The extremal index for a random tessellation

Nicolas Chenavier

Université du Littoral Côte d'Opale, 50 rue Ferdinand Buisson, Calais, France.
nicolas.chenavier@lmpa.univ-littoral.fr

Abstract. Let m be a random tessellation in $\mathbf{R}^d$, $d \geq 1$, observed in the window $\mathbf{W}_\rho = \rho^{1/d}[0,1]^d$, $\rho > 0$, and let f be a geometrical characteristic. We investigate the asymptotic behaviour of the maximum of $f(C)$ over all cells $C \in m$ with nucleus in $\mathbf{W}_\rho$ as ρ goes to infinity. When the normalized maximum converges, we show that its asymptotic distribution depends on the so-called extremal index. Two examples of extremal indices are provided for Poisson-Voronoi and Poisson-Delaunay tessellations.

Keywords random tessellations, extreme values, Poisson point process.

1 Introduction

Random tessellations A tessellation of $\mathbf{R}^d$, $d \geq 1$, endowed with its Euclidean norm $|\cdot|$, is a countable collection of nonempty compact subsets, called *cells*, with disjoint interiors which subdivides the space and such that the number of cells intersecting any bounded subset of $\mathbf{R}^d$ is finite. The set $\mathbf{T}$ of tessellations is endowed with the σ-algebra generated by the sets $\left\{m \in \mathbf{T}, \bigcup_{C\in m} \partial C \cap K = \emptyset\right\}$ where ∂K is the boundary of K for any compact set K in $\mathbf{R}^d$. By a random tessellation m, we mean a random variable with values in $\mathbf{T}$. It is said to be stationary if its distribution is invariant under translations of the cells. For a complete account on random tessellations, we refer to the book [11].

Given a fixed realization of m, we associate with each cell $C \in m$ a point $z(C)$, which is called the *nucleus* of the cell, such that $z(C + x) = z(C) + x$ for all $x \in \mathbf{R}^d$. To describe the mean behaviour of the tessellation, the notions of intensity and typical cell are introduced as follows. Let $B \subset \mathbf{R}^d$ be a Borel subset.The *intensity* γ of the tessellation is defined as $\gamma = \frac{1}{\lambda_d(B)} \cdot \mathbf{E}\left[\#\{C \in m, z(C) \in B\}\right]$, where λ_d is the d-dimensional Lebesgue measure. We assume that $\gamma \in (0, \infty)$ and, without loss of generality, we take $\gamma = 1$. The *typical cell* $\mathcal{C}$ is a random polytope whose the distribution is given by

$$\mathbf{E}[f(\mathcal{C})] = \frac{1}{\lambda_d(B)} \cdot \mathbf{E}\left[\sum_{\substack{C\in m,\\ z(C)\in B}} f(C - z(C))\right] \tag{1}$$

for all $f : \mathcal{K}_d \to \mathbf{R}$ bounded measurable function on the set of convex bodies $\mathcal{K}_d$, i.e. convex compact sets, endowed with the Hausdorff topology.

Extremes in stochastic geometry We are interested in the following problem: only a part of the tessellation is observed in the window $\mathbf{W}_\rho = \rho^{1/d}[0,1]^d$. Let $f : \mathcal{K}_d \to \mathbf{R}$ be a translation invariant measurable function, i.e. $f(C+x) = f(C)$ for all $C \in \mathcal{K}_d$ and $x \in \mathbf{R}^d$. We denote by $M_{f,\mathbf{W}_\rho}$ the maximum of $f(C)$ over all cells $C \in m$ with nucleus $z(C)$ in $\mathbf{W}_\rho$, i.e.

$$M_{f,\mathbf{W}_\rho} = \max_{\substack{C\in m,\\ z(C)\in \mathbf{W}_\rho}} f(C).$$

In this paper, we investigate the limit behaviour of $M_{f,\mathbf{W}_\rho}$ when ρ goes to infinity.

To the best of our knowledge, one of the first application of extreme value theory in stochastic geometry was given by Penrose (see Chapters 6,7 and 8 in Penrose [8]). More recently, Schulte and Thäle [12] established a theorem to derive the order statistics of a general functional, $f_k(x_1, ..., x_k)$ of k points of a homogeneous Poisson point process. Calka and Chenavier [2] went on to provide a series of results for the extremal properties of cells in the Poisson-Voronoi tessellation, which were then extended in [3]. Besides, extremes for the inradius of a Poisson line tessellation are also considered in [4].

A general theorem Before stating our results, we first recall the main theorem in [3]. To do it, we consider a threshold v_ρ such that the mean number of exceedance cells converges to a limit denoted by $\tau \geq 0$, i.e

$$\rho \cdot \mathbf{P}\left(f(\mathcal{C}) > v_\rho\right) \underset{\rho\to\infty}{\longrightarrow} \tau.$$

Such an assumption is classical in extreme value theory. We also introduce two conditions on m and f.

The first one deals with R-dependence. To introduce this condition we partition $\mathbf{W}_\rho$ into a set V_ρ of $N_\rho = \left\lfloor \frac{\rho}{\log \rho} \right\rfloor$ sub-cubes of equal size. These sub-cubes are indexed by the set of $\mathbf{i} = (i_1, \ldots, i_d) \in \left[1, N_\rho^{1/d}\right]^d$. With a slight abuse of notation, we identify a cube with its index. Let us define a distance between sub-cubes $\mathbf{i}$ and $\mathbf{j}$ as $d(\mathbf{i},\mathbf{j}) = \max_{1\leq r\leq d}\{|i_r - j_r|\}$. Moreover, if A, B are two sets of sub-cubes, we let $d(A,B) = \min_{\mathbf{i}\in A,\mathbf{j}\in B} d(\mathbf{i},\mathbf{j})$. For each $\mathbf{i} \in V_\rho$, we denote by

$$M_{f,\mathbf{i}} = \max_{\substack{C\in m,\\ z(C)\in \mathbf{i}\cap\mathbf{W}_\rho}} f(C).$$

When $\{C \in m, z(C) \in \mathbf{i} \cap \mathbf{W}_\rho\}$ is empty, we take $M_{f,\mathbf{i}} = -\infty$. We are now prepared to introduce our first condition which is referred as the finite range condition (FRC):

Condition (FRC): *there exists an integer R and an event A_ρ with $\mathbf{P}(A_\rho) \underset{\rho\to\infty}{\longrightarrow} 1$ such that, conditional on A_ρ, the σ-algebras $\sigma\{M_{f,\mathbf{i}}, \mathbf{i} \in A\}$ and $\sigma\{M_{f,\mathbf{i}}, \mathbf{i} \in B\}$ are independent when $d(A,B) > R$.*

Our second condition deals with a local property of m and f and is referred as the local correlation condition (LCC).

CONDITION (LCC): *with the same notation as before, we have*

$$N_\rho \, \mathbf{E}\left[\sum_{\substack{(C_1,C_2)_{\neq}\in m^2,\\ z(C_1),z(C_2)\in \mathbf{W}_{\log\rho}}} \mathbb{1}_{f(C_1)>v_\rho, f(C_2)>v_\rho}\right] \underset{\rho\to\infty}{\longrightarrow} 0,$$

where $(C_1, C_2)_{\neq} \in m^2$ *means that* (C_1, C_2) *is a pair of distinct cells of* m.

This (local) condition means that, with high probability, two neighboring cells (in the sense that their nuclei belong to $\mathbf{W}_{\log\rho}$ which is small compared to $\mathbf{W}_\rho$) are not simultaneously exceedances. Under these assumptions, we have the following result (Theorem 1 in [3]):

Theorem 1. *Let m be a stationary random tessellation of intensity* 1 *such that* CONDITIONS (FRC) *and* (LCC) *hold. Then*

$$\mathbb{P}(M_{f,\mathbf{W}_\rho} \leq v_\rho) \underset{\rho\to\infty}{\longrightarrow} e^{-\tau}.$$

Theorem 1 can be extended in various directions: order statistics with a rate of convergence, random tessellations satisfying some β-mixing property and marked point processes. Numerous examples of this theorem can be derived such as the minimum of the Voronoi flowers, the maximum and minimum of areas of a planar Poisson-Delaunay tessellation and the maximum of inradius of a Gauss-Poisson Voronoi tessellation (see Sections 3, 4 and 5 in [3]).

The main difficulty is to apply Theorem 1 and to check CONDITION (LCC) since it requires delicate geometric estimates. Our main question is: does Theorem 1 remains true when this condition does not hold?

Extremal index and new results When CONDITION (LCC) does not hold, the exceedance locations can be divided into clusters. More precisely, we show that the behaviour of $M_{f,\mathbf{W}_\rho}$ can be deduced up to a constant according to the following new result:

Proposition 2. *Let m be a stationary random tessellation of intensity* 1 *such that* CONDITION (FRC) *holds. Let us assume that for all* $\tau \geq 0$, *there exists a deterministic function* $v_\rho(\tau)$ *depending on* ρ *such that* $\rho \cdot \mathbb{P}(f(\mathcal{C}) > v_\rho(\tau))$ *converges to* τ *as* ρ *goes to infinity. Then there exist constants* $\theta, \theta', 0 \leq \theta \leq \theta' \leq 1$ *such that, for all* $\tau \geq 0$,

$$\limsup_{\rho\to\infty} \mathbb{P}(M_{f,\mathbf{W}_\rho} \leq v_\rho(\tau)) = e^{-\theta\tau} \text{ and } \liminf_{\rho\to\infty} \mathbb{P}(M_{f,\mathbf{W}_\rho} \leq v_\rho(\tau)) = e^{-\theta'\tau}.$$

In particular, if $\mathbf{P}\left(M_{f,\mathbf{W}_\rho} \leq v_\rho(\tau)\right)$ *converges, then* $\theta = \theta'$ *and*

$$\mathbf{P}\left(M_{f,\mathbf{W}_\rho} \leq v_\rho(\tau)\right) \underset{\rho\to\infty}{\longrightarrow} e^{-\theta\tau}.$$

Proposition 2 is similar to a result due to Leadbetter for stationary sequences of real random variables (see Theorem 2.2 in [6]). Its proof relies notably on the adaptation to our setting of several arguments included in [6]. According to Leadbetter, we say that the random tessellation m has *extremal index* θ if, for each $\tau \geq 0$, $\rho \cdot \mathbf{P}\left(f(\mathcal{C}) > v_\rho(\tau)\right) \underset{\rho\to\infty}{\longrightarrow} \tau$ and $\mathbf{P}\left(M_{f,\mathbf{W}_\rho} \leq v_\rho(\tau)\right) \underset{\rho\to\infty}{\longrightarrow} e^{-\theta\tau}$.

For a sequence of real random variables, the extremal index has an interpretation due to Leadbetter [6] as the reciprocal of the mean cluster size. Except in specific cases, the extremal index cannot be made explicit. A lot of inferences was considered to estimate this parameter (e.g. [9, 13]). For a random tessellation, we think that the extremal index has a similar geometric interpretation. In a future work, we hope to develop a general method to estimate the extremal index.

The paper is organized as follows. In Section 2, we prove Proposition 2. As an illustration, we also provide two examples of extremal indices in Section 3.

2 Proof of Proposition 2

We only prove Proposition 2 for the limit superior since the limit inferior can be dealt with a similar method. To do it, for each $\tau \geq 0$, we define:

$$\psi(\tau) = \limsup_{\rho\to\infty} \mathbf{P}\left(M_{f,\mathbf{W}_\rho} \leq v_\rho(\tau)\right). \tag{2}$$

The key idea is to establish a functional equation for ψ. More precisely, for each $k \in \mathbf{N}_+$, we will show that:

$$\limsup_{\rho\to\infty} \mathbf{P}\left(M_{f,\mathbf{W}_{\rho/k^d}} \leq v_\rho(\tau)\right) = \psi(\tau/k^d), \tag{3}$$

$$\limsup_{\rho\to\infty} \mathbf{P}\left(M_{f,\mathbf{W}_{\rho/k^d}} \leq v_\rho(\tau)\right) = \psi^{1/k^d}(\tau). \tag{4}$$

The first convergence only depends on the sequence $v_\rho(\tau)$ while the second one is a consequence of Condition (FRC).

Proof of (3). Let us assume that $v_\rho(\tau) \geq v_{\rho/k^d}(\tau/k^d)$. Then

$$\begin{aligned}
&\left|\mathbf{P}\left(M_{f,\mathbf{W}_{\rho/k^d}} \leq v_\rho(\tau)\right) - \mathbf{P}\left(M_{f,\mathbf{W}_{\rho/k^d}} \leq v_{\rho/k^d}(\tau/k^d)\right)\right| \\
&\qquad \leq \mathbf{P}\left(\bigcup_{\substack{C\in m,\\ z(C)\in \mathbf{W}_{\rho/k^d}}} \left\{v_{\rho/k^d}(\tau/k^d) \leq f(C) \leq v_\rho(\tau)\right\}\right) \\
&\qquad \leq \mathbf{E}\left[\sum_{\substack{C\in m,\\ z(C)\in \mathbf{W}_{\rho/k^d}}} \mathbb{1}_{v_{\rho/k^d}(\tau/k^d) \leq f(C) \leq v_\rho(\tau)}\right].
\end{aligned}$$

This together with the corresponding inequality when $v_\rho(\tau) \le v_{\rho/k^d}(\tau/k^d)$ shows that

$$\begin{aligned} &\left|\mathbf{P}\left(M_{f,\mathbf{W}_{\rho/k^d}} \le v_\rho(\tau)\right) - \mathbf{P}\left(M_{f,\mathbf{W}_{\rho/k^d}} \le v_{\rho/k^d}(\tau/k^d)\right)\right| \\ &\qquad\qquad \le \frac{\rho}{k^d}\left|\mathbf{P}\left(f(\mathcal{C}) > v_{\rho/k^d}(\tau/k^d)\right) - \mathbf{P}\left(f(\mathcal{C}) > v_\rho(\tau)\right)\right| \\ &\qquad\qquad = \frac{\rho}{k^d}\left|\frac{\tau/k^d}{\rho/k^d} - \frac{\tau}{\rho} + o\left(\frac{1}{\rho}\right)\right| \underset{\rho\to\infty}{\longrightarrow} 0 \end{aligned} \tag{5}$$

according to (1) and the fact that $\mathbf{P}\left(f(\mathcal{C}) > v_\rho(\tau)\right)$ converges to τ for each $\tau \ge 0$. Moreover, from (2) we have

$$\limsup_{\rho\to\infty} \mathbf{P}\left(M_{f,\mathbf{W}_{\rho/k^d}} \le v_{\rho/k^d}(\tau/k^d)\right) = \psi(\tau/k^d).$$

We obtain (3) from the previous equality and (5). □

Proof of (4). The main idea is to apply the following adaptation of Lemma 4 in [3]:

Lemma 3. *Let $L \ge 1$ and let $B^{(1)}, \dots, B^{(L)}$ be a L-tuple of Borel subsets included in W. Under the same assumptions as in Proposition 2, we have:*

$$\mathbf{P}\left(M_{f,\mathbf{W}_\rho} \le v_\rho\right) - \prod_{l=1}^{L} \mathbf{P}\left(M_{f,\mathbf{B}_\rho^{(l)}} \le v_\rho\right) \underset{\rho\to\infty}{\longrightarrow} 0,$$

where $\mathbf{B}_\rho^{(l)} = \rho^{1/d} B^{(l)}$, $1 \le l \le L$.

Partitioning $W = [0,1]^d$ into a set of k^d sub-cubes of equal volume $1/k^d$, say $B^{(1)}, \dots, B^{(k^d)}$, and applying Lemma 3, we get

$$\mathbf{P}\left(M_{f,\mathbf{W}_\rho} \le v_\rho(\tau)\right) - \prod_{l=1}^{k^d} \mathbf{P}\left(M_{f,\mathbf{B}_\rho^{(l)}} \le v_\rho(\tau)\right) \underset{\rho\to\infty}{\longrightarrow} 0.$$

Since $\mathbf{B}_\rho^{(l)}$ is a cube of volume ρ/k^d and since m is stationary, we have

$$\mathbf{P}\left(M_{f,\mathbf{W}_\rho} \le v_\rho(\tau)\right) - \mathbf{P}\left(M_{f,\mathbf{W}_{\rho/k^d}} \le v_\rho(\tau)\right)^{k^d} \underset{\rho\to\infty}{\longrightarrow} 0.$$

We obtain (4) from the previous convergence and (2). □

Proof of Proposition 2. For each $\tau \ge 0$ and $k \in \mathbf{N}_+$, it follows from (3) and (4) that $\psi(\tau/k^d) = \psi^{1/k^d}(\tau)$. Moreover, in the same spirit as in (5), we have

$$\mathbf{P}\left(M_{f,\mathbf{W}_{\rho/k^d}} \le v_\rho(\tau)\right) \ge 1 - \frac{\rho}{k^d}\mathbf{P}\left(f(\mathcal{C}) > v_\rho(\tau)\right) \underset{\rho\to\infty}{\longrightarrow} 1 - \frac{\tau}{k^d}.$$

Hence, taking the k^{th} powers and applying Lemma 3, we deduce that

$$\liminf_{\rho\to\infty} \mathbf{P}\left(M_{f,\mathbf{W}_\rho} \le v_\rho(\tau)\right) = \liminf_{\rho\to\infty} \mathbf{P}\left(M_{f,\mathbf{W}_{\rho/k^d}} \le v_\rho(\tau)\right)^{k^d} \ge \left(1 - \frac{\tau}{k^d}\right)^{k^d}.$$

Letting $k \to \infty$, we obtain $\liminf_{\rho\to\infty} \mathbf{P}\left(M_{f,\mathbf{W}_\rho} \le v_\rho(\tau)\right) \ge e^{-\tau}$. In particular, this shows that $\psi(\tau) > 0$. Since ψ is also non-increasing and since the only solution of the functional equation $\psi(\tau/k^d) = \psi^{1/k^d}(\tau)$ which is strictly positive and non-increasing is an exponential function, we have $\psi(\tau) = e^{-\theta\tau}$ for some $\theta \ge 0$. This concludes the proof of Proposition 2. □

3 Examples

We provide below two examples where the extremal index differs from 1.

The minimum of inradii of a Poisson-Voronoi tessellation Let $\mathbf{X}$ be a Poisson point process in $\mathbf{R}^d$ of intensity 1. For all $x \in \mathbf{X}$, we denote by $C_{\mathbf{X}}(x)$ the Voronoi cell of nucleus x:

$$C_{\mathbf{X}}(x) = \{y \in \mathbf{R}^d : |x - y| \le |x' - y|, x' \in \chi\}.$$

The family $m_{PVT} = \{C_{\mathbf{X}}(x), x \in \mathbf{X}\}$ is the so-called Poisson-Voronoi tessellation. Such a model is extensively used in many domains such as astrophysics [15] and telecommunications [1], see also the reference books [7, 11].

In this example, for each cell $C = C_{\mathbf{X}}(x)$, $x \in \mathbf{X}$, we take $z(C_{\mathbf{X}}(x)) = x$ and $f(C_{\mathbf{X}}(x)) = r(C_{\mathbf{X}}(x)) = \max\{r \ge 0 : B(x,r) \subset C_{\mathbf{X}}(x)\}$ to denote the inradius of the cell. First we notice that the distribution of $r(\mathcal{C})^d$, with $r(\mathcal{C}) = r(C_{\mathbf{X}\cup\{0\}}(0))$, is exponential with parameter $2^d\kappa_d$, where κ_d denotes the volume of the unit ball. Indeed, for any $v \ge 0$, we have $r(\mathcal{C}) \le v$ if and only if $\mathbf{X} \cap B(0, 2v) \ne \emptyset$. In particular, for any $t \ge 0$, we get

$$\rho \cdot \mathbf{P}\left(r(\mathcal{C})^d \le \left(2^d\kappa_d\rho\right)^{-1} t\right) \underset{\rho\to\infty}{\longrightarrow} t.$$

Moreover, according to the convergence (2b) in [2], we know that

$$\mathbf{P}\left(\min_{x\in\mathbf{X}\cap\mathbf{W}_\rho} r(C_{\mathbf{X}}(x))^d \ge \left(2^d\kappa_d\rho\right)^{-1} t\right) \underset{\rho\to\infty}{\longrightarrow} e^{-\frac{1}{2}\cdot t}.$$

Let us notice that the convergence was established in [2] for a fixed window and for a Poisson point process such that the intensity goes to infinity. By scaling property of the Poisson point process, the result of [2] can be re-written as above for a fixed intensity and for a window $\mathbf{W}_\rho$ where $\rho \to \infty$. This allows us to provide a first example of extremal index:

Example 1. The extremal index of the minimum of inradius of a Poisson-Voronoi tessellation exists and is $\theta = \frac{1}{2}$.

It can be also explained by a trivial heuristic argument. Indeed, if a cell minimizes the inradius, one of its neighbors has to do the same. Hence the mean cluster size of exceedances is 2 which implies that $\theta = 1/2$.

The maximum of circumradii of a Poisson-Delaunay tessellation Let $\mathbf{X}$ be a Poisson point process in $\mathbf{R}^d$ of intensity β_d^{-1}, where

$$\beta_d = \frac{(d^3+d^2)\Gamma\left(\frac{d^2}{2}\right)\Gamma^d\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d^2+1}{2}\right)\Gamma^d\left(\frac{d+2}{2}\right)2^{d+1}\pi^{\frac{d-1}{2}}}.$$

We connect two points $x, x' \in \mathbf{X}$ by an edge if and only if $C_{\mathbf{X}}(x) \cap C_{\mathbf{X}'}(x) \neq \emptyset$. The set of these edges defines a random tessellation m_{PDT} of $\mathbf{R}^d$ into simplices with intensity 1 (e.g. Theorem 10.2.8 in [11]) which is the so-called Poisson-Delaunay tessellation. Such a model is extensively used in medical image segmentation [14] and is a powerful tool for reconstructing a $3D$ set from a discrete point set [10].

Here we take $z(C)$ and $f(C) = R(C)$ as the circumcenter and the circumradius of any cell $C \in m_{PDT}$ respectively. A Taylor expansion of $\mathbf{P}\left(R(\mathcal{C}) > v\right)$ (e.g. Equation (3.14) in [3]), as v goes to infinity, shows that for each $t \in \mathbf{R}$

$$\rho \cdot \mathbf{P}\left(R(\mathcal{C})^d \geq \delta_d^{-1} \cdot \left(\log\left([(d-1)!]^{-1}\rho\log(\beta_d\rho)^{d-1}\right) + t\right)\right) \underset{\rho\to\infty}{\longrightarrow} e^{-t},$$

where $\delta_d = \beta_d\kappa_d$. Moreover, with standard arguments, we easily show that the maximum of circumradii of Delaunay cells $\max_{\substack{C\in m,\\ z(C)\in\mathbf{W}_\rho}} R(C)$ has the same asymptotic behaviour as the maximum of circumradii of the associated Voronoi cells $\max_{x\in\mathbf{X}\cap\mathbf{W}_\rho} R(C_{\mathbf{X}}(x))$. Besides, thanks to (2c) in [2], we know that

$$\mathbf{P}\left(\max_{x\in\mathbf{X}\cap\mathbf{W}_\rho} R(C_{\mathbf{X}}(x))^d \leq \delta_d^{-1}\left(\log\left(\alpha_d\beta_d\rho\log(\beta_d\rho)^{d-1}\right) + t\right)\right) \underset{\rho\to\infty}{\longrightarrow} e^{-e^{-t}},$$

where $\alpha_d := \frac{1}{d!}\left(\frac{\pi^{1/2}\Gamma\left(\frac{d}{2}+1\right)}{\Gamma\left(\frac{d+1}{2}\right)}\right)^{d-1}$. It follows that

$$\mathbf{P}\left(\max_{\substack{C\in m_{PDT},\\ z(C)\in\mathbf{W}_\rho}} R(C)^d \leq \delta_d^{-1} \cdot \left(\log\left([(d-1)!]^{-1}\rho\log(\beta_d\rho)^{d-1}\right) + t\right)\right) \underset{\rho\to\infty}{\longrightarrow} e^{-\theta_d \cdot e^{-t}},$$

where

$$\theta_d = \alpha_d\beta_d(d-1)! = \frac{(d^3+d^2)\Gamma\left(\frac{d^2}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}{2^{d+1}d\Gamma\left(\frac{d^2+1}{2}\right)\Gamma\left(\frac{d+2}{2}\right)}.$$

This allows us to provide a second example of extremal index:

Example 2. The extremal index of the maximum of circumradius of a Poisson-Delaunay tessellation exists and $\theta = \theta_d$. In particular, when $d = 1, 2, 3$, the extremal indices are $\theta = 1$, $\theta = 1/2$ and $\theta = 35/128$ respectively.

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