About the Density of Optimal Packings of Ellipses in a Square

P. Honvault
Density of Optimal Packings of Three Ellipses in a Square

Pascal Honvault
LMPA J.Liouville, BP 699, F-Calais
Univ. Lille Nord de France, F-59000 Lille, France
email address: honvault@lmpa.univ-littoral.fr

Abstract. We prove that the densest packing of three non-overlapping congruent ellipses of aspect ratio $E \in [0, 1]$ in a square is reached for $E = 1/3$, with density equal to $\pi/4$. This result was already known for two ellipses (for $E = 1/2$), but is no longer true for an arbitrary number of non-overlapping congruent ellipses.

Key Words: packing, ellipse, density
AMS classification: 51F, 51N

1. Statement of the results

Let $n \in \mathbb{N}^*$ and $\mathcal{E}_1, \ldots, \mathcal{E}_n$ be $n$ non-overlapping congruent ellipses of the same aspect ratio $E \in [0, 1]$. We can deal without loss of generality with unit ellipses, i.e. we suppose that the semi-major axis is equal to 1 (thus the common semi-minor axis is $E$). For each such configuration, there is a square of minimal side length $S_n(E)$ containing these ellipses, and we denote by $s_n(E) = \inf S_n(E)$ the smallest value of $S_n(E)$ under all such possible configurations. In this paper, we are interested in the density of these optimal packings. We will denote by $d_n(E) = \frac{n\pi E}{s_n^2(E)}$ and we focus on its maximum $d_n = \max \{d_n(E)/E \in [0, 1]\}$. It is well known that $d_1(E) = \pi E/2(1 + E^2)$, thus $d_1 = \pi/4$ is reached for a circle, and we proved in [2] that $d_2 = \pi/4$ again, the optimal being obtained for two “vertical” ellipses of aspect ratio $E = 1/2$.

It turns out that this density result remains true for three ellipses:

Theorem 1 The optimal density of three non-overlapping congruent ellipses of the same aspect ratio is $d_3 = \pi/4$. Moreover, this maximum is achieved for three “vertical” ellipses of aspect ratio $E = 1/3$.

Unfortunately, this cannot be generalized for all values of $n$. An easy way to see this is to consider hexagonal packings of unit circles in a rectangle (see for example [1], [4], or [5] for 1433-8157/$ 2.50 \copyright$ YEAR Heldermann Verlag, Berlin
packing problems). In point of fact, let \( n, m \) be two integers. If we pack \( m \) lines of \( n \) tangent unit circles like in the Fig.2, we obtain a rectangle of size \((2n + 1)\) per \((2 + (m - 1)\sqrt{3})\).

We apply a vertical stretching of shape \( E = (2n + 1)/(2 + (m - 1)\sqrt{3}) \) in order to have a square. The condition \( E < 1 \) will be realized if \( m > (2n + \sqrt{3} - 1)/\sqrt{3} \). The previous circles are transformed into ellipses of semi-axis 1 and \( E \), and the density of the new packing becomes:

\[
\frac{m.n.\pi E}{(2n + 1)^2} = \frac{n\pi(2n + 1 - (2 - \sqrt{3})E)}{\sqrt{3}(2n + 1)^2}.
\]

which has the limit \( \pi/2\sqrt{3} \) as \( n \) tends to infinity. More precisely, it is easy to see that, for each \( E \) fixed in \( [0, 1] \), this density is greater than \( \pi/4 \) for \( n \geq 5 \) and \( m \geq (2n + 1 - E(2 - \sqrt{3}))/E\sqrt{3} \). Consequently, the density of the packing will also be greater than \( \pi/4 \).
2. Proof of the theorem

Let us recall the data: Three non-overlapping congruent unit ellipses $E_1, E_2, E_3$ of the same aspect ratio $E$ contained in a square of length side $S_3(E)$. We must show that, for each configuration and each $E \in ]0, 1]$, 
\[
\frac{3\pi E}{S_3^2(E)} \leq \frac{\pi}{4},
\]
or equivalently:
\[
S_3(E) \geq \sqrt{12E}.
\]

Each of these ellipses is tangent to another one, otherwise we could reduce the length side of the square, and the tangent lines cut this square in four polygons in general (see Fig.3). Three of these polygons contain $E_1, E_2, E_3$ and we divide the proof in two parts, as these polygons are quadrilateral (or even triangles) or pentagons.

![Figure 3: Pentagonal and quadrilateral cases](image)

2.1. The quadrilateral case

It is a simple case, for if a quadrilateral ( or a triangle) contains a unit ellipse $E$ of aspect ratio $E$, then its area is bigger than $4E$. One can see it for instance by stretching $E$ into a unit circle and observe that a unit square containing it has the smallest area among quadrangles. Of course, a one dimensional stretching preserves the ratio of areas. Thus we conclude that the area $S_3(E)$ of the square containing $E_1, E_2, E_3$ is greater than $3 \times 4E = 12E$, and (1) is verified.

\[\square\]

2.2. The pentagonal case

Before starting the proof, let us notice that it is possible to have a configuration of two pentagons and one triangle, but this case is also easy. In fact, the minimal area of a pentagon containing a unit circle is $5 \tan(\pi/5)$ whereas the minimal area of a triangle containing a unit circle is $3\sqrt{3}$. Thus, if the common tangent lines of the three ellipses cut a square into two pentagons and one triangle, its area is greater than $(10 \tan(\pi/5) + 3\sqrt{3})E$, and this is greater than $12E$ again.
The difficult case is the one of a single pentagon, because a pentagon containing $E$ may have an area less than $4E$. So, we have to look at things closer. In order to see in details what takes place in each polygon, we shall use the following lemma (see Fig.4):

**Lemma 1** Let $E_\alpha$ be the ellipse of semi-axes $1$ and $E \in [0, 1]$, tangent to the Cartesian semi-axes $[O, x)$ and $[O, y)$, tilted at the angle $\alpha \in [-\pi/2, \pi/2]$ from the horizontal direction. Then $E_\alpha$ is parametrized by:

\[
t \in [0, 2\pi] \mapsto \begin{cases} 
  x(t) &= \cos(\alpha) \cos(t) - E \sin(\alpha) \sin(t) + \lambda_\alpha \\
  y(t) &= \sin(\alpha) \cos(t) + E \cos(\alpha) \sin(t) + \mu_\alpha 
\end{cases} \\
\text{where } \lambda_\alpha = \sqrt{\cos^2(\alpha) + E^2 \sin^2(\alpha)}, \quad \mu_\alpha = \sqrt{\sin^2(\alpha) + E^2 \cos^2(\alpha)} \text{ are the coordinates of the centre } \Omega_\alpha \text{ of } E_\alpha.
\]

**Proof:** It is clear that $E_\alpha$ admits a parametrization of the form:

\[
t \mapsto \begin{pmatrix} \cos(\alpha) - \sin(\alpha) \\
  \sin(\alpha) \cos(\alpha) 
\end{pmatrix} \begin{pmatrix} \cos(t) \\
  E \sin(t) \end{pmatrix} + \begin{pmatrix} \lambda_\alpha \\
  \mu_\alpha \end{pmatrix}
\]

where $\Omega_\alpha(\lambda_\alpha, \mu_\alpha)$ is the centre of the ellipse. Let us compute the optimum of the function $x(t)$ when $\alpha \neq \pm \pi/2$:

\[
x'(t) = -\cos(\alpha) \sin(t) - E \sin(\alpha) \cos(t) = 0 \iff \tan(t) = -E \tan(\alpha)
\]

This leads in particular to the minimal value $x_m$ of $x(t)$:

\[
x_m = \cos(\alpha) \cos(t)(1 - E \tan(\alpha) \tan(t)) + \lambda_\alpha
\]

and $x_m = 0$ give the result for $\lambda_\alpha$. The same holds for $\alpha = \pm \pi/2$ and a similar proof can be done for $\mu_\alpha$. \hfill \square

**Lemma 2** The ellipse $E_\alpha$ is tangent to the $x$-axis at a point $(x_\alpha, 0)$ with

\[
x_\alpha = \lambda_\alpha - \sin(2\alpha)(1 - E^2)/2\mu_\alpha
\]

and to the $y$-axis at a point $(0, y_\alpha)$ with

\[
y_\alpha = \mu_\alpha - \sin(2\alpha)(1 - E^2)/2\lambda_\alpha.
\]

Moreover, the tangent line to $E_\alpha$ from a point $(L_\alpha, 0)$, $L_\alpha \geq x_\alpha$, has the slope:

\[
m_\alpha = \frac{2\mu_\alpha(L_\alpha - \lambda_\alpha) + \sin(2\alpha)(1 - E^2)}{L_\alpha(2\lambda_\alpha - L_\alpha)}
\]

and the tangent line to $E_\alpha$ from a point $(0, l_\alpha)$, $l_\alpha \geq y_\alpha$, has the slope:

\[
m'_\alpha = \frac{l_\alpha(2\mu_\alpha - l_\alpha)}{\sin(2\alpha)(1 - E^2) - 2\lambda_\alpha(\mu_\alpha - l_\alpha)}.
\]
Proof: We have seen in the previous lemma that the minimal value $x_m = 0$ occurs when $\tan(t) = -E.\tan(\alpha)$. This yields to:

$$y_\alpha = \cos(\alpha)\cos(t)(\tan(\alpha) + E \tan(t)) + \mu(\alpha) = \mu_\alpha - \sin(2\alpha)(1 - E^2)/(2\lambda_\alpha).$$

A similar proof holds for $x_\alpha$.

On the other hand, a line: $y = m_\alpha(x - L_\alpha)$ passing through the point $(L_\alpha, 0)$ is tangent to $E_\alpha$ when the intersection is a double point (see Fig. 4). So the equation:

$$\sin(\alpha)\cos(t) + E\cos(\alpha)\sin(t) + \mu_\alpha = m_\alpha(\cos(\alpha)\cos(t) - E\sin(\alpha)\sin(t) + \lambda(\alpha) - L_\alpha)$$

has a double root. It is well known that an equation of the form $A\cos(t) + B\sin(t) = C$ has a double root if and only if $A^2 + B^2 = C^2$. In our case, we have:

$$A = \sin(\alpha) - m_\alpha \cos(\alpha), B = E(\cos(\alpha) + m_\alpha \sin(\alpha)), C = \mu_\alpha - m_\alpha(\lambda_\alpha - L_\alpha)$$

and this gives the result by applying the relation $A^2 + B^2 = C^2$ and simplifying by $m_\alpha$. Similar calculation for $m'_\alpha$. \qed

The same result is obtained for a unit ellipse $E_\beta$ of aspect ratio $E$, tilted at the angle $-\beta \in [0, \pi/2]$ from the $x - axis$ (see Fig. 5).

This time, the slopes of the tangent lines are:

$$m_\beta = -\frac{2\mu_\beta(L_\beta - \lambda_\beta) + \sin(2\beta)(1 - E^2)}{L_\beta(2\lambda_\beta - L_\beta)}$$

$$m'_\beta = -\frac{l_\beta(2\mu_\beta - l_\beta)}{\sin(2\beta)(1 - E^2) - 2\lambda_\beta(\mu_\beta - l_\beta)}.$$
Now our purpose is (i) to put together the two previous ellipses and (ii) to see if there is still some room available for a third ellipse $E_\gamma$ (see Fig.6).

(i) The slopes $m'_\alpha$ and $m'_\beta$ must be equal. This leads to:

$$l_\beta = \lambda_\beta m'_\alpha + \mu_\beta + \sqrt{\lambda_\beta^2 m'_\alpha^2 + \mu_\beta^2 + m'_\alpha \sin(2\beta)(1 - E^2)}.$$

Thus we know the length side of the square (denoted by $c$):

$$c = l_\alpha + l_\beta = l_\alpha + \lambda_\beta m'_\alpha + \mu_\beta + \sqrt{\lambda_\beta^2 m'_\alpha^2 + \mu_\beta^2 + m'_\alpha \sin(2\beta)(1 - E^2)}.$$

(ii) Let us search for a non negative $\epsilon$ and an angle $\gamma$ such that we can put the ellipse $E_\gamma$ in the pentagon at the bottom of the square (see figure 6). By the use of lemma 2, we have the following system of slopes for this pentagon:

$$\begin{align*}
\left\{ \begin{array}{l}
2\mu_\gamma(c - \epsilon + m_\beta(c - L_\beta) - \lambda_\gamma) + \sin(2\gamma)(1 - E^2) \\
\lambda_\gamma^2 - (c - \epsilon + m_\beta(c - L_\beta) - \lambda_\gamma)^2 \\
\mu_\gamma^2 - (c - L_\alpha - \epsilon/m_\alpha - \mu_\gamma)^2 \\
\sin(2\alpha)(1 - E^2) - 2\lambda_\gamma(\mu_\gamma - (c - L_\alpha - \epsilon/m_\alpha))
\end{array} \right.
\end{align*}$$

must be $-1/m_\beta$. $R$

Due to (1), the following proposition gives us a proof of the theorem:

**Proposition 1** If we suppose $c < \sqrt{12E}$, then the system (Σ) is impossible.

**Proof:** The system (Σ) is equivalent to the following one:

$$\begin{align*}
\left\{ \begin{array}{l}
\epsilon = c + m_\beta(c - L_\beta) - (\lambda_\gamma + m_\beta\mu_\gamma) - \epsilon_\beta.R \\
\epsilon = -\lambda_\alpha + m_\alpha(c - L_\alpha - \mu_\gamma) - \epsilon_\alpha.R.
\end{array} \right.
\end{align*}$$
where $\varepsilon_\alpha = \pm 1$, $\varepsilon_\beta = \pm 1$, and $R = \sqrt{m_\beta^2 \mu_\gamma^2 + \lambda_\gamma^2 + m_\beta \sin(2\gamma)(1 - E^2)}$.

Moreover, $\varepsilon_\beta = 1$ because $c - \varepsilon + m_\beta(c - L_\beta) \geq \lambda_\gamma + m_\beta \mu_\gamma$ (one can see that by drawing a straight line parallel to the one of slope $-1/m_\beta$ passing through the point $(\lambda_\gamma, \mu_\gamma)$), and $\varepsilon_\alpha = -1$ because $c - L_\alpha - \varepsilon/m_\alpha \geq \mu_\gamma + \lambda_\gamma/m_\alpha$ (draw the straight line parallel to the one of slope $-1/m_\alpha$ passing through the point $(\lambda_\gamma, \mu_\gamma)$).

For a sake of simplicity, we will note $M = m_\beta$, $\lambda = \lambda_\gamma$ and $\mu = \mu_\gamma$. By a symmetry argument, we can restrict our attention to $\gamma \in [0, \pi/2]$. We will show that the function:

$$\varphi(\gamma) = c + M(c - L_\beta) - \sqrt{M^2 \mu^2 + \lambda^2 + M \sin(2\gamma)(1 - E^2)} - (M \mu + \lambda)$$

is negative, which is a contradiction with $\varphi(\gamma) = \varepsilon \geq 0$ by the preceding system.

It can easily be seen by using a computer, but we can prove this strictly with the help of the function $\varphi$. We have, on $[0, \pi/2]$:

$$\varphi'(\gamma) = \frac{\sin(2\gamma)(1 - E^2)}{2} \left( \frac{1}{\lambda} - \frac{M}{\mu} \right) - \frac{M^2 - 1 + 2 \cot(2\gamma)}{\sqrt{M^2 \mu^2 + \lambda^2 + M \sin(2\gamma)(1 - E^2)}}$$
\[
\phi_1(\gamma) = \sin(2\gamma)(1 - E^2) \quad \text{where} \quad \phi_1(\gamma) \text{ is the expression in the bracket}
\]
and \[
\phi_2'(\gamma) = \sin(2\gamma)(1 - E^2) \quad \text{where} \quad \phi_2(\gamma) = \frac{1}{\lambda^3} + \frac{M}{\mu^3}
\]
\[
\phi_3(\gamma) = \frac{8M(1 + \cot^2(2\gamma))(M^2\mu^2 + \lambda^2 + M\sin(2\gamma)(1 - E^2))}{2\sqrt{M^2\mu^2 + \lambda^2 + M\sin(2\gamma)(1 - E^2)^2}}.
\]
and \[
\phi_4(\gamma) = \sin(2\gamma)(1 - E^2)(M^2 - 1 + 2\cot(2\gamma))^2 \quad \text{where} \quad \phi_4(\gamma) = \sin(2\gamma)(1 - E^2)(M^2 - 1 + 2\cot(2\gamma))^2.
\]
thus \(\phi_1'(\gamma)\) is non negative, and \(\phi_1(\gamma)\) is increasing on \([0, \pi/2]\). Moreover we have \(\lim_{\gamma \to 0} \phi_1(\gamma) = -\infty\) and \(\lim_{\gamma \to \pi/2} \phi_1(\gamma) = \infty\), thus there exists a unique value \(\gamma_0\) of \(\gamma\) such that \(\phi_1(\gamma)\) decreases on \([0, \gamma_0]\) and increases on \([\gamma_0, \pi/2]\).

**Proof for A:** We have \(A = c + M(c - L_\beta) - (1 + M.E) - \sqrt{1 + M^2.E^2}\).

In view of equation (2), we have:

\[
M.L_\beta = M.\lambda_\beta + \mu_\beta + \sqrt{M^2.\lambda^2_\beta + \mu^2_\beta + \sin(2\beta)(1 - E^2)}.
\]

Thus we have to prove that \(\psi(\beta) > c\) where the function \(\psi\) is defined by:

\[
\psi(\beta) = \frac{\sqrt{M^2.\lambda^2_\beta + \mu^2_\beta + \sin(2\beta)(1 - E^2)} + \sqrt{1 + M^2.E^2} + M(E + \lambda_\beta) + 1 + \mu_\beta}{1 + M}.
\]

But \(\frac{\partial \psi}{\partial M} \geq 0\), thus if we see \(\psi\) as a function of \(M \in [0, +\infty[\), we have:

\[
\psi(M) \geq \psi(0) = 2(\mu_\beta + 1).
\]

Finally, \(c < \sqrt{12E}\) by hypothesis implies that \(2(\mu_\beta + 1) > c\), which is the result.

**Proof for B:** We have \(B = c + M(c - L_\beta) - (E + M) - \sqrt{E^2 + M^2}\).

So we have \(B < 0\) if and only if \(\Phi(\beta) > c\) where \(\Phi\) is defined by:

\[
\Phi(\beta) = \frac{\sqrt{M^2.\lambda^2_\beta + \mu^2_\beta + \sin(2\beta)(1 - E^2)} + M.\lambda_\beta + \mu_\beta + E + M + \sqrt{E^2 + M^2}}{1 + M}.
\]

Moreover one can see that \(\Phi\) is an increasing function on \([0, +\infty[\) and that \(M \geq M_0\) where \(M_0\) is the value of \(M\) verifying \(L_\beta M_0 = c\). We have:

\[
\Phi(M_0) = \frac{c + E + M_0 + \sqrt{E^2 + M_0^2}}{1 + M_0}
\]
therefore

\[ \Phi(\beta) > c \iff \frac{E + M_0 + \sqrt{E^2 + M_1^2}}{M_0} > c. \]  

(4)

- This is clearly true if \( \sqrt{12E} \leq 2 \) (i.e. \( E \leq 1/3 \)).
- Otherwise, we can notice that the function defined by a ratio in equation (4) decreases, which proves that it’s enough to show that \( M_0 \leq M_1 \) where \( M_1 \) is the positive real number verifying:

\[ \frac{E + M_1 + \sqrt{E^2 + M_1^2}}{M_1} = \sqrt{12E}. \]

We have: \( M_0 \leq M_1 \) if and only if

\[ \frac{2\mu_\beta(\sqrt{12E} - \lambda_\beta) + \sin(2\beta)(1 - E^2)}{\sqrt{12E}(\sqrt{12E} - 2\lambda_\beta)} = \Lambda(\beta) \leq \frac{2E(\sqrt{12E} - 1)}{\sqrt{12E}(\sqrt{12E} - 2)} = \Lambda(0). \]

We study once again the variations of the function \( \Lambda \) on the interval \([-\pi/2, 0]\): There exists \( \beta_0 \in ]-\pi/2, 0[ \) such that \( \Lambda \) is decreasing on \([-\pi/2, \beta_0] \) and is increasing on \([\beta_0, 0] \). Moreover:

\[ \Lambda(-\pi/2) = \frac{2(\sqrt{12E} - E)}{\sqrt{12E}(\sqrt{12E} - 2E)} \leq \Lambda(0) = \frac{2E(\sqrt{12E} - 1)}{\sqrt{12E}(\sqrt{12E} - 2)} \]

which ends the proof.

\[ \Box \]

For the convenience of the reader, we join the picture of the best packings found by Thierry Gensane\(^1\) (personal communication) for three unit ellipses with aspect ratio \( E \) varying from 0.01 to 1 with step 0.01, and the corresponding density graph (see Fig.7, Fig.8). I would like to thank him for his contribution and valuable discussions on the subject. He used a stochastic algorithm based on a inflation formula (see for instance [3] ) which was already implemented in [2] to verify our theoretical results for two ellipses.

Even if there is some discontinuity on these drawings between 0.38 and 0.39, the density function is continuous (but not differentiable!)

Finally, this algorithm enables us to conjecture that the maximal density for the best packing of four ellipses in a square is again \( \pi/4 \), and it would be interesting to know from which value it remains true.

---

\(^1\) LMPA Université du Littoral Côte d’Opale.
Figure 7: Best packings found by the stochastic algorithm

Figure 8: Density graph for three ellipses
References


Received
### Année 2015

**LMPA 517 :** About the Density of Optimal Packings of Ellipses in a Square  
**P. Honvault**, Septembre 2015

**LMPA 516 :** $L^2$-stability of a finite element- finite volume discretization of convection-diffusion-reaction equations with nonhomogeneous mixed boundary conditions.  
**D. Deuring and R. Eymard**, Septembre 2015

**LMPA 515 :** A finite element - finite volume discretization of convection-diffusion-reaction equations with nonhomogeneous mixed boundary conditions : error estimates.  
**D. Deuring**, Septembre 2015

**LMPA 514 :** Asymptotic structure of viscous incompressible flow around a rotating body, with nonvanishing flow field at infinity.  
**D. Deuring**, S. Kračmar and Šárka Nečasová, Septembre 2015

### Année 2014

**LMPA 513 :** A structural aspect of the category of quandles  
**D. Bourn**, Décembre 2014

**LMPA 512 :** Robin boundary condition and shock problem for the focusing nonlinear Schrödinger equation  
**Spyridon Kamvissis, D. Shepelsky and L. Zielinski**, Novembre 2014

**LMPA 511 :** The Riemann-Hilbert problem approach to the short pulse equation  
**D. Shepelsky, A. Tiuridlo and L. Zielinski**, Novembre 2014

**LMPA 510 :** Exponential decay of the vorticity of an incompressible viscous flow in a rotating frame  
**P. Deuring and G. P. Galdi**, Septembre 2014

**LMPA 509 :** Leading terms of velocity and its gradients of the stationary rotational viscous incompressible flows with non zero velocity at infinity  
**P. Deuring, S. Kračmar and Š. Nečasová**, Septembre 2014

**LMPA 508 :** A variational approach to relativistic fluid balls  
**J. von Below and H. Kaul**, Août 2014

**LMPA 507 :** Discrete calculus of variation for homographic configurations in celestial mechanics  

**LMPA 506 :** Matrix polynomial and epsilon-type extrapolation methods with applications  

**LMPA 505 :** Block Arnoldi Based Methods for Large Nonsymmetric Algebraic Riccati Problems  
**A.H. Bentbib, K. Jbilou and El Sadek**, Mai 2014

**LMPA 504 :** Asymptotic Normality of a kernel estimators of density and mode for censored and associated data  
**Y. Ferrani, E. Ould-Saïd and A. Tatachak**, Mai 2014

**LMPA 503 :** Instability of stationary solutions of evolution equations on graphs under dynamical node transition  
**Joachim von Below and B. Vasseur**, Avril 2014
Année 2012

LMPA 482 : Kernel density and mode estimates for censored associated data.
Y. Ferrani, E. Ould-Saïd and A. Tatachak , Novembre 2012
LMPA 481 : Optimal packings of two ellipses in a square.
T. Gensane and P. Honvault , Octobre 2012
LMPA 480 : A Resonant-Superlinear Elliptic Problem Revisited.
M. Cuesta and C. De Coster , Septembre 2012
LMPA 479 : Strong uniform consistency of the robust nonparametric regression estimation for quasi-associated vectorial processes.
S. Attaoui, A. Laksaci and E. Ould-Saïd , Septembre 2012.
LMPA 477 : Pointwise decay of stationary rotational viscous incompressible flows with nonzero velocity at infinity.
LMPA 476 : Some asymptotic results on the nonparametric conditional density estimate in the single index for quasi-associated Hilbertian processes.
S. Attaoui, A. Laksaci and E. Ould-Saïd , Juin 2012.
LMPA 475 : A parallel implementation of the CMRH method for dense linear systems.
S. Duminil , Juin 2012.
LMPA 474 : Pointwise spatial decay of time-dependent Oseen flows : the case of data with noncompact support.
P. Deuring , Avril 2012.
W. Horrigue and E. Ould-Saïd , Mars 2012.
LMPA 472 : Some remarks on the eigenvalue multiplicities of the Laplacian on infinite locally finite trees.
LMPA 471 : A meshless method for the numerical computation of the solution of steady Burgers-type equations.
A. Bouhamidi, M. Hached and K. Jbilou , Mars 2012.
LMPA 470 : A note on the Lynden-Bell estimator under association.
O. Benrabah, E. Ould-Saïd and A. Tatachak , Mars 2012.
LMPA 468 : Approximation of eigenvalues for unbounded Jacobi matrices using finite submatrices.
A. Boutet de Monvel and L. Zielinski , Février 2012.

Année 2011

LMPA 467 : An Inexact Newton Block Arnoldi Method for Discrete-Time Algebraic Riccati Equations.
A. Bouhamidi and K. Jbilou , Novembre 2011.
LMPA 466 : Random number sequences and the first digit phenomenon.
B. Massé and D. Schneider , Novembre 2011.
LMPA 465 : Nonlinear Convection in Reaction-Diffusion Equations under Dynamical Boundary Conditions.
LMPA 464 : Spatial decay of time-dependent incompressible Navier-Stokes flows with nonzero
velocity at infinity.

P. Deuring, Septembre 2011.

LMPA 463: The Cauchy problem for the homogeneous time-dependent Oseen system in $\mathbb{R}^3$: spatial decay of the velocity.

P. Deuring, Septembre 2011.

LMPA 462: A representation formula for the velocity part of 3D time-dependent Oseen flows.

P. Deuring, Septembre 2011.

LMPA 461: On the central limit theorem for a conditional mode estimator of a randomly censored time series.


LMPA 460: Symmetry and Cauchy completion of quantaloid-enriched categories.


LMPA 459: Quadratic choreographies.


LMPA 458: Fast solvers for discretized Navier-Stokes problems using vector extrapolation.

S. Duminil, H. Sadok and D. Silvester, Mai 2011.


LMPA 456: Maximal inflation of two ellipses.

P. Honvault, Mars 2011.

LMPA 455: Interpolation on the hypersphere with Thiele type rational interpolants.

T. Gensane, Mars 2011.

LMPA 454: Eigenvalue multiplicities and asymptotics for second order elliptic operators on networks.


LMPA 453: Elementary characterisation of small quantaloids of closed cribles.

H. Heymans and I. Stubbé, Mars 2011.


LMPA 451: The Hierarchy of Weighted Densities and the First Digit Phenomenon.

B. Massé and D. Schneider, Mars 2011.

LMPA 450: Towards Stone duality for topological theories.


LMPA 449: 'Hausdorff distance' via conical cocompletion.

I. Stubbé, Février 2011.

LMPA 448: On the problem of Molluzzo for the modulus 4.


LMPA 447: On the mantissa distribution of powers of natural and prime number.

S. Eliahou, B. Massé and D. Schneider, Janvier 2011.