}$$

under the additional assumption

$$d(n) \xrightarrow[n \to \infty]{} \infty, \tag{1.6}$$

which ensures that A has compact resolvent, i.e., there exists an orthonormal basis $(v_n)_{n \in \mathbb{N}^*}$ such that $Av_n = \lambda_n(A)v_n$ holds for $n \in \mathbb{N}^*$, $\lambda_n(A) \to \infty$ as $n \to \infty$ and $(\lambda_n(A))_{n \in \mathbb{N}^*}$ is arranged increasingly, i.e., $\lambda_n(A) \le \lambda_{n+1}(A)$ for any $n \in \mathbb{N}^*$.

Assumption (H1). The off-diagonal entries $a_l(n)$, $1 \le l \le m$ satisfy the asymptotics

$$a_l(n) = c_l n^{\delta_l} + \mathrm{O}(n^{\delta - 1}) \text{ as } n \to \infty, \tag{1.7}$$

where δ_l, c_l, $l = 1, \dots, m$ are some fixed real numbers and $\delta \ge \max\{\delta_1, \dots, \delta_m\}$.

Assumption (H2). The diagonal entries $d(n)$ satisfy the asymptotics

$$d(n) = c_0 n^{\delta_0} + c n^{\delta_0 - 1} + \mathrm{O}(n^{\delta_0 - 2}) \text{ as } n \to \infty, \tag{1.8}$$

where $\delta_0 > 0$, $c_0 > 0$ and $c \in \mathbb{R}$ are fixed.

Our main result is the following

Theorem 1.1. *Let $A = D + A'$ be defined by* (1.1)–(1.6). *If both assumptions* (H1), (H2) *hold and if $\kappa := \delta_0 - \delta > 0$, then*

$$\lambda_n(A) = d(n) + \mathrm{O}(n^{\delta - \kappa}) \text{ as } n \to \infty. \tag{1.9}$$

1.3. **Comments.**

a. If (1.7) is replaced by the weaker condition

$$a_l(n) = \mathrm{O}(n^{\delta})$$

then the min-max principle allows us (see Theorem 3.1) to prove

$$\lambda_n(A) = d(n) + \mathrm{O}(n^{\delta}) \text{ as } n \to \infty. \tag{1.10}$$

The main purpose of Theorem 1.1 is to show that it is possible to replace the estimate (1.10) by the improved estimate (1.9).

b. For any fixed $j \in \mathbb{Z}$ the assumptions of Theorem 1.1 imply

$$\frac{a_l(n+j)}{d(n)} = \mathrm{O}(n^{-\kappa}) \text{ as } n \to \infty. \tag{1.11}$$

We observe that the assertion of Theorem 1.1 holds for any fixed $\kappa > 0$ while all papers [1, 2, 5–7] assume $\kappa > 1$.

c. We observe that

$$d(n+1) - d(n) \sim \delta_0 c_0 n^{\delta_0 - 1} \text{ as } n \to \infty \tag{1.12}$$

and we can treat the case $0 < \delta_0 < 1$ when $d(n+1) - d(n) \to 0$ as $n \to \infty$, while the papers [1, 5–7] assume that

$$\liminf_{n\to\infty} \big(d(n+1) - d(n)\big) > 0.$$

d. Theorem 1.1 will be obtained as a special case of more general estimates described in Section 5 (see Theorems 5.1 and 5.2). In Section 5.4 we give asymptotic estimates of eigenvalues for some cases of not power-like entries.

1.4. **Contents.** In Section 2 we check that the operator A is well defined under the assumption (1.4) and its resolvent is compact under assumption (1.6).

In Section 3 we show how the min-max principle ensures the estimate (1.10) if (1.7) is replaced by the weaker condition $a_l(n) = \mathrm{O}(n)$.

In Section 4 we present basic ingredients of our approach based on a construction of the unitary conjugation of A with smaller off-diagonal entries. A similar idea is often used to investigate spectral asymptotics of self-adjoint problems defined by a linear PDE, e.g., in relation with the semi-classical approximation in Quantum Mechanics.

In Section 5.1 we state a general estimate (Theorem 5.1) and in Section 5.2 we derive Theorem 5.2 which is an application of the general estimate to power-like entries. In Section 5.3 we easily check that Theorem 1.1 is a special case of Theorem 5.2 and in Section 5.4 we apply (Theorem 5.1) to treat some cases of a different asymptotic behaviour of the entries. We complete the proof of the general estimate in Section 6.

The main result (Theorem 1.1) stated above derives from Theorem 5.1 as follows:

$$\left.\begin{array}{c}\text{Theorem 5.1}\\ \text{Lemma 4.2}\end{array}\right\} \Longrightarrow \text{Theorem 5.2} \Longrightarrow \text{Theorem 1.1.}$$

2. The operator A

Proposition 2.1. *Let $A = A' + D$ be the operator defined by* (1.1)-(1.3) *and* (1.5).

(i) *If* (1.4) *holds, then A is self-adjoint.*

(ii) *If moreover* (1.6) *holds, then A has compact resolvent.*

It is well known (see [8]) that (1.5) defines a self-adjoint operator $\mathcal{D} \to l^2$ if A' has zero relative bound with respect to D, i.e., if for any $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that

$$\|A'x\|_{l^2} \leq \varepsilon \|Dx\|_{l^2} + C_\varepsilon \|x\|_{l^2} \text{ for } x \in \mathcal{D}. \tag{2.1}$$

Before starting the proof of Proposition 2.1 we introduce some notations. We recall that the scalar product in l^2 is defined by $\langle x, y\rangle = \sum_{k=1}^{\infty} \bar{x}_k y_k$ and we denote by $(\mathrm{e}_n)_{n=1}^{\infty}$ the canonical basis of l^2, i.e., $\mathrm{e}_n(j) = \delta_{j,n}$ where $\delta_{n,n} = 1$ and $\delta_{j,n} = 0$ for $j \neq n$.

Then we observe (see [8]) that it suffices to show (2.1) for $x \in c_{00}$, where c_{00} is the linear subspace of l^2 generated by the canonical basis, i.e.,

$$x \in c_{00} \Longleftrightarrow \#\{j : x(j) \neq 0\} < \infty. \tag{2.2}$$

We denote by $\mathcal{B}(l^2)$ the algebra of bounded linear operators on l^2 with the norm

$$\|T\| = \sup_{\|x\|_{l^2} \leq 1} \|Tx\|_{l^2}. \tag{2.3}$$

The shift operator $S \in \mathcal{B}(l^2)$ is defined by

$$S\,\mathrm{e}_n = \mathrm{e}_{n+1} \tag{2.4}$$

and for any $a \colon \mathbb{N}^* \to \mathbb{C}$ we denote by $a(\Lambda)$ the closed operator in l^2 given by

$$(a(\Lambda)x)(j) = a(j)x(j) \text{ for } x \in c_{00}. \tag{2.5}$$

We can then rewrite the definition of A' in the form

$$A'x = \sum_{1 \leq l \leq m} \left(S^l a_l(\Lambda) + a_l(\Lambda) S^{l*}\right) x \tag{2.6}$$

where $x \in c_{00}$ and S^{l*} is the adjoint of S^l.

Proof of (i). It suffices to show (2.1) for $x \in c_{00}$. For $l = 1, \ldots, m$ we denote

$$A'_l \coloneqq S^l a_l(\Lambda) + a_l(\Lambda) S^{l*}. \tag{2.7}$$

For arbitrary $x \in c_{00}$ we can write

$$\|A'_l x\|_{l^2}^2 \leq \left(||S^l a_l(\Lambda)x||_{l^2} + ||a_l(\Lambda) S^{*l} x||_{l^2}\right)^2. \tag{2.8}$$

Since $S^l a_l(\Lambda) S^{*l} = a_l(\Lambda - l)$ with the convention that $a_l(j - l) = 0$ if $j \leq l$ and S^l is the isometry, the right-hand side of (2.8) can be estimated from above by

$$2||S^l a_l(\Lambda)x||_{l^2}^2 + 2||a_l(\Lambda) S^{*l} x||_{l^2}^2 = 2||a_l(\Lambda)x||_{l^2}^2 + 2||a_l(\Lambda - l)x||_{l^2}^2. \tag{2.9}$$

Then taking $y = (\mathrm{i} + D)x$ we obtain

$$||A'_l (\mathrm{i} + D)^{-1} y||_{l^2}^2 \leq \sum_j |b_l(j) y(j)|^2 \tag{2.10}$$

with

$$b_l(j) = \left(2\,\frac{a_l(j)^2 + a_l(j-l)^2}{1 + d(j)^2}\right)^{1/2} \xrightarrow[j \to \infty]{} 0. \tag{2.11}$$

For $N \in \mathbb{N}^*$ we denote by Π_N the orthogonal projection onto $\{\mathrm{e}_n\}_{1\le n\le N}$ and $\Pi'_N := I-\Pi_N$. Then (2.11) implies $||A'_l(\mathrm{i}+D)^{-1}\Pi'_N|| \to 0$ as $N\to\infty$. Thus for a given $\varepsilon>0$ we can find $N(\varepsilon)\in\mathbb{N}$ such that

$$||A'_l(\mathrm{i}+D)^{-1}\Pi'_{N(\varepsilon)}y||_{l^2} \le \varepsilon||y||_{l^2}$$

and we deduce $||A'_lx||_{l^2} \le \varepsilon||(D+\mathrm{i})\Pi'_{N(\varepsilon)}x||_{l^2} + ||A'_l\Pi_{N(\varepsilon)}||\,||x||_{l^2}$. □

Proof of (ii). The operator $A'_l(\mathrm{i}+D)^{-1}$ is compact as a limit of finite rank operators $A'_l(\mathrm{i}+D)^{-1}\Pi_N$ in the norm of $\mathcal{B}(l^2)$. Then $(\mathrm{i}+A)^{-1}A'_l(\mathrm{i}+D)^{-1} = (\mathrm{i}+D)^{-1}-(\mathrm{i}+A)^{-1}$ is compact and compactness of $(\mathrm{i}+D)^{-1}$ (due to (1.6)), implies that $(\mathrm{i}+A)^{-1}$ is compact. □

3. Application of the min-max principle

3.1. **Statement.** In what follows for a sequence $x(n)$ we will use the notation

$$(\Delta x)(n) := x(n+1)-x(n). \tag{3.1}$$

The purpose of this section is to prove the following

Theorem 3.1. *Let $A = D + A'$ be given by (1.1)-(1.6). Assume moreover that there exist $C>0$, $\delta\in\mathbb{R}$, $\kappa>0$ satisfy $\delta+\kappa>0$ and $n_0\in\mathbb{N}$ such that*

$$(\Delta d)(n) \ge C^{-1}n^{\delta+\kappa-1} \text{ for } n>n_0, \tag{3.2}$$

and, for $l=1,\dots,m$,

$$a_l(n) = \mathrm{O}(n^\delta) \text{ as } n\to\infty. \tag{3.3}$$

Then one has the large n asymptotic formula

$$\lambda_n(A) = d(n) + \mathrm{O}(n^\delta). \tag{3.4}$$

We observe that due to (3.2) there exists $c>0$ and n_1 such that

$$d(n) \ge cn^{\delta+\kappa} \text{ for } n>n_1 \tag{3.5}$$

and (3.3) with (3.5) imply (1.11).

3.2. **Auxiliary estimates.**

Lemma 3.2. *Assume that $\alpha\colon \mathbb{N}^*\to\mathbb{R}$ satisfying the two conditions*

$$\alpha(j) \ge \sum_{1\le l\le m}(|a_l(j)|+|a_l(j-l)|), \tag{3.6}$$

$$\frac{\alpha(j)}{d(j)} \to 0 \text{ as } j\to\infty. \tag{3.7}$$

Then for every $n\in\mathbb{N}^*$ the estimate

$$d_n^- \le \lambda_n(A) \le d_n^+ \tag{3.8}$$

holds with

$$d_n^- := \inf_{j\ge n}\{d(j)-\alpha(j)\}, \tag{3.9}$$

$$d_n^+ := \sup_{j\le n}\{d(j)+\alpha(j)\}. \tag{3.10}$$

If moreover there exists $j_0 \in \mathbb{N}^*$ such that

$$|(\Delta\alpha)(j)| \leq (\Delta d)(j) \textit{ for } j \geq j_0, \tag{3.11}$$

then there exists $n_1 \in \mathbb{N}^*$ such that

$$|\lambda_n(A) - d(n)| \leq \alpha(n) \textit{ for } n \geq n_1. \tag{3.12}$$

Proof. Let V_n denote the linear subspace generated by $\{e_j\}_{1\leq j\leq n}$ and $V_n^\perp$ denote its orthogonal complement in l^2. Then suitable versions of the min-max principle give

$$\inf_{\substack{x\in\mathcal{D}\cap V_{n-1}^\perp\\ \|x\|_2\leq 1}} \langle Ax, x\rangle \leq \lambda_n(A) \leq \sup_{\substack{x\in V_n\\ \|x\|_2\leq 1}} \langle x, Ax\rangle. \tag{3.13}$$

Let A'_l be as in (2.7). Then writing

$$\langle x, A'_l x\rangle = \sum_j a_l(j)x(j+l)\overline{x(j)} + \sum_k a_l(k-l)x(k-l)\overline{x(k)} \tag{3.14}$$

with $k = j + l$ we can estimate $|\langle x, A'_l x\rangle|$ by

$$\sum_j 2|a_l(j)|\,|x(j+l)x(j)| \leq \sum_j |a_l(j)|(|x(j+l)|^2 + |x(j)|^2). \tag{3.15}$$

Therefore the right-hand side of (3.15) can be written in the form

$$\sum_k |a_l(k)||x(k+l)|^2 + \sum_j |a_l(j)||x(j)|^2 = \sum_j (|a_l(j-l)| + |a_l(j)|)|x(j)|^2 \tag{3.16}$$

and we obtain

$$|\langle x, A'x\rangle| \leq \sum_j \alpha(j)|x(j)|^2. \tag{3.17}$$

Next we observe that (3.17) implies

$$\langle x,\, (d(\Lambda) - \alpha(\Lambda))x\rangle \leq \langle x,\, Ax\rangle \leq \langle x,\, (d(\Lambda) + \alpha(\Lambda))x\rangle \tag{3.18}$$

and using (3.18) we can estimate the right-hand side of (3.13) by

$$\sup_{\substack{x\in V_n\\ \|x\|_2\leq 1}} \langle x, (d(\Lambda) + \alpha(\Lambda))x\rangle = d_n^+.$$

To complete the proof of (3.8) note that

$$\inf_{\substack{x\in\mathcal{D}\cap V_{n-1}^\perp\\ \|x\|_2\leq 1}} \langle x, (d(\Lambda) - \alpha(\Lambda))x\rangle = d_n^-$$

is smaller than the left-hand side of (3.13). In order to show (3.12) we observe that (3.11) implies

$$d(j) - \alpha(j) \leq d(j+1) - \alpha(j+1) \text{ for } j \geq n_0, \tag{3.19}$$

$$d(j) + \alpha(j) \leq d(j+1) + \alpha(j+1) \text{ for } j \geq n_0, \tag{3.20}$$

and consequently $d_n^\pm = d(n) \pm \alpha(n)$ for $n \geq n_1$, i.e., (3.12) follows from (3.8). □

3.3. **Proof of Theorem 3.1.** Due to (1.10) there exists $C_0 > 0$ such that (3.6), (3.7) hold with

$$\alpha(j) \coloneqq C_0 j^\delta \tag{3.21}$$

and (1.9) ensures the estimate

$$|(\Delta\alpha)(j)| \sim |\delta| C_0 j^{\delta-1} \leq |\delta|\, C_0 C j^{-\kappa} (\Delta d)(j) \text{ for } j \geq j_0. \tag{3.22}$$

Since $|\delta|\, C_0 C j^{-\kappa} \to 0$ as $j \to \infty$, it is clear that (3.22) implies (3.11) if j_0 is chosen large enough. Thus (3.12) holds with $\alpha(j)$ given by (3.21) and the proof of (1.12) is complete.

4. Basic ingredients of the approach

4.1. **Main ideas.** We write the following formal development of the conjugation

$$B_n \coloneqq \mathrm{e}^{-\mathrm{i}P_n} A \mathrm{e}^{\mathrm{i}P_n} = A + [A, \mathrm{i}P_n] + \frac{1}{2}[[A, \mathrm{i}P_n], \mathrm{i}P_n] + \dots \tag{4.1}$$

where P_n is self-adjoint and of finite rank for simplicity. Then $\lambda_n(A) = \lambda_n(B_n)$ and we want to determine P_n so that B_n is close to a diagonal operator at least for the entries with indices ranging between $n - \tau_n$ and $n + \nu\tau_n$ where $\nu \in \mathbb{N}^*$ is fixed and $(\tau_n)_{n=1}^\infty$ is a sequence of positive integers satisfying

$$\tau_n \leq \tau_{n+1} \text{ for } n \in \mathbb{N}^* \tag{4.2}$$

$$\tau_n \xrightarrow[n\to\infty]{} \infty \tag{4.3}$$

$$n - \mu\tau_n \xrightarrow[n\to\infty]{} \infty \tag{4.4}$$

with a certain $\mu > 1$ fixed. We remark that in the proof of Theorem 1.1 we take $\nu = 1$, $\mu = 2$ and $\tau_n = [\frac{1}{4}n]$, where $[s] = \max\{k \in \mathbb{Z} : k \leq s\}$)

Further on we fix $\nu \in \mathbb{N}^*$, $\mu > 1$ and a cut-off function $\chi \in C^1(\mathbb{R})$ satisfying

$$0 \leq \chi \leq 1,$$
$$\chi(s) = 1 \text{ for } s \in [-1, \nu]$$
$$\chi(s) = 0 \text{ for } s \notin [-\mu, \nu\mu].$$

Then we write the decomposition

$$a_l(j) = a_{n,l}(j) + \tilde{a}_{n,l}(j) \tag{4.5}$$

with

$$a_{n,l}(j) \coloneqq a_l(j)\, \chi\!\left(\frac{j-n}{\tau_n}\right), \tag{4.6}$$

$$\tilde{a}_{n,l}(j) \coloneqq a_l(j)\, (1-\chi)\!\left(\frac{j-n}{\tau_n}\right) \tag{4.7}$$

and the corresponding decomposition

$$A' = A_n + \tilde{A}_n, \tag{4.8}$$

where

$$A_n = \sum_{1\le l\le m} \left(S^l a_{n,l}(\Lambda) + a_{n,l}(\Lambda)S^{l*}\right), \tag{4.9}$$

$$\tilde{A}_n = \sum_{1\le l\le m} \left(S^l \tilde{a}_{n,l}(\Lambda) + \tilde{a}_{n,l}(\Lambda)S^{l*}\right). \tag{4.10}$$

Using (4.8) we rewrite (4.1) in the form

$$B_n = \mathrm{e}^{-\mathrm{i}P_n} A \mathrm{e}^{\mathrm{i}P_n} = D + \tilde{A}_n + A_n + [D, \mathrm{i}P_n] + W_n, \tag{4.11}$$

where W_n is considered as a lower order error. However due to

$$n - \tau_n \le j \le n + \nu\tau_n \ \Rightarrow\ \tilde{a}_{n,l}(j) = 0 \tag{4.12}$$

it is easy to see that for n large enough we have $(D + \tilde{A}_n)\mathrm{e}_n = D\mathrm{e}_n = d(n)\mathrm{e}_n$, i.e. $d(n)$ is an eigenvalue of $D + \tilde{A}_n$. Then in Section 4.2, Lemma 4.2 we show that $d(n)$ is the n-th eigenvalue of $D + \tilde{A}_n$ provided n is large enough and the entries are sufficiently regular. Next we choose P_n satisfying the commutator equation

$$A_n + [D, \mathrm{i}P_n] = 0, \tag{4.13}$$

hence the expression (4.11) takes the form

$$B_n = \mathrm{e}^{-\mathrm{i}P_n} A \mathrm{e}^{\mathrm{i}P_n} = D + \tilde{A}_n + W_n$$

and using $\lambda_n(A) = \lambda_n(B_n)$, $d(n) = \lambda_n(D + \tilde{A}_n)$ with the min-max principle we obtain

$$|\lambda_n(A) - d(n)| = |\lambda_n(B_n) - \lambda_n(D + \tilde{A}_n)| \le \|B_n - (D + \tilde{A}_n)\| \tag{4.14}$$

for $n > \tilde{n}_0$.

Lemma 4.1. *Let P_n be a finite rank self-adjoint operator satisfying $A_n = \mathrm{i}[P_n, D]$. If $A' = A_n + \tilde{A}_n$ and $B_n = \mathrm{e}^{-\mathrm{i}P_n} A \mathrm{e}^{\mathrm{i}P_n}$, then*

$$\|B_n - (D + \tilde{A}_n)\| \le \|[P_n, \tilde{A}_n]\| + \frac{1}{2}\|[P_n, A_n]\|. \tag{4.15}$$

All our results will follow from suitable estimates of the right-hand side of (4.15), i.e., estimates of the norms of commutators. The statement of a general estimate is given in Section 5 and the norms of commutators from the right-hand side of (4.15) are estimated in Section 6.

Proof. We introduce

$$\tilde{B}_n := \mathrm{e}^{-\mathrm{i}P_n} \tilde{A}_n \mathrm{e}^{\mathrm{i}P_n} - \tilde{A}_n \tag{4.16}$$

and we observe that

$$\tilde{B}_n = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\left(\mathrm{e}^{-\mathrm{i}sP_n} \tilde{A}_n \mathrm{e}^{\mathrm{i}sP_n}\right) \mathrm{d}s = \int_0^1 \mathrm{e}^{-\mathrm{i}sP_n} \mathrm{i}[\tilde{A}_n, P_n] \mathrm{e}^{\mathrm{i}sP_n} \mathrm{d}s.$$

Since for $s \in \mathbb{R}$ the operators $\mathrm{e}^{\mathrm{i}sP_n}$ are unitary, $\|\mathrm{e}^{\mathrm{i}sP_n}\| = 1$ and we find

$$\|\tilde{B}_n\| \le \|[\tilde{A}_n, P_n]\|. \tag{4.17}$$

Next for $s \in \mathbb{R}$ we introduce

$$G_n(s) := \mathrm{e}^{-\mathrm{i}sP_n}(D + \mathrm{i}[sP_n, D])\mathrm{e}^{\mathrm{i}sP_n} - D \tag{4.18}$$

and we observe that the differentiation of $s \mapsto G_n(s)$ gives

$$\frac{\mathrm{d}}{\mathrm{d}s} G_n(s) = \mathrm{e}^{-\mathrm{i}sP_n}\big(\mathrm{i}\,[D + \mathrm{i}[sP_n, D], P_n] + \mathrm{i}[P_n, D]\big)\mathrm{e}^{\mathrm{i}sP_n}$$
$$= \mathrm{e}^{-\mathrm{i}sP_n} s\,[[D, P_n], P_n]\,\mathrm{e}^{\mathrm{i}sP_n}.$$

Therefore

$$G_n(1) = \int_0^1 \mathrm{e}^{-\mathrm{i}sP_n} s\,[[D, P_n], P_n]\,\mathrm{e}^{\mathrm{i}sP_n}\mathrm{d}s$$

and we can estimate

$$\|G_n(1)\| \le \int_0^1 s\,\|[[D, P_n], P_n]\|\mathrm{d}s = \frac{1}{2}\|[[D, P_n], P_n]\|. \tag{4.19}$$

However

$$B_n = \mathrm{e}^{-\mathrm{i}P_n}(D + A_n)\mathrm{e}^{\mathrm{i}P_n} + \mathrm{e}^{-\mathrm{i}P_n}\tilde{A}_n\mathrm{e}^{\mathrm{i}P_n} = (G_n(1) + D) + (\tilde{B}_n + \tilde{A}_n),$$

hence

$$||B_n - (D + \tilde{A}_n)|| = ||\tilde{B}_n + G_n(1)|| \le ||\tilde{B}_n|| + ||G_n(1)||. \tag{4.20}$$

To complete the proof it remains to estimate the right-hand side of (4.20) using (4.17)-(4.19). □

4.2. **Equality** $d(n) = \lambda_n(D + \tilde{A}_n)$**.** In this section we give sufficient conditions to ensure the equality $d(n) = \lambda_n(D + \tilde{A}_n)$ used in the estimate (4.14). We consider fixed numbers $\mu > 1$, n_0, $\nu \in \mathbb{N}^*$ and a sequence of positive integers $(\tau_n)_{n=1}^{\infty}$ satisfying (4.2)-(4.4). We assume that the inequalities

$$d(n) < d(n+1), \tag{4.21}$$
$$k \le n - \tau_n \implies 4m|a_l(k)| \le d(n) - d(k+m), \tag{4.22}$$
$$k \ge n + \nu\tau_n - m \implies 4m|a_l(k)| \le d(k) - d(n) \tag{4.23}$$

hold for $n \ge n_0$ and $l = 1, \dots, m$.

Lemma 4.2. *Assume that* (4.21), (4.22), (4.23) *hold for* $n \ge n_0$. *If* $\tilde{A}_n$ *are defined by means of* $\tilde{a}_{n,l}$ *and* χ *as in Section 4.1, then there is* $\tilde{n}_0 \in \mathbb{N}$ *such that*

$$d(n) = \lambda_n(D + \tilde{A}_n) \text{ for } n > \tilde{n}_0. \tag{4.24}$$

Proof. We introduce

$$\tilde{\alpha}_n(j) := \sum_{1 \le l \le m} (|\tilde{a}_{n,l}(j)| + |\tilde{a}_{n,l}(j-l)|) \tag{4.25}$$

and observe that Lemma 3.2 allows us to estimate

$$\tilde{d}_n^- \le \lambda_n(D + \tilde{A}_n) \le \tilde{d}_n^+ \tag{4.26}$$

with

$$\tilde{d}_n^- = \inf_{j \ge n}\{d(j) - \tilde{\alpha}_n(j)\}, \tag{4.27}$$

$$\tilde{d}_n^+ = \sup_{j \le n}\{d(j) + \tilde{\alpha}_n(j)\}. \tag{4.28}$$

For $n \ge n_0$ and $0 \le i \le m$ we use (4.22) with $k = j - i$ and write have

$$j \le n - \tau_n \implies 4m|a_l(j-i)| \le d(n) - d(j - i + m).$$

If $j \geq n_0$ then $i \leq m \Rightarrow d(j-i+m) \geq d(j)$ follows fom (4.21) and using the inequality $|\tilde{a}_{n,l}(j-i)| \leq |a_l(j-i)|$ we deduce

$$n_0 \leq j \leq n - \tau_n \implies |\tilde{a}_n(j)| \leq d(n) - d(j).$$

Similarly for $n \geq n_0 + m$ and $0 \leq i \leq m$ we have

$$j \geq n + \nu\tau_n \implies 4m|a_l(j-i)| \leq d(j-i) - d(n)$$

and consequently

$$j \geq n + \nu\tau_n \implies |\tilde{a}_n(j)| \leq d(j) - d(n).$$

Next we observe that by definition $n - \tau_n \leq j \leq n + \nu\tau_n \Rightarrow \tilde{a}_n(j) = 0$, hence for $n \geq n_0 + m$ we have

$$n_0 \leq j \leq n \implies \tilde{a}_n(j) \leq d(n) - d(j) \implies d(j) + \tilde{\alpha}_n(j) \leq d(n), \tag{4.29}$$

$$j \geq n \implies \tilde{a}_n(j) \leq d(j) - d(n) \implies d(n) \leq d(j) - \tilde{\alpha}_n(j). \tag{4.30}$$

It is clear that (4.30) implies $\tilde{d}_n^- = d(n)$ for $n \geq n_0 + m$. Morover choosing $\tilde{n}_0 \geq n_0 + m$ such that $d(\tilde{n}_0) \geq \tilde{d}_{n_0}^+$ we ensure $j \leq n_0 \Rightarrow d(j) + \tilde{\alpha}_n(j) \leq d(n)$ for $n \geq \tilde{n}_0$ and due to (4.29) we obtain $\tilde{d}_n^+ = d(n)$ for $n \geq \tilde{n}_0$. Thus (4.24) follows from (4.26). □

4.3. **Comments on** (4.22) **and** (4.23)**.** Assume that (4.21) holds for $n \geq n_0$ and denote

$$\eta_1 := \limsup_{n\to\infty} \frac{d(n - \tau_n + m)}{d(n)},$$

$$\eta_2 := \limsup_{n\to\infty} \frac{d(n)}{d(n + \nu\tau_n - m)}.$$

We claim that $\eta_1 < 1$ implies (4.22) and $\eta_2 < 1$ implies (4.23) for $n \geq n_0'$. Indeed, assume $\eta_1 < \eta_1' < 1$ and $n - \tau_n \geq k \geq n_0$. Then choosing n_1 large enough we ensure

$$n \geq n_1 \implies \eta_1'\, d(n) \geq d(n - \tau_n + m) \geq d(k+m),$$

where the last inequality holds due to (4.21). Due to (1.4) we can find k_1 such that

$$k \geq k_1 \implies 4m|a_l(k)| \leq (1 - \eta_1')d(k) \leq (1 - \eta_1')d(n) \leq d(n) - d(k+m),$$

i.e. (4.22) holds for $n \geq n_1'$. Next we assume $\eta_2 < \eta_2' < 1$ and $n + \nu\tau_n - m \leq k$. Then choosing n_2 large enough we ensure

$$n \geq n_2 \implies d(n) \leq \eta_2'\, d(n + \nu\tau_n - m) \leq \eta_2'\, d(k),$$

where the last inequality holds due to (4.21). Due to (1.4) we can find k_2 such that

$$k \geq k_2 \implies 4m|a_l(k)| \leq (1 - \eta_2')d(k) \leq d(k) - d(n),$$

i.e. (4.23) holds for $n \geq n_2'$.

5. A general estimate

5.1. **Statement.** We fix $\mu > 1$ and a sequence of positive integers $(\tau_n)_{n=1}^{\infty}$ satisfying (4.2)-(4.4). For $s \geq 0$ we denote

$$\alpha_s(j) \coloneqq \max_{\substack{1\leq l\leq m \\ |i|\leq s}} |a_l(j+i)|, \tag{5.1}$$

$$\tilde{\alpha}_{n,s}(j) \coloneqq \frac{2\alpha_s(j)}{(\mu-1)\tau_n} + \max_{\substack{1\leq l\leq m \\ |i|\leq s}} |(\Delta a_l)(j+i)|, \tag{5.2}$$

$$\gamma_s(j) \coloneqq \min_{|i|\leq s+1} (\Delta d)(j+i), \tag{5.3}$$

$$\tilde{\gamma}_s(j) \coloneqq \max_{|i|\leq s} |(\Delta^2 d)(j+i)|, \tag{5.4}$$

where $(\Delta^2 d)(n) = (\Delta d)(n+1) - (\Delta d)(n) = d(n+2) - 2d(n+1) + d(n)$.

Theorem 5.1. *Let A be defined by* (1.1)-(1.6)*. Let $\mu > 1$, $(\tau_n)_{n=1}^{\infty}$, α_s, γ_s, $\tilde{\alpha}_{n,s}$, $\tilde{\gamma}_s$ be as above and $n_0, \nu \in \mathbb{N}^*$ such that* (4.21), (4.22), (4.23) *hold for $n \geq n_0$. If*

$$\rho_n(j) \coloneqq \frac{2\tilde{\alpha}_{n,4m}(j)\alpha_{4m}(j)}{\gamma_{4m}(j)} + \frac{m\tilde{\gamma}_{5m}(j)\alpha_{4m}(j)^2}{\gamma_{4m}(j)^2}, \tag{5.5}$$

then there is n_1 such that for $n \geq n_1$ one has the estimate

$$|\lambda_n(A) - d(n)| \leq 15m^3 \sup_{-\tau_n^- \leq i \leq \tau_n^+} \rho_n(n+i) \tag{5.6}$$

with $\tau_n^- := \mu\tau_n + 4m$ and $\tau_n^+ := \mu\nu\tau_n + 4m$.

This general estimate will be proved in Section 6.

5.2. **Application.** We check that Theorem 5.1 implies

Theorem 5.2. *Let A be defined by* (1.1)-(1.6)*. Assume that there exist $C > 0$, $\delta \in \mathbb{R}$, $\kappa > 0$ satisfying $\delta + \kappa > 0$ and $n_0 \in \mathbb{N}$ such that*

$$C^{-1} n^{\delta+\kappa-1} \leq (\Delta d)(n) \leq C n^{\delta+\kappa-1} \quad \text{for } n \geq n_0, \tag{5.7}$$

$$(\Delta^2 d)(n) = \mathrm{O}(n^{\delta+\kappa-2}) \quad \text{as } n \to \infty, \tag{5.8}$$

$$a_l(n) = \mathrm{O}(n^{\delta}) \quad \text{as } n \to \infty,\ l = 1, \dots, m. \tag{5.9}$$

$$(\Delta a_l)(n) = \mathrm{O}(n^{\delta-1}) \quad \text{as } n \to \infty,\ l = 1, \dots, m. \tag{5.10}$$

Then one has the estimate

$$\lambda_n(A) = d(n) + \mathrm{O}(n^{\delta-\kappa}) \text{ as } n \to \infty. \tag{5.11}$$

Proof. Due to (5.7) and (5.8) there exist $C_0 > c_0 > 0$ and $n_1 \in \mathbb{N}$ satisfying

$$c_0 n^{\delta+\kappa} \leq d(n) \leq C_0 n^{\delta+\kappa} \text{ for } n \geq n_1. \tag{5.12}$$

In order to ensure (4.24) we will check that the assumptions of Lemma 4.2 hold if $\tau_n = n + m - [n\varepsilon_0]$ where $\varepsilon_0 > 0$ is fixed sufficiently small and ν sufficiently large. Indeed, using the comments of Section 4.3 we observe that we can choose ε_0 such that

$$\frac{d(n-\tau_n+m)}{d(n)} = \frac{d([n\varepsilon_0])}{d(n)} \leq \frac{C_0}{c_0}\varepsilon_0^{\delta+\kappa} < 1. \tag{5.13}$$

Similarly we can estimate

$$\frac{d(n)}{d(n+\nu\tau_n-m)} \leq \frac{C_0 n^{\delta+\kappa}}{c_0(n+\nu\tau_n-m)^{\delta+\kappa}} \xrightarrow[n\to\infty]{} \frac{C_0}{c_0(1+\nu(1-\varepsilon_0))^{\delta+\kappa}} \tag{5.14}$$

and the limit is smaller than 1 if ν is chosen large enough.

Thus all assumptions of Lemma 4.2 hold and it remains to apply Theorem 5.1. Using (5.7)-(5.10) we can find a constant C_1 such that

$$\sup_{-\tau_n^- \leq i \leq \tau_n^+} \gamma_{4m}(n+i) \geq C_1^{-1} n^{\delta+\kappa-1}, \tag{5.15}$$

$$\sup_{-\tau_n^- \leq i \leq \tau_n^+} \tilde{\gamma}_{5m}(n+i) \leq C_1 n^{\delta+\kappa-2}, \tag{5.16}$$

$$\sup_{-\tau_n^- \leq i \leq \tau_n^+} \alpha_{4m}(n+i) \leq C_1 n^{\delta}, \tag{5.17}$$

$$\sup_{-\tau_n^- \leq i \leq \tau_n^+} \tilde{\alpha}_{n,4m}(n+i) \leq C_1 n^{\delta-1} \tag{5.18}$$

with γ_s, $\tilde{\gamma}_s$, α_s, $\tilde{\alpha}_{n,s}$, given by (5.1)-(5.4). Therefore,

$$\tilde{\alpha}_{n,4m}(j)\frac{\alpha_{4m}(j)}{\gamma_{4m}(j)} \leq C_1^3 n^{\delta-1}\frac{n^{\delta}}{n^{\delta+\kappa-1}} = C_1^3 n^{\delta-\kappa}, \tag{5.19}$$

$$\tilde{\gamma}_{5m}(j)\frac{\alpha_{4m}(j)^2}{\gamma_{4m}(j)^2} \leq C_1^5 n^{\delta+\kappa-2}\frac{n^{2\delta}}{n^{2(\delta+\kappa-1)}} = C_1^5 n^{\delta-\kappa} \tag{5.20}$$

hold when $-\tau_n^- \leq j-n \leq \tau_n^+$ and it is clear that

$$\sup_{-\tau_n^- \leq i \leq \tau_n^+} \rho_n(n+i) = \mathrm{O}(n^{\delta-\kappa}) \tag{5.21}$$

if $\rho_n(j)$ is given by (5.5). We conclude that (5.11) follows from (5.6) and (5.21). □

5.3. **Proof of Theorem 1.1.**

Proof. Let $\delta_0 = \delta+\kappa$. Then the assumptions (H1) and (H2) imply (5.7)-(5.10). Consequently Theorem 1.1 follows from Theorem 5.2. □

5.4. **Other examples.** In this section we use $\nu = 1$, $\mu = 2$ and $(\tau_n)_{n=1}^{\infty}$ adapted to $d(n)$ satisfying $d(n) \sim \omega(n)$ where $\omega :]0,\infty[\to]0,\infty[$ belongs to some special classes of weight functions.

(a) We fix $\kappa > 0$, $\kappa' \in \mathbb{R}$ and assume

$$|a_l(n)| + n|\Delta a_l(n)| = O(n^{-\kappa}(\ln n)^{-\kappa'}\omega(n)) \tag{5.22}$$

$$d(n) = \omega(n)\,\left(1+O(n^{-2})\right) \tag{5.23}$$

where

$$\omega(\lambda) = c_0\lambda^{\delta_0}(\ln\lambda)^{\delta_0'} \tag{5.24}$$

holds with some $c_0 > 0$, $\delta_0 > 0$, $\delta_0' \in \mathbb{R}$. We observe that the derivatives satisfy $\omega^{(k)}(\lambda) \sim c_k\lambda^{-k}\omega(\lambda)$ as $\lambda \to \infty$. Since $\Delta\omega(n) = \omega'(n+r_n)$ holds with some $r_n \in [0,1]$ and $\omega'(n+r_n) \sim \omega'(n)$ as $n \to \infty$ we easily deduce

$$\gamma_s(n) \sim \delta_0 n^{-1}\omega(n) \quad (n\to\infty) \tag{5.25}$$

$$\tilde{\gamma}_s(n) = O(n^{-2}\omega(n)) \quad (n\to\infty) \tag{5.26}$$

Using $\tau_n = [n/4]$ we find that Theorem 5.1 gives the estimate

$$\lambda_n(A) = d(n)\left(1 + O(n^{-2\kappa}(\ln n)^{-2\kappa'})\right) \tag{5.27}$$

It is easy to see that (5.27) still holds when $\kappa = 0$ and $\kappa' > 0$.

(b) We assume that (5.22) holds with some $\kappa > 0$, $\kappa' \in \mathbb{R}$,

$$d(n) = \omega(n)\ \left(1 + O(n^{-2}(\ln\lambda)^{-1})\right) \tag{5.28}$$

where

$$\omega(\lambda) = c_0(\ln\lambda)^{\delta_0'} \tag{5.29}$$

holds with some $c_0 > 0$, $\delta_0' > 0$. Then computing the derivatives of ω we find

$$\gamma_s(n) \sim \delta_0 n^{-1}(\ln n)^{-1}\omega(n) \quad (n \to \infty) \tag{5.30}$$

$$\tilde{\gamma}_s(n)\ = O(n^{-2}(\ln n)^{-1}\omega(n)\,) \quad (n \to \infty) \tag{5.31}$$

and using $\tau_n = [n/4]$ in Theorem 5.1 we obtain the estimate

$$\lambda_n(A) = d(n)\left(1 + O(n^{-2\kappa}(\ln n)^{1-\kappa'})\right) \tag{5.32}$$

Moreover (5.32) holds when $\kappa = 0$ and $\kappa' > 1$. Indeed, using $\tau_n = [n/4]$ and the asymptotic formula $\omega(n \pm n/4) - \omega(n) \sim \pm\delta_0'(4\ln(n))^{-1}\omega(n)$ it is easy to see that (4.22) and (4.23) follow from (5.22) with $\kappa = 0$ and $\kappa' > 1$.

(c) We assume that $\kappa > 0$, $0 < \theta < 1$ and

$$|a_l(n)| + n^{1-\theta}|\Delta a_l(n)| = O(n^{-\kappa}\omega(n)) \tag{5.33}$$

$$d(n) = \omega(n)\ \left(1 + O(n^{\theta-2})\right), \tag{5.34}$$

where

$$\omega(\lambda) = c_0\lambda^{\delta_0}\,\mathrm{e}^{c\lambda^\theta} \tag{5.35}$$

holds for some $\delta_0 \in \mathbb{R}$, $c_0 > 0$, $c > 0$. The derivatives satisfy $\omega^{(k)}(\lambda) \sim c_k\lambda^{-k(1-\theta)}\omega(\lambda)$ as $\lambda \to \infty$ and we deduce

$$\gamma_s(n) \sim c\theta n^{-(1-\theta)}\omega(n) \quad (n \to \infty) \tag{5.36}$$

$$\tilde{\gamma}_s(n)\ = O(n^{-2(1-\theta)}\omega(n)\,) \quad (n \to \infty) \tag{5.37}$$

Let $\tau_n = [n^{1-\theta}C_0]$ with $C_0 > 1/(c\theta)$. Then (4.22), (4.23) hold and Theorem 5.1 ensures

$$\lambda_n(A) = d(n)\left(1 + O(n^{-2\kappa})\right) \tag{5.38}$$

6. Proof of Theorem 5.1

6.1. **Auxiliary estimates of commutators.** For $n \in \mathbb{N}^*$, $l \in \mathbb{Z}$ we consider $p_{n,l}$, $a_{n,l}\colon \mathbb{Z} \to \mathbb{R}$ satisfying

$$p_{n,-l}(j) = p_{n,l}(j), \qquad a_{n,-l}(j) = a_{n,l}(j), \tag{6.1}$$

$$p_{n,l}(j) = a_{n,l}(j) = 0 \text{ when } j \leq 0, \tag{6.2}$$

$$p_{n,l}(j) = a_{n,l}(j) = 0 \text{ when } |l| > m \tag{6.3}$$

where $m \in \mathbb{N}^*$ is fixed. We fix $\nu \in \mathbb{N}^*$, $\mu > 1$ and assume moreover

$$p_{n,l}(j) \neq 0 \implies -\mu\tau_n \leq j - n \leq \mu\nu\tau_n, \tag{6.4}$$

where $(\tau_n)_{n=1}^{\infty}$ is as before and consider finite rank self-adjoint operators

$$P_n \coloneqq \sum_{1\le l\le m} \left(S^l p_{n,l}(\Lambda) + p_{n,l}(\Lambda) S^{l*}\right), \tag{6.5}$$

$$R_n \coloneqq \mathrm{i}[P_n, A_n] = \mathrm{i}(P_n A_n - A_n P_n), \tag{6.6}$$

where A_n is given by (4.9). For $s \ge 0$, $j \in \mathbb{Z}$ define

$$\alpha_{n,s}(j) \coloneqq \max_{\substack{|l|\le m \\ |i|\le s}} |a_{n,l}(j+i)|, \tag{6.7}$$

$$\beta_{n,s}(j) \coloneqq \max_{\substack{|l|\le m \\ |i|\le s}} |p_{n,l}(j+i)|, \tag{6.8}$$

$$\alpha'_{n,s}(j) \coloneqq \max_{\substack{|l|\le m \\ |i|\le s}} |(\Delta a_{n,l})(j+i)|, \tag{6.9}$$

$$\beta'_{n,s}(j) \coloneqq \max_{\substack{|l|\le m \\ |i|\le s}} |(\Delta p_{n,l})(j+i)|. \tag{6.10}$$

Lemma 6.1. *Let A_n, P_n, R_n, $\alpha_{n,s}$, $\beta_{n,s}$, $\alpha'_{n,s}$, $\beta'_{n,s}$ be as above and*

$$\rho_{n,s}(j) \coloneqq \alpha_{n,s}(j)\beta'_{n,s}(j) + \alpha'_{n,s}(j)\beta_{n,s}(j). \tag{6.11}$$

Then one has

$$\|R_n\| \le 10m^3 \sup_{-\tau_n^- \le i \le \tau_n^+} \rho_{n,4m}(n+i). \tag{6.12}$$

Proof. Our notations allow us to express

$$R_n = r_{n,0}(\Lambda) + \sum_{1\le k\le 2m} \left(S^k r_{n,k}(\Lambda) + r_{n,k}(\Lambda) S^{k*}\right), \tag{6.13}$$

where for $k \ge 0$ we have

$$r_{n,k}(j) = \langle \mathrm{e}_{j+k}, R_n\, \mathrm{e}_j\rangle = \mathrm{i}\langle \mathrm{e}_{j+k}, P_n A_n\, \mathrm{e}_j\rangle - \mathrm{i}\langle \mathrm{e}_{j+k}, A_n P_n\, \mathrm{e}_j\rangle. \tag{6.14}$$

For $s \in \mathbb{R}$ we write $s_+ = \max\{s, 0\}$ and $s_- = (-s)_+$. Then

$$\langle \mathrm{e}_j, P_n\, \mathrm{e}_i\rangle = \langle \mathrm{e}_i, P_n\, \mathrm{e}_j\rangle = p_{n,i-j}(i - (i-j)_+), \tag{6.15}$$

$$\langle \mathrm{e}_j, A_n\, \mathrm{e}_i\rangle = \langle \mathrm{e}_i, A_n\, \mathrm{e}_j\rangle = a_{n,i-j}(i - (i-j)_+) \tag{6.16}$$

and using (6.15), (6.16) in

$$\langle \mathrm{e}_{j+k}, P_n A_n\, \mathrm{e}_j\rangle = \sum_l \langle \mathrm{e}_{j+k}, P_n\, \mathrm{e}_{j+l}\rangle\, \langle \mathrm{e}_{j+l}, A_n\, \mathrm{e}_j\rangle,$$

$$\langle \mathrm{e}_{j+k}, A_n P_n\, \mathrm{e}_j\rangle = \sum_l \langle \mathrm{e}_{j+k}, A_n\, \mathrm{e}_{j+k-l}\rangle \langle \mathrm{e}_{j+k-l}, P_n\, \mathrm{e}_j\rangle,$$

we find

$$r_{n,k}(j) = \mathrm{i} \sum_{1\le |l|\le m} r_{n,k,l}(j) \tag{6.17}$$

with

$$\begin{aligned} r_{n,k,l}(j) = {} & p_{n,k-l}(j+k-(k-l)_+)a_{n,l}(j-l_-) \\ & - a_{n,l}(j+k-l_+)p_{n,k-l}(j-(k-l)_-). \end{aligned}$$

Moreover $p_{n,k-l} \neq 0 \implies |k-l| \leq m$ and we claim that

$$r_{n,k,l}(j) \neq 0 \implies -2m - \mu\tau_n \leq j - n \leq \mu\nu\tau_n + 2m. \tag{6.18}$$

Indeed, it suffices to use $0 \leq k \leq 2m$ and $|k-l| \leq m$ in

$$p_{n,k-l}(j-(k-l)_-) \neq 0 \implies n - \mu\tau_n \leq j-(k-l)_- \leq n + \mu\nu\tau_n,$$
$$p_{n,k-l}(j+k-(k-l)_+) \neq 0 \implies n - \mu\tau_n \leq j+k-(k-l)_+ \leq n + \mu\nu\tau_n.$$

Then reasoning as in Section 2 we can estimate

$$\|R_n\| \leq ||r_{n,0}(\Lambda)|| + \sum_{1\leq k\leq 2m} \big(2||r_{n,k}(\Lambda)||^2 + 2||r_{n,k}(\Lambda-k)||^2\big)^{1/2} \tag{6.19}$$

and the right-hand side of (6.19) can be estimated by

$$(4m+1) \sup_{j\geq 1} \max_{0\leq i\leq k\leq 2m} |r_{n,k}(j-i)|. \tag{6.20}$$

However for $i, i' \in \mathbb{Z}$ such that $i' < i$ we have the expression

$$a_{n,l}(j+i) - a_{n,l}(j+i') = \sum_{i'\leq j'\leq i-1} (\Delta a_{n,l})(j+j') \tag{6.21}$$

and using $|k - l_+ + l_-| = |k-l| \leq m$ we obtain the estimate

$$|a_{n,l}(j+k-l_+) - a_{n,l}(j-l_-)| \leq m\alpha'_{n,s}(j), \tag{6.22}$$

with $s = \max\{|l|, |k-l_+|\} \leq \max\{m, k\} \leq 2m$. Thus (6.22) holds with $s = 2m$ and

$$|p_{n,k-l}(j+k-(k-l)_+) - p_{n,k-l}(j-(k-l)_-)| \leq m\,\beta'_{n,2m}(j) \tag{6.23}$$

follows similarly. Since $r_{n,k,l}(j) = r'_{n,k,l}(j) + r''_{n,k,l}(j)$ holds with

$$r'_{n,k,l}(j) = (p_{n,k-l}(j+k-(k-l)_+) - p_{n,k-l}(j-(k-l)_-))a_{n,l}(j-l_-) \tag{6.24}$$
$$r''_{n,k,l}(j) = p_{n,k-l}(j-(k-l)_-)(a_{n,l}(j-l_-) - a_{n,l}(j+k-l_+)), \tag{6.25}$$

we obtain $|r_{n,k,l}(j)| \leq m\,\rho_{n,2m}(j)$ and $\|R_n\|$ can be estimated by

$$2m^2(4m+1) \sup_{-\tau_n^- \leq j-n\leq \tau_n^+} \max_{0\leq i\leq 2m} |\rho_{n,2m}(j-i)|. \tag{6.26}$$

To complete the proof of (6.12) it remains to use $\rho_{n,2m}(j-i) \leq \rho_{n,2m+|i|}(j)$. □

6.2. **Proof of Theorem 5.1.** Let n_0 be as in (4.21) and for $j \geq 1-l \geq n_0$ denote

$$d'_l(j) \coloneqq d(j+l) - d(j). \tag{6.27}$$

Consider A_n, P_n given by (4.9), (4.10) with $a_{n,l}$ as in (4.6) and

$$p_{n,l}(j) = \langle \mathrm{e}_{j+l},\, P_n\, \mathrm{e}_j\rangle = \mathrm{i}\,\frac{a_{n,l}(j)}{d'_l(j)} \text{ for } l = 1,\dots,m. \tag{6.28}$$

Then $R_n \coloneqq [\mathrm{i}P_n,\, D]$ coincides with A_n due to

$$\begin{aligned}\mathrm{i}\langle \mathrm{e}_{j+l},\, R_n\, \mathrm{e}_j\rangle &= \mathrm{i}\langle \mathrm{e}_{j+l},\, P_n D\, \mathrm{e}_j\rangle - \mathrm{i}\langle D\, \mathrm{e}_{j+l},\, P_n\, \mathrm{e}_j\rangle \\ &= \mathrm{i}(d(j) - d(j+l))\langle \mathrm{e}_{j+l},\, P_n\, \mathrm{e}_j\rangle \\ &= a_{n,l}(j) = \langle \mathrm{e}_{j+l}, A_n\, \mathrm{e}_j\rangle \text{ for } l \geq 0.\end{aligned}$$

Thus due to (4.14) and (4.15)

$$|\lambda_n(A) - d(n)| \leq ||[\tilde{A}_n, P_n]|| + \frac{1}{2}\,||[A_n, P_n]|| \text{ for } n \geq n_0. \tag{6.29}$$

We consider $\alpha_{n,s}$, $\beta_{n,s}$, $\alpha'_{n,s}$, $\beta'_{n,s}$ given by (6.7)-(6.10) and in order to apply Lemma 6.1 we will check that

$$\rho_{n,4m}(j) \leq \rho_n(j) \text{ for } n \geq n_0 + 5m \tag{6.30}$$

holds with ρ_n given by (5.5) and $\rho_{n,s}$ by (6.11). Indeed, it is clear that

$$\alpha_{n,s}(j) \leq \alpha_s(j), \tag{6.31}$$

$$\beta_{n,s}(j) \leq \beta_s(j) := \frac{\alpha_s(j)}{\gamma_s(j)} \text{ for } j > n_0 + m + s \tag{6.32}$$

hold with γ_s, α_s given by (5.3), (5.1). Then we observe that the function χ considered in Section 4.1 can be chosen such that

$$\|\chi'\|_\infty := \sup_{t\in\mathbb{R}} |\chi'(t)| \leq \frac{2}{\mu - 1},$$

where χ' denotes the derivative of χ and consequently

$$|\chi((j+1-n)/\tau_n) - \chi((j-n)/\tau_n)| \leq \frac{2}{(\mu-1)\tau_n},$$

hence taking $\tilde{\alpha}_{n,s}$ as indicated in (5.2) we obtain

$$\alpha'_{n,s}(j) \leq \tilde{\alpha}_{n,s}(j) \tag{6.33}$$

due to (4.6). Finally we need to estimate $\beta'_{n,s}(j)$ and for this purpose we first observe that

$$\begin{aligned} j \geq n_0, \quad & l > 0 \implies d'_l(j) \geq d(j+1) - d(j) > 0, \\ j \geq n_0 - l, \quad & l < 0 \implies -d'_l(j) \geq d(j) - d(j-1) > 0 \end{aligned}$$

hold due to (4.21). Further on we assume $j \geq n_0 + m$. Thus for $|l| \leq m$ we have

$$\min\{|d'_l(j)|, |d'_l(j+1)|\} \geq \gamma_0(j) \tag{6.34}$$

and writing $(\Delta d'_l)(j) = \sum_{i\in\mathcal{I}(l)} \Delta^2 d(j+i)$ where $\mathcal{I}(l) = [0,\, l-1] \cap \mathbb{Z}$ when $l > 0$ and $\mathcal{I}(l) = [l+1,\, 0] \cap \mathbb{Z}$ when $l < 0$, we obtain

$$|(\Delta d'_l)(j)| \leq |l| \tilde{\gamma}_l(j) \tag{6.35}$$

and consequently

$$\left| \frac{1}{d'_l(j+1)} - \frac{1}{d'_l(j)} \right| = \left| \frac{(\Delta d'_l)(j)}{d'_l(j+1) d'_l(j)} \right| \leq \frac{|l| \tilde{\gamma}_l(j)}{\gamma_0(j)^2}. \tag{6.36}$$

However by definition (6.28) we have

$$|\Delta p_{n,l}(j)| \leq \frac{|\Delta a_{n,l}(j)|}{|d'_l(j+1)|} + |a_{n,l}(j)| \left| \Delta \left(\frac{1}{d'_l(j)} \right) \right|, \tag{6.37}$$

hence using $|l| \leq m$, (6.34) and (6.36) to estimate the right-hand side of (6.37) we obtain

$$\beta'_{n,s}(j) \leq \tilde{B}_{n,s}(j) := \frac{\tilde{\alpha}_{n,s}(j)}{\gamma_s(j)} + \alpha_s(j) \frac{m \tilde{\gamma}_{m+s}(j)}{\gamma_s(j)^2} \tag{6.38}$$

for $n \geq n_0 + m + s$. Now it is clear that (6.30) follows from (6.31)-(6.33) and (6.38). Then using (6.30) in Lemma 3.2 we obtain

$$||[A_n, P_n]|| \leq 10m^3 \sup_{-\tau_n^- \leq i \leq \tau_n^+} \rho_n(n+i) \text{ for } n \geq n_1. \tag{6.39}$$

To complete the proof it remains to observe that one can use α_s and $\tilde{\alpha}_{n,s}$ as above if $\tilde{a}_{n,l}$ replaces $a_{n,l}$, hence the norm $||[\tilde{A}_n, P_n]||$ can be estimated in a similar manner. Thus we can conclude that the right-hand side of (6.29) can be estimated by the right-hand side of (5.6).

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