The double power monad is the composite power monad

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Abstract

We give a somewhat more conceptual proof of a theorem due to U. Höhle [2013]: the double power monad and the composite power monad, on the category of quantaloid-enriched categories, are the same; its algebras are the completely co-distributive complete categories, and the homomorphisms between such algebras are the bicontinuous functors.

1. Introduction

If $P = (P, \leq)$ is an ordered set, then its downclosed subsets form a sup-lattice $(\mathsf{Dwn}(P), \subseteq)$, and the order-preserving inclusion $P \longrightarrow \mathsf{Dwn}(P)\colon x \mapsto \downarrow x$ has a left adjoint if and only if P has all suprema. Dually, taking the upclosed subsets of P produces an inf-lattice $(\mathsf{Up}(P), \supseteq)$ (note that upsets are ordered by containment, whereas downsets are ordered by inclusion), and the order-preserving inclusion $P \longrightarrow \mathsf{Up}(P)\colon x \mapsto \uparrow x$ has a right adjoint if and only if P has all infima. Of course, P is a sup-lattice if and only if it is an inf-lattice, and then it is said to be a 'complete lattice'.

These two object correspondences can be made functorial in several ways, and the resulting functors interact in at least two ways. For starters, the inverse image of an order-preserving function $f\colon P \longrightarrow Q$ is a new order-preserving function $f^{-1}\colon \mathsf{Dwn}(Q) \longrightarrow \mathsf{Dwn}(P)$. This action on objects and morphisms defines a 2-functor on the locally ordered category Ord of ordered sets which reverses arrows and local order; for the sake of this introduction, let us write it as $\mathcal{L}\colon \mathsf{Ord} \longrightarrow \mathsf{Ord}^{\mathsf{coop}}$. It then so happens that this is a left 2-adjoint, and that the action of its right 2-adjoint $\mathcal{R}\colon \mathsf{Ord}^{\mathsf{coop}} \longrightarrow \mathsf{Ord}$ on objects is $Q \mapsto \mathsf{Up}(Q)$. As a result, the induced 2-monad $\mathcal{T} := \mathcal{R}\mathcal{L}\colon \mathsf{Ord} \longrightarrow \mathsf{Ord}$ acts on objects as $P \mapsto \mathsf{Up}(\mathsf{Dwn}(P))$: it is the **double power monad** on Ord.

On the other hand it is well-known that the locally ordered category Sup of sup-lattices and sup-morphisms is included in Ord by a forgetful 2-functor $\mathcal{U}\colon \mathsf{Sup} \longrightarrow \mathsf{Ord}$, right 2-adjoint to an $\mathcal{F}\colon \mathsf{Ord} \longrightarrow \mathsf{Sup}$ whose action on objects is $P \mapsto \mathsf{Dwn}(P)$; a 2-monad $\mathsf{Dwn} = \mathcal{U}\mathcal{F}\colon \mathsf{Ord} \longrightarrow \mathsf{Ord}$ results, and its action on objects is $P \mapsto \mathsf{Dwn}(P)$. In a similar manner, because the forgetful 2-functor $\mathcal{V}\colon \mathsf{Inf} \longrightarrow \mathsf{Ord}$ admits a left 2-adjoint $\mathcal{G}\colon \mathsf{Ord} \longrightarrow \mathsf{Inf}$, their composition produces a 2-monad $\mathsf{Up} = \mathcal{V}\mathcal{G}\colon \mathsf{Ord} \longrightarrow \mathsf{Ord}$, whose action on objects is $Q \mapsto \mathsf{Up}(Q)$. Now it turns out that the composition of these 2-monads, $\mathcal{S} := \mathsf{Up}\mathsf{Dwn}\colon \mathsf{Ord} \longrightarrow \mathsf{Ord}$, is again a 2-monad, and its action on objects is thus $P \mapsto \mathsf{Up}(\mathsf{Dwn}(P))$: it is the **composite power monad** on Ord.

In this short note we show how **the double power monad and the composite power monad are the same**. We prove this in the generality of quantaloid-enriched categories, of which not only ordered sets but also metric spaces [Lawvere, 1973], partial metric spaces [Höhle and Kubiak, 2011; Stubbe, 2013], sheaves [Walters, 1982], fuzzy sets [Stubbe, 2013], and several other structures, are instances. In this generality, the downsets/upsets of an orderd set must be replaced by contravariant/covariant presheaves on a quantaloid-enriched category. This result is due to U. Höhle [2013] (see his Theorem 5.10), for whom it is a stepping stone towards a definition of 'quantaloid-enriched topological spaces', but our (perhaps somewhat more conceptual) proof technique is different from his[1]. Putting the **classifying property** of enriched presheaf categories central, our treatment of the double power monad follows easily from R. Street's [2012] characterisation of the **core of an adjunction**; and the theory surrounding the composite power monad is, quite naturally, that of J. Beck's [1969] **distributive laws**. A simple description (in fact, several equivalent descriptions, thanks to [Stubbe, 2007]) of the algebras of the double/composite power monad follows from all this: they are precisely the completely co-distributive complete categories; and the homomorphisms between such algebras are precisely the bicontinuous functors.

2. Notations

Throughout this note we shall use the definitions and notations for quantaloid-enriched categories, distributors and functors, as in [Stubbe, 2005]. All preliminaries for this note can be found either in that paper or in [Stubbe, 2007]. For the sake of readability, we quickly recall a few notational conventions.

We fix a base **quantaloid** $\mathcal{Q}$ and assume it to be small. Its arrows are typically $f\colon A\longrightarrow B$, $g\colon B\longrightarrow C$; composites are written as $g\circ f\colon A\longrightarrow C$; identities are $1_A\colon A\longrightarrow A$. Local suprema are written as $\bigvee_i f_i\colon A\longrightarrow B$, and recall that composition distributes on both sides over suprema. The resulting right adjoints to composition are denoted by $f\circ -\dashv [f,-]$ and $-\circ f\dashv \{f,-\}$.

A $\mathcal{Q}$**-category** $\mathbb{A}$ is determined by a set $\mathbb{A}_0$ of objects, a type-function $t\colon \mathbb{A}_0\longrightarrow \mathcal{Q}_0$, and $\mathbb{A}_0\times\mathbb{A}_0$ hom-arrows $\mathbb{A}(a',a)\colon ta\longrightarrow ta'$ in $\mathcal{Q}$; this data is subject to transitivity and reflexivity axioms. A **distributor** $\Phi\colon \mathbb{A}\mathrel{-\!\!\circ\!\!\rightarrow}\mathbb{B}$ is an $\mathbb{B}_0\times\mathbb{A}_0$ matrix of arrows $\Phi(b,a)\colon ta\longrightarrow tb$ in $\mathcal{Q}$, subject to action axioms. The composition of $\Phi\colon \mathbb{A}\mathrel{-\!\!\circ\!\!\rightarrow}\mathbb{B}$ and $\Psi\colon \mathbb{B}\mathrel{-\!\!\circ\!\!\rightarrow}\mathbb{C}$ is written as $\Psi\otimes\Phi\colon \mathbb{A}\mathrel{-\!\!\circ\!\!\rightarrow}\mathbb{C}$ and computed with a "matrix multiplication formula"; and distributors are ordered "elementwise". A (large) quantaloid $\mathsf{Dist}(\mathcal{Q})$ of $\mathcal{Q}$-enriched categories and distributors results; here too, adjoints to composition are written as $\Phi\otimes -\dashv[\Phi,-]$ and $-\otimes\Phi\dashv\{\Phi,-\}$. A **functor** $F\colon \mathbb{A}\longrightarrow\mathbb{B}$ is a type-preserving function on objects satisfying a functoriality axiom. Functors compose in the obvious manner, so a category $\mathsf{Cat}(\mathcal{Q})$ results. Every functor $F\colon \mathbb{A}\longrightarrow\mathbb{B}$ between $\mathcal{Q}$-categories

[1] As I read a preprint version of Höhle's paper, which he had sent to me in July 2012 prior to submitting it for publication, I found it hard to follow his quite computational proof. Of course the setting – free (co)completions of quantaloid-enriched categories – appealed to me, so I decided that, to fully understand Höhle's theorem, I had to work out "my own proof" in a style and language more suited to my background. The result of my work is contained in this note, of which I sent an earlier version (essentially the current text minus its Introduction) to Höhle in October 2012. Today, almost a year later, I have decided to "prepublish" this note in the *Cahiers du LMPA*: for I believe that the presentation of the material here lends itself to interesting future developments.

determines an adjunction in $\mathsf{Dist}(\mathcal{Q})$:

$$\begin{tikzcd} \mathbb{A} \arrow[rr, bend left, "\mathbb{B}(-,F-)", "\circ" marking] \arrow[rr, phantom, "\perp"] & & \mathbb{B} \arrow[ll, bend left, "\mathbb{B}(F-,-)", "\circ" marking] \end{tikzcd}.$$

The assignment $F \mapsto \mathbb{B}(-,F-)$ (resp. $F \mapsto \mathbb{B}(F-,-)$) extends to a covariant (contravariant) inclusion of $\mathsf{Cat}(\mathcal{Q})$ in $\mathsf{Dist}(\mathcal{Q})$. In doing so, $\mathsf{Cat}(\mathcal{Q})$ inherits the 2-cells of $\mathsf{Dist}(\mathcal{Q})$: explicitly, for $F, G\colon \mathbb{A} \longrightarrow \mathbb{B}$ we put

$$F \leq G \stackrel{\text{def.}}{\iff} \mathbb{B}(-,F-) \leq \mathbb{B}(-,G-) \iff \forall a \in \mathbb{A} : 1_{ta} \leq \mathbb{B}(Fa, Ga).$$

In particular is the underlying order of a category $\mathbb{A}$ defined by $a \leq a' \iff 1_{ta} \leq \mathbb{A}(a,a')$.

A **contravariant presheaf** of type X on $\mathbb{A}$ is a distributor $\phi\colon \mathbb{1}_X \mathrel{\circ\!\!\rightarrow} \mathbb{A}$; here, $\mathbb{1}_X$ is the one-object $\mathcal{Q}$-category whose single hom-arrow is 1_X. The $\mathcal{Q}$-category of all contravariant presheaves is $\mathcal{P}\mathbb{A}$, with hom-arrows $\mathcal{P}\mathbb{A}(\psi,\phi) = [\psi,\phi]$. The Yoneda embedding is the functor $Y_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathcal{P}\mathbb{A}\colon a \mapsto \mathbb{A}(-,a)$. The Yoneda lemma says that, for any $a \in \mathbb{A}$ and $\phi \in \mathcal{P}\mathbb{A}$, $\mathcal{P}\mathbb{A}(Y_{\mathbb{A}}a, \phi) = \phi(a)$.

A **covariant presheaf** of type X on $\mathbb{A}$ is a distributor $\phi\colon \mathbb{A} \mathrel{\circ\!\!\rightarrow} *_X$. The $\mathcal{Q}$-category of all covariant presheaves is $\mathcal{P}^\dagger\mathbb{A}$, with hom-arrows $\mathcal{P}^\dagger\mathbb{A}(\psi,\phi) = \{\phi,\psi\}$. Note the permutation of the arguments—it is necessary to make $\mathcal{P}^\dagger\mathbb{A}$ a $\mathcal{Q}$-category (and not a $\mathcal{Q}^{\mathsf{op}}$-category)—which implies in particular that the underlying order of $\mathcal{P}^\dagger\mathbb{A}$ is "odd": $\psi \leq \phi$ as objects of $\mathcal{P}^\dagger\mathbb{A}$ if and only if $\phi \leq \psi$ as arrows in $\mathsf{Dist}(\mathcal{Q})$. The Yoneda embedding is $Y^\dagger_{\mathbb{A}}\colon \mathbb{A} \longrightarrow \mathcal{P}^\dagger\mathbb{A}\colon a \mapsto \mathbb{A}(a,-)$, and the Yoneda lemma says that $\mathcal{P}^\dagger\mathbb{A}(\phi, Y^\dagger_{\mathbb{A}}a) = \psi(a)$.

Proposition 2.1 *Every functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ determines adjunctions*

$$\begin{tikzcd} \mathcal{P}\mathbb{A} \arrow[rr, bend left, "\mathcal{P}_l F"] \arrow[rr, bend right, "\mathcal{P}_* F"'] & & \mathcal{P}\mathbb{B} \arrow[ll, "\mathcal{P}_r F"', "\perp", "\perp"'] \end{tikzcd} \qquad \textit{and} \qquad \begin{tikzcd} \mathcal{P}^\dagger\mathbb{A} \arrow[rr, "\mathcal{P}^\dagger_r F", "\top"', "\top"] & & \mathcal{P}^\dagger\mathbb{B} \arrow[ll, bend right, "\mathcal{P}^\dagger_* F"'] \arrow[ll, bend left, "\mathcal{P}^\dagger_l F"] \end{tikzcd}$$

in $\mathsf{Cat}(\mathcal{Q})$*, where*

$$\begin{array}{ll} \mathcal{P}_l F(\phi) := \mathbb{B}(-,F-) \otimes \phi & \mathcal{P}^\dagger_r F(\phi) := \phi \otimes \mathbb{B}(F-,-) \\ \mathcal{P}_r F(\psi) := \mathbb{B}(F-,-) \otimes \psi & \mathcal{P}^\dagger_l F(\psi) := \psi \otimes \mathbb{B}(-,F-) \\ \mathcal{P}_* F(\phi) := [\mathbb{B}(F-,-),\phi] & \mathcal{P}^\dagger_* F(\phi) := \{\mathbb{B}(-,F-),\phi\} \end{array}$$

These actions on arrows in $\mathsf{Cat}(\mathcal{Q})$ *determine 2-functors*

$$\begin{array}{ll} \mathcal{P}_l\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q}) & \mathcal{P}^\dagger_l\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})^{\mathsf{coop}} \\ \mathcal{P}_r\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})^{\mathsf{coop}} & \mathcal{P}^\dagger_r\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q}) \\ \mathcal{P}_*\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q}) & \mathcal{P}^\dagger_*\colon \mathsf{Cat}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})^{\mathsf{coop}} \end{array}$$

that satisfy the following equalities:

$$\begin{array}{lll}
\mathcal{P}_l\mathcal{P}_r = \mathcal{P}_r\mathcal{P}_l & \mathcal{P}_*\mathcal{P}_l = \mathcal{P}_r\mathcal{P}_r = \mathcal{P}_l\mathcal{P}_* & \mathcal{P}_r\mathcal{P}_* = \mathcal{P}_*\mathcal{P}_r \\
\mathcal{P}^\dagger_l\mathcal{P}_r = \mathcal{P}^\dagger_r\mathcal{P}_l & \mathcal{P}^\dagger_*\mathcal{P}_l = \mathcal{P}^\dagger_r\mathcal{P}_r = \mathcal{P}^\dagger_l\mathcal{P}_* & \mathcal{P}^\dagger_r\mathcal{P}_* = \mathcal{P}^\dagger_*\mathcal{P}_r \\
\mathcal{P}_l\mathcal{P}^\dagger_r = \mathcal{P}_r\mathcal{P}^\dagger_l & \mathcal{P}_*\mathcal{P}^\dagger_l = \mathcal{P}_r\mathcal{P}^\dagger_r = \mathcal{P}_l\mathcal{P}^\dagger_* & \mathcal{P}_r\mathcal{P}^\dagger_* = \mathcal{P}_*\mathcal{P}^\dagger_r \\
\mathcal{P}^\dagger_l\mathcal{P}^\dagger_r = \mathcal{P}^\dagger_r\mathcal{P}^\dagger_l & \mathcal{P}^\dagger_*\mathcal{P}^\dagger_l = \mathcal{P}^\dagger_r\mathcal{P}^\dagger_r = \mathcal{P}^\dagger_l\mathcal{P}^\dagger_* & \mathcal{P}^\dagger_r\mathcal{P}^\dagger_* = \mathcal{P}^\dagger_*\mathcal{P}^\dagger_r
\end{array}$$

Proof : First we perform some generally valid computations with liftings/extensions through left/right adjoints, in the quantaloid $\mathsf{Dist}(\mathcal{Q})$. Given $\phi \in \mathcal{P}\mathbb{A}$ and $\psi \in \mathcal{P}\mathbb{B}$, we have that

$$\phi \otimes \mathbb{B}(-, F-) \leq \psi \iff \phi \leq [\mathbb{B}(-, F-), \psi].$$

Because $\mathbb{B}(-, F-) \dashv \mathbb{B}(F-, -)$ in $\mathsf{Dist}(\mathcal{Q})$, it furthermore follows that

$$[\mathbb{B}(-, F-), \psi] = \mathbb{B}(F-, -) \otimes \psi,$$

and therefore also

$$\mathbb{B}(F-, -) \otimes \psi \leq \phi \iff \psi \leq [\mathbb{B}(F-, -), \phi].$$

This proves exactly that $\mathcal{P}_lF \dashv \mathcal{P}_rF \dashv \mathcal{P}_*F$. The results for the covariant presheaves are dual.

The 2-functoriality of $\mathcal{P}_l$ etc. is easy to see; one must only be a bit careful with the direction of 2-cells and remember that $\mathcal{P}^\dagger\mathbb{A}$ and $\mathcal{P}^\dagger\mathbb{B}$ have an "odd" underlying order. Each of these 2-functors therefore preserves adjunctions. Applying $\mathcal{P}_l$ to $\mathcal{P}_lF \dashv \mathcal{P}_rF \dashv \mathcal{P}_*F$ gives $\mathcal{P}_l\mathcal{P}_lF \dashv \mathcal{P}_l\mathcal{P}_rF \dashv \mathcal{P}_l\mathcal{P}_*F$; but on the other hand we know that $\mathcal{P}_l\mathcal{P}_lF \dashv \mathcal{P}_r\mathcal{P}_lF \dashv \mathcal{P}_*\mathcal{P}_lF$ too; so uniqueness of adjoints implies that $\mathcal{P}_l\mathcal{P}_rF = \mathcal{P}_r\mathcal{P}_lF$ and $\mathcal{P}_l\mathcal{P}_*F = \mathcal{P}_*\mathcal{P}_lF$. Similar arguments work for the other equations. $\square$

Note how both Yoneda lemmas can be written in terms of the 2-functors from Proposition 2.1:

Lemma 2.2 *For any $\mathcal{Q}$-category $\mathbb{A}$ we have that $\mathcal{P}_rY_{\mathbb{A}} \circ Y_{\mathcal{P}\mathbb{A}} = 1_{\mathcal{P}\mathbb{A}}$ and $\mathcal{P}^\dagger_l Y^\dagger_{\mathbb{A}} \circ Y^\dagger_{\mathcal{P}^\dagger\mathbb{A}} = 1_{\mathcal{P}^\dagger\mathbb{A}}$.*

3. Double power monad

Both $\mathcal{P}\mathbb{A}$ and $\mathcal{P}^\dagger\mathbb{A}$ enjoy a universal property: they classify distributors, but each in a different manner. Precisely:

1. $\mathsf{Dist}(\mathcal{Q})(\mathbb{B}, \mathbb{A}) \longrightarrow \mathsf{Cat}(\mathcal{Q})(\mathbb{B}, \mathcal{P}\mathbb{A})\colon (\Phi\colon \mathbb{B} \mathrel{-\!\!\circ\!\!\to} \mathbb{A}) \mapsto (F_\Phi\colon \mathbb{B} \longrightarrow \mathcal{P}\mathbb{A}\colon b \mapsto \Phi(-, b))$ is an isomorphism of ordered sets, with inverse $(F\colon \mathbb{B} \longrightarrow \mathcal{P}\mathbb{A}) \mapsto (\Phi_F\colon \mathbb{B} \mathrel{-\!\!\circ\!\!\to} \mathbb{A}, \Phi(a, b) = F(b)(a))$,
2. $\mathsf{Dist}(\mathcal{Q})(\mathbb{B}, \mathbb{A}) \longrightarrow \mathsf{Cat}(\mathcal{Q})(\mathbb{A}, \mathcal{P}^\dagger\mathbb{B})^{\mathsf{op}}\colon (\Psi\colon \mathbb{B} \mathrel{-\!\!\circ\!\!\to} \mathbb{A}) \mapsto (G_\Psi\colon \mathbb{A} \longrightarrow \mathcal{P}^\dagger\mathbb{B}\colon a \mapsto \Psi(a, -))$ is an isomorphism of ordered sets, with inverse $(G\colon \mathbb{A} \longrightarrow \mathcal{P}^\dagger\mathbb{B}) \mapsto (\Psi_G\colon \mathbb{B} \mathrel{-\!\!\circ\!\!\to} \mathbb{A}, \Psi(a, b) = G(a)(b))$.

Composing (and turning upside down) these isomorphisms of ordered sets, we get an isomorphism

$$\pi_{\mathbb{A},\mathbb{B}}\colon \mathsf{Cat}(\mathcal{Q})^{\mathsf{coop}}(\mathcal{P}\mathbb{A}, B) = \mathsf{Cat}(\mathcal{Q})(\mathbb{B}, \mathcal{P}\mathbb{A})^{\mathsf{op}} \cong \mathsf{Dist}(\mathcal{Q})(\mathbb{B}, \mathbb{A})^{\mathsf{op}} \cong \mathsf{Cat}(\mathcal{Q})(\mathbb{A}, \mathcal{P}^\dagger\mathbb{B})$$

where "coop" means (as usual) taking formally opposite 1-cells and 2-cells. Explicitly,

$$\pi_{\mathbb{A},\mathbb{B}}(F\colon \mathbb{B} \longrightarrow \mathcal{P}\mathbb{A}) = (\pi(F)\colon \mathbb{A} \longrightarrow \mathcal{P}^\dagger\mathbb{B}\colon a \mapsto F(-)(a))$$

$$\pi^{-1}_{\mathbb{A},\mathbb{B}}(G\colon \mathbb{A} \longrightarrow \mathcal{P}^\dagger\mathbb{B}) = (\pi^{-1}_{\mathbb{A},\mathbb{B}}(G)\colon \mathbb{B} \longrightarrow \mathcal{P}\mathbb{A}\colon b \mapsto G(-)(b)).$$

Proposition 3.1 *The object functions $\mathbb{A} \mapsto \mathcal{P}\mathbb{A}$ and $\mathbb{A} \mapsto \mathcal{P}^\dagger\mathbb{A}$ together with the family of order-isomorphisms $\pi_{\mathbb{A},\mathbb{B}} \colon \mathsf{Cat}(\mathcal{Q})^{\mathsf{coop}}(\mathcal{P}\mathbb{A}, B) \longrightarrow \mathsf{Cat}(\mathcal{Q})(\mathbb{A}, \mathcal{P}^\dagger\mathbb{B})$ consitute the* **core of an adjunction** *in the sense of [Street, 2012]. This means that these object functions uniquely extend to a pair of functors, making the family of order-isomorphisms natural (in $\mathbb{A}$ and $\mathbb{B}$) so that it expresses the adjunction of this pair of functors. In fact, this adjunction turns out to be precisely*

$$\begin{tikzcd} \mathsf{Cat}(\mathcal{Q}) \arrow[rr, bend left, "\mathcal{P}_r"] \arrow[rr, phantom, "\perp"] & & \mathsf{Cat}(\mathcal{Q})^{\mathsf{coop}} \arrow[ll, bend left, "\mathcal{P}_l^\dagger"] \end{tikzcd},$$

with unit $\beta_\mathbb{A} \colon \mathbb{A} \longrightarrow \mathcal{P}^\dagger\mathcal{P}\mathbb{A} \colon a \mapsto 1_{\mathcal{P}\mathbb{A}}(-)(a)$ and counit $\alpha_\mathbb{A} \colon \mathbb{A} \longrightarrow \mathcal{P}\mathcal{P}^\dagger\mathbb{A} \colon a \mapsto 1_{\mathcal{P}^\dagger\mathbb{A}}(-)(a)$.

Proof : We begin with some definitions (using the notations of [Street, 2012]):

1. $\beta_\mathbb{A} \in \mathsf{Cat}(\mathcal{Q})(\mathbb{A}, \mathcal{P}^\dagger\mathcal{P}\mathbb{A})$ is $\pi_{\mathbb{A},\mathcal{P}\mathbb{A}}(1_{\mathcal{P}\mathbb{A}})$,

2. $\alpha_\mathbb{A} \in \mathsf{Cat}(\mathcal{Q})^{\mathsf{coop}}(\mathcal{P}\mathcal{P}^\dagger\mathbb{A}, \mathbb{A})$ is $\pi_{\mathcal{P}^\dagger\mathbb{A},\mathbb{A}}(1_{\mathcal{P}^\dagger\mathbb{A}})$,

3. $\mathcal{U}_{\mathbb{A},\mathbb{B}} \colon \mathsf{Cat}(\mathcal{Q})^{\mathsf{coop}}(\mathbb{A}, \mathbb{B}) \longrightarrow \mathsf{Cat}(\mathcal{Q})(\mathcal{P}^\dagger\mathbb{A}, \mathcal{P}^\dagger\mathbb{B})$ sends $F \colon \mathbb{B} \longrightarrow \mathbb{A}$ to $\pi_{\mathcal{P}^\dagger\mathbb{A},\mathbb{B}}(\alpha_\mathbb{A} \circ F)$,

4. $\mathcal{F}_{\mathbb{A},\mathbb{B}} \colon \mathsf{Cat}(\mathcal{Q})(\mathbb{A}, \mathbb{B}) \longrightarrow \mathsf{Cat}(\mathcal{Q})^{\mathsf{coop}}(\mathcal{P}\mathbb{A}, \mathcal{P}\mathbb{B})$ sends $F \colon \mathbb{A} \longrightarrow \mathbb{B}$ to $\pi^{-1}_{\mathbb{A},\mathcal{P}\mathbb{B}}(\beta_\mathbb{B} \circ F)$.

It is fairly easy to compute that, for any $G \colon \mathbb{C} \longrightarrow \mathbb{B}$ and $F \colon \mathbb{B} \longrightarrow \mathcal{P}\mathbb{A}$,

$$\pi_{\mathbb{A},\mathbb{C}}(F \circ G) = \mathcal{U}_{\mathbb{C},\mathbb{B}}(G) \circ \pi_{\mathbb{A},\mathbb{B}}(F),$$

and (equivalently), for any $F \colon \mathbb{A} \longrightarrow \mathbb{B}$ and $G \colon \mathbb{B} \longrightarrow \mathcal{P}^\dagger\mathbb{C}$,

$$\pi^{-1}_{\mathbb{A},\mathbb{C}}(G \circ F) = \mathcal{F}_{\mathbb{A},\mathbb{B}}(F) \circ \pi^{-1}_{\mathbb{B},\mathbb{B}}(G).$$

Street's [2012] general theorem then implies that there is an order-enriched adjunction $\mathcal{F} \dashv \mathcal{U}$ with unit β and counit α. Explicit computations furthermore show that, in the case at hand, $\mathcal{F} = \mathcal{P}_r$ and $\mathcal{U} = \mathcal{P}_l^\dagger$. $\Box$

The unit and counit of the ajunction in the above Proposition can be understood in terms of Yoneda embeddings:

Lemma 3.2 *With notations as in the proof of Proposition 3.1 we have:*

- $\beta_\mathbb{A} = Y^\dagger_{\mathcal{P}\mathbb{A}} \circ Y_\mathbb{A} = \mathcal{P}_r^\dagger(Y_\mathbb{A}) \circ Y^\dagger_\mathbb{A}$,

- $\alpha_\mathbb{A} = Y_{\mathcal{P}^\dagger\mathbb{A}} \circ Y^\dagger_\mathbb{A} = \mathcal{P}_l(Y^\dagger_\mathbb{A}) \circ Y_\mathbb{A}$.

Proof : For $a \in \mathbb{A}$ and $\phi \in \mathcal{P}\mathbb{A}$ we have by definition that $\beta_\mathbb{A}(a)(\phi) = \phi(a)$, which is equal to $\mathcal{P}^\dagger\mathbb{A}(\phi, Y^\dagger_\mathbb{A} a)$ by Yoneda, hence further equal to $Y_{\mathcal{P}^\dagger\mathbb{A}}(Y^\dagger_\mathbb{A}(a))(\phi)$. The equality $Y^\dagger_{\mathcal{P}\mathbb{A}} \circ Y_\mathbb{A} = \mathcal{P}^\dagger_r(Y_\mathbb{A}) \circ Y^\dagger_\mathbb{A}$ follows from naturality[2] of $Y^\dagger$. (Similar for $\alpha_\mathbb{A}$.) $\Box$

By "abstact nonsense", the above adjunction determines a monad like so:

[2]Indeed, the Yoneda embeddings form a natural transformation, as recalled in the beginning of Section 4.

Theorem 3.3 *There is an order-enriched monad* $(\mathcal{T}, u, m)$ *on* $\mathsf{Cat}(\mathcal{Q})$ *as follows:*

- *the functor is* $\mathcal{T} := \mathcal{P}_l^\dagger \mathcal{P}_r = \mathcal{P}_r^\dagger \mathcal{P}_l$,
- *the multiplications are* $m_\mathbb{A} := \mathcal{P}_l^\dagger(\alpha_{\mathcal{P}\mathbb{A}}) = \mathcal{P}_l^\dagger(Y_{\mathcal{P}\mathbb{A}}^\dagger) \circ \mathcal{P}_l^\dagger(Y_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}}) = \mathcal{P}_l^\dagger(Y_{\mathcal{P}\mathbb{A}}) \circ \mathcal{P}_l^\dagger(\mathcal{P}_l(Y_{\mathcal{P}\mathbb{A}}^\dagger))$,
- *the units are* $u_\mathbb{A} := \beta_\mathbb{A} = Y_{\mathcal{P}\mathbb{A}}^\dagger \circ Y_\mathbb{A} = \mathcal{P}_r^\dagger(Y_\mathbb{A}) \circ Y_\mathbb{A}^\dagger$.

4. Composite of power monads

Writing $\mathsf{Cocont}(\mathcal{Q})$ for the 2-category of *cocomplete* $\mathcal{Q}$*-categories and cocontinuous functors*, the forgetful $\mathcal{U}\colon \mathsf{Cocont}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$ has a left 2-adjoint, namely $\mathcal{P}_l$. The unit for this adjunction is the Yoneda embedding $Y_\mathbb{A}\colon \mathbb{A} \longrightarrow \mathcal{P}\mathbb{A}$. The counit is given by "supremum": if $\mathbb{B}$ is a cocomplete category, then $\sup_\mathbb{B}\colon \mathcal{P}\mathbb{B} \longrightarrow \mathbb{B}$ is the map that sends $\phi \in \mathcal{P}\mathbb{B}$ to the enriched colimit[3] $\mathrm{colim}(\phi, 1_\mathbb{B})$. The 2-monad on $\mathsf{Cat}(\mathcal{Q})$ determined by this adjunction is:

- the functor $\mathcal{P}_l$,
- with multiplications $\sup_{\mathcal{P}\mathbb{A}}\colon \mathcal{P}\mathcal{P}\mathbb{A} \longrightarrow \mathcal{P}\mathbb{A}$,
- and units $Y_\mathbb{A}\colon \mathbb{A} \longrightarrow \mathcal{P}\mathbb{A}$.

It is a so-called KZ-doctrine [Kock, 1995]: the inequality $\mathcal{P}_l(Y_\mathbb{A}) \le Y_{\mathcal{P}\mathbb{A}}$ holds. This implies in particular that the following statements are equivalent[4]:

1. $\mathbb{B}$ is cocomplete,
2. $\mathbb{B}$ is injective[5] wrt. fully faithful functors in $\mathsf{Cat}(\mathcal{Q})$,
3. $Y_\mathbb{B}$ has a left inverse in $\mathsf{Cat}(\mathcal{Q})$,
4. $Y_\mathbb{B}$ has a left adjoint in $\mathsf{Cat}(\mathcal{Q})$;

in particular is the left adjoint/left inverse to $Y_\mathbb{B}$ then exactly $\sup_\mathbb{B}$. Indeed, the equivalence $(1 \Leftrightarrow 4)$ is in [Stubbe, 2005, 6.10], so it suffices to prove that $(1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4)$. First, consider functors between $\mathcal{Q}$-categories $F\colon \mathbb{A} \longrightarrow \mathbb{C}$ and $G\colon \mathbb{A} \longrightarrow \mathbb{B}$, with F fully faithful and $\mathbb{B}$ cocomplete. The $\mathbb{B}(F-,-)$-weighted colimit of G provides a functor $H\colon \mathbb{B} \longrightarrow \mathbb{B}$, and from the general rules for computing a weighted colimit it follows – with the aid of F's fully faithfulness in the second equality – that, for any $a \in \mathbb{A}$, $H(Fa) = \mathrm{colim}(\mathbb{C}(F-, Fa), G) = \mathrm{colim}(\mathbb{A}(-, a), G) \cong Ga$. So

[3]In general, given $\Phi\colon \mathbb{A} \,-\!\!\!\circ\!\!\!\rightarrow \mathbb{B}$ and $F\colon \mathbb{B} \longrightarrow \mathbb{C}$, the Φ-weighted colimit of F is, whenever it exists, the essentially unique functor $\mathrm{colim}(\Phi, F)\colon \mathbb{A} \longrightarrow \mathbb{C}$ determined by the universal property that $[\Phi, \mathbb{C}(F-,-)] = \mathbb{C}(\mathrm{colim}(\Phi, F)-,-)$.

[4]This is the content of a two-page note by the author, entitled "Cocomplete $\mathcal{Q}$-categories are precisely the injectives wrt. fully faithful functors", distributed privately in June 2006; for completeness' sake we repeat the argument here.

[5]By definition, $\mathbb{B}$ is *injective wrt. fully faithful functors* in $\mathsf{Cat}(\mathcal{Q})$ if, for every fully faithful $F\colon \mathbb{A} \longrightarrow \mathbb{C}$ and any other $G\colon \mathbb{A} \longrightarrow \mathbb{B}$, there exists a (not necessarily unique) $H\colon \mathbb{C} \longrightarrow \mathbb{B}$ such that $H \circ F \cong G$. It is general "abstract nonsense" that any retract of an injective is injective; so we could have added a line here saying that "$\mathbb{B}$ is a retract of an injective".

$\mathbb{B}$ is injective wrt. fully faithful functors. Next, from the mere injectivity of $\mathbb{B}$ and $Y_{\mathbb{B}}$'s fully faithfulness, the diagram

$$\begin{tikzcd}
\mathbb{B} \arrow[r, "Y_{\mathbb{B}}"] \arrow[dr, "1_{\mathbb{B}}"'] & \mathcal{P}\mathbb{B} \arrow[d, dotted] \\
 & \mathbb{B}
\end{tikzcd}$$

exhibits a left inverse to $Y_{\mathbb{B}}$. Finally, suppose that $L \circ Y_{\mathbb{B}} \cong 1_{\mathbb{B}}$. For any $\phi \in \mathcal{P}\mathbb{B}$ we then have that $\phi = \mathcal{P}\mathbb{B}(Y_{\mathbb{B}}-, \phi) \leq \mathbb{B}(LY_{\mathbb{B}}-, L\phi) = \mathbb{B}(-, L\phi) = Y_{\mathbb{B}}L\phi$, hence $Y_{\mathbb{B}} \circ L \geq 1_{\mathcal{P}\mathbb{B}}$, which suffices to show that $L \dashv Y_{\mathbb{B}}$.

A similar story can be told about $\mathsf{Cont}(\mathcal{Q})$, the 2-category of *complete $\mathcal{Q}$-categories and continuous functors*, the inclusion $\mathcal{U}^{\dagger} \colon \mathsf{Cont}(\mathcal{Q}) \longrightarrow \mathsf{Cat}(\mathcal{Q})$ and its left 2-adjoint $\mathcal{P}^{\dagger}_{r}$. The result is a 2-monad on $\mathsf{Cat}(\mathcal{Q})$ with:

- the functor $\mathcal{P}^{\dagger}_{r}$,

- with multiplications $\inf_{\mathcal{P}^{\dagger}\mathbb{A}} \colon \mathcal{P}^{\dagger}\mathcal{P}^{\dagger}\mathbb{A} \longrightarrow \mathcal{P}^{\dagger}\mathbb{A}$,

- and units $Y^{\dagger}_{\mathbb{A}} \colon \mathbb{A} \longrightarrow \mathcal{P}^{\dagger}\mathbb{A}$.

It is a co-KZ-doctrine, because $\mathcal{P}^{\dagger}_{r}(Y^{\dagger}_{\mathbb{A}}) \geq Y^{\dagger}_{\mathcal{P}^{\dagger}\mathbb{A}}$. This gives many characterisations of complete categories, entirely dual to the case of cocomplete categories.

The Yoneda lemmas, as stated in Lemma 2.2, thus have the following incarnation:

Lemma 4.1 *For any $\mathcal{Q}$-category $\mathbb{A}$,* $\sup_{\mathcal{P}\mathbb{A}} = \mathcal{P}_r Y_{\mathbb{A}}$ *and* $\inf_{\mathcal{P}^{\dagger}\mathbb{A}} = \mathcal{P}^{\dagger}_{l} Y^{\dagger}_{\mathbb{A}}$.

Proof: The free $\mathcal{P}$-algebra $\mathcal{P}\mathbb{A}$ is necessarily cocomplete, so $\sup_{\mathcal{P}\mathbb{A}}$ is the left adjoint/inverse to $Y_{\mathcal{P}\mathbb{A}}$. The statement in Lemma 2.2, saying that $\mathcal{P}_r Y_{\mathbb{A}}$ is left inverse to $Y_{\mathcal{P}\mathbb{A}}$, is thus equivalent to the statement that $\sup_{\mathcal{P}\mathbb{A}} = \mathcal{P}_r Y_{\mathbb{A}}$. Dual for the other statement. □

A $\mathcal{Q}$-category $\mathbb{B}$ is complete if and only if it is cocomplete [Stubbe, 2005]; indeed, defining $U_{\mathbb{B}} \colon \mathcal{P}\mathbb{B} \longrightarrow \mathcal{P}^{\dagger}\mathbb{B} \colon \phi \mapsto [\phi, \mathbb{B}]$ and $L_{\mathbb{B}} \colon \mathcal{P}^{\dagger}\mathbb{B} \longrightarrow \mathcal{P}\mathbb{B} \colon \psi \mapsto \{\psi, \mathbb{B}\}$, it follows that $\inf_{\mathbb{B}} = \sup_{\mathbb{B}} \circ L_{\mathbb{B}}$ and $\sup_{\mathbb{B}} = \inf_{\mathbb{B}} \circ U_{\mathbb{B}}$, if either $\sup_{\mathbb{B}}$ or $\inf_{\mathbb{B}}$ exists. Moreover, it is easily checked with an explicit computation that, for any $\mathbb{A}$, $U_{\mathbb{A}} = \mathcal{P}^{\dagger}_{l} Y_{\mathbb{A}} \circ Y^{\dagger}_{\mathcal{P}\mathbb{A}}$ and similarly $L_{\mathbb{A}} = \mathcal{P}_r Y^{\dagger}_{\mathbb{A}} \circ Y_{\mathcal{P}^{\dagger}\mathbb{A}}$. This implies that:

Lemma 4.2 *For any (co)complete $\mathbb{B}$,*

$$\begin{array}{llll}
\mathcal{P}^{\dagger}_{l} Y_{\mathbb{B}} = \mathcal{P}^{\dagger}_{r} \sup_{\mathbb{B}} & \mathcal{P}_{l} Y_{\mathbb{B}} = \mathcal{P}_{r} \sup_{\mathbb{B}} & \mathcal{P}_{r} Y_{\mathbb{B}} = \mathcal{P}_{*} \sup_{\mathbb{B}} & \mathcal{P}^{\dagger}_{r} Y_{\mathbb{B}} = \mathcal{P}^{\dagger}_{*} \sup_{\mathbb{B}} \\
\mathcal{P}^{\dagger}_{r} Y_{\mathbb{B}} = \mathcal{P}^{\dagger}_{l} \inf_{\mathbb{B}} & \mathcal{P}_{r} Y_{\mathbb{B}} = \mathcal{P}_{l} \inf_{\mathbb{B}} & \mathcal{P}_{*} Y_{\mathbb{B}} = \mathcal{P}_{r} \inf_{\mathbb{B}} & \mathcal{P}^{\dagger}_{*} Y_{\mathbb{B}} = \mathcal{P}^{\dagger}_{r} \inf_{\mathbb{B}}
\end{array}$$

Proof: We know that $\sup_{\mathbb{B}} \dashv Y_{\mathbb{B}}$, and by 2-functoriality of (say) $\mathcal{P}_l$ it follows that $\mathcal{P}_l \sup_{\mathbb{B}} \dashv \mathcal{P}_l Y_{\mathbb{B}}$ too. However, $\mathcal{P}_l \sup_{\mathbb{B}} \dashv \mathcal{P}_r \sup_{\mathbb{B}}$ too, so by uniqueness of adjoints we find that $\mathcal{P}_r \sup_{\mathbb{B}} = \mathcal{P}_l Y_{\mathbb{B}}$. Similar for the other equations. □

It is furthermore well known that, for $\mathbb{A}$ cocomplete, a functor $F \colon \mathbb{A} \longrightarrow \mathbb{B}$, is cocontinous if and only if it has a right adjoint: the right adjoint in case is exactly the pointwise left Kan extension of $1_{\mathbb{A}}$ along F, namely $F^{*} \colon \mathbb{B} \longrightarrow \mathbb{A} \colon b \mapsto \sup_{\mathbb{A}}(\mathbb{B}(F-, b))$. Therefore, if $F \colon \mathbb{A} \longrightarrow \mathbb{B}$

is any functor between $\mathcal{Q}$-categories, applying the 2-functor $\mathcal{P}^\dagger_r\colon \mathsf{Cat}(\mathcal{Q})\longrightarrow\mathsf{Cat}(\mathcal{Q})$ produces a cocontinuous functor $\mathcal{P}^\dagger_r F\colon \mathcal{P}^\dagger\mathbb{A}\longrightarrow\mathcal{P}^\dagger\mathbb{B}$ between cocomplete categories (with right adjoint $\mathcal{P}^\dagger_* F$). In particular does $\mathcal{P}^\dagger_r$ lift to the full subcategory $\mathsf{Cocont}(\mathcal{Q})$ of $\mathsf{Cat}(\mathcal{Q})$:

$$\begin{tikzcd}
\mathsf{Cat}(\mathcal{Q}) \arrow[r, "\mathcal{P}^\dagger_r"] & \mathsf{Cat}(\mathcal{Q}) \\
\mathsf{Cocont}(\mathcal{Q}) \arrow[u, "\mathcal{U}"] \arrow[r, dotted, "\mathcal{P}^\dagger_r"'] & \mathsf{Cocont}(\mathcal{Q}) \arrow[u, "\mathcal{U}"']
\end{tikzcd} \tag{1}$$

Because $\mathcal{P}_l$ is a KZ-doctrine, the forgetful functor $\mathcal{U}\colon \mathsf{Cocont}(\mathcal{Q})\longrightarrow\mathsf{Cat}(\mathcal{Q})$ is an embedding, so there can be at most one lifting of $\mathcal{P}^\dagger_r$ to $\mathsf{Cocont}(\mathcal{Q})$. In the speak of the theory of *distributive laws* [Beck, 1969], the existence of this (necessarily unique) lifting is equivalent to the existence of a (necessarily unique) distributive law of the *monad* $\mathcal{P}_l$ over the *endofunctor* $\mathcal{P}^\dagger_r$: an order-enriched natural transformation $\lambda\colon \mathcal{P}_l\circ\mathcal{P}^\dagger_r\Longrightarrow\mathcal{P}^\dagger_r\circ\mathcal{P}_l$ satisfying the commutativity of

$$\begin{tikzcd}
 & \mathcal{P}\mathcal{P}^\dagger\mathbb{A} \arrow[dd, "\lambda_{\mathbb{A}}"] & \mathcal{P}\mathcal{P}\mathcal{P}^\dagger\mathbb{A} \arrow[l, "\sup_{\mathcal{P}\mathcal{P}^\dagger\mathbb{A}}"'] \arrow[d, "\mathcal{P}_l\lambda_{\mathbb{A}}"] \\
\mathcal{P}^\dagger\mathbb{A} \arrow[ur, "Y_{\mathcal{P}^\dagger\mathbb{A}}"] \arrow[dr, "\mathcal{P}^\dagger_r Y_{\mathbb{A}}"'] & & \mathcal{P}\mathcal{P}^\dagger\mathcal{P}\mathbb{A} \arrow[d, "\lambda_{\mathcal{P}\mathbb{A}}"] \\
 & \mathcal{P}^\dagger\mathcal{P}\mathbb{A} & \mathcal{P}^\dagger\mathcal{P}\mathcal{P}\mathbb{A} \arrow[l, "\mathcal{P}^\dagger_r\sup_{\mathcal{P}\mathbb{A}}"]
\end{tikzcd} \tag{2}$$

Indeed, the commutative diagram in (1) determines (via the "calculus of mates") the distributive law as follows[6]:

$$\mathcal{P}\mathcal{P}^\dagger\mathbb{A}\xrightarrow{\mathcal{P}_l\mathcal{P}^\dagger_r Y_{\mathbb{A}}}\mathcal{P}\mathcal{P}^\dagger\mathcal{P}\mathbb{A}\xrightarrow{\sup_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}}}\mathcal{P}^\dagger\mathcal{P}\mathbb{A}$$

And conversely, the distributive law λ determines the lifting of $\mathcal{P}^\dagger_r$ to $\mathsf{Cocont}(\mathcal{Q})$ in the following sense: for any cocomplete $\mathbb{A}$, i.e. $\mathcal{P}_l$-algebra $\sup_{\mathbb{A}}\colon\mathcal{P}\mathbb{A}\longrightarrow\mathbb{A}$, the $\mathcal{P}_l$-algebra structure on the $\mathcal{P}^\dagger_r$-algebra $\inf_{\mathcal{P}^\dagger\mathbb{A}}\colon\mathcal{P}^\dagger\mathcal{P}^\dagger\mathbb{A}\longrightarrow\mathcal{P}^\dagger\mathbb{A}$ is

$$\mathcal{P}\mathcal{P}^\dagger\mathbb{A}\xrightarrow{\lambda_{\mathbb{A}}}\mathcal{P}^\dagger\mathcal{P}\mathbb{A}\xrightarrow{\mathcal{P}^\dagger_r\sup_{\mathbb{A}}}\mathcal{P}^\dagger\mathbb{A}$$

In other words, if $\mathbb{A}$ is cocomplete then $\sup_{\mathcal{P}^\dagger\mathbb{A}}=\mathcal{P}^\dagger_r\sup_{\mathbb{A}}\circ\lambda_{\mathbb{A}}$; in still other words, if $\mathbb{A}$ is cocomplete then $Y_{\mathcal{P}^\dagger\mathbb{A}}$ is right adjoint to $\mathcal{P}^\dagger_r\sup_{\mathbb{A}}\circ\lambda_{\mathbb{A}}$.

So, with a minimal effort, we exhibited a distributive law λ of the *monad* $\mathcal{P}_l$ over the *endofunctor* $\mathcal{P}^\dagger_r$: λ is compatible with the multiplication and the unit of the monad $\mathcal{P}_l$, as expressed by the commutativity of the diagram in (2). It takes a bit more effort to show that λ is also compatible with the multiplication and unit of the monad $\mathcal{P}^\dagger_r$.

[6]Note that $\lambda_{\mathbb{A}}$ is left adjoint to $\mathcal{P}_r\mathcal{P}^\dagger_r Y_{\mathbb{A}}\circ Y_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}}$.

Lemma 4.3 *The transformation λ satisfies the commutativity of*

$$
\begin{tikzcd}
 & \mathcal{P}\mathcal{P}^\dagger\mathbb{A} \arrow[dd, "\lambda_{\mathbb{A}}"] & & \mathcal{P}\mathcal{P}^\dagger\mathcal{P}^\dagger\mathbb{A} \arrow[ll, "\mathcal{P}_l \inf_{\mathcal{P}^\dagger\mathbb{A}}"'] \arrow[d, "\lambda_{\mathcal{P}^\dagger\mathbb{A}}"] \\
\mathcal{P}\mathbb{A} \arrow[ur, "\mathcal{P}_l Y^\dagger_{\mathbb{A}}"] \arrow[dr, "Y^\dagger_{\mathcal{P}\mathbb{A}}"'] & & & \mathcal{P}^\dagger\mathcal{P}\mathcal{P}^\dagger\mathbb{A} \arrow[d, "\mathcal{P}^\dagger_r \lambda_{\mathbb{A}}"] \\
 & \mathcal{P}^\dagger\mathcal{P}\mathbb{A} & & \mathcal{P}^\dagger\mathcal{P}^\dagger\mathcal{P}\mathbb{A} \arrow[ll, "\inf_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}}"]
\end{tikzcd}
$$

Proof: Using naturality of $Y^\dagger$ in the upper trapezoid, cocontinuity[7] of $Y^\dagger_{\mathcal{P}\mathbb{A}}$ in the lower trapezoid, the unit axiom for the monad $(\mathcal{P}_l, \sup_{\mathcal{P}\_}, Y)$ in the left-hand triangle, and the definition of λ in the right-hand triangle, the following diagram is seen to commute:

$$
\begin{tikzcd}
\mathcal{P}\mathbb{A} \arrow[rrr, "\mathcal{P}_l Y^\dagger_{\mathbb{A}}"] \arrow[dr, "\mathcal{P}_l Y_{\mathbb{A}}"] \arrow[dd, "1_{\mathcal{P}\mathbb{A}}"'] & & & \mathcal{P}\mathcal{P}^\dagger\mathbb{A} \arrow[dl, "\mathcal{P}_l \mathcal{P}^\dagger_r Y_{\mathbb{A}}"'] \arrow[dd, "\lambda_{\mathbb{A}}"] \\
 & \mathcal{P}\mathcal{P}\mathbb{A} \arrow[r, "\mathcal{P}_l Y^\dagger_{\mathcal{P}\mathbb{A}}"'] \arrow[dl, "\sup_{\mathcal{P}\mathbb{A}}"] & \mathcal{P}\mathcal{P}^\dagger\mathcal{P}\mathbb{A} \arrow[dr, "\sup_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}}"'] & \\
\mathcal{P}\mathbb{A} \arrow[rrr, "Y^\dagger_{\mathcal{P}\mathbb{A}}"'] & & & \mathcal{P}^\dagger\mathcal{P}\mathbb{A}
\end{tikzcd}
$$

The outer square is exactly the triangle in the statement of the proposition.

As for the rectangle in the statement of the proposition, we break it down in two parts. First we use the definition in the left-hand triangle, continuity of (the right adjoint) $\mathcal{P}^\dagger_r Y_{\mathbb{A}}$ in the upper trapezoid and cocontinuity[8] of $\inf_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}}$ in the lower trapezoid to check the commutativity of the following diagram:

$$
\begin{tikzcd}
\mathcal{P}^\dagger\mathcal{P}\mathbb{A} \arrow[dr, "\mathcal{P}_l \mathcal{P}^\dagger_r Y_{\mathbb{A}}"] \arrow[dd, "\lambda_{\mathbb{A}}"'] & & \mathcal{P}\mathcal{P}^\dagger\mathcal{P}^\dagger\mathbb{A} \arrow[ll, "\mathcal{P}_l \inf_{\mathcal{P}^\dagger\mathbb{A}}"'] \arrow[d, "\mathcal{P}_l \mathcal{P}^\dagger_r \mathcal{P}^\dagger_r Y_{\mathbb{A}}"] \\
 & \mathcal{P}\mathcal{P}^\dagger\mathcal{P}\mathbb{A} \arrow[dl, "\sup_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}}"] & \mathcal{P}\mathcal{P}^\dagger\mathcal{P}^\dagger\mathcal{P}\mathbb{A} \arrow[l, "\mathcal{P}_l \inf_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}}"] \arrow[d, "\sup_{\mathcal{P}^\dagger\mathcal{P}^\dagger\mathcal{P}\mathbb{A}}"] \\
\mathcal{P}^\dagger\mathcal{P}\mathbb{A} & & \mathcal{P}^\dagger\mathcal{P}^\dagger\mathcal{P}\mathbb{A} \arrow[ll, "\inf_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}}"]
\end{tikzcd}
$$

[7]Because $\mathcal{P}\mathbb{A}$ is (co)complete, $Y^\dagger_{\mathcal{P}\mathbb{A}}$ has a right adjoint, namely $\inf_{\mathcal{P}\mathbb{A}}$.

[8]Because $\inf_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}} = \mathcal{P}^\dagger_l Y^\dagger_{\mathcal{P}\mathbb{A}}$ by Lemma 4.1, it is a left adjoint, with right adjoint $\mathcal{P}^\dagger_r Y^\dagger_{\mathcal{P}\mathbb{A}}$.

Next, by naturality of Y in the upper-left square, cocontinuity[9] of $\mathcal{P}_r^\dagger\mathcal{P}_l\mathcal{P}_r^\dagger Y_\mathbb{A}$ in the lower-left square, cocontinuity[10] of $\mathcal{P}_r^\dagger\sup_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}}$ in the lower-right square, and the unit axiom for the monad $(\mathcal{P}_l, \sup_{\mathcal{P}-}, Y)$ in the upper-right triangle, and finally the definition of λ for the remaining bent arrows, we obtain the commutativity of:

$$\begin{tikzcd}
\mathcal{P}\mathcal{P}^\dagger\mathcal{P}^\dagger\mathbb{A} \arrow[r, "\mathcal{P}_l\mathcal{P}_r^\dagger\mathcal{P}_r^\dagger Y_\mathbb{A}"] \arrow[d, "\mathcal{P}_l\mathcal{P}_r^\dagger Y_{\mathcal{P}^\dagger\mathbb{A}}"] \arrow[dd, bend right=60, "\lambda_{\mathcal{P}^\dagger\mathbb{A}}"'] & \mathcal{P}\mathcal{P}^\dagger\mathcal{P}^\dagger\mathcal{P}\mathbb{A} \arrow[d, "\mathcal{P}_l\mathcal{P}_r^\dagger Y_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}}"] \arrow[dr, bend left, "1_{\mathcal{P}\mathcal{P}^\dagger\mathcal{P}^\dagger\mathcal{P}\mathbb{A}}"] & \\
\mathcal{P}\mathcal{P}^\dagger\mathcal{P}\mathcal{P}^\dagger\mathbb{A} \arrow[r, "\mathcal{P}_l\mathcal{P}_r^\dagger\mathcal{P}_l\mathcal{P}_r^\dagger Y_\mathbb{A}"] \arrow[d, "\sup_{\mathcal{P}^\dagger\mathcal{P}\mathcal{P}^\dagger\mathbb{A}}"] & \mathcal{P}\mathcal{P}^\dagger\mathcal{P}\mathcal{P}^\dagger\mathcal{P}\mathbb{A} \arrow[r, "\mathcal{P}_l\mathcal{P}_r^\dagger\sup_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}}"] \arrow[d, "\sup_{\mathcal{P}^\dagger\mathcal{P}\mathcal{P}^\dagger\mathcal{P}\mathbb{A}}"] & \mathcal{P}\mathcal{P}^\dagger\mathcal{P}^\dagger\mathcal{P}\mathbb{A} \arrow[d, "\sup_{\mathcal{P}^\dagger\mathcal{P}^\dagger\mathcal{P}\mathbb{A}}"] \\
\mathcal{P}^\dagger\mathcal{P}\mathcal{P}^\dagger\mathbb{A} \arrow[r, "\mathcal{P}_r^\dagger\mathcal{P}_l\mathcal{P}_r^\dagger Y_\mathbb{A}"'] \arrow[rr, bend right, "\mathcal{P}_r^\dagger\lambda_\mathbb{A}"'] & \mathcal{P}^\dagger\mathcal{P}\mathcal{P}^\dagger\mathcal{P}\mathbb{A} \arrow[r, "\mathcal{P}_r^\dagger\sup_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}}"'] & \mathcal{P}^\dagger\mathcal{P}^\dagger\mathcal{P}\mathbb{A}
\end{tikzcd}$$

Pasting these two commutative diagrams together along their common side $\sup_{\mathcal{P}^\dagger\mathcal{P}^\dagger\mathcal{P}\mathbb{A}} \circ \mathcal{P}_l\mathcal{P}_r^\dagger\mathcal{P}_r^\dagger Y_\mathbb{A}$ produces a big diagram whose contour is precisely the rectangle in the statement of the proposition. $\square$

The above lemma establishes the distributive law λ between the monads $\mathcal{P}_l$ and $\mathcal{P}_r^\dagger$; in the next one we shall compute the so-called λ-algebras, i.e. those that will turn out to be the algebras of the composite monad.

Lemma 4.4 *If $\mathbb{B}$ is a (co)complete category, then the diagram*

$$\begin{tikzcd}
\mathcal{P}\mathcal{P}^\dagger\mathbb{B} \arrow[r, "\mathcal{P}_l\inf_\mathbb{B}"] \arrow[d, "\lambda_\mathbb{B}"'] & \mathcal{P}\mathbb{B} \arrow[dr, "\sup_\mathbb{B}"] & \\
 & & \mathbb{B} \\
\mathcal{P}^\dagger\mathcal{P}\mathbb{B} \arrow[r, "\mathcal{P}_r^\dagger\sup_\mathbb{B}"'] & \mathcal{P}^\dagger\mathbb{B} \arrow[ur, "\inf_\mathbb{B}"'] &
\end{tikzcd}$$

commutes if and only if $\inf_\mathbb{B} \colon \mathcal{P}^\dagger\mathbb{B} \longrightarrow \mathbb{B}$ *is cocontinuous*[11].

[9] Because $\mathcal{P}_r^\dagger\mathcal{P}_l\mathcal{P}_r^\dagger Y_\mathbb{A} = \mathcal{P}_l^\dagger\mathcal{P}_r\mathcal{P}_r^\dagger Y_\mathbb{A}$ by Proposition 2.1, it is a left adjoint, with right adjoint $\mathcal{P}_r^\dagger\mathcal{P}_r\mathcal{P}_r^\dagger Y_\mathbb{A}$.

[10] Because $\mathcal{P}^\dagger\mathcal{P}\mathbb{A}$ is (co)complete, $\mathcal{P}_r^\dagger\sup_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}} = \mathcal{P}_l^\dagger Y_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}}$ by Lemma 4.2, so it is a left adjoint, with right adjoint $\mathcal{P}_r^\dagger Y_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}}$.

[11] In [Stubbe, 2007, 4.1, 5.4] it is proved that, for a cocomplete $\mathcal{Q}$-category $\mathbb{B}$, the following are equivalent

Proof : It was indicated before that $\mathcal{P}_r^\dagger \sup_{\mathbb{B}} \circ \lambda_{\mathbb{B}} = \sup_{\mathcal{P}^\dagger\mathbb{B}}$, so the diagram in the statement of the lemma is identical to

$$\begin{tikzcd} \mathcal{P}\mathcal{P}^\dagger\mathbb{B} \arrow[r, "\mathcal{P}_l \inf_{\mathbb{B}}"] \arrow[d, "\sup_{\mathcal{P}^\dagger\mathbb{B}}"'] & \mathcal{P}\mathbb{B} \arrow[d, "\sup_{\mathbb{B}}"] \\ \mathcal{P}^\dagger\mathbb{B} \arrow[r, "\inf_{\mathbb{B}}"'] & \mathbb{B} \end{tikzcd}$$

Its commutativity amounts to the cocontinuity of $\inf_{\mathbb{B}}$. □

The theory of distributive laws now has the following conclusion for us:

Theorem 4.5 *The order-enriched transformation* $\lambda\colon \mathcal{P}_l\mathcal{P}_r^\dagger \Longrightarrow \mathcal{P}_r^\dagger\mathcal{P}_l$ *with components*

$$\lambda_{\mathbb{A}} = \sup_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}} \circ\ \mathcal{P}_l\mathcal{P}_r^\dagger Y_{\mathbb{A}}$$

is a distributive law of the 2-monad $\mathcal{P}_l$ over the 2-monad $\mathcal{P}_r^\dagger$. *This means in particular that we can speak of the* composite 2-monad *of* $(\mathcal{P}_l, \sup_{\mathcal{P}-}, Y)$ *and* $(\mathcal{P}_r^\dagger, \inf_{\mathcal{P}^\dagger-}, Y^\dagger)$*:*

- *the functor is* $\mathcal{S} := \mathcal{P}_r^\dagger\mathcal{P}_l$*,*
- *the multiplication is* $\mu := (\inf_{\mathcal{P}^\dagger-} * \sup_{\mathcal{P}-}) \circ (1^\dagger * \lambda * 1)$*,*
- *and the units are* $\eta := \mathcal{P}_r^\dagger Y \circ Y^\dagger$*.*

A $\mathcal{Q}$*-category* $\mathbb{A}$ *is an* $\mathcal{S}$*-algebra if and only if it is (co)complete and* $\inf_{\mathbb{A}}$ *is cocontinuous; and an* $\mathcal{S}$*-homomorphism between* $\mathcal{S}$*-algebras* $\mathbb{A}$ *and* $\mathbb{B}$ *is a* bicontinuous *(i.e. both cocontinuous and continuous) functor* $F\colon \mathbb{A} \longrightarrow \mathbb{B}$*.*

5. Double power monad = composite of power monads

With notations as in the previous two sections we may now conclude:

Theorem 5.1 *The double power monad* $(\mathcal{T}, m, u)$ *is equal to the composite power monad* $(\mathcal{S}, \mu, \eta)$*.*

Proof : The underlying functors of $\mathcal{T} = \mathcal{P}_l^\dagger\mathcal{P}_r$ and $\mathcal{S} = \mathcal{P}_r^\dagger\mathcal{P}_l$ are identical thanks to Proposition 2.1, and the units of both monads are trivially identical. The only thing to verify, is the equality of the multiplications. But recall that

$$m_{\mathbb{A}} = \mathcal{P}_l^\dagger \alpha_{\mathcal{P}\mathbb{A}} = \mathcal{P}_l^\dagger Y_{\mathcal{P}\mathbb{A}}^\dagger \circ \mathcal{P}_l^\dagger Y_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}}$$

whereas

$$\mu_{\mathbb{A}} = \inf_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}} \circ \mathcal{P}_r^\dagger\mathcal{P}_r^\dagger \sup_{\mathcal{P}\mathbb{A}} \circ \mathcal{P}_r^\dagger\lambda_{\mathcal{P}\mathbb{A}} = \inf_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}} \circ \mathcal{P}_r^\dagger \sup_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}} = \inf_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}} \circ \mathcal{P}_l^\dagger Y_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}},$$

---

conditions: (1) $\mathbb{B}$ is completely distributive, i.e. $\sup_{\mathbb{B}}$ has a left adjoint, (2) $\sup_{\mathbb{B}}$ is continuous, (3) $\sup_{\mathbb{B}}$ has a cocontinuous right inverse, (4) $\mathbb{B}$ is a projective object in $\mathsf{Cocont}(\mathcal{Q})$, (5) $\mathbb{B}$ is totally continuous, i.e. for the distributor $\Theta_{\mathbb{B}} := \{\mathbb{B}(-, \sup_{\mathbb{B}} -), \mathcal{P}\mathbb{B}(Y_{\mathbb{B}}-, -)\}$ we have that $\sup_{\mathbb{B}}(\Theta_{\mathbb{B}}(-, b)) = b$ for all $b \in \mathbb{B}$; and then $T_{\mathbb{B}}\colon \mathbb{B} \longrightarrow \mathcal{P}\mathbb{B}\colon b \mapsto \Theta_{\mathbb{B}}(-, b)$ is the left adjoint/cocontinuous splitting of $\sup_{\mathbb{B}}$. Replacing $\mathcal{Q}$ by $\mathcal{Q}^{\mathsf{op}}$ we find the dual result: so there are several equivalent expressions for the fact that $\inf_{\mathbb{B}}$ is a cocontinuous functor.

so it suffices to see that $\inf_{\mathcal{P}^\dagger\mathcal{P}\mathbb{A}} = \mathcal{P}_l^\dagger Y_{\mathcal{P}\mathbb{A}}^\dagger$. This is an incarnation of the Yoneda lemma, cf. Lemma 4.1. $\square$

Note in particular that each $m_{\mathbb{A}} = \mu_{\mathbb{A}}$ is a right adjoint, and so is each functor in the image of $\mathcal{T}$. It follows that Höhle's [2013] 'Condition (R)' is satisfied (see his Definition 4.3 and Theorem 5.13), which is important for his study of 'regular topological $\mathcal{T}$-spaces'.

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