Internal equivalence relations, modular formula and Goursat condition in non-regular context

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Abstract

We introduce a notion of Goursat category valid in the non-regular context and such that: 1) in the regular context, it coincides with the notion introduced by Carboni-Kelly-Pedicchio, 2) any Mal'tsev category is a Goursat category, 3) this notion is characterized by a property of the fibration of points, 4) when $\mathbb{T}$ is an algebraic theory giving rise to a Goursat variety $\mathbb{V}(\mathbb{T})$, the category $\mathbb{T}(\mathbb{E})$ of internal $\mathbb{T}$-algebras in $\mathbb{E}$ is itself a Goursat category.

Introduction

The notion of Goursat category was introduced in the regular context by Carboni-Kelly-Pedicchio in [8] after a generalization of an old result of Goursat ([14] 1889) on relations in the category of groups stated in [9] (see also [22]). There were defined as those regular categories which are 3-permutable, namely such that, given any pair (R, S) of equivalence relations on an object X, we get $R \circ S \circ R = S \circ R \circ S$. Many further investigations about this notion can be found in [21], [5], [20], [15], [16].

The aim of this work is to introduce a definition valid in the non-regular context and such that: 1) of course, in the regular context, it coincides with the pioneering notion of Carboni-Kelly-Pedicchio, 2) any Mal'tsev category satisfies this definition, 3) this definition is characterized by a property of the fibration of points $¶_{\mathbb{E}} : Pt(\mathbb{E}) \to \mathbb{E}$, as it is the case for Mal'tsev categories [3], 4) when $\mathbb{T}$ is an algebraic theory giving rise to a Goursat variety $\mathbb{V}(\mathbb{T})$, the category $\mathbb{T}(\mathbb{E})$ of internal $\mathbb{T}$-algebras $\mathbb{E}$ is itself a Goursat category, for any finitely complete category $\mathbb{E}$.

Actually two directions emerged: an analytical way coming from the investigation of the notion of supremum of pairs of equivalence relations, and eventually a synthetical one which gets straight to the three first conditions but without satisfying Condition 4, just based on the discriminating notion of hyperextremal epimorphism in the category $Pt(\mathbb{E})$, an intermediate class of epimorphisms between extremal epimorphisms and split epimorphisms. However the two ways satisfying Condition 1, they both coincide, in the regular context, with the original notion of [8].

Incidentally, this discriminating notion of hyperextremal epimorphism allowed the author to fullfil the old project to understand the main properties of the regular categories regarding the equivalence relations without any resort to Metatheorems and to deal uniquely with diagrammatic proofs, see Corollary 2.1 via Lemma 1.7 and Proposition 2.2.

On the first way, the main observations are: 1) Proposition 3.3 following which a category $\mathbb{E}$ is such that any pair (R, S) of equivalence relations has a supremum $R \bigvee S$ if and only if the category $Equ\mathbb{E}$ of internal equivalence relations has regular epimorphisms with any domain above split epimorphisms in $\mathbb{E}$, and 2) Proposition 3.5 following which the modular formula for internal equivalence relations holds if and only if these regular epimorphisms are stable under pullbacks along maps in the fibers of the fibration $\mathcal{O}_{\mathbb{E}} : Equ\mathbb{E} \to \mathbb{E}$. They lead, first, to a charaterization of the circumstances under which the category $Equ\mathbb{E}$ is a regular category (Theorem 3.3), then to the Definition 4.1 of Goursat categories related with the notion of supremum of pairs of equivalence relations.

The second way, free from any relationship with suprema or composition of relations leads to the Definition 1.2 of extremal categories. In the regular context, as said above, both types of categories (extremal and Goursat) coincide with the notion introduced by Carboni-Kelly-Pedicchio which itself coincides with those regular categories $\mathbb{E}$ which give rise to a characteristic kind of regular category $Equ\mathbb{E}$ (Theorem 3.5).

In [11], there is a description of the *Polin variety*, showing it to be 4-permutable, but not congruence modular. Accordingly this variety $\mathbb{V}$ is such that $Equ\mathbb{V}$ is not regular. The last section is devoted to some applications of Proposition 3.3 to n-permutable regular categories. In particular the $(2n+1)$-permutable regular categories are characterized as those regular categories which are such that, for any regular epimorphism $f : X \twoheadrightarrow Y$ and any equivalence relation S on X, the reflexive and symmetric relation $f(S)^n$ is actually an equivalence relation.

The article is organized along the following lines. Section 1 is devoted to recalling basic facts about extremal epimorphims and internal equivalence relations; it introduces the notion of hyperextremal epimorphism and of extremal category. Section 2 is devoted to recalling basic facts about regular categories, but without any resort to Metatheorems; it gives rise to a characterization of regular extremal categories. Section 3 characterizes the existence of supremum of pairs of equivalence relations, the validity of the congruence formula and the circumstances under which the category $Equ\mathbb{E}$ is regular. Section 4 introduces a notion of Goursat category in the non-regular context which satisfies the four conditions described above. Finally Section 5 enlarges the applications of the results of Section 3 to n-permutable regular categories.

1 Aspects of equivalence relations

1.1 Preliminaries on extremal and regular epimorphisms

Any category $\mathbb{E}$ will be supposed finitely complete. Recall that an *extremal epimorphism* is a map $f : X \to Y$ such that any decomposition $f = \bar{f}.m$ with a monomorphism m implies that m is an isomorphism. A regular epimorphism (i.e. a map which is the coequalizer of its kernel equivalence relation) is an extremal epimorphism. Both classes of maps are preserved by functors having a right adjoint; in particular they are stable under pushout along any map.

It is straighforward that the underlying functor of a fibration $U : \mathbb{F} \to \mathbb{E}$ is faithful if and only if the fibers are preorders. In this case, any map in a fibre is a monomorphism; any square with x and y in the fibers is commutative if and only if we have $U(f) = U(f')$:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "x"'] & Y \arrow[d, "y"] \\
X' \arrow[r, "f'"'] & Y'
\end{tikzcd}$$

When any fibre above U has a greatest element $I(U)$ which is stable under cartesian maps, it defines a right adjoint $I : \mathbb{E} \to \mathbb{F}$ to the functor U; we shall denote by $\eta_X : X \rightarrowtail IU(X)$ the induced unique map.

Theorem 1.1. *Let $U : \mathbb{F} \to \mathbb{E}$ be a faithful fibration for which each fiber has a greatest element stable under cartesian maps. Given any commutative square in $\mathbb{F}$:*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "x"'] & Y \arrow[d, "y"] \\
X' \arrow[r, "f'"'] & Y'
\end{tikzcd}$$

with x and y inside a fibre and with f an extremal epimorphism, it is a pushout if and only if f' is an extremal epimorphism. In this situation f' is a regular epimorphism as soon as so is f.

Proof. We just recalled it is a necessary condition. Let us show it is sufficient. For that, consider the following diagram in $\mathbb{F}$, where we have $h.x = g.f$ $(*)$ and $U(f) = \phi = U(f')$:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, tail, "x"'] & Y \arrow[d, tail, "y"] \arrow[rrd, bend left, "g"] & & \\
X' \arrow[r, "f'"] \arrow[rr, dotted, bend right=10, "\psi"] \arrow[rrr, bend right=20, "h"'] \arrow[d] & Y' \arrow[d] & g^{-1}(Z) \arrow[r, "g^Z"] \arrow[d] & Z \arrow[d] \\
IU(X) \arrow[r, "I(\phi)"'] & IU(Y) \arrow[r, equal] & IU(Y) \arrow[r, "IU(g)"'] & IU(Z)
\end{tikzcd}
\qquad
\begin{tikzcd}
X' \arrow[r, "\psi"] \arrow[d, "f'"'] & g^{-1}(Z) \arrow[d, tail, "\eta_{g^{-1}(Z)}"] \\
Y' \arrow[r, tail, "\eta_{Y'}"'] \arrow[ru, dotted] & IU(Y)
\end{tikzcd}$$

From (*), we get $U(h) = U(g).\phi$. Let g^Z be the cartesian map above $U(g)$. Then we get a unique factorization ψ such that $g^Z.\psi = h$ and $U(\psi) = \phi$. Accordingly the right hand side square commutes, and we get the dotted desired factorization, since f' is an extremal epimorphism and $\eta_{g^{-1}(Z)}$ is a monomorphism. □

We have also the following very easy and useful:

Lemma 1.1. *Suppose $U : \mathbb{F} \to \mathbb{E}$ is a left exact fibration. Given any commutative square in $\mathbb{F}$ where f' is a cartesian map:*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "x"'] & Y \arrow[d, "y"] \\
X' \arrow[r, "f'"'] & Y'
\end{tikzcd}$$

and its image by U is a pullback, it is a pullback if and only if f is cartesian.

1.2 Reflexive and equivalence relations

We denote by $Pt(\mathbb{E})$ the category whose objects are the split epimorphisms and whose maps are the commutative squares of split epimorphisms:

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[d, shift left, "f"] & X' \arrow[d, shift left, "f'"] \\
Y \arrow[r, "y"'] \arrow[u, shift left, "s"] & Y' \arrow[u, shift left, "s'"]
\end{tikzcd}$$

and by $\P_{\mathbb{E}} : Pt(\mathbb{E}) \to \mathbb{E}$ the functor associating with any split epimorphism (f, s) its codomain Y. It is a fibration whose cartesian maps are the pullbacks of split epimorphisms. It is called the *fibration of points* and the fibre above Y is denoted by $Pt_Y(\mathbb{E})$, see [3].

Lemma 1.2. *A morphism (y, x) in $Pt(\mathbb{E})$ as above is an extremal (resp. regular) epimorphism if and only if both y and x are extremal (resp. regular) epimorphisms in $\mathbb{E}$.*

Proof. It is clear that if x and y are extremal (resp. regular) epimorphisms in $\mathbb{E}$, the morphism $(y, x) : (f, s) \to (f', s')$ is an extremal (resp. regular) epimorphism in $Pt(\mathbb{E})$. Conversely suppose it is an extremal (resp. regular) epimorphism in $Pt(\mathbb{E})$. Then $y = \P_{\mathbb{E}}(y, x)$ is an extremal (resp. regular) epimorphism since $\P_{\mathbb{E}}$ has a right adjoint. Suppose (y, x) is an extremal epimorphism $x = \bar{x}.n$ with a monomorphism n:

$$\begin{tikzcd}
X \arrow[r, "\bar{x}"'] \arrow[rr, bend left, "x"] \arrow[d, shift left, "f"] & U \arrow[r, tail, "n"] \arrow[dr] & X' \arrow[d, shift left, "f'"] \\
Y \arrow[rr, "y"'] \arrow[u, shift left, "s"] & & Y' \arrow[u, shift left, "s'"] \arrow[ul, dotted, bend left, "\bar{s}'"]
\end{tikzcd}$$

Since y is an extremal epimorphism and n a monomorphism, we get a factorization $\bar{s}'$ such that $n.\bar{s}' = s'$ and $\bar{s}'.y = \bar{x}.s$. Whence a decomposition $(y,x) = (1_Y, n).(y, \bar{x})$ in $Pt(\mathbb{E})$, so that n is an isomorphism in $\mathbb{E}$, and x an extremal epimorphism. Suppose now (y,x) is a regular epimorphism and we have a map $\phi : X' \to T$ in $\mathbb{E}$ which coequalizes $R[x]$, namely such that $R[x] \subset R[\phi]$. Since y is a regular epimorphism, there is a unique map $\sigma : Y' \to T$ such that $\sigma.y = \phi.s$. Now, consider the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "x"] \arrow[rr, bend left, "(f'.x{,}\phi)"] \arrow[d, shift left, "f"] & X' \arrow[r, dotted, "(f'{,}\tau)"] \arrow[d, shift left, "f'"] & Y' \times T \arrow[d, shift left, "p_{X'}"] \\
Y \arrow[r, "y"'] \arrow[u, shift left, "s"] & Y' \arrow[r, equal] \arrow[u, shift left, "s'"] & Y' \arrow[u, shift left, "(1_{Y'}{,}\sigma)"]
\end{tikzcd}$$

We have $R[x] \subset R[f'.x] \cap R[\phi] = R[(f'.x, \phi)]$, and since (y,x) is a regular epimorphism, we get a unique factorization (f', τ) in $Pt(\mathbb{E})$ which provides the desired factorization $\tau : X' \to T$ in $\mathbb{E}$. $\square$

We shall denote by $Ref\mathbb{E}$ (resp. $Equ\mathbb{E}$) the category of internal reflexive relations (resp. equivalence relations) in $\mathbb{E}$ and by $\nabla : \mathbb{E} \to Ref\mathbb{E}$, the fully faithful functor associating with any object X, the undiscrete equivalence relation:

$$\begin{tikzcd}
X \times X \arrow[r, shift left=2, "p_0^X"] \arrow[r, shift right=2, "p_1^X"'] & X \arrow[l, "s_0^X" description]
\end{tikzcd}$$

It has a left exact left adjoint $\mathcal{O}_{\mathbb{E}} : Ref\mathbb{E} \to \mathbb{E}$ associating with any reflexive relation its underlying object. This functor is actually a left exact fibration: given any reflexive relation S on Y and any map $f : X \to Y$, the cartesian map above f is given by the following pullback in $Ref\mathbb{E}$:

$$\begin{tikzcd}
f^{-1}(S) \arrow[r, "\tilde{f}^S"] \arrow[d, "(\bar{d}_0^S{,}\bar{d}_1^S)"'] & S \arrow[d, "(d_0^S{,}d_1^S)"] \\
\nabla_X \arrow[r, "\nabla_f"'] & \nabla_Y
\end{tikzcd}$$

The fibres of $\mathcal{O}_{\mathbb{E}} : Ref\mathbb{E} \to \mathbb{E}$ are preorders. We noticed that it is equivalent to the fact that the functor $\mathcal{O}_{\mathbb{E}}$ is faithful. Accordingly, given any diagram where R and S are reflexive relations:

$$\begin{tikzcd}
R \arrow[r, dotted, "\hat{f}"] \arrow[d, shift right=2, "d_0^R"'] \arrow[d, shift left=2, "d_1^R"] & S \arrow[d, shift right=2, "d_0^S"'] \arrow[d, shift left=2, "d_1^S"] \\
X \arrow[r, "f"'] \arrow[u] & Y \arrow[u]
\end{tikzcd}$$

there exists at most one a factorization $\hat{f}$, i.e. at most one map in $Ref\mathbb{E}$, above f; it is the case if and only if $R \subset f^{-1}(S)$. Given any map $f : X \to Y$ and any reflexive relation S on X, the unique map $(f, \check{f}) : S \to \nabla_Y$ above f will be call called the *paraterminal map* above f. The subcategory $Equ\mathbb{E}$ is stable in $Ref\mathbb{E}$ under finite limits and cartesian maps. Accordingly, both $\mathcal{O}_{\mathbb{E}} : Ref\mathbb{E} \to \mathbb{E}$ and its restriction $\mathcal{O}_{\mathbb{E}} : Equ\mathbb{E} \to \mathbb{E}$ satisfy the conditions of Theorem 1.1 and Lemma 1.1. A morphism$(f, \hat{f}) : R \to S$ in $Equ\mathbb{E}$ is called *fibrant* when any of the commutative squares indexed by 0 or 1 is a pullback.

The functor $\mathcal{O}_{\mathbb{E}} : Ref\mathbb{E} \to \mathbb{E}$ has a left adjoint functor Δ as well, given by the discrete equivalence relation:

$$\begin{tikzcd} X \arrow[r, shift left=2, "1_X"] \arrow[r, shift right=2, "1_X"'] & X \arrow[l, "1_X" description] \end{tikzcd}$$

The *kernel equivalence relation* of a map $f : X \to Y$ is the domain of the cartesian map with codomain Δ_Y above f:

$$\begin{tikzcd} R[f] \arrow[r, "\tilde{f}^{\Delta_Y}"] \arrow[d, tail, "(d_0^f, d_1^f)"'] & \Delta_Y \arrow[d, tail, "(1_Y, 1_Y)"] \\ \nabla_X \arrow[r, "\nabla_f"'] & \nabla_Y \end{tikzcd}$$

An equivalence relation R on X is said to be *effective* when there is some map $f : X \to Y$ such that $R = R[f]$. In the category *Set* of sets, a reflexive relation R on X is an equivalence relation if and only if it satisfies the *horn-filler condition*, namely: xRx' and xRx'' imply $x'Rx''$, $\forall (x, x', x'') \in X^3$. Or equivalently if and only if we have $R[d_0^R] \subset d_1^{-1}(R)$, or equivalently if and only if there is a morphism $(d_1^R, d_2^R) : R[d_0^R] \to R$ above d_1^R in $Ref\mathbb{E}$ defined by $d_2^R(xRx', xRx'') = x'Rx''$ *which is necessarily fibrant.* The same characterization holds in any category $\mathbb{E}$.

Lemma 1.3. *An extremal epimorphism $(f, \hat{f}) : S \to T$ in $Ref\mathbb{E}$ (resp. $Equ\mathbb{E}$) is such that its underlying map f in $\mathbb{E}$ is an extremal epimorphism. When f is an extremal epimorphism in $\mathbb{E}$, the map $(f, \hat{f}) : S \to T$ is extremal in $Ref\mathbb{E}$ (resp. $Equ\mathbb{E}$) if and only if it is cocartesian above f. The extremal epimorphism $(f, \hat{f}) : S \to T$ is a regular epimorphism in $Ref\mathbb{E}$ (resp. $Equ\mathbb{E}$) if and only if its underlying map $f : X \to Y$ is a regular epimorphism in $\mathbb{E}$.*

Proof. The first point is a consequence of the fact that the functor $\mathcal{O}_{\mathbb{E}}$ has a right adjoint. As for the second one: it is clear that if $(f, \hat{f})$ is cocartesian, it is extremal.

Conversely suppose it is an extremal epimorphism. Since $\mathcal{O}_{\mathbb{E}}$ is a fibration, to show it is cocartesian, it is sufficient to check the universal property for the only maps above f. So consider any map $(f, f') : S \to T'$ in $Ref\mathbb{E}$. It determines a unique map $(f, f'') : S \to T \cap T'$ in $Ref\mathbb{E}$ making the following diagram commute:

$$\begin{tikzcd} S \arrow[r, "{(f,f'')}"] \arrow[d, "{(f,\hat{f})}"'] & T\cap T' \arrow[r, tail, "{(1_Y,i')}"] \arrow[d, tail, "{(1_Y,i)}"] & T' \\ T \arrow[r, equal] & T & \end{tikzcd}$$

Since $(f,\hat{f})$ is an extremal epimorphism in $Ref\mathbb{E}$, the monomorphism $(1_Y,i)$ is an isomorphism, and the map $i'.i^{-1}$ produces the desired factorization.

In the same way, if $(f,\hat{f})$ is a regular epimorphism in $Ref\mathbb{E}$, the underlying map f is a regular epimorphism in $\mathbb{E}$. Suppose now f is a regular epimorphism, and $(f,\hat{f}): S \to T$ an extremal epimorphism in $Ref\mathbb{E}$. It is straighforward that it is a regular epimorphism in $Ref\mathbb{E}$ as soon as it is cocartesian above f. $\square$

Given a pair (R,S) of equivalence relations on an object X in a category $\mathbb{E}$, we denote by $R\square S$ the inverse image of the equivalence relation $S\times S$ along the inclusion $(d_0^R,d_1^R): R \rightarrowtail X\times X$. This defines a double equivalence relation

$$\begin{tikzcd} R\square S \arrow[r, shift left=2, "\delta_1^S"] \arrow[r, shift right=2, "\delta_0^S"'] \arrow[d, shift right=2, "\delta_0^R"'] \arrow[d, shift left=2, "\delta_1^R"] & S \arrow[l] \arrow[d, shift right=2, "d_0^S"'] \arrow[d, shift left=2, "d_1^S"] \\ R \arrow[r, shift left=2, "d_1^R"] \arrow[r, shift right=2, "d_0^R"'] \arrow[u] & X \arrow[l] \arrow[u] \end{tikzcd}$$

which is actually the largest double equivalence relation relating R and S. In set-theoretical terms, this double relation $R\square S$ is the subset of elements (u,v,u',v') of X^4 such that the quadratic set of relations $uRu', vRv', uSv, u'Sv'$ holds:

$$\begin{tikzcd} u \arrow[r, dotted, "S"] \arrow[d, dotted, "R"'] & v \arrow[d, dotted, "R"] \\ u' \arrow[r, dotted, "S"'] & v' \end{tikzcd}$$

The following observation is straightforward and nevertheless meaningful:

Lemma 1.4. *Suppose we have $R = R[f]$ with $f: X \to Y$ in $\mathbb{E}$. Then the kernel equivalence relation in equ$\mathbb{E}$ of the paraterminal map $(f,\check{f}): S \to \nabla_Y$ above f is given by the following diagram:*

$$\begin{tikzcd} R[f]\square S \arrow[r, shift left=2, "\delta_1^S"] \arrow[r, shift right=2, "\delta_0^S"'] & S \arrow[l] \arrow[r, "{(f,\check{f})}"] & \nabla_Y \end{tikzcd}$$

It is the kernel equivalence relation of any other map with domain S above f. Accordingly this equivalence relation is the unique effective equivalence relation on S in Equ$\mathbb{E}$ above $R[f]$.

Let us denote by ξ_R^S the inclusion $R\square S \rightarrowtail (d_0^R)^{-1}(S)$.

Lemma 1.5. *Let (R, S) be any pair of equivalence relations on an object X in a category $\mathbb{E}$. The following commutative diagram:*

$$
\begin{tikzcd}
R\square S \arrow[rr, "\delta_1^S"] \arrow[d, tail, "\xi_R^S"'] & & S \arrow[d, tail, "(d_0^S, d_1^S)"] \\
(d_0^R)^{-1}(S) \arrow[rr, "\check{d}_1^R"'] & & X \times X
\end{tikzcd}
$$

is a pullback, where the lower map is paraterminal.

Proof. Thanks to the Yoneda embedding, it is enough to check it in Set. The elements of $(d_0^R)^{-1}(S)$ are given by the left hand side diagrams, while their images by the lower horizontal map is (y, y'):

$$
\begin{tikzcd}
x \arrow[r, dotted, "R"] \arrow[d, dotted, "S"'] & y & & y \arrow[d, dotted, bend left, "S"] \\
x' \arrow[r, dotted, "R"'] & y' & & y'
\end{tikzcd}
$$

So, as soon as we have ySy', we get an element of $R\square S$. $\square$

Lemma 1.6. *Let $\mathbb{E}$ be a category and (R, S) any pair of equivalence relations on X. Then the map ξ_R^S is an isomorphism or, equivalently, the following morphism of equivalence relations:*

$$
\begin{tikzcd}
R\square S \arrow[r, shift left=2, "\delta_1^S"] \arrow[r, shift right=2, "\delta_0^S"'] \arrow[d, tail, "(\delta_0^R, \delta_1^R)"'] & S \arrow[l] \arrow[d, tail, "(d_0^S, d_1^S)"] \\
R \times R \arrow[r, shift left=2, "d_1^R \times d_1^R"] \arrow[r, shift right=2, "d_0^R \times d_0^R"'] & X \times X \arrow[l]
\end{tikzcd}
$$

is fibrant if and only if we have $R \subset S$.

Proof. Again, it is enough to check our assertion in Set. Suppose $R \subset S$. Then from the following left hand side diagram:

$$
\begin{tikzcd}
x \arrow[r, dotted, "S"] \arrow[d, dotted, "R"'] & y \arrow[d, dotted, "R"] & & x \arrow[r, dotted, "S"] \arrow[d, dotted, "S"'] & y \arrow[d, dotted, "S"] \\
x' & y' & & x' & y'
\end{tikzcd}
$$

we can deduce the right hand side one, and thus $x'Sy'$, which shows that the square indexed by 0 is a pullback (which means that the morphism of equivalence relations in question is fibrant). Conversely suppose this square is a pullback.

From the following left hand side diagram drawn from xRx', we can deduce the right hand side one:

$$\begin{tikzcd} x \arrow[r, dotted, "S"] \arrow[d, dotted, "R"'] & x \arrow[d, dotted, "R"] \\ x & x' \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, dotted, "S"] \arrow[d, dotted, "R"'] & x \arrow[d, dotted, "R"] \\ x \arrow[r, dotted, "S"'] & x' \end{tikzcd}$$

so that we get xSx' and $R \subset S$. □

This lemma shows that the left hand side equivalence relation on R is both the inverse image of S along d_0^R and d_1^R

1.3 Extremal categories

The following tools and observations will appear to be very discriminating:

Definition 1.1. *A morphism (y, x) in $Pt(\mathbb{E})$ is said to be a hyperextremal (resp. hyperregular) epimorphism when it is an extremal (resp. regular) epimorphism such that the factorization $R(x) : R[f] \to R[f']$ is an extremal (resp. regular) epimorphism in $\mathbb{E}$. A morphism in $Ref\mathbb{E}$:*

$$\begin{tikzcd} R \arrow[r, "\hat{f}"] \arrow[d, shift right=2, "d_0^R"'] \arrow[d, shift left=2, "d_1^R"] & S \arrow[d, shift right=2, "d_0^S"'] \arrow[d, shift left=2, "d_1^S"] \\ X \arrow[u] \arrow[r, "f"'] & Y \arrow[u] \end{tikzcd}$$

is said to be a hyperextremal (resp. hyperregular) epimorphism when the underlying morphism in $Pt(\mathbb{E})$ indexed by 0 is hyperextremal (resp. hyperregular).

It is clear that, in both categories $Pt(\mathbb{E})$ and $Ref\mathbb{E}$, the class of hyperextremal (resp. hyperregular) epimorphisms is an intermediate class between the class of extremal (resp. regular) epimorphisms and the class of split epimorphims. In both categories, the hyperextremal epimorphisms are stable under composition. When $\mathbb{E}$ is regular (see Section 2) extremal epimorphisms and regular epimorphisms coincide in $\mathbb{E}$, so that hyperextremal epimorphisms and hyperregular epimorphisms coincide in $Pt(\mathbb{E})$ (resp. $Ref\mathbb{E}$). The main justification of the previous definition is the following:

Lemma 1.7. *Given any category $\mathbb{E}$, the hyperextremal epimorphisms in $Ref\mathbb{E}$ reflect the equivalence relations.*

Proof. We have to show that a hyperextremal epimorphism $(f, \hat{f})$ in $Ref\mathbb{E}$ whose domain R is an equivalence relation makes its codomain S an equivalence relation as well:

$$\begin{tikzcd}
R \arrow[r, "\hat{f}"] \arrow[d, shift right=2, "d_0^R"'] \arrow[d, shift left=2, "d_1^R"] & S \arrow[d, shift right=2, "d_0^S"'] \arrow[d, shift left=2, "d_1^S"] \\
X \arrow[u] \arrow[r, "f"'] & Y \arrow[u]
\end{tikzcd}$$

For that, consider the following commutative square in $\mathbb{E}$:

$$\begin{tikzcd}
R[d_0^R] \arrow[r, "R(\hat{f})"] \arrow[d, "d_2^R"'] & R[d_0^S] \arrow[r, tail] \arrow[d, dotted, "d_2^S"] & Y \times Y \times Y \arrow[d, "p_2^Y"] \\
R \arrow[r, "\hat{f}"'] & S \arrow[r, tail, "(d_0^S, d_1^S)"'] & Y \times Y
\end{tikzcd}$$

where the upper unlabelled horizontal map associates (a, b, c) with (aSb, aSc). Since $R(\hat{f})$ is an extremal epimorphism in $\mathbb{E}$ and (d_0^S, d_1^S) is a monomorphism, we get the dotted desired factorization d_2^S. $\square$

Definition 1.2. *Let $\mathbb{E}$ be a pointed category. It is said to be punctually extremal when any extremal epimorphism in $Pt(\mathbb{E})$ above a terminal map:*

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, shift left=2, "f"] & Z \arrow[d, shift left=2] \\
Y \arrow[u, "s"] \arrow[r] & 1 \arrow[u]
\end{tikzcd}$$

is hyperextremal. When $\mathbb{E}$ is any category, it is said to be extremal when any fiber $Pt_Y(\mathbb{E})$ is punctually extremal and to be hyperextremal when any extremal epimorphism in $Pt(\mathbb{E})$ is hyperextremal.

Recall that a category $\mathbb{E}$ is unital [3] when it is pointed and, given any pair (A, B) of objects:

$$\begin{tikzcd}
A \arrow[r, shift right=1, "(1_A,0)"'] & A \times B \arrow[l, shift right=1, "p_A"'] \arrow[r, shift left=1, "p_B"] & B \arrow[l, shift left=1, "(0,1_B)"]
\end{tikzcd}.$$

the morphisms $(1_A, 0)$ and $(0, 1_B)$ are jointly extremally epimorphic. The category Mon of monoids is unital. A Mal'tsev category ([9],[10], see also [25] and [13]) is a category in which any reflexive relation is an equivalence relation; it was shown in [3] that a category $\mathbb{E}$ is a Mal'tsev one if and only if any fibre $Pt_Y\mathbb{E}$ is unital.

Proposition 1.1. *Any unital category is punctually extremal. Any Mal'tsev category is extremal.*

Proof. Consider the following left hand side diagram where g is an extremal epimorphism, and we have $R(g) = m.\gamma$ in $\mathbb{E}$ with a monomorphism m:

$$\begin{tikzcd}
R[f] \arrow[r, "\gamma"'] \arrow[rr, bend left, "R(g)"] \arrow[d, shift right=2, "d_0^f"'] \arrow[d, shift left=2, "d_1^f"] & S \arrow[r, tail, "m"'] \arrow[dr, shift left] \arrow[dr, shift right] & Z \times Z \arrow[d, shift right=2, "p_0^Z"'] \arrow[d, shift left=2, "p_1^Z"] \\
X \arrow[u] \arrow[rr, two heads, "g"'] \arrow[d, shift left, "f"] & & Z \arrow[u] \arrow[d, shift left] \\
Y \arrow[u, shift left, "s"] \arrow[rr] & & 1 \arrow[u, shift left]
\end{tikzcd}
\qquad
\begin{tikzcd}
R[f] \arrow[r, "\gamma"'] \arrow[rr, bend left, "R(g)"] & S \arrow[r, tail, "m"'] & Z \times Z \\
X \arrow[u, shift right=2, "s_{-1}^f"] \arrow[u, shift left=2, "s_1^f"'] \arrow[rr, two heads, "g"'] & & Z \arrow[u, shift right=2, "{(1_Z,0)}"] \arrow[u, shift left=2, "{(0,1_Z)}"'] \arrow[ul, dotted, shift left] \arrow[ul, dotted, shift right]
\end{tikzcd}$$

The splitting s produces a pair of morphims $(s_{-1}^f, s_1^f) : X \rightrightarrows R[f]$ which commutes with the pair $((1_Z, 0), (0, 1_Z))$. We have to show that $R(g)$ is an extremal epimorphism, namely that the monomorphism m is an isomorphism. Since g is an extremal epimorphism, the commutative square induced by the pair $(s_{-1}^f, (1_Z, 0))$ produces a factorization of $(1_Z, 0)$ through m, while the commutative square induced by the pair $(s_1^f, (0, 1_Z))$ produces a factorization of $(0, 1_Z)$ through m. Since the pair $(1_Z, 0), (0, 1_Z)$ is jointly extremally epimorphic, the monomorphism m is an isomorphism, and $R(g)$ is an extremal epimorphism. The final assertion is then straightforward. $\square$

Let $U : \mathbb{F} \to \mathbb{E}$ be a conservative functor preserving pullbacks and extremal epimorphisms. The category $\mathbb{F}$ is an extremal (resp. hyperextremal) category as soon as so is $\mathbb{E}$. In particular these notions are stable under passage to slice categories $\mathbb{E}/Y$, coslice categories $Y/\mathbb{E}$ and fibres $Pt_Y(\mathbb{E})$. When moreover $\mathbb{F}$ is pointed and U preserves the terminal object, the category $\mathbb{F}$ is punctually extremal as soon as so is $\mathbb{E}$.

1.4 Split reflexive relations

Definition 1.3. *A reflexive relation R in $\mathbb{E}$ is said to be split (or normalized in [2]) when, in addition, there is a map s_1^R:*

$$\begin{tikzcd}
R \arrow[r, shift left=2, "d_0^R"] \arrow[r, shift right=2, "d_1^R"'] & X \arrow[l, "s_0^R" description] \arrow[l, bend left=60, "s_1^R"]
\end{tikzcd}$$

such that $d_1^R.s_1^R = 1_X$ and the map $d_0^R.s_1^R$ coequalizes the pair (d_0^R, d_1^R), namely: $d_0^R.s_1^R.d_0^R = d_0^R.s_1^R.d_1^R$.

This implies that the endomorphsim $d_0^R.s_1^R$ is an idempotent. It is clear that when a map f is split by s, the kernel equivalence relation $R[f]$ is split by a map which we shall denote by s_1^f. In the set theoretical context, the map s_1^R imposes the choice of a specific element in each equivalence classe.

Lemma 1.8. *When an equivalence relation R on X in $\mathbb{E}$ is split by s_1^R, then we get $R \simeq R[d_0^R.s_1^R]$.*

Proof. The fact that $d_0^R.s_1^R$ coequalizes the pair (d_0^R, d_1^R) implies that we have $R \subset R[d_0^R.s_1^R]$, while the following commutative diagram:

$$\begin{tikzcd}
R[d_0^R.s_1^R] \arrow[r, "R(s_1^R)"] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & R[d_0^R] \arrow[r, "d_2^R"] \arrow[d, shift right=2, "d_0^R"'] \arrow[d, shift left=2, "d_1^R"] & R \arrow[d, shift right=2, "d_0^R"'] \arrow[d, shift left=2, "d_1^R"] \\
X \arrow[u] \arrow[r, "s_1^R"'] & R \arrow[u] \arrow[r, "d_0^R"'] & X \arrow[u]
\end{tikzcd}$$

shows that we have $R[d_0^R.s_1^R] \subset R$. $\square$

We shall need later on the following Lemma 2.2 from [4]:

Lemma 1.9. *Consider any vertical fibrant morphism of equivalence relations in $\mathbb{E}$ where f is split by s:*

$$\begin{tikzcd}
S \arrow[r, shift left=2, "d_0^S"] \arrow[r, shift right=2, "d_1^S"'] \arrow[d, "\hat{g}"'] & U \arrow[l] \arrow[l, dotted, bend left=60, "s_1^S"] \arrow[r, dotted, shift left, "\phi"] \arrow[d, "g"] & V \arrow[l, dotted, shift left, "\sigma"] \arrow[d, dotted, "\gamma"] \\
R[f] \arrow[r, shift left=2, "d_0^f"] \arrow[r, shift right=2, "d_1^f"'] & X \arrow[l] \arrow[l, bend left=60, "s_1^f"] \arrow[r, shift left, "f"] & Y \arrow[l, shift left, "s"]
\end{tikzcd}$$

Then the unique section s_1^S of d_1^S above s_1^f makes S a split equivalence relation on U. Moreover the split epimorphism (ϕ, σ) associated with the idempotent $d_0^S.s_1^S$ produces the right hand side leftward pullback.

We shall need also:

Proposition 1.2. *Consider any extremal epimorphism (h, g) in $Pt(\mathbb{E})$:*

$$\begin{tikzcd}
R[f] \arrow[r, "R(g)"] \arrow[d, shift right=2, "d_0^f"'] \arrow[d, shift left=2, "d_1^f"] & R[f'] \arrow[d, shift right=2, "d_0^{f'}"'] \arrow[d, shift left=2, "d_1^{f'}"] \\
X \arrow[u] \arrow[u, bend right=60, "s_1^f"'] \arrow[r, "g"] \arrow[d, shift left, "f"] & X' \arrow[u] \arrow[u, bend right=60, "s_1^{f'}"'] \arrow[d, shift left, "f'"] \\
Y \arrow[u, shift left, "s"] \arrow[r, "h"'] & Y' \arrow[u, shift left, "s'"]
\end{tikzcd}$$

then $(g, R(g)) : R[f] \to R[f']$ is an extremal epimorphism in $Equ\mathbb{E}$. Consider any hyperextremal epimorphism $(g, \hat{g}) : R[f] \to S$ in $Ref\mathbb{E}$:

$$\begin{tikzcd}
R[f] \arrow[r, "\hat{g}"] \arrow[d, shift right=2, "d_0^f"'] \arrow[d, shift left=2, "d_1^f"] & S \arrow[d, shift right=2, "d_0^S"'] \arrow[d, shift left=2, "d_1^S"] \\
X \arrow[r, "g"] \arrow[u] & X' \arrow[u]
\end{tikzcd}$$

then we get $S \simeq R[f']$.

Proof. Suppose g is an extremal epimorphism in $\mathbb{E}$. Consider now any decomposition of $(g, R(g))$ in $Equ\mathbb{E}$, where m is a monomorphism:

$$\begin{tikzcd}
R[f] \arrow[r, "\hat{g}"] \arrow[d, shift right=2, "d_0^f"'] \arrow[d, shift left=2, "d_1^f"] & S \arrow[r, tail, "m"] \arrow[d, shift right=2, "d_0^S"'] \arrow[d, shift left=2, "d_1^S"] & R[f'] \arrow[dl, shift right=1] \arrow[dl, shift left=1] \\
X \arrow[r, "g"] \arrow[u] \arrow[u, bend right=60, "s_1^f"'] \arrow[d, shift left=1, "f"] & X' \arrow[u] \arrow[ur] \arrow[ur, bend right=20, "s_1^{f'}"'] \arrow[d, shift left=1, "f'"] & \\
Y \arrow[r, "h"'] \arrow[u, shift left=1, "s"] & Y' \arrow[u, shift left=1, "s'"] &
\end{tikzcd}$$

Since g is an extremal epimorphism in $\mathbb{E}$, the splittings s_1^f and $s_1^{f'}$ determined by s and s' produce a splitting s_1^S of the equivalence relation S such that $m.s_1^S = s_1^{f'}$ and consequently $d_0^S.s_1^S = s'.f'$. Now, since S is a split equivalence relation, we have: $S = R[d_0^S.s_1^S] \simeq R[s'.f'] = R[f']$.

Given any hyperextremal epimorphism $(g, \hat{g}) : R[f] \to S$ in $Ref\mathbb{E}$, it is cocartesian above g by Lemma 1.3, and thus, up to isomorphism, equal to $(g, R(g))$. So we get $S \simeq R[f']$. $\square$

Not only any Mal'tsev category is extremal, but we get also the following:

Corollary 1.1. *Any Mal'tsev category $\mathbb{E}$ is hyperextremal.*

Proof. Consider any decomposition $R(g) = \gamma.m$ with a monomorphism m in $\mathbb{E}$:

$$\begin{tikzcd}
R[f] \arrow[r, "\gamma"] \arrow[d, shift right=2, "d_0^f"'] \arrow[d, shift left=2, "d_1^f"] & S \arrow[r, tail, "m"] \arrow[d, shift right=2, "d_0^S"'] \arrow[d, shift left=2, "d_1^S"] & R[f'] \arrow[dl, shift right=1] \arrow[dl, shift left=1] \\
X \arrow[r, "g"] \arrow[u] \arrow[u, bend right=60, "s_1^f"'] \arrow[d, shift left=1, "f"] & X' \arrow[u] \arrow[ur] \arrow[ur, bend right=20, "s_1^{f'}"'] \arrow[d, shift left=1, "f'"] & \\
Y \arrow[r, "h"'] \arrow[u, shift left=1, "s"] & Y' \arrow[u, shift left=1, "s'"] &
\end{tikzcd}$$

When g is an extremal epimorphism, this produces a reflexive relation S, which is an equivalence relation since $\mathbb{E}$ is a Mal'tsev category. Now since $(g, R(g))$ is extremal in $Equ\mathbb{E}$ by the previous proposition, then m is an isomorphism, and $R(g)$ is extremal in $\mathbb{E}$. $\square$

1.5 Inverse image along split epimorphims

Theorem 1.2. *Given any split epimorphism $(f,s) : X \rightleftarrows Y$, the inverse image $f^{-1} : Equ_Y\mathbb{E} \to Equ_X\mathbb{E}$ induces a preorder bijection between the equivalence relations on Y and the equivalence relations on X containing $R[f]$. It inverse mapping is given by the restriction of s^{-1}.*

Proof. It is clear that f^{-1} takes values among the equivalence relations on X containing $R[f]$ and is a preorder homomorphism. Now if S is an equivalence relation on X containing $R[f]$, consider the following diagram:

$$\begin{tikzcd}
R|[f]\square S \arrow[r, shift left=2, "\delta_0^S"] \arrow[r, shift right=2, "\delta_1^S"'] \arrow[d] & S \arrow[l] \arrow[l, dotted, bend left=50, "s_1^S"] \arrow[r, dotted, shift left, "\phi"] \arrow[d] & s^{-1}(S) \arrow[l, dotted, shift left, "\sigma"] \arrow[d, dotted] \\
R[f]\times R[f] \arrow[r, shift left=2, "d_0^f\times d_0^f"] \arrow[r, shift right=2, "d_1^f\times d_1^f"'] & X\times X \arrow[l] \arrow[l, bend left=50, "s_1^f\times s_1^f"] \arrow[r, shift left, "f\times f"] & Y\times Y \arrow[l, shift left, "s\times s"]
\end{tikzcd}$$

Its left hand side part is a fibrant morphism of equivalence relations by Lemma 1.6. According to Lemma 1.9, it produces the right hand side pullback. $\square$

2 Regular categories

2.1 Basic facts

A category $\mathbb{E}$ is regular [1] when the regular epimorphisms are stable under pullbacks and any effective equivalence relation has a coequalizer. In this case regular epimorphisms coincide with extremal epimorphisms. The direct image along a regular epimorphism $f : X \twoheadrightarrow Y$ of an equivalence relation S on X is given by the codomain of the coequalizer $\bar{f}$ in $\mathbb{E}$ of the left hand side part of the following upper row which, as we noticed, is an effective equivalence relation in the category $\mathbb{E}$:

$$\begin{tikzcd}
R[f]\square S \arrow[r, shift left=2, "\delta_1^S"] \arrow[r, shift right=2, "\delta_0^S"'] \arrow[d, tail, "(\delta_0^f,\delta_1^f)"'] & S \arrow[l] \arrow[r, dotted, two heads, "\bar{f}"] \arrow[d, tail, "(d_0^S,d_1^S)"] & f(S) \arrow[d, tail] \\
R[f]\times R[f] \arrow[r, shift left=2, "d_1^f\times d_1^f"] \arrow[r, shift right=2, "d_0^f\times d_0^f"'] & X\times X \arrow[l] \arrow[r, two heads, "f\times f"'] & Y\times Y
\end{tikzcd}$$

in general it only gives rise to a reflexive and symmetric relation $f(S)$ on Y. The following proposition is very important; the point (v) and the corollary are classical, but, here, we do not use Metatheorems and introduce very straighforward categorical proofs:

Proposition 2.1. *Let $\mathbb{E}$ be a regular category:*
(i) any hyperextremal epimorphism in $Pt(\mathbb{E})$ is hyperregular
(ii) any cartesian extremal epimorphism in $Pt(\mathbb{E})$ is hyperextremal
(iii) the hyperextremal epimorphisms are stable under pullbacks in the categories $Pt(\mathbb{E})$ and $Ref\mathbb{E}$
(iv) if f is a regular epimorphism in $\mathbb{E}$, the following morphism in $Pt(\mathbb{E})$:

$$\begin{tikzcd} X \times X \arrow[r, two heads, "f\times f"] \arrow[d, shift left, "p_0^X"] & Y \times Y \arrow[d, shift left, "p_0^Y"] \\ X \arrow[r, two heads, "f"'] \arrow[u, shift left, "s_0^X"] & Y \arrow[u, shift left, "s_0^Y"] \end{tikzcd}$$

is a hyperextremal epimorphism; accordingly ∇_f is a hyperextremal epimorphism in $Ref\mathbb{E}$ and any cartesian extremal epimorphism in $Ref\mathbb{E}$ is hyperextremal
(v) the inverse image f^{-1} of reflexive relations along a regular epimorphism $f : X \twoheadrightarrow Y$ reflects the equivalence relations.

Proof. The first point is sraightforward. The second and third ones are direct consequences of the fact that pullbacks in $Pt(\mathbb{E})$ and $Ref\mathbb{E}$ are levelwise and that extremal (=regular) epimorphisms are stable under pullbacks. The fourth one is a direct consequence of the fact that regular epimorphisms are stable under products.

As for the fifth point: given any reflexive relation T on Y, the inverse image along the regular epimorphism f is given by the following pullback in $Ref\mathbb{E}$:

$$\begin{tikzcd} f^{-1}(T) \arrow[r, "\tilde{f}^T"] \arrow[d, tail, "(d_0,d_1)"'] & T \arrow[d, tail, "(d_0^T,d_1^T)"] \\ \nabla_X \arrow[r, "\nabla_f"'] & \nabla_Y \end{tikzcd}$$

According to the point (iv), the upper horizontal morphism is a hyperextremal epimorphism in $Ref\mathbb{E}$; by Lemma 1.7, the reflexive relation T is then an equivalence relation as soon as so is $f^{-1}(T)$. Whence (v). $\square$

Corollary 2.1. *Let $\mathbb{E}$ be a regular category, $f : X \twoheadrightarrow Y$ a regular epimorphism and S any equivalence relation on X. Then we have $R[f] \subset S$ if and only if $S = f^{-1}(f(S))$. In this case, the reflexive relation $f(S)$ is actually an equivalence relation. In other words, when f is a regular epimorphism, the mapping f^{-1} induces a preorder bijection between the equivalence relations on Y and the equivalence relations on X containing $R[f]$.*

Proof. Consider the following diagram:

$$
\begin{tikzcd}
R[f]\square S \arrow[r, shift left=2, "\delta_1^S"] \arrow[r, shift right=2, "\delta_0^S"'] \arrow[d, tail, "{(\delta_0^f,\delta_1^f)}"'] & S \arrow[l] \arrow[r, two heads, "\bar{f}"] \arrow[d, tail, "{(d_0^S,d_1^S)}"] & f(S) \arrow[d, tail] \\
R[f]\times R[f] \arrow[r, shift left=2, "d_1^f\times d_1^f"] \arrow[r, shift right=2, "d_0^f\times d_0^f"'] & X\times X \arrow[l] \arrow[r, two heads, "f\times f"'] & Y\times Y
\end{tikzcd}
$$

By Lemma 1.6, we know that we have $R[f]\subset S$ if and only if the left hand side morphism of equivalence relations is fibrant. By the Barr-Kock theorem, since $\bar{f}$ is a regular epimorphism, this is the case if and only if the right hand side square is a pullback, i.e. if and only if $S = f^{-1}(f(S))$. The last assertion is a consequence of the point (v) of the previous proposition. $\square$

We shall close this section by the following:

Proposition 2.2. *Let $\mathbb{E}$ be a regular category, $f : X \twoheadrightarrow Y$ a regular epimorphism and (T, S) any pair of equivalence relations on X such that $R[f]\subset T$. Then we have $f(T)\cap f(S) = f(T\cap S)$.*

Proof. Consider the following diagram:

$$
\begin{tikzcd}
T\cap S \arrow[r, tail] \arrow[d, tail] & T \arrow[r, two heads, "\bar{f}^T"] \arrow[d, tail] & f(T) \arrow[d, tail] \\
S \arrow[r, tail] & X\times X \arrow[r, two heads, "f\times f"'] & Y\times Y
\end{tikzcd}
$$

where the left hand side square is a pullback by definition and the right hand side one since we have $R[f]\subset T$. The following whole rectangle is the same as the previous one and consequently is a pullback:

$$
\begin{tikzcd}
T\cap S \arrow[r, two heads, "\bar{f}^{T\cap S}"] \arrow[d, tail] & f(T\cap S) \arrow[r, tail] \arrow[d, tail] & f(T) \arrow[d, tail] \\
S \arrow[r, two heads, "\bar{f}^S"'] & f(S) \arrow[r, tail] & Y\times Y
\end{tikzcd}
$$

Since the upper right hand side horizontal map is a monomorphism, the left hand side square is a pullback as well. Since it has horizontal regular epimorphisms and $\mathbb{E}$ is regular, the right hand side square is a pullback as well, which means that $f(T\cap S) = f(T)\cap f(S)$. $\square$

2.2 Regular extremal categories

Proposition 2.3. *Let $\mathbb{E}$ be a regular category. TFAE:*
(i) $\mathbb{E}$ is extremal
(ii) $\mathbb{E}$ is hyperextremal
(iii) the direct image of any equivalence relation is an equivalence relation
(iv) $Equ\mathbb{E}$ is a regular category in which the regular epimorphisms are levelwise regular epimorphisms in the category $\mathbb{E}$.

Proof. Let $\mathbb{E}$ be a regular category. In the regular context, extremal (resp. hyperextremal) epimorphisms coincide with regular (resp. hyperregular) epimorphisms. Now, suppose (i) and consider any regular epimorphism in $Pt(\mathbb{E})$ as in the left hand side diagram:

$$\begin{tikzcd}
X \arrow[r, two heads, "g"] \arrow[d, shift right, "f"'] & X' \arrow[d, shift right, "f'"'] \\
Y \arrow[r, two heads, "h"'] \arrow[u, shift right, "s"'] & Y' \arrow[u, shift right, "s'"']
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, two heads, "\bar{d}_0"] \arrow[drr, dotted, two heads, "\bar{d}_1"'] \arrow[ddr, "\phi"'] & X \arrow[drr, "g"] \arrow[ddr, "f"] & & \\
& & \bar{X} \arrow[r, two heads, "\bar{h}"] \arrow[d, shift right, "\bar{f}"'] & X' \arrow[d, shift right, "f'"'] \\
& R[h] \arrow[r, shift left=2, "d_0^h"] \arrow[r, shift right=2, "d_1^h"'] & Y \arrow[l] \arrow[r, two heads, "h"'] \arrow[u, shift right, "\bar{s}"'] & Y' \arrow[u, shift right, "s'"']
\end{tikzcd}$$

Then complete the diagram by the pullbacks $(\bar{f}, \bar{s})$ of (f', s') along h and (ϕ, σ) of (f, s) along d_0^h. There is the dotted factorization $\bar{d}_1$ making the upper quadrangle a pullback, and $\bar{d}_1$ a regular epimorphism since so is g. Since the category $\mathbb{E}$ is extremal and $(d_1^h, \bar{d}_1)$ is a regular epimorphism in $Pt(\mathbb{E})$ above the split epimorphism d_1^h, it is hyperregular. The right hand side pullback produces a cartesian hyperregular epimorphism $(h, \bar{h})$. So, the morphism $(h, g).(d_0^h, \bar{d}_0) = (h, \bar{h}).(d_1^h, \bar{d}_1)$ is hyperregular as a composition of hyperregular epimorphisms; accordingly so is (h, g) and we get (ii). Since (i) is a particular case of (ii), we get [(i) $\iff$ (ii)]. Now suppose (ii). The direct image of the equivalence relation R along the regular epimorphism f:

$$\begin{tikzcd}
R \arrow[r, two heads, "\tilde{f}"] \arrow[d, shift right=2, "d_0^R"'] \arrow[d, dotted, shift left=2, "d_1^R"] & f(R) \arrow[d, shift right=2, "d_0"'] \arrow[d, dotted, shift left=2, "d_1"] \\
X \arrow[r, two heads, "f"'] \arrow[u] & Y \arrow[u]
\end{tikzcd}$$

produces a regular epimorphism in $Pt(\mathbb{E})$ and, by (ii), a hyperregular epimorphism in $Ref\mathbb{E}$. According to Lemma 1.7, since R is an equivalence relation, so is the reflexive relation $f(R)$; whence (iii). Now suppose (iii); it is clear that the regular epimorphisms in $Equ\mathbb{E}$ are levelwise in $\mathbb{E}$, and $\mathbb{E}$ being regular, so is $Equ\mathbb{E}$. Whence (iv). Now suppose (iv); consider any regular epimorphism (h, g)

in $Pt(\mathbb{E})$ as above. We know by Proposition 1.2, that $(g, R(g)) : R[f] \to R[f']$ is an extremal (=regular) epimorphism in $Equ\mathbb{E}$. From (iv) we deduce that $R(g)$ is a regular epimorphism in $\mathbb{E}$; whence (ii). $\square$

This result arises the question to know when, more generally, the category $Equ\mathbb{E}$ is a regular category. We shall answer it in Section 3.6.

3 Aspects of the modular law

3.1 Suprema of pairs of equivalence relations

We shall begin by the two following very general observations which show that, in some relatively simple circumstances, there are suprema of pairs of equivalence relations:

Proposition 3.1. *Consider any commutative square of split epimorphisms in a category $\mathbb{E}$:*

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, shift left, "f"] & X' \arrow[d, shift left, "f'"] \\ Y \arrow[u, shift left, "s"] \arrow[r, "h"'] & Y' \arrow[u, shift left, "s'"] \end{tikzcd}$$

then the equivalence relation $g^{-1}(R[f'])$ is the supremum $R[g] \bigvee R[f]$ of the equivalence relations $R[f]$ and $R[g]$.

Proof. By the Yoneda embedding, it is enough to prove it in Set. We get $R[f] \subset g^{-1}(R[f'])$ and $R[g] \subset g^{-1}(R[f'])$ from $g^{-1}(R[f']) = R[f'.g] = R[h.f]$. We have moreover:

$$xg^{-1}(R[f'])x' \iff f'g(x) = f'g(x') \iff s'f'g(x) = s'f'g(x')$$

So that we get: $xg^{-1}(R[f'])x' \iff gsf(x) = gsf(x')$. From the following left hand side diagram:

$$\begin{tikzcd} x \arrow[d, dotted, "R[f]"'] & x' \arrow[d, dotted, "R[f]"] \\ sf(x) \arrow[r, dotted, "R[g]"'] & sf(x') \end{tikzcd} \qquad \begin{tikzcd} x \arrow[d, dotted, "T"'] & x' \arrow[d, dotted, "T"] \\ sf(x) \arrow[r, dotted, "T"'] & sf(x') \end{tikzcd}$$

we get the right hand side one as soon as T contains $R[f]$ and $R[g]$. Since T is an equivalence relation, we get xTx'. $\square$

Proposition 3.2. *A map $(g, \hat{g}) : R \to T$ is an extremal epimorphism in $Equ\mathbb{E}$ above the split epimorphism $(g, t) : X \rightleftarrows Z$ in $\mathbb{E}$ (and thus a regular epimorphism in $Equ\mathbb{E}$) if and only if we have $g^{-1}(T) = R[g] \bigvee R$.*

Proof. Being an extremal epimorphism above the split epimorphism g is equivalent to being cocartesian above g by Lemma 1.3. Accordingly $(g, \hat{g})$ is characterized by the fact that, for any other morphism $(g, \bar{g}) : R \to S$ above g, we have $T \subset S$, or equivalently by the fact that, for any equivalence relation S on Z, we have:

$$R \subset g^{-1}(S) \iff T \subset S \iff g^{-1}(T) \subset g^{-1}(S)$$

the second equivalence being given by Theorem 1.2. Now, the bijection of this same theorem characterizes T by: for any equivalence U above X containing $R[g]$, we have: $R \subset U \iff g^{-1}(T) \subset U$ or, in other words, by $g^{-1}(T) = R[g] \bigvee R$. $\square$

Here is now the main observation of the article:

Theorem 3.1. *Let (R, S) be any pair of equivalence relations on X in a category $\mathbb{E}$. Suppose that T is another equivalence relation T on X which, in addition, contains S. We have $R \subset T$ if and only if we have $(d_0^R)^{-1}(S) \subset (d_1^R)^{-1}(T)$ or, equivalently, if and only if there is a morphism $(d_1^R, \bar{d}_1^T) : (d_0^R)^{-1}(S) \to T$ in $Equ\mathbb{E}$ making the following diagram commute:*

$$\begin{tikzcd}
R\square S \arrow[r, shift left=2, "\delta_1^S"] \arrow[r, dotted, shift right=2, "\delta_0^S"'] \arrow[d, tail, "\xi_R^S"'] & S \arrow[l, dotted, "\sigma_0^S"description] \arrow[d, tail, "i"] \\
(d_0^R)^{-1}(S) \arrow[r, "\bar{d}_1^T"'] & T
\end{tikzcd}$$

which, then, is necessarily a pullback. Given any equivalence relation T on X containing R and S, we have $T = R \bigvee S$ if and only if the previous commutative diagram provides us with a pushout in $Equ\mathbb{E}$.

Proof. By the Yoneda embedding it is enough to check the first assertion in *Set*. The objects of $(d_0^R)^{-1}(S)$ are given by the left hand side diagrams while the objects of $(d_1^R)^{-1}(T)$ are given by the right hand side diagrams:

$$\begin{tikzcd}
x \arrow[r, dotted, "S"] \arrow[d, dotted, "R"'] & y \arrow[d, dotted, "R"] \\
x' & y'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
x \arrow[d, dotted, "R"'] & x' \arrow[d, dotted, "R"] \\
x' \arrow[r, dotted, "T"'] & y'
\end{tikzcd}$$

It is clear that if we have $S \subset T$ and $R \subset T$, the left hand side diagram implies $x'Ty'$ and the validity of the right hand side diagram. Conversely if we have the inclusion $(d_0^R)^{-1}(S) \subset (d_1^R)^{-1}(T)$, the following left hand side diagram, drawn from xRy, implies the right hand side one which implies that $R \subset T$:

$$\begin{tikzcd}
x \arrow[r, dotted, "S"] \arrow[d, dotted, "R"'] & x \arrow[d, dotted, "R"] \\
x & y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
x \arrow[d, dotted, "R"'] & x \arrow[d, dotted, "R"] \\
x \arrow[r, dotted, "T"'] & y
\end{tikzcd}$$

Now, saying $(d_0^R)^{-1}(S) \subset (d_1^R)^{-1}(T)$ is equivalent to the existence of a morphism $\bar{d}_1^T : (d_1^R)^{-1}(S) \to T$ above d_1^R, which, in turn, is equivalent to the commutativity in $Equ\mathbb{E}$ of the diagram in question. It fits inside the following commutative diagram:

$$\begin{tikzcd}
R\Box S \arrow[r, shift left, "\delta_1^S"] \arrow[d, tail, "\xi_R^S"'] & S \arrow[l, dotted, shift left, "\sigma_0^S"] \arrow[d, tail, "i"] \arrow[dr] & \\
(d_0^R)^{-1}(S) \arrow[r, "\bar{d}_1^T"'] & T \arrow[r, tail, "(d_0^T, d_1^T)"'] & X \times X
\end{tikzcd}$$

The quadrangle is a pullback by Lemma 1.5. Its squared part is a pullback as well because of the factorization (d_0^T, d_1^T).

Let us show that $T = R \bigvee S$ if and only if this square is a pushout. If T' is another equivalence relation containing R and S, we get another commutative diagram:

$$\begin{tikzcd}
R\Box S \arrow[r, shift left=2, "\delta_1^S"] \arrow[r, dotted, shift right=2, "\delta_0^S"'] \arrow[d, tail, "\xi_R^S"'] & S \arrow[l, dotted] \arrow[d, tail, "i'"] \\
(d_1^R)^{-1}(S) \arrow[r, "\bar{d}_1^{T'}"'] & T'
\end{tikzcd}$$

If the diagram with T provides us with a pushout in $Equ\mathbb{E}$, we get the inclusion $T \subset T'$ and $T = R \bigvee S$. Conversely if we have $T = R \bigvee S$, we get the inclusion $T \subset T'$ which shows that the square with T provides us with a pushout in the category $Equ\mathbb{E}$. $\square$

Corollary 3.1. *Let $\mathbb{E}$ be a category and (R, S) any pair of equivalence relations on X. Given any other equivalence relation T on X, the following condition are equivalent:*
(i) T is the supremum of R and S
(ii) there is a regular epimorphism $(d_1^R, \bar{d}_1^T)$ in $Equ\mathbb{E}$ above d_1^R:

$$\begin{tikzcd}
(d_0^R)^{-1}(S) \arrow[r, "\bar{d}_1^T"] \arrow[d, shift right=2, "\bar{\delta}_0^R"'] \arrow[d, shift left=2, "\bar{\delta}_1^R"] & T \arrow[d, shift right=2, "d_0^T"'] \arrow[d, shift left=2, "d_1^T"] \\
R \arrow[r, "d_1^R"'] \arrow[u] & X \arrow[u]
\end{tikzcd}$$

In this case there is also a regular epimorphism in $Equ\mathbb{E}$ above d_1^S:

$$\begin{tikzcd}
(d_0^S)^{-1}(R) \arrow[r, "\bar{\bar{d}}_1^T"] \arrow[d, shift right=2, "\bar{\delta}_0^S"'] \arrow[d, shift left=2, "\bar{\delta}_1^S"] & T \arrow[d, shift right=2, "d_0^T"'] \arrow[d, shift left=2, "d_1^T"] \\
S \arrow[r, "d_1^S"'] \arrow[u] & X \arrow[u]
\end{tikzcd}$$

Proof. According to the previous theorem, Condition (i) is equivalent to the fact that the following diagram provides us with a pushout in $Equ\mathbb{E}$:

$$\begin{tikzcd}
R\Box S \arrow[r, shift left=2, "\delta_1^S"] \arrow[r, dotted, shift right=2, "\delta_0^S"'] \arrow[d, tail, "\xi_R^S"'] & S \arrow[l, dotted] \arrow[d, tail, "i"] \\
(d_1^R)^{-1}(S) \arrow[r, "\overline{d_1^T}"'] & T
\end{tikzcd}$$

Since δ_1^S is a split epimorphism in $Equ\mathbb{E}$, and thus a regular epimorphism, this square provides us with a pushout in $Equ\mathbb{E}$ if and only if $(d_1^R, \overline{d_1^T})$ is a regular epimorphism as well, according to Theorem 1.1. The last assertion comes from the symmetry of $R \bigvee S$. $\square$

3.2 Existence of suprema for pairs of equivalence relations

Proposition 3.3. *The category $\mathbb{E}$ is such that any pair (R, S) of equivalence relations has a supremum $R \bigvee S$ if and only if $Equ\mathbb{E}$ has regular epimorphism-swith any domain above split epimorphisms in $\mathbb{E}$. Under these assumptions, a morphism of equivalence relation is regular in $Equ\mathbb{E}$:*

$$\begin{tikzcd}
S \arrow[r, "\hat{f}"] \arrow[d, shift right=2, "d_0^S"'] \arrow[d, shift left=2, "d_1^S"] & V \arrow[d, shift right=2, "d_0^V"'] \arrow[d, shift left=2, "d_1^V"] \\
X \arrow[u] \arrow[r, shift right, "f"'] & Y \arrow[u] \arrow[l, shift right, "s"']
\end{tikzcd}$$

if and only if we have $f^{-1}(V) = R[f] \bigvee S$.

Proof. By Corollary 3.1, if $Equ\mathbb{E}$ has regular epimorphisms with any domain above split epimorphism, then the regular epimorphism with domain $(d_0^R)^{-1}(S)$ above the split epimorphism (d_1^R, s_0^R) provides us with $R \bigvee S$. Conversely suppose that $\mathbb{E}$ is such that any pair (R, S) of equivalence relations has a supremum $R \bigvee S$. Let $(f, s) : X \rightleftarrows Y$ be a split epimorphism in $\mathbb{E}$ and S an equivalence relation on X. Then $R[f] \subset R[f] \bigvee S$, so that, by Theorem 1.2 we know that there is a split epimorphism $R[f] \bigvee S \to s^{-1}(R[f] \bigvee S)$ in $Equ\mathbb{E}$ above (f, s). Then the morphism $S \rightarrowtail R[f] \bigvee S \rightleftarrows s^{-1}(R \bigvee S)$ is the regular epimorphism above f with codomain S; indeed, given any other map $(f, \hat{f}) : S \to T$, we get $S \subset f^{-1}(T)$ and $R[f] \subset f^{-1}(T)$, whence $R[f] \bigvee S \subset f^{-1}(T)$ and $s^{-1}(R[f] \bigvee S) \subset s^{-1}(f^{-1}(T)) = T$ which produces the desired factorization. $\square$

The following proposition will provide us with examples of categories in which $Equ\mathbb{E}$ has regular epimorphisms with any domain above split epimorphisms.

Proposition 3.4. *Suppose that $\mathbb{E}$ is such that the partial order associated with the preorder determined by any fibre $Equ_Y\mathbb{E}$ has infima. Then the fibration $\mathcal{O}_{\mathbb{E}} : Equ\mathbb{E} \to \mathbb{E}$ is a cofibration as well. Accordingly it has regular epimorphisms with any domain above regular epimorphisms and a fortiori above split epimorphisms.*

Proof. Let $f : X \to Y$ be any morphism in $\mathbb{E}$ and S an equivalence relation on X. Then the cocartesian map with domain S above it is determined by the infimum W of the equivalence relations V on Y such that $S \subset f^{-1}(V)$. Indeed, by commutation of limits, the infima are preserved by inverse image, and we get $S \subset f^{-1}(W)$, whence a map $(f, f^\sharp) : S \to W$ above f. Let us show that it is the cocartesian map above f. Since $\mathcal{O}_{\mathbb{E}}$ is fibration, it is enough to prove the universal property for the maps above f. Given any other map $(f, \hat{f}) : S \to T$, we get $S \subset f^{-1}(T)$, and then we have $W \subset T$ which produces the desired factorization. The last assertion is a consequence of Lemma 1.3. $\square$

Any variety $\mathbb{V}$ of Universal Algebra obviously satisfies the condition of the previous proposition.

3.3 The modular formula

Recall that the *modular formula* for equivalence relations is: given any triple (R, S, T) of equivalence relations on X, then we get $(R \bigvee S) \cap T = R \bigvee (S \cap T)$ provided that we have $R \subset T$.

Proposition 3.5. *Suppose that $Equ\mathbb{E}$ has regular epimorphisms with any domain above split epimorphisms in $\mathbb{E}$ (or equivalently that any pair (R, S) of equivalence relations has a supremum $R \bigvee S$). Then the modular formula holds if and only if these regular epimorphisms are stable under pullbacks along maps in the fibers of $\mathcal{O}_{\mathbb{E}}$.*

Proof. Suppose that the regular epimorphisms in question are stable under pullbacks along maps in the fibres and that we have $R \subset T$. Then consider the following commutative diagram:

$$\begin{tikzcd}
(d_0^R)^{-1}(S \cap T) \arrow[r, dotted, "\bar{d}_1"] \arrow[d, tail, "(d_0^R)^{-1}()"'] \arrow[rr, bend left, "\bar{d}_1^T"] & (R \bigvee S) \cap T \arrow[r, tail] \arrow[d, tail] & T \arrow[d, tail, "(d_0^T, d_1^T)"] \\
(d_0^R)^{-1}(S) \arrow[r, "(d_1^R)^\sharp"'] \arrow[rr, bend right, "\check{d}_1^R"'] & R \bigvee S \arrow[r, tail] & X \times X
\end{tikzcd}$$

The right hand side square is a pullback with vertical maps in a fibre. The whole diagram is a pullback in any category $\mathbb{E}$: consider the following diagram:

$$\begin{tikzcd}
x \arrow[r, dotted, "R"] \arrow[d, dotted, "S"'] & y & & y \arrow[d, dotted, "T"] \\
x' \arrow[r, dotted, "R"'] & y' & & y'
\end{tikzcd}$$

since we have $R \subset T$, we get xTx' and consequently $x(T \cap S)x'$.

So there is a dotted factorization $\bar{d}_1$ making the left hand side square a pullback. Since $(d_1^R)^\sharp$ provides us with a regular epimorphism above d_1^R, so does this dotted factorization. Accordingly its codomain $(R \bigvee S) \cap T$ is $R \bigvee (S \cap T)$.

Conversely suppose the modular formula holds. Consider the following diagram where f is a split epimorphism and any square is a pullback:

$$\begin{tikzcd}
S' \arrow[r, tail] \arrow[d, tail, "j"'] & f^{-1}(U) \arrow[r, two heads, "\tilde{f}_U"] \arrow[d, tail, "f^{-1}(i)"'] & U \arrow[d, tail, "i"] \\
S \arrow[r, tail] \arrow[dr] & f^{-1}(V) \arrow[r, two heads, "\tilde{f}_V"] \arrow[d, tail] & V \arrow[d, tail] \\
 & X \times X \arrow[r, two heads, "f \times f"'] & Y \times Y
\end{tikzcd}$$

The map $S \to V$ is a regular epimorphism in $Equ\mathbb{E}$ if and only if we get $f^{-1}(V) = R[f] \bigvee S$. We have $f^{-1}(U) \subset f^{-1}(V) = R[f] \bigvee S$, and so we get:

$$f^{-1}(U) = (R[f] \bigvee S) \cap f^{-1}(U) = R[f] \bigvee (S \cap f^{-1}(U)) = R[f] \bigvee S'$$

the second identity by the modular law since $R[f] \subset f^{-1}(U)$, the third one since the left hand side upper square is a pullback. Accordingly the map $S' \to U$ is a regular epimorphism in $Equ\mathbb{E}$. $\square$

The previous characterization induces the following:

Definition 3.1. *A category $\mathbb{E}$ is said to be equi-modular when $Equ\mathbb{E}$ has regular epimorphisms with any domain above split epimorphisms in $\mathbb{E}$ and these regular epimorphisms are stable under pullbacks along maps in the fibers of $\mathcal{O}_{\mathbb{E}}$. It is said to be extremally equi-modular when, in addition, these regular epimorphisms in $Equ\mathbb{E}$ are levelwise extremal epimorphims in $\mathbb{E}$.*

Proposition 3.6. *Given any regular category $\mathbb{E}$, TFAE:*
(i) $\mathbb{E}$ is extremal
(ii) $\mathbb{E}$ is extremally equi-modular.

Proof. According to the point (iv) in Proposition 2.3, when $\mathbb{E}$ is regular and extremal, it is extremally equi-modular. Conversely suppose (ii). Consider any morphism in $Pt(\mathbb{E})$ with an extremal epimorphism g and a split epimorphism (h,t) as in the left hand side commutative diagram:

$$\begin{tikzcd}
X \arrow[r, two heads, "g"] \arrow[d, shift right, "f"'] & X' \arrow[d, shift right, "f'"'] \\
Y \arrow[u, shift right, "s"'] \arrow[r, shift left, "h"] & Y' \arrow[u, shift right, "s'"'] \arrow[l, shift left, "t"]
\end{tikzcd}
\qquad
\begin{tikzcd}
(d_0^g)^{-1}(R[f]) \arrow[r, "\bar{d}_0"] \arrow[ddr, shift right, "d_0"'] \arrow[ddr, shift left, "d_1"] \arrow[drr, dotted, "\bar{d}_1"] & R[f] \arrow[drr, "R(g)"] \arrow[ddr, shift right] \arrow[ddr, shift left] & & \\
 & & g^{-1}(R[f']) \arrow[r, two heads, "\gamma"] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & R[f'] \arrow[d, shift right, "d_0^{f'}"'] \arrow[d, shift left, "d_1^{f'}"] \\
 & R[g] \arrow[r, shift left, "d_0^g"] \arrow[r, shift right, "d_1^g"'] & X \arrow[l] \arrow[r, two heads, "g"] & X'
\end{tikzcd}$$

Complete the right hand side diagram by the equivalence relations $g^{-1}(R[f'])$ and $(d_0^g)^{-1}(R[f])$. There is the dotted factorization $\bar{d}_1$ making the upper quadrangle a pullback. On the one hand, since we have $g^{-1}(R[f']) = R[g] \bigvee R[f]$ by Proposition 3.1 and since $\mathbb{E}$ is extremally equi-modular, the factorization $\bar{d}_1$ is an extremal epimorphism. On the other hand since $\mathbb{E}$ is regular and g is an extremal epimorphism, so is the factorization γ. Now, $R(g).\bar{d}_0 = \gamma.\bar{d}_1$ is an extremal epimorphism, and so is $R(g)$. $\square$

3.4 The categorical shifting property

In [17], Gumm characterized the varieties of Universal Algebra satisfying the congruence formula by the validity of the *Shifting Lemma*: given any triple of equivalence relations (R,S,T) such that $S \cap T \subset R$ the following left hand side situation implies the right hand side one:

$$\begin{tikzcd}
x \arrow[r, dotted, "S"] \arrow[d, dotted, "T"'] \arrow[d, dotted, bend right=60, "R"'] & y \arrow[d, dotted, "T"] \\
x' \arrow[r, dotted, "S"'] & y'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
y \arrow[d, dotted, bend left=60, "R"] \\
y'
\end{tikzcd}$$

The categorical description of the shifting property was given in [6] and is the following one: given any triple of equivalence relations (R,S,T) on X such that $S \cap T \subset R \subset T$, the following morphism of equivalence relations is fibrant:

$$\begin{tikzcd}
S\square R \arrow[r, shift left, "\delta_1^R"] \arrow[r, shift right, "\delta_0^R"'] \arrow[d, tail, "S\square i"'] & R \arrow[l] \arrow[d, tail, "i"] \\
S\square T \arrow[r, shift left, "\delta_1^T"] \arrow[r, shift right, "\delta_0^T"'] & T \arrow[l]
\end{tikzcd}$$

Definition 3.2. *[4][7] A category $\mathbb{E}$ is said to be a Gumm category when the categorical shifting property holds.*

In [4], it is shown that the regular Mal'tsev categories and the exact category $\mathbb{E}$ with coequalizers of reflexive pairs in which the modular formula holds are Gumm categories. In [7], it was noticed that, since the conservative functors which preserve pullbacks reflect them as well, given any conservative functor $U : \mathbb{F} \to \mathbb{E}$ preserving pullbacks, the category $\mathbb{F}$ is a Gumm category as soon as so is $\mathbb{E}$. In particular it was noticed that this notion is stable under passage to slice categories $\mathbb{E}/Y$, coslice categories $Y/\mathbb{E}$ and fibres $Pt_Y(\mathbb{E})$. In this section, we shall enlarge a bit the range of examples of Gumm categories.

First, we can add the following important observation. Let $\mathbb{T}$ be an algebraic theory in the sense of Universal algebra and $\mathbb{V}(\mathbb{T})$ the corresponding variety of $\mathbb{T}$-algebras. Then, since the pullbacks in $\mathcal{F}(\mathbb{E}, \mathbb{V}(\mathbb{T}))$ are componentwise, any functor category $\mathcal{F}(\mathbb{E}, \mathbb{V}(\mathbb{T}))$ is a Gumm category as soon as $\mathbb{V}(\mathbb{T})$ is a congruence modular variety. Now let $\mathbb{T}(\mathbb{E})$ be the category of internal $\mathbb{T}$-algebras in $\mathbb{E}$. Then the corresponding Yoneda embedding $Y^{\mathbb{T}} : \mathbb{T}(\mathbb{E}) \to \mathcal{F}(\mathbb{E}^{op}, \mathbb{V}(\mathbb{T}))$ being left exact and conservative, the category $\mathbb{T}(\mathbb{E})$ is a Gumm category as soon as $\mathbb{V}(\mathbb{T})$ is a congruence modular variety.

Then, we shall need the following categorical description of the modular formula:

Lemma 3.1. *Suppose that $Equ\mathbb{E}$ has supremum of equivalence relations. The modudar formula holds if and only if, considering any commutative diagram in the fibre $Equ_X\mathbb{E}$ where the whole rectangle is a pullback:*

$$\begin{tikzcd}
S \cap T \arrow[r, tail, dotted] \arrow[d, tail] \arrow[rr, bend left] & R \arrow[r, tail] \arrow[d, tail] & T \arrow[d, tail] \\
S \arrow[r, tail] & R \bigvee S \arrow[r, tail] & T \bigvee S
\end{tikzcd}$$

the right hand side square is a pullback if and only if there is a dotted factorization.

Proof. Suppose the modular law holds. We get:

$$(R \bigvee S) \cap T = R \bigvee (S \cap T) = R$$

when we have $S \cap T \subset R$. Conversely suppose that we have $R \subset T$ and consider the following commutative diagram in $Equ_X\mathbb{E}$:

$$\begin{tikzcd}
S \cap T \arrow[r, tail] \arrow[d, tail] & R \bigvee (S \cap T) \arrow[r, tail] \arrow[d, tail] & T \arrow[d, tail] \\
S \arrow[r, tail] & R \bigvee S \arrow[r, tail] & T \bigvee S
\end{tikzcd}$$

where the upper right hand side horizontal map is determined by the inclusion $R \subset T$. Then notice that $R \bigvee (S \cap T) \bigvee S = R \bigvee S$ since $S \cap T \subset R \bigvee S$. Accordingly the right hand side square is a pullback and we get $R \bigvee (S \cap T) = (R \bigvee S) \cap T$. $\square$

Proposition 3.7. *Any equi-modular category $\mathbb{E}$ is a Gumm category.*

Proof. Suppose that $\mathbb{E}$ is equi-modular and we have $S \cap T \subset R \subset T$; then consider the following commutative diagram:

$$\begin{tikzcd}
S\square R \arrow[rr, shift left=2, "\delta_1^R"] \arrow[rr, shift right=2, "\delta_0^R"'] \arrow[dr] \arrow[dd] & & R \arrow[ll] \arrow[dr] \arrow[dd] & \\
& S\square(R\bigvee S) \arrow[rr, shift left=2] \arrow[rr, shift right=2] \arrow[dd] & & R\bigvee S \arrow[ll] \arrow[dd, "i\bigvee S"] \\
S\square T \arrow[rr, shift left=2, "\delta_1^T"] \arrow[rr, shift right=2, "\delta_0^T"'] \arrow[dr] & & T \arrow[ll] \arrow[dr] & \\
& S\square(T\bigvee S) \arrow[rr, shift left=2] \arrow[rr, shift right=2] & & T\bigvee S \arrow[ll]
\end{tikzcd}$$

The right hand side vertical square is a pullback by the modular law, and so is the left hand side vertical one since inverse image preserves intersections. The front morphism of equivalence relations is fibrant since both $R \bigvee S$ and $T \bigvee S$ contain S. So that the back morphism of equivalence relations is fibrant as well. $\square$

Now it would remain the difficult task to understand why the equational context of Universal Algebra implies the converse. Besides we have the following:

Proposition 3.8. *Any protomodular category $\mathbb{E}$ is a Gumm category.*

Proof. Consider the following diagram:

$$\begin{tikzcd}
S\square(S\cap T) \arrow[r, tail, "S\square j"] \arrow[d, shift right=2, "\delta_0^{S\cap T}"'] \arrow[d, shift left=2, "\delta_1^{S\cap T}"] & S\square R \arrow[r, tail, "S\square i"] \arrow[d, shift right=2, "\delta_0^R"'] \arrow[d, shift left=2, "\delta_1^R"] & S\square T \arrow[d, shift right=2, "\delta_0^T"'] \arrow[d, shift left=2, "\delta_1^T"] \\
S\cap T \arrow[r, tail, "j"'] \arrow[u] & R \arrow[r, tail, "i"'] \arrow[u] & T \arrow[u]
\end{tikzcd}$$

Clearly, the whole rectangle indexed by 0 is a pullback (check it in Set). Since $S\square i$ is a monomorphism, so is the left hand side square. Accordingly, when $\mathbb{E}$ is protomodular, so is the right hand side one as well. $\square$

3.5 Pullbacks along cartesian maps in $Equ\mathbb{E}$

We characterized the stability of regular epimorphisms in $Equ\mathbb{E}$ under pullbacks along maps in the fibres of the fibration $\mathcal{O}_{\mathbb{E}} : Equ\mathbb{E} \to \mathbb{E}$. In this section we shall characterize the stability of regular epimorphisms in $Equ\mathbb{E}$ under pullbacks along cartesian maps. For that we need the following obervations:

Proposition 3.9. *Given any fibrant morphism $(g, \hat{g}) : R \to R'$ of equivalence relations in a category $\mathbb{E}$, for any equivalence relation S on the codomain X' of* *g, the following square is a pullback:*

$$\begin{tikzcd}
(d_0^R)^{-1}(g^{-1}(S)) \arrow[r, "\check{d}_1^R"] \arrow[d, "(d_0)^{-1}(\tilde{g})"'] & X \times X \arrow[d, "g\times g"] \\
(d_0^{R'})^{-1}(S) \arrow[r, "\check{d}_1^{R'}"'] & X' \times X'
\end{tikzcd}$$

or, in other words, given any cartesian map $(g, \tilde{g}) : \Sigma \to S$, *the unique factorization* $(\hat{g}, (d_0)^{-1}(\tilde{g})) : (d_0^{R'})^{-1}(\Sigma) \to (d_0^{R'})^{-1}(S)$ *is cartesian as well.*

Proof. Let us check this in *Set*. Consider the following diagram:

$$\begin{tikzcd}
x \arrow[r, dotted, "R'"] \arrow[d, dotted, "S"'] & y & y = g(b) \\
x' \arrow[r, dotted, "R'"'] & y' & y' = g(b')
\end{tikzcd}$$

Since the morphism $(g, \hat{g}) : R \to R'$ is fibrant, there are unique a and a' determining the following diagram:

$$\begin{tikzcd}
a \arrow[r, dotted, "R"] & b & g(a) = x \arrow[d, dotted, "S"] \\
a' \arrow[r, dotted, "R"'] & b' & g(a') = x'
\end{tikzcd}$$

which implies $ag^{-1}(S)a'$; accordingly the square in question is a pullback. The following commutative diagram:

$$\begin{tikzcd}
(d_0^R)^{-1}(g^{-1}(S)) \arrow[r, tail] \arrow[d, "(d_0)^{-1}(\tilde{g})"'] \arrow[rr, bend left, "\check{d}_1^R"] & R \times R \arrow[r, "d_1^R \times d_1^R"] \arrow[d, "\hat{g}\times\hat{g}"] & X \times X \arrow[d, "g\times g"] \\
(d_0^{R'})^{-1}(S) \arrow[r, tail] \arrow[rr, bend right, "\check{d}_1^{R'}"'] & R' \times R' \arrow[r, "d_1^{R'} \times d_1^{R'}"'] & X' \times X'
\end{tikzcd}$$

shows that the left hand side square is a pullback since so is the right hand side one, the morphism $R \to R'$ being fibrant. Accordingly the morphism $(\hat{g}, (d_0)^{-1}(\tilde{g}))$ is cartesian. □

Proposition 3.10. *Suppose* $(g, \bar{g}) : \Sigma \to S$ *is a fibrant morphism above the split epimorphism* $(g, t) : X \rightleftarrows Z$. *Then we have* $\Sigma \subset g^{-1}(T)$ *if and only if* $S \subset T$ *for any equivalence relation* T *on* Z. *Suppose that the supremum* $S \bigvee T$ *does exist. Then we have* $\Sigma \bigvee g^{-1}(T) = g^{-1}(S \bigvee T)$.

Proof. The implication $[S \subset T \Rightarrow \Sigma \subset g^{-1}(T)]$ is straightforward. We shall check the inverse in Set. Suppose we have zSz'. Since $(g, \bar{g}) : \Sigma \to S$ is a fibrant morphism, there is a unique $x' \in X$ such that $t(z)\Sigma x'$ and $g(x') = z'$. Whence $z = g(t(z))Tg(x') = z'$ when we have $\Sigma \subset g^{-1}(T)$.

Suppose now that the supremum $S \bigvee T$ does exist. Let W be any equivalence relation X containing Σ and $g^{-1}(T)$. The second inclusion implies that we have $R[g] \subset W$, so that we get $W = g^{-1}(t^{-1}(W))$ and $T \subset t^{-1}(W)$. Then the first inclusion gives us $S \subset t^{-1}(W)$ according to the first point of this proposition. Whence $S \bigvee T \subset t^{-1}(W)$, and $g^{-1}(S \bigvee T) \subset g^{-1}(t^{-1}(W)) = W$. Accordingly we get $g^{-1}(S \bigvee T) = \Sigma \bigvee g^{-1}(T)$. $\square$

Theorem 3.2. *Suppose that $Equ\mathbb{E}$ has regular epimorphisms with any domain above split epimorphisms. Then these regular epimorphisms are stable under pullbacks along cartesian maps in $Equ\mathbb{E}$ if and only if any fibrant morphism $(g, \hat{g}) : R \to R'$ of equivalence relations is such that, given any equivalence relation S on the codomain Z of g, we get: $g^{-1}(R' \bigvee S) = R \bigvee g^{-1}(S)$. So, these regular epimorphisms are stable under any pullback in $Equ\mathbb{E}$ if and only if the category $\mathbb{E}$ is equi-modular and the previous condition holds.*

Proof. Suppose the our condition holds and we have a pullback in $Equ\mathbb{E}$:

$$\begin{tikzcd}
g^{-1}(S) \arrow[r, "(f,\hat{f})"] \arrow[d, "(g,\tilde{g})"'] & h^{-1}(V) \arrow[d, "(h,\tilde{h})"] \\
S \arrow[r, "(f',(f')^\sharp)"'] & V
\end{tikzcd}$$

where the right hand side vertical map is cartesian. So is the left hand side vertical one. Moreover the morphism $R(g) : R[f] \to R[f']$ is fibrant since the image of this square by $\mathcal{O}_\mathbb{E}$ is a pullback. By our condition we have $g^{-1}(R[f'] \bigvee S) = R[f] \bigvee g^{-1}(S)$. On the other hand, we have $(f')^{-1}(V) = R[f'] \bigvee S$ since the lower map is a regular epimorphism. So we get:

$$f^{-1}(h^{-1}(V)) = g^{-1}((f')^{-1}(V)) = g^{-1}(R[f'] \bigvee S) = R[f] \bigvee g^{-1}(S)$$

which means that $(f, \hat{f})$ is a regular epimorphism. Conversely suppose that the regular epimorphisms are stable under pullbacks along cartesian maps. Consider the following diagram:

$$\begin{tikzcd}
(d_0^R)^{-1}(g^{-1}(S)) \arrow[r, dotted] \arrow[d, "(d_0)^{-1}(\tilde{g})"'] \arrow[rr, bend left, "\check{d}_1^R"] & g^{-1}(R' \bigvee S) \arrow[r, tail] \arrow[d, tail] & X \times X \arrow[d, "g \times g"] \\
(d_0^{R'})^{-1}(S) \arrow[r, "(d_1^R)^\sharp"'] \arrow[rr, bend right, "\check{d}_1^{R'}"'] & R' \bigvee S \arrow[r, tail] & Z \times Z
\end{tikzcd}$$

where $(g, \hat{g}) : R \to R'$ is a fibrant morphism of equivalence relations and S an equivalence relation on the codomain Z of g. The right hand side square is a pullback by definition of the inverse image. The whole diagram is a pullback by Proposition 3.9. Accordingly there is a dotted factorization making the left hand side square a pullback; it is a pullback along the vertical right hand side cartesian map. Accordingly this dotted arrow provides us with a regular epimorphism above d_1^R in $Equ\mathbb{E}$ and its codomain $g^{-1}(R' \bigvee S)$ is $R \bigvee g^{-1}(S)$. These regular epimorphisms are stable under any pullbacks if and only if they are stable under pullbacks along maps in the fibres and pullbacks along cartesian maps; the first point is equivalent to the equi-modularity while the second one is equivalent to our condition. $\square$

3.6 When is $Equ\mathbb{E}$ regular?

We are now ready to answer this question. First, the category $Equ\mathbb{E}$ must have coequalizers of effective equivalence relations.

Proposition 3.11. *Suppose $Equ\mathbb{E}$ has coequalizers of effective relations. Then so has $\mathbb{E}$. Accordingly, saying that $Equ\mathbb{E}$ has coequalizers of effective relations is equivalent to saying that $Equ\mathbb{E}$ has regular epimorphisms above regular epimorphisms in $\mathbb{E}$. In particular, the category $Equ\mathbb{E}$ has certainly regular epimorphisms with any domain above split epimorphisms.*

Proof. Since the functor $\mathcal{O}_{\mathbb{E}} : Equ\mathbb{E} \to \mathbb{E}$ has a right adjoint ∇ and is left exact, it preserves the regular epimorphisms and their kernel equivalence relations; whence our first point. The second point is a consequence of Lemma 1.4. $\square$

See Proposition 3.4 for examples of such categories $Equ\mathbb{E}$.
Secondly, the category $\mathbb{E}$ must be itself regular, since $\Delta_{\mathbb{E}} : \mathbb{E} \rightarrowtail Equ\mathbb{E}$ makes $\mathbb{E}$ equivalent to a fully faithful subcategory of $Equ\mathbb{E}$, stable under pullbacks and subobjects. From that, we shall need the further observation:

Proposition 3.12. *Let $\mathbb{E}$ be a regular category, $f : X \twoheadrightarrow Y$ a regular epimorphism and S an equivalence relation on X. The direct image $f(S)$ is an equivalence relation if and only if $f^{-1}(f(S))$ is the supremum $R[f] \bigvee S$ in $Equ\mathbb{E}$. If it is the case, $(f, \bar{f}) : S \to f(S)$ is a regular epimorphism in $Equ\mathbb{E}$ above f.*

Proof. If we have $f^{-1}(f(S)) \simeq R[f] \bigvee S$, the reflexive relation $f^{-1}(f(S))$ is an equivalence relation, and so is $f(S)$ by Proposition 2.1. Conversely suppose that $f(S)$ is an equivalence relation and consider the following diagram in $\mathbb{E}$:

$$\begin{tikzcd}
(d_0^f)^{-1}(S) \arrow[r, "\bar{\delta}_0^S"] \arrow[drrr, dotted, two heads, "\bar{d}_1"'] \arrow[ddrr] & S \arrow[rrrd, "\bar{f}"] \arrow[ddrr] & & & \\
& & & f^{-1}(f(S)) \arrow[r, two heads, "\bar{f}"'] \arrow[d, tail] & f(S) \arrow[d, tail] \\
& & R[f] \times R[f] \arrow[r, shift left=2, "d_0^f \times d_0^f"] \arrow[r, shift right=2, "d_1^f \times d_1^f"'] & X \times X \arrow[l] \arrow[r, two heads, "f \times f"'] & Y \times Y
\end{tikzcd}$$

There is a dotted factorization $\bar{d}_1$ making the upper quadrangle a pullback. Since $\bar{f}$ is a regular epimorphism, so is $\bar{d}_1$. Accordingly it produces a levelwise regular epimorphic map in $\mathbb{E}$, and, as such, a regular epimorphism $(d_1^f, \bar{d}_1) : (d_0^f)^{-1}(S) \to f^{-1}(f(S))$ in $Equ\mathbb{E}$ above d_1^f. By Corollary 3.1, we get $f^{-1}(f(S)) = R[f] \bigvee S$. Now, the morphism $(f, \bar{f}) : S \to f(S)$ in $Equ\mathbb{E}$ is a levelwise regular epimorphic map in $\mathbb{E}$, and, as such, a regular epimorphism in $Equ\mathbb{E}$ above f. $\square$

Proposition 3.13. *Let $\mathbb{E}$ be a regular category. TFAE:*
(i) $Equ\mathbb{E}$ has regular epimorphisms with any domain above split epimorphisms
(ii) $Equ\mathbb{E}$ has coequalizers of effective relations
(iii) $Equ\mathbb{E}$ has regular epimorphisms above regular epimorphisms in $\mathbb{E}$.
In this case a morphism $(f, \hat{f}) : S \to T$ is regular in $Equ\mathbb{E}$ if and only if f is regular in $\mathbb{E}$ and $f^{-1}(T) = R[f] \bigvee S$. The modular formula holds if and only if the regular epimorphisms in $Equ\mathbb{E}$ are stable under pullbacks along maps in the fibres.

Proof. We already noticed that [(ii)⇒(i)]. Conversely suppose (i). Then the supremum $R[f] \bigvee S$ does exists and we have $R[f] \subset R[f] \bigvee S$, so that the direct image $f(R[f] \bigvee S)$ is an equivalence relation. Then the map:

$$S \rightarrowtail R[f] \bigvee S \twoheadrightarrow f(R[f] \bigvee S)$$

is the regular epimorphism above f by the same proof as the one of Proposition 3.3. So we get (ii). Now by Proposition 3.5 it is clear then that if the regular epimorphisms in $Equ\mathbb{E}$ are stable under pullbacks along maps in the fibres, the modular formula holds. The proof of the converse is obtained as in Proposition 3.5. [(ii) $\iff$ (iii)] is given by the last but one proposition. $\square$

Theorem 3.3. *Given any category $\mathbb{E}$, TFAE:*
(i) the category $Equ\mathbb{E}$ is regular
(ii) the category $\mathbb{E}$ is equi-modular, regular and such that:
(*) we get: $g^{-1}(R' \bigvee S) = R \bigvee g^{-1}(S)$ for any fibrant morphism $(g, \hat{g}) : R \to R'$ and any equivalence relation S on the codomain Z of g.

Proof. (i) implies that $\mathbb{E}$ is regular and $Equ\mathbb{E}$ has coequalizers of effective relations. So that any pair of equivalence relations on X has a supremum. Since the regular epimorphisms with any domain above split epimorphisms are stable, in particular, under pullbacks along maps in the fibres, the modular formula holds by Proposition 3.5 and $\mathbb{E}$ is equi-modular. And by Theorem 3.2, since the regular epimorphisms with any domain above split epimorphisms are stable under pullbacks along cartesian maps in $Equ\mathbb{E}$, given any fibrant morphism $(g, \hat{g}) : R \to R'$ of equivalence relations and for any equivalence relation S on the codomain Z of g, we get: $g^{-1}(R' \bigvee S) = R \bigvee g^{-1}(S)$.

Now suppose (ii). By the previous proposition, we know that $Equ\mathbb{E}$ has coequalizers of effective relations and that these regular epimorphisms are stable under pullbacks along maps in the fibres. It remains to show they are stable under pullbacks along cartesian maps in $Equ\mathbb{E}$ when the condition (*) holds. Suppose this condition holds and that we have a pullback in $Equ\mathbb{E}$:

$$\begin{tikzcd}
g^{-1}(S) \arrow[r, "{(f,\hat{f})}"] \arrow[d, "{(g,\tilde{g})}"'] & h^{-1}(V) \arrow[d, "{(h,\tilde{h})}"] \\
S \arrow[r, "{(f',(f')^{\sharp})}"'] & V
\end{tikzcd}$$

where the right hand side vertical map is cartesian. So is the left hand side vertical one. Moreover the morphism $R(g) : R[f] \to R[f']$ is fibrant since the image of this square by $\mathcal{O}_{\mathbb{E}}$ is a pullback. By the condition (*) we have $g^{-1}(R[f'] \bigvee S) = R[f] \bigvee g^{-1}(S)$. On the other hand, we have $(f')^{-1}(V) = R[f'] \bigvee S$ since the lower map is a regular epimorphism. So we get

$$f^{-1}(h^{-1}(V)) = g^{-1}((f')^{-1}(V)) = g^{-1}(R[f'] \bigvee S) = R[f] \bigvee g^{-1}(S)$$

which means that $(f, \hat{f})$ is a regular epimorphism. $\square$

The previous characterization induces the following:

Definition 3.3. *A category $\mathbb{E}$ is said to be equi-regular when any of the previous equivalent conditions holds.*

There is now a natural question: how far is a regular category $\mathbb{E}$ from satisfying the condition (*).

Lemma 3.2. *Let $\mathbb{E}$ be a regular category. When m is a monomorphism, f a regular epimorphism and the morphism $(m.f, \phi) : R \to T$ in $Equ\mathbb{E}$ is fibrant:*

$$\begin{tikzcd}
R \arrow[r, two heads, "\hat{f}"] \arrow[rr, bend left, "\phi"] \arrow[d, shift right=2, "d_0^R"'] \arrow[d, shift left=2, "d_1^R"] & S \arrow[r, tail, "\hat{m}"] \arrow[d, shift right=2, "d_0^S"'] \arrow[d, shift left=2, "d_1^S"] & T \arrow[d, shift right=2, "d_0^T"'] \arrow[d, shift left=2, "d_1^T"] \\
X \arrow[r, two heads, "f"'] \arrow[u] & Y \arrow[r, tail, "m"'] \arrow[u] & Z \arrow[u]
\end{tikzcd}$$

both $(m, \hat{m})$ and $(f, \hat{f})$ are so, where $\phi = \hat{m}.\hat{f}$ is the canonical decomposition.

Proof. It is clear that S is a reflexive relation on Y. Since the whole rectangle indexed by 0 is a pullback and $\hat{m}$ is a monomorphism, so is the left hand side square. Accordingly we get a hyperextremal epimorphism $(f, \hat{f}) : R \to S$ in $Ref\mathbb{E}$ and, since R is an equivalence relation, so is S by Lemma 1.7, and the left hand side morphism is fibrant. Now, since f is a regular epimorphism and since the whole rectangle and the left hand side square indexed by 0 are pullbacks, so is the right hand side one, the category $\mathbb{E}$ being regular. Accordingly the right hand side morphism is fibrant as well. $\square$

Proposition 3.14. *Let $\mathbb{E}$ be a regular category with suprema of pairs of equivalence relations, and $(g, \hat{g}) : R \to R'$ a fibrant morphism where g is a regular epimorphism. Then for any equivalence relation S on the codomain Z of g, we get $g^{-1}(R' \bigvee S) = R \bigvee g^{-1}(S)$.*

Proof. We have $R[g] \subset g^{-1}(S) \subset R \bigvee g^{-1}(S)$. By Corollary 2.1, since g is a regular epimorphism and $\mathbb{E}$ a regular category, we get an equivalence relation W on Z such that $g^{-1}(W) = R \bigvee g^{-1}(S)$. Let us show that $W = R' \bigvee S$. Let V be any equivalence relation on Z containing R' and S. So $g^{-1}(V)$ contains $g^{-1}(S)$ and $g^{-1}(R')$, and consequently R as well. Accordingly $g^{-1}(V)$ contains $R \bigvee g^{-1}(S) = g^{-1}(W)$. According to the same Corollary 2.1, we get $W \subset V$. $\square$

Whence the following:

Theorem 3.4. *Let $\mathbb{E}$ be a regular and equi-modular category. It is equi-regular if and only if the condition (*) holds for any monomorphic fibrant morphism $(m, \hat{m}) : R \rightarrowtail R'$ in $Equ\mathbb{E}$.*

It would be very interesting to be able to characterize those varieties which are such that this condition (*) holds for any monomorphic fibrant morphism $(m, \hat{m}) : R \rightarrowtail R'$. On the other hand, we can ask ourselves whether there exist equi-regular categories in which the regular epimorphisms in $Equ\mathbb{E}$ are not the levelwise regular epimorphisms in $\mathbb{E}$. If it was not the case, this would mean that any equi-regular category is a Goursat regular category, see Theorem 3.5 below. Now, any variety $\mathbb{V}$ is exact, and the category $Equ\mathbb{V}$ is equivalent to the category $Reg\mathbb{V}$ of regular epimorphisms. It is shown in [12], thanks to a result of [19], that, when $\mathbb{V}$ is an ideal determined variety, the category $Reg\mathbb{V}(\simeq Equ\mathbb{V})$ is regular. But, in [24], it is shown that ideal determined varieties need not be congruence 3-permutable.

3.7 Goursat regular category according to Carboni-Kelly-Pedicchio

In a regular category $\mathbb{E}$, relations $U \rightarrowtail X \times Y$ between X and Y, understood as morphims $X - - > Y$, are composable and the composition is associative up to isomorphism [9]. The reflexive relations are clearly stable under composition. It is not the case neither for reflexive and symmetric relations nor for equivalence relations. However, if R and S are reflexive relations, the reflexive $R \circ S \circ R$ is necessarily symmetric as soon as R and S are symmetric, since we have $(R \circ S \circ R)^{op} = R^{op} \circ S^{op} \circ R^{op} \simeq R \circ S \circ R$.

Proposition 3.15. *Let $\mathbb{E}$ be a regular category and (R, S) a pair of equivalence relations on X. Then the reflexive and symmetric relation $d_1^R((d_0^R)^{-1}(S))$ is nothing but $R \circ S \circ R$. So, $d_1^R((d_0^R)^{-1}(S))$ is an equivalence relation if and only if $R \circ S \circ R$ is an equivalence relation and, in this case, we get: $d_1^R((d_0^R)^{-1}(S)) = R \circ S \circ R = R \bigvee S$. Then, given any other equivalence relation T such that $R \subset T$, the modular formula holds: $(R \bigvee S) \cap T = R \bigvee (S \cap T)$.*

Proof. We noticed that the objects of $(d_0^R)^{-1}(S)$ are given by the following diagrams in $\mathbb{E}$:

$$\begin{tikzcd} x \arrow[r, dotted, "S"] \arrow[d, dotted, "R"'] & y \arrow[d, dotted, "R"] \\ x' & y' \end{tikzcd}$$

Accordingly, when $\mathbb{E}$ is a regular category, the reflexive and symmetric relation $d_1^R((d_0^R)^{-1}(S))$ is nothing but $R \circ S \circ R^{op} \simeq R \circ S \circ R$. Whence the first point. Suppose that $d_1^R((d_0^R)^{-1}(S))$ is an equivalence relation, then $(d_1^R, \bar{d}_1^R) : (d_0^R)^{-1}(S) \to R \circ S \circ R$ is a levelwise regular epimorphism in $\mathbb{E}$, and, as such, a regular epimorphism in $Equ\mathbb{E}$. According to Corollary 3.1, we get $R \circ S \circ R = R \bigvee S$, whence the second point. If, in addition, we have $R \subset T$, we know by Lemma 1.6 that $(d_0^R)^{-1}(T) = R\square T = (d_1^R)^{-1}(T)$, so that:

$$(d_0^R)^{-1}(S \cap T) = (d_0^R)^{-1}(S) \cap (d_0^R)^{-1}(T) = (d_0^R)^{-1}(S) \cap (R\square T)$$

On the other hand, since $R\square T = (d_1^R)^{-1}(T)$, we get $R[d_1^R] \subset R\square T$, so that:

$$d_1^R((d_0^R)^{-1}(S) \cap (R\square T)) = d_1^R((d_0^R)^{-1}(S)) \cap d_1^R(R\square T) = d_1^R((d_0^R)^{-1}(S)) \cap T$$

the first equality by Proposition 2.2, and the second one since $d_1^R(R\square T) = d_1^R((d_1^R)^{-1}(T)) = T$. So, we get:

$$d_1^R((d_0^R)^{-1}(S \cap T)) = d_1^R((d_0^R)^{-1}(S)) \cap T = (R \bigvee S) \cap T$$

which is an equivalence relation; accordingly $(R \bigvee S) \cap T = R \bigvee (S \cap T)$ by our second point. $\square$

In [8], a regular category was said to be a *Goursat category* (from an old result of [14] (1889) concerning relations in the category of groups), when the composition of equivalence relations is 3-permutable, namely when, for any pair (R, S) of equivalence relation on X, we get $R \circ S \circ R = S \circ R \circ S$.

In the varietal context, Mitschke [23] gave an example of a strict 3-permutable variety with the notion of *implication algebra*, while Hagemann-Mitschke [18] gave a characterization of 3-permutable varieties, and another example of this notion with the *right-complemented semi-groups*: a variety is 3-permutable if and only if its algebraic theory contains two ternary operations r and s such that $r(x, y, y) = x$, $r(x, x, y) = s(x, y, y)$ and $s(x, x, y) = y$.

In a Goursat regular category, when (R, S) is a pair of equivalence relations, $R \circ S \circ R$ becomes an equivalence relation since we have $(R \circ S \circ R) \circ (R \circ S \circ R) = R \circ S \circ R \circ R \circ S \circ R = R \circ S \circ R \circ S \circ R = R \circ R \circ S \circ R \circ R = R \circ S \circ R$. From Proposition 3.15, we know that $R \circ S \circ R = d_1((d_0^R)^{-1}(S))$. So, we get immediately:

Lemma 3.3. *In a Goursat regular category $\mathbb{E}$, given any pair (R, S) of equivalence relations on X, we have:*
(i) the reflexive relation $d_1((d_0^R)^{-1}(S))$ is an equivalence relation

(ii) $d_1((d_0^R)^{-1}(S))$ *is* $R \bigvee S$
(iii) the modular formula holds for any triple (R, S, T) *of equivalence relations on* X *such that* $R \subset T$.

Whence the following characterizations:

Theorem 3.5. *Let* $\mathbb{E}$ *be a regular category. TFAE:*
(i) $\mathbb{E}$ *is a Goursat regular category*
(ii) for any pair (R, S) *of equivalence relations on* X, $d_1^R((d_0^R)^{-1}(S))$ *is an equivalence relation*
(iii) the direct image along a regular epimorphism in $\mathbb{E}$ *of any equivalence relation is an equivalence relation*
(iv) $\mathbb{E}$ *is a equi-regular category such that the regular epimorphims in* $Equ\mathbb{E}$ *are the levelwise regular epimorphisms in* $\mathbb{E}$
(v) $\mathbb{E}$ *is an extremal category*
(vi) $\mathbb{E}$ *is a hyperextremal category*
(vii) $\mathbb{E}$ *is an extremally equi-modular category*

Proof. We get $[(i) \Rightarrow (ii)]$ from the previous lemma. And we get $[(ii) \Rightarrow (i)]$ as well: indeed if for any pair (R, S) of equivalence relations on X, $d_1^R((d_0^R)^{-1}(S))$ is an equivalence relation, this is the case of $d_1^S((d_0^S)^{-1}(R))$ as well, and we have: $d_1^R((d_0^R)^{-1}(S)) = R \bigvee S = d_1^S((d_0^S)^{-1}(R))$. So we get: $R \circ S \circ R^{op} = S \circ R \circ S^{op}$ and then $R \circ S \circ R \simeq S \circ R \circ S$. Accordingly the regular category $\mathbb{E}$ is a Goursat one.

Let us check $[(ii) \Rightarrow (iii)]$. Let $f : X \twoheadrightarrow Y$ be a regular epimorphism and S an equivalence relation on X. Then, considering the following diagram in $\mathbb{E}$, there is a dotted factorization $\bar{d}_1$ making the upper quadrangle a pullback, and since $\bar{f}$ is a regular epimorphism, so is this factorization $\bar{d}_1$:

$$\begin{tikzcd}
(d_0^f)^{-1}(S) \arrow[r, "\bar{\delta}_0^S"] \arrow[drrr, dotted, two heads, "\bar{d}_1"'] \arrow[ddrr] & S \arrow[drrr, two heads, "\bar{f}"] \arrow[ddrr] & & & \\
& & & f^{-1}(f(S)) \arrow[r, two heads, "\tilde{f}"'] \arrow[d, tail] & f(S) \arrow[d, tail] \\
& & R[f] \times R[f] \arrow[r, shift left=2, "d_0^f \times d_0^f"] \arrow[r, shift right=2, "d_1^f \times d_1^f"'] & X \times X \arrow[l] \arrow[r, two heads, "f \times f"'] & Y \times Y
\end{tikzcd}$$

Accordingly $f^{-1}(f(S))$ is the direct image of the equivalence relation $(d_0^f)^{-1}(S)$ along d_1^f, namely $d_1^f((d_0^f)^{-1}(S))$, and thus $f^{-1}(f(S))$ is an equivalence relation which implies that $f(S)$ is an equivalence relation as well.

We already proved $[(\text{iii}) \iff (\text{iv}) \iff (\text{v}) \iff (\text{vi})]$ in Proposition 2.3 and $[(\text{v}) \iff (\text{vii})]$ in Proposition 3.6. It is clear that $[(iv) \Rightarrow (ii)]$ since, when we have (iv), the codomain of a regular epimorphism in $Equ\mathbb{E}$ coincides with the direct image of its domain. $\square$

The characterization $[(i) \iff (iii)]$ was already given by Theorem 6.8 in the pioneering paper [8] and a characterization equivalent to $[(i) \iff (vi)]$ was already given by Theorem 1 in [15]; both results were proved by means of calculus of relations (namely using Metatheorems and internal logic); here we introduced straighforward categorical proofs.

4 Goursat condition in non-regular context

4.1 Definition

Inspired by Propositions 3.1, 3.12 and Theorem 3.5, we shall introduce now a Goursat condition valid in the non-regular context.

Definition 4.1. *Given any category $\mathbb{E}$, it is said to be a Goursat category when, given any morphism in $Pt(\mathbb{E})$ as in the left hand side diagram:*

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, shift right, "f"'] & X' \arrow[d, shift right, "f'"'] \\
Y \arrow[r, "h"'] \arrow[u, shift right, "s"'] & Y' \arrow[u, shift right, "s'"']
\end{tikzcd}
\qquad
\begin{tikzcd}
(d_0^g)^{-1}(R[f]) \arrow[r, "\bar{d}_0^g"] \arrow[ddr, tail] \arrow[drr, dotted, "\bar{d}_1^g"'] & R[f] \arrow[drr, "R(g)"] \arrow[ddr, tail] & & \\
 & & g^{-1}(R[f']) \arrow[r, two heads, "\bar{g}"] \arrow[d, tail] & R[f'] \arrow[d, tail] \\
 & R[g] \times R[g] \arrow[r, shift left, "d_0^g \times d_0^g"] \arrow[r, shift right, "d_1^g \times d_1^g"'] & X \times X \arrow[r, "g \times g"'] & X' \times X'
\end{tikzcd}$$

the unique dotted factorization $\bar{d}_1^g$ is an extremal epimorphism.

In set-theoretical terms, the map $\bar{d}_1^g$ associates (x, x') with the following left hand side diagram:

$$\begin{tikzcd}
t \arrow[r, dotted, "R[g]"] \arrow[d, dotted, "R[f]"'] & x & x & g(x) \arrow[d, dotted, "R[f']"] \\
t' \arrow[r, dotted, "R[g]"'] & x' & x' & g(x')
\end{tikzcd}$$

Since the extremal epimorphisms are reflected by the conservative functors which preserve pullbacks, given any conservative functor $U : \mathbb{F} \to \mathbb{E}$ preserving pullbacks, the category $\mathbb{F}$ is a Goursat category as soon as so is $\mathbb{E}$. In particular this notion is stable under passage to slice categories $\mathbb{E}/Y$, coslice categories $Y/\mathbb{E}$ and fibres $Pt_Y(\mathbb{E})$.

Let $\mathbb{T}$ be an algebraic theory in the sense of Universal algebra and $\mathbb{V}(\mathbb{T})$ the corresponding variety of $\mathbb{T}$-algebras. Then any functor category $\mathcal{F}(\mathbb{E}, \mathbb{V}(\mathbb{T}))$ being regular, its extremal epimorphims are those natural transformations which are componentwise extremal epimorphisms. So $\mathcal{F}(\mathbb{E}, \mathbb{V}(\mathbb{T}))$ is clearly a Goursat category in our sense as soon as $\mathbb{V}(\mathbb{T})$ is a Goursat variety. Now let $\mathbb{T}(\mathbb{E})$

be the category of internal $\mathbb{T}$-algebras in $\mathbb{E}$. Then consider the corresponding Yoneda embedding $Y^{\mathbb{T}} : \mathbb{T}(\mathbb{E}) \to \mathcal{F}(\mathbb{E}^{op}, \mathbb{V}(\mathbb{T}))$; it is left exact and conservative. Accordingly the category $\mathbb{T}(\mathbb{E})$ is a Goursat category is our sense as soon as $\mathbb{V}(\mathbb{T})$ is a Goursat variety.

Proposition 4.1. *Suppose $\mathbb{E}$ is an extremally equi-modular category. Then $\mathbb{E}$ is a Goursat category.*

Proof. In the diagram above, since $g^{-1}(R[f']) = R[g] \bigvee R[f]$ by Proposition 3.1, the map $(d_1^g, \bar{d}_1^g)$ is a regular epimorphism in $Equ\mathbb{E}$ above the split epimorphism d_1^g by Corollary 3.1. According to our assumption, the map $\bar{d}_1^g$ is an extremal epimorphism and $\mathbb{E}$ is a Goursat category. $\square$

4.2 The regular context

First we have to check that, in the regular context, our definition coincides with the Carboni-Kelly-Pedicchio one.

Proposition 4.2. *Let $\mathbb{E}$ be a regular category. Then it is a Goursat one according to the previous definition if and only if it is a Goursat regular category in the sense of [8].*

Proof. Suppose $\mathbb{E}$ is a Goursat regular category in the sense of [8]. According to Condition (vii) of Theorem 3.5, the category $\mathbb{E}$ is a Goursat category by the previous proposition.

Conversely suppose $\mathbb{E}$ is regular and a Goursat category in our sense. We are going to show it is a hyperextremal category which will mean a Goursat category in the sense of [8] by the same theorem. So suppose that g is a regular epimorphism (which implies that so is h); since $\mathbb{E}$ is regular, then $g \times g$, and therefore $\tilde{g}$, are regular epimorphisms in $\mathbb{E}$. Now $\bar{d}_1^g$ is a regular epimorphism by the previous definition. So that $R(g).\bar{d}_0^g = \tilde{g}.\bar{d}_1^g$ is a regular epimorphim in $\mathbb{E}$, which implies that so is the factorization $R(g)$. Accordingly the morphism (h, g) is hyperregular in $Pt(\mathbb{E})$, and the category $\mathbb{E}$ is hyperextremal. $\square$

4.3 Goursat categories and fibration of points

Let us introduce the following:

Definition 4.2. *Let $\mathbb{E}$ be a pointed category. It is said to be a punctually Goursat category when, given any morphism in $Pt(\mathbb{E})$ above the terminal map:*

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, shift left, "f"] & Z \arrow[d, shift left] \\ Y \arrow[r] \arrow[u, shift left, "s"] & 1 \arrow[u, shift left] \end{tikzcd}$$

the paraterminal map:

$$\begin{tikzcd}
(d_0^g)^{-1}(R[f]) \arrow[r, two heads, "\bar{d}_1^g"] \arrow[d, shift right=2, "\bar{\delta}_0^f"'] \arrow[d, shift left=2, "\bar{\delta}_1^f"] & X \times X \arrow[d, shift right=2, "p_0^X"'] \arrow[d, shift left=2, "p_1^X"] \\
R[g] \arrow[u] \arrow[r, two heads, "d_1^g"'] & X \arrow[u] \arrow[l, dotted, shift right=2, "s_0^g"']
\end{tikzcd}$$

is such that $\bar{d}_1^g$ is an extremal epimorphism in $\mathbb{E}$.

In set theoretical terms, this means that, for any pair of elements $(x, x') \in X \times X$, there is a pair $(t, t') \in X \times X$ satisfying the following diagram:

$$\begin{tikzcd}
t \arrow[r, dotted, "R[g]"] \arrow[d, dotted, "R[f]"'] & x \\
t' \arrow[r, dotted, "R[g]"'] & x'
\end{tikzcd}$$

Similarly to above, given any left exact conservative functor $U : \mathbb{F} \to \mathbb{E}$, the pointed category $\mathbb{F}$ is a punctually Goursat category as soon as so is $\mathbb{E}$.

Proposition 4.3. *Any unital category $\mathbb{E}$ is a punctually Goursat category.*

Proof. Suppose $\mathbb{E}$ is unital and consider any decomposition $R(g) = m.\gamma$ in $\mathbb{E}$ with a monomorphism m. It produces a relation S on X. We have to show that S coincides with ∇_X.

$$\begin{tikzcd}
& S \arrow[dr, tail, "m"] \arrow[ddr, dotted, shift right] \arrow[ddr, dotted, shift left] & \\
(d_0^g)^{-1}(R[f]) \arrow[ur, "\gamma"] \arrow[rr, two heads, "\check{d}_1^g"] \arrow[d, shift right=2, "\bar{\delta}_0^f"'] \arrow[d, shift left=2, "\bar{\delta}_1^f"] & & X \times X \arrow[d, shift right=2, "p_0^X"'] \arrow[d, shift left=2, "p_1^X"] \\
R[g] \arrow[u] \arrow[rr, two heads, "d_1^g"'] & & X \arrow[u]
\end{tikzcd}$$

For that consider the following diagram where $R[f.d_0^g] = (d_0^g)^{-1}(R[f])$:

$$\begin{tikzcd}
R[f.d_0^g] \arrow[r, "\gamma"'] \arrow[rr, bend left, "\check{d}_1^g"] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & S \arrow[r, tail, "m"'] \arrow[dr, shift right] \arrow[dr, shift left] & X \times X \arrow[d, shift right=2, "p_0^X"'] \arrow[d, shift left=2, "p_1^X"] \\
R[g] \arrow[u] \arrow[rr, two heads, "d_1^g"'] \arrow[d, "f.d_0^g"] & & X \arrow[u] \arrow[d, shift left] \\
Y \arrow[u, dotted, shift left=2, "{(s,0)}"] \arrow[rr] & & 1 \arrow[u, shift left]
\end{tikzcd}
\qquad
\begin{tikzcd}
R[f.d_0^g] \arrow[r, "\gamma"'] \arrow[rr, bend left, "\check{d}_1^g"] & S \arrow[r, tail, "m"'] & X \times X \\
R[g] \arrow[u, shift right=2, "\sigma_{-1}"] \arrow[u, shift left=2, "\sigma_1"'] \arrow[rr, two heads, "d_1^g"'] & & X \arrow[u, shift right=2] \arrow[u, shift left=2, "{(0,1_X)}"'] \arrow[ul, dotted, "{(1_X,0)}"]
\end{tikzcd}$$

The morphism $f.d_0^g$ is split by $(s,0)$ since $g.s = 0$ and makes the lower square a morphism in $Pt(\mathbb{E})$. In turn, this splitting produces a pair of morphims $(\sigma_{-1}, \sigma_1) : X \rightrightarrows R[f.d_0^g]$ which commutes with the pair $((1_X, 0), (0, 1_X))$. Since the map d_1^g is a split epimorphism, the commutative square induced by the pair $(\sigma_{-1}, (1_X, 0))$ produces a factorization of $(1_X, 0)$ through m, while the commutative square induced by the pair $(\sigma_1, (0, 1_X))$ produces a factorization of $(0, 1_X)$ through m. Since the pair $(1_X, 0), (0, 1_X)$ is jointly extremally epimorphic, the monomorphism m is an isomorphism, and $\check{d}_1^g$ is an extremal epimorphism in the category $\mathbb{E}$. $\square$

Theorem 4.1. *Let $\mathbb{E}$ be any category. It is a Goursat category if and only if any fibre $Pt_Y(\mathbb{E})$ is a punctually Goursat category. Accordingly any Mal'tsev category is a Goursat category in our sense.*

Proof. It is clear that the punctual Goursat axiom for the fibre $Pt_{Y'}(\mathbb{E})$ is the particular case of the Goursat axiom for $\mathbb{E}$ where the lower map h in Definition 4.1 is split. Conversely, starting with any morphism $(h, g) : (f, s) \to (f', s')$ in $Pt(\mathbb{E})$:

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, shift right, "f"'] & X' \arrow[d, shift right, "f'"'] \\
Y \arrow[r, "h"'] \arrow[u, shift right, "s"'] & Y' \arrow[u, shift right, "s'"']
\end{tikzcd}
\qquad
\begin{tikzcd}
R \arrow[r, "h_0"] \arrow[ddr, "\phi"'] \arrow[drr, dotted, "h_1"] & X \arrow[ddr, "f"] \arrow[drr, "g"] & & \\
 & & \bar{X} \arrow[r, "\bar{h}"] \arrow[d, shift right, "\bar{f}"'] & X' \arrow[d, shift right, "f'"'] \\
 & R[h] \arrow[r, shift left=2, "d_0^h"] \arrow[r, shift right=2, "d_1^h"'] & Y \arrow[l] \arrow[r, "h"'] \arrow[u, shift right, "\bar{s}"'] & Y' \arrow[u, shift right, "s'"']
\end{tikzcd}$$

complete the diagram by the pullbacks $(\bar{f}, \bar{s})$ of (f', s') along h and (ϕ, σ) of (f, s) along d_0^h. Then there is the dotted factorization h_1 making the upper quadrangle (*) a pullback. As soon as the fibre $Pt_Y(\mathbb{E})$ is a punctually Goursat category, the morphism $(d_1^{h_1}, \bar{d}_1^{h_1}) : (d_0^{h_1})^{-1}(R[\phi]) \to (h_1)^{-1}(R[\bar{f}])$ in $Equ\mathbb{E}$ is a levelwise extremal epimorphism. Now consider the following commutative diagram in $Equ\mathbb{E}$:

$$\begin{tikzcd}
(d_0^{h_1})^{-1}(R[\phi]) \arrow[r, "{(d_0^{h_1}, \bar{d}_0^{h_1})}"] \arrow[d, "{(R(h_0), \psi_0)}"'] & R[\phi] \arrow[r, "{(h_1, R(h_1))}"] \arrow[d, "{(h_0, R(h_0))}"'] & R[\bar{f}] \arrow[d, "{(\bar{h}, R(\bar{h}))}"] \\
(d_0^g)^{-1}(R[f]) \arrow[r, "{(d_0^g, \bar{d}_0^g)}"'] & R[f] \arrow[r, "{(g, R(g))}"'] & R[f']
\end{tikzcd}$$

The right hand side square is a pullback since so is the square (*); the left hand side square is a pullback as well since so is its image by the functor $\mathcal{O}_{\mathbb{E}}$ and the parallel horizontal maps are cartesian in $Equ\mathbb{E}$. This defines ψ_0 as the pullback of $R(h_0)$ along $\bar{d}_0^g$. The previous whole rectangle is the following one as well:

$$\begin{tikzcd}
(d_0^{h_1})^{-1}(R[\phi]) \arrow[r, "(d_1^{h_1},\bar{d}_1^{h_1})"] \arrow[d, "(R(h_0),\psi_0)"'] & (h_1)^{-1}(R[\bar{f}]) \arrow[r, "(h_1,\tilde{h}_1)"] \arrow[d, "(h_0,\chi_0)"'] & R[\bar{f}] \arrow[d, "(\bar{h},R(\bar{h}))"] \\
(d_0^g)^{-1}(R[f]) \arrow[r, "(d_1^g,\bar{d}_1^g)"'] & g^{-1}(R[f']) \arrow[r, "(g,\tilde{g})"'] & R[f']
\end{tikzcd}$$

The right hand side square is a pullback since so is its image by the functor $\mathcal{O}_{\mathbb{E}}$ and the parallel horizontal maps are cartesian in $Equ\mathbb{E}$. Since the vertical right hand side map is clearly fibrant, as produced from a cartesian map in $Pt(\mathbb{E})$, so is the middle vertical one. So, certainly χ_0 is a split epimorphism in $\mathbb{E}$, since so is the morphism h_0. Now $\bar{d}_1^g.\psi_0 = \chi_0.\bar{d}_1^{h_1}$ is an extremal epimorphism in $\mathbb{E}$ as a composition of extremal epimorphisms. Accordingly the map $\bar{d}_1^g$ is an extremal epimorphism in $\mathbb{E}$, and $\mathbb{E}$ is a Goursat category in our sense. The last assertion comes from the previous proposition and the fact that a category $\mathbb{E}$ is a Mal'tsev one if and only any fibre $Pt_Y(\mathbb{E})$ is unital [3]. $\square$

4.4 Goursat and Gumm categories

In this section, we shall clarify the relationship between Goursat and Gumm categories, via the fibration of points $¶_{\mathbb{E}} : Pt(\mathbb{E}) \to \mathbb{E}$.

Proposition 4.4. *Let $\mathbb{E}$ be a regular punctually Goursat category. Consider any regular epimorphism in $Pt(\mathbb{E})$:*

$$\begin{tikzcd}
X \arrow[r, two heads, "g"] \arrow[d, shift left, "f"] & Z \arrow[d, shift left] \\
Y \arrow[u, shift left, "s"] \arrow[r] & 1 \arrow[u, shift left]
\end{tikzcd}$$

Then it is hyperregular. Suppose, in addition, that (f, g) is jointly monic. Given any equivalence relation R on X such that we have $R \subset R[f]$, the direct image $g(R)$ is an equivalence relation.

Proof. Given any morphism (τ_Y, g) in $Pt(\mathbb{E})$ above the terminal map, we have $(g \times g).\bar{d}_1^g = R(g).\bar{d}_0^g$. When $\mathbb{E}$ is regular and g a regular epimorphism, so is $g \times g$, and $\bar{d}_1^g$ as well when $\mathbb{E}$ is a punctually Goursat category. Accordingly so is $R(g)$ and the regular epimorphism in question is hyperregular. Consider now the following regular epimorphism in $Pt(\mathbb{E})$:

$$\begin{tikzcd}
R \arrow[r, two heads, "\bar{g}"] \arrow[d, shift left, "f.d_0^R"] & g(R) \arrow[d, shift left] \\
Y \arrow[u, shift left, "s_0^R.s"] \arrow[r] & 1 \arrow[u, shift left]
\end{tikzcd}$$

it is hyperrregular by our first point, and the factorization:

$$R(\bar{g}) : R[f.d_0^R] = R[f]\square R \to g(R) \times g(R)$$

is a regular epimorphism in $\mathbb{E}$, the identity on the left hand side being clear in Set, since we have $R \subset R[f]$. Consider now the following diagram in $\mathbb{E}$:

$$\begin{tikzcd}
R[d_0^R] \arrow[rr, tail] \arrow[dr, "R(\bar{g})"'] \arrow[dd] & & R[f]\square R \arrow[dr, "R(\bar{g})"] \arrow[dd] & \\
& R[d_0^{gR}] \arrow[rr, tail] \arrow[dd] & & g(R) \times g(R) \arrow[dd, "d_0 \times d_0"] \\
R^2[f] \arrow[rr, tail, "\gamma_f"'] \arrow[dr] & & R[f]\square R[f] \arrow[dr, "R(g).\delta_0^{R[f]}"] & \\
& Z \arrow[rr, tail, "s_0^Z"'] & & Z \times Z
\end{tikzcd}$$

The front vertical square is a pullback. The back vertical square, where γ_f associates the right hand side diagram with the left hand side one:

$$\begin{tikzcd}
& x' \arrow[dr, dotted, "R[f]"] & \\
x \arrow[ur, dotted, "R[f]"] \arrow[rr, dotted, "R[f]"'] & & x''
\end{tikzcd}
\qquad
\begin{tikzcd}
x \arrow[r, dotted, "R[f]"] \arrow[d, dotted, "R[f]"'] & x' \arrow[d, dotted, "R[f]"] \\
x \arrow[r, dotted, "R[f]"'] & x''
\end{tikzcd}$$

is a pullback as well. Now, since (f, g) is jointly monic, the lower horizontal square is a pullback; indeed given any diagram:

$$\begin{tikzcd}
x \arrow[r, dotted, "R[f]"] \arrow[d, dotted, "R[f]"'] & t \arrow[d, dotted, "R[f]"] \\
x' \arrow[r, dotted, "R[f]"'] & t'
\end{tikzcd}
\qquad
\begin{tikzcd}
g(x) \arrow[d, equal] \\
g(x')
\end{tikzcd}$$

we get $x(R[f] \cap R[g])x'$ and thus $x = x'$. Accordingly the upper horizontal square is a pullback as well and the map $R(\bar{g}) : R[d_0^R] \to R[d_0^{gR}]$ is a regular epimorphism, since so is $R(\bar{g}) : R[f]\square R \to g(R) \times g(R)$ in the regular category $\mathbb{E}$. The map $(g, \bar{g}) : R \to g(R)$ is therefore a hyperextremal epimorphism in $Ref\mathbb{E}$, and by Lemma 1.7 the reflexive relation $g(R)$ is an equivalence relation. $\square$

Definition 4.3. *[4] Let $\mathbb{E}$ be a pointed category. It is said to be punctually congruence hyperextensible, when, given any punctual span:*

$$\begin{tikzcd} Y \arrow[r, shift right, "s"'] & X \arrow[l, shift right, "f"'] \arrow[r, shift left, "g"] & Z. \arrow[l, shift left, "t"] \end{tikzcd}$$

and any equivalence relation R on X such that we have $R[f] \cap R[g] \subset R$, we get $R[f] \cap g^{-1}(t^{-1}(R)) \subset R$.

Clearly we can suppose moreover that $R \subset R[f]$: if it is not the case, choose $R' = R \cap R[f]$. The main interest of this notion comes from the fact that, as shown in [4], a category $\mathbb{E}$ is a Gumm category if and only if any fibre $Pt_Y\mathbb{E}$ is punctually congruence hyperextensible.

Proposition 4.5. *Any regular punctually Goursat category $\mathbb{E}$ is punctually congruence hyperextensible. Any regular Goursat category is a Gumm category.*

Proof. Since $\mathbb{E}$ is regular, it is enough to check the condition on punctual relations, see [4]. So let us consider a punctual relation:

$$\begin{tikzcd} Y \arrow[r, shift right, "s"'] & X \arrow[l, shift right, "f"'] \arrow[r, shift left, "g"] & Z. \arrow[l, shift left, "t"] \end{tikzcd}$$

and a relation R on X such that $R \subset R[f]$. According to Proposition 4.4, the direct image $g(R)$ is an equivalence relation; so that, in this regular context, we get $g^{-1}(g(R)) = R \bigvee R[g]$. The section t of g gives rise to the inclusion $t^{-1}(R) \subset g(R)$. Then we get:

$$g^{-1}(t^{-1}(R)) \cap R[f] \subset g^{-1}(g(R)) \cap R[f]$$

But this last term is:

$$(R \bigvee R[g]) \cap R[f] = R \bigvee (R[g] \cap R[f]) = R \bigvee \Delta_X = R$$

by Proposition 3.15, since $R \subset R[f]$.

As recalled above, it was shown in [4] that a category $\mathbb{E}$ is a Gumm one if and only if any fibre $Pt_Y\mathbb{E}$ is punctually congruence hyperextensible; whence the final assertion. $\square$

The relationship, in the regular context, between Goursat and Gumm categories was already stated in [16], but again still using Metatheorems.

5 The case of n-permutable regular categories

Following [8], given any pair (R, S) of reflexive relations on an object X in a regular category $\mathbb{E}$, let us denote by $(R, S)_n, n \geq 2$ the alternate composition $R \circ S \circ R \circ S...$ of length n which is a reflexive relation as well. Clearly we have: $(R, S)_n \subset (R, S)_{n+1}$ and $(S, R)_n \subset (R, S)_{n+1}$. Then call *$n$-permutable* a regular category satisfying $(R, S)_n = (S, R)_n$ for all pairs (R, S) of equivalence relations. Let us recall also a direct consequence of the Theorem 3.1 in [8]:

Theorem 5.1. *Given a regular category $\mathbb{E}$, TFAE:*
(i) the regular category $\mathbb{E}$ is n-permutable
(ii) for any pair (R, S) of equivalence relations in $\mathbb{E}$, $(R, S)_n$ is an equivalence relation as well.
In this case we get $(R, S)_n = R \bigvee S$ in $Equ\mathbb{E}$.

Since a n-permutable regular category $\mathbb{E}$ admits suprema of pairs of equivalence relations, Proposition 3.13 guarantees that $Equ\mathbb{E}$ has regular epimorphisms above regular epimorphisms in $\mathbb{E}$. In [11], there is a description of the *Polin variety*, showing it to be 4-permutable, but not congruence modular. Accordingly, in this variety $\mathbb{V}$, these regular epimorphisms in $Equ\mathbb{V}$ are not stable under pullback along maps in the fibres of $\mathcal{O}_{\mathbb{V}}$ and the category $Equ\mathbb{V}$ is not regular.

Proposition 5.1. *Given any $2n$-permutable or $(2n+1)$-permutable regular category $\mathbb{E}$, any regular epimorphism $f : X \twoheadrightarrow Y$ and any equivalence relation S on X, then the reflexive relation $f(S)^n$ is an equivalence relation in $\mathbb{E}$.*

Proof. In the first case, we have $(R[f], S)_{2n} = R[f] \bigvee S$, while, in the second we have $(R[f], S)_{2n+1} = R[f] \bigvee S$. Accordingly, by Corollary 2.1, in the first case $f((R[f], S)_{2n})$ is an equivalence relation, while so is $f((R[f], S)_{2n+1})$ in the second one. That, in both cases $m = 2n$ and $m = 2n + 1$, we have $f((R[f], S)_m) = f(S)^n$ is a direct consequence of the following lemma by:
$f((R[f], S)_{2n}) \subset (\Delta_Y, f(S))_{2n} = f(S)^n = f((S \circ R[f])_{2n-1}) \subset f((R[f], S)_{2n})$, and by: $f((R[f], S)_{2n+1}) \subset (\Delta_Y, f(S))_{2n+1} = f(S)^n = f((S \circ R[f])_{2n-1})$ with $f((S \circ R[f])_{2n-1}) \subset f((R[f], S)_{2n+1})$. □

Lemma 5.1. *Let $\mathbb{E}$ be a regular category and $f : X \twoheadrightarrow Y$ a regular epimorphism.*
(i) given any pair (S, T) of reflexive relations on X, we have $f(S) \circ f(T) = f((S \circ R[f]) \circ T)$.
(ii) for any reflexive relation S on X, we have $f(S)^n = f((S, R[f])_{2n-1})$.

Proof. In any regular category, we have $f(S \circ T) \subset f(S) \circ f(T)$ for any pair (S, T) of reflexive relations; so we get: $f((S \circ R[f]) \circ T) \subset f(S) \circ f(T)$.

By the Metatheorems [1], it is enough to check the converse in Set. Suppose that we have $yf(S) \circ f(T)y'$ in the set Y. So there an element $t \in Y$ such that $yf(T)t$ and $tf(S)y'$. This means that, in the set X, there are pairs of elements $(x, u), (v, x')$ such that we have: $f(x) = y, f(u) = t$ and xTu, $f(v) = t, f(x') = y'$ and vSx'. Accordingly we have $xTuR[f]vSx'$, namely $xS \circ R[f] \circ Tx'$ and $yf(S \circ R[f] \circ T)y'$.

In particular we have $f(S)^2 = f((S, R[f])_3)$. The end of the proof is made by induction. Suppose $f(S)^k = f((R[f], S)_{2k-1}), \forall k < n$. Then: $f(S)^n = f(S)^{n-1} \circ f(S) = f((S, R[f])_{2n-3}) \circ f(S) = f((S, R[f])_{2n-3} \circ R[f] \circ S) = f((S, R[f])_{2n-1}$. □

Theorem 5.2. *Given any regular category $\mathbb{E}$, TFAE:*
(i) the regular category $\mathbb{E}$ is $(2n+1)$-permutable

(ii) for any regular epimorphism $f : X \twoheadrightarrow Y$ in $\mathbb{E}$ and any equivalence relation S on X, then $f(S)^n$ is an equivalence relation.

Proof. It remains to show [(ii)⇒(i)]. So suppose (ii). We noticed in Proposition 3.15 that the direct image $d_1^R((d_0^R)^{-1}(S))$ is $R \circ S \circ R$. In presence of (ii), $(d_1^R((d_0^R)^{-1}(S))^n = (R \circ S \circ R)^n = (R, S)_{2n+1}$ is an equivalence relation. Accordingly the regular category $\mathbb{E}$ is $(2n+1)$-permutable. □

The previous result holds, in particular, for any variety of Universal Algebra.

References

[1] M. Barr – *Exact categories*, Springer L.N. in Math., **236** (1971), 1-120.

[2] D. Bourn – *Lower dimensional geometry of the notion of choice*, CMS Conférence Proc., **13**, Category Theory (1991), 55-73.

[3] D. Bourn – *Mal'cev categories and fibration of pointed objects*, Appl. Categ. Struct. **4** (1996), 307-327.

[4] D. Bourn – *Fibration of points and congruence modularity*, Algebra Universalis **52** (2005), 403-429.

[5] D. Bourn – *Congruence distributivity in Goursat and Mal'cev categories*, Appl. Categ. Struct. **13** (2005), 101-111.

[6] D. Bourn and M. Gran, – *Categorical aspects of modularity*, in :Galois theory, Hopf algebras, and Semiabelian categories, G.Janelidze, B.Pareigis, W.Tholen editors, Fields Institute Communications, **43**, Amer. Math. Soc., 2004, 77-100.

[7] D. Bourn and M. Gran – *Normal sections and direct product decompositions*, Communications in Algebra, 32 (2004), 3825-3842.

[8] A. Carboni, G. M. Kelly and M. C. Pedicchio – *Some remarks on Mal'tsev and Goursat categories*, Appl. Categ. Struct. **1** (1993), 385-421.

[9] A. Carboni, J. Lambek, M.C. Pedicchio – *Diagram chasing in Mal'cev categories*, J. Pure Appl. Algebra **69** (1991), 271-284.

[10] A. Carboni, M.C. Pedicchio, N. Pirovano – *Internal graphs and internal groupoids in Mal'cev categories*, CMS Conference Proceedings **13** (1992), 97-109.

[11] A. Day and R. Freese – *A characterization of identities implying congruence modularity I*, Canad. J. Math., **32** (1980), 1140-1167.

[12] T. Everaert – *Effective descent morphisms of regular epimorphisms*, J. Pure Appl. Algebra. **216** (2012), 1896-1904.

[13] T.H. Fay – *On categorical conditions for congruence to commute*, Algebra Universalis, **8** (1978), 173-179.

[14] E. Goursat – *Sur les substitutions orhtogonales*, Ann. Sci. Ec. Norm. Sup. **3** (1889), 9-102.

[15] M. Gran and D. Rodelo – *A new characterization of Goursat categories*, Appl. Categ. Struct. **20** (2012), 229-238.

[16] M. Gran and D. Rodelo – *Beck-Chevalley condition and Goursat categories*, Pré-Publicações Dept. Matemàtica Univ. Coimbra, dec. 2015

[17] H.P. Gumm – *Geometrical methods in congruence modular algebras*, Mem. Amer. Math. Soc. **45** (1983).

[18] J. Hagemann and A. Mitschke – *On n-permutable congruences*, Algebra Universalis **3** (1973), 8-12.

[19] G. Janalidze, L. Márki, W. Tholen and A. Ursini – *Ideal determined categorie*, Cah. Topol. Géom. Diff. Catég. **51** (2010), 115-125.

[20] G. Janelidze and M. Sobral – *Descent for regular epimorphims in Barr exact Goursat categories*, Appl. Categ. Struct. **11** (2011), 271-276.

[21] S. Lack – *The 3 by 3 lemma for Goursat regular categories*, Homology, Homotopy and Applications **6** (2004), 1-3.

[22] J. Lambek – *Goursat's theorem and homological algebra*, Can. Math. Bull. **7** (1964), 597-608.

[23] A. Mitschke – *Implication algebras are 3-permutable and 2-distributive*, Algebra Universalis **1** (1971), 182-186.

[24] J.G. Raftery – *Ideal determined varieties need not be congruence* 3-*permutable*, Algebra Universalis **31** (1994), 293-297.

[25] J. D. H. Smith – *Mal'cev varieties*, Springer Verlag, Lect. Notes Math. **554**, Springer Verlag (1976).

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