# Semiparametric Efficiency, Distribution-Freeness, and Invariance

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**Abstract**

Semiparametric models typically involve a finite-dimensional parameter of interest $\boldsymbol{\theta} \in \boldsymbol{\Theta} \subseteq \mathbb{R}^k$, along with an infinite-dimensional nuisance parameter $f$. Quite often, the submodels corresponding to a fixed value of $\boldsymbol{\theta}$ possess a group structure that induces a maximal invariant $\sigma$-field $\mathcal{B}(\boldsymbol{\theta})$. In classical examples, where $f$ denotes the density of some i.i.d. innovations, $\mathcal{B}(\boldsymbol{\theta})$ is the $\sigma$-field generated by the ranks of the residuals associated with the parameter value $\boldsymbol{\theta}$. It is shown that semiparametrically efficient (at given $f$) conditionally distribution-free inference procedures can generally be constructed from parametrically optimal ones by conditioning on $\mathcal{B}(\boldsymbol{\theta})$. The same procedures, when combined with a suitable estimation of the underlying nuisance density $f$, yield conditionally distribution-free semiparametrically efficient inference methods, e.g., semiparametrically efficient permutation tests. Remarkably, this is achieved without any explicit tangent space or efficient score computations and without any sample-splitting device. By means of several examples, including both i.i.d. and time-series models, we show how these results apply in models for which rank-based inference or permutation tests seldom have been considered so far.

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## 1 Introduction

Semiparametric models typically involve a finite-dimensional parameter of interest $\boldsymbol{\theta} \in \boldsymbol{\Theta} \subseteq \mathbb{R}^k$, along with an infinite-dimensional nuisance parameter $f$—often, the unspecified density of some white noise underlying the data generating process. Classical examples include

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