Mal'tsev reflection, S-Mal'tsev and S-protomodular categories

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Abstract

We show that the fibers of the fibration $(\ )_0 : Cat\mathbb{E} \to \mathbb{E}$ associated with the internal categories in $\mathbb{E}$ are endowed with a structure partially dealing with the Mal'tsev and protomodular contexts, and we investigate the properties of their associated partial Mal'tsev and protomodular concepts.

Introduction

It was showed in [2] that the fibers of the fibration $(\ )_0 : Grd\mathbb{E} \to \mathbb{E}$ associated with the internal groupoids in $\mathbb{E}$ are protomodular. No similar structural result existed for the fibers of the fibration $(\ )_0 : Cat\mathbb{E} \to \mathbb{E}$ associated with the internal categories in $\mathbb{E}$. A recent work about the category Mon of monoids [7] which is nothing but the fibre of this fibration above the singleton 1 focuses our attention on classes of split epimorphisms between monoids (called Schreier, left homogeneous and homogeneous split epimorphisms, see also [15]) satisfying partial aspects of the Mal'tsev and protomodular processes and properties. So it is quite natural to investigate whether the other fibers $Cat_Y\mathbb{E}$ would not satisfy some property of this kind. This would be all the more interesting since:
1) it would produce a similar conceptual situation in a non-pointed context
2) it would allow to better dissociate what belongs to partial Mal'tsevness and to partial protomodularity
3) it would allow to clarify the underlying relationship between partial Mal'tsevness or protomodularity and the "global" classical ones.

So the main point was to identify classes of split epimorphisms in the fibers $Cat_Y\mathbb{E}$ which would generalize the properties of the classes in question in Mon. It happened that there is a relatively simple answer with the class Σ of those split epimorphic functors in the fibers of $(\ )_0$ *which are split fibrations* as well. For a functor, being a fibration is a property; but being a split fibration needs moreover to specify a coherent choice of cartesian maps. However, in the case of a bijective on objects split epimorphic functor, being a split fibration becomes a property if you force the splitting to be itself the coherent choice of cartesian maps. In the fiber $Cat_1 = Mon$ such split epimorphisms coincide with the left homogenous split epimorphisms between monoids introduced in [7], where the definition was clearly refering to the set theoretical context, namely, was not intrinsic to the category Mon. Enlarging here the question to the whole category Cat allowed us to reach a definition which is only in concern with the category Cat itself, and consequently to extend it easily to any internal context $Cat\mathbb{E}$, and more genrally to any strongly representable 2-category in the sense of [11].

Starting from this specific investigation on the fibration $(\ )_0$, we are led to introduce a general concept of Mal'tsev reflection (Definition 2.2) which allows us on the one hand to define the Mal'tsev categories as those categories $\mathbb{C}$ which are such that the fibration of points $¶_{\mathbb{C}} : Pt\mathbb{C} \to \mathbb{C}$ is a Mal'tsev reflection, and on the other hand to restrict the Mal'tsev concept to a subclass S of split epimorphisms (see Definition 4.3 of S-Mal'tsev categories). In addition, and it is our first sructural point: the reflection $((\ )_0, \nabla) : Cat\mathbb{E} \rightleftarrows \mathbb{E}$ is itself a Mal'tsev one. In the same way, introducing the notion of strongly split epimorphism in the non-pointed context, we are able to restrict the protomodular concept to such a subclass S (again see Definition 4.3 of S-protomodular categories). Exactly as in the global context, a S-protomodular category is necessarily a S-Mal'tsev one (Theorem 4.1). The previous investigations in the category Cat allow us to assert our second structural point: any fibre $Cat_Y\mathbb{E}$ is endowed with a structure of Σ_Y-protomodular category, where Σ_Y denotes the class of those split epimorphic functors with fibrant splitting which belong to this fibre.

The article is organized along the following lines:
Section 1) is devoted to recalls and notations about internal categories; Section 2) is devoted to the definition of Mal'tsev reflections and their main properties; Section 3) is devoted to the investigation on the split epimorphic functors with fibrant splitting which leads to the structural properties of the fibres $Cat_Y\mathbb{E}$; Section 4) introduces the notions of S-Mal'tsev and S-protomodular categories and investigates how some well known results of the global Mal'tsev context remain valid in the S-relative one; Section 5) shortly gets back to the fibers $Cat_Y\mathbb{E}$ in order to examplify in this context some of the tools introduced in the conceptual section 4).

1 Brief recall about internal categories

1.1 Simplicial notations

Let $\mathbb{E}$ be finitely complete category. Recall that an internal reflexive graph in $\mathbb{E}$ is a diagram of the form:

$$X_1 \underset{d_1}{\overset{d_0}{\underset{\longleftarrow s_0 -}{\rightrightarrows}}} X_0$$

where we have $d_0 s_0 = 1_{X_0} = d_1 s_0$. A reflexive relation is a reflexive graph such that the pair (d_0, d_1) is jointly monomorphic.

An internal category in $\mathbb{E}$ is a reflexive graph as above such that the following pullback of split epimorphisms which defines the object X_2 as the internal "object of composable pairs":

$$\begin{array}{ccc} X_2 & \underset{d_0}{\overset{s_0}{\leftrightarrows}} & X_1 \\ {\scriptstyle d_2}\downarrow\uparrow{\scriptstyle s_1} & & {\scriptstyle d_1}\downarrow\uparrow{\scriptstyle s_0} \\ X_1 & \underset{d_0}{\overset{s_0}{\leftrightarrows}} & X_0 \end{array} \tag{1}$$

is endowed with a composition map $d_1 : X_2 \to X_1$ satisfying the further simplicial identities:
(1) $d_0d_1 = d_0d_0$, $d_1d_1 = d_1d_2$ (*incidence axioms*)
(2) $d_1s_0 = 1_{X_1}$, $d_1s_1 = 1_{X_1}$ (*composition with identities*)
In addition, this composition map must satisfy the *associativity axiom*; for that consider the following pullback of split epimorphisms which defines the object X_3 as the internal "object of composable triples":

$$\begin{array}{ccc} X_3 & \underset{d_0}{\overset{s_0}{\leftrightarrows}} & X_2 \\ {\scriptstyle d_3}\downarrow\uparrow{\scriptstyle s_2} & & {\scriptstyle d_2}\downarrow\uparrow{\scriptstyle s_1} \\ X_2 & \underset{d_0}{\overset{s_0}{\leftrightarrows}} & X_1 \end{array} \tag{2}$$

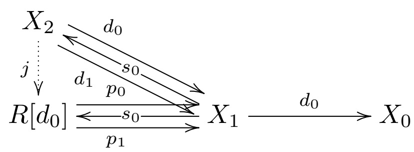
The composition map d_1 induces a unique couple of maps $(d_1, d_2)\colon X_3 \rightrightarrows X_2$ such that $d_0d_1 = d_0d_0$, $d_2d_1 = d_1d_3$ and $d_0d_2 = d_1d_0$, $d_2d_2 = d_2d_3$. The associativity axiom is given by the remaining simplicial axiom:
(3) $d_1d_1 = d_1d_2$.

An internal functor $f_\bullet : X_\bullet \to Y_\bullet$ is a natural transformation between 3-truncated simplicial objects. According to the pullbacks involved in the definition of an internal category, it is enough to have a natural transformation between the underlying 2-truncated simplicial objects.

An internal category $X_\bullet$ in a category $\mathbb{E}$ is *an internal groupoid* when, moreover, the following square determined by the composition map d_1 is actually a pullback:

$$\begin{array}{ccc} X_2 & \xrightarrow{d_1} & X_1 \\ {\scriptstyle d_0}\downarrow & & \downarrow{\scriptstyle d_0} \\ X_1 & \xrightarrow[d_0]{} & X_0, \end{array} \tag{3}$$

or, in other words, when the following vertical comparison map $j = (d_0, d_1)$ is an isomorphism:



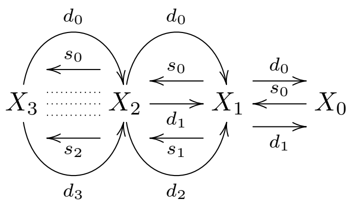
(4)

In this case we have a discrete fibration between groupoids:

$$\begin{array}{ccc} R[d_0] & \xrightarrow{d_2} & X_1 \\ {\scriptstyle d_0}\downarrow\uparrow\downarrow{\scriptstyle d_1} & & {\scriptstyle d_0}\downarrow\uparrow\downarrow{\scriptstyle d_1} \\ X_1 & \xrightarrow[d_1]{} & X_0 \end{array} \tag{5}$$

We shall denote by $Grd\mathbb{E}$ the full subcategory of $Cat\mathbb{E}$ whose objects are the internal groupoids.

In short, an internal category $X_\bullet$ is a 3-truncated simplicial object (for the general definition of a simplicial object, see [13] Chapter VII for instance, or more specifically [14]):



where the squares 1 and 2 are pullbacks; it is a groupoid if and only if any commutative square is a pullback.

1.2 The fibration of internal categories

The functor $(\ )_0 : Cat\mathbb{E} \to \mathbb{E}$ associating with any internal category $X_\bullet$ its "object of objects" X_0 is left exact and admits both a left and a right adjoint, given respectively, for any object $X \in \mathbb{E}$, by the indiscrete equivalence relation $\nabla_\bullet(X)$ and the discrete equivalence relation $\Delta_\bullet(X)$ on X. The left adjoint makes $(\ )_0$ a fibration whose cartesian maps are those internal functors $f_\bullet : X_\bullet \to Y_\bullet$ which are such that the following square is a pullback:

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ {\scriptstyle (d_0^X,d_1^X)}\downarrow & & \downarrow{\scriptstyle (d_0^Y,d_1^Y)} \\ X_0 \times X_0 & \xrightarrow[f_0 \times f_0]{} & Y_0 \times Y_0 \end{array}$$

namely those functors which are internally *fully faithful*. Accordingly any functor $h_\bullet : X_\bullet \to Y_\bullet$ has a canonical decomposition $X_\bullet \xrightarrow{\check{h}_\bullet} \tilde{X}_\bullet \xrightarrow{\tilde{h}_\bullet} Y_\bullet$ where $\tilde{h}_\bullet$ is fully faithful and $\check{h}_\bullet$ invertible on objects. This decomposition is stable under pullbacks in $Cat\mathbb{E}$. Any functor which is both $(\ )_0$-cartesian and invertible on objects is an isomorphism.

The fibre above X will be denoted by $Cat_X\mathbb{E}$; in this fibre $\Delta_\bullet(X)$ is an initial object, while $\nabla_\bullet(X)$ is a terminal one. The fibre $Cat_1\mathbb{E}$ is just the category $Mon\mathbb{E}$ of internal monoids in $\mathbb{E}$.

Proposition 1.1. *Any commutative square in $Cat\mathbb{E}$ where both $y_\bullet$ and $x_\bullet$ are $(\ )_0$-cartesian and both $t_\bullet$ and $t'_\bullet$ are invertible on objects:*

$$\begin{array}{ccc} X'_\bullet & \xrightarrow{x_\bullet} & X_\bullet \\ {\scriptstyle t'_\bullet}\downarrow & & \downarrow{\scriptstyle t_\bullet} \\ Y'_\bullet & \xrightarrow[y_\bullet]{} & Y_\bullet \end{array}$$

is a pullback.

Proof. Straightforward. □

1.3 The 2-category $Cat\mathbb{E}$

Equipped with the internal natural transformations (=homotopies between 2-truncated simplicial morphims), $Cat\mathbb{E}$ becomes a 2-category. In this context an internal functor $f_\bullet : X_\bullet \to Y_\bullet$ is fully faithful if and only if, given any natural tranformation $\gamma_\bullet : f_\bullet.g_\bullet \Rightarrow f_\bullet.g'_\bullet$, there exits a unique natural transformation $\beta_\bullet : g_\bullet \Rightarrow g'_\bullet$ such that $\gamma_\bullet = f_\bullet.\beta_\bullet$.

Actually the 2-category $Cat\mathbb{E}$ is *representable* [11], namely any internal category $X_\bullet$ admits a universal natural tranformation $\theta_\bullet(X)$ with codomain $X_\bullet$:

$$\theta_\bullet(X) : X^2_\bullet \underset{\delta^1_\bullet(X)}{\overset{\delta^0_\bullet(X)}{\rightrightarrows}} X_\bullet \quad (\Downarrow)$$

which gives rise to a left exact functor $(\ )^2 : Cat\mathbb{E} \to Cat\mathbb{E}$. Actually $Cat\mathbb{E}$ is even *strongly representable* [11] in the sense that the natural tranformation $\theta_\bullet(X)$ is 2-universal, or equivalently, that the functor $(\ )^2$ is a 2-functor.

This gives us a mean to characterize internal groupoids among internal categories.

Proposition 1.2. *An internal category $X_\bullet$ is an internal groupoid if and only if the internal functor $\delta^1_\bullet(X) : X^2_\bullet \to X_\bullet$ (resp. $\delta^0_\bullet(X)$) is $(\ )_0$-cartesian.*

Proof. Thanks to the Yoneda embedding it is sufficient to check it in Cat. That, for a groupoid $\mathbb{X}$, the functor $\delta^1(\mathbb{X})$ is fully faithful is straighforward. Conversely suppose it is fully faithful. Let $\phi : x \to x'$ be any map in $\mathbb{X}$. Consider the pair $(1_{x'}, \phi)$ of objects in $\mathbb{X}^2$, and the map $1_{x'} : \delta^1(\mathbb{X})(1_{x'}) = x' \to x' = \delta^1(\mathbb{X})(\phi)$. So there exists a unique map in $\mathbb{X}^2$ above it, namely a commutative square:

$$\begin{array}{ccc} x' & \xrightarrow{\psi} & x \\ {\scriptstyle 1_{x'}}\downarrow & & \downarrow{\scriptstyle \phi} \\ x' & \xrightarrow[1_{x'}]{} & x' \end{array}$$

This means that any map ϕ has a right inverse, and consequently an inverse. □

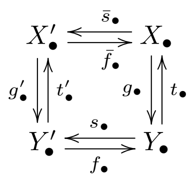
1.4 Cartesian split epimorphisms

Proposition 1.3. *A split epimorphism* $(f_\bullet, s_\bullet) : X_\bullet \rightleftarrows Y_\bullet$ *is* $(\ )_0$*-cartesian in* $Cat\mathbb{E}$ *if and only if there is moreover a natural isomorphism* $\gamma_\bullet : 1_{X_\bullet} \Rightarrow s_\bullet.f_\bullet$ *such that* $1_{f_\bullet} = f_\bullet.\gamma_\bullet$ *and* $1_{s_\bullet} = \gamma_\bullet.s_\bullet$.

Proof. Suppose $f_\bullet$ is cartesian (i.e. fully faithful) and split by $s_\bullet$. Accordingly, from the identity natural isomorphism from $f_\bullet$ to $f_\bullet = f_\bullet.s_\bullet.f_\bullet$, we get a natural isomorphism $\gamma_\bullet : 1_{X_\bullet} \Rightarrow s_\bullet.f_\bullet$ such that $1_{f_\bullet} = f_\bullet.\gamma_\bullet$. From that we have $f_\bullet.(\gamma_\bullet.s_\bullet) = f_\bullet.s_\bullet = 1_{Y_\bullet} = f_\bullet.1_{s_\bullet}$, whence $1_{s_\bullet} = \gamma_\bullet.s_\bullet$.

Conversely suppose we have a natural isomorphism $\gamma_\bullet : 1_{X_\bullet} \Rightarrow s_\bullet.f_\bullet$ satisfying the asserted equations. Starting with any natural transformation $\tau_\bullet : f_\bullet.h_\bullet \Rightarrow f_\bullet.h'_\bullet$, the natural transformation $\bar{\tau}_\bullet = (\gamma_\bullet h'_\bullet)^{-1}.s_\bullet\tau_\bullet.\gamma_\bullet h_\bullet : h_\bullet \Rightarrow h'_\bullet$ is the unique one such that $f_\bullet.\bar{\tau}_\bullet = \tau_\bullet$. □

Proposition 1.4. *Any commutative square of split epimorphisms in* $Cat\mathbb{E}$*:*



where both $f_\bullet$ *and* $\bar{f}_\bullet$ *are* $(\ )_0$*-cartesian and both* $g_\bullet$ *and* $g'_\bullet$ *are invertible on objects is such that the pair* $(t'_\bullet, \bar{s}_\bullet)$ *is jointly extremally epic.*

Proof. According to the previous proposition, this square is a pullback. By the Yoneda embedding it is enough to check the assertion in Cat. For sake of simplicity, we can suppose both $g_\bullet$ and $g'_\bullet$ bijective on objects. Accordingly, with the notation of the previous proposition, the natural isomorphisms $\gamma_\bullet : 1_{Y'_\bullet} \Rightarrow s_\bullet.f_\bullet$ and $\bar{\gamma}_\bullet : 1_{X'_\bullet} \Rightarrow s_\bullet.f_\bullet$ are necessarily such that $\bar{\gamma}_\bullet = t'_\bullet.\gamma_\bullet.g'_\bullet$. Now let $X''_\bullet \rightarrowtail X'_\bullet$ a subcategory "containing" $X_\bullet$ and $Y'_\bullet$. Let $\phi : \alpha \to \beta$ any map in $X'_\bullet$; we have. $\phi = \bar{\gamma}_\bullet(\beta)^{-1}.\bar{s}_\bullet\bar{f}_\bullet(\phi).\bar{\gamma}_\bullet(\alpha) = t'_\bullet(\gamma_\bullet(g'_\bullet\beta)^{-1}).\bar{s}_\bullet\bar{f}_\bullet(\phi).t'_\bullet(\gamma_\bullet(g'_\bullet\alpha))$. According to the assumption on X'', the map ϕ is in $X''_\bullet$. □

The property of the functor $(\ )_0$ described in the proposition above recaptures in dimension 1 some invertible aspects of the 2-categorical nature of $Cat\mathbb{E}$.

2 Fibered Mal'tsev reflection

In this section, we shall emphasize some formal aspects of the previous property of the fibration $(\ )_0 : Cat\mathbb{E} \to \mathbb{E}$.

2.1 Reflection, fibered reflection and Mal'tsev reflection

A *reflection* $(U,T) : \mathbb{C} \rightleftarrows \mathbb{D}$ is a functor U together with a fully faithful right adjoint $T : \mathbb{D} \rightarrowtail \mathbb{C}$. A map $\phi \in \mathbb{C}$ is said to be *U-invertible* when $U(\phi)$ is invertible and *U-cartesian* [1] when the following diagram is a pullback in $\mathbb{C}$:

$$\begin{array}{ccc} X & \xrightarrow{\ \phi\ } & Y \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ TU(X) & \xrightarrow[TU(\phi)]{} & TU(Y) \end{array}$$

This last definition was based upon the fact that this kind of maps is necessarily *hypercartesian* with respect to the functor U. The same kind of morphisms is called U-trivial in [12].

Definition 2.1. *A reflection $(U,T) : \mathbb{C} \rightleftarrows \mathbb{D}$ is be said to be a fibered reflection when, in addition, U is fibration and, for any object $D \in \mathbb{D}$, $T(D)$ is a terminal object in the fibre above D, so that we have $U.T = 1_{\mathbb{D}}$. A fibered reflection is said to be finitely complete when, moreover, the fibers are finitely complete and the change of base functors are left exact. It is said to be left exact when, furthermore, the category $\mathbb{D}$ is finitely complete.*

When the reflection is a left exact fibered one, then $\mathbb{C}$ is finitely complete as well, the U-cartesian maps are stable under pullbacks, and the functor U is a left exact functor. It was clearly observed that the reflection $((\ )_0, \nabla) : Cat\mathbb{E} \rightleftarrows \mathbb{E}$ is a left exact fibered one. For any fibered reflection, the unit $\tau_X : X \to TU(X)$ can be chosen as the terminal map in the fibre above $U(X)$, so that we have $U(\tau_X) = 1_{U(X)}$, and a map is cartesian with respect to U if and only if it U-cartesian as above. Now, from Proposition 1.4, let us introduce the following:

Definition 2.2. *Let be given a reflection $(U,T) : \mathbb{C} \rightleftarrows \mathbb{D}$. It will be said to be a Mal'tsev reflection when any square of split epimorphisms in $\mathbb{C}$:*

$$\begin{array}{ccc} X' & \overset{\bar{s}}{\underset{\bar{f}}{\leftrightarrows}} & X \\ {\scriptstyle g'}\downarrow\uparrow{\scriptstyle t'} & & {\scriptstyle g}\downarrow\uparrow{\scriptstyle t} \\ Y' & \overset{s}{\underset{f}{\leftrightarrows}} & Y \end{array}$$

where both f and $\bar{f}$ are U-cartesian and both g and g' are U-invertible (which implies that this square is necessarily a pullback), is such that the pair $(t', \bar{s})$ is jointly extremally epic.

According to Proposition 3.1, and **it is our first structural point: the reflection $((\ )_0, \nabla) : Cat\mathbb{E} \rightleftarrows \mathbb{E}$ is a Mal'tsev one**. The same holds for its restriction to internal groupoids $((\ )_0, \nabla) : Grd\mathbb{E} \rightleftarrows \mathbb{E}$.

Lemma 2.1. *Let be given a Mal'tsev reflection $(U, S) : \mathbb{C} \rightleftarrows \mathbb{D}$ and a square as in the definition above. Any commutative square of split epimorphisms in $\mathbb{C}$:*

$$\begin{array}{ccc} X'' & \underset{f''}{\overset{s''}{\leftrightarrows}} & X \\ {\scriptstyle g''}\downarrow\uparrow{\scriptstyle t''} & & {\scriptstyle g}\downarrow\uparrow{\scriptstyle t} \\ Y' & \underset{f}{\overset{s}{\leftrightarrows}} & Y \end{array}$$

is such that the factorization $h : X'' \to X'$ is an extremal epimorphism.

Proof. There is a factorization $h : X'' \to X'$ since, as we noticed, the square given in the definition is necessarily a pullback. Let us suppose now there is a monomorphism $h'' : Z \rightarrowtail X'$ such that $h = h''.h'$. Then, by the jointly extremal epic property produced by the Mal'tsev assumption, the factorizations $h'.s'' : X \to Z$ (such that $h''.(h'.s'') = \bar{s}$) and $h'.t'' : Y' \to Z$ (such that $h''.(h'.t'') = t'$) make $h'' : Z \rightarrowtail X'$ an isomorphism. □

2.2 Mal'tsev and unital categories

In this section we shall justify the previous terminology by Corollary 2.1 below. When $\mathbb{E}$ is a finitely complete category, recall that $Pt(\mathbb{E})$ denotes the category whose objects are the split epimorphisms in $\mathbb{E}$ and whose arrows are the commuting squares between such split epimorphisms, and that $¶_{\mathbb{E}}: Pt(\mathbb{E}) \to \mathbb{E}$ denotes the functor associating with any split epimorphism its codomain: it is *the fibration of points*. The fibre $Pt_Y(\mathbb{E})$ above Y is a pointed category whose zero object $I_{\mathbb{E}}(Y) = (1_Y, 1_Y) : Y \rightleftarrows Y$ is given functorially: it makes $(¶_{\mathbb{E}}, I_{\mathbb{E}}) : Pt\mathbb{E} \rightleftarrows \mathbb{E}$ a left exact fibered reflection.

Lemma 2.2. *Let $(U, T) : \mathbb{C} \rightleftarrows \mathbb{D}$ be a fibered reflection with pointed fibers. Then it is a Mal'tsev one if and only if, for any U-cartesian split epimorphism (f, s), the pair (s, α_X) is jointly extremally epic:*

$$\begin{array}{ccc} X & \underset{f}{\overset{s}{\leftrightarrows}} & Y \qquad E \\ {\scriptstyle \tau_X}\downarrow\uparrow{\scriptstyle \alpha_X} & & {\scriptstyle \tau_Y}\downarrow\uparrow{\scriptstyle \alpha_Y} \\ TU(X) & \underset{TU(f)}{\overset{TU(s)}{\leftrightarrows}} & TU(Y) \end{array}$$

Proof. When it is a Mal'tsev reflection, the previous square is one of those in the definition, and the pair (s, α_X) is jointly extremally epic. Conversely consider a diagram of split epimorphisms in $\mathbb{C}$ where both f and $\bar{f}$ are U-cartesian and both g and g' are U-invertible:

$$\begin{array}{ccc} X' & \underset{\bar{f}}{\overset{\bar{s}}{\leftrightarrows}} & X \\ {\scriptstyle g'}\downarrow\uparrow{\scriptstyle t'} & & {\scriptstyle g}\downarrow\uparrow{\scriptstyle t} \\ Y' & \underset{f}{\overset{s}{\leftrightarrows}} & Y \end{array}$$

To show that $(\bar{s}, t')$ is jointly extremally epic, it is enough to show that $(\bar{s}, t'.\alpha_{Y'})$ is so. But we have $(\bar{s}, t'.\alpha_{Y'}) = (\bar{s}, \alpha_{X'}.TU(t'))$, where $TU(t')$ is an isomorphism and $(\bar{s}, \alpha_{X'})$ is jointly extremally epic since $(\bar{f}, \bar{s})$ is a U-cartesian split epimorphism. □

It was showed in [3] that a finitely complete category $\mathbb{C}$ is a Mal'tsev one, namely a category in which any reflexive relation is an equivalence relation [9] [10], if and only if any fibre $Pt_Y\mathbb{C}$ is unital, a property which is tautologically equivalent to saying that the fibration $\P_{\mathbb{C}}$ is a Mal'sev one. Whence:

Corollary 2.1. *A finitely complete category $\mathbb{C}$ is a Mal'tsev one if and only if the (fibered) reflection of points $(\P_{\mathbb{C}}, I_{\mathbb{C}})$ is a Mal'tsev one.*

Unital categories can be also defined via a Mal'tsev reflection: let us denote by Π the class of split epimorphims which are, up to isomorphisms, the canonically split direct product projections:

$$X \times Y \underset{\langle 1_X, 0\rangle}{\overset{p_X}{\rightleftarrows}} X,$$

It is clear that this class is stable under pullbacks. Accordingly the full subcategory $j : \Pi Pt_{\mathbb{C}} \rightarrowtail Pt(\mathbb{C})$ whose objects are those which are in Π determines a cartesian subfibration:

$$\begin{array}{ccc} \Pi Pt\mathbb{C} & \xrightarrow{\ j\ } & Pt\mathbb{C} \\ & {}_{\P^{\Pi}_{\mathbb{C}}}\searrow \quad \swarrow_{\P_{\mathbb{C}}} & \\ & \mathbb{C} & \end{array}$$

The fact that the isomorphisms belong to Π makes $(\P^{\Pi}_{\mathbb{C}}, I^{\Pi}) : \Pi Pt\mathbb{C} \rightleftarrows \mathbb{C}$ a (pointed) fibered reflection.

Proposition 2.1. *Let $\mathbb{C}$ be a finitely complete pointed category. The category $\mathbb{C}$ is unital if and only if the pointed reflection $(\P^{\Pi}_{\mathbb{C}}, I^{\Pi})$ is a Mal'tsev one.*

Proof. Recall that a pointed category $\mathbb{C}$ is unital when, in the following pullback diagram:

$$\begin{array}{ccc} X \times Y & \underset{p_Y}{\overset{\langle 0, 1_Y\rangle}{\leftrightarrows}} & Y \\ {\scriptstyle p_X}\downarrow\uparrow{\scriptstyle \langle 1_X, 0\rangle} & & {\scriptstyle \tau_Y}\downarrow\uparrow{\scriptstyle \alpha_Y} \\ X & \underset{\tau_X}{\overset{\alpha_X}{\leftrightarrows}} & 1 \end{array}$$

the pair $(\langle 1_X, 0\rangle, \langle 0, 1_Y\rangle)$ is jointly extremally epic. This is clearly the case when the pointed fibered reflection is question is a Mal'tsev one. Conversely take any pullback diagram of split epimorphisms as the left hand side one below:

$$\begin{array}{ccccc}
X' \times Y & \underset{f\times Y}{\overset{s\times Y}{\rightleftarrows}} & X \times Y & \underset{p_Y}{\overset{\langle 0,1_Y\rangle}{\rightleftarrows}} & Y \\
{\scriptstyle p_{X'}}\downarrow\uparrow{\scriptstyle \langle 1_{X'},0\rangle} & & {\scriptstyle p_X}\downarrow\uparrow{\scriptstyle \langle 1_X,0\rangle} & & {\scriptstyle \tau_Y}\downarrow\uparrow{\scriptstyle \alpha_Y} \\
X' & \underset{f}{\overset{s}{\rightleftarrows}} & X & \underset{\tau_X}{\overset{\alpha_X}{\rightleftarrows}} & 1
\end{array}$$

We have to show that the pair $(\langle 1_{X'}, 0\rangle, s \times Y)$ is jointly extremally epic. It is enough to show it for the pair $(\langle 1_{X'}, 0\rangle, s\times Y.\langle 0, 1_Y\rangle)$, namely $(\langle 1_{X'}, 0\rangle, \langle 0, 1_Y\rangle)$. This is the case as soon as $\mathbb{C}$ is unital. □

2.3 Functors which are saturated on subobjects

It was also showed in [3] that a finitely complete category $\mathbb{C}$ is a Mal'tsev one if and only any fibre $Pt_Y\mathbb{C}$ is strongly unital. which is equivalent to saying that the change of base functor with respect to $\P_{\mathbb{C}}$ along any split epimorphism is *saturated on subobjects*. In this section, we are going to show that the same characterisation holds for any left exact fibered Mal'tsev reflection. Recall the following:

Definition 2.3. *A left exact functor $F : \mathbb{C} \to \mathbb{D}$ between finitely complete categories is said to be saturated on subobjects, when any monomorphism in $\mathbb{C}$ is cartesian and when any monomorphism $A \rightarrowtail F(B)$ in $\mathbb{D}$ admits a hypercartesian map above it.*

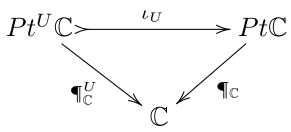
Recall that a hypercartesian map above a monomorphism is necessarily a monomorphism. So, when a left exact functor is saturated on subobjects, any subobject of $F(B)$ comes from a unique, up to isomorphism, subobject of B.

Lemma 2.3. *Suppose that the left exact saturated on subobjects functor F has a left exact left inverse S. Then:*
1) any monomorphism $m : A \rightarrowtail F(B)$ such that $S(m)$ is an isomorphism is itself an isomorphism
2) the functor F is fully faithful
3) any map $\phi : F(B) \to E$ such that $S(\phi)$ is a monomorphism is itself a monomorphism.

Proof. 1) Since F is saturated on subobjects there is a monomorphism $\bar{m} : \bar{A} \rightarrowtail B$ such that $F(\bar{m}) = m$. So $\bar{m} = SF(\bar{m}) = S(m)$ is an isomorphism, and so is $m = F(\bar{m})$.
2) F is faithful since it has a left inverse. Now let $\phi : F(B) \to F(B')$ be any map in $\mathbb{D}$; so we get $S(\phi) : B \to B'$. We have to compare ϕ and $FS(\phi)$. For that take the equalizer $k : J \rightarrowtail F(B)$ of this pair. Since S is left exact, $S(k)$ is an isomorphism. According to 1), k is itself an isomorphism, and $\phi = FS(\phi)$.
3) Let $\phi : F(B) \to E$ be such that $S(\phi)$ is a monomorphism. Consider its kernel equivalence relation $(d_0^\phi, d_1^\phi) : R[\phi] \rightarrowtail F(B) \times F(B) = F(B \times B)$. Since

F is saturated on subobjects, there is a relation $(\delta_0, \delta_1) : R \rightarrowtail B \times B$ such that $F(R) = R[\phi]$ and $F(\delta_0, \delta_1) = (d_0^\phi, d_1^\phi)$, whence $(\delta_0, \delta_1) = SF(\delta_0, \delta_1) = S(d_0^\phi, d_1^\phi)$. Since $S(\phi)$ is a monomorphism and S is left exact, $S(d_0^\phi, d_1^\phi)$ is up to isomorphism the diagonal $s_0^B : B \rightarrowtail B \times B$. So $(d_0^\phi, d_1^\phi) = F(\delta_0, \delta_1)$ is up to isomorphism the diagonal $F(B) \rightarrowtail F(B) \times F(B)$ and ϕ is a monomorphism. □

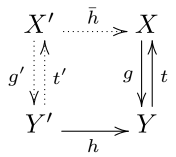
Let us start now with a fibered reflection (U, S). Let us denote by $Pt^U\mathbb{C}$ the full subcategory of $Pt\mathbb{C}$ whose objects are the split epimorphisms in the fibers of U and by $¶_{\mathbb{C}}^U$ the functor obtained by the following composition:



The zero object $(1_Y, 1_Y)$ in the fibre $Pt_Y\mathbb{C}$ lying in $Pt^U\mathbb{C}$, we get a pointed fibered reflection $(¶_{\mathbb{C}}^U, I_{\mathbb{C}}^U) : Pt^U\mathbb{C} \rightleftarrows \mathbb{C}$.

Lemma 2.4. *When (U, S) is a finitely complete (resp. left exact) fibered reflection, so is $(¶_{\mathbb{C}}^U, I_{\mathbb{C}}^U)$, and the functor ι_U is cartesian.*

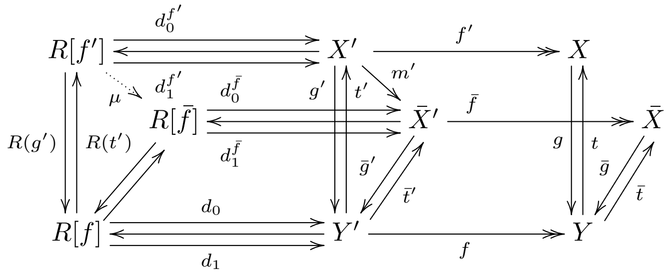
Proof. Condider the following plain arrows in $\mathbb{C}$ where (g, t) is a split epimorphism in the fibre above $U(Y)$:



When h is in a fibre, the $¶_{\mathbb{C}}^U$-cartesian map above it is given by the pullback in the fibre above $U(Y)$, and it is preserved by ι_U. When h is U-cartesian, take the U-cartesian map $\bar{h}$ above $U(h)$, it determines a factorisation g' in the fibre above $U(Y')$ which makes the square a pullback in $\mathbb{C}$. The $¶_{\mathbb{C}}^U$-cartesian map is completed by the splitting t'. Saying that we have a pullback in $\mathbb{C}$ is saying that this cartesian map is preserved by ι_U. Since (U, S) is a finitely complete, so is $(¶_{\mathbb{C}}^U, I_{\mathbb{C}}^U)$. It is left exact, when so is (U, S). □

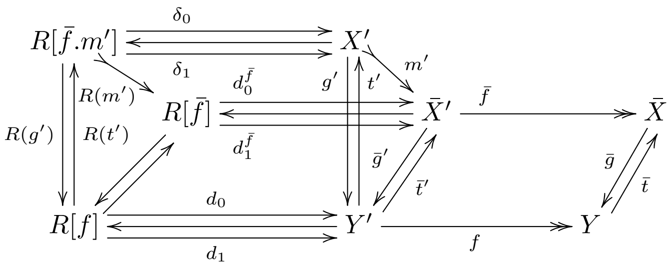
Theorem 2.1. *Let $(U, T) : \mathbb{C} \rightleftarrows \mathbb{D}$ be a left exact fibered Mal'tsev reflection. The change of base functor with respect to the fibration $¶_{\mathbb{C}}^U$ along any U-cartesian pullback stable regular epimorphism f is fully faithful; it is saturated on subobjects when f is a U-cartesian split epimorphism.*

Proof. 1) Full faithfulness. Consider the U-cartesian pullback stable regular epimorphism f and the following right hand side pullbacks of split epimorphims in the fibers of U along f; suppose in addition we have a map m' of split epimorphisms in the fibre above $U(Y')$; then complete the diagram on the left hand side with the kernel equivalence relations:



since all the commutative squares are pullbacks, there is a unique factorization $\mu : R[f'] \to R[\bar{f}]$ such that we have $d_0^{\bar{f}}.\mu = m'.d_0^{f'}$ and $R(\bar{g}').\mu = R(g')$; then certainly we get $\mu.s_0^{f'} = s_0^{\bar{f}}.m'$ and $\mu.R(t') = R(\bar{t}')$. We can check that $d_1^{\bar{f}}.\mu = m'.d_1^{f'}$ as well by composition with the pair $(s_0^{f'}, R(t'))$ since it is jointly extremally epic under the assumption of the Malt'ev property. Then the pair (m', μ) is underlying an internal functor $R[f'] \rightarrowtail R[\bar{f}]$. Since f is a pullback stable regular epimorphism, so is f'. Accordingly there is a unique map $m : X \to \bar{X}$ such that $m.f' = \bar{f}.m'$; we get also $\bar{g}.m = g$, and m is a map of split epimorphisms in the fibre above $U(Y)$.

2) Saturation on subobjects. Suppose now the U-cartesian morphism f is split by s; then it is a pullback stable regular epimorphism and the change of base functor along f in question is certainly fully faithful according to 1). Now consider the following diagram where the right hand side quadrangle is a pullback of split epimorphisms in the fibers of U along f and m' a subobject in the fibre above $U(Y')$:



Complete it on the left hand side with the kernel equivalence relations. According to Lemma 2.1, the induced factorization μ from $R[\bar{f}.m']$ to the domain of the pullback of the U-invertible split epimorphism g' along the U-cartesian split epimorphism d_0 is an extremal epimorphism, while it is a monomorphism since $R(m')$ is so. Accordingly, it is an isomorphism, which means that the left hand side downward vertical square indexed by 0 is a pullback. This square makes the pair $(g', R(g'))$ underlying a discrete cofibration between equivalence relations; it is consequently a discrete fibration which means that the left hand side downward vertical square indexed by 1 is a pullback as well. Since the lower left hand side quadrangles are pullbacks, this makes the pair $(m', R(m'))$ underlying a discrete (co-)fibration between the equivalence relations $R[\bar{f}.m']$ and $R[\bar{f}]$ as well.

Since f is split, certainly $\bar{f}$ is split and is certainly the quotient of $R[\bar{f}]$. It remains to show that we have the quotient of $R[\bar{f}.m']$ as well. More precisely, the splitting s of f determines: 1) a splitting s_1 of d_1 such that $d_0.s_1 = s.f$, 2) a splitting $\bar{s}$ of $\bar{f}$ such that $\bar{g}'.\bar{s} = s.\bar{g}$; whence a splitting $\bar{s}_1$ of $d_1^{\bar{f}}$ such that $d_0^{\bar{f}}.\bar{s}_1 = \bar{s}.\bar{f}$ and $R(\bar{g}').\bar{s}_1 = s_1.\bar{g}'$; and finally, according to the last paragraph, 3) a splitting σ_1 of δ_1 such that $R(g').\sigma_1 = s_1.g'$ and $R(m').\sigma_1 = \bar{s}_1.m'$. Since m' is a monomorphism and since we have $d_0^{\bar{f}}.\bar{s}_1.d_0\bar{f} = d_0^{\bar{f}}.\bar{s}_1.d_1\bar{f}$, we get $\delta_0.\sigma_1.\delta_0 = \delta_0.\sigma_1.\delta_1$ (*). This implies that $\delta_0.\sigma_1$ is an idempotent.

Take the equalizer $s' : X \to X'$ of $\delta_0.\sigma_1$ and $1_{X'}$, it determines a map $f' : X' \to X$ such that $f'.s' = 1_X$ and $s'.f' = \delta_0.\sigma_1$. The equation (*) implies that f' is the coequalizer of δ_0 and δ_1, then certainly we have $R[\bar{f}.m'] \subset R[f']$. Actually we have $R[\bar{f}.m'] \simeq R[f']$ because of the map σ_1. As a consequence, the monomorphism m' determines a map $m : X \to \bar{X}$ such that the following diagram commute:

$$\begin{array}{ccccc} R[\bar{f}.m'] = R[f'] & \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} & X' & \overset{f'}{\twoheadrightarrow} & X \\ \downarrow{\scriptstyle R(m')} & & \downarrow{\scriptstyle m'} & & \downarrow{\scriptstyle m} \\ R[\bar{f}] & \underset{d_1^{\bar{f}}}{\overset{d_0^{\bar{f}}}{\rightrightarrows}} & \bar{X}' & \underset{\bar{f}}{\twoheadrightarrow} & \bar{X} \end{array}$$

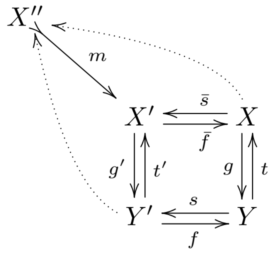
By the Barr-Kock Theorem, since the left hand side squares are pullbacks, so is the right hand side one, and the map m is a monomorphism, since so is the map m'; on the other hand the map f' is now U-cartesian since so is $\bar{f}$. In addition, the following square given by the sections is a pullback with s, and consequently s', U-cartesian:

$$\begin{array}{ccc} X' & \overset{s'}{\leftarrow} & X \\ {\scriptstyle m'}\downarrow & & \downarrow{\scriptstyle m} \\ \bar{X}' & \underset{\bar{s}}{\leftarrow} & \bar{X} \end{array}$$

in this way, since the map m' is in a fibre of U, so is m which becomes a morphism of split epimorphisms in the fibre above $U(Y)$. This map m is hypercartesian above m' with respect to the change of base along f since this change of base functor is already known to be fully faithful. □

Corollary 2.2. *Given any left exact fibered reflection $(U, T) : \mathbb{C} \rightleftarrows \mathbb{D}$, it is a Mal'tsev one if and only if the change of base functors with respect to the fibration $¶_{\mathbb{C}}^{U}$ along any U-cartesian split epimorphism (f, s) are saturated on subobjects.*

Proof. We just showed it is a necessary condition. Conversely consider the following diagram of split epimorphisms in $\mathbb{C}$ where both f and $\bar{f}$ are U-cartesian and both g and g' in a fibre:



Suppose m is a monomorphism with the two dotted factorizations. Then certainly $U(m)$ is an isomorphism. Taking the U-cartesian maps above $U(m)$ allows us to consider the same situation as above, with now m in a fibre. The "vertical" dotted factorization makes m a monomorphism in the fibre above $U(Y')$, while the "horizontal" dotted factorization insures that $f^*(m)$ is an isomorphism. Since the change of base functor f^* is saturated on subobjects, then, following the first point of Lemma 2.3, m is itself an isomorphism. □

3 Functors with fibrant splitting

In this section we shall introduce and study a special class of split epimorphisms in $Cat\mathbb{E}$. Although a bit technical, this section is the most important one since this class will be showed to satisfy a very strong structural property in the fibers $Cat_Y\mathbb{E}$ which will lead us to the notion of S-Mal'tsev and S-protomodular categories.

3.1 A glance at Cat

Let us focus momentarily our attention on the category Cat of categories.

Definition 3.1. *A split epimorphic functor $(F, S) : \mathbb{X} \rightleftarrows \mathbb{Y}$ in Cat will be said to have a fibrant splitting when it is bijective on objects and such that, for any map $\phi \in \mathbb{Y}$, the map $S(\phi)$ is a cartesian map in $\mathbb{X}$.*

In this way, the cartesian maps being composable, the functor F becomes a fibration with a coherent choice of cartesian maps, namely (following the usual terminology) a *split fibration.* In other words, a split epimorphism with a fibrant splitting is nothing but *a bijective on objects split fibration.* Accordingly *the split epimorphisms with fibrant splitting are stable under composition* since so are the two notions of split fibrations and bijective on objects split epimorphisms. Let us denote by $\Sigma Pt_{\mathbb{Y}}$ the category of functors with a given fibrant splitting above the category $\mathbb{Y}$ and by $\Sigma PtCat$ the full subcategory of $PtCat$ whose objects are those which belong to Σ. It is clear that $\Sigma PtCat$ is a category with finite limits which are preserved by the inclusion in $PtCat$.

Lemma 3.1. *For any category $\mathbb{Y}$, the category $\Sigma Pt_{\mathbb{Y}}$ is equivalent to the functor category $\mathcal{F}(\mathbb{Y}^{op}, Mon)$, where Mon is the category of monoids.*

Proof. The classical Grothendiek construction associates with any functor $\phi : \mathbb{Y}^{op} \to Mon$ a bijective on objects split fibration $F : \mathbb{X} \to \mathbb{Y}$. Because of the bijection on objects, the coherent choice of cartesian maps coincide with a splitting: $S : \mathbb{Y} \to \mathbb{X}$. Conversely, according to the previous remark, any functor with a fibrant splitting gives rise to a split fibration which, itself, by the inverse of the Grothendiek construction, produces a functor $\phi : \mathbb{Y}^{op} \to Cat$. Now, F being bijective on objects, its fibers have only one object and are monoids; accordingly ϕ takes its values in Mon. □

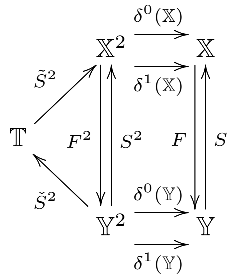
Proposition 3.1. *Let $\mathbb{Y}$ be a groupoid. Any bijective on objects split epimorphism (F, S) has a fibrant splitting.*

Proof. Given a map $\phi \in \mathbb{Y}$, we have to show that the map $S(\phi)$ is cartesian. Starting with a map $\psi \in \mathbb{X}$ such that $F(\psi) = \phi$, the desired unique factorization through $S(\phi)$ is certainly $S(\phi^{-1}).\psi$. □

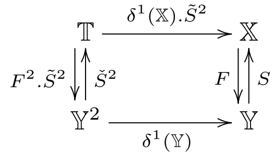
We recalled that the fibre above 1 of the fibration $(\ )_0 : Cat \to Set$ is the category Mon of monoids. The split epimorphisms with a fibrant splitting in this fibre coincide with those split epimorphims in Mon which are called a *left homogeneous* in [7], namely such that, for any element $b \in B$, the map $\mu_b \colon K[f] \to f^{-1}(b)$ defined by the multiplication on the left by $s(b)$, as $\mu_b(k) = s(b) \cdot k$, is bijective. They were extensively studied in [7], see also [15].

We give now a categorical characterization inside Cat:

Proposition 3.2. *A split epimorphic functor $(F, S) : \mathbb{X} \rightleftarrows \mathbb{Y}$ has a fibrant splitting if and only if, from the following diagram, where the triangle on the left hand side gives the canonical decomposition of the functor: $S^2 : \mathbb{Y}^2 \rightarrowtail \mathbb{X}^2$ associated with the fibration $(\ )_0 : Cat \to Set$*



we get a downward pullback of invertible on objects split epimorphic functors:



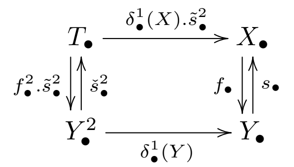
Proof. Let us denote by $(G, T) : \mathbb{P} \rightleftarrows \mathbb{Y}^2$ the pullback of $(F, S) : \mathbb{X} \rightleftarrows \mathbb{Y}$ along $\delta^1(\mathbb{Y})$. Our condition means that the natural comparison functor $H : \mathbb{T} \to \mathbb{P}$ is an isomorphism. Since H is necessarily invertible on objects, it is equivalent to say that it is fully faithful.

In set theoretical terms, the objects of $\mathbb{T}$ are the maps α of $\mathbb{Y}$, and a map $\alpha \to \alpha'$ in $\mathbb{T}$ is a pair (ϕ, ψ) of maps in $\mathbb{X}$ such that $S(\alpha').\phi = \psi.S(\alpha)$. The objects of $\mathbb{P}$ are the same as those of $\mathbb{T}$, and a map $\alpha \to \alpha'$ in $\mathbb{P}$ is a pair (β, ψ) of maps in $\mathbb{Y} \times \mathbb{X}$ such that $\alpha'.\beta = F(\psi).\alpha$. The comparison functor $H : \mathbb{T} \to \mathbb{P}$ is defined by $H(\phi, \psi) = (F(\phi), \psi)$. Saying that it is fully faithful is saying given any map $(\beta, \psi) : \alpha \to \alpha'$ in $\mathbb{P}$, there is a unique map ϕ in $\mathbb{X}$ such that $F(\phi) = \beta$ and $S(\alpha').\phi = \psi.S(\alpha)$. Taking ψ equal to an identity map insures that any map $S(\alpha)$ is hypercartesian and thus cartesian. Conversely suppose that any $S(\alpha)$ is cartesian; since such maps are stable under composition, they are hypercartesian, and clearly make the functor H fully faithful. □

3.2 Back to $Cat\mathbb{E}$

It is clear that the condition described above makes sense in any representable 2-category and in particular in the 2-category $Cat\mathbb{E}$ when $\mathbb{E}$ is finitely complete.

Definition 3.2. *A split epimorphic functor* $(f_\bullet, s_\bullet) : X_\bullet \rightleftarrows Y_\bullet$ *in* $Cat\mathbb{E}$ *will be said* to have a fibrant splitting *when it is in a fibre of* $Cat\mathbb{E}$ *(i.e.* f_0 *is an identity map) and such that the condition described in the previous proposition holds, namely when the following diagram:*



is a pullback in $Cat\mathbb{E}$.

Clearly any isomorphism in a fibre produces a split epimorphism with a fibrant splitting. Thank to the Yoneda embedding we get the following:

Proposition 3.3. *Let* $Y_\bullet$ *be a internal groupoid in* $\mathbb{E}$. *Any split epimorphism* $(f, s) : X_\bullet \rightleftarrows Y_\bullet$ *in a fibre of* $(\ )_0 : Cat\mathbb{E} \to \mathbb{E}$ *has a fibrant splitting.*

In particular any functor $f_\bullet : X_\bullet \to \Delta_\bullet(Y)$ in the fibre $Cat_Y\mathbb{E}$, being split since the groupoid $\Delta_\bullet(Y)$ is an initial object in this fibre, produces a split epimorphism with a fibrant splitting. In the same way, any split epimorphism $(f, s) : X_\bullet \rightleftarrows \nabla_\bullet(Y)$ in the fibre $Cat_Y\mathbb{E}$ has a fibrant splitting since $\nabla_\bullet(Y)$ is a groupoid as well.

3.3 Stability properties

We shall denote by Σ the class of split epimorphic functors with fibrant splitting in $Cat\mathbb{E}$, by $\Sigma PtCat\mathbb{E}$ the full subcategory of $PtCat\mathbb{E}$ whose objects are in Σ and by Σ_Y the class of split epimorphic functors with fibrant splitting in the fibre $Cat_Y\mathbb{E}$. By the Yoneda embedding, we get from the same result in Cat the following:

Proposition 3.4. *The class Σ is stable under composition in $Cat\mathbb{E}$. The class Σ_Y is stable under finite limits in $Cat_Y\mathbb{E}$.*

Starting with the functor $U = (\ )_0 : Cat\mathbb{E} \to \mathbb{E}$ and following the construction of Lemma 2.4 we get the cartesian subfibration on the right hand side:

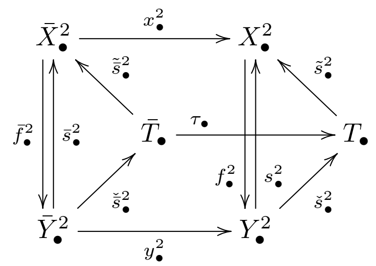
$$\begin{array}{ccccc}
\Sigma PtCat\mathbb{E} & \xrightarrow{j_0} & Pt^0Cat\mathbb{E} & \xrightarrow{\iota_0} & PtCat\mathbb{E} \\
\downarrow{\scriptstyle ¶^{\Sigma}_{Cat\mathbb{E}}} & & \downarrow{\scriptstyle ¶^{0}_{Cat\mathbb{E}}} & & \downarrow{\scriptstyle ¶_{Cat\mathbb{E}}} \\
Cat\mathbb{E} & = & Cat\mathbb{E} & = & Cat\mathbb{E}
\end{array}$$

Proposition 3.5. *Let $\mathbb{E}$ be a finitely complete category. The functor $¶^{\Sigma}_{Cat\mathbb{E}}$ is a cartesian subfibration of $¶^{0}_{Cat\mathbb{E}}$*

Proof. Let $(f_\bullet, s_\bullet)$ be in Σ. Consider the following cartesian map in $Pt^0Cat\mathbb{E}$, namely a pullback in $Cat\mathbb{E}$ such that $(\bar{f}_\bullet, \bar{s}_\bullet)$ is an identity on the objects:

$$\begin{array}{ccc}
\bar{X}_\bullet & \xrightarrow{x_\bullet} & X_\bullet \\
{\scriptstyle \bar{f}_\bullet}\downarrow\uparrow{\scriptstyle \bar{s}_\bullet} & & {\scriptstyle f_\bullet}\downarrow\uparrow{\scriptstyle s_\bullet} \\
\bar{Y}_\bullet & \xrightarrow[y_\bullet]{} & Y_\bullet
\end{array}$$

The functor $(\ )^2 : Cat\mathbb{E} \to Cat\mathbb{E}$ being left exact the following downward commutative square is a pullback while, the canonical decomposition associated with the fibration $(\ )_0$ being stable under pullbacks as well, there is a functor $\tau_\bullet$ making pullbacks the following upward commutative quadrangles:



Now the whole following commutative rectangles are the same:

$$
\begin{array}{ccccc}
\bar{T}_\bullet & \xrightarrow{\tau_\bullet} & T_\bullet & \xrightarrow{\delta(X)^1_\bullet.\tilde{s}^2_\bullet} & X_\bullet \\
{\scriptstyle \bar{f}^2_\bullet.\tilde{\bar{s}}^2_\bullet}\downarrow\uparrow{\scriptstyle \check{\bar{s}}^2_\bullet} & & {\scriptstyle f^2_\bullet.\tilde{s}^2_\bullet}\downarrow\uparrow{\scriptstyle \check{s}^2_\bullet} & & {\scriptstyle f_\bullet}\downarrow\uparrow{\scriptstyle s_\bullet} \\
\bar{Y}^2_\bullet & \xrightarrow[y^2_\bullet]{} & Y^2_\bullet & \xrightarrow[\delta(Y)^1_\bullet]{} & Y_\bullet
\end{array}
\qquad
\begin{array}{ccccc}
\bar{T}_\bullet & \xrightarrow{\delta(\bar{X})^1_\bullet.\tilde{\bar{s}}^2_\bullet} & \bar{X}_\bullet & \xrightarrow{x_\bullet} & X_\bullet \\
{\scriptstyle \bar{f}^2_\bullet.\tilde{\bar{s}}^2_\bullet}\downarrow\uparrow{\scriptstyle \check{\bar{s}}^2_\bullet} & & {\scriptstyle \bar{f}_\bullet}\downarrow\uparrow{\scriptstyle \bar{s}_\bullet} & & {\scriptstyle f_\bullet}\downarrow\uparrow{\scriptstyle s_\bullet} \\
\bar{Y}^2_\bullet & \xrightarrow[\delta(\bar{Y})^1_\bullet]{} & \bar{Y}_\bullet & \xrightarrow[y_\bullet]{} & Y_\bullet
\end{array}
$$

The left hand side rectangle is a pullback as made of two pullbacks: the right hand side one since $(f_\bullet, s_\bullet)$ has a fibrant splitting, and the left hand side one since we started with a pullback. So, the right hand side rectangle is a pullback. Its right hand side part is a pullback by assumption. Accordingly, its left hand part is a pullback, which means that the split epimorphic functor $(\bar{f}_\bullet, \bar{s}_\bullet)$ has a fibrant splitting as well. □

So, from Proposition 3.3, we get the following class of examples:

Corollary 3.1. *Suppose that $Y_\bullet$ is an internal groupoid and $h_\bullet : \bar{Y}_\bullet \to Y_\bullet$ any internal functor. Let $(f_\bullet, s_\bullet)$ be any split epimorphism in the fibre $Cat_{Y_0}\mathbb{E}$. Consider the following pullback in $Cat\mathbb{E}$ where $(\bar{f}_\bullet, \bar{s}_\bullet)$ is in the fibre $Cat_{\bar{Y}_0}\mathbb{E}$:*

$$
\begin{array}{ccc}
\bar{X}_\bullet & \xrightarrow{x_\bullet} & X_\bullet \\
{\scriptstyle \bar{f}_\bullet}\downarrow\uparrow{\scriptstyle \bar{s}_\bullet} & & {\scriptstyle f_\bullet}\downarrow\uparrow{\scriptstyle s_\bullet} \\
\bar{Y}_\bullet & \xrightarrow[y_\bullet]{} & Y_\bullet
\end{array}
$$

Then $(\bar{f}_\bullet, \bar{s}_\bullet)$ has a fibrant splitting.

Proposition 3.6. *Let $\mathbb{E}$ be a finitely complete category. The category $\Sigma PtCat\mathbb{E}$ is finitely complete and the functor j_0 (and consequently the functor $\iota_0.j_0$) is left exact.*

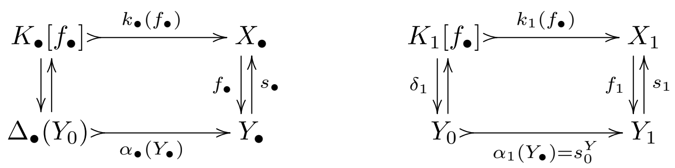
Proof. Since the fibration $(\ )_0 : Cat\mathbb{E} \to \mathbb{E}$ is left exact, products and pullbacks in $Cat\mathbb{E}$ commute with the canonical decomposition. Moreover, they commute with $(\ )^2$ since this functor is left exact. Accordingly products and pullbacks commute with any construction allowing the definition of a split epimorphic functor with fibrant splitting. So these split epimorphic functors are stable under products and pullbacks in $Pt^0(Cat\mathbb{E})$ and $Pt(Cat\mathbb{E})$. □

Finally, the zero object $(1_{Y_\bullet}, 1_{Y_\bullet})$ in the fibre $Pt_{Y_\bullet}Cat\mathbb{E}$ lying in Σ, we get a fibered reflection $(\P^{\Sigma}_{Cat\mathbb{E}}, I^{\Sigma}_{Cat\mathbb{E}}) : \Sigma PtCat\mathbb{E} \rightleftarrows Cat\mathbb{E}$. Whence the following:

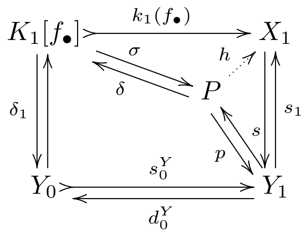
Corollary 3.2. *Let $\mathbb{E}$ be a finitely complete category. The fibered reflection $(\P^{\Sigma}_{Cat\mathbb{E}}, I^{\Sigma}_{Cat\mathbb{E}})$ is left exact.*

3.4 A characterisation in $\mathbb{E}$

In this section, on the model of the definition of left homogeneous split epimorphisms in *Mon* given in [7] which are actually the split epimorphisms with fibrant splitting in the fibre Cat_1, we shall give a characterisation of the split epimorphic internal functors with a fibrant splitting by a diagrammatic property in the ground category $\mathbb{E}$. This will lead to our main observation, Theorem 3.1. Let $(f_\bullet, s_\bullet) : X_\bullet \rightleftarrows Y_\bullet$ be any split epimorphism in the fibre $Cat_{Y_0}\mathbb{E}$. Let us consider the following left hand side pullback in the fibre $Cat_{Y_0}\mathbb{E}$, and its "level 1" part in $\mathbb{E}$ on the right hand side which is a pullback in $\mathbb{E}$ as well:



The category $K_\bullet[f_\bullet]$ is the subcategory of $X_\bullet$ consisting in those "endomorphisms" whose image by $f_\bullet$ is an identity map. Now let us consider the leftward pullback of split epimorphims along $d_0^Y : Y_1 \to Y_0$ in $\mathbb{E}$:



The object P is the object of "internal pairs" $(\gamma, \phi) :\ y \xrightarrow{\gamma} y \in X_1,\ y \xrightarrow{\phi} y' \in Y_1$ of maps where γ is an endomap such that $f_1(\gamma) = 1_Y$. There is a canonical factorization $h : P \to X_1$ corresponding to the composition $s_1(\phi).\gamma$ which is such that $h.\sigma = k_1(f)$, $s_1.h = p$ and $h.s = s_1$.

Proposition 3.7. *The split epimorphism $(f_\bullet, s_\bullet) : X_\bullet \rightleftarrows Y_\bullet$ in the fibre $Cat_{Y_0}\mathbb{E}$ has a fibrant splitting if and only if the map h is an isomorphism in $\mathbb{E}$.*

Proof. Thanks to the Yoneda embedding it is sufficient to check it in *Cat*. We have to show that we have a map $q_{(f,s)} : X_1 \to K_1[f_\bullet]$ satisfying $\delta_1.q_{(f,s)} = d_1^Y.f_1$, and such that we have, for any map $\psi : y \to y'$ in X_1, $\psi = s_1 f_1(\psi).q_{(f,s)}(\psi)$, and, for any pair $(\gamma, \phi) :\ y \xrightarrow{\gamma} y \in K_1[f_\bullet],\ y \xrightarrow{\phi} y' \in Y_1,\ q_{(f,s)}(s_1(\phi).\gamma) = \gamma$. This is clearly characterizing $s_1(\phi)$ as a cartesian map in X_1. □

We get now our main observation:

Theorem 3.1. *Let $\mathbb{E}$ be a finitely complete category. Given any split epimorphic functor $(f_\bullet, s_\bullet)$ with fibrant splitting, the following pullback in $Cat_{Y_0}\mathbb{E}$:*

$$\begin{array}{ccc} K_\bullet[f] & \xrightarrow{k_\bullet(f)} & X_\bullet \\ \downarrow\uparrow & & f_\bullet \downarrow\uparrow s_\bullet \\ \Delta_\bullet(Y_0) & \xrightarrow[\alpha_\bullet(Y)]{} & Y_\bullet \end{array}$$

is such that the pair $(k_\bullet(f), s_\bullet)$ is jointly extremally epic.

Proof. Thank to the Yoneda embedding it is enough to check the result in Cat. Now the previous isomorphism h says that any map $\psi : y \to y' \in X_1$ is a composition $s_1(\phi).\gamma$ of a map in the image of the functor $s_\bullet$ and a map in $K_\bullet[f]$. Accordingly the pair $(k_\bullet(f), s_\bullet)$ is jointly extremally epic. □

3.5 Functors with cofibrant splitting

By duality we can define the class of split epimorphic functors with cofibrant splitting. If we denote this class by Σ^{co} and by Σ^{co}_Y the class of split epimorphic functors with cofibrant splitting in the fibre $Cat_Y\mathbb{E}$, they have exactly the same properties as Σ and Σ_Y as described above. In the fibre $Cat_1 = Mon$ the class Σ^{co}_1 coincides with the class of *Schreier split epimorphisms* between monoids introduced in [15], see also [7].

4 S-Mal'tsev and S-protomodular categories

We are going now to enter into the details of some new structural aspects of the fibers $Cat_Y\mathbb{E}$.

Definition 4.1. *A split epimorphism (f, s) in a finitely complete category $\mathbb{C}$ will be called strongly split, when, given any pullback:*

$$\begin{array}{ccc} \bar{X} & \xrightarrow{x} & X \\ \bar{f} \downarrow\uparrow \bar{s} & & f \downarrow\uparrow s \\ \bar{Y} & \xrightarrow[y]{} & Y \end{array}$$

the pair (x, s) is jointly extremally epic.

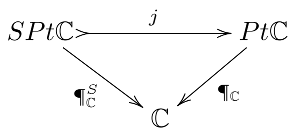
When the category $\mathbb{C}$ is pointed the previous definition is equivalent to the fact that, in the following pullback:

$$\begin{array}{ccc} K[f] & \xrightarrow{k_f} & X \\ \downarrow\uparrow & & f \downarrow\uparrow s \\ 1 & \xrightarrow[\alpha_Y]{} & Y \end{array}$$

the pair (k_f, s) is jointly extremally epic; accordingly, in the pointed context, the previous definition is equivalent to the one given in [5] and in [15]. Proposition 3.1 implies that **the split epimorphic functors with fibrant splitting are strongly split in the fibre $Cat_Y\mathbb{E}$**. By duality, **the split epimorphic functors with cofibrant splitting are strongly split in the fibre $Cat_Y\mathbb{E}$**. Recall that a category $\mathbb{C}$ is protomodular [2] when any split epimorphism is strongly split.

Definition 4.2. *A class S of split epimorphisms in a category $\mathbb{C}$ is said to be point-congruous when 1) it is stable under pullbacks along any map, 2) stable under finite limits in $Pt\mathbb{C}$ and 3) contains the terminal object $1 \rightleftarrows 1$ of $Pt\mathbb{C}$.*

Let us denote by $SPt\mathbb{C}$ the full subcategory of $Pt\mathbb{C}$ whose objects are those which are in S. When S is point-congruous, the point 1) means that it determines a subfibration $\P_{\mathbb{C}}^S$ of the fibration of points:



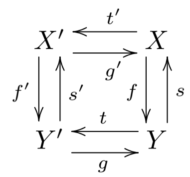
The point 2) implies that any fibre $SPt_X\mathbb{C}$ is stable under finite limits in the fibre $Pt_X\mathbb{C}$ and that any change of base functor with respect to $\P_{\mathbb{C}}^S$ is left exact. The point 3), in presence of the first one, is equivalent to the fact that S contains the class of isomorphisms and, accordingly, that any fibre $SPt_X\mathbb{C}$ is pointed. This last point makes the reflection $(\P_{\mathbb{C}}^S, I_{\mathbb{C}}^S) : SPt\mathbb{C} \rightleftarrows \mathbb{C}$ a (pointed) fibered one, while the second point makes it a finitely complete fibered reflection, which becomes left exact as soon as $\mathbb{C}$ is finitely complete. If we denote by Σ_Y the class of split epimorphic functors with fibrant splitting in $Cat_Y\mathbb{E}$, Section 3.3 shows that **the class Σ_Y is point-congruous in the fibre $Cat_Y\mathbb{E}$**.

Definition 4.3. *Let $\mathbb{C}$ be a finitely complete category; it will be said to be a S-Mal'tsev category when:*
1) the class S is point-congruous;
2) the reflection $(\P_{\mathbb{C}}^S, I_{\mathbb{C}}^S) : SPt\mathbb{C} \rightleftarrows \mathbb{C}$ is a Mal'tsev one.
It will be said to be S-protomodular when:
1) the class S is point-congruous;
2) any split epimorphism in S is strongly split.

It is clear that, when $\mathbb{C}$ is S-Mal'tsev, any fibre $SPt_X\mathbb{C}$ is unital. On the other hand, and **here is our second structural point**: according to Theorem 3.1 and Proposition 3.6, **any fibre $Cat_Y\mathbb{E}$ is Σ_Y-protomodular**. By duality **any fibre $Cat_Y\mathbb{E}$ is Σ_Y^{co}-protomodular**. In the pointed case, i.e. when $\mathbb{C}$ is a pointed category, our definition of S-protomodular categories coincides with the one of [7] coming from the investigation of left homogeneous split epimorphisms in $Mon = Cat_1$, see also [8].

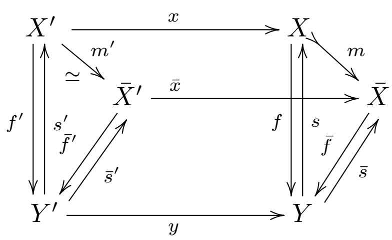
Theorem 4.1. *Let $\mathbb{C}$ be a finitely complete category and S a class of split epimorphisms. Then:*
1) when $\mathbb{C}$ is S-protomodular, it is a S-Mal'tsev category,
2) when $\mathbb{C}$ is S-protomodular, any change of base functor with respect to the fibration $¶_{\mathbb{C}}^{S}$ is conservative.

Proof. 1) Consider the following $¶_{\mathbb{C}}^{S}$-cartesian split epimorphim, namely rightward pullback of split epimorphisms in $\mathbb{C}$ where the vertical parts are in S:



The split epimorphism (f', s') is then a strongly split epimorphism. Now, because of the splitting t, the leftward square is still a pullback of split epimorphisms, and the pair (t', s) is certainly joinly strongly epic.

2) Since any change of base functor with respect to the fibration $¶_{\mathbb{C}}^{S}$ is left exact, it is enough to prove that it is conservative on monomorphisms. So let us consider the following diagram where all the quadrangles are pullbacks and all the split epimorphisms are in $SPt\mathbb{C}$:

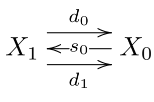


Suppose moreover that the factorization m' is an isomorphism. Since the split epimorphim $(\bar{f}, \bar{s})$ is a strongly split epimorphism the pair $(\bar{x}, \bar{s})$ is jointly extremally epic; accordingly, since m' is an isomorphism, so is the pair $(\bar{x}.m', \bar{s})$, and m is necessarily an isomorphism. □

4.1 Partial Mal'tsev aspects of S-Mal'tsev categories

In this section, we shall analyse how some partial aspects of Mal'tsev categories still hold in the S-Mal'tsev context. For that we need the following:

Definition 4.4. *An internal reflexive graph (resp. category, groupoid) in the S-Mal'tsev category $\mathbb{C}$*

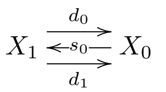


is said to be a S-reflexive graph (resp. S-category, S-groupoid) if the split epimorphism (d_0, s_0) is in S.

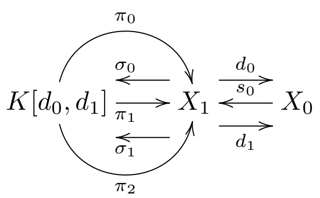
Since S is point-congruous, S-reflexive graphs are closed under finite limits inside the category of internal reflexive graphs. The same is true for S-categories and S-groupoids as well. By definition, in a Mal'tsev category any reflexive relation is an equivalence relation. Here we have only:

Proposition 4.1. *Let $\mathbb{C}$ be a S-Mal'tsev category. Any S-reflexive relation is transitive.*

Proof. Let us consider the following S-reflexive relation:

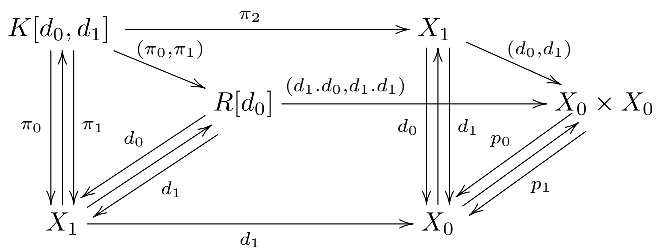


Let us recall that its simplicial kernel is the upper part of the universal 2-simplicial object associated with it:

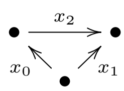


(6)

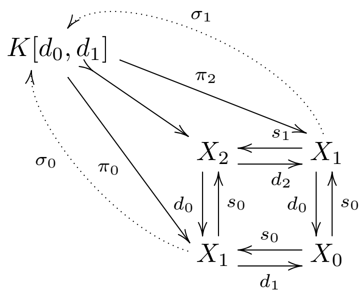
Since $\mathbb{C}$ is finitely complete, it is obtained by the following pullback of reflexive graphs:



In *Set*-theoretical terms, $K[d_0, d_1]$ is the set of triple arrows $(x_0, x_1, x_2) \in X_1$ whose incidence conditions are given by the following drawing:



The vertical square indexed by 0 in the diagram above determines a factorization $(\bar{\pi}_0, \bar{\pi}_2) : K[d_0, d_1] \to X_2$ to the following vertical pullback:

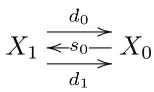


Actually it is a monomorphism since $(d_0, d_1) : X_1 \rightarrowtail X_0 \times X_0$ is a relation. The right hand side split epimorphism $(d_0, s_0) : X_2 \rightleftarrows X_1$ is in S since we have a S-reflexive relation, accordingly the pair (s_0, s_1) is jointly extremally epic, and the dotted factorizations σ_0 and σ_1 make the factorization $(\bar{\pi}_0, \bar{\pi}_2) : K[d_0, d_1] \to X_2$ an isomorphism; accordingly the map $X_2 \overset{(\bar{\pi}_0, \bar{\pi}_2)^{-1}}{\longrightarrow} K[d_0, d_1] \overset{\pi_1}{\to} X_1$ produces the desired transitivity map. $\square$

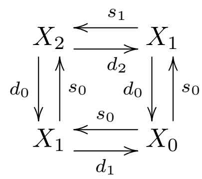
There are examples of S-reflexive relations which are not equivalence relation, see the fibre $Cat_1 = Mon$ in Section 5. In a Mal'cev category, on a reflexive graph there is at most one structure of internal category, and any internal category is a groupoid [10]. Here we have only:

Proposition 4.2. *Let $\mathbb{C}$ be a S-Mal'tsev category. On a S-reflexive graph there is at most one structure of category. It is sufficient to have the composition map $d_1 : X_2 \to X_1$ with axiom (2), the axioms (1) and (3) come for free.*

Proof. Let us consider the following S-reflexive graph:



Consider now the pullback 1 of the definition of an internal category:



Since the right hand side split epimorphism is in S, the pair $(s_0, s_1) : X_1 \rightrightarrows X_2$ is jointly extremally epic. So there is atmost one map d_1 satisfying Axiom 2). Axiom 1) comes by composition with the jointly extremally epic pair (s_0, s_1) and Axiom 3) comes with composition with the pair (s_0, s_2) in the diagram 2 of the definition of an internal category, since it is jointly extremally epic as well. $\square$

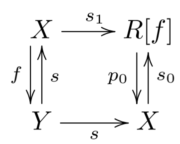
Now, in order to characterize S-equivalence relations and S-groupoids later on, we shall need the following:

Definition 4.5. *A map $f : X \to Y$ in a S-Mal'tsev category $\mathbb{C}$ will be called S-special when the kernel equivalence relation $R[f]$ is a S-equivalence relation. An object X will be called S-special when the terminal map $\tau_X : X \to 1$ is S-special.*

Clearly, in a S-Mal'tsev category, the class of S-special maps is stable under pullbacks since so is S, and the full subcategry $S^\sharp\mathbb{C} \subset \mathbb{C}$ of S-special objects is stable under finite limits in $\mathbb{C}$ thanks to the Axiom 2) of a point-congruous class. Any split epimorphism in S is not necesserily S-special. However we have a kind of converse:

Lemma 4.1. *Let $\mathbb{C}$ be a S-Mal'tsev category. A split S-special map $f : X \to Y$ is a split epimorphism in S.*

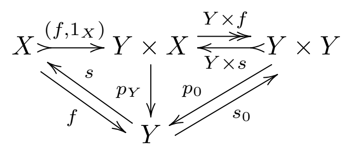
Proof. Let s be the splitting. Consider the following pullback:



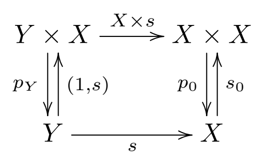
So, the map f being S-special if and only if the split epimorphism (p_0, s_0) is in S, the split epimorphism (f, s) is in S by stability under pullbacks. □

Proposition 4.3. *Let $\mathbb{C}$ be a S-Mal'tsev category. Any split epimorphism between S-special objects is in S, and consequently any map between S-special objects is a S-special morphism. The subcategory $S^\sharp\mathbb{C}$ of S-special objects is a Mal'tsev one. When $\mathbb{C}$ is S-protomodular, $S^\sharp\mathbb{C}$ is protomodular.*

Proof. Let us recall that any split epimorphism $(f, s) : X \rightleftarrows Y$ produces a kernel diagram in the fibre $Pt_Y\mathbb{C}$:

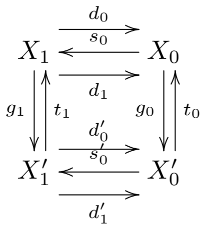


When Y is in $S^\sharp\mathbb{C}$, the right hand side split epimorphism is in S by definition. The following pullback:



shows that, when X is in S, the middle split epimorphism is in S. Since the fibre $SPt_Y\mathbb{C}$ is stable under finite limits, so is the kernel (f, s). On the other hand, since $S^\sharp\mathbb{C}$ is stable under finite limits, the kernel equivalence relation of f lies in $S^\sharp\mathbb{C}$, and the split epimorphism $(p_0, s_0) : R[f] \rightleftarrows X$ is in S. Accordingly $R[f]$ is a S-equivalence relation, and f is a special map. The same argument holds for any map between S-special objects. It follows that any fibre $Pt_Y S^\sharp\mathbb{C}$ is unital so that $S^\sharp\mathbb{C}$ is Mal'tsev category [3]. When, moreover, $\mathbb{C}$ is S-protomodular, then, according to Theorem 4.1, the change of base functor with respect to the fibration $Pt(S^\sharp\mathbb{C})$ is conservative, and consequently $S^\sharp\mathbb{C}$ is protomodular. □

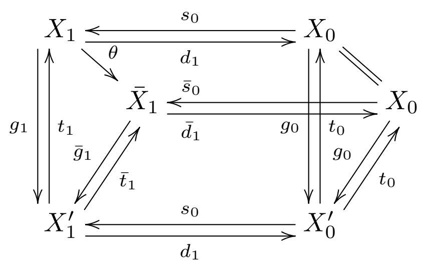
In a Malt'sev category we have also the following useful result, see [3]: given any split epimorphism of reflexive graphs,



the commutative square with maps d_1 is a pullback as soon as so is the square with maps d_0. Here we have:

Proposition 4.4. *Let $\mathbb{C}$ be a S-Mal'tsev category. Given a split epimorphism of reflexive graphs in $\mathbb{C}$, if the split epimorphism (g_0, t_0) is in S, the commutative square with maps d_1 is a pullback as soon as so is the square with maps d_0.*

Proof. If the square with the d_0 is a pullback and the split epimorphism (g_0, t_0) is in S, so is the split epimorphism (g_1, t_1), and the pullback $(\bar{g}_1, \bar{t}_1)$ of (g_0, t_0) along d_1 as well, in the following diagram:

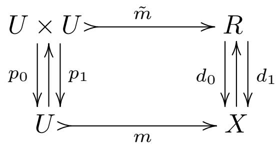


Let θ be the induced factorization. Since $\mathbb{C}$ is a S-Mal'tsev category, the change of base functor along the split epimorphism d_0 is saturated on subobjects according to Theorem 2.1. Thus, by the point 3) of Lemma 2.3, the map θ is a monomorphism since $s_0^*(\theta) = 1_{X_1}$ is an isomorphism. Still the change of base functor along the split epimorphism d_1 is saturated on subobjects. According to the point 1) of this same Lemma 2.3, the map θ is an isomorphism again since $s_0^*(\theta)$ is an isomorphism, which means that the commutative square with maps d_1 is a pullback as well. □

Observe that, when the square with the d_0 is a pullback and the split epimorphism (g_0, t_0) is in S, there is at most one map $d_1 : X_1 \to X_0$ completing the diagram, since the pair (t_1, s_0) is jointly extremally epic. Finally, concerning the normal monomorphisms, we have the following:

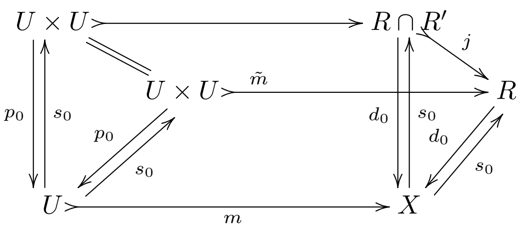
Proposition 4.5. *Let $\mathbb{C}$ be a S-Mal'tsev category. When $m : U \rightarrowtail X$ is a monomorphism which is normal to a S-equivalence relation R on X, the object U is S-special. When $\mathbb{C}$ is S-protomodular, a monomorphism m is normal to at most one S-equivalence relation.*

Proof. Saying that the monomorphism m is normal to R is saying that $m^{-1}(R) = \nabla_U$ and that the induced following functor is a discrete fibration, i.e. that any of the following commutative squares is a pullback:



Accordingly, when R is a S-equivalence relation, so is ∇_U, and U is S-special.

Now suppose that $\mathbb{C}$ is S-protomodular, and m normal to two equivalence relations R and R'. Then m is normal to $R \cap R'$. Now consider the following diagram:



When R is a S equivalence relation, the downward quadrangle being a pullback, the pair $(\tilde{m}, s_0)$ is extremally epic, so that the monomorphism j is an isomorphism, and we get $R \subset R'$. If R' is a S-equivalence relation as well, we have $R = R'$.

□

4.2 Commutation in the fibers $Pt_Y\mathbb{C}$

We recalled that a category $\mathbb{C}$ is a Mal'tsev one, if and only if the fibers $Pt_Y\mathbb{C}$ are unital, and consequently it allows a notion of commutation inside these fibers [4]. The definition of a S-Mal'tsev category also allows some observations about commutation in the fibers $Pt_Y\mathbb{C}$ and, in the same way, about centralization of reflexive relations. Consider any pullback:

$$\begin{array}{ccc} X' & \overset{t'}{\underset{g'}{\leftrightarrows}} & X \\ {\scriptstyle f'}\downarrow\uparrow{\scriptstyle s'} & & {\scriptstyle f}\downarrow\uparrow{\scriptstyle s} \\ Y' & \overset{t}{\underset{g}{\leftrightarrows}} & Y \end{array}$$

where the split epimorphism (f,s) is in S. Accordingly, the pair (t',s') is jointly extremally epic. Consider now two maps having same codomain in the fibre $Pt_Y\mathbb{C}$:

$$\begin{array}{ccccc} X & \xrightarrow{h} & V & \xleftarrow{k} & Y' \\ & {\scriptstyle f}\searrow\nwarrow{\scriptstyle s} & \downarrow\uparrow{\scriptstyle g} & \swarrow\nearrow{\scriptstyle t} & \\ & & Y & & \end{array}$$

Definition 4.6. *Let $\mathbb{C}$ be a S-Mal'tsev category and (f,s) a split epimorphism in S. The pair (h,k) is said to commute in $Pt_Y\mathbb{C}$ when there is a (necessarily unique) map $\phi : X' \to V$ such that $\phi.t' = h$ and $\phi.s' = k$. The map ϕ is called the* cooperator *of this pair, see [4].*

The unicity of ϕ comes from the fact that the pair (t',s') is jointly extremally epic and makes its existence a property for the pair (h,k). Now we get the following characterization:

Proposition 4.6. *Let $\mathbb{C}$ be a S-Mal'tsev category and a S-reflexive graph:*

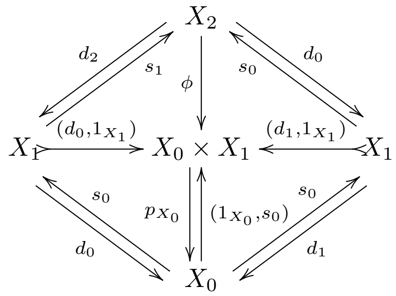
$$X_1 \underset{d_1}{\overset{d_0}{\underset{\longleftarrow s_0 \longrightarrow}{\rightrightarrows}}} X_0$$

It is an internal category if and only if the following subobjects commute in $Pt_{X_0}\mathbb{C}$:

$$\begin{array}{ccccc} X_1 & \overset{(d_0,1_{X_1})}{\rightarrowtail} & X_0 \times X_1 & \overset{(d_1,1_{X_1})}{\leftarrowtail} & X_1 \\ & {\scriptstyle d_0}\searrow\nwarrow{\scriptstyle s_0} & {\scriptstyle p_{X_0}}\downarrow\uparrow{\scriptstyle (1_{X_0},s_0)} & \swarrow\nearrow{\scriptstyle s_0} {\scriptstyle d_1} & \\ & & X_0 & & \end{array}$$

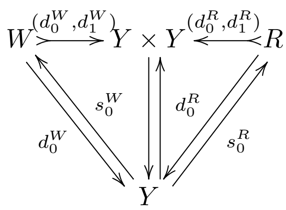
Proof. The two subobjects commute in $Pt_{X_0}\mathbb{C}$ if and only if they have a cooperator $\phi\colon X_2 \to X_0 \times X_1$, i.e a morphism satisfying $\phi.s_0 = (d_1, 1_{X_1})$ and

$\phi.s_1 = (d_0, 1_{X_1})$:

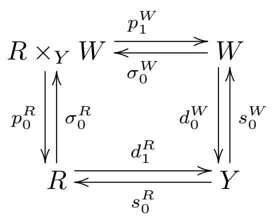


where the whole quadrangle is the pullback 1 of the definition of an internal category. The morphism ϕ is a pair $(d_0.d_2, d_1)$, where $d_1 : X_2 \to X_1$ is such that $d_1.s_0 = 1_{X_1}$ and $d_1.s_1 = 1_{X_1}$. Since the morphism d_1 satisfies these two identities, it makes the reflexive graph an internal category by Proposition 4.2. Conversely, the composition map $d_1 : X_2 \to X_1$ of an internal category satisfies the previous two identities and produces the cooperator $\phi = (d_0.d_2, d_1)$. $\square$

Now consider a pair (R, W) of reflexive relations $(d_0^R, d_1^R) : R \rightrightarrows Y$ and $(d_0^W, d_1^W) : W \rightrightarrows Y$ on Y, with W a S-reflexive one, we shall say that *the two reflexive relations R and W centralize each other* (which we shall denote by $[R, W] = 0$ as usual) when the following subobjects commute in $Pt_Y\mathbb{C}$ where the central object below is the split epimorphism $(p_0, s_0) : Y \times Y \rightleftarrows Y$:



The cooperator $\phi : R \times_Y W \to Y \times Y$ in the fibre $Pt_Y\mathbb{C}$, where $R \times_Y W$ is defined by the following pullback with the pair (σ_0^R, σ_0^W) jointly extremally epic:



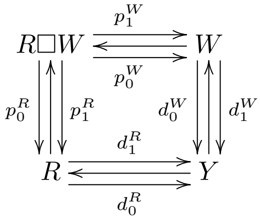
(7)

is necessarily of the form $\phi(xRyWz) = (x, p(xRyWz))$, with p satisfying the two equations $p(xRxWy) = y$ and $p(xRyWy) = x$ since we have $\sigma_0^W(yWz) = (yRyWz)$ and $\sigma_0^R(xRy) = (xRyWy)$). The morphism $p : R \times_Y W \to X$ which, with these equations, characterises the property that the reflexive relations R and W centralise each other in $\mathbb{C}$ is called the *connector* between R and W, (see

[6] and also [10]). We know that, in a Malt'sev category, two reflexive relations (i.e. two equivalence relations) R and S on Y commute as soon as $R \cap S = \Delta_Y$. Here, in a S-Mal'tsev category, we have as well:

Proposition 4.7. *Let $\mathbb{C}$ be a S-Mal'tsev category. The reflexive relation R and the S-reflexive relation W on the object Y commute in $\mathbb{C}$ as soon as $R \cap W = \Delta_Y$.*

Proof. Let us denote by $R\Box W$ the inverse image of the reflexive relation $W \times W$ along $(d_0^R, d_1^R) : R \rightarrowtail Y \times Y$. This produces a double relation:

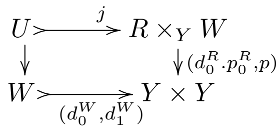


Since we have $R \cap W = \Delta_Y$, the factorization $\theta : R\Box W \to R \times_Y W$ is necessarily a monomorphism. Since W is a S-reflexive relation the split epimorphism (d_0^W, s_0^W) is in S, and according to Lemma 2.1 this same factorization is an extremal epimorphism. Accordingly it is an isomorphism, which says that the square above which is downward indexed by 0 and rightward indexed by 1 is a pullback. Accordingly the map $R\Box W \overset{p_0^W}{\to} W \overset{d_1^W}{\to} Y$ produces the desired connector. $\Box$

When we have $[R, W] = 0$ with W a S-reflexive relation, we can recover a well-known result in Mal'cev categories:

Proposition 4.8. *Let $\mathbb{C}$ be a S-Mal'tsev category. Suppose the reflexive relation R and the S-reflexive relation W on Y commute in $\mathbb{C}$, then we have necessarily $xWp(xRyWz)$ and $p(xRyWz)Rz$.*

Proof. Let us consider the following pullback:



It defines U as the subobject of those $xRyWz \in R \times_Y W$ such that we have $xWp(xRyWz)$. For any $yWz \in W$, the element $yRyWz \in R \times_Y W$ belongs to U since we have $yWp(yRyWz)$ by $p(yRyWz) = z$. This means that σ_0^W factors through U. In the same way, for any $xRy \in R$, the element $xRyWy \in R \times_Y W$ belongs to U since we have $xWp(xRyWy)$ by $p(xRyWy) = x$. This means that σ_0^R factors through U. Since the pair (σ_0^R, σ_0^W) is jointly extremally epic, the map j is an isomorphism, and for every $xRyWz \in R \times_Y W$ we have $xWp(xRyWz)$.

We have a similar result concerning the subobject $V \rightarrowtail R \times_Y W$ defines by the following pullback:

$$\begin{array}{ccc} V & \overset{j}{\rightarrowtail} & R \times_Y W \\ \downarrow & & \downarrow {\scriptstyle (p, d_1^W . p_1^W)} \\ R & \underset{(d_0^R, d_1^R)}{\rightarrowtail} & Y \times Y \end{array}$$

This give us $p(xRyWz)Rz$ for any $xRyWz \in R \times_Y W$. $\square$

In set theoretical terms, the previous proposition means that, with any triple $xRyWz$, we can associate a square of related elements:

$$\begin{array}{ccc} x & \overset{W}{\dashrightarrow} & p(x, y, z) \\ {\scriptstyle R}\downarrow & & \vdots\downarrow {\scriptstyle R} \\ y & \underset{W}{\longrightarrow} & z. \end{array}$$

More acutely, this says that any connected pair of reflexive relations (R, W) on the object Y, with W a S-reflexive relation, produces the following diagram of reflexive relations in $\mathbb{C}$:

$$\begin{array}{ccc} R \times_X W & \overset{p_1^W}{\underset{(d_0^R . p_0^R, p)}{\rightleftarrows\!\!\rightarrow}} & W \\ {\scriptstyle p_0^R}\downarrow\uparrow\downarrow {\scriptstyle (p, d_1^W . p_1^W)} & & {\scriptstyle d_0^W}\downarrow\uparrow\downarrow {\scriptstyle d_1^W} \\ R & \overset{d_1^R}{\underset{d_0^R}{\rightleftarrows\!\!\rightarrow}} & Y \end{array}$$

It is called the *centralizing double relation* associated with the connector and it characterizes the commutation $[R, W] = 0$. Moreover, W being a S-reflexive relation, all the downward squares with indexation 0 are pullbacks according to Proposition 4.4. When, in addition, R is an equivalence relation, the upper row becomes an equivalence relation as well. When R and W are equivalence relations, all the reflexive relations in this diagram are equivalence relations, and, moreover, any commutative square is a pullback.
Finally we get the following characterization in the S-Mal'tsev setting:

Proposition 4.9. *Let $\mathbb{C}$ be a S-Mal'tsev category. Consider a S-reflexive graph:*

$$X_1 \overset{d_0}{\underset{d_1}{\rightrightarrows}} X_0 \quad (s_0 : X_0 \to X_1)$$

The following conditions are equivalent:
1) this graph is underlying a S-groupoid
2) the map d_0 is S-special and $[R[d_0], R[d_1]] = 0$
A S-reflexive relation is an equivalence relation if and only if the map d_0 is S-special.

Proof. The fact that $[R[d_0], R[d_1]] = 0$ for any groupoid is a very general fact and 1) $\Rightarrow$ 2) is a consequence of the diagram 5 relative to groupoids. Conversely suppose that d_0 is S-special, then $[R[d_0], R[d_1]] = 0$ makes sense and the structure of groupoid is defined like in Theorem 3.6 in [10].

Now suppose that the S-reflexive graph is actually a relation. The pair $(d_0, d_1) : X_1 \rightrightarrows X_1$ being jointly monic, we get $R[d_0] \cap R[d_1] = \Delta_{X_0}$. According to Proposition 4.7, we have $[R[d_0], R[d_1]] = 0$ in a S-Mal'tsev setting. So that the reflexive relation is an equivalence relation. □

5 Some examples in the fibres Cat_Y

In this section we shall examplify some of the previous conceptual notions in the fibres Cat_Y. Given a set Y, the fibre Cat_Y is the category whose objects are the categories having Y as set of objects and whose maps are those functors which respect these objects. We showed that this fibre is Σ_Y-protomodular where Σ_Y is the class of split epimorphic functors with fibrant splitting in the fibre Cat_Y.

An internal category in Cat_Y is a 2-category with Y as object of objects, while an internal reflexive graph is a category equipped with 2-cells between parallel pairs of morphims, but only endowed with a horizontal composition (i.e. without requiring the interchange law which characterizes the 2-categories). In this section we shall briefly describe what are the Σ_Y-reflexive graphs, the Σ_Y-reflexive relations and the Σ_Y-categories.

Suppose $Y = 1$ is the singleton. Then Cat_1 is the category Mon of monoids. We already noticed that Σ_1 coincides with the class S of left homogeneous split epimorphisms between as described in [7]. A major example of S-reflexive relation in Mon is the preorder $\mathcal{O}_{\mathbb{N}}$ on the natural numbers:

$$\mathcal{O}_{\mathbb{N}} \underset{p_1}{\overset{p_0}{\underset{\longleftarrow s_0 -}{\rightrightarrows}}} \mathbb{N}$$

where

$$\mathcal{O}_{\mathbb{N}} = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid x \leq y\}$$

It is clearly not an equivalence relation. Any split epimorphism above a group being in S, any ordered group G produces in the same way a S-reflexive relation in Mon. We shall denote by $O_S Mon$ the category of S-preordered monoids and order-preserving homomorphisms, which contains the category OGp of preordered groups. On the other hand, we know that a monoid is S-special if and only if it is a group [7].

We shall call 2-*monoid* an internal category in Mon and a 2_S-*monoid* an internal S-category in Mon. We shall denote by 2-Mon and 2_S-Mon the respective associated categories.

Suppose now Y is any set. A reflexive graph in Cat_Y:

$$\mathbb{X}_1 \underset{d_1}{\overset{d_0}{\underset{\longleftarrow s_0 -}{\rightrightarrows}}} \mathbb{X}_0$$

is a Σ_Y-reflexive graph when, for any 2-cell:

$$\theta \ : x \underset{\psi}{\overset{\phi}{\underset{\Downarrow}{\rightrightarrows}}} x'$$

there is a unique 2-cell:

$$\gamma \ : x \underset{\chi}{\overset{1_x}{\underset{\Downarrow}{\rightrightarrows}}} x$$

such that $\theta = \phi.\gamma$.

Finally, a category $\mathbb{Y}$ is a Σ_Y-special object in Cat_Y, when for any pair of parallel maps $(f, f') : x \rightrightarrows x'$, there exists a unique map $g : x \to x$ such that $f' = f.g$. This implies that any $Hom(x, x)$ is a group which induces a simply transitive right action on any set $Hom(x, x')$. We can check at first sight a consequence of Proposition 3.1, namely that any groupoid in Cat_Y is a Σ_Y-special object.

Let us end this section by introducing a class of examples of Σ_Y-reflexive relation (which, according to Proposition 4.1, will be necesssarily a preorder). Start with a functor $M : \mathbb{X}^{op} \to O_S Mon$ with $\mathbb{X} \in Cat_Y$. From it, you get a Σ_Y-reflexive relation in Cat_Y in the following way. Let $\mathbb{X}_M$ be the category in Cat_Y determined by the Grothendiek construction associated with the underlying functor $M : \mathbb{X}^{op} \to Mon$. Its set of object is Y, while a map $y \to y'$ is a pair (ϕ, a) with $\phi : y \to y' \in \mathbb{X}$ and $a \in M(y)$. The composition is given by $(\phi', a').(\phi, a) = (\phi'.\phi, M(\phi)(a').a)$. This category is endowed with a preorder in Cat_Y with $(\phi, a) \leq (\bar{\phi}, \bar{a}) : y \rightrightarrows y'$ if and only if $\phi = \bar{\phi}$ and $a \leq \bar{a}$. It is a Σ_Y-reflexive relation since the monoid $M(y)$ is endowed with a S-preorder: starting from a pair $(\phi, a) \leq (\phi, \bar{a})$, we have $a \leq \bar{a}$ in $M(y)$; whence a unique $\alpha \in M(y)$ such that $1_y \leq \alpha$ and $\bar{a} = a.\alpha$. So that $(1_Y, \alpha)$ is the unique map in $\mathbb{X}_M$ such that $(\phi, \bar{a}) = (\phi, a).(1_Y, \alpha)$.
By the same type of construction, one obtains a Σ_Y-internal category from any functor: $M : \mathbb{X}^{op} \to 2_S\text{-}Mon$. This provides us with a large class of examples.

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