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A structural aspect of the category of quandles

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A STRUCTURAL ASPECT OF THE CATEGORY OF QUANDLES

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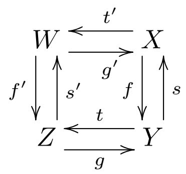
Abstract. We show that the category of quandles satisfies a Maltsev property relative to a certain class of split epimorphisms. This stuctural aspect produces a class of congruences, called *puncturing*, which have the property to permute with any other congruence.

Introduction

A Maltsev category is a category in which any reflexive relation is an equivalence relation, see [9]. The categories *Gp* of groups and *K-Lie* of Lie algebras are major examples of Maltsev categories. A soon as a Maltsev category is regular, any pair (R, S) of equivalence relations on an object X does permute (i.e. $R \circ S = S \circ R$).

In [3], Maltsev categories was characterized in terms of split epimorphism: a left exact category $\mathbb{D}$ is a Maltsev one if and only if any commutative square of split epimorphisms in $\mathbb{D}$:



is such that the factorization $\phi : W \to X \times_Y Z$, where $X \times_Y Z$ is the domain of the pullback of f along g, is an extremal epimorphism. More recently was introduced a similar notion but only requested for a certain class S of split epimorphisms (f, s) which is stable under pullback and contains isomorphisms (S-Maltsevness in [5]). In this last context, when $\mathbb{D}$ is regular, only a certain class of equivalence relations does permute with any other equivalence relation. Moreover, under stricter conditions on S, any S-Maltsev category $\mathbb{D}$ produces a subcategory $\mathbb{D}_\sharp$ which is necessarily a Maltsev one and is called the Maltsev *core* of this relative Maltsev structure. These stricter conditions were shown to be fulfilled, inside a richer structure, by the class of *Schreier split epimorphisms* in the categories Mon of monoids and $SRng$ of semi-rings, see [7].

On the other hand, a certain class $\simeq_N$ of congruences in the category Qnd of quandles, introduced in [8], was shown to permute with any other congruence in [12]. Whence a natural question: is there a class Σ of split epimorphisms in Qnd which could explain this phenomenon.

This work is born as an attempt to answer to this question. It only partially answers to it: effectively it exhibits a class Σ of split epimorphisms satisfying the relative Maltsev property which produces a class of permuting congruences (namely the *puncturing* congruences, see Section 3.3), but the equivalence relation $\simeq_N$ does not belong to this class in general. However the following question remains valid:

is any congruence $\simeq_N$ *generated* by the class of puncturing congruences. It seems to be a rather uneasy question which it is not answered to here; only suggestions are given: the meaning of "generated" in Section 3.8 and a significant structural fact in Section 4.10.

Anyway, this Σ-Maltsevness is a solid structural point which determines a series of properties ([5]) which are partially detailled here for the category *Qnd*. We exhibit moreover a stricter class $\Sigma^+ \subset \Sigma$ which fulfils the conditions which leads to a Maltsev core. We characterize this Maltsev core: it coincides with the subcategory of what is known under the name of *latin quandles*. Actually, in the finite context, the classes Σ and Σ^+ coincide. We are also led to characterize what are the internal group structures in *Qnd*. Incidentally we produce an example of two distinct internal group structures on the same quandle, see Section 5.28. This article is organized along the following lines:
Section 1 is devoted to a review of basic facts in *Qnd* and to the introduction of the notion of *puncturing and acupuncturing* element which is the driving force of our structural investigation.
Section 2 is devoted to the study of the split epimorphisms in *Qnd*, the split epimorphisms being, in the algebraic contexts, the source of very important structural facts, see the classification table at the end of the book [2].
Section 3 is devoted to the proof of the Σ-Maltsevness and to the properties of the puncturing congruences.
Section 4 is devoted to the study of the Maltsev core associated with the Σ^+-Maltsevness.
Section 5 gives a characterization of the internal groups in *Qnd*.
Section 6 is devoted to investigate the subcategories L_Y of the slice categories Qnd/Y whose objects are the latin homomorphism above Y which propagate the Maltsev property and allow us to desbribe the Baer sum for the abelian latin extensions.
I should like to thank V. Even who shared with me part of his references.

1. Quandles

A quandle is a set X endowed with a binary idempotent operation $\triangleright : X\times X \to X$ such that for any object x the translation $-\triangleright x : X \to X$ is an automorphism with respect to the law $\triangleright$ whose inverse will be denoted by $-\triangleright^{-1} x$. A homomorphism of quandles is an application $f : (X, \triangleright) \to (Y, \triangleright)$ which respects the binary operation. The notion was independantly introduced in [13] and [16], see [17] for a survey and [11] for a first attempt of classification in the finite context.

As a variety of universal algebra the category *Qnd* is finitely complete and Barr-exact [1]. As in any variety, the regular epimorphisms and the extremal epimorphisms coincide with the surjective homomorphisms. The initial object is $\emptyset$ and the terminal one is the singleton 1. The fact that the operation is idempotent produces, with any element x, a homomorphism $x : 1 \to X$; so that, on the one hand, any fibre $f^{-1}(y)$ of a homomorphism $f : X \to Y$ is still a quandle and, on the other hand, the constant applications are homomorphisms.

When $(X, \triangleright)$ is a quandle, so is $(X, \triangleright^{-1})$; this defines an involutive functor $(\,)^* : Qnd \to Qnd$. The fixed points of this functor are called *involutive*. We shall denote by $IQnd$ the full subcategory of *Qnd* whose objects are the involutive quandles. It is stable in Qnd under monomorphism and product, so that it is stable under finite limit; it is stable under regular epimorphism as well.

A quandle is said to be *symmetric* when the law $\triangleright$ is symmetric. We shall denote by $SymQd$ the full subcategory of Qnd whose objects are the symmetric quandles. Similarly as above, it is stable in Qnd under monomorphism and product, so that it is stable under finite limit; it is stable under regular epimorphism as well. We shall set $SIQnd = IQnd \cap SymQd$.

1.1. **Examples.** With any group G is associated a canonical quandle structure Q_G defined by $x \triangleright y = y^{-1}.x.y$; so, the automorphism $- \triangleright 1$ is the identity automorphism Id_G, while the application $1 \triangleright -$ constant on 1. This construction defines a functor $Q : Gp \to Qnd$ from the category Gp of groups which makes the following diagram commute:

$$\begin{array}{ccc} Gp & \xrightarrow{Q} & Qnd \\ {\scriptstyle ()^{op}}\downarrow & & \downarrow{\scriptstyle ()^*} \\ Gp & \xrightarrow[Q]{} & Qnd \end{array}$$

The functor Q is faithful left exact and conservative (i.e. reflects the isomorphisms); it preserves and reflects the regular epimorphisms.

On the other hand, with any group G is associated an involutive quandle Ξ_G defined by $x \triangleright y = y.x^{-1}.y$. We get $- \triangleright 1 = (-)^{-1}$ and $1 \triangleright - = (-)^2$. This defines a functor:

$$\Xi : Gp \to IQnd$$

which is faithful left exact, conservative and which preserves and reflects the regular epimorphisms.

A quandle X is called *trivial* when the law is given by the first projection, i.e. when we have $x \triangleright y = x$, namely when we have $- \triangleright y = Id_X$ for all y; we shall denote this quandle operation by $\triangleright_0$. Clearly a trivial quandle is involutive. This defines a functor:

$$T : Set \to IQnd$$

which is injective, fully faithful, left exact and which preserves and reflects the regular epimorphisms. It makes the following diagram commute, where U is the forgetful functor from the category Ab of abelian groups and i is the inclusion:

$$\begin{array}{ccccc} Ab & \rightarrowtail & \xrightarrow{i} & & Gp \\ {\scriptstyle U}\downarrow & & & & \downarrow{\scriptstyle Q} \\ Set & \xrightarrow[T]{} & IQnd & \rightarrowtail & Qnd \end{array}$$

Finally, starting with any abelian group $(A, +)$ endowed with a automorphism g, the Alexander quandle $(A, \triangleright_g)$ is defined by $a \triangleright_g b = g(a) + b - g(g)$; we get $- \triangleright_g 0 = g$ and $0 \triangleright_g - = Id_A - g$. Let us denote by $AutAb$ the category whose objects are the pairs $((A, +), g)$ of an abelian group and an automorphism and whose morphisms are the group homomorphisms commuting with these chosen automorphisms. It is an abelian category. There is an involutive functor $(\,)^* : AutAb \to AutAb$ defined by $((A, +), g)^* = ((A, +), g^{-1})$. The Alexander quandles produce a functor:

$$Al : AutAb \to Qnd$$

which is left exact, conservative and which preserves and reflects the regular epimorphisms. Since it is straighforward that $\triangleright_g^{-1} = \triangleright_{g^{-1}}$, this functor makes the following diagram commute:

$$\begin{array}{ccc} AutAb & \xrightarrow{Al} & Qnd \\ {\scriptstyle ()^*}\downarrow & & \downarrow{\scriptstyle ()^*} \\ AutAb & \xrightarrow[Al]{} & Qnd \end{array}$$

Proposition 1.2. *Let $((A,+),g)$ be in $AutAb$. The quandle $(A,\triangleright_g)$ is involutive if and only if the automorphism g is involutive.*

Proof. The quandle $(A,\triangleright_g)$ is involutive if and only if: $g(x)+y-g(y) = g^{-1}(x)+y-g^{-1}(y)$ which is equivalent to: $g(x-y) = g^{-1}(x-y)$, i.e. g is involutive. □

If we denote by $IAutAb$ the full subcategory of $AutAb$ whose objects are those $((A,+),g)$ which are such that g is involutive, we have a restriction functor $Al^I : IAutAb \to IQnd$. Introducing the functor $J : Ab \to IAutAb$ defined by $J(A,+) = ((A,+),-Id_A)$, we get the following commutative diagram:

$$\begin{array}{ccc} Ab & \rightarrowtail^{i} & Gp \\ {\scriptstyle J}\downarrow & & \downarrow{\scriptstyle \Xi} \\ IAutAb & \xrightarrow[Al^I]{} & IQnd \end{array}$$

Proposition 1.3. *Let $((A,+),g)$ be in $AutAb$. The quandle $(A,\triangleright_g)$ is symmetric if and only if $2g = Id_A$ or, in other words, $2Id_A$ is an automorphism and g is its inverse.*

Proof. The quandle $(A,\triangleright_g)$ is symmetric if and only if we have $g(x)+y-g(y) = g(y)+x-g(x)$, i.e. $Id_A - 2g(x) = Id_A - 2g(y)$. This is equivalent to $2g = Id_A$. □

If we denote by $SyautAb$ the full subcategory of $AutAb$ whose objects are those $((A,+),g)$ which are such that $2g = Id_A$, we have a restriction functor $Al^S : SyautAb \to SymQd$. Let us denote by $SIautAb$ the subcategory of $AutAb$ whose objects are those $((A,+),g)$ where g is involutive and satisfies $2g = Id_A$. We get a restriction functor $Al^{SI} : SIautAb \to SIQnd$.

Lemma 1.4. *The only objects of the category $SIautAb$ are the objects $((A,+),-Id_A)$ with $3Id_A = 0$.*

Proof. The objects of $SIautAb$ are such that $g^2 = Id_A$ and $2g = Id_A$. Whence $g^{-1} = 2Id_A = g$ with $4Id_A = Id_A$ or $3Id_A = 0$ or $2id_A = -Id_A$ as well. □

Let us denote by Ab_3 the full subcategory of Ab whose objects are the abelian groups $(A,+)$ such that $3Id_A = 0$. Any additive group of a field with characteristic 3 belongs to Ab_3. The subcategory Ab_3 is stable in Ab under product and monomorphism, and consequently under finite limit. It is stable under regular epimorphim as well. So it is an abelian category. The previous lemma shows the restriction $J_3 : Ab_3 \to SIautAb$ of the functor $J : Ab \to AutAb$ defined above by $J(A) = ((A,+),-Id_A)$ is an isomorphism of categories.

Corollary 1.5. *Let $(A,+)$ be an abelian group. The involutive quandle Ξ_A is symmetric if and only if $3Id_A = 0$.*

Proof. We have $\Xi_A = Al((A,+),-Id_A)$. The automorphism $g = -Id_A$ is involutive. The condition $2g = Id_A$ becomes $-2Id_A = Id_A$, namely $3Id_A = 0$. □

1.6. **Autonomous quandles.** In this section we shall introduce the notion of puncturing and acupuncturing element which will be the driving force of our structural investigation.

Definition 1.7. *An element x in a quandle X is said to be neutral when the application $- \triangleright x : X \to X$ is Id_X, puncturing when the application $x \triangleright - : X \to X$ is surjective and acupuncturing when the application $x \triangleright - : X \to X$ bijective.*

So, a quandle X is trivial when any element is neutral. In a finite quandle an element is acupuncturing if and only if it is puncturing. When the element x is acupuncturing, we denote by $x \,\tilde{\triangleright}- : X \to X$ the inverse of $x \triangleright - : X \to X$.

Lemma 1.8. *Given a quandle $(X, \triangleright)$, the two following conditions are equivalent:*
1) $(x \triangleright x') \triangleright (y \triangleright y') = (x \triangleright y) \triangleright (x' \triangleright y')$, i.e. the binary operation $\triangleright$ is a homomorphism of quandles
2) $(x \triangleright x') \triangleright^{-1} (y \triangleright y') = (x \triangleright^{-1} y) \triangleright (x' \triangleright^{-1} y')$, i.e. the binary operation $\triangleright^{-1}$ is a homomorphism of quandles.

Proof. First let us show that 2) implies 1). From 2) we get:

$$((x \triangleright y) \triangleright (x' \triangleright y')) \triangleright^{-1} (y \triangleright y') = ((x \triangleright y) \triangleright^{-1} y) \triangleright ((x' \triangleright y') \triangleright^{-1} y') = x \triangleright x'$$

Whence 1). Conversely from 1), we get:

$$((x \triangleright^{-1} y) \triangleright (x' \triangleright^{-1} y')) \triangleright (y \triangleright y') = ((x \triangleright^{-1} y) \triangleright y) \triangleright ((x' \triangleright^{-1} y') \triangleright y') = x \triangleright x'$$

Whence 2). □

Definition 1.9. *A quandle $(X, \triangleright)$ is said to be autonomous when any of the two previous conditions is satisfied. It is said to be pseudo-leftassociative when, for all (x, y, t, z), we have: $((x \triangleright y) \triangleright^{-1} t) \triangleright z = ((x \triangleright z) \triangleright^{-1} t) \triangleright y$.*

Let us denote by AQd the full subcategory of Qnd whose objects are the autonomous quandles. It is stable in Qnd under monomorphism and product, so that it is stable under finite limit; it is stable under regular epimorphism as well. Any trivial quandle is autonomous and pseudo-leftassociative, so that, actually, the functor T takes its values in $AIQd = IQnd \cap AQd$; and we have $T : Set \to AIQd$. We shall set $ASQd = SymQd \cap AQd$ and $ASIQd = IQnd \cap SymQd \cap AQd$.

Proposition 1.10. *An autonomous quandle $(X, \triangleright)$ is such that any translation $e \triangleright - : X \to X$ is a quandle homomorphism.*

Proof. We get: $(e \triangleright a) \triangleright (e \triangleright b) = (e \triangleright e) \triangleright (a \triangleright b) = e \triangleright (a \triangleright b)$. □

Proposition 1.11. *Given any autonomous quandle $(X, \triangleright)$, the subset of neutral elements is a subquandle; the subset of puncturing element is a subquandle as well.*

Proof. Suppose x and x' neutral. Then:

$$y \triangleright (x \triangleright x') = (y \triangleright y) \triangleright (x \triangleright x') = (y \triangleright x) \triangleright (y \triangleright x') = y \triangleright y = y$$

$$y \triangleright (x \triangleright^{-1} x') = (y \triangleright^{-1} y) \triangleright (x \triangleright^{-1} x') = (y \triangleright^{-1} x) \triangleright (y \triangleright^{-1} x') = y \triangleright y = y$$

Suppose x and x' puncturing. Given any element b we have to produce an element a such that $(x \rhd x') \rhd a = b$. We know elements α and α' such that $x \rhd \alpha = b$ and $x' \rhd \alpha' = b$. Using the autonomous property we get

$$(x \rhd x') \rhd (\alpha \rhd \alpha') = (x \rhd \alpha) \rhd (x' \rhd \alpha') = b \rhd b = b$$

In the same way we get: $(x \rhd^{-1} x') \rhd (\alpha \rhd^{-1} \alpha') = (x \rhd \alpha) \rhd^{-1} (x' \rhd \alpha') = b \rhd^{-1} b = b$. $\square$

Proposition 1.12. *Any Alexander quandle $(X, \rhd_g)$ is autonomous and pseudo-leftassociative.*

Proof. Let us start from $((A, +), g)$, then we have:

$$(a \rhd_g b) \rhd_g (a' \rhd_g b') = (g(a) + b - g(b)) \rhd_g (g(a') + b' - g(b'))$$

$$= g^2(a) + g(b) - g^2(b) + g(a') + b' - g(b') - g^2(a') - g(b') + g^2(b')$$

and in the same way:

$$(a \rhd_g a') \rhd_g (b \rhd_g b') = (g(a) + a' - g(a')) \rhd_g (g(b) + b' - g(b'))$$

$$= g^2(a) + g(a') - g^2(a') + g(b) + b' - g(b') - g^2(b) - g(b') + g^2(b')$$

We have also:

$$((x \rhd_g y) \rhd_g^{-1} t) \rhd_g z = ((g(x) + y - g(y)) \rhd_{g^{-1}} t) \rhd_g z = (x + g^{-1}(y) - y + t - g^{-1}(t)) \rhd_g z$$

$$= g(x) + y - g(y) + g(t) - t + z - g(z) = ((x \rhd_g z) \rhd_g^{-1} t) \rhd_g y$$

since we can clearly exchange y and z in the last but one term. $\square$

So, the functor Al is such that $Al : AutAb \to AQd$; in the same way we get functors $Al^I : IAutAb \to AIQd$ and $Al^S : SyautAb \to ASQd$, $Al^{SI} : SIautAb \to ASIQd$.

Proposition 1.13. *When $g \neq Id_A$, there is no neutral element in the Alexander quandle $(A, \rhd_g)$.*

Proof. Suppose e neutral. Then, for all a we have $g(a) + e - g(e) = a$; i.e. $a - g(a) = e - g(e)$, or equivalently $Id_A - g = 0$ or $g = Id_A$. $\square$

Proposition 1.14. *In an Alexander quandle $(A, \rhd_g)$, any element is puncturing when $Id_A - g$ is surjective. Any element is acupuncturing when $Id_A - g$ is bijective; in this case we have $a \tilde{\rhd} v = (Id_A - g)^{-1}(v - a) + a$. When $Id_A - g$ is not surjective, this quandle has no puncturing element.*

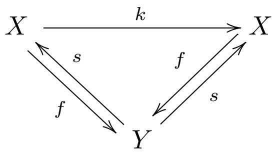
Proof. We noticed that $0 \rhd_g - = Id_A - g$; so that $Id_A - g$ is surjective if and only if the element 0 is puncturing. Let us check that, in this case, any element is puncturing. Let a be an element of A. Given any $v \in A$, we have to show that there is an element $u \in A$ such that $v = a \rhd_g u = g(a) + (Id_A - g)(u)$; or equivalently $v - g(a) = (Id_A - g)(u)$ which has a solution when $Id_A - g$ is surjective. The solution is unique as soon as $Id_A - g$ is an isomorphism and this defines $a \tilde{\rhd} v$. Conversely suppose that the element a is puncturing, then for all v there is a u such that $v - g(a) = (Id_A - g)(u)$, so that $Id_A - g$ is surjective. $\square$

2. Split epimorphisms in *Qnd*

2.1. **The fibration of points.** Given any homomorphism $f : (X, \triangleright) \to (Y, \triangleright)$ of quandles, we shall denote by $R[f]$ the kernel congruence: $xR[f]x'$ if $f(x) = f(x')$. From now on, split epimorphism will mean split epimorphism with a given splitting. Recall from [2] that, for any category $\mathbb{D}$, $\mathrm{Pt}(\mathbb{D})$ denotes the category whose objects are the split epimorphisms (=the "genereralized points") of $\mathbb{D}$ and whose arrows are the commuting squares between such split epimorphisms, and that $\P_{\mathbb{D}} : \mathrm{Pt}(\mathbb{D}) \to \mathbb{D}$ denotes the functor associating with each split epimorphism its codomain.

The functor $\P_{\mathbb{D}} : \mathrm{Pt}(\mathbb{D}) \to \mathbb{D}$ is a fibration (the so-called *fibration of points*) whenever $\mathbb{D}$ has pullbacks of split epimorphisms. The $\P_{\mathbb{D}}$-cartesian maps are precisely pullbacks of split epimorphisms. Given any morphism $f : X \to Y$ in $\mathbb{D}$, base-change along f with respect to the fibration $\P_{\mathbb{D}}$ (=taking pullback) is denoted by $f^* : \mathrm{Pt}_Y(\mathbb{D}) \to \mathrm{Pt}_X(\mathbb{D})$. Recall also that the category $\mathbb{D}$ is said to be *protomodular* when these base-changes are conservative. The categories *Gp* of groups and *K-Lie* of Lie algebras are protomodular.

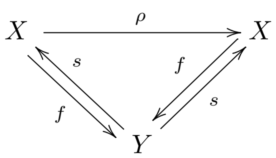
The fibre $f^{-1}(y)$ of any split epimorphism $(f, s) : X \rightleftarrows Y$ in *Qnd* is a quandle pointed by the element $s(y)$. The automorphisms $- \triangleright s(y) : X \to X$ and $- \triangleright^{-1} s(y) : X \to X$ are necessarily stable on this fibre since the operation $\triangleright$ is idempotent. Let us denote the restriction to the fibre of the latter by k_y. So we get an application $k : X \to X$ defined by $k(x) = k_{f(x)}(x)$ satisfying $x = k(x) \triangleright sf(x)$ and making the following diagram commute:



2.2. **Puncturing and acupuncturing split epimorphisms.**

Definition 2.3. *A split epimorphism will be said to be puncturing when, for any object $y \in Y$, the object $s(y)$ is puncturing in the fiber $f^{-1}(y)$, and acupuncturing when, for any object $y \in Y$, the object $s(y)$ is acupuncturing in the fiber $f^{-1}(y)$. In this last case we shall denote by ρ_y the inverse of $s(y) \triangleright -$ in this fibre.*

For any acupuncturing split epimorphism, we have an application $\rho : X \to X$ defined by $\rho(x) = \rho_{f(x)}(x)$ satisfying $x = sf(x) \triangleright \rho(x)$ and making the following diagram commute:



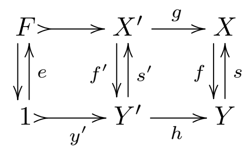
Lemma 2.4. *Let (f, s) be an acupuncturing split epimorphism. Then we get: $\rho(x \triangleright s(t)) = \rho(x) \triangleright s(t)$ for any $(t, x) \in Y \times X$.*

Proof. We have: $s(f(x) \triangleright t) \triangleright (\rho(x) \triangleright s(t)) = (sf(x) \triangleright s(t)) \triangleright (\rho(x) \triangleright s(t)) = (sf(x) \triangleright \rho(x)) \triangleright s(t) = x \triangleright s(t)$. Since the application $s(f(x) \triangleright t) \triangleright -$ is bijective on the fibre $f^{-1}(f(x) \triangleright t)$, we get the asserted identity. $\square$

We shall denote by Σ and Σ^+ the classes of puncturing and acupuncturing split epimorphisms, by ΣPt and $\Sigma^+ Pt$ the induced full subcategories of $Pt(Qnd)$. Clearly, the classes Σ and Σ^+ contain the isomorphisms.

Proposition 2.5. *1) The classes Σ and Σ^+ are stable under pullback along any map in Qnd.*
2) The full subcategories ΣPt and $\Sigma^+ Pt$ are stable under products in $Pt(Qnd)$.
3) The subcategory $\Sigma^+ Pt$ is stable under equalizer in $Pt(Qnd)$.

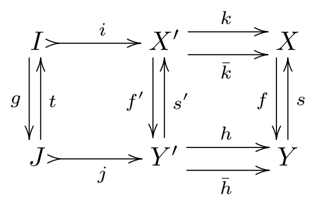
Proof. Consider the following diagram where the split epimorphism (f, s) is puncturing (resp. acupuncturing) and any square is a pullback:



(F, e) is, at the same time and up to isomorphism, the pointed fibre $(f'^{-1}(y'), s'(y'))$ and the pointed fibre $(f^{-1}(h(y')), s \circ h(y'))$. So they are equivalently punctured or acupunctured.

Given a pair (f, s), $(\bar{f}, \bar{s})$ of split epimorphisms, the pointed fibre of their product above the element $(y, \bar{y})$ is the product $(f^{-1}(y), s(y)) \times (\bar{f}^{-1}(\bar{y}), \bar{s}(\bar{y}))$ of the pointed fibers. This product is (acu)punctured is any of its component is so.

Now consider the following diagram of split epimorphisms (f, s) where the two horizontal rows are equalizers:

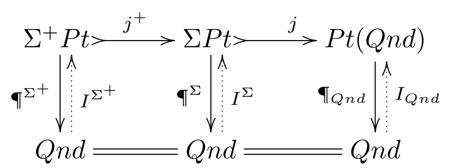


Given any y' such that $h(y') = \bar{h}(y') = \gamma$, we have to show that the pointed fibre $(g^{-1}(y'), t(y')) = (g^{-1}(y'), s'(y'))$ is acupunctured; namely: given any x' in this fibre (i.e. satisfying $k(x') = \bar{k}(x')$), there is a unique ξ' in this fibre such that $t(y') \triangleright \xi' = s'(y') \triangleright \xi' = x'$. We know that such an element ξ' exits in the fibre $f'^{-1}(y')$ when (f', s') is acupunctured. We have to show that ξ' is actually in I. We have:

$$k(s'(y')) \triangleright k(\xi') = k(x') = \bar{k}(x') = \bar{k}(s'(y')) \triangleright \bar{k}(\xi')$$

So in the pointed fibre $(f^{-1}(\gamma), s(\gamma))$ we get: $s(\gamma) \triangleright k(\xi') = s(\gamma) \triangleright \bar{k}(\xi')$. If (f, s) is acupunctured as well, we get $k(\xi') = \bar{k}(\xi')$, and ξ' is in J and consequently in the fibre $g^{-1}(y')$ as desired. □

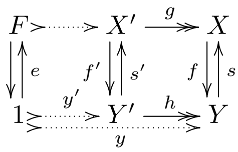
According to the first point of the previous proposition we get two subfibrations of the fibration of points:



Actually we get two sub-reflections as well, defined respectively by $I^{\Sigma}(X) = (1_X, 1_X)$ and $I^{\Sigma^+}(X) = (1_X, 1_X)$. On the other hand, the class Σ^+ is *point-congruous* in the sense of, since it contains the isomorphisms and satisfies the three conditions of the Proposition 2.5.

Proposition 2.6. *The puncturing split epimorphims are stable under regular epimorphism in $Pt(Qnd)$. The acupuncturing split epimorphims are stable under cartesian regular epimorphism in $Pt(Qnd)$.*

Proof. Consider the following right hand side square where the split epimorphism (f', s') is puncturing and the homomorphisms g and h are surjective:



We have to show that (f, s) is puncturing. Let x be an element of X. We are looking for an element $\rho(x)$ in the same fibre as x such that $x = sf(x) \triangleright \rho(x)$. There is an element x' in X' and an element $\rho(x')$ in the same fibre as x' such that $x = g(x')$ and $x' = s'f'(x') \triangleright \rho(x')$. Accordingly we get $x = g(x') = sf(x) \triangleright g(\rho(x'))$ with $g(\rho(x'))$ in the same fibre as x.

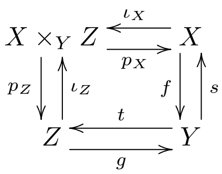
Suppose now the right hand side square is a pullback and the split epimorphism (f', s') is acupuncturing. Let y be an element of Y we have to show that the fibre $f^{-1}(y)$ is acupunctured by $s(y)$. Since h is surjective, there is an element $y' \in Y'$ such that $h(y') = y$. Consider now the left hand side pullback which produces at the same time the pointed fibre $(f'^{-1}(y'), s'(y'))$ and the pointed fibre $(f^{-1}(y), s(y))$. So they are equivalently acupunctured. □

3. Relative Maltsevness

In this section we shall show that the category Qnd satisfies a Maltsev property relatively to the classes Σ and Σ^+, see [5].

Theorem 3.1. *The category Qnd is a Σ-Maltsev category and a Σ^+-Maltsev category.*

Proof. We have to show that the two previous reflections are Maltsev reflections in the sense of [5]. It means that, given any pullback of split epimorphisms in Qnd with (f, s) in Σ (resp. Σ^+):



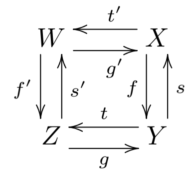
the pair (ι_X, ι_Z) of morphisms is jointly strongly epic, or equivalently that the quandle $X \times_Y Z$ is the supremum of the subquandles $\iota_X(X)$ and $\iota_Z(Z)$. Now suppose $(x, z) \in X \times_Y Z$. We have $z = k(z) \triangleright tg(z) = k(z) \triangleright tf(x)$ with $g(k(z)) = g(z)$ and, since (f, s) is in Σ, $x = sf(x) \triangleright \rho = sg(z) \triangleright \rho$ with $f(\rho) = f(x)$ for some ρ. Whence:

$$(x, z) = (sg(z), k(z)) \triangleright (\rho, tf(\rho)) = \iota_Z(k(z)) \triangleright \iota_X(\rho)$$

Accordingly the quandle $X \times_Y Z$ is certainly the supremum of the subquandles $\iota_X(X)$ and $\iota_Z(Z)$. This is a fortiori the case when the split epimorphism (f, s) belongs to Σ^+. □

In this relative Maltsev context, we can partially recover (see [5]) a series of aspects of Maltsev categories given in [9] and [10].

Corollary 3.2. *Suppose (f, s) is in Σ. Any commutative squares of split epimorphisms in Qnd:*



is such that the canonical factorization $\phi : W \to X \times_Y Z$ to the pullback of f along g is a surjective homomorphism.

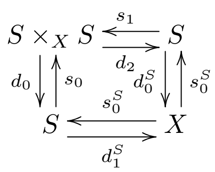
Proof. Decompose ϕ as $m.\bar{\phi}$ with $\bar{\phi}$ surjective and $m : \phi(W) \hookrightarrow X \times_Y Z$ the inclusion. Then $\phi(W)$ contains the subquandles $\iota_X(X)$ and $\iota_Z(Z)$; so that $\phi(W) = X \times_Y Z$. □

3.3. **Puncturing and acupuncturing relations.**

Definition 3.4. *A relation $(d_0, d_1) : R \rightrightarrows X$ in Qnd is said to be a puncturing (resp. acupuncturing) relation when it is reflexive and such that the induced split epimorphism $(d_0, s_0) : R \rightleftarrows X$ is puncturing (resp. acupuncturing), namely such that for all xRy there is a (resp. unique) $\rho(x, y)$ such that $xR\rho(x, y)$ and $y = x \triangleright \rho(x, y)$.*

Proposition 3.5. *Let S be any puncturing relation on a quandle $(X, \triangleright)$. It is necessarily transitive.*

Proof. Consider the following pullback:

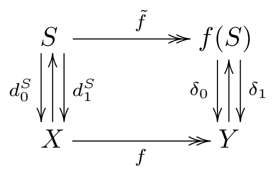


where $S \times_X S$ is the quandle of pairs $(x_1 S x_2, x_2 S x_3)$. Concerning the split epimorphism (d_1^S, s_0^S), we know from $x_1 S x_2$ that there is a unique t such that tSx_2 and $x_1 = t \triangleright x_2$. Concerning the puncturing split epimorphism (d_0^S, s_0^S), we know from $x_2 S x_3$ that there is a unique τ such that $x_2 S \tau$ and $x_3 = x_2 \triangleright \tau$. Whence $(tSx_2) \triangleright (x_2 S \tau) = x_1 S x_3$. □

The direct image along a surjective homomorphism of a reflexive (resp. symmetric) relation is reflexive (resp. symmetric). In general the direct image along a surjective homomorphism of a transitive relation is no longer transitive.

Proposition 3.6. *The direct image along a surjective homomorphism of a puncturing preorder (resp. equivalence relation) is a puncturing preorder (resp. equivalence relation).*

Proof. Let S be a puncturing preorder on the quandle $(X, \triangleright)$ and $f : X \twoheadrightarrow Y$ a surjective homomorphism. Consider the following diagram:

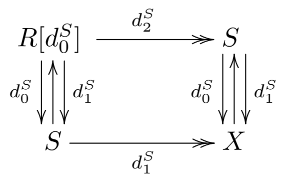


By Proposition 2.6, since f and $\tilde{f}$ are regular epimorphism, and (d_0^S, s_0^S) is puncturing, so is (δ_0, σ_0); and $f(S)$ is a transitive relation. □

Finally we are able to characterize the Σ-equivalence relations:

Proposition 3.7. *A reflexive relation S on a quandle $(X, \triangleright)$ is underlying a Σ-equivalence relation if and only if $d_0^S : S \to X$ is such that $R[d_0^S]$ is a puncturing relation, i.e. d_0^S is a Σ-special homomorphism according to the terminology of Section 4 below.*

Proof. If S is an equivalence relation, then we get the following diagram in which any commutative square is a pullback:

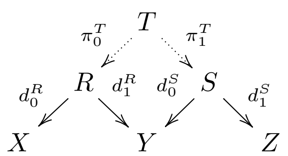


Accordingly, if S is Σ-relation, so is $R[d_0^S]$. Conversely suppose the reflexive relation S satisfies the property in question. We have to show it is an equivalence relation. Since S is reflexive, it is equivalent to show that for all (x, y, z) such that xSy and xSz we get ySz. From the assumption on S and, from xSy, xSz, we know that there is an element t such that xSt and $z = y \triangleright t$. Now we have $(ySy) \triangleright^{-1} (xSy) = (y \triangleright^{-1} x)S(y \triangleright^{-1} y) = (y \triangleright^{-1} x)Sy$. Then we get:

$$((y \triangleright^{-1} x)Sy) \triangleright (xSt) = ((y \triangleright^{-1} x) \triangleright x)S(y \triangleright t) = ySz$$

□

3.8. **Suprema of puncturing equivalence relations.** The binary relations can be composed in Qnd, with $R \circ S = \{(x, x')/\exists t,\ xRt,\ tSx'\}$. In the categorical way, consider the two binary relations R and S:



take the pullback of d_0^S along d_1^R and decompose the factorization $(d_0^R.p_0^T, d_1^S.p_1^T) : T \to X \times Z$ into an injection and a surjection: $T \twoheadrightarrow R \circ S \hookrightarrow X \times Z$. Two relations R and S on a same quandle $(X, \triangleright)$ *permute* when $R \circ S = S \circ R$. Recall that, if R and S is a pair of preorders (resp. equivalence relations), then $R \circ S$ is a preorder (resp. an equivalence relation) when R and S permute. Moreover $R \circ S$ becomes the supremum of R and S among the preorders (resp. the equivalence relations).

Theorem 3.9. *Given any pair of a reflexive relation R and a puncturing symmetric relation S on a quandle $(X, \triangleright)$, the two relations permute.*

Proof. Since S is puncturing and symmetric both split epimorphisms (d_0^S, s_0^S) and (d_1^S, s_0^S) are in Σ. Let us denote by $R\square S$ the inverse image of the reflexive relation $S \times S$ along $(d_0^R, d_1^R) : R \rightarrowtail Y \times Y$. This produces a double relation:

$$\begin{array}{ccc} R\square S & \overset{p_1^S}{\underset{p_0^S}{\rightleftarrows}} & S \\ p_0^R \Big\downarrow\Big\uparrow\Big\downarrow p_1^R & & d_0^S \Big\downarrow\Big\uparrow\Big\downarrow d_1^S \\ R & \overset{d_1^R}{\underset{d_0^R}{\rightleftarrows}} & X \end{array}$$

which is the largest double equivalence relation on X relating R and S. In set theoretical terms, $R\square S$ defines the subset of elements (u, v, u', v') of X^4 such that we have:

$$\begin{array}{ccc} u & \overset{S}{\dashrightarrow} & v \\ R \downarrow & & \downarrow R \\ u' & \underset{S}{\dashrightarrow} & v' \end{array}$$

Now let us denote by T the domain of the pullback of d_0^S along d_1^R. Then, according to Corollary 3.2, the canonical factorization $\phi : R\square S \to T$ is surjective, so that $R \circ S$ coincides with the decomposition of $(d_0^R.p_0^R, d_1^S.p_1^S) : R\square S \to X \times X$ (which associates (u, v') with the previous square) into an injection and a surjection. If we denote by $\bar{T}$ the domain of the pullback of d_0^R along d_1^S, the canonical factorization $\psi : R\square S \to \bar{T}$ is surjective as well, so that $S \circ R$ coincides with the decomposition of the morphism $(d_0^S.p_0^S, d_1^R.p_1^R) : R\square S \to X \times X$ (which, again, associates (u, v') with the previous square) into an injection and a surjection. Since the map $(d_0^R.p_0^R, d_1^S.p_1^S)$ and $(d_0^S.p_0^S, d_1^R.p_1^R)$ coincide, we get $R \circ S = S \circ R$. $\square$

Let $(S_i)_{i\in E}$ be a family of puncturing preorders (resp. equivalence relations) on a quandle $(X, \triangleright)$ indexed by the set E. If E is finite, it is clear that the composition of all the preorders (resp. equivalence relations) does not depend on the order of the composition since they all permute with each other, and that this composition is the supremum $\bigvee_E S_i$ of all the preorders (resp. equivalence relations) of this family. It is clear that, from $E' \subset E$, we get $\bigvee_{E'} S_i \subset \bigvee_E S_i$. The following lemma is straighforward:

Lemma 3.10. *Let $(S_i)_{i\in E}$ be a finite family of puncturing preorders (resp. equivalence relations) on a quandle $(X, \triangleright)$. Then the supremum $\bigvee_E S_i$ of this family permute with any preorder (resp. equivalence relation) on $(X, \triangleright)$.*

Lemma 3.11. *Let $(S_i)_{i\in E}$ be a family of puncturing preorders (resp. equivalence relations) on a quandle $(X, \triangleright)$ indexed by any set E. Then the supremum $\bigvee_E$ of this family among the preorders (resp. equivalence relations) permute with any preorder (resp. equivalence relation) on $(X, \triangleright)$.*

Proof. The supremum $\bigvee_E$ of this family can be described in the following way: denote by FE the set of the finite subsets of E and denote by $\bigvee_E$ the union of the congruences $\bigvee_A$ for $A \in FE$. In other words, we have $x \bigvee y$ if there is a finite part $A \subset E$ such that $x \bigvee_A y$. We have to show that $\bigvee_E$ is a preorder (resp. an equivalence relation). It is clear that $\bigvee_E$ is reflexive (resp. reflexive and symmetric) when any S_i is preorder (resp. an equivalence relation). Let us show it is transitive. Suppose $x \bigvee_E y$ and $y \bigvee_E z$. There exists finite parts A and B such that $x \bigvee_A y$ and $y \bigvee_B z$. This implies $x \bigvee_{A\cup B} y$ and $y \bigvee_{A\cup B} z$ whence $x \bigvee_E z$. We have to check it is a congruence. Suppose $x \bigvee_E y$ and $x' \bigvee_E y'$. There exist finite parts A and B such that $x \bigvee_A y$ and $x' \bigvee_B y'$. This implies $x \bigvee_{A\cup B} y$ and $x' \bigvee_{A\cup B} y'$, and therefore $(x \triangleright x') \bigvee_{A\cup B} (y \triangleright y')$; so that we get $(x \triangleright x') \bigvee_E (y \triangleright y')$.

Now let R be any preorder (resp. equivalence relations) on $(X, \triangleright)$. Then $R \circ \bigvee_E$ is defined by the pairs (x, y) such that there is an element t and a finite part A of E such that xRt and $t \bigvee_A y$; so it is nothing the union of the $R \circ \bigvee_A$ for all $A \in EF$. The symmetric result holds for $\bigvee_E \circ R$. Now by the previous lemma we get $R \circ \bigvee_A = \bigvee_A \circ R$ for all A. Whence $R \circ \bigvee_E = \bigvee_E \circ R$. □

3.12. **Action congruences.** When $(X, \triangleright)$ is a quandle any element x produces an automorphism of quandles $- \triangleright x : X \to X$. The subgroup of $AutX$ generated by these automorphisms is denoted by $InnX$. Any subgroup N of $InnX$ determines an equivalence relation $\simeq_N$ on X given by $x \simeq_N x'$ if and only if x and x' lie in the same orbit of the action of N. Recall the following theorem from [8]:

Theorem 3.13. *The equivalence relation $\simeq_N$ is a congruence if and only if N is a normal subgroup of $InnX$.*

and the following lemma from [12]:

Lemma 3.14. *When N is a normal subgroup of $InnX$, the congruence $\simeq_N$ on X permutes with any other congruence R on X.*

It is easy to check that the congruence $\simeq_{InnX}$ is not in general a puncturing relation. However the following question still makes sense: is any action congruence $\simeq_N$ generated by the class of puncturing congruences, in the sense of Lemma 3.11?

4. Latin quandles

Definition 4.1. *A homomorphism $h : X \to Y$ is said to be* Σ-special *(resp.* latin*) when its kernel congruence $R[f]$ is puncturing (resp. acupuncturing). A quandle $(X, \triangleright)$ is said to be Σ-special (resp. latin) when its terminal map τ_X is Σ-special (resp. latin).*

So a quandle $(X, \triangleright)$ is Σ-special (resp. latin) if and only if any element x is puncturing (resp. acupuncturing). It is clear that no trivial quandle can be Σ-special. It is clear also that when f has finite fibers, a homomorphism is latin as soon as it is Σ-special. The terminology *latin* comes from the fact that the table of a finite latin quandle is a latin square, namely a square in which any element appears exactly once in any row and any column. When a quandle is latin we shall denote by $x \tilde{\triangleright} -$ the inverse of the application $x \triangleright -$. A homomorphism $f : X \to Y$ of quandles is latin if and only if any non-empty fibre $f^{-1}(y)$ is a latin quandle.

Lemma 4.2. *In a latin quandle we have: $(x \tilde{\triangleright} y) \triangleright t = (x_{\triangleright} t) \tilde{\triangleright} (y \triangleright t)$.*

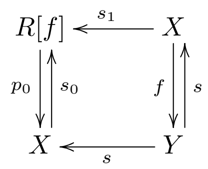
Proof. We have: $(x \triangleright t) \triangleright ((x \tilde{\triangleright} y) \triangleright t) = (x \triangleright (x \tilde{\triangleright} y)) \triangleright t = y \triangleright t$. □

Let us denote by LQd of the full subcategory of Qnd whose objects are the latin quandles. This category is a variety of universal algebra. It is stable under finite limit in Qnd. The regular epimorphisms of LQd are those surjective homomorphisms of Qnd which belong to LQd. Since the class Σ^+ is point-congruous, i.e. contains the isomorphisms and satisfies the three conditions of Proposition 2.5, the category LQd coincides with what is called the Maltsev *core* of the Σ^+-Malcev category, see [5], and is consequently a Maltsev category.

A split epimorphism which is in Σ^+ is not necessarily a latin homomorphism; however we have the following:

Lemma 4.3. *Any latin split epimorphism (f, s) is in Σ^+.*

Proof. It is a consequence of the fact that the following square is a pullback:



□

4.4. **Examples.**

Proposition 4.5. *Any symmetric quandle X is latin. So, we have an inclusion $SymQd \hookrightarrow LQd$. A latin quandle X is symmetric if and only if for all (x, y) we have $y \tilde{\triangleright} x = x \triangleright^{-1} y$.*

Proof. Suppose X symmetric. Since we have $x \triangleright - = - \triangleright x$, any object is acupuncturing. Moreover, for all (x, y) we have

$$x \triangleright (y \triangleright^{-1} x) = (y \triangleright^{-1} x) \triangleright x = y = x \triangleright (x \tilde{\triangleright} y)$$

whence $y \triangleright^{-1} x = x \tilde{\triangleright} y$. Conversely suppose that X is latin and $x \triangleright^{-1} y = y \tilde{\triangleright} x$. We shall have $x \triangleright y = y \triangleright x$ if and only if $(x \triangleright y) \triangleright^{-1} x = y$. Now we have

$$(x \triangleright y) \triangleright^{-1} x = x \tilde{\triangleright} (x \triangleright y) = y$$

by definition of the operation $\tilde{\triangleright}$. □

Proposition 4.6. *An Alexander quandle $(A, \triangleright_g)$ is a latin quandle if and only if $Id_A - g$ is an isomorphism. In this case, on the one hand we have $a \tilde{\triangleright} v = (Id_A - g)^{-1}(v - a) + a$ and on the other hand the automorphism g has no non-trivial fixed point. When the group A is finite, this last condition is characteristic.*

Proof. The first part of this proposition is given by Proposition 1.14. The last part of the second assertion is the consequence of the fact that $Ker(Id_A - g)$ is the subgroup of the fixed points of g. □

Let us denote by $LautAb$ the subcategory of $AutAb$ whose objects $((A, +), g)$ are such that the homomorphism $Id_A - g$ is an automorphism. We get a restriction functor: $Al^L : LautAb \to LQd$. On the other hand, we have an inclusion $SyauAb \hookrightarrow LautAb$, since, from $2g = Id_A$, we get $Id_A - g = g$.

Corollary 4.7. *Let $(A, +)$ be an abelian group. Beyond the symmetric case, the involutive quandle Ξ_A is a latin quandle if and only if the homomorphism $2Id_A : A \to A$ is an automorphism.*

Proof. We noticed that the quandle Ξ_A is an Alexander quandle with $g = -Id_A$ and $Id_A - g = 2Id_A$. □

Corollary 4.8. *When n is odd, any dihedral quandle $\Xi_{\mathbb{Z}/n\mathbb{Z}}$ is a latin quandle. When $n \neq 3$, it is non symmetric*

Proof. Suppose $n = 2p-1$. Then we have $\bar{2}.\bar{p} = \bar{1}$ in the ring $\mathbb{Z}/n\mathbb{Z}$; so $\bar{2}$ is invertible in this ring which implies that $2Id$ is an automorphism. This quandle is in LQd according to the previous proposition. More precisely we have $\bar{x}\,\tilde{\triangleright}\bar{y} = \overline{p(x+y)}$. On the other hand we have $x \triangleright y = y \triangleright x$ if and only if $\overline{3x} = \overline{3y}$. So this quandle is symmetric if and only if $\overline{3t} = 0$ for all $t \in \mathbb{N}$ which is not true for $n > 3$. $\square$

Example 4.9. The quandle $\Xi_{\mathbb{Z}/3\mathbb{Z}}$ is an example of a symmetric and involutive latin quandle:

	a	b	c
a	a	c	b
b	c	b	a
c	b	a	c

4.10. **Connected quandles.** A quandle $(X, \triangleright)$ is said to be connected when it has only one orbit under the action of $InnX$. Writing inner automorphisms to the right of the element they are being applied to, this means that, for any pair (x, x') of elements of X, there is some n-uple $(a_1, .., a_n)$ of elements such that: $x' = x \triangleright^{\epsilon_n} a_1 ... \triangleright^{\epsilon_n} a_n$ with $\epsilon_i \in \{1, -1\}$. It is clear that a latin quandle is a very special case of connected quandle.

Proposition 4.11. *Let $f : X \twoheadrightarrow Y$ be a surjective Σ-special homomorphism (or a fortiori a surjective latin homomorphism). If Y is connected, so is X.*

Proof. Let (x, x') a pair of elements of X. Since Y is connected, there is some n-uple $(a_1, .., a_n)$ of elements of Y such that $f(x') = f(x) \triangleright^{\epsilon_n} a_1 ... \triangleright^{\epsilon_n} a_n$. Since f is surjective, any a_i has an antecedent α_i. The element $\xi = x \triangleright^{\epsilon_n} \alpha_1 ... \triangleright^{\epsilon_n} \alpha_n$ is such that $f(\xi) = f(x')(= y')$. Since the kernel congruence $R[f]$ of f is puncturing, there is an element t in the fibre of y' such that $x' = \xi \triangleright t$, whence: $x' = x \triangleright^{\epsilon_n} \alpha_1 ... \triangleright^{\epsilon_n} \alpha_n \triangleright^1 t$. $\square$

4.12. **Affine latin quandles.** In any category $\mathbb{E}$, an object X is said to have an affine structure when its is endowed with an internal Maltsev operation, namely a ternary operation $p : X \times X \times X \to X$ such that $p(x, y, y) = x$ and $p(x, x, y) = y$ which is commutative, i.e. such that $p(x, y, z) = p(z, y, x))$ and left associative, i.e. such that $p(p(x, y, z), z, t) = p(x, y, t)$. It is clear that the commutativity implies the right associativity: $p(x, y, p(y, z, t)) = p(x, z, t)$.

Any internal abelian group structure $(A, +, 0)$ in $\mathbb{E}$ gives to the object A an affine structure defined by $p(a, b, c) = a - b + c$. Conversely any global element $e : 1 \to X$ gives to any affine structure (X, p) an abelian group structure $(X, +_e, e)$, just setting $x +_e y = p(x, e, y)$. We then get $x -_e y = p(x, y, e)$. Moreover any pair (e, e') of global elements gives rise to an internal group isomorphism: $p(e', e, -) : (A, +_e, e) \to (A, +_{e'}, e')$. Any element e of a non-empty quandle $(X, \triangleright)$ determining a global element $e : 1 \to X$ in the category Qnd, there is a bijection between the internal abelian groups and the non-empty affine objects in Qnd.

The distinctive feature of a Maltsev category $\mathbb{D}$ is that an object X admits *at most one* internal Maltsev operation which is necessarily commutative and left associative, see [10] and [6]; it implies that there is at most one affine structure on an object X. So being affine, in a Maltsev context, becomes a property. As in any Malc'ev category we have:

Proposition 4.13. *Any internal group in LQd is abelian. In other words, the inclusion $Ab(LQd) \hookrightarrow Gp(LQd)$ is actually an isomorphism of categories.*

Proposition 4.14. *If a latin quandle $(X, \triangleright)$ is affine, then we have necessarily $p(x,y,z) = (x \triangleright^{-1} y)) \triangleright (y \tilde{\triangleright} z) = (z \triangleright^{-1} y)) \triangleright (y \tilde{\triangleright} x)$.*

Proof. Suppose $(X, \triangleright)$ is affine and p is the associated Maltsev operation on X in Qnd. Then we have:

$$p(x,y,z) = p((x \triangleright^{-1} y) \triangleright y, y \triangleright y, y \triangleright (y \tilde{\triangleright} z)) = p(x \triangleright^{-1} y, y, y) \triangleright p(y, y, y \tilde{\triangleright} z)$$

The second equality of the proposition holds since p is necessarily commutative. □

Remark 4.15. According to the famous result of Maltsev [15] characterizing the 2-permutable varieties of universal algebra, now so-called Maltsev varieties [18], any quandle in LQd is endowed with a Maltsev operation generated by the operations of the theory; we can obviously choose the previous formula $p(x,y,z) = (x \triangleright^{-1} y)) \triangleright (y \tilde{\triangleright} z)$.

Proposition 4.16. *A latin Alexander quandle $(A, \triangleright_g)$ is necessarily affine.*

Proof. Here the calculation of the Maltsev formula of Proposition 4.14 gives for any automorphism g:

$$p(a,b,c) = a - b + c$$

Then to show that it is a homomorphism of quandles is straightforward. □

Corollary 4.17. *When n is odd, any dihedral quandle $\Xi_{\mathbb{Z}/n\mathbb{Z}}$ is affine. It is a non symmetric affine quandle when $n \neq 3$.*

Proposition 4.18. *If a symmetric quandle $(X, \triangleright)$ is affine, its Maltsev operation is defined by $p(x,y,z) = (x \triangleright z) \triangleright^{-1} y$. A symmetric object X is affine if and only if the following conditions hold:*
1) $(X, \triangleright)$ is autonomous
2) $(X, \triangleright)$ is pseudo-leftassociative.

Proof. According to the previous observations the ternary operation must be:

$$p(x,y,z) = (x \triangleright^{-1} y)) \triangleright (y \tilde{\triangleright} z) = (x \triangleright^{-1} y)) \triangleright (z \triangleright^{-1} y) = (x \triangleright z) \triangleright^{-1} y$$

In particular, we have $p(x,y,x) = x \triangleright^{-1} y$. If X is affine, from $p(x \triangleright x', y \triangleright y', x \triangleright x') = p(x,y,x) \triangleright p(x', y', x')$ we get immediately:

$$(x \triangleright x') \triangleright^{-1} (y \triangleright y') = (x \triangleright^{-1} y) \triangleright (x' \triangleright^{-1} y')$$

which means that $(X, \triangleright)$ is autonomous. On the other hand we recalled above that, when X is affine, we have necessarily $p(p(x,y,z),z,t) = p(x,y,t)$ which gives us:

$$(((x \triangleright z) \triangleright^{-1} y) \triangleright t) \triangleright^{-1} z = (x \triangleright t) \triangleright^{-1} y$$

which is equivalent to:

$$((x \triangleright z) \triangleright^{-1} y) \triangleright t = ((x \triangleright t) \triangleright^{-1} y) \triangleright z$$

Conversely suppose the two conditions hold. We have to show that p is a homomorphism of quandles, namely that:

$$((x \triangleright x') \triangleright (z \triangleright z')) \triangleright^{-1} (y \triangleright y') = ((x \triangleright z) \triangleright^{-1} y) \triangleright ((x' \triangleright z') \triangleright^{-1} y')$$

Let us denote respectively by (a) and (b) the two formula. From 2) we can exchange z and $(x' \triangleright z') \triangleright^{-1} y'$ in the term (b):

$$(b) = ((x \triangleright ((x' \triangleright z') \triangleright^{-1} y')) \triangleright^{-1} y) \triangleright z = ([((x' \triangleright z') \triangleright^{-1} y') \triangleright x] \triangleright^{-1} y) \triangleright z$$

Now, again from 2) we can exchange x' and z' inside the square brackets:

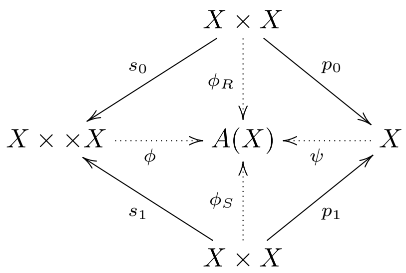
$$= ([((x \triangleright x') \triangleright^{-1} y') \triangleright z'] \triangleright^{-1} y) \triangleright z = ((z' \triangleright ((x \triangleright x') \triangleright^{-1} y')) \triangleright^{-1} y) \triangleright z$$

Again from 2), we can exchange $(x \triangleright x') \triangleright^{-1} y'$ and z:

$$= ((z' \triangleright z) \triangleright^{-1} y) \triangleright ((x \triangleright x') \triangleright^{-1} y') = ((z \triangleright z') \triangleright^{-1} y) \triangleright ((x \triangleright x') \triangleright^{-1} y')$$

Since $(X, \triangleright)$ is autonomous, this last term is equal to the following one which itself is clearly equal to (a): $((z \triangleright z') \triangleright (x \triangleright x')) \triangleright^{-1} (y \triangleright y')$. □

4.19. **Associated affine latin quandle.** Let us denote by $AfLQd$ the full subcategory of LQd whose objects are the affine latin quandles. Since the category LQd is a Barr-exact Maltsev category the inclusion $AfLQq \hookrightarrow LQd$ has a left adjoint; in other words, any latin quandle has a universal associated affine latin quandle; it is given by the colimit $A(X)$ of the following quadrangle in LQd, see [2]:



where, according to the simplicial notations, we have $p_0(x, x') = x$, $p_1(x, x') = x'$, $s_0(x, x') = (x, x', x')$ and $s_1(x, x') = (x, x, x')$.

5. Internal groups in Qnd

Given any category $\mathbb{E}$, we shall denote by $Gp(\mathbb{E})$ (resp. $Ab(\mathbb{E})$) the category of internal groups (resp. abelian groups) in $\mathbb{E}$, see [14]. From what was recalled above, any element e in a non-empty affine quandle X gives rise to an internal abelian group structure $(X, +_e, e)$ in Qnd. It is then natural to investigate what are exactly the internal groups in Qnd.

Remark 5.1. All the functors given in Section 1.1 being left exact, they transfer the internal group structures.
1) Whence a first left exact functor which produces examples of non-abelian group structures in Qnd:

$$Gp(T) : Gp \to Gp(AQd)$$

Any object in an additive category $\mathbb{A}$ being naturally endowed with an internal abelian group stucture, the forgetful functor $U : Ab(\mathbb{A}) \to \mathbb{A}$ is an isomorphism.
2) So the functor Al takes actually its values in $Ab(AQd)$:

$$Al : AutAb \to Ab(AQd)$$

since $((A, \triangleright_g), +)$ is automatically an internal abelian group in Qnd. This explains why any Alexander quandle is necessarily affine as soon as it is latin (see Proposition 4.16).

The aim of this section is to characterize the internal groups as well as the affine latin objects in Qnd.

Proposition 5.2. *Let $(G, \cdot, e)$ be an internal group on the object $(G, \triangleright)$ in Qnd. Then we have necessarily:*

$$x \triangleright y = (x \triangleright e) \cdot (e \triangleright y)$$

and the applications $- \triangleright e : G \to G$ and $e \triangleright - : G \to G$ are group homomorphisms. In particular we get : $(x \triangleright e)^{-1} = (x^{-1}) \triangleright e$

Proof. Saying that the group law is a quandle homomorphism is saying that for all (x, x', y, y') in G we have:

$$(x \triangleright x') \cdot (y \triangleright y') = (x \cdot y) \triangleright (x' \cdot y')$$

Setting $y = e = x'$, we get $(x \triangleright e) \cdot (e \triangleright y') = x \triangleright y'$. Setting $x = y = e$, we get $(e \triangleright x') \cdot (e \triangleright y') = e \triangleright (x' \cdot y')$, while setting $x' = e = y'$ we get $(x \triangleright e) \cdot (y \triangleright e) = (x \cdot y) \triangleright e$. ☐

5.3. **Characterization of the internal abelian groups in Qnd.**

Theorem 5.4. *The functor $Al : AutAb \to Ab(Qnd)$ is an isomorphism of categories which makes the following diagram commutative:*

$$\begin{array}{ccc} AutAb & \xrightarrow{Al} & Ab(Qnd) \\ {\scriptstyle U}\downarrow\uparrow{\scriptstyle I} & & {\scriptstyle Ab(U)}\downarrow\uparrow{\scriptstyle Ab(T)} \\ Ab & =\!=\!= & Ab \end{array}$$

where U is the functor forgetting the automorphism, $Ab(U)$ the functor forgetting the quandle structure and I is defined by $I(A, +) = ((A, +), Id_A)$.

Proof. The inverse functor is defined by $\Psi((A, \triangleright), +) = ((A, +), - \triangleright 0)$. Since we observed that $- \triangleright_g 0 = g$, it is clear that $\Psi \circ Al = Id$. Now thanks to the previous proposition, starting from an abelian group stucture $((X, \triangleright), +)$, we get:

$$x - (x \triangleright 0) = (x \triangleright x) + (-x \triangleright 0)) = (x + (-x)) \triangleright (x + 0) = 0 \triangleright x$$

So that the quandle operation on $Al(((A, +), - \triangleright 0))$ is given by:

$$x \bar{\triangleright} y = (x \triangleright 0) + (y - y \triangleright 0) = (x \triangleright 0) + (0 \triangleright y) = (x + 0) \triangleright (0 + y) = x \triangleright y$$

Accordingly we have $Al \circ \Psi = Id$. The commutativity of the square is straighforward. ☐

Let us denote $\bar{A}Qd$ the full subcategory of AQd whose objects are the autonomous quandles which are pseudo-leftassociative and by $i_{\bar{A}}$ the inclusion functor: $\bar{A}Qd \hookrightarrow Qnd$.

Corollary 5.5. *The functor $Ab(i_{\bar{A}}) : Ab(\bar{A}Qd) \to Ab(Qnd)$ is an isomorphism of categories. Any abelian group $((A, \triangleright), +)$ in Qnd makes the quandle $(A, \triangleright)$ autonomous and pseudo-leftassociative.*

Proof. Since any Alexander quandle is autonomous and pseudo-leftassociative, we have a dotted factorization:

$$\begin{array}{ccc} AutAb & \xrightarrow{Al} & Ab(Qnd) \\ & \searrow \cdots & \nearrow {\scriptstyle Ab(i_{\bar{A}})} \\ & Ab(\bar{A}Qd) & \end{array}$$

Since Al is an isomorphim, and $Ab(i_{\bar{A}})$ a monomorphism, $Ab(i_{\bar{A}})$ is actually an isomorphism. □

Corollary 5.6. *The restriction functor:* $Al^I : IAutAb \to Ab(IQd)$ *is an isomorphism of categories. The categories* $Ab(AIQd)$ *and* $Ab(IQd)$ *are isomorphic.*

Proof. Straighforward from Theorem 5.4 and Proposition 1.2. □

Corollary 5.7. *The restriction functor:* $Al^S : SyautAb \to Ab(SQd)$ *is an isomorphism of categories. The categories* $Ab(ASQd)$ *and* $Ab(SQd)$ *are isomorphic.*

Proof. Straighforward from Theorem 5.4 and Proposition 1.3. □

Corollary 5.8. *The restriction functor* $\Xi : Ab_3 \to Ab(SIQd)$ *is an isomorphism of categories. The only internal abelian group structures on the involutive and symmetric quandles are the ones of the quandles* Ξ_A *such that* $3Id_A = 0$*.*

Proof. Straighforward from Theorem 5.4, Proposition 1.4. □

Corollary 5.9. *The restriction functor:* $Al^L : LautAb \to Ab(LQd)$ *is an isomorphism of categories. The categories* $Ab(LAQd)$ *and* $Ab(LQd)$ *are isomorphic, where* $LAQd$ *is the full subcategory of* LQd *whose objects are autonomous latin quandles.*

Proof. Straighforward from Theorem 5.4 and Proposition 4.6. □

The following commutative diagram summarizes the previous observations:

$$\begin{array}{ccccccccc}
Ab_3 & \overset{J_3}{\rightarrowtail} & SyautAb & \rightarrowtail & LautAb & \rightarrowtail & AutAb & \leftarrowtail & IAutAb \\
{\scriptstyle \Xi}\downarrow{\scriptstyle \simeq} & & {\scriptstyle Al^S}\downarrow{\scriptstyle \simeq} & & {\scriptstyle Al^L}\downarrow{\scriptstyle \simeq} & & {\scriptstyle Al}\downarrow{\scriptstyle \simeq} & & {\scriptstyle Al}\downarrow{\scriptstyle \simeq} \\
Ab(SIQd) & \rightarrowtail & Ab(SymQd) & \rightarrowtail & Ab(LQd) & \rightarrowtail & Ab(Qnd) & \leftarrowtail & Ab(IQnd)
\end{array}$$

5.10. **Characterization of the affine latin quandles.**

Proposition 5.11. *The affine latin quandles are either the empty set or a quandle* $Al((A,+),g)$ *where* g *is an automorphism such that* $Id_A - g$ *is an automorphism.*

Proof. We already noticed at the end of the second paragraph of Section 4.12 that there is a bijection between the internal abelian groups and the non-empty affine objects in Qnd. □

We can now give the following precision about Proposition 4.18:

Corollary 5.12. *The only non-empty affine symmetric quandles are the quandles* $Al((A,+),g)$ *with* $2g = Id_A$*; the only non-empty affine symmetric involutive quandles are the quandles* Ξ_A *with* $3Id_A = 0$*.*

5.13. **Non-additive Alexander functor.** Now let $(G,\cdot,1)$ be a group, and (g,h) a pair of an endomorphism g and an application $h : G \to G$ such that $h(1) = 1$. Let us introduce the binary operation $x \triangleright_{g,h} y = g(x) \cdot h(y)$ on G. We are going to investigate when this operation is underlying a quandle operation. We have $1 \triangleright y = h(y)$ and $x \triangleright 1 = g(x)$. So, first, g must be an automorphism.

Lemma 5.14. *When* g *is a group automorphism, then for any* y *the application* $- \triangleright_{g,h} y : G \to G$ *is bijective.*

Proof. Given any $v \in G$, we have to show that there is a unique $u \in G$ such that $v = u \triangleright y$, namely $v = g(u) \cdot h(y)$, or equivalently $g(u) = v \cdot h(y)^{-1}$. When g is an automorphism, this gives us: $v \triangleright^{-1} y = g^{-1}(v \cdot h(y)^{-1}) = g^{-1}(v) \cdot g^{-1}(h(y)^{-1})$. □

This law is idempotent if and only if, for all x, we have $x = g(x) \cdot h(x)$, namely $h(x) = g(x)^{-1} \cdot x$. So the application h is determined by g and satisfies $h(1) = 1$. Since g is a group homomorphism, we get $g \circ h(x) = g^2(x^{-1}) \cdot g(x) = h \circ g(x)$.

Proposition 5.15. *If g is a group automorphism, the binary operation $x \triangleright y = g(x) \cdot g(y)^{-1} \cdot y = g(x \cdot y^{-1}) \cdot y$ is a quandle operation. We have moreover $\triangleright_g^{-1} = \triangleright_{g^{-1}}$.*

Proof. It remains to show the left distributivity. Setting $h(y) = g(y)^{-1} \cdot y$, we get: $(x \triangleright y) \triangleright z = (g(x) \cdot h(y)) \triangleright z = g^2(x) \cdot g \circ h(y) \cdot h(z)$, while:

$$(x \triangleright z) \triangleright (y \triangleright z) = (g(x) \cdot h(z)) \triangleright (g(y) \cdot h(z)) = g^2(x) \cdot g \circ h(z) \cdot h(g(y) \cdot h(z))$$

So that the two terms are equal if and only if, for all (y, z) we have:

$$g \circ h(y) \cdot h(z) = g \circ h(z) \cdot h(g(y) \cdot h(z))$$

Now, we have:

$$h(g(y) \cdot h(z)) = g(g(y) \cdot h(z))^{-1} \cdot g(y) \cdot h(z) = (g \circ h(z))^{-1} \circ (g^2(y))^{-1} \cdot g(y) \cdot h(z)$$

So it remains to check: $g \circ h(y) = g \circ h(z) \cdot (g \circ h(z))^{-1} \circ (g^2(y))^{-1} \cdot g(y)$, namely $g \circ h(y) = (g^2(y))^{-1} \cdot g(y)$, which is true. The last assertion is straightforward. $\square$

In this way, the notion of Alexander quandle is extended to the non-abelian groups. Define $AutGp$ as the category whose objects are pairs $((G, \cdot), g)$ of a group and group automorphism g and whose morphisms are group homomorphisms commuting with these automorphism. The forgetful functor $AutGp \to Gp$ forgetting the automorphism being conservative and Gp being protomodular, so is $AutGp$ which is consequently a Maltsev category. Its subcategory of affine objects is $AutAb$. The previous proposition produces a left exact conservative functor which preserves and reflects the regular epimorphisms:

$$Al : AutGp \to Qnd$$

whose restriction to $AutAb$ is the previously defined functor Al. On $AutGp$, we have an involutive functor defined by $((G, \cdot), g)^* = (G, \cdot), g^{-1})$, which makes the following diagram commute:

$$\begin{array}{ccc} AutGp & \xrightarrow{Al} & Qnd \\ {\scriptstyle ()^*}\downarrow & & \downarrow{\scriptstyle ()^*} \\ AutGp & \xrightarrow[Al]{} & Qnd \end{array}$$

Proposition 5.16. *Let $((G, \cdot), g)$ be in $AutGp$. The quandle $(G, \triangleright_g)$ is involutive if and only if the automorphism g is involutive.*

Proof. The quandle $(G, \triangleright)$ is involutive if and only if we have $g(x) \cdot g(y^{-1}) \cdot y = g^{-1}(x) \cdot g^{-1}(y^{-1}) \cdot y$ which is equivalent to $g(x \cdot y^{-1}) = g^{-1}(x \cdot y^{-1})$, i.e. g is involutive. $\square$

So, if we denote by $IAutGp$ the full subcategory of $AutGp$ whose objects $((G, \cdot), g)$ are such that g is involutive, we get a restriction functor $Al^I : IAutGp \to IQnd$.

Proposition 5.17. *Let $((G, \cdot), g)$ be in $AutGp$. The quandle $(G, \triangleright_g)$ is symmetric if and only if G is abelian and $2g = Id_G$ or, in other words, $2Id_G$ is an automorphism and g is its inverse.*

Proof. The quandle $(G, \triangleright_g)$ is symmetric if and only if we have $g(x) \cdot g(y^{-1}) \cdot y = g(y) \cdot g(x^{-1}) \cdot x$. With $y = 1$ this implies that $g(x) = g(x^{-1}) \cdot x$ which, since g is bijective, is equivalent to $x = x^{-1} \cdot g^{-1}(x)$ or $g^{-1}(x) = x^2$. Saying that $x \mapsto x^2$ is a group homomorphism is saying that G is abelian. In additive notation, the last identity is $g^{-1} = 2Id_G$ which is equivalent to $2g = Id_G$. The converse is straightforward since $2g = Id_G$ is equivalent to $h = Id_G - g = g$ which obviously implies the symmetry of $\triangleright_g$. □

Proposition 5.18. *Let $((G, \cdot), g)$ be in AutGp. The quandle $(G, \triangleright_g)$ is latin if and only if the application defined by $h(x) = g(x^{-1}) \cdot x$ is bijective.*

Proof. For the same reason as in Lemma 5.14, any application $x \triangleright_g -$ is bijective as soon as $1 \triangleright_g - = h(-)$ is bijective. □

So, if we denote by $LautGp$ the subcategory of those objects $((G, \cdot), g)$ of $AutGp$ which are such that the application defined by $h(x) = g(x^{-1}) \cdot x$ is bijective, we get a restriction functor $Al^L : LautGp \to LQnd$.

5.19. **Characterization of the internal groups in Qnd.**

Proposition 5.20. *Let $((G, \cdot), g)$ be in AutGp. The application $h(x) = g(x^{-1}) \cdot x$ is a group homomorphism if and only if $g(x^{-1}) \cdot x$ is in the center $Z(G)$ of G. In this case the group operation $\cdot$ gives to the quandle $(G, \triangleright_g)$ an internal group structure in Qnd.*

Proof. The application h is a group homomorphism if and only if, for all (x, y), we have $g(x^{-1}) \cdot x \cdot g(y^{-1}) \cdot y = g((x \cdot y)^{-1}) \cdot x \cdot y$ which is equivalent to $g(x^{-1}) \cdot x \cdot g(y^{-1}) = g(y^{-1}) \cdot g(x^{-1}) \cdot x$, namely to: $h(x) \cdot g(y^{-1}) = g(y^{-1}) \cdot h(x)$ which means that g and h commute. Since g is an automorphism, it is equivalent to: for all (x, z) we have $g(x^{-1}) \cdot x \cdot z = z \cdot g(x^{-1}) \cdot x$, namely to: $g(x^{-1}) \cdot x$ is in the center $Z(G)$ of G. Now since g and h commute, the quandle operation $\triangleright_g$ is a group homomorphism which is equivalent to the fact that the group operation $\cdot$ is a quandle homomorphism. □

Corollary 5.21. *In the category Qnd there are non-trivial quandles on which there is a non-abelian internal group structure and consequently a non-commutative internal Maltsev operation.*

Proof. Take any non-abelian group G, consider the group $\mathbb{Z}/3\mathbb{Z} \times G$ and the automorphism g defined by $g(\epsilon, x) = (-\epsilon, x)$. Then $h(\epsilon, x) = g((\epsilon, x)^{-1}) \cdot (\epsilon, x) = g(-\epsilon, x^{-1}) \cdot (\epsilon, x) = (\epsilon, x^{-1}) \cdot (\epsilon, x) = (-\epsilon, 1)$ is in the center of this group. Accordingly there is on the set $\mathbb{Z}/3\mathbb{Z} \times G$ an involutive quandle binary operation defined by $(\epsilon, x) \triangleright (\epsilon', y) = (-(\epsilon + \epsilon'), x)$ which determines a non-abelian group structure in Qnd. □

Let us denote by $ZautGp$ the category whose objects are the pairs $((G, \cdot), g)$ of a group and an automorphism g such that for all x the element $g(x^{-1}) \cdot x$ is in the center of G, and whose morphisms are the group homomorphisms commuting with these automorphisms. The forgetful functor $U : ZautGp \to Gp$ being conservative and the category Gp being protomodular, the category $ZautGp$ is protomodular as well and consequently a Maltsev category. Its full subcategory of affine objects is nothing but $AutAb$. Proposition 5.20 produces a functor $Al^Z : ZautGp \to Gp(Qnd)$ whose restriction $AutAb \to Ab(Qnd)$ is nothing but the classical Alexander functor Al. Notice that if $((G, \cdot), g)$ is in $ZautGp$ so is $((G, \cdot), g^{-1})$. As previously, this defines an involutive functor $(\,)^* : ZautGp \to ZautGp$. According to the identity $\triangleright_g^{-1} = \triangleright_{g^{-1}}$, the following diagram commute:

$$\begin{array}{ccc} ZautGp & \xrightarrow{Al} & Gp(Qnd) \\ {\scriptstyle (\,)^*}\downarrow & & \downarrow{\scriptstyle Gp(\,)^*} \\ ZautGp & \xrightarrow[Al]{} & Gp(Qnd) \end{array}$$

Remark 5.22. Let us denote by $I : Gp \to AutGp$ the functor defined by $I(G,\cdot) = ((G,\cdot), Id_G)$; since we have $x^{-1}\cdot x = 1$, it factorizes through $ZautGp$ and moreover makes the following diagram commute:

$$\begin{array}{ccc} Gp & \xrightarrow{I} & ZautGp \\ \| & & \downarrow{\scriptstyle Al} \\ Gp & \xrightarrow[Gp(T)]{} & Gp(Qnd) \end{array}$$

Proposition 5.23. *Suppose $((G,\cdot),g)$ be in $ZautGp$. Then the quandle $(G,\triangleright_g)$ is autonomous and pseudo-leftassociative.*

Proof. Let us start from $((G,\cdot),g)$, then we have:

$$(x \triangleright_g y) \triangleright_g (x' \triangleright_g y') = (g(x)\cdot g(y)^{-1}\cdot y) \triangleright_g (g(x')\cdot g(y')^{-1}\cdot y')$$

$$= g^2(x)\cdot g^2(y^{-1})\cdot g(y)\cdot g(y')^{-1}\cdot g^2(y')\cdot g^2(x'^{-1})\cdot g(x')\cdot g^{(}y')\cdot y'$$

and in the same way:

$$(x \triangleright_g x') \triangleright_g (y \triangleright_g y') = (g(x)\cdot g(x'^{-1})\cdot x') \triangleright_g (g(y)\cdot g(y')^{-1}\cdot y')$$

$$= g^2(x)\cdot g^2(x'^{-1})\cdot g(x')\cdot g(y'^{-1})\cdot g^2(y')\cdot g^2(y^{-1})\cdot g(y)\cdot g(y')^{-1}\cdot y'$$

The two terms are equal if and only if the two following ones are equal:

$$(g^2(y^{-1})\cdot g(y))\cdot(g(y')^{-1}\cdot g^2(y'))\cdot(g^2(x'^{-1})\cdot g(x'))$$

$$(g^2(x'^{-1})\cdot g(x'))\cdot(g(y'^{-1})\cdot g^2(y'))\cdot(g^2(y^{-1})\cdot g(y))$$

which is true, since any of the three terms between brackets is in the centre $Z(G)$. We have also:

$$((x \triangleright_g y) \triangleright_g^{-1} t) \triangleright_g z = ((g(x)\cdot g(y)^{-1}\cdot y) \triangleright_{g^{-1}} t) \triangleright_g z = (x\cdot y^{-1}\cdot g^{-1}(y)\cdot g^{-1}(t^{-1})\cdot t) \triangleright_g z$$

$$= g(x)\cdot(g(y)^{-1}\cdot y)\cdot(t^{-1}\cdot g(t))\cdot(g(z)^{-1}\cdot z) = ((x \triangleright_g z) \triangleright_g^{-1} t) \triangleright_g y$$

since we can permute the tree last terms between brackets which, once again, belong to the centre $Z(G)$. □

We can now produce a characterization of the internal groups in Qnd:

Theorem 5.24. *The functor $Al^Z : ZautGp \to Gp(Qnd)$ is an isomorphism of categories which makes commutative the following diagram:*

$$\begin{array}{ccc} ZautGp & \xrightarrow{Al^Z} & Gp(Qnd) \\ {\scriptstyle U}\downarrow\uparrow{\scriptstyle I} & & {\scriptstyle Gp(U)}\downarrow\uparrow{\scriptstyle Gp(T)} \\ Gp & = & Gp \end{array}$$

Proof. The proof is the same as in Proposition 5.4. The inverse functor is defined by $\Psi((G, \triangleright), \cdot) = ((G, \cdot), - \triangleright 1)$. Since we observed that $- \triangleright_g 1 = g$, it is clear that $\Psi \circ Al^Z = Id$. Now thanks to Proposition 5.2, starting from an internal group stucture $((X, \triangleright), \cdot)$, we get:

$$(x \triangleright 1)^{-1} \cdot x = (x^{-1} \triangleright 1)) \cdot (x \triangleright x) = (x^{-1} \cdot x) \triangleright (1 \cdot x) = 1 \triangleright x$$

So that the quandle structure arising from $((X, \cdot), - \triangleright 1)$ is given by:

$$x \bar{\triangleright} y = (x \triangleright 1) \cdot (y \triangleright 1)^{-1} \cdot y = (x \triangleright 1) \cdot (1 \triangleright y) = (x \cdot 1) \triangleright (1 \cdot y) = x \triangleright y$$

Accordingly we have $Al^Z \circ \Psi = Id$. □

In the same way as in the abelian case, we get the following:

Corollary 5.25. *The functor $Gp(i_{\bar{A}}) : Gp(\bar{A}Qd) \to Gp(Qnd)$ is an isomorphism of categories. Any internal group $((G, \triangleright), \cdot)$ in Qnd makes the quandle $(G, \triangleright)$ autonomous and pseudo-leftassociative.*

Let $IZautGp$ denote the full subcategory of $ZautGp$ whose objects $((G, \cdot), g)$ are such that g is involutive.

Corollary 5.26. *The restriction functor $Al^{ZI} : IZautGp \to Gp(IQnd)$ is an isomorphism of categories.*

According to Proposition 4.13, we know that $Gd(LQd)$ is isomorphic to $Ab(LQd)$. So, we can expect the following:

Proposition 5.27. *We have: $ZautGp \cap LautGp = AutAb$. In other words, if $((G, \cdot), g)$ is in $ZautGp$ and if the homomorphism h define by $h(x) = g(x^{-1}) \cdot x$ is an automorphism, then the group G is necessarily abelian. When the group G is finite, the second condition is reduced to: g has no non-trivial fixed point.*

Proof. If h is an automorphism, the quandle $(G, \triangleright_g)$ is latin by Proposition 5.18. Now, since $((G, \cdot), g)$ is in $ZautGp$, the internal group $Al((G, \cdot), g)$ is abelian by Proposition 4.13, which implies by Theorem 5.4 that the group G is itself abelian. By definition of h, $Kerh$ is the subgroup of the fixed points of g. When G is finite, the last assertion implies that h is an automorphism. □

5.28. **Non-latin affine quandles.** As recalled at the beginning of Section 4.12, one distinctive feature of the Maltsev context is the unicity of an affine structure. So, a priori, there can be two distinct affine structures on a non-latin quandle $(X, \triangleright)$ which means, as soon as it is non empty, two distinct abelian group structures with same unit. It would be very interesting to produce explicit examples of such a situation.

In this short section we shall succeed only to produce two distinct group structures on a same non-latin quandle. For that consider an abelian group K endowed with an action of an abelian group A. On the set $A \times K$ there are two group structures, namely the the abelian one $+$ on the direct product and the non-abelian one $\oplus$ on the semi-direct product. Introduce now an equivariant automorphism g on K, and consider the application $\gamma : A \times K \to A \times K$ defined by $\gamma(a, k) = (a, g(k))$. It is clear that γ is an automorphism for the two laws.

Proposition 5.29. *Under the conditions of the previous paragraph, suppose moreover that the equivariant automorphism g is such that the action of A becomes trivial on the subgroup $Im(Id_K - g)$. Then the Alexander quandles associated with*

$((A\times K,+),\gamma)$ and $((A\times K,\oplus),\gamma)$ are the same. And the two internal group structures $Al((A\times K,+),\gamma)$ and $Al((A\times K,\oplus),\gamma)$ in Qnd are distinct.

Proof. Let us denote by $\triangleright^+$ and $\triangleright^\oplus$ the two respective quandle laws. On the one hand we have:

$$(a,k)\triangleright^+(a',k')=(a,g(k))-(a',g(k'))+(a',k')=(a,g(k)-g(k')+k')$$

On the other hand, we get:

$$(a,k)\triangleright^\oplus(a',k')=(a,g(k))\oplus(-a',{}^{-a'}g(-k'))\oplus(a',k')$$
$$=(a,g(k))\oplus(0,{}^{-a'}g(-k')+{}^{-a'}k')=(a,g(k)+{}^{-a'}(-g(k')+k'))$$

And this last term is $(a,g(k)-g(k')+k')$ under our assumption on Id_G-g. □

5.30. **A remark about the fibers of the functor $Gp(U):Gp(Qnd)\to Gp$.** Let $U:AutGp\to Gp$ be the functor forgetting the automorphism and $(G,\cdot)$ be any group. It is clear that if g and g' are two automorphisms, so is $g'\circ g$. This defines a "tensor product" on the objects of the fibre $U_G=U^{-1}(G,\cdot)$ defined by $((G,\cdot),g')\otimes((G,\cdot),g)=((G,\cdot),g'\circ g)$. However this construction is not functorial. This operation is strictly associative and $I(G)=((G,\cdot),Id_G)$ is a strict unit. Moreover the functor $(\,)^*:AutGp\to AutGp$ gives to this "tensor product" a strict dual since we get $((G,\cdot),g)\otimes((G,\cdot),g^{-1})=((G,\cdot),Id_G)$ and $((G,\cdot),g^{-1})\otimes((G,\cdot),g)=((G,\cdot),Id_G)$. Accordingly this operation $\otimes$ gives a group structure to the set of objects of the fibre $U^{-1}(G,\cdot)$.

Proposition 5.31. *This "tensor product" is stable on $ZautGp$.*

Proof. Suppose $((G,\cdot),g)$ and $((G,\cdot),g')$ are in $ZautGp$. Then we have:

$$g'\circ g(x)^{-1}\cdot x=(g'\circ g(x^{-1})\cdot g(x))\cdot(g(x)^{-1}\cdot x)$$

The two terms on the right hand side are in the centre $Z(G)$ of G, and so is their product. We already noticed that $I(G)=((G,\cdot),Id_G)$ is in $ZautGp$ and that the functor $(\,)^*$ is stable on $ZautGp$. □

The isomorphism of Theorem 5.24 extends this property to the fibers of the functor $Gp(U):Gp(Qnd)\to Gp$. It gives to any fibre $Gp(U)^{-1}(G,\cdot)$ a group structure defined by $(G,\triangleright_{g'})\otimes(G,\triangleright_g)=(G,\bar{\triangleright})$ where $x\bar{\triangleright}y=(x\triangleright_{g'}y)\triangleright_g y$. Starting from any pair $(X,\triangleright'),(X,\triangleright)$ of quandles this formula produces an idempotent operation $\bar{\triangleright}$ such that $\bar{\triangleright}^{-1}$ is defined by $x\bar{\triangleright}^{-1}y=(x\triangleright^{-1}y)\triangleright'^{-1}y$, however it is not a quandle operation in general.

6. The fibres L_Y

In this section, we shall investigate the properties of the latin homomorphisms. It is clear that the latin homomorphisms are stable under pullback in Qnd since the class Σ^+ is so. In the same way, if Qnd^2 denote the category whose objects are the homomorphisms of Qnd and whose morphisms are the commutative squares in Qnd, the latin homomorphism are stable under product and under finite limit in Qnd^2 since Σ^+ is point-congruous. Let us begin with giving examples of such homomorphisms.

Lemma 6.1. *Let ϕ be a morphism $((G,\cdot),g)\to((G',\cdot),g')$ in the category $AutGp$. Then $Al(\phi)$ is a latin homomorphism in Qnd if and only the resriction to the kernel $Ker\phi$ of the application $h(x)=g(x)^{-1}\cdot x$ is bijective.*

Proof. If $Al(\phi)$ is a latin homomorphism, its fibre $\phi^{-1}(0) = Ker\phi$ is a latin quandle, whence the property on $Ker\phi$. Conversely suppose $Ker\phi$ satisfies this property. Let (x, x') be a pair of elements of G such that $\phi(x) = \phi(x')$. We are looking for an element a such that $\phi(a) = \phi(x)$ and $x' = x \triangleright_g a$, namely $x' = g(x) \cdot g(a)^{-1} \cdot a$. Or equivalently such that $x' \cdot x^{-1} = h(x \cdot a^{-1})$. Since $x' \cdot x^{-1}$ is in $Ker\phi$, there is a unique $\alpha \in Ker\phi$ such that $x' \cdot x^{-1} = h(\alpha)$. Accordingly $a = \alpha^{-1} \cdot x$ is the desired unique solution. □

Given any category $\mathbb{E}$ and any object $Y \in \mathbb{E}$, the slice category $\mathbb{E}/Y$ is the category whose objects are the maps with codomain Y and the morphisms are the commutative triangles above Y. When $(Y, \triangleright)$ is a quandle, let us denote by L_Y the full subcategory of the slice category Qnd/Y whose objects are the latin homomorphisms with codomain Y.

6.2. **Maltsev property.**

Theorem 6.3. *Let h be any morphism in L_Y:*

$$\begin{array}{ccc} X & \xrightarrow{\;h\;} & X' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & Y & \end{array}$$

Then h is a latin homomorphism in Qnd. Consequently the category L_Y is a Maltsev category.

Proof. The kernel congruence $R[h]$ is given by the following pullback in the category $CongQ$ of congruences in Qnd:

$$\begin{array}{ccc} R[h] & \xrightarrow{\;h\;} & \Delta_{X'} \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \iota} \\ R[f] & \xrightarrow[\;R(h)\;]{} & R[f'] \end{array}$$

where $\Delta_{X'}$ is the discrete equivalence relation on X' which is obviously acupuncturing. Since we are in L_Y, the equivalence relations $R[f]$ and $R[f']$ are acupuncturing. Since the pullbacks in $CongQ$ are levelwise, and the class Σ^+ is point-congruous, the congruence $R[h]$ is acupuncturing as well. So, h is a latin homomorphism. According to Proposition 4.3 any split epimorphism in L_Y is in Σ^+. So that the category L_Y satisfies the property on commutative squares of split epimorphisms recalled in the Introduction which asserts that it is a Maltsev category. □

Proposition 6.4. *The category L_Y is stable in Qnd under the decomposition in surjection and injection. Accordingly the category L_Y is regular.*

Proof. Let h be any morphism in L_Y and consider its decomposition in surjection and injection in Qnd:

$$\begin{array}{ccccc} X & \xrightarrow[\;\bar{h}\;]{} & U & \xrightarrow[\;m\;]{} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'.m} & & \downarrow{\scriptstyle f'} \\ Y & = & Y & = & Y \end{array} \qquad (h = m \circ \bar{h})$$

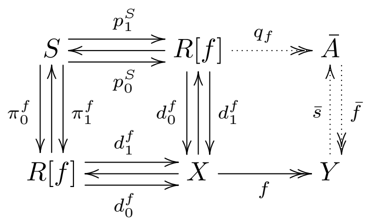
Then, since $\bar{h}$ is surjective and $R[f]$ is (acu)puncturing, the equivalence relation $R[f'.m]$ is puncturing. So given any pair (x, x') of elements of U such that $f(x) = f(x')$, we know that there is an element t in U such that $f(t) = f(x)$ and $x' = x \triangleright t$. This t is unique since it is unique in X', the homomorphism f' being a latin homomorphism. Now the slice category Qnd/Y is Barr-exact since so is Qnd, and consequently the subcategory L_Y is regular. □

6.5. **Baer sum of abelian latin extensions.** We shall call *affine* (resp. *abelian latin extension*) any latin (resp. surjective latin) homomorphism $f : (X, \triangleright) \to (Y, \triangleright)$ equipped with a (unique possible) homomorphic ternary operation: $p : R^2[f] \to X$ satisfying the Maltsev identities: $p(xRyRy) = x$ and $p(xRxRy) = y$, writing R instead of $R[f]$ for sake of simplicity.

Lemma 6.6. *A homomorphism $f : (X, \triangleright) \to (Y, \triangleright)$ is affine if and only the application: $xRx'Rx'' \mapsto (x \triangleright^{-1} x') \triangleright (x' \tilde{\triangleright}_y x'')$ is a homomorphism of quandles, where $\tilde{\triangleright}_y$ is the operation associated with the latin fibre above $y = f(x) = f(x') = f(x'')$.*

Proof. For the same reason as in Proposition 4.14 the unique possible ternary operation satisfying the Maltsev identities is: $p(xRx'Rx'') = (x \triangleright^{-1} x') \triangleright (x' \tilde{\triangleright}_y x'')$. □

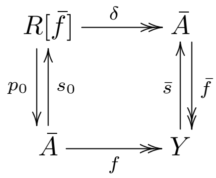
Since L_Y is a Maltsev category we have necessarily the coherence identities $xRp(xRySz)$ and $p(xRySz)Rz$, the right and left associativity $p(xRyRp(yRzSt)) = p(xRzRt)$ and $p(p(xRyRz)RzRt) = p(xRyRt)$ and the commutativity condition $p(x, y, z) = p(z, y, x)$, see [6]. This allows to define a congruence S on the quandle $R[f]$ by $(x, x')S(t, t')$ if and only if $x' = p(x, t, t')$, see [4]. Let us consider now the left hand side part of following diagram where the congruence S is the upper horizontal one and any commutative square is a pullback:



with $\pi_0^f((x, x')S(t, t')) = (x, t)$ while $\pi_1^f((x, x')S(t, t')) = (x', t')$. Since Qnd is a Barr-exact category and since the left hand side square indexed by 0 is a discrete fibration between congruences, the upper congruence has a quotient q_f which makes the right hand side square a pullback, where the split epimorphism $(\bar{f}, \bar{s})$ is the induced one.

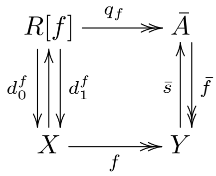
Proposition 6.7. *The split epimorphism $(\bar{f}, \bar{s})$ is latin and is an abelian extension. It is called the* direction *of the abelian latin extension f.*

Proof. Since the right hand side square is a pullback, the split epimorphism $(\bar{f}, \bar{s})$ is in Σ^+ by Proposition 2.6. Moreover the vertical right hand side part is necessarily a group as a quotient of the groupoid (actually an equivalence relation) $R[f]$. Let us denote by $m : R[\bar{f}] \to \bar{A}$ the group operation in the slice category Qnd/Y and $\delta : R[\bar{f}] \to \bar{A}$ the induced division. The following square is a pullback of split epimorphisms as in any group structure:

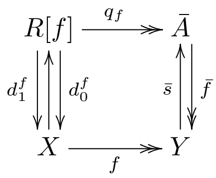


Accordingly (p_0, s_0) is in Σ^+ and the homomorphism $\bar{f}$ is latin. This group is necessarily abelian since L_Y is a Maltsev category; consequently $\bar{f}$ has an abelian kernel relation. □

Suppose now given any abelian group $(\bar{f}, \bar{s}) : \bar{A} \rightleftarrows Y$ in L_Y. For sake of simplicity we shall denote this whole abelian structure by the only symbol $\bar{A}$. We shall call $\bar{A}$-torsor a pair $(f, R[f], q_f)$ of a surjective latin homomorphism together with a discrete fibration from its kernel congruence $R[f]$:



According to the previous proposition any abelian latin extension gives rise to a $\bar{A}$-torsor. A morphism of $\bar{A}$-torsors $(f, R[f], q_f) \to (f', R[f'], q_{f'})$ is given by a map $h : X \to X'$ such that $f'.h = f$ and $q_{f'}.R(h) = q_f$. Any $\bar{A}$-torsor $(f, R[f], q_f)$ produces another one, namely $(f, R^{op}[f], q_f)$ where the projections are twisted:

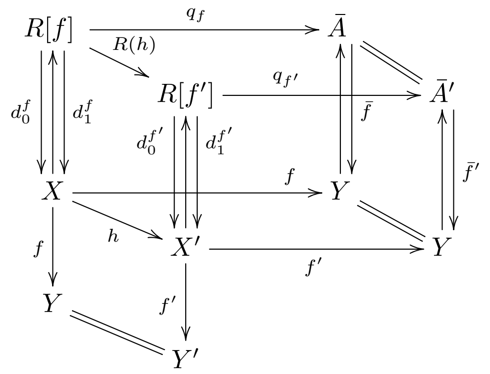


The groupoid structure given on $R[f]$ implies that $q_f(a, b) + q_f(b, a) = q_f(a, a) = 0$, in other words, that $q_f(b, a) = -q_f(a, b)$, or equivalently that $q_f.tw = -1_{\bar{A}}.q_f$, where $tw : R[f] \to R^{op}[f]$ is the twisting isomorphism. Accordingly, the homomorphism induced on $\bar{A}$ by tw is $-1_{\bar{A}}$.

Let us denote by $EXT(\bar{A}, Y)$ the category whose objects are the $\bar{A}$-torsors and whose morphisms h are the morphisms which induce $1_{\bar{A}}$ on $\bar{A}$. As usual we get

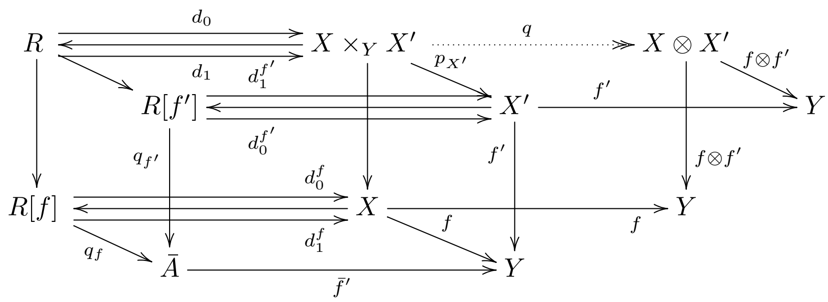
Corollary 6.8. *Any morphism of $\bar{A}$-torsors is an isomorphism.*

Proof. Consider the following diagram where h is a morphism of $\bar{A}$-extensions:



Then the left hand side quadrangle is a pullback. By the Barr-Kock theorem, the lower square is a pulback as well, and h is an isomorphism. □

Accordingly the category $EXT(\bar{A}, Y)$ is a groupoid. Let us denote by $Ext(\bar{A}, Y)$ of set of connected components of this groupoid. Classically there is a symmetric tensor product on the $\bar{A}$-torsors. This tensor product allows us to define the *Baer sum* on the set $Ext(\bar{A}, Y)$ which gives it an abelian group structure. Starting with two $\bar{A}$-torsors f and f' in $Ext(\bar{A}, Y)$, the Baer sum is given by the following construction; take the following quadrangled pullbacks, paying attention to the fact that the upper relation is $R^{op}[f']$ while the lower one is $R[f]$:



These pullbacks define a congruence R on $X \times_Y X'$, define by $(x, x')R(t, t')$ if $q_f(x, t) + q_{f'}(x', t') = 0$. Since the back right hand side part of the diagram is underlying a discrete fibrations between equivalence relations and the category Qnd is exact, the quotient q of the congruence R produces a unique factorization $f \otimes f'$ such that $f \otimes f'.q = p_X.f = p_{X'}.f'$ which closes the two squares as pullbacks. This is the Baer sum of f and f', see [4]. The inverse of $(f, R[f], q_f)$ for the abelian group structure on $Ext(\bar{A}, Y)$ is precisely $(f, R^{op}[f], q_f)$.

When $Y = 1$ is the terminal object, the group $Ext(\bar{A}, 1)$ remains invisible, i.e. we have $Ext(\bar{A}, 1) = \{0\}$, since, for any non-empty quandle $(X, \triangleright)$, its terminal map τ_X is always split.

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