A note on the notion of characteristic subobject in the Mal'tsev and protomodular settings

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Abstract

We introduce a categorical definition of characteristic subobjects which is alternative to the one given in [14] for semi-abelian categories and applies to the wider context of Mal'tsev categories. It allows, on the other hand, to set clearly this notion and to produce examples in the category Rg of non unitary rings and in the category $TopGp$ of topological groups.

Introduction

In a recent work [14] A. Cigoli and A. Montoli investigated the notion of characteristic subobject in the context of semi-abelian categories. Mimicking strictly what happens in the categories Gp of groups and R-Lie of Lie R-algebras, their approach was based upon the notion of internal action as described in [4]. In any semi-abelian category the internal actions are in bijection with the split epimorphisms and, actually, it is possible to get rid of the concept of internal action and to get straight to the notion of characteristic subobject just using split epimorphisms and hypercartesian morphisms related to the change of base functors associated with the fibration of points (Definition 2.1). This alternative approach has, among others, the following benefits:
1) it allows us to extend the notion to the conceptually lighter and wider context of Mal'cev categories
2) it gives rise to a conceptual proof of the characterization of the characteristic subobjects as those normal subobjects which are stable under composition by normal subobjects on the left hand side (Theorem 2.2)
3) it enlarges in a clear way the notion of characteristic subobjects to the category Rg of non unitary rings or $TopGp$ of topological groups where the question was up to now far from being cleared (Sections 2.3 and 2.4)
4) it shows that the notion is stable under the passage to the fibres $Pt_Y\mathbb{C}$ of the fibration of points (Theorem 2.1)
5) it produces some unexpected situations dealing with objects whose any subobject is a characteristic one (Corollary 3.1)
6) in the exact action representative contexts, such as Gp and R-Lie, it allows to characterize the characteristic subobjects as those normal subobjects whose quotient maps have a (unique) extension to the action groupoids (Corollary 3.2)

7) it allows to characterize in simple terms those pairs of characteristic subobjects whose commutator is characteristic as well (Theorem 5.1)
8) it brings some new enlightments about the notion of peri-abelian category (Section 5.1).

The article is organized along the following lines. Section 1) gives some recalls about hypercartesian monomorphisms. Section 2) introduces the alternative definition, the first stability properties and the first investigations in Rg and $TopGp$. Section 3) gives conceptual insights on the classical definition of the characteristic subobjects in groups and in Lie R-algebras dealing with the group $AutX$ and the algebra $DerX$ through the investigation of the characteristic subobjects in the action representative context. Section 4) is dealing with the centralizer of characteristic subobjects while Section 5) is dealing with the commutator of such pairs.

1 Hypercartesian morphisms

Let $U : \mathbb{E} \to \mathbb{F}$ be any functor. Recall that a map $f : X \to Y$ in $\mathbb{E}$ is hypercartesian with respect to U when, given any map $g : X' \to Y$ with a factorization $h : U(X) \to U(X')$ in $\mathbb{F}$ such that $u(g) = U(f).h$, there is a unique factorization $\bar{h} : X \to X'$ in $\mathbb{E}$ such that $g = \bar{h}.f$ and $U(\bar{h}) = h$. Accordingly any hypercartesian map whose image by U is an isomorphism is an isomorphism.

Proposition 1.1. *1) The hypercartesian maps with respect to U are stable under composition; a hypercartesian map above a monomorphism is itself a monomorphism.*
2) When $\mathbb{E}$ has pullbacks and U preserves them, the monomorphic hypercartesian maps are stable under pullback; accordingly they are stable under finite intersection.
3) Under the conditions 2) the functor U is conservative if and only if any monomorphism is hypercartesian.

Proof. 1) The first assertion of 1) is classical. Suppose now that f is a hypercartesian map such that $U(f)$ is a monomorphism, and that $(l, l') : T \rightrightarrows X$ is a pair of maps such that $f.l = f.l'$. Since $U(f)$ is a monomorphism, we get $U(l) = U(l')$, hence $l = l'$.
2) It is classical as well that, when U preserves pullback, the hypercartesian maps are stable under pullback.
3) When U preserves pullbacks, the functor U is conservative if and only if any monomorphism whose image by U is an isomorphism, is an isomorphism as well. Now, it is clear that if any monomorphism is hypercartesian with respect to U, any monomorphism with invertible image is itself invertible. Conversely suppose that U left exact and conservative. Let $m : X \rightarrowtail Y$ be any monomorphism. Let $g : X' \to Y$ be a map with a factorization $h : U(X) \to U(X')$ in $\mathbb{F}$

such that $u(g) = U(m).h$. Consider the following pullback in $\mathbb{E}$:

$$\begin{array}{ccc} \bar{X} & \overset{\bar{m}}{\rightarrowtail} & X' \\ {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle g} \\ X & \underset{m}{\rightarrowtail} & Y \end{array}$$

It is preserved by U, and the factorization h makes the monomorphism $U(\bar{m})$ an isomorphism. Since U is conservative, m is itself an isomorphism, and $\gamma.\bar{m}^{-1}$ produces the factorization making m hypercartesian. □

As usual we shall call subobject a class of monomorphisms up to isomorphism.

Proposition 1.2. *Let $U : \mathbb{C} \to \mathbb{D}$ be any left exact conservative functor. It reflects the inclusion of subobjects. When in addition $\mathbb{C}$ and $\mathbb{D}$ are pointed, regular and protomodular (i.e.* homological*) and U preserves the regular epimorphisms, then any regular epimorphisms in $\mathbb{C}$ is hypercocartesian.*

Proof. Let u and v two subobjects of X such that $U(u) \subset U(v)$. This is equivalent to say that the pullback of $U(v)$ along $U(u)$ is an isomorphism. Since U is left exact and conservative, the same is true for the pullback of v along u, and so u is a subobject of v. Now starting from a regular epimorphism $f : X \twoheadrightarrow Y$ in the homological category $\mathbb{C}$, suppose that there is a map $g : X \to Z$ and a map $h : U(Y) \to U(Z)$ in $\mathbb{D}$ such that $U(g) = h.U(f)$. Then the kernel $K[U(f)]$ is a subobject of the kernel $K[U(g)]$; accordingly the kernel $K[f]$ is a subobject of the kernel $K[g]$, and since f is a regular epimorphism and, consequently, the cokernel of its kernel, we get a unique factorization $\bar{h} : Y \to Z$ such that $g = \bar{h}.f$. Whence $U(g) = U(\bar{h}).U(f)$, and since U preserves the regular epimorphisms, $U(\bar{h}) = h$. □

2 Characteristic monomorphisms

From now on, any category $\mathbb{C}$ will be assumed to be at least a pointed finitely complete Mal'tsev category, namely such that any reflexive graph is an equivalence relation, see [12] and [13]. We denote by $\alpha_B : 1 \rightarrowtail B$ the initial map and by $\alpha_B^* : Pt_B\mathbb{C} \to \mathbb{C}$ the associated change of base functor with respect to the fibration of point, i.e. the functor given by the pullback of split epimorphisms along α_B, in other words the functor associating its kernel with any split epimorphisms above B.

2.1 Definition

Let us introduce the following:

Definition 2.1. *Let $\mathbb{C}$ be a pointed Mal'tsev category. A monomorphism $j : J \rightarrowtail X$ is said to be characteristic when, for any object B and any split epimorphism $(g, t) : A \rightleftarrows B$ such that $\alpha_B^*(g, t) = X$ (i.e. with kernel X), there is a* *hypercartesian map $j_{(g,t)}$ in $Pt_B\mathbb{C}$ above j which is preserved as a hypercartesian map by any change of base functor.*

Since α_B^* is left exact the hypercartesian map $j_{(g,t)}$ is necessarily a monomorphism in $Pt_B\mathbb{C}$. On the other hand, it is clear that the monomorphism $1_X : X \rightarrowtail X$ is characteristic.

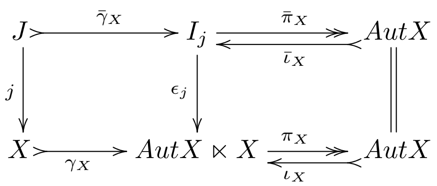
Recall that a protomodular category is such that any change of base functor with respect to the fibration of points is conservative [2], and that a protomodular category is a Mal'tsev one. The categories *Gp* of groups, *Rg* of non unitary rings and *R-Lie* of Lie *R*-algebras are protomodular.

Proposition 2.1. *When $\mathbb{C}$ is a pointed protomodular category a monomorphism $j : J \rightarrowtail X$ is characteristic if and only if, for any object B and any split epimorphism $(g,t) : A \rightleftarrows B$ such that $\alpha_B^*(g,t) = X$, there is a monomorphism $j_{(g,t)}$ in $Pt_B\mathbb{C}$ above j. In particular, any initial monomorphism $\alpha_X : 1 \rightarrowtail X$ is characteristic.*

Proof. It is a straightforward consequence of the point 3) of Proposition 1.1. □

Proposition 2.2. *A subgroup $j : J \rightarrowtail X$ is characteristic in the classical sense if and only if it is characteristic with respect to the previous definition in the category Gp of groups.*

Proof. Suppose that the subgroup J of X is characteristic in the classical sense, and let $(g,t) : A \rightleftarrows B$ be any split epimorphism in *Gp* with kernel X. Given any element $b \in B$, the isomorphism $\iota_b : X \to X$ defined by $\iota_b(x) = s(b).x.s(b)^{-1}$ is stable on J, so that the set $\bar{J} = \{\gamma.s(b)/\gamma \in J,\ b \in B\}$ is a subgroup of A which determines by restriction a plit epimorphism $(\bar{g},\bar{t}) : \bar{J} \rightleftarrows B$ whose kernel is J.
Conversely suppose that $j : J \rightarrowtail X$ is characteristic with respect to the previous definition in the category *Gp* of groups. Consider the split epimorphism $(\pi_X, \iota_X) : AutX \ltimes X \rightleftarrows AutX$. Its kernel is the group X. Consider now the hypercartesian map ϵ_j above j:



The fact that J is the kernel of $\bar{\pi}_X$ means that $I_j = AutX \ltimes J$. Accordingly the identity $(\phi, 1).(Id_X, \gamma) = (\phi, \phi(\gamma))$ in I_j for any automorphism ϕ and any $\gamma \in J$ shows that $\phi(\gamma)$ is in J. □

Proposition 2.3. *A subobject $j : J \rightarrowtail X$ is characteristic in the classical sense in the category R-Lie of Lie R-algebras if and only if it is characteristic with respect to the previous definition in the category R-Lie.*

Proof. The proof is the same as the previous one, replacing the group $AutX$ by the algebra of derivations $DerX$. $\square$

When $\mathbb{C}$ is semi-abelian, the notion of characteristic subobject here coincides with the one in [14] via our Proposition 2.1 and their Proposition 2.2.

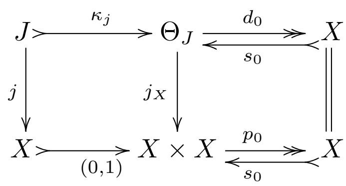
2.2 Basic properties of the characteristic monomorphisms

Proposition 2.4. *Let $\mathbb{C}$ be a pointed Mal'tsev category. The characteristic monomorphisms are stable under composition and finite intersection.*

Proof. These points are consequences of the properties 1) and 2) of the hypercartesian maps recalled in Proposition 1.1.. $\square$

Proposition 2.5. *Let $\mathbb{C}$ be a pointed Mal'tsev category. A characteristic monomorphism j is normal to an equivalence relation $\Theta_J \rightrightarrows X$ which is the largest equivalence relation on X whose normalization is contained in j; it is called the* characteristic relation *of j.*

Proof. Let $j : J \rightarrowtail X$ be a characteristic monomorphism. Consider the following hypercartesian map with codomain $(p_0, s_0) : X \times X \rightleftarrows X$ above j:

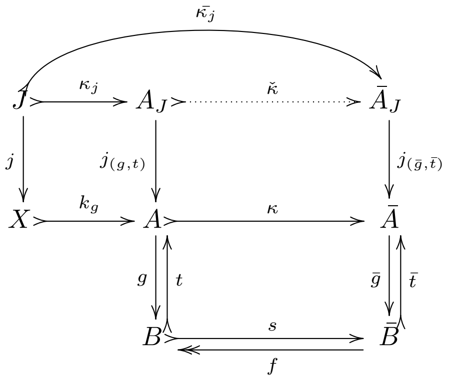


It makes Θ_J a reflexive relation and thus an equivalence relation on X since $\mathbb{C}$ is a Mal'tsev category. The left hand side pullback shows that its associated normal monomorphism (i.e. its *normalization*) is j. The fact that it is the largest equivalence relation on X whose normalization is contained in j is a consequence of the fact that the map j_X is hypercartesian in the fibre $Pt_X\mathbb{C}$. $\square$

Clearly, from the proof of the previous proposition, the characteristic relation of the characteristic monomorphism 1_X is the indiscrete equivalence relation ∇_X; and, in the protomodular context, the characteristic relation of the characteristic monomorphism α_X is the discrete equivalence relation Δ_X.

Theorem 2.1. *Let $\mathbb{C}$ be a pointed Mal'tsev category and j a characteristic monomorphism. Given any split epimorphism $(g, t) : A \rightleftarrows B$ such that $\alpha_B^*(g, t) = X$, the associated hypercartesian map $j_{(g,t)}$ in $Pt_B\mathbb{C}$ above j is characteristic in this fibre.*

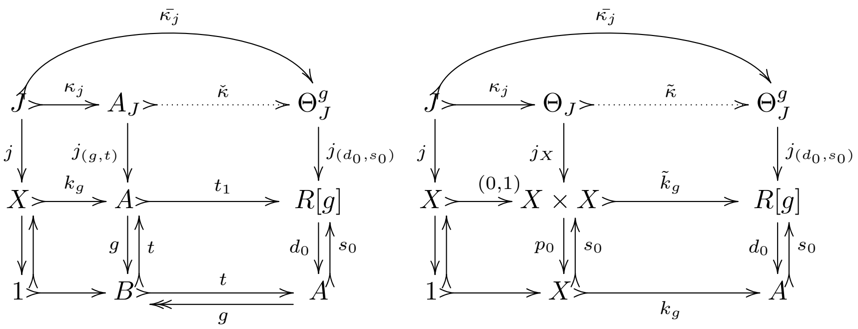
Proof. Let us consider the following diagram where $(\bar{g}, \bar{t})$ is a split epimorphism in the fibre $Pt_B\mathbb{C}$ with codomain (f, s) and whose kernel in this fibre is (g, t). This last point means that the lower rightward square is a pullback:



Accordingly the map $\kappa.k_g$ is the kernel of $\bar{g}$ in $\mathbb{C}$. Since j is characteristic, there is a hypercartesian map $j_{(\bar{g},\bar{t})}$ in the fibre $Pt_{\bar{B}}\mathbb{C}$ above j. Since the change of base functor $s^* : Pt_{\bar{B}}\mathbb{C} \to Pt_B\mathbb{C}$ preserves the hypercartesian map above j by definition, there is a factorization $\check{\kappa}$ which makes the upper right hand side square a pullback. It is straightforward to check that this map $j_{(\bar{g},\bar{t})}$ is the desired hypercartesian map above $j_{(g,t)}$ with respect to the change of base functor s^*. □

Corollary 2.1. *Let $\mathbb{C}$ be a pointed Mal'tsev category and j a characteristic monomorphism. The characteristic equivalence relation Θ_J^g associated with the characteristic map $j_{(g,t)}$ in the fibre $Pt_B\mathbb{C}$ is mapped by the change of base functor α_B^* on the characteristic relation Θ_J of j.*

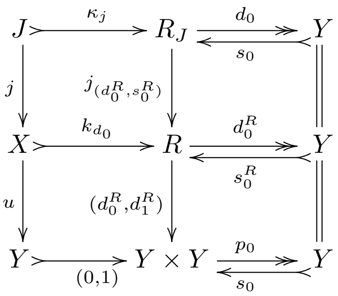
Proof. Consider the same construction as above where the split epimorphism $(\bar{g}, \bar{t})$ in question is now $(d_0, s_0) : R[g] \rightleftarrows A$:



The two global diagram above are the same. So the hypercartesian map with codomain (d_0, s_0) above $j_(g, t)$ in the fibre $Pt_B\mathbb{C}$, which is the hypercartesian map with codomain (d_0, s_0) above j in $\mathbb{C}$, is mapped on j_X. □

Proposition 2.6. *Let $\mathbb{C}$ be a pointed Mal'tsev category. Let $j : J \rightarrowtail X$ be a characteristic monomorphism and $u : X \rightarrowtail Y$ a monomorphism normal to an equivalence relation $(d_0^R, d_1^R) : R \rightrightarrows Y$. Then $u.j$ is normal to an equivalence subrelation R_J of R which is the largest equivalence subrelation of R whose normalization is contained in j.*

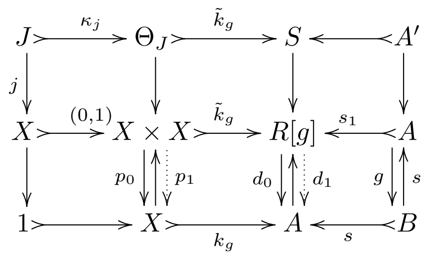
Proof. Consider the following diagram where $j_{(d_0^R, s_0^R)}$ is the hypercartesian map with codomain $(d_0^R, s_0^R) : R \rightleftarrows Y$ above j:



It defines R_J as an equivalence subrelation of R whose normalization is $u.j$. It is the largest equivalence subrelation of R whose normalization is contained in j since $j_{(d_0^R, s_0^R)}$ is hypercartesian. □

Theorem 2.2. *Let $\mathbb{C}$ be a pointed protomodular category. A monomorphism $j : J \rightarrowtail X$ is characteristic if and only if, given any normal monomorphism $u : X \rightarrowtail Y$, the monomorphism $u.j$ is normal.*

Proof. By the previous proposition, we know that it is a necessary condition. Now, let j be a monomorphism satisfying this condition. Given $(g,t) : A \rightleftarrows B$ a split epimorphism such that $\alpha_B^*(g,t) = X$, we denote by S the equivalence relation to which $k_g.j$ is normal. Since $\mathbb{C}$ is protomodular, from $J \subset X$ we get $S \subset R[g]$. Then the inverse image $k_g^{-1}(S)$ is normal to j and, since $\mathbb{C}$ is protomodular, the inclusion $S \rightarrowtail R[g]$ is the hypercartesian map above j and we get $k_g^{-1}(S) = \Theta_J$. Let us consider now the following diagram where any square is a pullback:



Then the image of $S \rightarrowtail R[g]$ by the change of base $s^* : Pt_A\mathbb{C} \to Pt_B\mathbb{C}$ (on the vertical right hand side) produces the desired (hypercartesian) map above

j in the fibre $Pt_B\mathbb{C}$ (the map $A' \to B$ being split by the inverse image of the subdiagonal: $s_0 : A \rightarrowtail S$). $\square$

In a regular Mal'tsev category two equivalence relations (R, S) on an object X can be composed and $R{\cdot}S$ is the supremum of R and S among the equivalence relations on X. So when $\mathbb{C}$ is homological any pair of normal monomorphisms on X has a supremun among the normal monomorphisms on X. Moreover any left exact functor which preserves the regular epimorphism does preserves these suprema.

Proposition 2.7. *Let $\mathbb{C}$ be a homological category. The supremum of two characteristic monomorphisms among the normal monomorphisms is characterisctic as well.*

Proof. Let $j : J \rightarrowtail X$ and $j' : J' \rightarrowtail X$ be a pair of characteristic monomorphisms. Their supremum l of among the normal monomorphisms is the normalization of the equivalence relation $\Theta_J \cdot \Theta_{J'}$. Given any any split epimorphism $(g, t) : A \rightleftarrows B$ with kernel X, it is clear that the normalization of $\Theta^g_J \cdot \Theta^g_{J'}$ which is the supremum of $j_{(g,t)}$ and $j'_{(g,t)}$ in the fibre $Pt_B\mathbb{C}$ is mapped to l by the change of base α^*_B thanks to Corollary 2.1. $\square$

2.3 Characteristic subrings

We denote by *Rg* the category of non unitary rings. The normal subobjects in this category are the bilateral ideals. The notion of characteristic subobject in *Rg* was not quite clear up to now. The characterization described in Theorem 2.2 gives now its full meaning to the notion in *Rg*: they are those bilateral ideals which are stable under composition by bilateral ideals on the left hand side. There is not the same kind of characterization as in the categories *Gp* and *R-Lie* because, in *Rg*, there are not split extension classifiers, see Section 3 below and the proof of Proposition 2.2. We shall give explicit examples of characteristic subrings in the next sections.

2.4 Characteristic topological subgroups

We are now able to give examples of characteristic toplogical subgroups as well:

Proposition 2.8. *Given (X, T_X) a topological group, any characteristic subgroup $j : J \rightarrowtail X$ produces a characteristic subobject $(J, T^J_X) \rightarrowtail (X, T_X)$ in the category $TopGp$ of topological groups, where T^J_X is the induced topology.*

Proof. Recall (see [6]) that a subobject $m : (H, T_H) \rightarrowtail (G, T_G)$ is normal in $TopGp$ if and only if H is a normal subgroup of G and the function $d^m_* : H \times G \to H;\ (h, g) \mapsto g.h.g^{-1}$ is continuous from the topological product

$(H, T_H) \times (G, T_G)$ to (H, T_H). Now let $u : (X, T_X) \rightarrowtail (Y, T_Y)$ be a normal topological subgroup. Consider the following diagram:

$$\begin{array}{ccc} (J, T_J^X) \times (Y, T_Y) & \stackrel{j \times Y}{\rightarrowtail} & (X, T_X) \times (Y, T_Y) \\ {\scriptstyle d_*^j} \downarrow & & \downarrow {\scriptstyle d_*^u} \\ (J, T_J^X) & \underset{j}{\rightarrowtail} & (X, T_X) \end{array}$$

The function d_*^u is continuous; its restriction d_*^j to subobjects in $TopGp$ which are endowed with the induced topologies is continuous as well. □

Whence, now, the following question: is any characteristic subobject of (X, T_X) in $TopGp$ of this kind?

2.5 Regular epimorphisms with characteristic kernel

Let us introduce now the following:

Definition 2.2. *Given a homological category $\mathbb{C}$, we shall call characteristic a regular epimorphism which has a characteristic kernel.*

Proposition 2.9. *Let $\mathbb{C}$ be a pointed exact protomodular category. A regular epimorphism $f : X \twoheadrightarrow Y$ is charateristic if and only if there is a hypercocartesian map above f with respect to any change of base functor: $\alpha_B^* : Pt_B\mathbb{C} \to \mathbb{C}$. Accordingly, in this context, the characteristic epimorphims are stable under composition.*

Proof. Suppose that f is characteristic. Let us denote by j the kernel of f. Given any split epimorphism $(g, t) : A \rightleftarrows B$ with kernel X, let us denote by j_A the hypercartesian monomorphism above it in the fibre $Pt_B\mathbb{C}$ which is pointed exact and protomodular as well. This j_A is characteristic in this fibre, and thus a normal monomorphism, which has a cokernel $\bar{f}$ in the pointed exact protomodular fibre $Pt_B\mathbb{C}$. Since the change of base α_B^* is left exact and preserves the regular epimorphism, the image of $\bar{f}$ by this change of base is f. According to Proposition 1.2, $\bar{f}$ is necessarily hypercocartesian.
Conversely suppose that, given any split epimorphism $(g, t) : A \rightleftarrows B$ such that $\alpha_B^*(g, t) = X$, the regular epimorphism f produces a hypercocartesian map $\bar{f}$ above f with respect to α_B^*. Then the kernel of $\bar{f}$ in $Pt_B\mathbb{C}$ will be the desired hypercartesian map above j. □

Proposition 2.10. *Let $\mathbb{C}$ be a pointed exact protomodular category. Then the characteristic monomorphisms are stable under pullback along characteristic epimorphisms.*

Proof. Consider the following diagram where the left hand side square is a pullback, f a characteristic epimorphism and j a characteristic monomorphism:

$$\begin{array}{ccccccc} X' & \overset{\bar{j}}{\rightarrowtail} & X & \overset{k_g}{\rightarrowtail} & A & \underset{g}{\overset{t}{\leftrightarrows}} & B \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \check{f}} & & \| \\ Y' & \underset{j}{\rightarrowtail} & Y & \underset{k_g}{\rightarrowtail} & A' & \underset{g'}{\overset{t'}{\leftrightarrows}} & B \end{array}$$

Let (g,t) be a split epimorphism such that $\alpha_B^*(g,t) = X$. We have to find a map in $Pt_B\mathbb{C}$ above $\bar{j}$. Since f is characteristic, there is a regular epimorphism $\check{f}$ in $Pt_B\mathbb{C}$ above it. Since j is characteristic, there a monomorphism $j_{A'}$ in the fibre $Pt_B\mathbb{C}$ with codomain A' above j. The pullback j_A of $j_{A'}$ along $\check{f}$ produces the desired map above $\bar{j}$. □

3 The case of action representative categories

This section is devoted to give a conceptual insight on the classical aspects of the characteristic monomorphisms in groups and in Lie algebras. Recall that a pointed protomodular category $\mathbb{C}$ is action representative ([3],[5]) when, for any object X, there is a split extension classifier:

$$X \overset{\gamma_X}{\rightarrowtail} D_1(X) \underset{\sigma_0 X}{\overset{\delta_0 X}{\rightleftarrows}} D_0(X)$$

for the split exact sequences whose kernel is this object X. The categories Gp of groups is action representative, and for any group X, the split extension classifier is the one which is used in the proof of Proposition 2.2:

$$X \overset{\gamma_X}{\rightarrowtail} AutX \ltimes X \underset{\iota_X}{\overset{\pi_X}{\rightleftarrows}} AutX$$

The same holds for the category K-Lie of Lie algebras where the group $AutX$ is replaced by the Lie algebra $DerX$. The split extension classifier is actually underlying a groupoid structure which determines the following discrete fibration, since this groupoid classifies the internal groupoids whose normalization has the object X as domain, which implies that we have $j_X = \delta_1 X.\gamma_X$, see [3]:

$$\begin{array}{ccc} X \times X & \overset{\tilde{j}_X}{\longrightarrow} & D_1(X) \\ {\scriptstyle p_0^X}\downarrow\uparrow\downarrow{\scriptstyle p_1^X} & & {\scriptstyle \delta_0 X}\downarrow\uparrow\downarrow{\scriptstyle \delta_1 X} \\ X & \underset{j_X}{\longrightarrow} & D_0(X) \end{array}$$

In the case of groups j_X associates with any object x its associated inner automorphism ι_x, while $\delta_1 X(\phi, x) = \iota_x.\phi$.

Proposition 3.1. *Let $\mathbb{C}$ be a pointed action representative protomodular category. The characteristic subobjects of X are in bijection with the subobjects of the split extension classifier $(\delta_0 X, \sigma_0 X) : D_1(X) \rightleftarrows D_0(X)$ in the fibre $Pt_{D_0(X)}\mathbb{C}$.*

Proof. Consider any subobject ι of the split extension classifier in the fibre $Pt_{D_0(X)}\mathbb{C}$:

$$\begin{array}{ccccc} J & \stackrel{k}{\rightarrowtail} & I & \underset{s_0^I}{\overset{d_0^I}{\rightleftarrows}} & D_0(X) \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \iota} & & \| \\ X & \underset{\gamma_X}{\rightarrowtail} & D_1(X) & \underset{\sigma_0 X}{\overset{\delta_0 X}{\rightleftarrows}} & D_0(X) \end{array}$$

Let us denote by k the kernel of d_0^I and j the induced factorization. Let us show that j is characteristic. Let $(g,t) : A \rightleftarrows B$ be any split epimorphism such that $\alpha_B^*(g,t) = X$, and let us denote by $\phi : B \to D_0(X)$ its classifying map:

$$\begin{array}{ccccc} X & \stackrel{k_g}{\rightarrowtail} & A & \underset{t}{\overset{g}{\rightleftarrows}} & B \\ \| & & \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle \phi} \\ X & \underset{\gamma_X}{\rightarrowtail} & D_1(X) & \underset{\sigma_0 X}{\overset{\delta_0 X}{\rightleftarrows}} & D_0(X) \end{array}$$

Then the image of ι by the change of base functor $\phi^* : Pt_{D_0(X)}\mathbb{C} \to Pt_B\mathbb{C}$ will produce the desired monomorphism with codomain (g,t) above j.

Conversely a characteristic monomorphism $j : J \rightarrowtail X$ will produce a (unique hypercartesian) map ι_X above it:

$$\begin{array}{ccccc} J & \stackrel{k}{\rightarrowtail} & D_J(X) & \underset{s_0}{\overset{d_0}{\rightleftarrows}} & D_0(X) \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \iota_X} & & \| \\ X & \underset{\gamma_X}{\rightarrowtail} & D_1(X) & \underset{\sigma_0}{\overset{\delta_0}{\rightleftarrows}} & D_0(X) \end{array}$$

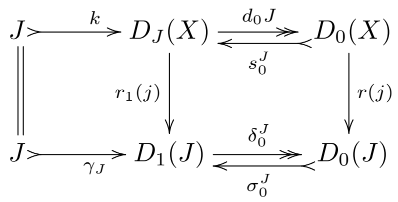
□

Corollary 3.1. *Let $\mathbb{C}$ be a pointed protomodular category which is action representative. Then, in any any pointed protomodular fibre $Pt_{D_0(X)}\mathbb{C}$, the object $(\delta_0 X, \sigma_0 X) : D_1(X) \rightleftarrows D_0(X)$ is such that any of its monomorphisms is characteristic. If, in addition, $\mathbb{C}$ is exact, any regular epimorphism with this domain is characteristic in $Pt_{D_0(X)}\mathbb{C}$.*

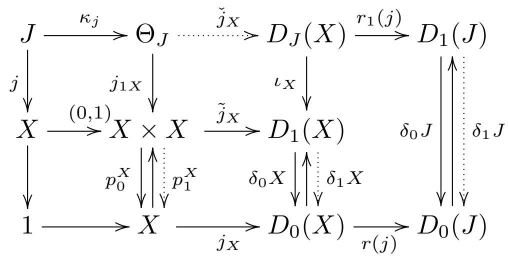
Remark. We recalled that the split extension classifier associated with X is actually underlying a structure of internal groupoid in $\mathbb{C}$; accordingly any monomorphism ι_X as above determines a subgroupoid $D_J(X)$ of this groupoid (since in a Mal'tsev category any subgraph of a groupoid is a groupoid as well).

Proposition 3.2. *Let $\mathbb{C}$ be a pointed action representative protomodular category. Any characteristic monomorphism $j : J \rightarrowtail X$ produces a comparison map $r(j) : D_0(X) \to D_0(J)$ such that the map $r(j).j_X : X \to D_0(J)$ classifies the equivalence relation Θ_j, in other words such that "the restriction map $r(j)$ respects the inner automorphisms".*

Proof. In turn, the upper split exact sequence at the end of the proof of the previous proposition above produces a classifying map:



Now let us consider the following diagram where any plain square is a pullback:



The fact that ι_X is the hypercartesian map above j and that $\tilde{j}_X.(0,1) = \gamma_X$ induces a map $\check{j}_X$ which shuts the central upper square and makes it a pullback since the change of base along j_X preserves the hypercartesian maps, and we have moreover $r_1(j).\check{j}_X.\kappa_j = \gamma_J$. We shall show that $(r(j), r_1(j))$ is a morphism of groupoids, namely that we have $\delta_1 J.r_1(j) = r(j).\delta_1 X.\iota_X$, by composition with the jointly strongly epic pair $(\kappa_j.\check{j}_X, s_0^J)$. The composition by s_0^J is straightforward; the composition by $\kappa_j.\check{j}_X$ gives rise to $r(j).j_X.j$ and j_J which are equal since they both classify the indiscrete equivalence relation ∇_J. Accordingly any square in this diagram is commutative and a pullback, and the map $r(j).j_X : X \to D_0(J)$ classifies the equivalence relation Θ_j. $\square$

Proposition 3.3. *Let $\mathbb{C}$ be a pointed exact action representative protomodular category. The class up to isomorphism of the characteristic epimorphisms with domain X are in bijection with the class up to isomorphism of the regular epimorphisms with domain $(\delta_0 X, \sigma_0 X) : D_1(X) \rightleftarrows D_0(X)$ in the fibre $Pt_{D_0(X)}\mathbb{C}$. Any characteristic epimorphism $f : X \twoheadrightarrow Y$ has a unique extension to the level* *of the action groupoids:*

$$
\begin{array}{ccccc}
X & \overset{\gamma_X}{\rightarrowtail} & D_1(X) & \underset{\delta_0 X}{\overset{\delta_1 X}{\rightleftarrows}} & D_0(X) \\
{\scriptstyle f}\downarrow & & {\scriptstyle r_{1f}}\downarrow & & \downarrow{\scriptstyle r_f} \\
Y & \underset{\gamma_Y}{\rightarrowtail} & D_1(Y) & \underset{\delta_0 Y}{\overset{\delta_1 Y}{\rightleftarrows}} & D_0(Y)
\end{array}
$$

making the left hand side square commute, which is equivalent to $r_f.j_X. = j_Y.f$; i.e. the map r_f respects "the inner automorphisms".

Proof. Let ρ be a regular epimorphism in the fibre $Pt_{D_0(X)}\mathbb{C}$ with domain $(\delta_0 X, \sigma_0 X) : D_1(X) \rightleftarrows D_0(X)$:

$$
\begin{array}{ccccc}
X & \overset{\gamma_X}{\rightarrowtail} & D_1(X) & \underset{s_0^X}{\overset{\delta_0 X}{\rightleftarrows}} & D_0(X) \\
{\scriptstyle f}\downarrow & & {\scriptstyle \rho}\downarrow & & \| \\
Y & \underset{k_{d_0Q}}{\rightarrowtail} & Q & \underset{s_0 Q}{\overset{d_0 Q}{\rightleftarrows}} & D_0(X)
\end{array}
$$

Denote k_{d_0Q} the kernel of d_0Q and f the induced regular epimorphism. Let us show that f is characteristic. So, let $(g,t) : A \rightleftarrows B$ be any split epimorphism with kernel X, and $\phi : B \to D_0(X)$ its classifying map. Then the image of ρ by $\phi^* : Pt_{D_0(X)}\mathbb{C} \to Pt_B\mathbb{C}$ will produce the desired regular epimorphism with domain (g,t) above f.

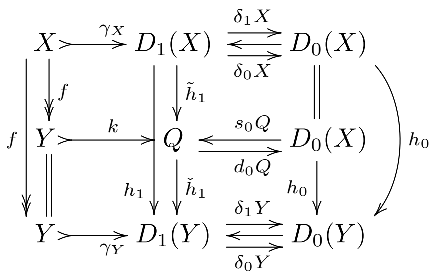
Conversely, according to Proposition 2.9, any characteristic epimorphism f produces a (unique hypercocartesian) epimorphism ρ_f above it:

$$
\begin{array}{ccccc}
X & \overset{\gamma_X}{\rightarrowtail} & D_1(X) & \underset{s_0^X}{\overset{\delta_0 X}{\rightleftarrows}} & D_0(X) \\
{\scriptstyle f}\downarrow & & {\scriptstyle \rho_f}\downarrow & & \| \\
Y & \underset{\gamma_f}{\rightarrowtail} & D_f(Y) & \underset{\sigma_0 f}{\overset{\delta_0 f}{\rightleftarrows}} & D_0(X) \\
\| & & {\scriptstyle \check{r}_f}\downarrow & & \downarrow{\scriptstyle r_f} \\
Y & \underset{\gamma_Y}{\rightarrowtail} & D_1(Y) & \underset{\sigma_0 Y}{\overset{\delta_0 Y}{\rightleftarrows}} & D_0(Y)
\end{array}
$$

which, in turn, produces a classifying map r_f. Showing that $r_f.j_X. = j_Y.f$ is showing that they classify the same split extension with kernel Y. The second one classifies the pullback along f of the split epimorphism (p_0^Y, s_0^Y), namely the following right hand side split extension:

$$
\begin{array}{ccc}
X \times X & \xrightarrow{X \times f} & X \times Y \\
{\scriptstyle p_0^X}\searrow\nwarrow{\scriptstyle s_0^X} & & {\scriptstyle p_X}\swarrow\nearrow{\scriptstyle (1,f)} \\
& X &
\end{array}
$$

while the first one classifies the codomain of the pullback of the regular epimorphism ρ_f along $j_X : X \to D_0(X)$; now this pullback is necessarily the hypercocartesian map above f with domain (p_0^X, s_0^X), namely the one given by the triangle above since its image by α_X^* is precisely f. From that, it is straightforward that $(r_f, \check{r}_f.\rho_f)$ is underlying an internal functor, i.e. that the square with the δ_1 do commute.
It remains to show that it is the unique internal functor inducing f. So let (h_0, h_1) be another internal functor of this kind. Consider the following diagram where the lower right hand side square is a pullback:



Now, since the two upper lines are split exact sequences and f is a regular epimorphism, so is the factorization $\tilde{h}_1$ which is consequently hypercocartesian above f. So that there is an isomorphism $\varepsilon : D_f(Y) \to Q$ in the fibre $Pt_{D_0(X)}\mathbb{C}$, which makes the middle split exact sequence above the same up to isomorphism as the one defining $D_f(Y)$. Accordingly the maps h_0 and r_0 classify the same split extension with kernel Y, and so they are equal. $\square$

Corollary 3.2. *Given a pointed exact action representative protomodular category $\mathbb{C}$, the characteristic epimorphisms $X \twoheadrightarrow Y$ are in bijection with those internal functors $\underline{D}_1(X) \to \underline{D}_1(Y)$ which induce a regular epimorphism $X \twoheadrightarrow Y$.*

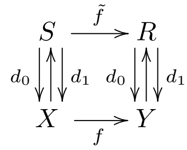
Proof. Starting with an internal functor $(h_0, h_1) : \underline{D}_1(X) \to \underline{D}_1(Y)$ as above, let us denote by $f : X \to Y$ the induced factorization. Again let us consider the decomposition through the pullback of h_0 along $\delta_0 Y$ as above. Then, when f is a regular epimorphism so is $\check{h}_1$, which makes f a characteristic epimorphism in $\mathbb{C}$ whose associated internal functor is precisely $(h_0, h_1) : \underline{D}_1(X) \to \underline{D}_1(Y)$. $\square$

4 Centralization of a characteristic subobjet

Recall that a Mal'tsev $\mathbb{C}$ is said to be action distinctive when, in the category $Pt\mathbb{C}$, any object (f, s) admits a largest ¶-cartesian equivalence relation on it, called its *action distinctive* equivalence relation [8]; a Mal'tsev category is action distinctive if and only if any equivalence relation R on X has a centralizer $Z[R]$, namely a largest equivalence relation among those which centralize R.

An action distinctive category $\mathbb{C}$ is said to be *functorially action distinctive* when, in addition, there is a (unique) functorial extension of any ¶-cartesian map up to the level of the ¶-distinctive equivalence relations. An action distinctive

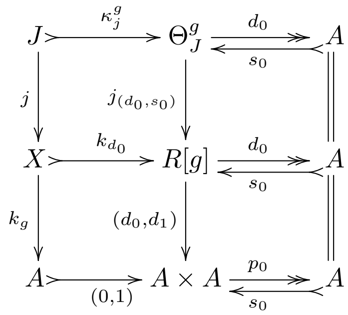
Mal'tsev category is functorially action distinctive if and only if, given any discrete fibration between equivalence relations:



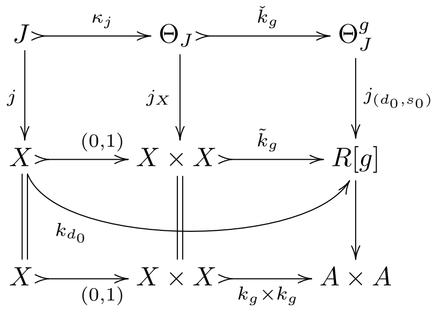
we have $Z[S] = f^{-1}(Z[R])$. In this section we shall produce some further examples of characteristic monomorphisms, among which the centralizers of the characteristic monomorphisms.

Theorem 4.1. *Let $\mathbb{C}$ be a pointed protomodular category which is functorially action distinctive. Given a characteristic monomorphism $j : J \rightarrowtail X$, the normalization of the centralizer $Z[\Theta_J]$ of the characteristic relation Θ_J is characteristic. Accordingly, given any object X, the normalization of the largest central equivalence relation $Z[X]$ on X is characteristic.*

Proof. Let $(g, t) : A \rightleftarrows B$ be any split epimorphism such that $\alpha_B^*(g, t) = X$. Let us consider the following diagram where $j_{(d_0,s_0)}$ is the hypercartesian map above j in $Pt_X\mathbb{C}$, see Corollary 2.1:



It makes Θ_J^g the equivalence subrelation of $R[g]$ which is normal to $k_g.j$. We know that the inverse image $k_g^{-1}(\Theta_J^g)$ is Θ_J, whence the following diagram where any square is a pullback:



So, the following right hand side morphism of equivalence relations is a discrete fibration:

$$\begin{array}{ccccc} J & \overset{\kappa_j}{\rightarrowtail} & \Theta_J & \overset{\check{k}_g}{\rightarrowtail} & \Theta_J^g \\ \downarrow & & d_0^S \downarrow\uparrow\downarrow d_1^S & & d_0 \downarrow\uparrow\downarrow d_1 \\ 1 & \rightarrowtail & X & \underset{k_g}{\rightarrowtail} & A \end{array}$$

because the two d_0 has the same kernel J. Since $\mathbb{C}$ is functorially action distinctive we get $Z[\Theta_J] = k_g^{-1}(Z[\Theta_J^g])$. Let us consider now the equivalence relation $\Gamma_g = Z[\Theta_J^g] \cap R[g]$; then we have $k_g^{-1}(\Gamma_g) = k_g^{-1}(Z[\Theta_J^g]) \cap k_g^{-1}(R[g]) = Z[\Theta_J] \cap \nabla_X = Z[\Theta_J]$. Next, consider the following diagram where any square is a pullback:

$$\begin{array}{ccccccc} Z_X^J & \overset{\tilde{\kappa}}{\rightarrowtail} & Z[\Theta_J] & \overset{\tilde{k}_g}{\rightarrowtail} & \Gamma_g & \leftarrowtail & A' \\ \zeta\downarrow & & \downarrow & & \downarrow & & \downarrow \\ X & \overset{(0,1)}{\rightarrowtail} & X \times X & \overset{\tilde{k}_g}{\rightarrowtail} & R[g] & \overset{s_1}{\leftarrowtail} & A \\ \downarrow & & p_0 \downarrow\uparrow\downarrow p_1 & & d_0 \downarrow\uparrow\downarrow d_1 & & g \downarrow\uparrow s \\ 1 & \rightarrowtail & X & \underset{k_g}{\rightarrowtail} & A & \underset{s}{\leftarrowtail} & B \end{array}$$

It defines the map ζ as the normalization of $Z[\Theta_J]$. We have to show that it is characteristic. The diagram shows that the inclusion $\Gamma_g \rightarrowtail R[g]$ is the (hypercartesian) map above ζ as well. Accordingly its image by the change of base $s^* : Pt_B\mathbb{C} \to Pt_A\mathbb{C}$ (on the vertical right hand side) produces the desired (hypercartesian) map above ζ (the map $A' \to B$ being split by the inverse image of the subdiagonal: $s_0 : A \rightarrowtail \Gamma_g$). The last assertion comes from the fact that the indiscrete equivalence relation ∇_X is the denormalization of the characteristic monomorphism $1_X : X \rightarrowtail X$. □

Any action accessible protomodular category [9] is functorially action distinctive [8]; consequently any action representative category or any category of interest [18] is an example of functorially action distinctive category by [17]. Finally we get:

Corollary 4.1. *Let $\mathbb{C}$ be a homological category which is action accesible. The centralizer of any characteristic subobject is characteristic as well. In particular, given any object X, the center $Z_X \rightarrowtail X$ of X is characteristic.*

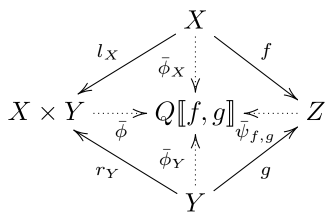
Proof. Any action accessible homological category is groupoid accessible, and in any groupoid accessible category the normalization of the centralizer $Z[R]$ of an equivalence relation R is the centralizer of the normalization I_R, see Proposition 5.2 in [9]. □

5 Commutator of two characteristic subobjects

It is well known that in the category Gp of groups the commutator of two characteristic subobjects is characteristic as well. In this section, we shall see under which conditions the commutator of two characteristic monomorphisms is characteristic. For that we have to recall the conceptual context in which the notion of commutator does exist. Actually this notion has two levels.

5.1 The general context

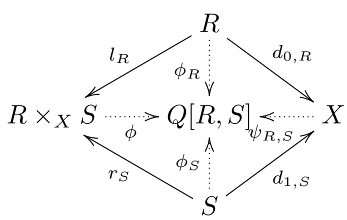
Let $\mathbb{C}$ be a pointed regular Mal'tsev category which is finitely cocomplete and let us recall from [2] the following constructions and results:
1) Given any pair $f : X \to Z$, $g : Y \to Z$ of coterminal maps, consider the following diagram, where $Q[\![f,g]\!]$ is the colimit of the diagram made of the plain arrows:



Clearly the maps $\bar{\phi}_X$ and $\bar{\phi}_Y$ are completely determined by the pair $(\bar{\phi}, \bar{\psi}_{f,g})$, and clearly the map $\bar{\phi}$ is the cooperator of the pair $(\bar{\psi}_{f,g}.f, \bar{\psi}_{f,g}.g)$ which assures the commutation of the two maps. On the other hand, the map $\bar{\psi}_{f,g}$ is a regular epimorphism which measures the lack of cooperation of the pair (f,g) [2]:

Proposition 5.1. *The map $\bar{\psi}_{f,g}$ is the universal morphism which, by composition, makes the pair (f,g) cooperate. The map $\bar{\psi}_{f,g}$ is an isomorphism if and only if the pair (f,g) cooperates.*

Accordingly the kernel relation $R[\bar{\psi}_{f,g}]$ is the relational commutator of the pair (f,g). When $\mathbb{C}$ is homological, the kernel $[\![f,g]\!]$ of $\bar{\psi}_{f,g}$ measures of the lack cooperation and is called the commutator of the pair (f,g).
2) On the other hand, the context of Mal'tsev categories is the right one to deal with the notion of centralization of equivalence relations, see [19] and [10]. In the finitely cocomplete regular context, given any pair (R,S) of equivalence relations on X, we denote by $Q[R,S]$ the colimit of the plain arrows in the following diagram:



The maps ϕ_R and ϕ_S are completely determined by the pair $(\phi, \psi_{R,S})$. The map $\psi_{R,S}$ is a regular epimorphism which measures the lack of centralization of R and S [2]:

Proposition 5.2. *The map $\psi_{R,S}$ is the universal regular epimorphism which makes the direct images $\psi_{R,S}(R)$ and $\psi_{R,S}(S)$ centralizing each other. The equivalence relations R and S centralize each other if and only if $\psi_{R,S}$ is an isomorphism.*

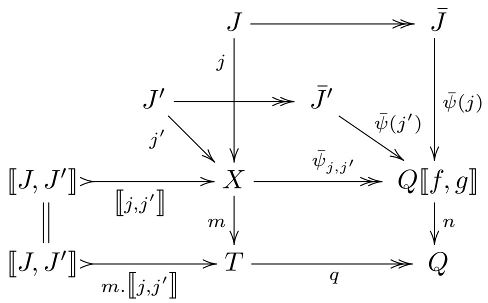
Accordingly the kernel relation $R[\psi_{R,S}]$ is the relational commutator of the pair (R, S) and we have $R[\psi_{R,S}] \subset R \cap S$. When $\mathbb{C}$ is homological, the kernel $[R, S]$ of $\psi_{R,S}$ which measures of the lack of centralization of (R, S) is the commutator of the pair (R, S). Given a pair (R, S) of equivalence relations, we denote by (I_R, I_S) the pair of their normalizations. Then we have $[\![I_R, I_S]\!] \subset [R, S] \subset I_R \cap I_S$, but the first two commutators do not coincide in general. So, in a pointed protomodular category, when (m, m') is a pair of normal monomorphisms, there are two kinds of commutator: the commutator $[\![m, m']\!]$ and the commutator $[m, m'] = [R^\sharp_m, R^\sharp_{m'}]$ where $R^\sharp_m$ denotes the denormalization of the normal monomorphism m. So we have $[\![m, m']\!] \subset [m, m']$. Recall also that a pointed Mal'tsev category is said to satisfy the condition SH ("Smith=Huq") when R and S centralize each other as soon as the normalizations of R and S commute, i.e. as soon as we have $[\![I_R, I_S]\!] = 0$. Under this condition, in the protomodular context, we have necessarily $[\![I_R, I_S]\!] = [R, S]$ for any pair of equivalence relations, and $[\![m, m']\!] = [m, m']$ for any pair of normal monomorphism. The categories *Gp*, *Rg* and *R-Lie* satisfy SH.

5.2 Commutator of two characteristic monomorphisms

Theorem 5.1. . *Let $\mathbb{C}$ be a pointed exact protomodular category with finite colimits. The following conditions are equivalent: given a pair (j, j') of characteristic monomorphisms of X,*
1) the regular epimorphism $\bar{\psi}_{j,j'}$ is characteristic
2) the commutator $[\![j, j']\!]$ is characteristic
3) for any normal monomorphism $m : X \rightarrowtail T$, we have $m.[\![j, j']\!] = [\![m.j, m.j']\!]$.

Proof. 1) $\Rightarrow$ 2) is straighforward by definition of a characteristic epimorphism. Suppose that condition 2) holds. Given any normal monomorphism $m : X \rightarrowtail T$, the monomorphism $m.[\![j, j']\!]$ is normal and admits a cokernel $q : T \twoheadrightarrow Q$ whose it is the kernel, since $\mathbb{C}$ is exact protomodular. Accordingly there is a factorization

$n : Q[\![f,g]\!] \to Q$:



This factorization makes the right hand side lower square a pullback and thus a pushout ($\mathbb{C}$ being protomodular); moreover, since m is a normal monomorphism, so is n. We have to show now that q is $\bar{\psi}_{m.j,m.j'}$.

For that, let us denote by $\bar{\psi}(j)$ and $\bar{\psi}(j')$ the direct images of j and j' along $\bar{\psi}_{j,j'}$. They are normal monomorphism as well. The right hand side square above implies that $n.\bar{\psi}(j)$ and $n.\bar{\psi}(j')$ are the direct images of the normal monomorphism $m.j$ and $m.j'$ along q; these images commute since $\bar{\psi}(j)$ and $\bar{\psi}(j')$ commute. Now let be given a regular epimorphism $q' : T \twoheadrightarrow Q'$ and let us denote by $q'(m)$ the direct image of m along q', and by $\bar{q} : X \twoheadrightarrow \bar{Q}$ the regular epimorphism such that $q'.m = q'(m).\bar{q}$. Suppose moreover that the direct images $q'(m).\bar{q}(j)$ and $q'(m).\bar{q}(j')$ of $m.j$ and $m.j'$ along q' do commute. Since $q'(m)$ is a monomorphism, $\bar{q}(j)$ and $\bar{q}(j')$ do commute as well. Accordingly there is a factorization $\bar{\chi} : Q[\![j,j']\!] \to \bar{Q}$ such that $\bar{\chi}.\bar{\psi}_{j,j'} = \bar{q}$. So that $q'(m).\bar{\chi}.\bar{\psi}_{j,j'} = q'(m).\bar{q} = q'.m$, and since the right hand side square above is a pushout, we get a factorization $\chi : Q \to Q'$ such that $\chi.q = q'$ and $\chi.n = q'(m).\bar{\chi}$. The first equality shows that q is nothing but $\psi_{m.j,m.j'}$ and its kernel $m.[\![j,j']\!]$ nothing but $[\![m.j,m.j']\!]$. Since a commutator is a kernel, it is a normal monomorphism; so the condition 3) implies that $[\![j,j']\!]$ is characteristic which means that $\bar{\psi}_{j,j'}$ is characteristic. □

These conditions are satisfied for any pair of characteristic subobjects in the categories *Gp* of groups and *R-Lie* of Lie R-algebras, and this gives in particular the reason why the lower central series in *Gp* and *R-Lie* are made of characteristic subobjects. In the category *Rg* of non unitary rings, the commutator $[\![J,J']\!]$ of two ideals is the ideal $JJ' + J'J$ and, from that, it is easy to check that, in *Rg*, any pair of characteristic subobjects satisfies the condition 3) of the theorem, which provides us with a large class of examples of characteristic monomorphisms in *Rg*. In *TopGp*, the subobject $[\![m.j,m.j']\!]$, being a kernel, is endowed with the induced topology, while the subobject $m.[\![j,j']\!]$ is not necessarily so unless the monomorphism m itself comes from an induced topology. We have also the following straightforward observation, but under very rather heavy conditions:

Proposition 5.3. *Let $\mathbb{C}$ be a pointed exact protomodular category with finite colimits. Suppose moreover that any change of base functor $\alpha_B^* : Pt_B\mathbb{C} \to \mathbb{C}$ preserves the colimits. Then if $j : J \rightarrowtail X$ and $j' : J' \rightarrowtail X$ are two characteristic monomorphisms:*
1) the regular epimorphism $\bar{\psi} : X \twoheadrightarrow Q[\![j, j']\!]$ is characteristic
2) the regular epimorphism $\psi : X \twoheadrightarrow Q[\Theta_J, \Theta_{J'}]$ is characteristic.
Accordingly the commutators $[\![j, j']\!]$ and $[J, J']$ are characteristic.

Proof. 1) Let $(g, t) : A \rightleftarrows B$ be any split epimorphism such that $\alpha_B^*(g, t) = X$. Consider now the hypercartesian maps $j_{(g,t)}$ and $j'_{(g,t)}$ in $Pt_B\mathbb{C}$ above j and j' respectively. Since α_B^* preserves the colimits, the regular epimorphism $\bar{\psi}_B : A \twoheadrightarrow Q[\![j_{(g,t)}, j'_{(g,t)}]\!]$ in this fibre is mapped to $\bar{\psi} : X \twoheadrightarrow Q[\![j, j']\!]$.
2) Denote by $\Theta_{J'}$ and $\Theta^g_{J'}$ the characteristic relations of the characteristic maps $j_{(g,t)}$ and $j'_{(g,t)}$ in $Pt_B\mathbb{C}$. Since α_B^* preserves the colimits, the regular epimorphism $\psi_B : A \twoheadrightarrow Q[\Theta^g_J, \Theta^g_{J'}]$ in $Pt_B\mathbb{C}$ is mapped to the regular epimorphism $\psi : X \twoheadrightarrow Q[\Theta_J, \Theta_{J'}]$. □

The change of base functors $\alpha_B^* : Pt_B\mathbb{C} \to \mathbb{C}$ preserves the colimits as soon as they have coadjoints, namely when the pointed protomodular category are *locally algebraically cartesian closed* in the sense of [11], as are *Gp* and *R-Lie* (see [16]), but neither *Rg*. Any pointed locally algebraically cartesian closed Mal'tsev category satisfies SH.

5.3 Peri-abelian categories

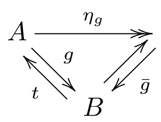
In particular, given any object X in a finitely cocomplete regular Mal'tsev category $\mathbb{C}$, the previous construction 1) applied to the pair $(1_X, 1_X)$ of normal monomorphisms amounts to the following coequalizer in $\mathbb{C}$ which produces the associated abelian object $A(X)$ [2]:

$$X \underset{\iota_1 X}{\overset{\iota_0 X}{\rightrightarrows}} X \times X \xrightarrow{\eta_X} A(X)$$

Since a Mal'tsev category $\mathbb{C}$ is such that any fibre $Pt_B\mathbb{C}$ is Mal'tsev as well, then it is possible, in each fibre $Pt_B\mathbb{C}$, to associate with any object a universal abelian object. Recall from [7] the following definition which was introduced for cohomological purpose:

Definition 5.1. *A finitely cocomplete, regular, pointed Mal'tsev category $\mathbb{D}$ is said to be peri-abelian when the change of base functor along any map $Y' \to Y$ with respect to the fibration of points preserves the associated abelian object.*

When $\mathbb{C}$ is protomodular, it is enough to check that the property holds for the change of base functors along the initial maps α_B. In the case $\mathbb{C} = Gp$ the category of groups, the associated abelian object in Pt_BGp is described by the following diagram where $\bar{A} = A/[\ker g, \ker g]$, since $[\ker g, \ker g]$ is a normal subgroup of A:



It is clear that $ker\bar{g} = \ker g/[\ker g, \ker g]$ is the universal abelian group $A(Kerg)$ associated with $Kerg$, so that the change of base with respect to the fibration of points along the initial map $\alpha_B : 1 \to B$ preserves the associated abelian object. So the category Gp of groups, is peri-abelian. By the same scheme of proof, the categories Rg of non unitary rings (where an abelian object in Rg is a trivial ring, namely a ring in which the multiplication is trivial) and $\mathbb{K}$-*Lie* of $\mathbb{K}$-Lie algebras are peri-abelian as well.

Proposition 5.4. *Given a finitely cocomplete, homological category $\mathbb{C}$ satisfying SH, the following conditions are equivalent:*
1) for any object object X, the unit $\eta_X : X \twoheadrightarrow A(X)$ of the abelianization functor is a characteristic regular epimorphism
2) $\mathbb{C}$ is peri-abelian.

Proof. If $\mathbb{C}$ is peri-abelian, and $(g,t) : A \rightleftarrows B$ is any object with kernel X in the fibre $Pt_B\mathbb{C}$, the unit $\eta_X : X \twoheadrightarrow A(X)$ has the unit $\eta_{(g,t)} : (g,t) \twoheadrightarrow A(g,t)$ in $Pt_B\mathbb{C}$ above it , and thus η_X is characteristic.
Conversely suppose that for any object object X, the unit $\eta_X : X \twoheadrightarrow A(X)$ of the abelianization functor is characteristic. Starting with any object $(g,t) : A \rightleftarrows B$ with kernel X in $Pt_B\mathbb{C}$ and taking the hypercocartesian map $\eta : (g,t) \twoheadrightarrow (\tilde{g}, \tilde{t})$ above it, the kernel of $\tilde{g}$ is the abelian group $A(X)$, and when $\mathbb{C}$ satisfies SH, $(\tilde{g}, \tilde{t})$ is an abelian object in $Pt_B\mathbb{C}$. Now let us denote $h : A(g,t) \to (\tilde{g}, \tilde{t})$ the induced factorization in $Pt_B\mathbb{C}$ from the universal abelian object associated with (g,t). It is certainly mapped by α_B^* on a isomorphism, so the kernel of the underlying split epimorphism of $A(g,t)$ is isomorphic to $A(X)$ and $\mathbb{C}$ is peri-abelian. □

Since any pointed locally algebraically cartesian closed Malt'sev category satisfies SH, any finitely cocomplete locally algebraically cartesian closed homological category is pari-abelian.

Corollary 5.1. *Given a finitely cocomplete, exact, pointed protomodular category $\mathbb{C}$ satisfying SH, the following conditions are equivalent:*
1) for any object object X, the unit $\eta_X : X \twoheadrightarrow A(X)$ of the abelianization functor is a characteristic regular epimorphism
2) the commutator $[\![1_X, 1_X]\!] \rightarrowtail X$ is characteristic
3) for any normal monomorphism $m : X \rightarrowtail T$, we have $m.[\![1_X, 1_X]\!] = [\![m, m]\!]$
4) $\mathbb{C}$ is peri-abelian.

Proof. It is a straightforward consequence of Theorem 5.1 and of the previous proposition. □

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