

Normalizers in the non-pointed case

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Abstract

The aim of this work is to point out a structural phenomenon hidden behind the existence of normalizers through the investigation of this property in the non-pointed case: given any category \mathbb{E} , a certain property of the fibration of points $\mathbf{Pt}_{\mathbb{E}}: Pt(\mathbb{E}) \rightarrow \mathbb{E}$ guarantees the existence of normalizers. This property becomes a characterization of this existence when \mathbb{E} is quasi-pointed and protomodular. This property is also showed to be equivalent to a property of the category $Grd\mathbb{E}$ of internal groupoids in \mathbb{E} which is a kind of dual, for the monomorphic internal functors, of the comprehensive factorization.

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Introduction

The first place where the question of the existence of normalizers was studied in a conceptual way, namely outside some specific contexts as groups, rings or Lie algebras, but more generally inside a semi-abelian context (which is a pointed context) is [19]. Modulo a slight shifting in the requirement of the involved universal property, it was showed in [13] that, in the pointed protomodular context, the existence of normalizers is unexpectedly equivalent to a much larger phenomenon, namely to the fact that the functor $K : Kt\mathbb{C} \rightarrow \mathbb{C}$ from the category $Kt\mathbb{C}$ (whose objects are the split extensions in \mathbb{C} :

$$0 \longrightarrow Kerf \xrightarrow{k_f} X \begin{array}{c} \xleftarrow{s} \\ \xrightarrow{f} \end{array} Y \longrightarrow 0$$

and morphisms are the natural morphisms between split extensions) to the ground category \mathbb{C} associating with any split extension the domain $Kerf$ of its kernel map is fibrant on monomorphisms. And it was also showed that this property has two heavy structural consequences, namely that \mathbb{C} is *action accessible* in the sense of [14] and *fiberwise algebraically cartesian closed* in the sense of [12].

The notion of normal subobject having a plain meaning in a non-pointed context, the one of normalizer is straightforward. The aim of this work was first to investigate whether there was, in any non-pointed context, a condition which characterizes the existence of normalizers, or in other words to transfer the pointed characteristic condition of [13] to a non-pointed one. Actually we do better, introducing two other equivalent formulations which, here again, are far from being expected (even in the pointed context) and point out a much larger phenomenon:

1) the existence of a universal decomposition for the monomorphisms between split epimorphisms in \mathbb{E} ; namely, any monomorphism $(y, x) : (f' s') \rightarrow (f, s)$ produces a universal decomposition where the left hand side part is a pullback:

$$\begin{array}{ccc}
 X' \xrightarrow{x} X & & X' \xrightarrow{\bar{u}} \bar{X} \xrightarrow{\bar{w}} X \\
 f' \downarrow \uparrow s' & & f' \downarrow \uparrow s' \\
 Y' \xrightarrow{y} Y & & Y \xrightarrow{u} \bar{Y} \xrightarrow{w} Y \\
 & & \bar{f} \downarrow \uparrow \bar{s} \\
 & & f \downarrow \uparrow s
 \end{array}$$

$\begin{array}{c} \curvearrowright \\ x \\ \curvearrowleft \end{array}$
 $\begin{array}{c} \curvearrowright \\ y \\ \curvearrowleft \end{array}$

2) the existence of a universal decomposition in the category $Grd\mathbb{E}$ of internal groupoids in \mathbb{E} which is a kind of dual, for the monomorphic functors, of the comprehensive factorization [24], [3]; namely, any monomorphic functor between internal groupoids $(v_0, v_1) : \underline{U}_1 \rightarrow \underline{T}_1$ produces a universal decomposition where the internal functor (u_0, u_1) is a discrete fibration:

$$\begin{array}{ccc}
 U_1 \xrightarrow{v_1} T_1 & & U_1 \xrightarrow{u_1} X_1 \xrightarrow{w_1} T_1 \\
 d_0 \downarrow \uparrow d_1 & & d_0 \downarrow \uparrow d_1 \\
 U_0 \xrightarrow{v_0} T_0 & & U_0 \xrightarrow{u_0} X_0 \xrightarrow{w_0} T_0 \\
 & & d_0 \downarrow \uparrow d_1
 \end{array}$$

$\begin{array}{c} \curvearrowright \\ v_1 \\ \curvearrowleft \end{array}$
 $\begin{array}{c} \curvearrowright \\ v_0 \\ \curvearrowleft \end{array}$

The universal decompositions 1) and 2) in the category Gp of groups are described in detail at the end of Section 3.2.

In the same way as in the pointed case, these formulations imply that: 1) any equivalence relation R has a centralizer (i.e action distinctiveness) and 2) in the protomodular context any subobject in a fibre $Pt_Y \mathbb{E}$ has a centralizer (i.e fiberwise algebraic cartesian closedness), but this time in the non-pointed context and with different and more limpid proofs (see Propositions 7.1 and 7.2 for instance). In the exact protomodular context, the first point implies action accessibility; non-pointed action accessibility is all the more important that it brings the non-pointed Schreier-Mac Lane theorem, see Theorem 4.11 in [7].

Examples of non-pointed categories which satisfy this property are given with any slice or coslice category of the pointed protomodular categories with normalizers, and, in various circumstances, with any fibre $Grd_Y \mathbb{E}$ of the fibration $(\)_0 : Grd \mathbb{E} \rightarrow \mathbb{E}$ of internal groupoids, for instance and for quite different reasons, when \mathbb{E} is a Mal'tsev category or when \mathbb{E} is an elementary topos.

This new approach allows us also to clarify the relationship between existence of normalizers and Mal'tsevness or (strong) protomodularity of the ground category \mathbb{E} , see Lemma 2.1, Corollary 2.1 and Section 5. It sheds new light on the non-pointed additive setting (according to [6]) as well, giving rise to several subtle differentiations, see Section 6.

The article is organized along the following lines:

Section 1) is devoted to introducing the categorical conceptual setting which leads to the notion of the existence of normalizers, namely the categories which are Θ -coreflexive on monomorphisms.

Section 2) investigates the particular case of the categories which are \mathbb{N} -coreflexive on monomorphisms (namely which satisfy any of the two equivalent formulations), and it determines their first properties: in particular they are showed to be necessarily Mal'tsev categories.

Section 3) makes the bridge with the already investigated pointed case.

Section 4) is devoted to the stability under slicing and coslicing.

Section 5) is devoted to investigating the relationship between \mathbb{N} -coreflexeness and protomodularity.

Section 6) is devoted to the meaning of the \mathbb{N} -coreflexiveness in the non-pointed additive setting.

Section 7) is devoted to the proofs of the above mentioned strong structural consequences of the \mathbb{N} -coreflexiveness.

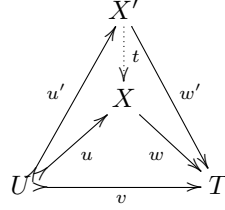
Section 8) extends many aspects of the previous results to the partial Mal'tsev context in the sense of [9].

Section 9) is devoted to proving that, when \mathbb{E} is locally cartesian closed [23] (and a fortiori a topos), any fibre $Grd_Y \mathbb{E}$ is \mathbb{N} -coreflexive on monomorphisms or, equivalently has normalizers, and to investigating what is remaining of the \mathbb{N} -coreflexiveness on the larger fibers $Cat_Y \mathbb{E}$.

1 Θ -coreflexive categories

Let Θ be a class of morphisms in a finitely complete category \mathbb{E} : it is said to be *quasi-proper* when it contains the isomorphisms, it is stable under composition with them, it is stable under product and pullback; it is said to be *proper* when it contains the isomorphisms, is stable under composition and pullback and is such that, whenever $g.f$ and g are in Θ , the map f is in Θ . It is clear that a proper class is quasi-proper. A class is said to satisfy the "*three out of two*" condition whenever, if a pair of maps among f , g and $g.f$ is in Θ , the third one is in Θ , and the "*monomorphic three out of two*" (resp. "*split monomorphic three out of two*") *condition* whenever the previous condition is reduced to the case where f is a monomorphism (resp. a split monomorphism).

Definition 1.1. Let Θ be any class in a category \mathbb{E} , and $v : U \rightarrow T$ a monomorphism. We say that $v = w.u$ with $u \in \Theta$ is an extremal decomposition (with respect to Θ), when any other decomposition $v = w'.u'$ with $u' \in \Theta$:



produces a unique factorization t .

Clearly the map u is a monomorphism, and a monomorphism $v : U \rightarrow T$ is in Θ if and only if $v = 1_T.v$ is an extremal decomposition.

Lemma 1.1. Suppose Θ quasi-proper. If $v = w.u$ is an extremal decomposition, then the coreflector w is necessarily a monomorphism.

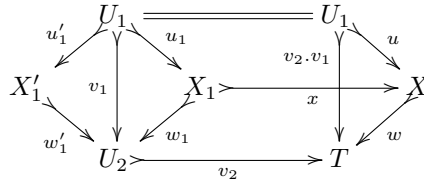
Proof. Complete the following right hand side square with the kernel equivalence relations $R[v]$ and $R[w]$ of the maps v and w :

$$\begin{array}{ccccc}
 R[v] = U & \xrightleftharpoons[1_U]{1_U} & U & \xrightarrow{v} & T \\
 R(u) \downarrow & & \downarrow u & & \parallel \\
 R[w] & \xrightleftharpoons[p_1]{p_0} & X & \xrightarrow{w} & T
 \end{array}$$

Since the left hand side part of the diagram is a joint pullback and u is in Θ which is quasi-proper, then $R(u)$ is in Θ . Accordingly the decomposition $v = (w.p_0).R(u)$ produces a unique factorization through w . So, we get $p_0 = p_1$, and w is a monomorphism. \square

Lemma 1.2. Suppose Θ is quasi-proper. Let $v = v_2.v_1$ be a monomorphic decomposition of v and $v = w.u$ an extremal decomposition. Then the extremal decomposition $v_1 = w_1.u_1$ is given by the pullback of the extremal decomposition $v = w.u$ along the monomorphism v_2 .

Proof. The map v_2 being a monomorphism, the following vertical rectangle is pullback. Introduce the lower quadrangle as a pullback and denote u_1 the canonical factorization:



So, the upper quadrangle is a pullback and u_1 is a monomorphism in Θ . Let $v_1 = w'_1.u'_1$ be a decomposition with u'_1 a monomorphism in Θ . Then the decomposition $v = v_2.v_1 = (v_2.w'_1).u'_1$ produces a factorization $t : X'_1 \rightarrow X$ satisfying $w.t = v_2.w'_1$ which assures the factorization through the vertex X_1 of the lower quadrangled pullback. \square

Definition 1.2. A monomorphism $v : U \rightarrow T$ in \mathbb{E} will be said to be a Θ -outsider when $v = v.1_U$ is an extremal decomposition.

It is then straightforward that a map in Θ which is also Θ -outsider is an isomorphism.

Lemma 1.3. Suppose $v : U \rightarrow T$ is a Θ -outsider. Then v has no other monomorphic decomposition $v = w.u$ with u in Θ than $v = v.1_U$, up to isomorphism.

Proof. Suppose $v = u'.w'$ with w' monomorphic, then the factorization τ :

$$\begin{array}{ccccc}
 & & X' & & \\
 & u' \nearrow & \downarrow \tau & \nwarrow w' & \\
 U & \xrightarrow{1_u} & U & \xrightarrow{v} & T
 \end{array}$$

is a monomorphism since so is w' ; then it is an isomorphism since it is split by u' and u' is an isomorphism as well. \square

Lemma 1.2 gives rise immediately to:

Lemma 1.4. Suppose Θ is quasi-proper. Let $w = m.w'$ be any monomorphic decomposition and w a Θ -outsider. Then w' is a Θ -outsider.

Lemma 1.5. If Θ is proper and $v = w.u$ is an extremal decomposition, the map w is Θ -outsider.

Here is the main tool of this work:

Definition 1.3. Let Θ be a quasi-proper class in a category \mathbb{E} . This category will be said Θ -coreflexive on monomorphisms when any monomorphism $v : U \rightarrow T$ has an extremal decomposition $v = w.u$ with respect to the class Θ . It will be said stably Θ -coreflexive on monomorphisms when these extremal decompositions are stable under pullbacks along maps in Θ .

When \mathbb{E} is stably Θ -coreflexive, the Θ -outsider monomorphisms are stable under pullback along maps in Θ .

Define a normal monomorphism in the category Gp of groups as an injective homomorphism $m : H \rightarrow G$ such that $m(H)$ is a normal subgroup of G . The class N of normal monomorphisms is quasi-proper in Gp but not proper.

Example 1.1. The category Gp is N -coreflexive on monomorphisms, but not stably N -coreflexive.

Proof. Starting from any monomorphism $n : H \rightarrow G$, its N -coreflection is given by $\tilde{n} : H \xrightarrow{\cong} n(H) \rightarrow N(n(H))$ where $N(G')$ is the normalizer of the subgroup $G' \rightarrow G$. \square

We shall give further examples of Θ -coreflexive categories in the next sections. We shall see later on in Section 2.3 how the above question relative to the existence of normalizers in Gp can be “embedded”, via the functor $\nabla : Gp \rightarrow EquGp$ (where $EquGp$ is the category of congruences in groups and ∇_G is the undiscrete congruence on G), into the question of the existence a class Θ in $EquGp$ which satisfies the three out of two condition and is stably Θ -coreflexive on monomorphisms in $EquGp$.

Remark 1.1. *Suppose that Θ is proper and \mathbb{E} is Θ -coreflexive on monomorphisms. Then, according to Lemma 1.1 and to the remark following immediately the definition of a Θ -outsider, these conditions produce an idempotent operator on subobjects of any object T whose “closed subobject part” is given by the Θ -outsiders with codomain T and “dense subobjects part” is given the monomorphisms with codomain T which are in Θ . However this idempotent operator does not preserve the inclusion of subobjects and therefore is not a closure operator.*

Lemma 1.6. *Suppose Θ is proper and \mathbb{E} is Θ -coreflexive on monomorphisms. When, in addition, Θ satisfies the monomorphic three out of two condition, any decomposition $v = w.u$ of the monomorphism v with u in Θ and w a Θ -outsider monomorphism is extremal.*

Proof. Suppose that $v = w'.u'$ is the extremal decomposition:

$$\begin{array}{ccccc}
 & & X' & & \\
 & u' \nearrow & \uparrow \bar{u} & \nwarrow w' & \\
 U & \xrightarrow{u} & X & \xrightarrow{w} & T
 \end{array}$$

There is a factorization \bar{u} which is a monomorphism since so is w and which, according to Lemma 1.4, is a Θ -outsider since so is w . Since, moreover, Θ satisfies the *monomorphic three out of two* condition, then \bar{u} is in Θ ; accordingly it is an isomorphism. \square

2 \blacktriangleright -coreflexive categories

Given a finitely complete category \mathbb{E} , recall [4] that $Pt\mathbb{E}$ denotes the category whose objects are the split epimorphisms (where split epimorphism means split epimorphism with a given splitting) in \mathbb{E} and whose arrows are the commuting squares between such split epimorphisms, and that $\blacktriangleright_{\mathbb{E}} : Pt\mathbb{E} \rightarrow \mathbb{E}$ denotes the functor associating with any split epimorphism its codomain: it is *the fibration of points*. The $\blacktriangleright_{\mathbb{E}}$ -cartesian maps are nothing but the pullbacks of split epimorphisms and determine a proper class in $Pt\mathbb{E}$ we shall denote by \blacktriangleright for short.

Recall also that a Mal'tsev category is a category in which any reflexive relation is an equivalence relation, see [16] and [17]. Let us begin by a first clarification:

Lemma 2.1. *Any finitely complete category \mathbb{E} in which any \mathbb{Q} -invertible morphism is a \mathbb{Q} -outsider is necessarily a Mal'tsev category.*

Proof. Let be given a reflexive relation $(d_0, d_1) : R \rightrightarrows X$ on the object X . Then consider the following left hand side commutative diagram in $Pt\mathbb{E}$:

$$\begin{array}{ccc}
 & & (d_0, d_1) \\
 & \curvearrowright & \\
 R & \xrightarrow{s_1} & R[d_0] \xrightarrow{(d_1 \cdot p_0, d_1 \cdot p_1)} X \times X \\
 \uparrow d_0 & \uparrow s_0 & \uparrow p_0 \quad \uparrow s_0 \\
 X & \xrightarrow{s_0} & R \xrightarrow{d_1} X \\
 \downarrow d_0 & \downarrow s_0 & \downarrow p_0 \quad \downarrow s_0 \\
 & & X \times X \\
 & \curvearrowleft & \\
 & & (d_0, d_1)
 \end{array}
 \qquad
 \begin{array}{ccc}
 R[d_0] & \xrightarrow{d_2} & R \\
 \uparrow d_0 & \uparrow s_0 & \uparrow d_0 \\
 X & \xrightarrow{d_1} & X \\
 \downarrow d_0 & \downarrow s_0 & \downarrow d_0 \\
 X & \xrightarrow{d_1} & X
 \end{array}$$

Then, since the whole rectangle is a \mathbb{Q} -outsider, there is a factorization given by the right hand side diagram such that $(d_0, d_1) \cdot d_2 = (d_1 \cdot p_0, d_1 \cdot p_1)$ which shows that we have $R[d_0] \subset (d_1)^{-1}(R)$, so that R is an equivalence relation. \square

2.1 Definition and first properties

Definition 2.1. *We shall say that the category \mathbb{E} is (resp. stably) \mathbb{Q} -coreflexive on monomorphisms when the category $Pt\mathbb{E}$ is (resp. stably) coreflexive on monomorphisms with respect to the class of $\mathbb{Q}_{\mathbb{E}}$ -cartesian maps.*

Proposition 2.1. *Let \mathbb{E} be $\mathbb{Q}_{\mathbb{E}}$ -coreflexive on monomorphisms. If the monomorphism $(\gamma, \bar{\gamma}) : (a', b') \rightarrow (a, b)$ in $Pt\mathbb{E}$ has a decomposition $(w', \bar{w}') \cdot (u', \bar{u}')$ where (w', \bar{w}') is $\mathbb{Q}_{\mathbb{E}}$ -invertible and (u', \bar{u}') is a $\mathbb{Q}_{\mathbb{E}}$ -cartesian monomorphism, then its extremal decomposition $(w, \bar{w}) \cdot (u, \bar{u})$ is such that (w, \bar{w}) is $\mathbb{Q}_{\mathbb{E}}$ -invertible as well. Accordingly any $\mathbb{Q}_{\mathbb{E}}$ -invertible monomorphism in $Pt\mathbb{E}$ is a $\mathbb{Q}_{\mathbb{E}}$ -outsider.*

Proof. Consider the following diagrams of split epimorphisms where w' is invertible and the right hand side diagram is an extremal decomposition in $Pt\mathbb{E}$:

$$\begin{array}{ccc}
 & & \bar{\gamma} \\
 & \curvearrowright & \\
 \bar{U} & \xrightarrow{\bar{a}'} & \bar{A}' \xrightarrow{\bar{w}'} \bar{T} \\
 \uparrow a' & \uparrow b' & \uparrow \bar{a}' \quad \uparrow \bar{b}' \\
 U & \xrightarrow{u'} & A' \xrightarrow{w'} T \\
 \downarrow a' & \downarrow b' & \downarrow \bar{a}' \quad \downarrow \bar{b}' \\
 & & U & \xrightarrow{u} & A \xrightarrow{w} T \\
 & \curvearrowleft & \\
 & & \gamma
 \end{array}$$

Then there is a factorization:

$$\begin{array}{ccc} \bar{A}' & \xrightarrow{\bar{t}} & \bar{A} \\ \bar{a}' \downarrow \uparrow \bar{b}' & & \bar{a} \downarrow \uparrow \bar{b} \\ A & \xrightarrow{t} & A \end{array}$$

such that $w.t = w'$. Now, since w' is an isomorphism and w is a monomorphism, the map w is an isomorphism. The last assertion is then straightforward. \square

Corollary 2.1. *Any category \mathbb{E} which is $\mathfrak{M}_{\mathbb{E}}$ -coreflexive on monomorphisms is a Mal'tsev category.*

Proof. It is a straightforward consequence of the previous Proposition and of Lemma 2.1. \square

2.2 Internal groupoids

We shall be also interested in the category $Grd\mathbb{E}$ of internal groupoids in \mathbb{E} and in the proper class DiF of those internal functors which are discrete fibrations (we shall call them fibrant morphisms). Recall that an internal groupoid \underline{Y}_1 is a reflexive graph $(d_0^Y, d_1^Y) : Y_1 \rightrightarrows Y_0$ endowed with a map $d_2^Y : R[d_0^Y] \rightarrow Y_1$ making the following diagram a 3-truncated simplicial object:

$$\underline{Y}_1 : \begin{array}{ccccc} & \overset{R(d_2^Y)}{\curvearrowright} & & \overset{d_2^Y}{\curvearrowright} & \\ & \xrightarrow{p_2} & R[d_0^Y] & \xrightarrow{p_1} & Y_1 \xrightarrow{d_1^Y} Y_0 \\ R[d_0^Y]^2 & \xrightarrow{p_1} & R[d_0^Y] & \xrightarrow{p_0} & Y_1 \xrightarrow{s_0} Y_0 \\ & \xrightarrow{p_0} & & & \xrightarrow{d_0^Y} \end{array}$$

In the set-theoretical context, we have $d_2^Y(\phi, \psi) = \psi \cdot \phi^{-1}$. We shall begin by the following observation:

Theorem 2.1. *The category \mathbb{E} is \mathfrak{M} -coreflexive (resp. stably \mathfrak{M} -coreflexive) on monomorphisms if and only if the category $Grd\mathbb{E}$ of internal groupoids in \mathbb{E} is DiF -coreflexive (resp. stably DiF -coreflexive) on monomorphisms, where DiF is the class of fibrant morphisms.*

This will be the consequence of the following general result:

Proposition 2.2. *Let $U : \mathbb{E} \rightarrow \mathbb{F}$ be a left exact functor, Θ a proper class in \mathbb{F} . Then $\Theta' = U^{-1}(\Theta)$ is a proper class in \mathbb{E} . Suppose that U has a left exact left adjoint G such that $G.U$ preserves the maps in Θ . Suppose moreover the natural transformation $\eta : 1_{\mathbb{F}} \Rightarrow U.G$ is in Θ . If the category \mathbb{E} is Θ' -coreflexive on monomorphisms (resp. stably Θ' -coreflexive), then the category \mathbb{F} is Θ -coreflexive on monomorphisms (resp. stably Θ -coreflexive) as well.*

If, moreover, the functor U is monadic, the converse is true, namely if the category \mathbb{F} is Θ -coreflexive on monomorphisms (resp. stably Θ -coreflexive), then the category \mathbb{E} is Θ' -coreflexive on monomorphisms (resp. stably Θ' -coreflexive) as well. Moreover the functor U preserves and reflects the extremal decompositions.

Proof. The fact that Θ' is proper as soon as Θ is proper is straightforward. Now let $v : S \rightarrow T$ be a monomorphism in \mathbb{F} . Since G is left exact, the map $G(v)$ is a monomorphism in \mathbb{E} . Let $m.n : G(S) \rightarrow W \rightarrow G(T)$ be its extremal decomposition in the category \mathbb{E} , the right hand side square below be a pullback in \mathbb{F} and u the induced factorization:

$$\begin{array}{ccccc}
 & & v & & \\
 & & \curvearrowright & & \\
 S & \xrightarrow{u} & X & \xrightarrow{w} & T \\
 \eta_S \downarrow & & l \downarrow & & \downarrow \eta_T \\
 U.G(S) & \xrightarrow{U(n)} & U(W) & \xrightarrow{U(m)} & U.G(T) \\
 & & \curvearrowleft & & \\
 & & U.G(v) & &
 \end{array}$$

The map η_T being in Θ , so is l ; the maps η_S and $U(n)$ being in Θ (since n is in $\Theta' = U^{-1}(\Theta)$), so is u .

Let us show that $w.u$ is extremal in \mathbb{F} . So, let $v = w'.u'$ with u' a monomorphism in Θ . The map $U.G(u')$ being in Θ , the map $G(u')$ is in Θ' and the decomposition $G(v) = G(u').G(w')$ produces a factorization $t : G(X') \rightarrow W$ in \mathbb{E} such that we have $m.t = G(w')$ and which, by adjonction, determines a map $\tau : X' \rightarrow U(W)$ such that $U(m).\tau = \eta_T.w'$; whence the desired factorization $\bar{\tau} : X' \rightarrow X$. Suppose moreover \mathbb{E} is stably Θ' -coreflexive. Starting with a map $\theta : T' \rightarrow T$ in Θ , the pullback along θ in \mathbb{F} preserves the previous construction, since $G(\theta)$ is in Θ' and U left exact. Accordingly the category \mathbb{F} is stably Θ -coreflexive.

Conversely we shall show that when the monad $(T = U.G, \eta, \mu)$ on \mathbb{F} is such that T is left exact and preserves the maps in Θ and such that $\eta : 1_{\mathbb{F}} \Rightarrow T$ is in Θ , the category Alg^T is Θ' -coreflexive on monomorphisms (resp. stably Θ' -coreflexive) as soon as the category \mathbb{F} is Θ -coreflexive on monomorphisms (resp. stably Θ -coreflexive) where Θ' is $(U^T)^{-1}(\Theta)$. So, let $v : (U, \alpha) \rightarrow (T, \beta)$ a monomorphism in Alg^T . Let us consider the following diagram, where the lower row is the extremal decomposition of v in \mathbb{E} :

$$\begin{array}{ccccc}
 & & T(v) & & \\
 & & \curvearrowright & & \\
 T(U) & \xrightarrow{T(u)} & T(X) & \xrightarrow{T(w)} & T(T) \\
 \alpha \downarrow & & \xi \downarrow & & \downarrow \beta \\
 U & \xrightarrow{u} & X & \xrightarrow{w} & T \\
 & & \curvearrowleft & & \\
 & & v & &
 \end{array}$$

Let us show that the object X of \mathbb{F} is endowed with a T -algebra structure ξ . The monomorphisms η_U and $T(u)$ are in Θ . So that $\beta.T(W).(T(u).\eta_U) =$

$\beta \cdot \eta_T \cdot w \cdot u = w \cdot u = v$ produces a factorization $\xi : T(X) \rightarrow X$ which is easily seen to be a T -algebra structure since w is monomorphic; this makes $v = w \cdot u$ a decomposition in Alg^T ; it is then straightforward to check that it is extremal for the class Θ' . This construction shows that the functor U preserves and reflects the extremal decompositions.

The pullback stable aspect of these results is also straightforward from the left exactness of the endofunctor T . \square

Theorem 2.1 is a consequence of the fact that $Grd\mathbb{E}$ is monadic on $Pt\mathbb{E}$ by a monad satisfying the assumptions of the previous theorem, relatively to the class of \mathbb{E} -cartesian maps, see [3]. So, \mathbb{E} is \mathbb{E} -coreflexive on monomorphisms if and only if any monomorphic internal functor $(v_0, v_1) : \underline{U}_1 \rightarrow \underline{T}_1$ between groupoids produces an extremal decomposition where the internal functor (u_0, u_1) is a fibrant morphism:

$$\begin{array}{ccc}
 U_1 \xrightarrow{v_1} T_1 & & U_1 \xrightarrow{u_1} X_1 \xrightarrow{w_1} T_1 \\
 d_0 \downarrow \uparrow d_1 \quad d_0 \downarrow \uparrow d_1 & & d_0 \downarrow \uparrow d_1 \quad d_0 \downarrow \uparrow d_1 \quad d_0 \downarrow \uparrow d_1 \\
 U_0 \xrightarrow{v_0} T_0 & & U_0 \xrightarrow{u_0} X_0 \xrightarrow{w_0} T_0
 \end{array}$$

$\begin{array}{c} \curvearrowright \\ v_1 \\ \curvearrowleft \\ v_0 \end{array}$

It is a kind of dual for monomorphic functors of the comprehensive factorization for internal functors between groupoids described in [24] and [3]. Actually *the result is even more precise*: if a monomorphism in $Pt\mathbb{E}$ is underlying a functor between groupoids as on the left hand side

$$\begin{array}{ccc}
 U_1 \xrightarrow{v_1} T_1 & & U_1 \xrightarrow{u_1} X_1 \xrightarrow{w_1} T_1 \\
 d_0 \downarrow \uparrow d_1 \quad d_0 \downarrow \uparrow d_1 & & d_0 \downarrow \uparrow d_1 \quad d_0 \downarrow \uparrow d_1 \quad d_0 \downarrow \uparrow d_1 \\
 U_0 \xrightarrow{v_0} T_0 & & U_0 \xrightarrow{u_0} X_0 \xrightarrow{w_0} T_0
 \end{array}$$

$\begin{array}{c} \curvearrowright \\ v_1 \\ \curvearrowleft \\ v_0 \end{array}$

then the extremal decomposition in $Pt\mathbb{E}$ provides the middle vertical part with a unique groupoid structure, making a fibrant the left hand side internal functor.

2.3 Abstract normalizers

Let us recall the following:

Definition 2.2. A monomorphism u in \mathbb{E} is said to be normal to an equivalence relation R when:

- i) we have: $u^{-1}(R) = \nabla_U$
- ii) the induced internal functor:

$$\begin{array}{ccc} U \times U & \xrightarrow{\tilde{u}} & R \\ p_0 \downarrow \uparrow p_1 & d_0 \downarrow \uparrow d_1 & \\ U & \xrightarrow{u} & X \end{array}$$

is a discrete fibration.

In the category *Set* of sets, when U is not empty, it is equivalent to saying that U is an equivalence class of R . Clearly, in this category, a monomorphism can be normal to many equivalence relations. In particular the inclusion $\emptyset \rightarrow X$ is normal to any equivalence relation R on X and in particular to ∇X . Recall now the following definition from [13]:

Definition 2.3. A category \mathbb{E} has normalizers when, for any monomorphism $v : U \rightarrow T$, there is a pair (u, R_v) with R_v normal to $u : U \rightarrow X$ and a factorization $w : X \rightarrow T$ such that $v = w.u$ which is universal with respect to this kind of specific decomposition of v .

By the universal property of a normalizer, this equivalence relation R_v is the largest equivalence relation R on X to which u is normal.

Proposition 2.3. Suppose \mathbb{E} is \mathfrak{N} -coreflexive on monomorphisms. Then any monomorphism $v : U \rightarrow T$ in \mathbb{E} has a normalizer in the previous sense. Any normal monomorphism $u : U \rightarrow X$ in \mathbb{E} admits a largest equivalence R_u on X to which u is normal.

Proof. It is a straightforward consequence of Theorem 2.1 and Proposition 2.1 applied to the following monomorphic morphism of equivalence relation:

$$\begin{array}{ccc} U \times U & \xrightarrow{v \times v} & T \times T \\ p_0^U \downarrow \uparrow p_1^U & p_0^T \downarrow \uparrow p_1^T & \\ U & \xrightarrow{v} & T \end{array}$$

□

2.4 Aspects of normal subobjects in Mal'tsev categories

In this section we shall show that in the Mal'tsev context there is a certain kind of monomorphisms which are normal in a unique way.

Proposition 2.4. *Suppose \mathbb{C} is a Mal'tsev category. Then a split epimorphism of equivalence relations as on the left hand side:*

$$\begin{array}{ccc}
 R_X & \xrightleftharpoons{d_0^X} & X \\
 R_f \uparrow & R_s \uparrow & \uparrow f \\
 R_Y & \xrightleftharpoons{d_1^Y} & Y
 \end{array}
 \qquad
 \begin{array}{ccc}
 R_X & \xleftarrow{s_0^X} & X \\
 R_f \uparrow & R_s \uparrow & \uparrow f \\
 R_Y & \xleftarrow{s_0^Y} & Y
 \end{array}$$

is fibrant if and only if the right hand side diagram is a pullback in $Pt\mathbb{E}$.

Proof. Obviously any fibrant morphism fulfils the condition of our assertion. Now, in any category, the pullback of R_f along s_0^Y is nothing but $R_X \cap R[f]$. So that the right hand side diagram is a pullback if and only if $R_X \cap R[f] = \Delta_X$. Suppose now \mathbb{C} is a Mal'tsev category and $R_X \cap R[f] = \Delta_X$. Consider the following diagram where the lower square is a pullback:

$$\begin{array}{ccc}
 & R_X & \\
 & \downarrow \phi & \swarrow d_0^X \\
 R_f \curvearrowright & P & \xrightleftharpoons{\pi_X} X \\
 & \downarrow \pi_f & \downarrow f \\
 & R_Y & \xrightleftharpoons{d_0^Y} Y
 \end{array}$$

Let ϕ be the natural factorization. Thanks to the Yoneda embedding, it is easy to check that, in any kind of category, $R_X \cap R[f] = \Delta_X$ implies that ϕ is a monomorphism. When, in addition, the category \mathbb{C} is a Mal'tsev category, the factorization ϕ , being involved in a pullback of split epimorphisms, is necessarily a strong epimorphism, see [5]. Accordingly ϕ is an isomorphism and the leg d_0 of the relation R in $Pt\mathbb{C}$ is \mathbb{R} -cartesian. The same holds for the leg d_1 . \square

Proposition 2.5. *Suppose \mathbb{C} is a Mal'tsev category. Given any split monomorphism (s, f) , there is at most one equivalence relation R on its codomain X such that $s^{-1}(R) = \nabla Y$ and $R \cap R[f] = \Delta X$. This equivalence relation R is the smallest among the equivalence relations S on X such that $s^{-1}(S) = \nabla Y$ and the monomorphism s is normal to R . Accordingly s is normal to at most one equivalence relation R such that $R \cap R[f] = \Delta X$.*

Proof. Suppose $s^{-1}(R) = \nabla Y$. Then consider the following diagram:

$$\begin{array}{ccccc}
 Y \times Y & \xrightarrow{\bar{s}} & R & \xrightarrow{(f, d_0^R, f, d_1^R)} & Y \times Y \\
 p_0^Y \downarrow \uparrow p_1^Y & & d_0^R \downarrow \uparrow d_1^R & & p_0^Y \downarrow \uparrow p_1^Y \\
 Y & \xrightarrow{s} & X & \xrightarrow{f} & Y
 \end{array}$$

We get a split epimorphism of equivalence relations $R \rightrightarrows \nabla X$. If, in addition, we have $R \cap R[f] = \Delta X$ and \mathbb{C} is a Mal'tsev category, it is a fibrant morphism according to the previous proposition, and this is also the case of its splitting $\nabla Y \rightarrow R$. This last point means that s is normal to R . On the other hand, the pair (s_0^R, \tilde{s}) is jointly strongly epic since any of the downward right hand side squares is a pullback. If S is another equivalence on X such that $s^{-1}(S) = \nabla Y$, then $s^{-1}(R \cap S) = \nabla Y$. Now consider the following factorizations:

$$\begin{array}{ccccc}
 & & R \cap S & & \\
 & \nearrow^{s_0^{R \cap S}} & \downarrow j & \nwarrow_{\tilde{s}_{R \cap S}} & \\
 X & \xrightarrow{s_0^R} & R & \xleftarrow{\tilde{s}} & Y \times Y
 \end{array}$$

they show that j is an isomorphism and $R \subset S$. Accordingly R is the smallest among the equivalence relations S on X such that $s^{-1}(S) = \nabla Y$. Whence its uniqueness and the last assertion. \square

Corollary 2.2. *Suppose \mathbb{C} is a Mal'tsev category. An object X is affine if and only if there is an equivalence relation R on $X \times X$ such that $(s_0^X)^{-1}(R) = \nabla X$ and $R \cap R[p_1^X] = \Delta(X \times X)$. In this case the equivalence relation in question is the following one:*

$$\begin{array}{ccc}
 & X \times p_0^X & \\
 X \times X \times X & \xrightarrow{\quad} & X \times X \\
 & \xleftarrow{X \times s_0^X} & \\
 & (p, p_1^X \cdot X \times p_1^X) &
 \end{array}$$

namely we have $(a, b)R(a', b')$ if and only if $a' = p(a, b, b')$.

Proof. In a Mal'tsev category an object X is endowed to at most one Mal'tsev operation $p : X \times X \times X \rightarrow X$; in this case X is said to be affine and the following middle vertical diagram:

$$\begin{array}{ccccc}
 X \times X & \xrightarrow{s_0^X \times X} & X \times X \times X & \xrightarrow{(p_1^X \cdot X \times p_0^X, p_1^X \cdot X \times p_1^X)} & X \times X \\
 p_0^X \downarrow \uparrow p_1^X & & X \times p_0^X \downarrow \uparrow (p, p_1^X \cdot X \times p_1^X) & & p_0^X \downarrow \uparrow p_1^X \\
 X & \xrightarrow{s_0^X} & X & \xrightarrow{p_1^X} & X
 \end{array}$$

produces the desired equivalence relation. Conversely, according to the previous proposition, the following diagram:

$$\begin{array}{ccccc}
 X \times X & \xrightarrow{\tilde{s}_0^X} & R & \xrightarrow{(p_1^X \cdot d_0^R, p_1^X \cdot d_1^R)} & X \times X \\
 p_0^X \downarrow \uparrow p_1^X & & d_0^R \downarrow \uparrow d_1^R & & p_0^X \downarrow \uparrow p_1^X \\
 X & \xrightarrow{s_0^X} & X & \xrightarrow{p_1^X} & X
 \end{array}$$

makes fibrant the right hand side morphism of equivalence relation. Accordingly the square indexed by 0 is a pullback and we get $R \simeq X \times X \times X$, so that $X \times X \times X \simeq R \xrightarrow{p_0^X, d_1^R} X$ produces the desired Mal'tsev operation \square

3 Examples of \mathbb{Q} -coreflexive categories

Recall that a category \mathbb{C} is protomodular if and only if the base-change functors with respect to the fibration of points $\mathbb{Q}_{\mathbb{C}}: Pt\mathbb{C} \rightarrow \mathbb{C}$ are conservative, or equivalently if and only if the class of $\mathbb{Q}_{\mathbb{C}}$ -cartesian maps in $Pt\mathbb{C}$ satisfies the three out of two condition, see [4]. What is interesting here is that, in a protomodular category, a monomorphism is normal to at most one equivalence relation; so that, for a monomorphism, being normal becomes a property. On the other hand any protomodular category is a Mal'tsev one [5].

The categories Gp of groups, Rg of (non-unitary) rings and $R-Lie$ of Lie algebras on a ring R are examples of protomodular categories. The aim of this section is to show that *the pointed protomodular categories Gp of groups, Rg of rings and $R-Lie$ of Lie algebras on a ring R are stably \mathbb{Q} -coreflexive on monomorphisms.* Let us begin by the following observation:

Proposition 3.1. *In a protomodular category \mathbb{E} : 1) any decomposition where the left hand side part is pullback is extremal:*

$$\begin{array}{ccccc} X' & \xrightarrow{x} & \bar{X} & \xrightarrow{\bar{x}} & X \\ f' \downarrow & \uparrow s' & \bar{f} \downarrow & \uparrow \bar{s} & f \downarrow s \\ Y' & \xrightarrow{y} & Y & \xlongequal{\quad} & Y \end{array}$$

2) any normal monomorphism is a normalizer.

Proof. 1) Consider any other decomposition with (v, u) $\mathbb{Q}_{\mathbb{E}}$ -cartesian:

$$\begin{array}{ccccc} & & U & & \\ & u \nearrow & \downarrow t & \searrow u' & \\ X' & \xrightarrow{x} & \bar{X} & \xrightarrow{\bar{x}} & X \\ f' \downarrow & \uparrow s' & \downarrow \bar{f} & \uparrow \bar{s} & f \downarrow s \\ & v \nearrow & V & \searrow v' & \\ Y' & \xrightarrow{y} & Y & \xlongequal{\quad} & Y \end{array}$$

Take the pullback \bar{u} of \bar{x} along u' ; it determines a monomorphism in $Pt_V \mathbb{E}$ whose image by the change of base functor v^* is the isomorphism $1_{X'}$ in $Pt_{Y'} \mathbb{E}$ since (v, u) is $\mathbb{Q}_{\mathbb{E}}$ -cartesian. Now, since \mathbb{E} is protomodular, \bar{u} is an isomorphism which produces the desired factorization $U \rightarrow \bar{X}$ above v' .

2) The second point is checked on the same model. Let v be any normal monomorphism and R_v the (unique) equivalence relation to which v is normal. Let $v = g.u$ be any decomposition of v with u a normal monomorphism (to R_u).

We have to show that R_u can be factorized through R_v , which is equivalent to $R_u \subset g^{-1}(R_v)$. Clearly $u^{-1}(g^{-1}(R_v)) = v^{-1}(R_v) = \nabla Y$. Then $R_u \cap g^{-1}(R_v)$ is normal to u since it is included in R_u and such that $u^{-1}(R_u \cap g^{-1}(R_v)) = \nabla Y$. Now, since \mathbb{E} is protomodular, we get $R_u \cap g^{-1}(R_v) \simeq R_u$ and consequently $R_u \subset g^{-1}(R_v)$. \square

3.1 Quasi-pointed categories

A category \mathbb{E} is said to be pointed when the terminal object is also initial, and quasi-pointed when the map $0 \rightarrow 1$ is a monomorphism, which implies that any initial map $\alpha_X : 0 \rightarrow X$ is a monomorphism. In this last case the fibre $Pt_0\mathbb{E}$ becomes a pointed full subcategory of \mathbb{E} .

Clearly the category *Set* of sets is quasi-pointed. Given any category \mathbb{E} , consider the fibration $()_0 : Grd\mathbb{E} \rightarrow \mathbb{E}$ associating with any internal groupoid \underline{Y}_1 its “object of objects” Y_0 ; the fibre $Grd_1\mathbb{E}$ above 1 is nothing but the category $Gp\mathbb{E}$ of internal groups in \mathbb{E} which is pointed; *any other fibre $Grd_Y\mathbb{E}$ is quasi-pointed and protomodular* [4].

Suppose now \mathbb{E} is quasi pointed; we call *kernel* of a map $f : X \rightarrow Y$ the upper horizontal arrow in the following left hand side pullback where α_Y is the initial map:

$$\begin{array}{ccc}
 K[f] \xrightarrow{k_f} X & & \\
 \downarrow & & \downarrow f \\
 0 \xrightarrow{\alpha_Y} Y & &
 \end{array}
 \qquad
 \begin{array}{ccc}
 EnX \xrightarrow{\epsilon_X} X & & \\
 \downarrow & & \downarrow \\
 0 \xrightarrow{\quad} 1 & &
 \end{array}
 \qquad
 \begin{array}{ccccc}
 & & \epsilon_X & & \\
 & & \curvearrowright & & \\
 EnX & \xrightarrow{\bar{\epsilon}_X} & X \times X & \xrightarrow{p_1} & X \\
 \downarrow & & \downarrow p_0 & & \downarrow \\
 0 & \xrightarrow{\alpha_X} & X & \longrightarrow & 1
 \end{array}$$

For the special case of the terminal map, we use the notations of the middle pullback and call this kernel the *endosome* of X , while we denote by $\bar{\epsilon}_X$ the unique factorization making the left hand side square of the right hand side diagram a pullback. Clearly the subobject ϵ_X is normal to ∇X . The middle pullback determines a left exact functor $En : \mathbb{E} \rightarrow Pt_0\mathbb{E}$ which is a right adjoint to the inclusion $Pt_0\mathbb{E} \hookrightarrow \mathbb{E}$.

Lemma 3.1. *Let \mathbb{E} be a quasi-pointed category. Then if a monomorphism $v : U \rightarrow X$ is normal, so is $u.\epsilon_U : EnU \rightarrow X$. When, in addition, \mathbb{E} is protomodular, the converse is true. In this case a monomorphism $v : U \rightarrow T$ has a normalizer as soon as $v.\epsilon_U : EnU \rightarrow X$ has one.*

Proof. Since any ϵ_U is normal to ∇U , if $u : U \rightarrow X$ is normal to R in any category, so is $u.\epsilon_U$. Conversely when $u.\epsilon_U$ is normal to R , then $u^{-1}(u.\epsilon_U) = \epsilon_U$ is normal to $u^{-1}(R)$. If, in addition, \mathbb{E} is protomodular, we get $u^{-1}(R) = \nabla_U$, and thanks to the three out of two conditions for the fibrant morphisms of equivalence relations, u is normal (to R). Suppose now that $v.\epsilon_U = w.\tilde{u}$ is the factorization through the normalizer $(\tilde{u}, R_{\tilde{u}})$ of the monomorphism $v.\epsilon_U$. Then,

since ϵ_U is normal, we get a factorization $u : U \twoheadrightarrow X$ where X is the codomain of \tilde{u} such that $w.u = v$ and $u.\epsilon_U = \tilde{u}$. According to the second equality, u is normal to $R_{\tilde{u}}$. From that and the first assertion, $(u, R_{\tilde{u}})$ is necessarily the normalizer of v . \square

When \mathbb{E} is quasi-pointed, we shall denote by $Kt\mathbb{E}$ the category of split exact sequences, namely of split epimorphisms with a chosen kernel, and by $K : Kt\mathbb{E} \rightarrow Pt_0\mathbb{E}$ the functor associating with any split exact sequence the domain of its kernel. Not only the functor K is left exact, but *it creates pullbacks and equalizers*. We shall denote by $J : \mathbb{E} \rightarrow Kt\mathbb{E}$ the functor associating with any X the following exact sequence:

$$EnX \xrightarrow{\bar{\epsilon}_X} X \times X \begin{array}{c} \xleftarrow{s_0^X} \\ \xrightarrow{p_0^X} \end{array} X$$

Clearly we have $K.J = En$.

On the other hand, it is clear that the forgetful functor $H : Kt\mathbb{E} \rightarrow Pt\mathbb{E}$ is a fully faithful and essentially surjective, namely that it determines a weak equivalence of categories, making the functor $\mathfrak{N}_{\mathbb{E}}.H$ a fibration. In order to produce the three examples asserted above, we shall show that, in the pointed protomodular case, the notion of category which is stably \mathfrak{N} -coreflexive on monomorphisms coincides with the notion of category with normalizers in the sense of Definition 2.3, see [13]. For that we need the following:

Definition 3.1. *A left exact functor $K : \mathbb{E} \rightarrow \mathbb{F}$ is said to be fibrant on monomorphisms when, given any monomorphism $u : U \twoheadrightarrow K(X)$, there is a monomorphic cartesian map $\tilde{u} : \tilde{U} \twoheadrightarrow X$ whose image is isomorphic to u .*

We have the straightforward following:

Lemma 3.2. *Given any left exact functor $K : \mathbb{E} \rightarrow \mathbb{F}$, TFAE:*

- 1) *K determines a bijection between the set of isomorphic classes of subobjects of $K(X)$ and the set of isomorphic classes of subobjects of X*
- 2) *K is conservative and fibrant on monomorphisms.*

Proof. This follows from the fact that any left exact functor is conservative as soon as it is conservative on monomorphisms, and that any left exact conservative functor is such that any monomorphism is hypercartesian. \square

Let us begin by the following lemma which is a simple adaptation of Proposition 2.4 in [12] from the pointed case to the non-pointed case:

Lemma 3.3. *Suppose \mathbb{E} is quasi-pointed. A monomorphism $v : U \twoheadrightarrow T$ with $U \in Pt_0\mathbb{E}$ has a normalizer in the sense of Definition 2.3 if and only if the monomorphism $Env : U \twoheadrightarrow EnT = KJ(T)$ admits a K -cartesian monomorphism above it.*

Proof. Suppose v has a normalizer (u, R_v) . We are going to show that the following right hand side map (w, \tilde{w}, v) with $\tilde{w} = (w.d_0^R, w.d_1^R)$ in $Kt\mathbb{E}$ is cartesian above Env :

$$\begin{array}{ccccc}
 & & Env & & \\
 & \curvearrowright & & \curvearrowleft & \\
 U & & U & \xrightarrow{Env} & EnT \\
 \downarrow k_f & \searrow x & \downarrow (0, u) & & \downarrow \bar{e}_T \\
 \bar{X} & \cdots \xrightarrow{\bar{x}} & R_v & \xrightarrow{\tilde{w}} & T \times T \\
 \downarrow f & \uparrow s & \downarrow d_0^R & \uparrow s_0^R & \downarrow p_0^T & \uparrow s_0^T \\
 \bar{Y} & \cdots \xrightarrow{\bar{y}} & X & \xrightarrow{w} & T \\
 & \curvearrowleft & & \curvearrowright & \\
 & & y & &
 \end{array}$$

Suppose now we have a map (y, x, Env) in $Kt\mathbb{E}$. We have to find a factorization $(\bar{y}, \bar{x}, 1_U)$. The map x is necessarily of the form $(y.f, t)$ for some $t : \bar{X} \rightarrow T$ such that $t.s = y$. Moreover the commutation at the upper level implies that $t.k_f = v$, where k_f is normal to the kernel equivalence relation $R[f]$. According to the universal property of a normalizer, there is a factorization $g : \bar{X} \rightarrow X$ such that $w.g = t$, $g.k_f = u$ and $R[f] \subset R_v$ which produces a morphism $(g, \tilde{g}) : R[f] \rightarrow R_v$ of equivalence relations. If we set $\bar{y} = g.s$ and $\bar{x} = \tilde{g}.(s.f, 1_{\bar{X}})$ we get a factorization of split epimorphisms. Then $\bar{x}.k_f = (0, u)$ follows from $g.k_f = u$.

Since the functor K creates the pullbacks, it is easy to check that any K -cartesian map above a monomorphism is itself a monomorphism, see Lemma 3.4 below. Now suppose that $Env : U \rightarrow EnT = K.J(T)$ has a K -cartesian map (w, \tilde{w}, Env) above it:

$$\begin{array}{ccccc}
 & & Env & & \\
 & \curvearrowright & & \curvearrowleft & \\
 U & & U & \xrightarrow{Env} & EnT \\
 \downarrow (0, k_{d_0}) & \searrow (\delta_1.p_0, \delta_1.p_1) & \downarrow k_{d_0} & & \downarrow \bar{e}_T \\
 R[d_0] & \cdots \xrightarrow{d_2} & R_v & \xrightarrow{\tilde{w}} & T \times T \\
 \downarrow p_0 & \uparrow s_0 & \downarrow d_0 & \uparrow s_0 & \downarrow p_0^T & \uparrow s_0^T \\
 R_v & \cdots \xrightarrow{d_1} & X & \xrightarrow{w} & T \\
 & \curvearrowleft & & \curvearrowright & \\
 & & \delta_1 & &
 \end{array}$$

Then \tilde{w} is necessarily of the form $(w.d_0, \delta_1)$ with $\delta_1 : R_v \rightarrow T$ such that $\delta_1.s_0 = w$ and $\delta_1.k_{d_0} = v$. Now let us consider the morphism $(\delta_1, (\delta_1.p_0, \delta_1.p_1), Env)$. It produces a factorization $(d_1, d_2, 1_U)$. We can check that $d_1.s_0 = 1_X$ by composition with the monomorphism w . This gives R_v the structure of reflexive graph. From $w.d_1 = \delta_1 = p_1^T.\tilde{w}$, the pair (w, \tilde{w}) is underlying a morphism

of reflexive graphs $R_v \rightrightarrows \nabla_Y$. Since (w, \tilde{w}) is a levelwise monomorphism, the reflexive graph R_v is a reflexive relation. Now the map d_2 shows that we have $R[d_0] \subset (d_1^{-1})(R_v)$ and that consequently R_v is an equivalence relation on X ; as such, it makes the lower left hand side square a pullback, and since $u = d_1.k_{d_0}$, it is normal to R_v . It remains to show that (u, R_v) has the universal property of a normalizer: this can be done similarly to the proof of Proposition 2.4 in [12]. \square

The following lemma is technical and straightforward:

Lemma 3.4. *Let $K : \mathbb{E} \rightarrow \mathbb{F}$ be a left exact functor between finitely complete categories which creates pullbacks. If $u : U \rightrightarrows T$ is a monomorphism in \mathbb{E} , and $t : T' \rightarrow T$ any morphism such that $K(t)$ is a monomorphism and there is map w satisfying $k(u) = K(t).w$, then there is a pullback $\bar{u} : \bar{U} \rightrightarrows T'$ of u along t such that $K(\bar{u}) = w$. If moreover u is K -cartesian, so is \bar{u} . In particular, if $u : U \rightrightarrows T$ is a monomorphic K -cartesian map in \mathbb{E} and $g : W \rightarrow T$ a map such that $K(g)$ is an identity map, then there exists a pullback \bar{u} of u along g such that $K(\bar{u}) = K(u)$ and \bar{u} is K -cartesian.*

Proposition 3.2. *Suppose \mathbb{E} is quasi-pointed and finitely complete. TFAE:*

- 1) *the functor $K : Kt\mathbb{E} \rightarrow E$ is fibrant on monomorphisms*
- 2) *any monomorphism $u : U \rightrightarrows T$ with $U \in Pt_0\mathbb{E}$ has a normalizer.*

Proof. We have 1) \Rightarrow 2) by the previous proposition. As for the converse, first notice that any split exact sequence can be embedded in some $J(T)$:

$$\begin{array}{ccccc}
K_f \rightrightarrows & \xrightarrow{Enk_f} & EnX & \xlongequal{\quad} & EnX \\
k_f \downarrow & & \downarrow (0, \epsilon_X) & & \downarrow \bar{\epsilon}_X \\
X \rightrightarrows & \xrightarrow{(f, 1)} & Y \times X & \xrightarrow{s \times X} & X \times X \\
f \downarrow \uparrow s & & p_0^Y \downarrow \uparrow (1, s) & & p_0^X \downarrow \uparrow s_0^X \\
Y & \xlongequal{\quad} & Y & \xrightarrow{s} & X
\end{array}$$

Given any monomorphism $m : U \rightrightarrows K_f$, in presence of 2), the monomorphism $Enk_f.m : U \rightrightarrows EnX = K.J(X)$ has a normalizer which means that there is K -cartesian map $(\beta, \alpha, Enk_f.m)$ above $Enk_f.m$ by the previous proposition. According to the previous lemma there is a pullback $(\bar{\beta}, \bar{\alpha}, m)$ of this map along $(s, (s.f, 1), Enk_f)$ which is necessarily K -cartesian above m . \square

In comparison with the pointed context, see Theorem 2.8 in [13], this is a rather poor result: the category Set of sets is an example of such a situation since the functor K , being a terminal functor, is trivially fibrant on monomorphisms.

3.2 The characterization theorem

However the protomodular context provides us with a much sharper observation:

Theorem 3.1. *Suppose \mathbb{E} is quasi-pointed. Then, if \mathbb{E} is \mathfrak{K} -coreflexive on monomorphisms, the functor K is fibrant on monomorphisms. When moreover \mathbb{E} is protomodular, TFAE:*

- 1) the category \mathbb{E} is \mathfrak{K} -coreflexive on monomorphisms
 - 2) the functor K is fibrant on monomorphisms
 - 3) the category \mathbb{E} has normalizers in the sense of Definition 2.3.
- In this case the category \mathbb{E} is stably \mathfrak{K} -coreflexive.

Proof. Suppose 1). Let $(a, b) : A \rightrightarrows B$ be a split epimorphism and $v : U \rightarrow K[a]$ any monomorphism. Let us consider the extremal decomposition of the monomorphism $(\alpha_B, v.k_a)$ where $k_a : K[a] \rightarrow A$ is the kernel of a :

$$\begin{array}{ccccc}
 & & v.k_a & & \\
 & & \curvearrowright & & \\
 U & \xrightarrow{\bar{u}} & \bar{A} & \xrightarrow{\alpha} & A \\
 \updownarrow & & \updownarrow \bar{a} & & \updownarrow a \\
 Q & \xrightarrow{\alpha_{\bar{B}}} & \bar{B} & \xrightarrow{\beta} & B \\
 & & \curvearrowleft & & \\
 & & \alpha_B & &
 \end{array}$$

The left hand side square being a pullback, the map \bar{u} is a kernel of \bar{a} . Let us show that the following left hand side monomorphism (β, α, v) in $Kt\mathbb{E}$ is K -cartesian above v :

$$\begin{array}{ccccccc}
 & & & & v & & \\
 & & & & \curvearrowright & & \\
 U & \xlongequal{\quad} & U & & U & \xrightarrow{v} & K[a] \\
 \parallel & & \downarrow k_f & & \downarrow \bar{u} & & \downarrow k_a \\
 U & \xrightarrow{k_f} & X & \xrightarrow{\bar{x}} & \bar{A} & \xrightarrow{\alpha} & A \\
 \updownarrow & & \updownarrow f & & \updownarrow \bar{a} & & \updownarrow a \\
 0 & \xrightarrow{\alpha_Y} & Y & \xrightarrow{\bar{y}} & \bar{B} & \xrightarrow{\beta} & B \\
 & & & & \curvearrowleft & & \\
 & & & & y & &
 \end{array}$$

So consider any map (y, x, v) in $Kt\mathbb{E}$. Now complete the diagram in $Kt\mathbb{E}$ by the map $(\alpha_Y, k_f, 1_U)$ on the left hand side; it produces a decomposition of the map $(\alpha_B, v.k_a)$ in $Pt\mathbb{E}$, whence the dotted factorization (\bar{y}, \bar{x}) such that (among other things) $\bar{x}.k_f = \bar{u}$, which shows that the factorization of (\bar{y}, \bar{x}) at the level of the kernels is 1_U . Accordingly the map $(\bar{y}, \bar{x}, 1_U)$ is the required factorization in $Kt\mathbb{E}$. Whence 2.

Suppose 2). Then \mathbb{E} has normalizer for any monomorphism $u : U \rightarrow X$ with $U \in Pt_0\mathbb{E}$. When \mathbb{E} is protomodular, then it has normalizer for any monomorphism by Lemma 3.1. Whence 3). And it is clear that 3) implies 2) by the previous proposition.

Let us check 2) \Rightarrow 1). First it is easy to check, by Lemma 3.4 that the K -cartesian maps above the monomorphisms are necessarily monomorphic. Consider any monomorphism $(y, x) : (f', s') \rightarrow (f, s)$ in $Pt\mathbb{E}$. Complete the diagram by the kernels and the factorization $K(x)$, then take the K -cartesian map $(\bar{y}, \bar{x}, K(x))$ above this monomorphism $K(x)$:

$$\begin{array}{c}
 \begin{array}{c}
 \text{K}(x) \\
 \curvearrowright \\
 K[f'] \equiv K[f'] \xrightarrow{K(x)} K[f] \\
 \downarrow k_{f'} \quad \downarrow k_{\bar{f}} \quad \downarrow k_f \\
 X' \xrightarrow{\underline{x}} \bar{X} \xrightarrow{\bar{x}} X \\
 \uparrow f' \quad \uparrow \bar{f} \quad \uparrow f \\
 \downarrow s' \quad \downarrow \bar{s} \quad \downarrow s \\
 Y' \xrightarrow{\underline{y}} \bar{Y} \xrightarrow{\bar{y}} Y \\
 \uparrow \\
 \text{y}
 \end{array}
 \end{array}$$

It determines a factorization $(\underline{y}, \underline{x}, 1_{K[f']})$. Since \mathbb{E} is protomodular, the isomorphic factorization $1_{K[f']}$ at the level of kernels implies that the map $(\underline{y}, \underline{x})$ in $Pt\mathbb{E}$ is underlying a pullback, namely that this map is $\mathfrak{N}_{\mathbb{E}}$ -cartesian.

It remains to show that the decomposition $(y, x) = (\bar{y}, \bar{x}).(\underline{y}, \underline{x})$ is extremal. So, consider another decomposition of $(y, x) = (\bar{b}, \bar{a}).(\underline{b}, \underline{a})$ with $(\underline{b}, \underline{a})$ monomorphic and $\mathfrak{N}_{\mathbb{E}}$ -cartesian; this implies that the map $\underline{a}.k_{f'}$ is a kernel of g . Complete the following diagram with the kernels:

$$\begin{array}{c}
 \begin{array}{c}
 \text{K}(x) \\
 \curvearrowright \\
 K[f'] \equiv K[f'] \equiv K[f'] \xrightarrow{K(x)} K[f] \\
 \downarrow k_{f'} \quad \downarrow \underline{a}.k_{f'} \quad \downarrow k_{\bar{f}} \quad \downarrow \bar{a} \quad \downarrow k_f \\
 X' \xrightarrow{\underline{a}} A \xrightarrow{\alpha} \bar{X} \xrightarrow{\bar{x}} X \\
 \uparrow f' \quad \uparrow g \quad \uparrow \bar{f} \quad \uparrow \bar{s} \quad \uparrow f \\
 \downarrow s' \quad \downarrow t \quad \downarrow \bar{s} \quad \downarrow s \\
 Y' \xrightarrow{\underline{b}} B \xrightarrow{\beta} \bar{Y} \xrightarrow{\bar{y}} Y \\
 \uparrow \\
 \bar{b}
 \end{array}
 \end{array}$$

The map $(\bar{b}, \bar{a}, K(x))$ in $Kt\mathbb{E}$ gives a unique factorization $(\beta, \alpha, 1_{K[f']})$ through the K -cartesian map $(\bar{y}, \bar{x}, K(x))$. The map (β, α) is actually the desired factorization in $Pt\mathbb{E}$, so that \mathbb{E} is \mathfrak{N} -coreflexive on monomorphisms. The fact that \mathbb{E} is

stably \mathbb{Q} -coreflexive is a consequence of the fact that the K -cartesian morphism are necessarily stable under pullbacks along $\mathbb{Q}_{\mathbb{E}}$ -cartesian morphisms, again according to Lemma 3.4, since any $\mathbb{Q}_{\mathbb{E}}$ -cartesian morphism can be extended into a morphism in $Kt\mathbb{E}$ whose image by K is an identity map. \square

Corollary 3.1. *Suppose the category \mathbb{C} is quasi-pointed, protomodular and \mathbb{Q} -coreflexive on monomorphisms. A monomorphism $(y, x) : (\bar{f}, \bar{s}) \rightarrow (f, s)$ is a \mathbb{Q} -outsider monomorphism in $Pt\mathbb{C}$ if and only if it determines a K -cartesian monomorphism in $Kt\mathbb{E}$ above $K(x)$:*

$$\begin{array}{ccc}
 K[\bar{f}] & \xrightarrow{K(x)} & K[f] \\
 k_{\bar{f}} \downarrow & & \downarrow k_f \\
 \bar{X} & \xrightarrow{x} & X \\
 \bar{f} \downarrow \uparrow \bar{s} & & f \downarrow \uparrow s \\
 \bar{Y} & \xrightarrow{y} & Y
 \end{array}$$

Example 3.1. *The pointed protomodular categories Gp of groups, Rg of rings and $R-Lie$ of Lie algebras on a ring R are categories which are stably \mathbb{Q} -coreflexive on monomorphisms.*

Proof. They are all pointed protomodular categories with normalizers. \square

Let us describe the extremal decomposition in $Pt(Gp)$. Given any subobject in this category:

$$\begin{array}{ccccc}
 & \curvearrowright & & \curvearrowleft & \\
 X' & \dashrightarrow & \bar{X} & \dashrightarrow & X \\
 f' \downarrow \uparrow s' & & \bar{f} \downarrow \uparrow \bar{s} & & f \downarrow \uparrow s \\
 Y' & \dashrightarrow & \bar{Y} & \dashrightarrow & Y \\
 & \curvearrowleft & & \curvearrowright &
 \end{array}$$

the subgroup \bar{Y} is $\{y \in Y/s(y).u.s(y)^{-1} \in X', \forall u \in Ker f'\}$ while the subgroup \bar{X} is $\{x \in X/x.sf(x)^{-1} \in X' \text{ and } sf(x).u.sf(x)^{-1} \in X', \forall u \in Ker f'\}$. The extremal decomposition in $Grg(Gp)$ is obtained exactly in the same way by the remark at the end of Section 2.2.

We have non-pointed examples of the situation with the following:

Proposition 3.3. *Let $()_0 : Grd \rightarrow Set$ be the forgetful functor from groupoids to sets. Then any fibre Grd_Y is stably \mathbb{Q} -coreflexive on monomorphisms.*

Proof. We recalled that any fibre Grd_Y is quasi-pointed and protomodular. A subobject $u_1 : \underline{U}_1 \hookrightarrow \underline{X}_1$ is normal in the fibre Grd_Y if and only if, for any arrow $\phi : y \rightarrow y'$ in \underline{X}_1 and any endomap τ on y in \underline{U}_1 , the endomap $\phi.\tau.\phi^{-1}$ is in \underline{U}_1 . The normalizer of any subobject $v_1 : \underline{U}_1 \hookrightarrow \underline{T}_1$ is defined by the subset

\underline{X}_1 of those arrows $\phi : y \rightarrow y'$ of \underline{T}_1 which are such that, for any endomap τ on y in \underline{U}_1 , the endomap $\phi.\tau.\phi^{-1}$ is in \underline{U}_1 , while for any endomap θ on y' in \underline{U}_1 , the endomap $\phi^{-1}.\theta.\phi$ is in \underline{U}_1 . \square

In Section 9, we shall extend the previous proposition to to any fibre $Grd_Y \mathbb{E}$ provided that the category is locally cartesian closed.

4 Slicing and coslicing

There are other ways to produce non-pointed examples. Recall that, given a category \mathbb{C} and any object Y in \mathbb{C} , the slice category \mathbb{C}/Y is the category whose objects are the maps with codomain Y and whose maps are the commutative triangles above Y . The coslice category Y/\mathbb{C} is defined by duality. The domain functor $dom : \mathbb{C}/Y \rightarrow \mathbb{C}$ is a discrete fibration while the codomain functor $cod : Y/\mathbb{C} \rightarrow \mathbb{C}$ is a discrete cofibration.

Proposition 4.1. *Let $U : \mathbb{E} \rightarrow \mathbb{F}$ be a left exact discrete fibration (resp. cofibration) and Θ a quasi-proper class in \mathbb{F} . Then $\Theta' = U^{-1}(\Theta)$ is a quasi-proper class. When \mathbb{F} is (resp. stably) Θ -coreflexive on monomorphisms, then \mathbb{E} is (resp. stably) Θ' -coreflexive on monomorphisms*

Proof. Given any monomorphism $v : Z \rightarrow T$ in \mathbb{E} , consider the extremal decomposition $U(v) = w.u$ in \mathbb{F} . When U is a discrete fibration (resp. cofibration) it determines a unique decomposition $v = \bar{w}.\bar{u}$ above it. The map \bar{u} is in Θ' since u is in Θ . It is straightforward that this decomposition is extremal. The fact that U is left exact and that we have $\Theta' = U^{-1}(\Theta)$, induces the assertion about the stable coreflexivity. \square

Corollary 4.1. *The two notions of categories (namely \heartsuit -coreflexive and stably \heartsuit -coreflexive on monomorphisms) are stable under slicing and coslicing; accordingly they are both stable under the passage to any fibre $Pt_Y \mathbb{E}$.*

Proposition 4.2. *If \mathbb{C} is stably \heartsuit -coreflexive on monomorphisms, the base-change functors with respect to the fibration $\heartsuit_{\mathbb{C}}$ preserve the extremal decompositions in the fibres.*

Proof. This comes from the two following observations:

- 1) given an object Y in \mathbb{C} the domain functor $dom : Pt_Y \mathbb{C} \rightarrow \mathbb{C}$ reflects the extremal \heartsuit -decompositions
- 2) given a map $h : Y' \rightarrow Y$ in \mathbb{C} , the base-change functor $h^* : Pt_Y \mathbb{C} \rightarrow Pt_{Y'} \mathbb{C}$ produces $\heartsuit_{\mathbb{C}}$ -cartesian maps in \mathbb{C} .

The conclusion follows from the assumed stable \heartsuit -coreflexivity in \mathbb{C} . \square

According to the previous section, the slice and coslice categories of the categories Gp of groups, R of rings and $R-Lie$ of Lie algebras on the ring R will produce first examples of non-pointed stably \heartsuit -coreflexive categories.

Corollary 4.2. *Let \mathbb{C} be stably \mathbb{Q} -coreflexive on monomorphisms. When \mathbb{C} is protomodular, it is strongly protomodular.*

Proof. A protomodular category \mathbb{C} is strongly protomodular when, in addition, any (conservative) base-change functor $h^* : Pt_Y \mathbb{C} \rightarrow Pt_{Y'} \mathbb{C}$ reflects the normal monomorphisms, see [2]. The following lemma and the previous proposition guarantee it when \mathbb{C} is stably \mathbb{Q} -coreflexive on monomorphisms. \square

Lemma 4.1. *Let $H : \mathbb{E} \rightarrow \mathbb{F}$ be a conservative left exact functor between protomodular categories which are \mathbb{Q} -coreflexive on monomorphisms. Then the functor F reflects the normal monomorphisms as soon as F preserves the extremal decompositions.*

Proof. We have to show that if the image by F of the monomorphism $v : U \rightarrow T$ is normal, then v is itself normal. Let $v = w.u$ the decomposition through the normalizer u . Since F is left exact and preserves the extremal decompositions and since $F(v)$ is normal, then $F(w)$ is an isomorphism. Accordingly, F being conservative, the morphism w is an isomorphism as well, and v is normal. \square

5 A structural observation

In this section we shall try to clarify the relationship between protomodularity and extremal decomposition in $Pt\mathbb{E}$. We noticed in Proposition 3.1 that, in a protomodular category \mathbb{C} any diagram in $Pt\mathbb{C}$ where the left hand side square is a pullback:

$$\begin{array}{ccccc} X' & \xrightarrow{x} & \bar{X} & \xrightarrow{\bar{x}} & X \\ f' \downarrow & \uparrow s' & \bar{f} \downarrow & \uparrow \bar{s} & f \downarrow s \\ Y' & \xrightarrow{y} & Y & \xlongequal{\quad} & Y \end{array}$$

is necessarily an extremal decomposition. Actually we can be more precise:

Proposition 5.1. *Let \mathbb{E} be any finitely complete category. TFAE:*

- 1) *any base-change functor with respect to the fibration $\mathbb{Q}_{\mathbb{E}}$ along a monomorphism y (resp. along a split monomorphism (y, g)) is conservative*
- 2) *any diagram in $Pt\mathbb{E}$ as above (resp. with a split monomorphism (y, g)) is an extremal decomposition with respect to the class of \mathbb{Q} -cartesian morphisms*
- 3) *the class of $\mathbb{Q}_{\mathbb{E}}$ -cartesian maps satisfies the monomorphic (resp. split monomorphic) three out of two condition in the category $Pt\mathbb{E}$.*

Proof. Suppose 1). Then consider any other decomposition with (v, u) $\mathbb{Q}_{\mathbb{E}}$ -cartesian:

$$\begin{array}{ccccc} & & U & & \\ & \nearrow u & \downarrow t & \searrow u' & \\ X' & \xrightarrow{x} & \bar{X} & \xrightarrow{\bar{x}} & X \\ f' \downarrow & \uparrow s' & \bar{f} \downarrow & \uparrow \bar{s} & f \downarrow s \\ & \nearrow v & V & \searrow v' & \\ Y' & \xrightarrow{y} & Y & \xlongequal{\quad} & Y \end{array}$$

Take the pullback \bar{u} of \bar{x} along u' ; it determines a monomorphism in $Pt_V\mathbb{E}$ whose image by the change of base functor v^* is the isomorphism $1_{X'}$ in $Pt_{Y'}\mathbb{E}$ since (v, u) is $\mathfrak{A}_{\mathbb{E}}$ -cartesian. Now, since v is a monomorphism (resp. $(v, g.v')$ is a split monomorphism), the map \bar{u} is itself an isomorphism which produces the desired factorization $U \rightarrow \bar{X}$ for 2).

Conversely, suppose 2). If, in addition, the whole rectangle is a pullback, namely a $\mathfrak{A}_{\mathbb{E}}$ -cartesian map, then the monomorphism \bar{x} has a section according to the universal property of an extremal decomposition and consequently is an isomorphism. Accordingly the change of base functor y^* with respect to the fibration $\mathfrak{A}_{\mathbb{E}}$ along the monomorphism y is conservative on monomorphisms; now, any left exact functor which is conservative on monomorphisms is conservative on any map. whence 1). The equivalence between 2) and 3) is a straightforward adaptation of the proof, given in [4], that protomodularity is equivalent to the three out of two condition for the $\mathfrak{A}_{\mathbb{E}}$ -cartesian maps \square

So, it seems worth introducing the following definition:

Definition 5.1. *Given any finitely complete category \mathbb{E} and any class J of morphisms in \mathbb{E} , this category will be said protomodular on the class J when the base-change functor with respect with the fibration of points $\mathfrak{A}_{\mathbb{E}}$ along any map in J is conservative.*

It is clear that:

- 1) a pointed category is protomodular if and only if it is protomodular on the class $Sp\mathbb{M}$ of split monomorphisms
- 2) a category is protomodular on the class $Sp\mathbb{M}$ of split monomorphisms if and only if any fibre $Pt_Y\mathbb{E}$ is protomodular
- 3) the previous proposition characterizes those categories which are protomodular on the class \mathbb{M} of monomorphisms (resp. the class $Sp\mathbb{M}$ of split monomorphisms)
- 4) any finitely complete \mathbb{E} is protomodular on the class $Sp\mathbb{E}$ of split epimorphisms
- 5) any finitely complete regular category \mathbb{E} is protomodular on the class Reg of regular epimorphisms
- 6) accordingly a regular category \mathbb{E} is protomodular is and only if it is protomodular on the class \mathbb{M} of monomorphisms
- 7) a quasi-pointed category \mathbb{E} is protomodular if and only if it is protomodular on the class \mathbb{M} of monomorphisms; accordingly the second part of Theorem 3.1 is valid under the weaker assumption of protomodularity on the class \mathbb{M} .

Examples. 1) A Mal'tsev operation p on a set X (namely a ternary operation $p : X \times X \times X \rightarrow X$ such that $p(x, y, y) = x = p(y, y, x)$) will be said right symmetric when, in addition, we have $p(x, y, p(y, x, z)) = z$. Let us denote by RSM the full subcategory of the category Mal of Mal'tsev operations whose objects are those sets X which are endowed with a right symmetric one. Then RSM is protomodular on $Sp\mathbb{M}$ without being protomodular.

Proof. Clearly the empty set prevents RSM to be protomodular. Now consider the following pullback in RSM with $Y' \neq \emptyset$:

$$\begin{array}{ccc} X' & \xrightarrow{k} & X \\ f' \downarrow & \uparrow s' & f \downarrow \\ Y' & \xrightarrow{t} & Y \\ & \xleftarrow{g} & \end{array}$$

Then X' is the subobject of the elements x of X satisfying $f(x) = tgf(x)$. Now let \bar{X} be any subobject of X containing X' and $s(Y)$. Given any x in X , we have $f(p(stgf(x), sf(x), x)) = p(tgf(x), f(x), f(x)) = tgf(x)$. Consequently the element $p(stgf(x), sf(x), x)$ belongs to X' , whence $\bar{X} = X$. Now we get $x = p(sf(x), stgf(x), p(stgf(x), sf(x), x))$; therefore x is in \bar{X} since any of the three terms belongs to \bar{X} . When $Y' = \emptyset$, the other sets in the diagram are empty, and t^* becomes an isomorphism; so it is trivially conservatif. \square

2) A metagroup is a set X endowed with a binary operation satisfying: i) the cancellation rule: $(x/y)/(z/y) = (x/z)$, ii) the weak quadratic neutrality $x/(y)/y = x$ and iii) the quadratic constancy $x/x = y/y$. Denote by MGP the category of metagroups. Clearly a non-empty metagroup is a group by setting $x/x = 1$ and the category MGP is just the category GP of groups completed with \emptyset . Accordingly MGP is protomodular on $\mathbb{S}p\mathbb{M}$ without being protomodular.

Proposition 5.2. *Suppose \mathbb{E} is finitely complete and protomodular on the class $\mathbb{S}p\mathbb{M}$ of split monomorphisms. Then \mathbb{E} is a Mal'tsev category, any split monomorphism is normal to at most one equivalence relation, any split normal monomorphism is a normalizer. When it is protomodular on the class \mathbb{M} of monomorphisms, then it is a Mal'tsev category, any monomorphism is normal to at most one equivalence relation, any normal monomorphism is a normalizer. In both cases, the fibres $Pt_Y \mathbb{E}$ are protomodular.*

Proof. The proofs are the same as the ones given for protomodular categories in [2] and in Lemma 3.1, since the only base-change functors which are used in the proof of Mal'tsevness (resp. of uniqueness of normality, existence of normalizers) are base-change along split monomorphisms (resp. monomorphisms). The last point is a consequence of the above point 2). \square

Proposition 5.3. *The notion of finitely complete category which is protomodular on the class $\mathbb{S}p\mathbb{M}$ of split monomorphisms (resp. on the class \mathbb{M} of monomorphisms) is stable under slicing and coslicing.*

Proof. The proof here is quite similar to the proof of Corollary 4.1. \square

6 \mathbb{Q} -coreflection and additive setting

A pointed category \mathbb{A} is additive if and only if any base-change functor $\alpha_X^* : Pt_X \mathbb{A} \rightarrow Pt \mathbb{A}$ along an initial map is an equivalence of category, which implies that any base-change functor with respect to the fibration of points is an equivalence of categories, see [4]. Accordingly it is protomodular. Moreover any monomorphism is normal, and consequently any additive category has normalizers by Lemma 3.1. More precisely we have the following characterizations:

Theorem 6.1. *Let \mathbb{E} be a pointed finitely complete category. TFAE:*

- 1) \mathbb{E} is additive
- 2) any base-change α_X^* along the initial map is fully faithful (and consequently an equivalence of categories since α_X is split by τ_X)
- 2') any base-change with respect the fibration $\mathbb{Q}_{\mathbb{E}}$ is an equivalence of categories
- 3) \mathbb{E} is protomodular and any base-change functor with respect to the fibration $\mathbb{Q}_{\mathbb{E}}$ is fibrant on monomorphisms
- 4) \mathbb{E} is protomodular, \mathbb{Q} -coreflexive on monomorphisms and any \mathbb{Q} -outsider monomorphism is \mathbb{Q} -invertible
- 5) \mathbb{E} is protomodular and any monomorphism m is a normal one
- 5') \mathbb{E} is protomodular and any split monomorphism (s, f) is a normal one.

Proof. The equivalences 1) \iff 2) \iff 2') are a consequence of Corollary in Section 4 of [4]. Suppose 2'); then any base-change is an equivalence of categories. Accordingly it is conservatif (and \mathbb{E} is protomodular) and fibrant on monomorphisms; whence 3). Suppose 3). So, consider the decomposition of any monomorphism (y, x) in $Pt \mathbb{E}$ given by the right hand side pullback of the left hand side diagram:

$$\begin{array}{ccc}
 & x & \\
 & \curvearrowright & \\
 X'' & \xrightarrow{\xi} & X' \xrightarrow{x'} X \\
 f'' \downarrow \uparrow s'' & & f' \downarrow \uparrow s' \quad f \downarrow \uparrow s \\
 Y' & \xrightarrow{y} & Y
 \end{array}
 \qquad
 \begin{array}{ccc}
 & x & \\
 & \curvearrowright & \\
 X'' & \xrightarrow{x''} & \bar{X} \xrightarrow{\bar{\xi}} X \\
 f'' \downarrow \uparrow s'' & & \bar{f} \downarrow \uparrow \bar{s} \quad f \downarrow \uparrow s \\
 Y' & \xrightarrow{y} & Y
 \end{array}$$

Then the y^* -cartesian monomorphism $\bar{\xi}$ above ξ produces the right hand side decomposition with a pullback which is extremal, by Proposition 3.1, since the category \mathbb{E} is protomodular; whence 4). Now suppose 4). Given any monomorphism $m : Y \rightarrow X$, consider the extremal decomposition of the monomorphism $(m, m \times m)$:

$$\begin{array}{ccc}
 & m \times m & \\
 & \curvearrowright & \\
 Y \times Y & \xrightarrow{\tilde{m}} & R \xrightarrow{(d_0^R, d_1^R)} X \times X \\
 p_0^Y \downarrow \uparrow s_0^Y \downarrow \uparrow p_1^Y & & d_0^R \downarrow \uparrow s_0^R \downarrow \uparrow d_1^R \quad p_0^X \downarrow \uparrow s_0^X \downarrow \uparrow p_1^X \\
 Y & \xrightarrow{m} & X \xrightarrow{=} X
 \end{array}$$

it produces a middle vertical equivalence relation, which is normal to m since the left hand side square indexed by 0 is a pullback; whence 5). The implication $5) \Rightarrow 5')$ is trivial. Now suppose 5'). Since any diagonal s_0^X is split, then it is normal, so any object X is an abelian object in the pointed protomodular category \mathbb{E} (see Proposition 3.2.15 in [2]) and therefore \mathbb{E} is additive. \square

In a non-pointed context there are many steps inside the “additive” setting. Recall the following table given by decreasing order of generality, see [6]:

Definition 6.1. *A category \mathbb{C} is:*

- 1) *a naturally Mal'tsev one when any fibre of the fibration \mathbb{C} is additive*
- 2) *antepenessentially affine when any base-change functor is fully faithful*
- 3) *penessentially affine when, in addition, any base-change functor is fibrant on monomorphisms*
- 4) *essentially affine when any base-change functor is an equivalence of categories.*

The largest level is the one of naturally Mal'tsev categories which were first defined in [22] as categories in which any object is endowed with a natural Mal'tsev operation, or equivalently in which any reflexive graph is endowed with a unique internal groupoid structure. We shall denote by $p_A : A \times A \times A \rightarrow A$ the natural Mal'tsev operation on any object A . Later on in [5], they were shown to be equivalent to those categories in which any fibre of the fibration $Pt_A \mathbb{C}$ is additive, which implies that *a pointed category is additive if and only if it is a naturally Mal'tsev one*. From [6] we know that:

- 1) any fibre $Grd_Y \mathbb{C}$ is essentially affine as soon as \mathbb{C} is additive
- 2) any fibre $Grd_Y \mathbb{C}$ is penessentially affine as soon as \mathbb{C} is a Mal'tsev category
- 3) any fibre $Grd_Y \mathbb{C}$ is antepenessentially affine as soon as \mathbb{C} is a Gumm category (in the varietal context, this means that the variety \mathbb{V} is congruence modular, see [20] and [11] for more details).

We have the following characterizations (see [5]) which enlarge Theorem 6.1:

Theorem 6.2. *Let \mathbb{E} be finitely complete category. TFAE:*

- i) *\mathbb{E} is a naturally Mal'tsev category*
- ii) *any base-change s^* along a split monomorphism (s, f) is fully faithful (and consequently an equivalence of categories since s is split by f)*
- iii) *\mathbb{E} is protomodular on SpM and any base-change s^* along a split monomorphism (s, f) is fibrant on monomorphisms*
- iv) *\mathbb{E} is protomodular on SpM and any monomorphism (y, x) in $Pt\mathbb{E}$ above a split monomorphism (y, g) determines an extremal decomposition of the follow-*

ing kind:

$$\begin{array}{ccccc}
 & & x & & \\
 & \curvearrowright & & \curvearrowleft & \\
 X' & \xrightarrow{\tilde{x}} & \bar{X} & \xrightarrow{\bar{x}} & X \\
 f' \downarrow & \uparrow s' & \bar{f} \downarrow & \uparrow \bar{s} & f \downarrow s \\
 Y' & \xrightarrow{y} & Y & \xlongequal{\quad} & Y \\
 & \uparrow & & & \\
 & \wedge & & & \\
 & & g & &
 \end{array}$$

v) \mathbb{E} is a protomodular on $\mathbb{S}p\mathbb{M}$ and any split monomorphism (s, f) is normal to a (unique) equivalence relation R ; this equivalence relation R is such that $R \cap Rf = \Delta X$.

Proof. Suppose i), then ii) is a consequence of Theorem 6.1 and of the fact that any fibre $Pt_Y \mathbb{E}$ is additive. Suppose ii). First, since any base-change s^* is an equivalence of categories, it is conservative and \mathbb{E} is protomodular on $\mathbb{S}p\mathbb{M}$; then ii) \Rightarrow iii) is a straightforward. The implication iii) \Rightarrow iv) is a straightforward adaptation of 3) \Rightarrow 4) in Theorem 6.1, thanks to Proposition 5.1. Suppose iv). Since \mathbb{E} is protomodular on $\mathbb{S}p\mathbb{M}$, it is a Mal'tsev category and any split monomorphism (s, f) is normal to at most one equivalence relation. Let (s, f) be a split monomorphism. Consider the following diagram where (d_0^R, s_0^R) in the left hand side part is given by the extremal decomposition of the monomorphism $(s, s \times s)$ in $Pt\mathbb{E}$:

$$\begin{array}{ccccccc}
 & & s \times s & & & & \\
 & \curvearrowright & & \curvearrowleft & & & \\
 Y \times Y & \xrightarrow{\bar{s}} & R & \xrightarrow{(d_0^R, d_1^R)} & X \times X & \xrightarrow{f \times f} & Y \times Y \\
 p_0^Y \downarrow & \uparrow s_0^Y & \downarrow p_1^Y & & p_0^X \downarrow & \uparrow s_0^X & \downarrow p_1^X \\
 d_0^R \downarrow & \uparrow s_0^R & \downarrow d_1^R & & p_0^X \downarrow & \uparrow s_0^X & \downarrow p_1^X \\
 Y & \xrightarrow{s} & X & \xlongequal{\quad} & X & \xrightarrow{f} & Y
 \end{array}$$

Then R is underlying an equivalence relation on X to which s is normal since the left hand side square indexed by 0 is a pullback. Again by Proposition 5.1, since s is split by f , the morphism $R \rightarrow \nabla Y$ is fibrant and we get $R \cap Rf = \Delta X$. Whence v). Now suppose v). Then \mathbb{E} is Mal'tsev category. Since any diagonal monomorphism s_0^X is split by p_1^X for any X , it is normal to an equivalence relation R on $X \times X$ such that $R \cap R[p_1^X] = \nabla(X \times X)$. By Corollary 2.2, any object X is affine and the Mal'tsev category \mathbb{E} is a naturally Mal'tsev one, whence 1). \square

The three last types of non-pointed additive categories given in Definition 6.1 are necessarily protomodular since a fully faithful functor is necessarily conservative; they are naturally Mal'tsev categories by the above condition ii). Moreover, in a penessentially affine category, any monomorphism is normal, see [6]. The equivalence 3) \iff 4) in Theorem 6.1 focuses attention on the following:

Proposition 6.1. *Given any finitely complete category \mathbb{E} , consider the following conditions:*

- 1) \mathbb{E} is \mathbb{Q} -coreflexive on monomorphisms and any \mathbb{Q} -outsider is \mathbb{Q} -invertible
 - 2) \mathbb{E} is a Mal'tsev category and any base-change y^* with respect to the fibration $\mathbb{Q}_{\mathbb{E}}$ along a monomorphism y is fibrant on subobjects
 - 3) \mathbb{E} is a Mal'tsev category and, for any monomorphism $m : Y' \rightarrow Y$ and any equivalence relation R on Y' , there is a largest equivalence relation on Y among those ones which are such that $m^{-1}(S)$ is R and the induced monomorphism $R \rightarrow S$ is fibrant
 - 4) \mathbb{E} is a Mal'tsev category and, for any monomorphism $m : Y' \rightarrow Y$, there is a largest equivalence relation R_m to which m is normal
 - α) \mathbb{E} is a Mal'tsev category with normalizers and any monomorphism is normal.
- Then we get $1) \Rightarrow 2) \Rightarrow 3) \Rightarrow 4)$ and $1) \Rightarrow \alpha) \Rightarrow 4)$.

If, in addition, \mathbb{E} is protomodular on the class \mathbb{M} of monomorphisms we get $1) \Leftrightarrow 2)$ and $\alpha) \Leftrightarrow 4)$. In both cases, \mathbb{E} is a naturally Mal'tsev category.

Proof. Suppose 1). By Corollary 2.1, we know that \mathbb{E} is a Mal'tsev category. Given any left hand side diagram where the square is a pullback:

$$\begin{array}{ccc}
 & \xrightarrow{x} & \\
 X'' & \xrightarrow{\xi} X' \xrightarrow{x'} & X \\
 \downarrow f'' \uparrow s'' & \downarrow f' \uparrow s' & \downarrow f \uparrow s \\
 Y' & \xrightarrow{y} & Y
 \end{array}
 \qquad
 \begin{array}{ccc}
 & \xrightarrow{x} & \\
 X'' & \xrightarrow{x''} \bar{X} \xrightarrow{\bar{\xi}} & X \\
 \downarrow f'' \uparrow s'' & \downarrow \bar{f} \uparrow \bar{s} & \downarrow f \uparrow s \\
 Y' & \xrightarrow{y} & Y
 \end{array}$$

the extremal decomposition of (y, x'') on the right hand side produces the desired m^* -cartesian map above ξ . Whence 2). Suppose 2). Now consider the y^* -cartesian map above (d_0^R, y, d_1^R) :

$$\begin{array}{ccc}
 & \xrightarrow{(y, d_0^R, y, d_1^R)} & \\
 R & \xrightarrow{(d_0^R, y, d_1^R)} Y' \times Y \xrightarrow{y \times Y} & Y \times Y \\
 \downarrow d_0^R \uparrow s_0^R & \downarrow p_0^{Y'} \uparrow (1, y) & \downarrow p_0^Y \uparrow s_0^Y \\
 Y' & \xrightarrow{y} & Y
 \end{array}
 \qquad
 \begin{array}{ccc}
 & \xrightarrow{(y, d_0^R, y, d_1^R)} & \\
 R & \xrightarrow{\tilde{y}} \Sigma \xrightarrow{(d_0^\Sigma, d_1^\Sigma)} & X \\
 \downarrow d_0^R \uparrow s_0^R & \downarrow d_0^\Sigma \uparrow s_0^\Sigma & \downarrow f \uparrow s \\
 Y' & \xrightarrow{y} & Y
 \end{array}$$

it produces an reflexive relation Σ on Y , which is an equivalence relation since \mathbb{E} is a Mal'tsev category, and a monomorphism $R \rightarrow \Sigma$ of equivalence relations which is fibrant since the square indexed by 0 is a pullback. Accordingly we get $y^{-1}(\Sigma) = R$. The universal property of the y^* -cartesian map shows that Σ is the largest equivalence relation on Y among those S which are such that $y^{-1}(S)$ is R and the induced monomorphism $R \rightarrow S$ is fibrant. Whence 3). The implication $3) \Rightarrow 4)$ is trivial. The implication $1) \Rightarrow \alpha)$ is a consequence of Proposition 2.3. Again the implication $\alpha) \Rightarrow 4)$ is trivial.

When, in addition, \mathbb{E} is protomodular on the class \mathbb{M} of monomorphisms, we get 2) \Rightarrow 1) by Proposition 5.1 and 4) \Rightarrow α) by Lemma 3.1. Moreover when \mathbb{E} is protomodular on the class \mathbb{M} of monomorphisms and satisfies 4), then any split monomorphism is normal, and, by the condition v) in Theorem 6.2, the category \mathbb{E} is a naturally Mal'tsev category. \square

So, it appears to be worthwhile to introduce the following definition selecting a new type of "non-pointed additive" category:

Definition 6.2. *Let \mathbb{E} be a finitely complete category. It will be said to be an ortho-naturally Mal'tsev category when it is protomodular on $\text{Sp}\mathbb{M}$ and such that any base-change with respect to the fibration $\mathbb{Q}_{\mathbb{E}}$ is fibrant on monomorphisms.*

By the condition iii) in Theorem 6.2, any ortho-naturally Mal'tsev category is a naturally Mal'tsev one.

Example. A metagroup is said to be commutative when, in addition, we have $x/(x/y) = y$. We shall denote by $CMGp$ the category of commutative metagroups. Clearly a non-empty commutative metagroup is an abelian group by setting $x/x = 1$ and the category $CMGp$ is just the category Ab of abelian groups completed with \emptyset . Accordingly $CMGp$ is an ortho-naturally Mal'tsev category without being protomodular: the base-change α_X^* is the terminal functor which is trivially fibrant on monomorphisms, while any other m^* is an equivalence of categories which is trivially fibrant on monomorphisms as well.

On the other hand, we get the following characterization which brings some precisions between the different levels of non-pointed additivity:

Proposition 6.2. *Let \mathbb{C} be a finitely complete category. TFAE:*

- 1) \mathbb{C} is penessentially affine
- 2) \mathbb{C} is an antepenessentially affine and ortho-naturally Mal'tsev category.

Proposition 6.3. *Let \mathbb{E} be a finitely complete category. Consider the following conditions:*

- 1) \mathbb{E} is an ortho-naturally Mal'tsev category
- 2) \mathbb{E} is \mathbb{Q} -coreflexive on monomorphisms and such that any \mathbb{Q} -outsider monomorphism is \mathbb{Q} -invertible.

When \mathbb{E} is protomodular on the class \mathbb{M} of monomorphism, then 1) \Rightarrow 2).

When \mathbb{E} is regular, then 2) \Rightarrow 1).

Accordingly when \mathbb{E} is regular and protomodular on the class \mathbb{M} of monomorphism, we have 1) \iff 2).

Proof. When \mathbb{E} is protomodular on the class \mathbb{M} of monomorphism and an ortho-naturally Mal'tsev category, it is \mathbb{Q} -coreflexive on monomorphisms and such that any \mathbb{Q} -outsider monomorphism is \mathbb{Q} -invertible by Proposition 6.1. When \mathbb{E} is regular, \mathbb{Q} -coreflexive on monomorphisms and such that any \mathbb{Q} -outsider monomorphism is \mathbb{Q} -invertible, then it is a Mal'tsev category by Corollary 2.1. So, any base-change with respect to the fibration $\mathbb{Q}_{\mathbb{E}}$ is fibrant on monomorphisms, since on the one hand it is true for any monomorphism by Proposition

6.1 and since on the other hand the base-change along any regular epimorphism is both conservative and fibrant on monomorphisms in any regular Mal'tsev category, according to the following lemma. \square

Lemma 6.1. *Let \mathbb{E} be any regular Mal'tsev category. Then the base-change functor g^* with respect to the fibration $\mathbf{Pt}\mathbb{E}$ is conservative and fibrant on monomorphisms when g is any regular epimorphism.*

Proof. 1) In any category \mathbb{E} the base-change functor g^* with respect to the fibration of points is conservative on monomorphism provided that f is a pullback stable regular epimorphism.

2) Consider the following diagram where the right hand side quadrangle is a pullback, (f', s') in $Pt\mathbb{E}$ and m is a monomorphism in $Pt_Y(\mathbb{E})$:

$$\begin{array}{ccccc}
R[\bar{g}.m] & \xrightleftharpoons{\delta_0} & X' & & \\
\downarrow R(f') & \nearrow R(m) & \downarrow f' & \nearrow m & \\
R[\bar{g}] & \xrightleftharpoons{\delta_1} & \bar{X}' & \xrightarrow{\bar{g}} & \bar{X} \\
\downarrow R(s') & \nearrow R(s') & \downarrow \bar{f}' & \nearrow \bar{s}' & \\
R[g] & \xrightleftharpoons{d_1} & Y' & \xrightarrow{g} & Y \\
& & \downarrow \bar{f} & \nearrow \bar{s} &
\end{array}$$

Complete the diagram with the kernel relation $R[\bar{g}.m]$. The factorization $\phi : R[\bar{g}.m] \rightarrow P$, where P is the domain of the pullback of the split epimorphism (f', s') along the split epimorphism (d_0, s_0) is a strong epimorphism since \mathbb{E} is a Mal'tsev category, and a monomorphism as well since $R(m)$ is a monomorphism. Accordingly ϕ is an isomorphism and the vertical left hand side squares produce a fibrant morphism between equivalence relations. Let $n.g'$ be the canonical decomposition of $\bar{g}.m$, where n is a monomorphism and g' is the quotient of the effective equivalence relation $R[\bar{g}.m]$:

$$\begin{array}{ccccc}
R[\bar{g}.m] & \xrightleftharpoons{\delta_0} & X' & \xrightarrow{g'} & X \\
\downarrow R(f') & \nearrow R(m) & \downarrow f' & \nearrow m & \downarrow n \\
R[\bar{g}] & \xrightleftharpoons{\delta_1} & \bar{X}' & \xrightarrow{\bar{g}} & \bar{X} \\
\downarrow R(s') & \nearrow R(s') & \downarrow \bar{f}' & \nearrow \bar{s}' & \downarrow f \\
R[g] & \xrightleftharpoons{d_1} & Y' & \xrightarrow{g} & Y \\
& & \downarrow \bar{f} & \nearrow \bar{s} & \downarrow \bar{f} \\
& & & & \downarrow \bar{s}
\end{array}$$

Now, by the Barr-Kock Theorem, the right hand side vertical square is a pullback since the two other left hand side vertical squares are so. Accordingly we get $m = g^*(n)$. Since g^* is conservative, the monomorphism n is necessarily g^* -cartesian. \square

Corollary 6.1. *For a quasi-pointed protomodular and regular category \mathbb{E} , TFAE:*

- 1) \mathbb{E} is \mathbb{Q} -coreflexive on monomorphisms and such that any \mathbb{Q} -outsider monomorphism is \mathbb{Q} -invertible
- 2) the functor $K : Kt\mathbb{E} \rightarrow Pt_0\mathbb{E}$ is fibrant on monomorphism, and any K -cartesian map is K invertible
- 3) any monomorphism in \mathbb{E} is a normal monomorphism
- 4) \mathbb{E} is an ortho-naturally Mal'tsev category.

Proof. When \mathbb{E} is quasi-pointed and protomodular we have 1) \iff 2) \iff 3) by Theorem 3.1 and Corollary 3.1. When \mathbb{E} is protomodular and regular we have 1) \iff 4) by Proposition 6.3. \square

7 Properties of the \mathbb{Q} -coreflexive categories in the Mal'tsev and protomodular settings

In this section, we shall generalize what is obtained in the pointed case in [13], but with much more straightforward proofs and in a wider context (see Propositions 7.1 and 7.2 for instance): we shall show that any category \mathbb{E} which is \mathbb{Q} -coreflexive on monomorphisms (and therefore is a Mal'tsev category) has centralizers of equivalence relations, and earnest centralizers of subobjects when it is pointed protomodular. When it is protomodular and exact in the sense of [1], then it is action accessible in the sense of [14] and [8]. This last point is all the more important that it implies the non-pointed Schreier-Mac Lane theorem, see Theorem 4.11 in [7].

7.1 Θ -distinctiveness and action distinctiveness

Given a proper class Θ in \mathbb{E} , recall that the category \mathbb{E} is said to be Θ -distinctive when any object X admits a largest equivalence relation D_X in Θ on it which is called *the Θ -distinctive equivalence relation* on X ; an object X is said to be Θ -eccentric when D_X is the discrete equivalence relation Δ_X ; the category \mathbb{E} is said to be *functorially Θ -distinctive* when, in addition, given any map $f : X \rightarrow Y$ in Θ , we have $f^{-1}(D_Y) = D_X$, see [8]. We shall show here that the Θ -distinctiveness is a consequence of the Θ -coreflexivity on monomorphisms. For that, let us introduce the following:

Definition 7.1. *A proper class Θ is said to be amenably proper when an equivalence relation $(d_0^R, d_1^R) : R \rightrightarrows X$ is in Θ (namely d_0^R (and therefore d_1^R) is in Θ) as soon as the subdiagonal $s_0^R : X \rightarrow R$ is in Θ .*

Example 7.1. *Proposition 2.4 show that in any Mal'tsev category \mathbb{C} the class of $\mathbb{Q}_{\mathbb{C}}$ -cartesian maps is amenably proper.*

The main interest of the amenably proper classes is the following:

Lemma 7.1. *Suppose Θ is an amenably proper class in \mathbb{C} . Then given any monomorphism of equivalence relation $S \rightarrow R$, the equivalence relation S is in Θ as soon as R is in Θ . In other words any sub-equivalence relation of an equivalence relation in Θ is in Θ .*

Proof. Consider the following left hand side diagram:

$$\begin{array}{ccc} S & \xrightleftharpoons{d_0^S} & X \\ \tilde{j} \downarrow & \begin{array}{c} d_1^S \\ \leftarrow \\ d_0^R \\ \rightarrow \\ d_1^R \end{array} & \downarrow j \\ R & \xrightleftharpoons{d_1^R} & Y \end{array} \quad \begin{array}{ccc} S & \xleftarrow{s_0^S} & X \\ \tilde{j} \downarrow & & \downarrow j \\ R & \xleftarrow{s_0^R} & Y \end{array}$$

The right hand side square is a pullback since $S \cap R[j] = R \cap \Delta X = \Delta X$; so s_0^S is in Θ as soon as so is s_0^R . \square

In the context of Θ -coreflexiveness, the Θ -distinctiveness becomes straightforward:

Proposition 7.1. *Let \mathbb{C} be a category and Θ an amenably proper class. If \mathbb{C} is Θ -coreflexive on monomorphisms, it is Θ -distinctive; if it is stably Θ -coreflexive, it is functorially Θ -distinctive. An object X is Θ -eccentric if and only if the diagonal $s_0 : X \rightarrow X \times X$ is a Θ -outsider monomorphism.*

Proof. Since \mathbb{C} is a Θ -coreflexive category, the extremal decomposition of the diagonal:

$$\begin{array}{ccc} & D_X & \\ s_0^{D_X} \nearrow & & \searrow (\delta_0, \delta_1) \\ X & \xrightarrow{s_0} & X \times X \end{array}$$

produces a reflexive relation D_X . If D_X is an equivalence relation, it is in Θ since Θ is amenably proper; and it is clearly the largest one. Let us show it is an equivalence relation. For that consider the following commutative diagram:

$$\begin{array}{ccccc} & & R[\delta_0] & & \\ & s_1^{D_X} \nearrow & \uparrow p_0 & \searrow \pi & \\ D_X & \xrightarrow{\quad} & X \times X \times X & & \\ \delta_0 \uparrow & \begin{array}{c} s_0^{D_X} \\ \leftarrow \\ \downarrow d_2 \\ \rightarrow \\ s_0^{D_X} \end{array} & \downarrow & \begin{array}{c} p_0^X \\ \leftarrow \\ \downarrow \\ \rightarrow \\ p_2^X \end{array} & \\ X & \xrightarrow{s_0^{D_X}} & D_X & \xrightarrow{(\delta_0, \delta_1)} & X \times X \\ & \searrow s_0 & \xrightarrow{\quad} & & \end{array}$$

where $\pi(xD_X y, xD_X z) = (x, y, z)$. Since the left hand side downward quadrangle is pullback and $s_0^{D_X}$ is in Θ , so is $s_1^{D_X}$. The following decomposition $s_0 = p_2^X \cdot \pi \cdot (s_1^{D_X} \cdot s_0^{D_X})$ produces a factorization d_2 which shows that D_X is an equivalence relation.

Suppose now that \mathbb{C} is stably Θ -coreflexive and that $f : X \rightarrow Y$ is a map in Θ , then the following pullbacks produce an extremal decomposition:

$$\begin{array}{ccc}
R[f] & \longrightarrow & Y \\
\iota \downarrow & & \downarrow s_0 \\
f^{-1}(D_Y) & \longrightarrow & D_Y \\
(d_0, d_1) \downarrow & & \downarrow (\delta_0, \delta_1) \\
X \times X & \xrightarrow{f \times f} & Y \times Y
\end{array}$$

with ι in Θ . On the other hand, since the equivalence relation D_Y and the map f are in Θ , so is $f^{-1}(D_Y)$; whence a factorization j making the following diagram commute:

$$\begin{array}{ccc}
f^{-1}(D_Y) & \xrightarrow{j} & D_X \\
& \searrow d_0 & \swarrow \delta_0 \\
& & X
\end{array}$$

Since Θ is a proper class, the map j is in Θ . Now the following factorization with $j \cdot \iota$ in Θ produces the dotted inverse to j :

$$\begin{array}{ccccc}
& & D_X & & \\
& \nearrow j \cdot \iota & \vdots & \searrow (\delta_0, \delta_1) & \\
R[f] & \xrightarrow{\iota} & f^{-1}(D_Y) & \xrightarrow{(d_0, d_1)} & X \times X
\end{array}$$

so that we get $D_X \simeq f^{-1}(D_Y)$ and the category \mathbb{C} is functorially Θ -distinctive. The last assertion is straightforward according to the construction of D_X . \square

A category \mathbb{C} is called *action distinctive* when $Pt\mathbb{C}$ is \mathfrak{A} -distinctive; a Mal'tsev category is action distinctive if and only if any equivalence relation R has a centralizer $Z[R]$, see [8]. So it is obviously the case for the categories Gp of groups, Rg of (non-unitary) rings and $R-Lie$ of Lie algebras on a ring R . We have more generally:

Corollary 7.1. *If \mathbb{C} is \mathfrak{A} -coreflexive on monomorphisms, it is action distinctive; if it is stably \mathfrak{A} -coreflexive, it is functorially action distinctive.*

Proof. We know by Corollary 2.1 that \mathbb{C} is a Mal'tsev category. We showed in Lemma 7.1 above that, in the Mal'tsev setting, the class of $\mathfrak{A}_{\mathbb{C}}$ -cartesian maps is amenably proper, so we can apply the previous proposition. \square

Accordingly any category \mathbb{C} which is \mathfrak{A} -coreflexive on monomorphisms has centralizers of equivalence relations. In particular, the normalizer of a diagonal $s_0^X : X \rightarrow X \times X$ is normal to the largest equivalence relation on X centralizing

∇X . We shall denote the \blacktriangleleft -distinctive equivalence relation $D[f, s]$ of a split epimorphism (f, s) by the following left hand side diagram in \mathbb{C} :

$$\begin{array}{ccc}
D_X[f, s] \begin{array}{c} \xrightarrow{\delta_0^X} \\ \xleftarrow{\delta_0^X} \\ \xrightarrow{\delta_1^X} \\ \xleftarrow{\delta_1^X} \end{array} X & & D_R[d_0^R, s_0^R] \begin{array}{c} \xrightarrow{\delta_0^R} \\ \xleftarrow{\delta_0^R} \\ \xrightarrow{\delta_1^R} \\ \xleftarrow{\delta_1^R} \end{array} R \\
D_f \downarrow \uparrow D_s \quad \delta_0^X \quad f \downarrow \uparrow s & & D_{d_0^R} \downarrow \uparrow D_{s_0^R} \quad \delta_0^R \quad d_0^R \downarrow \uparrow d_1^R \\
D_Y[f, s] \begin{array}{c} \xrightarrow{\delta_0^Y} \\ \xleftarrow{\delta_0^Y} \\ \xrightarrow{\delta_1^Y} \\ \xleftarrow{\delta_1^Y} \end{array} Y & & D_X[d_0^R, s_0^R] \begin{array}{c} \xrightarrow{\delta_0^X} \\ \xleftarrow{\delta_0^X} \\ \xrightarrow{\delta_1^X} \\ \xleftarrow{\delta_1^X} \end{array} X
\end{array}$$

while the lower row of the right hand side one determines the centralizer of the equivalence relation R on X . When \mathbb{C} is an action distinctive Mal'tsev category, an object $(f, s) : X \rightrightarrows Y$ of $Pt\mathbb{C}$ is eccentric if and only if $s^{-1}(Z[R[f]]) = \Delta_Y$, where $Z[R[f]]$ is the centralizer of the kernel equivalence relation $R[f]$ of the map f . In the category Gp , given a split epimorphism $(f, s) : X \rightrightarrows Y$, the equivalence relation $D_Y[f, s]$ is nothing but the kernel equivalence relation of its associated canonical action $\phi : Y \rightarrow AutK$, where K is the kernel of the homomorphism f . Accordingly a split epimorphism $(f, s) : X \rightrightarrows Y$ is \blacktriangleleft -eccentric in Gp if and only if ϕ is injective, namely if and only if its associated canonical action is faithful.

7.2 Facc and B-C facc properties

In this section we shall show that in any pointed protomodular category which is \blacktriangleleft -coreflexive on monomorphisms, any subobject has an earnest centralizer, i.e. a universal map commuting with it (and not only a universal monomorphism commuting with it). Such kind of category is called *algebraically cartesian closed* (acc) in [12], because it is equivalent to saying that any base change along a terminal map $\tau_Y^* : \mathbb{E} \rightarrow Pt_Y\mathbb{E}$ has a right adjoint Φ_Y .

A non-pointed category \mathbb{E} is called *fibrewise algebraically cartesian closed* (facc) when any fibre $Pt_Y\mathbb{C}$ is acc. It is said to be B-C facc in [8] when, in addition, the Beck-Chevalley condition holds, i.e.: for any pullback of split epimorphisms

$$\begin{array}{ccc}
X' & \xrightarrow{x} & X \\
f' \downarrow \uparrow s' & & f \downarrow \uparrow s \\
Y' & \xrightarrow{y} & Y
\end{array}$$

we get $y^*.\Phi_f \simeq \Phi_{f'}.x^*$. It is the case if and only if the change of base functor $y^* : Pt_Y\mathbb{C} \rightarrow Pt_{Y'}\mathbb{C}$ preserves the earnest centralizers of subobjects.

Recall from [5] that in a pointed Mal'tsev category \mathbb{E} the notion of commutation of morphisms is intrinsic; a pair (f, g) of maps with domains (X, Y) and common codomain Z is said to commute when there is a (necessarily unique) map $\chi : X \times Y \rightarrow Z$, called the co-operator of (u, f) , such that $\chi.\iota_0^X = f$ and $\chi.\iota_1^Y = g$.

Proposition 7.2. *Let \mathbb{E} be a pointed Mal'tsev category and $u : U \rightarrow X$ a subobject. A map $f : T \rightarrow X$ commutes with u if and only if there is a dotted*

arrow making the following left hand side diagram commute:

$$\begin{array}{ccccc}
& & \alpha_X \times (u, 1_U) & & \\
& \nearrow & & \searrow & \\
U & \xrightarrow{l_1^U} & T \times U & \cdots \cdots \rightarrow & X \times X \times U \\
\tau_U \downarrow \uparrow \alpha_U & & p_0^T \downarrow \uparrow l_0^T & & p_0^X \times \tau_U \downarrow \uparrow s_0^X \times \alpha_U \\
1 & \xrightarrow{\alpha_T} & T & \xrightarrow{f} & X
\end{array}
\quad
\begin{array}{ccccc}
& & \alpha_X \times (u, 1_U) & & \\
& \nearrow & & \searrow & \\
U & \xrightarrow{} & W_{(u,f)} & \cdots \cdots \rightarrow & X \times X \times U \\
\tau_U \downarrow \uparrow \alpha_U & & \downarrow \uparrow & & p_0^X \times \tau_U \downarrow \uparrow s_0^X \times \alpha_U \\
1 & \xrightarrow{\alpha_T} & T & \xrightarrow{f} & X
\end{array}$$

When \mathbb{E} is protomodular it is the case if and only if there a decomposition with a left hand side pullback, as on the right hand side.

A pointed protomodular category \mathbb{E} which is \mathfrak{A} -coreflexive on monomorphisms has earnest centralizers, i.e. is algebraically cartesian closed.

Proof. Suppose χ is the co-operator, then the dotted arrow is $(f.p_0^T, \chi, p_1^U)$. Now, complete the right hand side diagram in the following way:

$$\begin{array}{ccccccc}
& & \alpha_X \times (u, 1_U) & & & & \\
& \nearrow & & \searrow & & & \\
U & \xrightarrow{} & W_{(u,f)} & \cdots \cdots \rightarrow & X \times X \times U^{p_1^U} & \rightarrow & U \\
\tau_U \downarrow \uparrow \alpha_U & & \downarrow \uparrow & & p_0^X \times \tau_U \downarrow \uparrow s_0^X \times \alpha_U & & \tau_U \downarrow \uparrow \alpha_U \\
1 & \xrightarrow{\alpha_T} & T & \xrightarrow{f} & X & \xrightarrow{\tau_X} & 1
\end{array}$$

The rectangle made of the three squares is a pullback, the left hand side square is a pullback by assumption, so that, when \mathbb{E} is protomodular, the rectangle made of the two right hand squares is a pullback and we get $W_{(u,f)} \simeq T \times U$. Accordingly, the pair (u, f) commutes.

Suppose now that \mathbb{E} is \mathfrak{A} -coreflexive on monomorphisms, then the extremal decomposition of the monomorphism $(\alpha_X, (\alpha_X \times (u, 1_U)))$ in $Pt\mathbb{E}$:

$$\begin{array}{ccccc}
& & \alpha_X \times (u, 1_U) & & \\
& \nearrow & & \searrow & \\
U & \xrightarrow{} & \Theta & \rightarrow & X \times X \times U \\
\tau_U \downarrow \uparrow \alpha_U & & \downarrow \uparrow & & p_0^X \times \tau_U \downarrow \uparrow s_0^X \times \alpha_U \\
1 & \xrightarrow{\alpha_{Z_u}} & Z_u & \xrightarrow{\zeta_U} & X \\
& & & \searrow & \alpha_X
\end{array}$$

is such that any map f commuting with u factors through ζ_u . When, moreover, \mathbb{E} is protomodular, by the previous observation; the pair (u, ζ_u) commutes, and ζ_u is the earnest centralizer of u . \square

Proposition 7.3. *Let \mathbb{E} be a category which is protomodular on split epimorphisms and \mathfrak{A} -coreflexive on monomorphisms. Then it is facc, and B-C facc when \mathbb{E} is stably \mathfrak{A} -coreflexive on monomorphisms.*

Proof. Under our assumptions any fibre $Pt_Y\mathbb{E}$ is protomodular and \mathfrak{A} -coreflexive on monomorphisms; according to the previous proposition, it is acc. When, in

addition, \mathbb{E} is stably \mathfrak{M} -coreflexive on monomorphisms, then by Proposition 4.2 any base-change functor preserves the extremal decompositions in the fibers and consequently preserves the construction of the earnest centralizer given in the previous proposition. \square

7.3 Θ -accessibility and action accessibility

In this section, still on the pointed model of [13] but still with more transparent proofs, we shall show that, in the protomodular and exact context, any category which is \mathfrak{M} -coreflexive on monomorphisms is action accessible in the sense of [14].

Definition 7.2. *Given a proper class Θ in a category \mathbb{E} , we call:*

- 1) Θ -special any parallel pair $Y \rightrightarrows Y'$ of morphisms in Θ provided that they are equalized by a monomorphism in Θ
- 2) Θ -faithful any object X in \mathbb{E} such that, given any parallel pair $(h, h') : V \rightrightarrows X$ of morphisms in Θ , they are equal as soon as this pair is Θ -special
- 3) Θ -accessible any category \mathbb{E} which has “enough” Θ -faithful objects, namely such that any object Y admits a map in Θ toward a Θ -faithful object. Such a map (or, for short, its codomain) will be called a Θ -index of the object Y .

A Θ -accessible category is Θ -distinctive: to get the Θ -distinctive equivalence relation of an object X , just take the kernel equivalence relation of any of its Θ -index; actually any Θ -accessible category is functorially Θ -distinctive. In a Θ -accessible category, an object U is Θ -faithful if and only if it is Θ -eccentric.

Definition 7.3. *A proper class Θ is said to be regularly proper when any decomposition $f = g \cdot q$ of a map in Θ with q a regular epimorphism is such that g is in Θ as well.*

In this case q is in Θ as well since Θ is a proper class.

Example 7.2. *Given a regular Mal'tsev category \mathbb{C} , the class of $\mathfrak{M}_{\mathbb{C}}$ -cartesian maps is regularly proper in $Pt_{\mathbb{C}}$.*

Proof. It is exactly the meaning of Proposition 3.4 in [15]. \square

Lemma 7.2. *Let \mathbb{E} be a regular category where Θ is a regularly proper class. An object U is Θ -faithful as soon as the previous definition is only assumed for the jointly monic Θ -special pairs.*

Proof. Suppose that X is an object which satisfies the previous definition for the jointly monic pairs. Then take any Θ -special parallel pair (t_0, t_1) with codomain X :

$$\begin{array}{ccccc}
 U & \xrightarrow{j} & V & \begin{array}{c} \xrightarrow{t_1} \\ \xrightarrow{t_0} \end{array} & X \\
 \searrow g & & \searrow h & \begin{array}{c} \xrightarrow{\bar{t}_1} \\ \xrightarrow{\bar{t}_0} \end{array} & \nearrow \\
 \bar{U} & \xrightarrow{\bar{j}} & \bar{V} & & \\
 & & & \nearrow \bar{t}_0 &
 \end{array}$$

Let j any monomorphism in Θ which equalizes them. Take $(\bar{t}_0, \bar{t}_1).h$ the decomposition of $(t_0, t_1) : V \rightarrow X \times X$ where h is a regular epimorphism and $(\bar{t}_0, \bar{t}_1) : \bar{V} \rightarrow X \times X$ is a monomorphism. Since the class Θ is regularly proper and $\bar{t}_i.h = t_i$, $i \in \{0, 1\}$ is in Θ , so is \bar{t}_i . Accordingly (\bar{t}_0, \bar{t}_1) is a jointly monic parallel pair in Θ . Notice that h is in Θ and, so, $h.j$ as well. It remains to show that the pair (\bar{t}_0, \bar{t}_1) is Θ -special. Take $h.j = \bar{j}.g$ the canonical decomposition through a regular epimorphism and a monomorphism. Since g a regular epimorphism and $h.j$ is in Θ , the map \bar{j} is itself in Θ , and the pair (\bar{t}_0, \bar{t}_1) is Θ -special. According to the assumption we get $\bar{t}_0 = \bar{t}_1$, and consequently $t_0 = t_1$. \square

Here is the main result of this section:

Proposition 7.4. *Let \mathbb{E} be a regular category where Θ is an amenable regularly proper class satisfying the monomorphic three out of two condition. Suppose \mathbb{E} is Θ -coreflexive on monomorphisms. Then any Θ -special parallel pair with codomain X factors through the Θ -distinctive equivalence relation D_X . An object X is Θ -faithful if and only if its Θ -eccentric.*

Proof. By Proposition 7.1, the category \mathbb{E} is Θ -distinctive. Now start with a Θ -special pair (t_0, t_1) with codomain X and j a monomorphism in Θ which equalizes them. Consider the jointly monic Θ -special pair (\bar{t}_0, \bar{t}_1) given by the previous lemma and the following diagram in \mathbb{E} :

$$\begin{array}{ccccc}
 \bar{U} & \xrightarrow{\bar{t}_i.\bar{j}} & X & \xrightarrow{s_0^{D_X}} & D_X & \xrightarrow{(\delta_0^X, \delta_1^X)} & X \times X \\
 & \searrow \bar{j} & \bar{V} & \xrightarrow{(\bar{t}_0, \bar{t}_1)} & & & \\
 & & & \nearrow & & &
 \end{array}$$

The map $\bar{t}_i.\bar{j}$ is a monomorphism in Θ . The decomposition $s_0^X = (\delta_0^X, \delta_1^X).s_0^{D_X}$ is an extremal decomposition according to Proposition 7.1 and (δ_0^X, δ_1^X) is a Θ -outsider monomorphism. By Lemma 1.6, since \mathbb{E} is a Θ -coreflexive category and Θ satisfies the monomorphic three out of two condition, the decomposition $(\delta_0^X, \delta_1^X).(s_0^{D_X}.\bar{t}_i.\bar{j})$ is an extremal decomposition of $(\bar{t}_0, \bar{t}_1).\bar{j}$ where \bar{j} is a monomorphism in Θ ; accordingly there is a dotted factorization of the pair (\bar{t}_0, \bar{t}_1) , which, in turn, produces the desired factorization of the pair (t_0, t_1) . We already observed that any Θ -faithful object is Θ -eccentric. When X is Θ -eccentric, the map $s_0^{D_X}$ is an isomorphism, and we get $t_0 = t_1$; consequently X is Θ -faithful. \square

Corollary 7.2. *Let \mathbb{E} be a regular category which is protomodular on monomorphisms and \mathbb{E} -coreflexive on monomorphisms. Then any \mathbb{E} -special parallel pair with codomain the split epimorphism (f, s) factors through the \mathbb{E} -distinctive equivalence relation $\mathbb{D}[f, s]$. A split epimorphism (f, s) is eccentric if and only if it is faithful.*

Proof. Since \mathbb{E} is protomodular on monomorphisms, the class of \mathbb{E} -cartesian maps satisfies the monomorphic three out of two condition in $Pt\mathbb{E}$, and \mathbb{E} is

Mal'tsev as well. Accordingly $Pt\mathbb{E}$ is a regular Mal'tsev category, and the class of $\mathfrak{A}_{\mathbb{E}}$ -cartesian maps is amenably regular in $Pt\mathbb{E}$. So we can apply the previous proposition to $Pt\mathbb{E}$. \square

Proposition 7.5. *When \mathbb{E} is an exact Θ -distinctive category where Θ is a regularly proper class such that the quotient of any equivalence relation in Θ is in Θ , it has enough Θ -eccentric objects. When, in addition, it is Θ -coreflexive on monomorphisms and Θ is an amenably proper class which satisfies the monomorphic three out of two condition, the category \mathbb{E} is Θ -accessible.*

Proof. Starting with any object Y , take its Θ -distinctive equivalence relation and its quotient q which is in Θ :

$$D_Y \begin{array}{c} \xrightarrow{\delta_0^Y} \\ \xleftarrow{\delta_1^Y} \end{array} Y \xrightarrow{q} Q$$

Now consider the distinctive equivalence relation D_Q . In the regular context the distinctive equivalence relations have a functorial extension on the regular epimorphisms (Proposition 5.3 in [8]). Accordingly we have $q^{-1}(\Delta_Q) = R[q] = D_Y = q^{-1}(D_Q)$. Thus we get $D_Q = \Delta_Q$, since q is a regular epimorphism; and Q is eccentric. When, moreover, \mathbb{E} is Θ -coreflexive and Θ is amenable and satisfies the monomorphic three out of two condition, the object Q is Θ -faithful according to Proposition ???. \square

Action accessibility was first introduced in the pointed protomodular context [14], then enlarged to the non-pointed Mal'tsev context [8]. When \mathbb{E} is a Mal'tsev category, a split epimorphism $(f, s) : X \rightrightarrows Y$ is said to be *faithful* when it is faithful, as an object of $Pt\mathbb{E}$, with respect to the class of $\mathfrak{A}_{\mathbb{E}}$ -cartesian maps; \mathbb{E} is said to be *action accessible* [14], when $Pt\mathbb{E}$ is accessible with respect to this same class. The categories Gp of groups, Rg of non commutative rings, $R-Lie$ of Lie algebras and $GpTop$ of topological groups are examples of action accessible categories. Now we get:

Proposition 7.6. *When \mathbb{E} is an exact action distinctive Mal'tsev category, it has enough eccentric objects. When \mathbb{E} is an exact category which is protomodular on monomorphisms and \mathfrak{A} -coreflexive on monomorphisms, the category \mathbb{E} is action accessible.*

Proof. When \mathbb{E} is exact action distinctive Mal'tsev category, the quotient of any $\mathfrak{A}_{\mathbb{E}}$ -cartesian equivalence relation is $\mathfrak{A}_{\mathbb{E}}$ -cartesian maps, according to Proposition 5.4 in [13]. So that, in the Mal'tsev and protomodular contexts, we can apply Proposition 7.5 to $Pt\mathbb{E}$. \square

The proof given in the proposition above (via the construction of the quotient in Proposition 7.5) is precisely the way that the category Rg of non commutative rings was proved to be action accessible in [14], since the ideals used in this pioneering proof are precisely the ideals associated with the equivalence relations

$D_X[f, s]$ and $D_Y[f, s]$ associated with the split epimorphism (f, s) . According to the two previous sections, notice that, in the protomodular context, the stably \mathbb{Q} -coreflexiveness appears as an upper bound to the action accessibility and B-C face property.

8 Σ -Mal'tsev and Σ -protomodular categories

We observed in Proposition 3.3 that the fibers Grd_Y are \mathbb{Q} -coreflexive on monomorphisms. In this section we shall investigate which part of this kind of structure is remaining on the larger fibers Cat_Y of the fibration $(\)_0 : Cat \rightarrow Set$ associated with categories. Recall from [10]:

Definition 8.1. *Let Σ be a class of split epimorphisms; denote by $\Sigma(\mathbb{E})$ the full subcategory of $Pt(\mathbb{E})$ whose objects are in Σ . This class is said to be:*

- 1) *fibrational when Σ is stable under pullback and contains the isomorphisms*
- 2) *point-congruous when, in addition, $\Sigma(\mathbb{E})$ is stable under finite limits in $Pt(\mathbb{E})$.*

When Σ is fibrational, it determines a pointed subfibration $\mathbb{Q}_\Sigma : \Sigma(\mathbb{E}) \rightarrow \mathbb{E}$ of the fibration of points. Recall from [9]:

Definition 8.2. *Let Σ be a fibrational class of split epimorphisms in \mathbb{E} . Then \mathbb{E} is said to be a Σ -Mal'tsev category when, for any pullback of split epimorphisms above (g, t) , as in the left hand side diagram:*

$$\begin{array}{ccc}
 X \times_Y Z & \xleftarrow{i_X} & X \\
 p_Z \downarrow \uparrow i_Z & \xrightarrow{p_X} & \downarrow f \uparrow s \\
 Z & \xleftarrow{t} & Y \\
 & \xrightarrow{g} &
 \end{array}
 \qquad
 \begin{array}{ccc}
 X \times_Y Z & \xrightarrow{p_X} & X \\
 p_Z \downarrow \uparrow i_Z & & \downarrow f \uparrow s \\
 Z & \xrightarrow{g} & Y
 \end{array}$$

the pair (i_Z, i_X) is jointly extremally epic whenever the split epimorphism (f, s) belongs to the class Σ . It is said to be Σ -protomodular when, for any pullback of split epimorphisms above any map g as in the right hand side diagram, the pair (s, p_X) is jointly extremally epic whenever the split epimorphism (f, s) belongs to the class Σ .

Clearly any Σ -protomodular category is a Σ -Mal'tsev one.

8.1 Examples

1) Let Mon be the category of monoids. A split epimorphism $(f, s) : X \rightrightarrows Y$ will be called a *weakly Schreier* split epimorphism when, for any element $y \in Y$, the application $\mu_y : Ker f \rightarrow f^{-1}(y)$ defined by $\mu_y(k) = k \cdot s(y)$ is surjective. The class Σ' of weakly Schreier split epimorphisms is fibrational and *the category Mon is a Σ' -Mal'tsev category.*

1') In [10] a split epimorphism $(f, s) : X \rightrightarrows Y$ in Mon was called a *Schreier* split epimorphism when the application μ_y is bijective. This defines a subclass $\Sigma \subset \Sigma'$. It was shown to be point-congruous and the category Mon to be Σ -protomodular in [10]. Actually a split epimorphism $(f, s) : X \rightrightarrows Y$ is a Schreier one if and only if there is a function $q : X \rightarrow Kerf$ such that $x = q(x).sf(x)$, $\forall x \in X$ and $q(k.s(t)) = k$, $\forall (k, t) \in Kerf \times Y$. Any split epimorphism above a group is a Schreier one.

2) Suppose that $U : \mathbb{C} \rightarrow \mathbb{D}$ is a left exact functor. It is clear that if Σ is a fibrational (resp. point-congruous) class of split epimorphisms in \mathbb{D} , so is the class $\bar{\Sigma} = U^{-1}\Sigma$ in \mathbb{C} . When, in addition, the functor U is conservative (i.e. reflects the isomorphisms), then \mathbb{C} is a $\bar{\Sigma}$ -Mal'tsev (resp. Σ -protomodular) category as soon as so is \mathbb{D} .

3) Let SRg be the category of semi-rings. The forgetful functor $U : SRg \rightarrow CoM$ (where CoM is the category of commutative monoids) is left exact and conservative. We call *weakly Schreier* a split epimorphism in $\bar{\Sigma}'$. In [10] a split epimorphism in $\bar{\Sigma}$ was called a Schreier one. Thanks to the point 2), this gives us two other partial Mal'tsev (resp. protomodular) structures.

4) Given any finitely complete category \mathbb{E} , define a split epimorphism $(f, s) : X \rightrightarrows Y$ in the category $Mon\mathbb{E}$ of internal monoids as a Schreier one when there is a map $q : M \rightarrow Kerf$ in \mathbb{E} such that $m_X.(k_f.q, sf) = 1_M$ and $q.m_X.(k_k \times s) = p_0^{K_f}$, where m_X is the monoid operation of M . Denote Σ_1 the induced class. By the Yoneda embedding, this class Σ_1 is fibrational and point-congruous, and the category $Mon\mathbb{E}$ is Σ_1 -protomodular [10].

5) The category Mon is nothing but the fibre above the singleton 1 of the fibration $()_0 : Cat \rightarrow Set$ which associates with any category its set of objects. Let us denote by Cat_Y the fibre above the set Y . A bijective on objects split epimorphic functor $(F, S) : \mathbb{A} \rightrightarrows \mathbb{B}$ in Cat_Y is called *with cofibrant splittings* when any map $S(\phi)$ is hypercocartesian or equivalently when F is a split cofibration whose splitting is precised by the splitting S . The class Σ_Y of such split epimorphisms is fibrational and point-congruous, and the category Cat_Y is Σ_Y -protomodular. The Grothendieck construction concerning the split cofibrations shows that the fibre of Σ_Y above the category \mathbb{B} is isomorphic to the functor category $\mathcal{F}(\mathbb{B}, Mon)$. The fibre Cat_Y is quasi-pointed and a split epimorphism $(F, S) : \mathbb{A} \rightrightarrows \mathbb{B}$ in Cat_Y has cofibrant splittings if and only if there is a function $q : \mathbb{A} \rightarrow KerF$ such that $\phi = q(\phi).SF(\phi)$, $\forall \phi \in \mathbb{B}$ and $q(k.S(\psi)) = k$, $\forall (k, \psi) \in KerF \times \mathbb{B}$. Any split epimorphism above a groupoid has cofibrant splittings. In this case we have $q(\phi) = \phi.SF(\phi^{-1})$.

6) Given any finitely complete category \mathbb{E} , note by $()_0 : Cat\mathbb{E} \rightarrow \mathbb{E}$ the fibration associating with any internal category its object of objects, and by $Cat_Y\mathbb{E}$ the fibre above Y . Define a split epimorphic functor $(\underline{F}_1, \underline{S}_1) : \underline{X}_1 \rightrightarrows \underline{Y}_1$ as having cofibrant splittings when there is a map $q_1 : X_1 \rightarrow KerF_1$ in \mathbb{E} such that $m^{X_1}.(k_{F_1}.q_1, S_1F_1) = 1_{X_1}$ and $q_1.m_{X_1}.(k_{F_1}, S_1) = p_0^{K_{F_1}}$ where m_{X_1} denotes the composition law of the internal category \underline{X}_1 . Denote by Σ_Y the induced class. By the Yoneda embedding: 1) any split epimorphism above a groupoid has

cofibrant splittings, 2) the class Σ_Y is fibrational and point-congruous, and 3) any fibre $Cat_Y \mathbb{E}$ is Σ_Y -protomodular.

We shall need later on the following observation:

Lemma 8.1. *Given any split epimorphic functor $(F_1, S_1) : X_1 \rightrightarrows Y_1$ with cofibrant splittings in $Cat \mathbb{E}$, the following square is a pullback of split epimorphisms above d_1 in \mathbb{E} :*

$$\begin{array}{ccc}
 Ker F_1 & \xrightarrow{k_{F_1}} & X_1 \\
 \tilde{F}_1 \downarrow \uparrow \tilde{S}_1 & \xrightarrow{q_1} & F_1 \downarrow \uparrow \tilde{S}_1 \\
 Y & \xrightarrow{s_0} & Y_1 \\
 & \xleftarrow{d_1} &
 \end{array}$$

Proof. It is true in Set because of the uniqueness of the decomposition $\phi = u.S(\psi)$ with $(u, \psi) \in (Ker F_1 \times_Y Y_1)$. The Yoneda embedding produces the lifting to \mathbb{E} . \square

The aim of this section is to show that any fibre Cat_Y is stably \mathfrak{A}_{Σ_Y} -coreflexive on monomorphisms. We shall investigate the case of $Mon \mathbb{E}$ and $Cat_Y \mathbb{E}$ in the next section.

Proposition 8.1. *The category Mon is stably \mathfrak{A}_{Σ} -coreflexive on monomorphisms.*

Proof. Given any subobject in $Pt(Mon)$ between Schreier split epimorphisms:

$$\begin{array}{ccccc}
 & \curvearrowright & & \curvearrowleft & \\
 X' & \xrightarrow{\quad} & \bar{X} & \xrightarrow{\quad} & X \\
 f' \downarrow \uparrow s' & & \bar{f} \downarrow \uparrow \bar{s} & & f \downarrow \uparrow s \\
 Y' & \xrightarrow{\quad} & \bar{Y} & \xrightarrow{\quad} & Y \\
 & \curvearrowleft & & \curvearrowright &
 \end{array}$$

the submonoid \bar{Y} is $\{y \in Y/q(s(y).u) \in X', \forall u \in Ker f'\}$ while the submonoid \bar{X} is $\{x \in X/q(x) \in X', \forall u \in Ker f' \text{ and } f(x) \in \bar{Y}\}$. From this construction, checking the stability under pullback along \mathfrak{A}_{Σ} -cartesian morphisms is straightforward. \square

Proposition 8.2. *Any fibre Cat_Y is stably \mathfrak{A}_{Σ} -coreflexive on monomorphisms.*

Proof. Given any subobject in $Pt(Cat_Y)$ between split epimorphisms with fibrant splittings:

$$\begin{array}{ccccc}
 & & \curvearrowright & & \\
 & & \downarrow & & \downarrow \\
 \mathbb{X}' & \dashrightarrow & \bar{\mathbb{X}} & \dashrightarrow & \mathbb{X} \\
 \uparrow F' & & \uparrow F & & \uparrow F \\
 \mathbb{Y}' & \dashrightarrow & \bar{\mathbb{Y}} & \dashrightarrow & \mathbb{Y} \\
 \downarrow S' & & \downarrow \bar{S} & & \downarrow S \\
 & & \curvearrowleft & &
 \end{array}$$

the morphisms of the subcategory $\bar{\mathbb{Y}}$ are those morphisms $\psi \in \mathbb{Y}$ such that $q(S(\psi).u) \in \mathbb{X}'$, $\forall u \in Ker F'$ while the morphisms of the subcategory $\bar{\mathbb{X}}$ those morphisms $\phi \in \mathbb{X}$ such that $q(\phi) \in \mathbb{X}'$, $\forall u \in Ker F'$ and $F(\psi) \in \bar{\mathbb{Y}}$. From this construction, checking the stability under pullback along \mathfrak{A}_{Σ} -cartesian morphisms is straightforward. \square

Many results about the categories which are \mathfrak{A} -coreflexive on monomorphisms are still valid for the categories which are only \mathfrak{A}_{Σ} -coreflexive on monomorphisms provided that \mathbb{E} is a Σ -Mal'tsev category. For that we need the following straightforward translations:

Proposition 8.3. *Suppose \mathbb{E} is a Σ -Mal'tsev category. Then a split epimorphism of equivalence relations with (f, s) in Σ as on the left hand side:*

$$\begin{array}{ccc}
 R_X \xrightleftharpoons[d_0^X]{} X & & R_X \xleftarrow{s_0^X} X \\
 R_f \uparrow R_s \quad d_1^X \quad f \downarrow s & & R_f \uparrow R_s \quad f \downarrow s \\
 R_Y \xrightleftharpoons[d_1^Y]{} Y & & R_Y \xleftarrow{s_0^Y} Y
 \end{array}$$

is fibrant if and only if the right hand side diagram is a pullback in $Pt\mathbb{E}$. In particular, the class of \mathfrak{A}_{Σ} -cartesian morphisms is amenably proper.

Proof. In a Σ -Mal'tsev category the arguments of the proof of Proposition 2.4 still hold since the factorization ϕ still is a strong epimorphism as soon as (f, s) is in Σ . Accordingly the class of \mathfrak{A}_{Σ} -cartesian morphisms is amenably proper in the category $\Sigma(\mathbb{E})$. \square

Proposition 8.4. *The category Mon and any fibre Cat_Y are functorially \mathfrak{A}_{Σ} -distinctive.*

Proof. Just apply Proposition 7.1. \square

From that, we obtain in Mon and Cat_Y the existence of centralizers for a certain class of equivalence relations by the following proposition where a Σ -equivalence relation R on X is an equivalence relation such that the split epimorphism (d_0^R, s_0^R) belongs to Σ :

Proposition 8.5. *Let \mathbb{E} be a Σ -Mal'tsev category for a point-congruous fibrational class Σ . Suppose moreover that \mathbb{E} is \mathbb{N}_{Σ} -coreflexive on monomorphisms. Then any Σ -equivalence relation has a centralizer.*

Proof. Let R be a Σ -equivalence relation on X . Consider the extremal decomposition in $\Sigma(\mathbb{E})$:

$$\begin{array}{ccccc}
 R & \xrightarrow{\quad} & D_R[d_0^R, s_0^R] & \xrightarrow{(\delta_0^R, \delta_1^R)} & R \times R \\
 \begin{array}{c} \uparrow d_0^R \\ \downarrow d_1^R \end{array} & & \begin{array}{c} \uparrow D_{d_0^R} \\ \downarrow D_{s_0^R} \end{array} & \begin{array}{c} \curvearrowright d_1 \\ \curvearrowleft \end{array} & \begin{array}{c} \uparrow d_0^R \\ \downarrow d_1^R \end{array} \\
 X & \xrightarrow{\quad} & D_X[d_0^R, s_0^R] & \xrightarrow{(\delta_0^X, \delta_1^X)} & X \times X
 \end{array}$$

Then the middle part produces the largest \mathbb{N}_{Σ} -cartesian equivalence relation on R , namely makes $D_X[d_0^R, s_0^R]$ the largest equivalence relation on X centralizing the equivalence relation R . \square

9 The fibers $Cat_Y \mathbb{E}$ and $Grd_Y \mathbb{E}$ when \mathbb{E} is a topos

Here we shall be interested in the fibers $Cat_Y \mathbb{E}$ and $Grd_Y \mathbb{E}$ in the locally cartesian context. We know that in any category \mathbb{E} the fibers $Grd_Y \mathbb{E}$ are quasi-pointed and protomodular categories while the fibers $Cat_Y \mathbb{E}$ are quasi-pointed and Σ_Y -protomodular, where Σ_Y is the point-congruous class of internal split epimorphic functors with cofibrant splittings.

It is showed in [13] that, when a category \mathbb{E} is cartesian closed, the pointed category $Gp \mathbb{E}$ of internal groups in \mathbb{E} has normalizers, i.e. is stably \mathbb{N} -coreflexive on monomorphisms by Theorem 3.1. In this section, we shall show that, when \mathbb{E} is *locally cartesian closed* [23] and a fortiori a topos, see [21], any fibre $Cat_Y \mathbb{E}$ of the fibration $(\)_0 : Cat \mathbb{E} \rightarrow \mathbb{E}$ is stably \mathbb{N}_{Σ_Y} -coreflexive on monomorphisms. It will follow that the category $Mon \mathbb{E}$ of internal monoids is stably \mathbb{N}_{Σ_1} -coreflexive on monomorphisms and that any fibre $Grd_Y \mathbb{E}$ is stably \mathbb{N} -coreflexive on monomorphisms or, equivalently, has normalizers.

Recall that a category \mathbb{E} is *locally cartesian closed* when, for any map $y : Y' \rightarrow Y$, the pullback functor $y^* : \mathbb{E}/Y \rightarrow \mathbb{E}/Y'$ has a right adjoint π_y . When it is the case, the Beck-Chevalley commutation holds, saying that given any pullback diagram:

$$\begin{array}{ccc}
 X' & \xrightarrow{x} & X' \\
 h \downarrow & & \downarrow g \\
 Y' & \xrightarrow{y} & Y
 \end{array}$$

we get: $g^* \cdot \pi_y = \pi_x \cdot h^*$.

Given any pair (u, w) of subobjects of Y' , we define $u \trianglelefteq_y w$ as the subobject $\pi_{y,u}(i_u)$ of Y , where i_u is the monomorphism determined by the inclusion $u \wedge w \leq u$; it is the largest of those subobjects m of Y such that $u \wedge y^{-1}(m) \leq w$, so that

in particular we get $u \wedge y^{-1}(u \trianglelefteq_y w) \leq w$; clearly we get (1): $u \trianglelefteq_y u = 1_Y$. On the other hand, given any pullback diagram as above, from the Beck-Chevalley commutation, we get (2): $g^*(u \trianglelefteq_y w) = h^*(u) \trianglelefteq_x h^*(w)$.

Given any slit epimorphic functor $(F_1, \underline{S}_1) : \underline{X}_1 \rightarrow \underline{Y}_1$ having cofibrant splittings in $Cat_Y \mathbb{E}$, consider the following diagram in \mathbb{E} :

$$\begin{array}{ccccccc}
& & \xrightarrow{p_0} & & & & \\
& & \swarrow & & \xrightarrow{Ker F_1 \times_{d_0} p_0} & & \\
Ker F_1 & \xrightarrow{k_{F_1}} & X_1 & \xleftarrow{\mu} & Ker F_1 \times_{d_0} Y_1 & \xleftarrow{Ker F_1 \times_{d_0} m_{Y_1}} & Ker F_1 \times_{d_0} Y_1 \times_Y Y_1 \\
\uparrow \tilde{F}_1 & \uparrow \tilde{S}_1 & \uparrow F_1 & \uparrow S_1 & \uparrow \tilde{F}_1 \times_{d_0} Y_1 & \uparrow \tilde{S}_1 \times_{d_0} Y_1 & \uparrow \tilde{F}_1 \times_{d_0} Y_1 \times_Y Y_1 \\
Y = Y_0 & \xrightarrow{d_0} & Y_1 & \xleftarrow{d_1} & Y_1 & \xleftarrow{p_0} & Y_1 \times_Y Y_1 \\
& & \xrightarrow{d_1} & & \xrightarrow{m_{Y_1}} & & \xrightarrow{p_1}
\end{array}$$

where the left hand side quasi-rectangle is a pullback above d_0 . Let us denote by σ_0 is the splitting of p_0 induced by $s_0 : Y \rightarrow Y_1$. The dotted arrow μ is the arrow $m_{X_1} \cdot (S_1 \cdot (\tilde{F}_1 \times_{d_0} Y_1), k_{F_1} \cdot p_0)$ where m_{X_1} denotes the composition map of the internal category \underline{X}_1 . In set theoretical terms we have $\mu(u, \psi) = S_1(\psi) \cdot u$. We set $p_1^\diamond = q_1 \cdot \mu : Ker F_1 \times_{d_0} Y_1 \rightarrow Ker F_1$ which makes commute the quasi-rectangle above d_1 (in set theoretical terms we have $p_1^\diamond(u, \psi) = q_1(S_1(\psi) \cdot u)$) and produces a reflexive graph above Y_1 .

From that, we get $p_1^\diamond \times_Y p_1 = (p_1^\diamond \cdot (Ker F_1 \times_{d_0} p_0), d_1 \cdot \tilde{F}_1 \times_{d_0} Y_1 \times_Y Y_1)$. In set theoretical terms we have $p_1^\diamond \times_Y p_1(u, \psi, \psi') = (p_1^\diamond(u, \psi), \psi')$. We can check $p_1^\diamond \cdot m_{Y_1} = p_1^\diamond \cdot p_1^\diamond \times_Y p_1$ in \mathbb{E} by the set theoretical identity: $q_1(S_1(\psi') \cdot \psi) \cdot u = q_1(S_1(\psi') \cdot q_1(S_1(\psi) \cdot u))$. So that the upper row of the quasi-rectangle completed by the map p_1^\diamond is underlying an internal category above \underline{Y}_1 .

Lemma 9.1. *Let \mathbb{E} be a locally cartesian closed category. Suppose given any subobject $j : (g_1, t_1) \rightarrow (\tilde{F}_1, \tilde{S}_1)$ in the fibre $Pt_Y \mathbb{E}$, and denote H_1 the domain of the morphism g_1 . The subobject $i_Y = p_0^*(j) \trianglelefteq_{\tilde{F}_1 \times_{d_0} Y_1} (p_1^\diamond)^*(j) : \tilde{Y}_1 \rightarrow Y_1$ is underlying a subcategory of \underline{Y}_1 in $Cat_Y \mathbb{E}$. The subobject $i_X = (F_1)^*(i_Y) \wedge q_1^*(j) : \tilde{X}_1 \rightarrow X_1$ is underlying a reflexif subgraph of \underline{X}_1 in the fibre $Gph_Y \mathbb{E}$ of internal reflexive graphs above Y . Moreover there is a split epimorphic morphism of reflexive graphs $(\bar{F}_1, \bar{S}_1) : \bar{X}_1 \rightrightarrows \bar{Y}_1$ such that $Ker \bar{F}_1 = H_1$.*

If, in addition, j is a monoid homomorphism, the reflexive graph \bar{X}_1 is underlying a subcategory of \underline{X}_1 in $Cat_Y \mathbb{E}$ and the pair (\bar{F}_1, \bar{S}_1) is underlying a split epimorphic functor $(\bar{F}_1, \bar{S}_1) : \bar{X}_1 \rightrightarrows \bar{Y}_1$ having cofibrant splittings such that $Ker \bar{F}_1 = \underline{H}_1$.

Proof. Since, in the above diagram introducing $Ker F_1 \times_{d_0} Y_1$, the upward vertical quasi-rectangle indexed by 0 is a pullback, we get by the Beck-Chevalley commutation:

$$s_0^*(i_Y) = s_0^*(p_0^*(m) \trianglelefteq_{\tilde{F}_1 \times_{d_0} Y_1} p_0^*(m)) = \sigma_0^*(p_0^*(m)) \trianglelefteq_{\tilde{F}_1} \sigma_0^*((p_1^\diamond)^*(m)) = m \trianglelefteq_{\tilde{F}_1} m$$

And the last term is nothing but 1_Y by (1). This shows that $i : \bar{Y}_1 \rightarrow Y_1$ produces a reflexif subgraph of Y_1 we shall denote in the following way:

$$Y = Y_0 \begin{array}{c} \xleftarrow{\bar{d}_0} \\ \xrightarrow{\bar{s}_0} \\ \xleftarrow{\bar{d}_1} \end{array} \bar{Y}_1$$

It remains to show now that \bar{Y}_1 is stable under composition inside \underline{Y}_1 , or in other words that we have: $i_Y \times_Y i_Y \leq (m_{Y_1})^*(i_Y)$. Since we have:

$$(m_{Y_1})^*(i_Y) = (m_{Y_1})^*(p_0^*(j) \trianglelefteq_{\bar{F}_1 \times_{d_0} Y_1} ((p_1^\diamond)^*(j)))$$

and the following square is a pullback:

$$\begin{array}{ccc} KerF_1 \times_{d_0} Y_1 \times_Y \bar{Y}_1 & \xrightarrow{KerF_1 \times_{d_0} m_{Y_1}} & KerF_1 \times_{d_0} Y_1 \\ \bar{F}_1 \times_{d_0} Y_1 \times_Y Y_1 \downarrow \uparrow \tilde{s}_1 \times_{d_0} Y_1 \times_Y Y_1 & & \bar{F}_1 \times_{d_0} Y_1 \downarrow \uparrow \tilde{s}_1 \times_{d_0} Y_1 \\ Y_1 \times_Y Y_1 & \xrightarrow{m_{Y_1}} & Y_1 \end{array}$$

by (2) the subobject $(m_{Y_1})^*(i_Y)$ is nothing but:

$$(KerF_1 \times_{d_0} m_{Y_1})^*(p_0^*(j)) \trianglelefteq_{\bar{F}_1 \times_{d_0} Y_1 \times_{d_0} Y_1} (KerF_1 \times_{d_0} m_{Y_1})^*((p_1^\diamond)^*(j))$$

Now we have: $p_0.(KerF_1 \times_{d_0} m_{Y_1}) = p_0.(KerF_1 \times_{d_0} p_0)$

and: $p_1^\diamond.(KerF_1 \times_{d_0} m_{Y_1}) = p_1^\diamond.p_1^\diamond \times_Y p_1$, so that:

$$(KerF_1 \times_{d_0} m_{Y_1})^*(p_0^*(j)) = (KerF_1 \times_{d_0} p_0)^*(p_0^*(j)) = j \times_{d_0} Y_1 \times_Y Y_1$$

$$\text{while : } (KerF_1 \times_{d_0} m_{Y_1})^*((p_1^\diamond)^*(j)) = (p_1^\diamond \times_Y p_1)^*((p_1^\diamond)^*(j))$$

So we have:

$$(m_{Y_1})^*(i_Y) = \pi_{(\bar{F}_1 \times_{d_0} Y_1 \times_Y Y_1).(j \times_{d_0} Y_1 \times_Y Y_1)}(p_1^\diamond \times_Y p_1)^*((p_1^\diamond)^*(j))$$

Consequently the inequality $i_Y \times_Y i_Y \leq (m_{Y_1})^*(i_Y)$ is equivalent to:

$$((\tilde{F}_1 \times_{d_0} Y_1 \times_Y Y_1).(j \times_{d_0} Y_1 \times_Y Y_1))^*(i_Y \times_Y i_Y) \leq (p_1^\diamond \times_Y p_1)^*((p_1^\diamond)^*(j))$$

namely: $H_1 \times_{d_0} (i_Y \times_Y i_Y) \leq (j \times_{d_0} Y_1 \times_Y Y_1)^*(p_1^\diamond \times_Y p_1)^*((p_1^\diamond)^*(j))$. Accordingly we have now to find a factorization $H_1 \times_{d_0} \bar{Y}_1 \times_Y \bar{Y}_1 \rightarrow H_1$ above the map

$p_1^\circ.(p_1^\circ \times_Y p_1).(Ker F_1 \times_Y i_Y \times_Y i_Y)$. For that consider the following diagram:

$$\begin{array}{ccccc}
& & H_1 \times_{d_0} Y_1 \times_Y Y_1 & \xrightarrow{H_1 \times_{d_0} p_0} & H_1 \times_{d_0} Y_1 & \xrightarrow{p_0} & H_1 \\
& \nearrow^{H_1 \times_Y i_Y \times_Y i_Y} & \downarrow & \nearrow^{H_1 \times_Y i_Y} & \downarrow & \nearrow^{\bar{p}_1^{H_1}} & \downarrow j \\
H_1 \times_{\bar{d}_0} \bar{Y}_1 \times_Y \bar{Y}_1 & \xrightarrow{H_1 \times_{\bar{d}_0} \bar{d}_0} & H_1 \times_{\bar{d}_0} \bar{Y}_1 & \xrightarrow{j \times_{d_0} Y_1} & H_1 \times_{d_0} Y_1 & & \\
& \searrow^{\bar{p}_1^{H_1} \times_Y \bar{d}_1} & \downarrow & & \downarrow & & \\
& & Ker F_1 \times_{d_0} Y_1 \times_Y Y_1 & \xrightarrow{Ker F_1 \times_{d_0} p_0} & Ker F_1 \times_{d_0} Y_1 & \xrightarrow{p_0} & Ker F_1 \\
& \nearrow^{Ker F_1 \times_Y i_Y \times_Y i_Y} & \downarrow & \nearrow^{Ker F_1 \times_Y i_Y} & \downarrow & \nearrow^{p_1^\circ} & \downarrow \\
Ker F_1 \times_{\bar{d}_0} \bar{Y}_1 \times_Y \bar{Y}_1 & \xrightarrow{Ker F_1 \times_{\bar{d}_0} \bar{d}_0} & Ker F_1 \times_{\bar{d}_0} \bar{Y}_1 & \xrightarrow{Ker F_1 \times_{d_0} m_{Y_1}} & Ker F_1 \times_{d_0} Y_1 & \xrightarrow{p_0} & Ker F_1 \\
& \searrow^{\bar{p}_1 \times_Y \bar{d}_1} & \downarrow & \nearrow^{p_1^\circ \times_Y p_1} & \downarrow & \nearrow^{p_1^\circ} & \downarrow \\
& & Ker F_1 \times_{\bar{d}_0} \bar{Y}_1 & \xrightarrow{\bar{p}_1 \times_Y \bar{d}_1} & Ker F_1 \times_{d_0} Y_1 & \xrightarrow{p_0} & Ker F_1 \\
& & & & & & \downarrow \\
& & & & & & Ker F_1
\end{array}$$

where \bar{p}_1 is $p_1^\circ.(Ker F_1 \times_Y i_Y)$, so that the map $\bar{p}_1 \times_Y \bar{d}_1$ is above $p_1 \times_Y d_1$ and defined in set theoretical terms by $\bar{p}_1 \times_Y \bar{d}_1(u, \psi, \psi') = (q_1(S_1(\psi).u), \psi')$. Now let $\bar{p}_1^{H_1} : H_1 \times_{\bar{d}_0} \bar{Y}_1 \rightarrow H_1$ the map above \bar{p}_1 which is determined by the inclusion $(p_0)^*(j) \wedge (\bar{F}_1 \times_{d_0} Y_1)^*(i_Y) \leq (p_1^\circ)^*(j)$. From that, the map $\bar{p}_1^{H_1} \times_Y \bar{d}_1$ is defined in set theoretical terms by $\bar{p}_1^{H_1} \times_Y \bar{d}_1(u, \psi, \psi') = (q_1(S_1(\psi).u), \psi')$. Then the map $\bar{p}_1^{H_1} . (\bar{p}_1^{H_1} \times_Y \bar{d}_1)$ produces the desired factorization.

Now let us show that \bar{X}_1 determines a reflexive subgraph of X_1 . For that consider the following diagram:

$$\begin{array}{ccccc}
& & \bar{X}_1 & \xrightarrow{\iota} & (q_1)^*(H_1) & \xrightarrow{\bar{k}_{F_1}} & H_1 \\
& \nearrow^{j_{\bar{X}_1}} & \downarrow & \nearrow^{i_X} & \downarrow & \nearrow^{q_1^{H_1}} & \downarrow j \\
\bar{S}_1((F_1^*)(X_1)) & \xrightarrow{(F_1^*)(i_Y)} & X_1 & \xrightarrow{k_{F_1}} & Ker F_1 & & \\
& \searrow^{\bar{F}_1} & \downarrow & \nearrow^{q_1} & \downarrow & \nearrow^{\bar{F}_1} & \downarrow \bar{S}_1 \\
& & \bar{Y}_1 & \xrightarrow{i_Y} & Y_1 & \xrightarrow{s_0} & Y \\
& & & & & & \downarrow d_1 \\
& & & & & & \bar{Y}
\end{array}$$

$\left. \begin{array}{c} \bar{F}_1 \downarrow \bar{S}_1 \\ \bar{F}_1 \downarrow \bar{S}_1 \end{array} \right\} t_0$
 \bar{s}_0

where, by construction, the three outside downward squares are pullbacks. The subobject i_X is the upper diagonal map. The lower right hand side square is a pullback as well by Lemma 8.1 since (\bar{F}_1, \bar{S}_1) has cofibrant splittings; so that the big square is a pullback. From that, the splitting t_0 produces a splitting \bar{S}_1 which shows that $(S_1)^*(i_X) = i_Y$, and since $s_0^*(i_Y) = 1_Y$, we have $s_0^*((S_1)^*(i_X)) = 1_Y$ so that \bar{X}_1 is underlying a reflexive subgraph of X_1 . And the splitting \bar{s}_0 produces a splitting $k_{\bar{F}_1}$ which is the kernel of $\bar{F}_1 = \bar{F}_1 . j_{\bar{X}_1}$.

Suppose H_1 is a submonoid of $Ker F_1$ in the fibre $Pt_Y \mathbb{E}$; we have to show that \bar{X}_1 is a subcategory of X_1 . By the Yoneda embedding, this comes from

the fact that, in set theoretical terms, we get:

$$q_1(\phi' \cdot \phi) = q_1(\phi') \cdot q_1(S_1 F_1(\phi) \cdot q_1(\phi)), \quad \forall(\phi, \phi') \in X_1 \times_Y X_1$$

which shows when (ϕ, ϕ') is in $\bar{X}_1 \times_Y \bar{X}_1$, $q_1(\phi' \cdot \phi)$ is in H_1 as soon as H_1 is a submonoid. Moreover the map $\bar{q}_1 = q_1^{H_1} \cdot \iota$ gives cofibrant splittings to the split epimorphism (\bar{F}_1, \bar{S}_1) . \square

Lemma 9.2. *Let \mathbb{E} be a locally cartesian closed category. Suppose given any submonoid $j : (g_1, t_1) \rightrightarrows (\tilde{F}_1, \tilde{S}_1)$ in the fibre $Pt_Y \mathbb{E}$, the following diagram in $Cat_Y \mathbb{E}$ determines an extremal decomposition with respect to the class Σ_Y :*

$$\begin{array}{ccccc}
 & \curvearrowright & & \curvearrowleft & \\
 H_1 & \dashrightarrow & \bar{X}_1 & \dashrightarrow & X_1 \\
 \updownarrow & & \bar{F}_1 \downarrow \uparrow \bar{S}_1 & & E_1 \downarrow \uparrow S_1 \\
 \Delta Y & \dashrightarrow & \bar{Y}_1 & \dashrightarrow & Y_1 \\
 & \curvearrowleft & & \curvearrowright &
 \end{array}$$

When Y_1 and X_1 are groupoids and $j : (g_1, t_1) \rightrightarrows (\tilde{F}_1, \tilde{S}_1)$ is a subgroup in the fibre $Pt_Y \mathbb{E}$, there are subgroupoids producing an extremal decomposition in the fibre $Grd_Y \mathbb{E}$:

$$\begin{array}{ccccc}
 & \curvearrowright & & \curvearrowleft & \\
 H_1 & \dashrightarrow & \bar{X}_1^\# & \dashrightarrow & X_1 \\
 \updownarrow & & \bar{F}_1^\# \downarrow \uparrow \bar{S}_1^\# & & E_1 \downarrow \uparrow S_1 \\
 \Delta Y & \dashrightarrow & \bar{Y}_1^\# & \dashrightarrow & Y_1 \\
 & \curvearrowleft & & \curvearrowright &
 \end{array}$$

Proof. Suppose we have the following decomposition where the left hand side square is a pullback (which means that H_1 is the kernel of \hat{F}_1) and the middle split epimorphism has cofibrant splittings:

$$\begin{array}{ccccc}
 & \curvearrowright & & \curvearrowleft & \\
 H_1 & \xrightarrow{k_{\hat{F}_1}} & \hat{X}_1 & \xrightarrow{H_1} & X_1 \\
 \updownarrow & & \hat{F}_1 \downarrow \uparrow \hat{S}_1 & & E_1 \downarrow \uparrow S_1 \\
 \Delta Y & \longrightarrow & \hat{Y}_1 & \xrightarrow{G_1} & Y_1 \\
 & \curvearrowleft & & \curvearrowright &
 \end{array}$$

First we have to find a factorization $\hat{G}_1 : \hat{Y}_1 \rightarrow \bar{Y}_1$ through i_Y . By the Beck-Chevalley commutation, it is equivalent to find a factorization from

$(\tilde{F}_1.(j \times_{d_0} Y_1))^*(G_1)$ to $((p_0)^*(j))^*((p_1^\circ)^*(j))$ or, in other words, a factorization $H_1 \times_{\hat{d}_0} \hat{Y}_1 \rightarrow H_1$ above $p_1^\circ.(Ker F_1 \times_Y G_1)$. For that consider the following diagram:

$$\begin{array}{ccccc}
H_1 \times_{\hat{d}_0} \hat{Y}_1 & \xrightarrow{H_1 \times_Y G_1} & H_1 \times_{d_0} Y_1 & \begin{array}{c} \xrightarrow{p_0} \\ \xleftarrow{p_1^{H_1}} \end{array} & H_1 \\
\downarrow j \times_{\hat{d}_0} \hat{Y}_1 & & \downarrow j \times_{d_0} Y_1 & & \downarrow j \\
Ker F_1 \times_{\hat{d}_0} \hat{Y}_1 & \xrightarrow{Ker F_1 \times_Y G_1} & Ker F_1 \times_{d_0} Y_1 & \begin{array}{c} \xrightarrow{p_0} \\ \xleftarrow{p_1^\circ} \end{array} & Ker F_1 \\
\downarrow \tilde{F}_1 \times_{\hat{d}_0} \hat{Y}_1 & & \downarrow \tilde{F}_1 \times_{d_0} Y_1 & & \downarrow \tilde{F}_1 \\
\hat{Y}_1 & \xrightarrow{G_1} & Y_1 & \begin{array}{c} \xrightarrow{d_0} \\ \xleftarrow{d_1} \end{array} & Y
\end{array}$$

The desired factorization is given by the map $p_1^{H_1}.(H_1 \times_Y G_1)$. The end of the proof is a straightforward checking.

Suppose now \underline{Y}_1 and \underline{X}_1 are groupoids and $j : (g_1, t_1) \mapsto (\tilde{F}_1, \tilde{S}_1)$ is a subgroup in the fibre $Pt_Y \mathbb{E}$. The previous construction only produces subcategories $\underline{\hat{Y}}_1$ and $\underline{\hat{X}}_1$. In order to produce subgroupoids we have to introduce the action of the inverse mappings on \underline{Y}_1 and \underline{X}_1 . This can be done in the following way. Consider the following left hand side decomposition completed by the inverse mappings:

$$\begin{array}{ccc}
H_1^{op} & \xrightarrow{\quad} & \overline{X_1^{op}} \xrightarrow{\quad} X_1^{op} \xrightarrow{()^{-1}} X_1 \\
\updownarrow & & \updownarrow \begin{array}{c} \overline{F_1^{op}} \downarrow \overline{S_1^{op}} \\ E_1^{op} \downarrow S_1^{op} \\ E_1 \downarrow S_1 \end{array} \\
\Delta Y & \xrightarrow{\quad} & \overline{Y_1^{op}} \xrightarrow{\quad} Y_1^{op} \xrightarrow{()^{-1}} Y_1
\end{array}
\qquad
\begin{array}{ccc}
\overline{X_1^\#} & \xlongequal{\quad} & \overline{X_1} \cap \overline{X_1^{op}} \xrightarrow{\quad} X_1 \\
\updownarrow \begin{array}{c} \overline{F_1^\#} \downarrow \overline{S_1^\#} \\ E_1 \downarrow S_1 \end{array} & & \updownarrow \\
\overline{Y_1^\#} & \xlongequal{\quad} & \overline{Y_1} \cap \overline{Y_1^{op}} \xrightarrow{\quad} Y_1
\end{array}$$

and then take the intersection with the direct decomposition. \square

Theorem 9.1. *Let \mathbb{E} be a locally cartesian closed category and denote by $()_0 : Cat \mathbb{E} \rightarrow \mathbb{E}$ the forgetful functor. Then any quasi-pointed fibre $Cat_Y \mathbb{E}$ is stably $\blacktriangleleft_{\Sigma_Y}$ -coreflexive on monomorphisms and any quasi-pointed protomodular fibre $Grd_Y \mathbb{E}$ is stably \blacktriangleleft -coreflexive on monomorphisms.*

Proof. Let be given any subobject of split epimorphisms with cofibrant splittings as on the left hand side, and complete it with the kernel of \hat{F}_1 :

$$\begin{array}{ccc}
Ker \hat{F}_1 & \xrightarrow{k_{\hat{F}_1}} & \hat{X}_1 \xrightarrow{L_1} X_1 \\
\updownarrow & & \updownarrow \begin{array}{c} \hat{F}_1 \downarrow \hat{S}_1 \\ E_1 \downarrow S_1 \end{array} \\
\Delta Y & \xrightarrow{\quad} & \hat{Y}_1 \xrightarrow{K_1} Y_1
\end{array}
\qquad
\begin{array}{ccc}
Ker \hat{F}_1 & \xrightarrow{k_{\hat{F}_1}} & \overline{X_1} \xrightarrow{i_{X_1}} X_1 \\
\updownarrow & & \updownarrow \begin{array}{c} \overline{F_1} \downarrow \overline{S_1} \\ E_1 \downarrow S_1 \end{array} \\
\Delta Y & \xrightarrow{\quad} & \overline{Y_1} \xrightarrow{i_{Y_1}} Y_1
\end{array}$$

Then consider the construction given by the previous lemmas now based upon the monoid inclusion $\underline{Ker}(L)_1 : \underline{Ker} \hat{F}_1 \mapsto \underline{Ker} \hat{F}_1$ as on the right hand side.

It is an extremal decomposition according to the previous lemma, therefore it produces a factorization of split epimorphic functors with fibrant splittings:

$$\begin{array}{ccccc}
 \underline{Ker}\hat{F}_1 & \xrightarrow{k_{\hat{F}_1}} & \hat{X}_1 & \xrightarrow{\hat{L}_1} & \bar{X}_1 \\
 \updownarrow & & \hat{F}_1 \downarrow \hat{S}_1 & & \bar{F}_1 \downarrow \bar{S}_1 \\
 \Delta Y & \xrightarrow{\quad} & \hat{Y}_1 & \xrightarrow{\bar{K}_1} & \bar{X}_1
 \end{array}$$

It remains to show that the right hand side square is a pullback of split epimorphic functors with cofibrant splittings. We know that so are the whole rectangle and the left hand side square in $Cat_Y \mathbb{E}$ since both split epimorphisms have the same kernel $\underline{Ker}\hat{F}_1$. Since the fibre $Cat_Y \mathbb{E}$ is Σ_Y -protomodular, so is the right hand side square. The end of the previous lemma allows us to get the result regarding the fibre $Grd \mathbb{E}$ in a quite similar way. \square

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