Some properties on extremes for transient random walks in random sceneries

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Abstract

Let $(S_n)_{n\geq 0}$ be a transient random walk in the domain of attraction of a stable law and let $(\xi(s))_{s\in\mathbb{Z}}$ be a stationary sequence of random variables. In a previous work, under conditions of type $D(u_n)$ and $D'(u_n)$, we established a limit theorem for the maximum of the first n terms of the sequence $(\xi(S_n))_{n\geq 0}$ as n goes to infinity. In this paper we show that, under the same conditions and under a suitable scaling, the point process of exceedances converges to a Poisson point process. We also give some properties of $(\xi(S_n))_{n\geq 0}$.

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1 Introduction

In 2009, Franke and Saigo [4, 5] considered the following problem. Let $(X_k)_{k\geq 1}$ be a sequence of centered, integer-valued i.i.d. random variables and let $S_0 = 0$ a.s. and $S_n = X_1 + \cdots + X_n$, $n \geq 1$. Assume that, for any $x \in \mathbb{R}$,

$$\mathbb{P}\left(\frac{S_n}{n^{1/\alpha}} \leq x\right) \underset{n\to\infty}{\longrightarrow} F_\alpha(x),$$

where F_α is the distribution function of a stable law with characteristic function given by

$$\phi(\theta) = \exp(-|\theta|^\alpha(C_1 + iC_2 \operatorname{sgn}\theta)), \quad \alpha \in (0, 2].$$

Let $(\xi(s))_{s\in\mathbb{Z}}$ be a stationary sequence of $\mathbb{R}$-valued random variables which are independent of the sequence $(X_k)_{k\geq 1}$. The sequence $(\xi(S_n))_{n\geq 0}$ is referred to as a *random walk in a random scenery.* In [5], Franke and Saigo derive limit theorems for the random variable $\max_{i\leq n} \xi(S_i)$ as n goes to infinity when the $\xi(s)$'s are i.i.d.. The statements of their theorems depend on the value of α. When $\alpha < 1$ (resp. $\alpha > 1$), it is known that the random walk $(S_n)_{n\geq 0}$ is transient (resp. recurrent) [7, 8]. An important concept concerning random walks is the

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range. The latter is defined as the number of sites visited by the first n terms of the random walk, namely $R_n := \#\{S_1, \dots, S_n\}$. The following result, due to Le Gall and Rosen [8], deals with its asymptotic behavior.

Theorem 1 (LeGall and Rosen). *(i) If $\alpha < 1$, then*

$$\frac{R_{[nt]}}{n} \underset{n\to\infty}{\longrightarrow} qt \quad \mathbb{P} - a.s.$$

with $q := \mathbb{P}(S_k \neq 0, \forall k \geq 1)$.

(ii) If $\alpha = 1$, then

$$\frac{h(n)R_{[nt]}}{n} \underset{n\to\infty}{\longrightarrow} t \quad in \quad L^p(\mathbb{P}),$$

where $h(n) := 1 + \sum_{k=1}^{n} \mathbb{P}(S_k = 0)$.

(iii) If $1 < \alpha \leq 2$, then for any $L \in \mathbb{N}$ and any $t_1 < \dots < t_L$,

$$\frac{1}{n^{1/\alpha}} \left(R_{\lfloor nt_1 \rfloor}, \dots, R_{\lfloor nt_L \rfloor}\right) \underset{n\to\infty}{\longrightarrow} \left(m(Y(0,t_1)), \dots, m(Y(0,t_L))\right),$$

in distribution.

In the above result, $\{Y(t), t \in \mathbb{R}\}$ denotes the right-continuous $\alpha-$stable Lévy process with characteristic function given by $\phi(t\theta)$ and m is the Lebesgue measure on $\mathbb{R}$. One of the results of [5] is the following. If u_n is a threshold such that $n\mathbb{P}(\xi > u_n) \underset{n\to\infty}{\longrightarrow} \tau$ for some $\tau > 0$, with $\xi = \xi(1)$, and if the $\xi(s)$'s are i.i.d. then

$$\mathbb{P}\left(\max_{i\leq n} \xi(S_i) \leq u_n\right) \underset{n\to\infty}{\longrightarrow} e^{-\tau q}$$

for $\alpha < 1$. Such a result was generalized in [1] for sequences $(\xi(s))_{s\in\mathbb{Z}}$ which are not necessarily i.i.d., but which satisfy a slight modification of the classical $D(u_n)$ and $D'(u_n)$ conditions of Leadbetter (see [9, 10] for a statement of these conditions).

In this paper, we give a more precise treatment of the extremes of $(\xi(S_n))_{n\geq 0}$. To do it, we assume that the threshold is of the form $u_n = u_n(x) = a_n x + b_n$ ($a_n \in \mathbb{R}$, $b_n > 0$ and $x \in \mathbb{R}$) and that, for any $x \in \mathbb{R}$, the following term exists and is finite:

$$\nu(x, \infty) := \lim_{n\to\infty} n\mathbb{P}(\xi > u_n(x)). \tag{1}$$

The quantity ν defines a measure on some topological space E. According to the Gnedenko's theorem [6], if ξ is in the domain of attraction of an extreme value distribution G, then ν is of the form:

$$\nu(x,\infty) = \begin{cases} x^{-\beta}, & E = (0,\infty] \quad \text{if } G \text{ is a Fréchet distribution;} \\ (-x)^{-\delta}, & E = (-\infty, 0] \quad \text{if } G \text{ is a Weibull distribution;} \\ e^{-x}, & E = (-\infty, \infty] \quad \text{if } G \text{ is a Gumbel distribution;} \end{cases}$$

for some $\beta, \delta > 0$. Notice that if P_n denotes the distribution of $\frac{\xi - a_n}{b_n}$, then (1) can be rephrased as

$$nP_n(A) \underset{n\to\infty}{\longrightarrow} \nu(A), \tag{2}$$

for any Borel subset $A \subset \mathbb{R}$. Secondly, we assume that the (stationary) sequence $(\xi(s))_{s\in\mathbb{Z}}$ satisfies conditions of type $D(u_n)$ and $D'(u_n)$ in the same spirit as in [1]. To introduce the first one, we write for each $i_1 < \cdots < i_p$ and for each $u \in \mathbb{R}$,

$$F_{i_1,\ldots,i_p}(u) = \mathbb{P}\left(\xi(i_1) \le u, \ldots, \xi(i_p) \le u\right).$$

$\mathbf{D}(u_n)$ condition We say that $(\xi(s))_{s\in\mathbb{Z}}$ satisfies the $\mathbf{D}(u_n)$ condition if there exist a sequence $(\alpha_{n,\ell})_{(n,\ell)\in\mathbb{N}^2}$ and a sequence (ℓ_n) of positive integers such that $\alpha_{n,\ell_n} \underset{n\to\infty}{\longrightarrow} 0$, $\ell_n = o(n)$, and

$$|F_{i_1,\ldots,i_p,j_1,\ldots,j_{p'}}(u_n) - F_{i_1,\ldots,i_p}(u_n)F_{j_1,\ldots,j_{p'}}(u_n)| \le \alpha_{n,\ell}$$

for any integers $i_1 < \cdots < i_p < j_1 < \cdots < j_{p'}$ such that $j_1 - i_p \ge \ell$. Notice that the bound holds uniformly in p and p'. Roughly, the $\mathbf{D}(u_n)$ condition (see e.g. p29 in [11]) is a weak mixing property for the tails of the joint distributions.

The $\mathbf{D}'(u_n)$ condition (see e.g. p29 in [11]) is a local type property and precludes the existence of clusters of exceedances. To introduce it, we consider a sequence (k_n) such that

$$k_n \underset{n\to\infty}{\longrightarrow} \infty, \quad \frac{n^2}{k_n}\alpha_{n,\ell_n} \underset{n\to\infty}{\longrightarrow} 0, \quad k_n\ell_n = o(n), \tag{3}$$

where (ℓ_n) and $(\alpha_{n,l})_{(n,l)\in\mathbb{N}^2}$ are the same as in the $\mathbf{D}(u_n)$ condition.

$\mathbf{D}'(u_n)$ condition In conjunction with the $\mathbf{D}(u_n)$ condition, we say that $(\xi(s))_{s\in\mathbb{Z}}$ satisfies the $\mathbf{D}'(u_n)$ condition if there exists a sequence of integers (k_n) satisfying (3) such that

$$\lim_{n\to\infty} n \sum_{s=1}^{\lfloor n/k_n \rfloor} \mathbb{P}\left(\xi(0) > u_n, \xi(s) > u_n\right) = 0.$$

In the classical literature, the sequences $(\alpha_{n,l})_{(n,l)\in\mathbb{N}^2}$ and (k_n) only satisfy $k_n\alpha_{n,\ell_n} \underset{n\to\infty}{\longrightarrow} 0$ (see e.g. (3.2.1) in [11]) whereas in (3) we have assumed that $\frac{n^2}{k_n}\alpha_{n,\ell_n} \underset{n\to\infty}{\longrightarrow} 0$. In this sense, the $\mathbf{D}'(u_n)$ condition as written above is slightly more restrictive than the usual $D'(u_n)$ condition.

Our paper is organized as follows. In Section 2, we prove that under suitable scaling the so-called point process of exceedances converges to a Poisson point process in the transient case. In Section 3, we give some properties of the random walk in random scenery. More precisely, we show that the (stationary) sequence $(\xi(S_n))_{n\ge 0}$ satisfies the classical $D(u_n)$ condition of Leadbetter, but does not satisfy the $D'(u_n)$ condition. Our results generalize [5] for sequences $(\xi(s))_{s\in\mathbb{Z}}$ which are not i.i.d. but which only satisfy the $\mathbf{D}(u_n)$ and $\mathbf{D}'(u_n)$ conditions. We also give some remarks on the so-called extremal index and on the $D^{(k)}(u_n)$ condition.

2 Point process of exceedances

2.1 Poisson approximation

The main result of this section claims that the point process of exceedances converges to a Poisson point process in the transient case, i.e. $\alpha < 1$. To introduce it, we denote for any $k \geq 1$ by

$$\tau_k = \inf\{m \geq 0 : \#\{S_1, \ldots, S_m\} \geq k\}$$

the time at which the random walk visits its k-th site. The *point process of exceedances* is defined as

$$\Phi_n = \left\{ \left(\frac{\tau_k}{n}, \frac{\xi(S_{\tau_k}) - b_{m(n)}}{a_{m(n)}} \right) : \tau_k \leq n \right\}_{k \geq 1} \subset [0,1] \times \mathbb{R}, \tag{4}$$

where $m(n) = \lfloor qn \rfloor$.

Proposition 2. *Let $\alpha < 1$. Assume that the sequence $(\xi(s))_{s \in \mathbb{Z}}$ satisfies the $\mathbf{D}(u_n)$ and $\mathbf{D}'(u_n)$ conditions for any threshold $u_n = u_n(x) = a_n x + b_n$, $x \in \mathbb{R}$, satisfying Equation (1). Then Φ_n converges weakly to a Poisson point process Φ with intensity measure $m_{[0,1]} \otimes \nu$, where $m_{[0,1]}$ denotes the Lebesgue measure in $[0,1]$, i.e. for any Borel subsets $B_1, \ldots, B_K \subset [0,1] \times \mathbb{R}$ with $m_{[0,1]} \otimes \nu(\partial B_i) = 0$, $1 \leq i \leq K$,*

$$(\#\Phi_n \cap B_1, \ldots, \#\Phi_n \cap B_K) \xrightarrow[n \to \infty]{\mathcal{D}} (\#\Phi \cap B_1, \ldots, \#\Phi \cap B_K).$$

By using the Laplace functional, Franke and Saigo (Theorem 3 in [5]) obtained a similar result when the $\xi(s)$'s are i.i.d. Proposition 2 extends it and is based on Kallenberg's theorem. Our result is stated only in the transient case, i.e. for $\alpha < 1$. However, it remains true for $\alpha = 1$ by taking $m(n) = \left\lfloor \frac{n}{h(n)} \right\rfloor$. When $\alpha > 1$, the point process of exceedances is defined in the same spirit as (4) by taking this time $m(n) = \lfloor n^{1/\alpha} \rfloor$. In this case, similarly to Theorem 4 in [5], we can show by adapting the proof of Proposition 2 that Φ_n converges weakly to a Cox point process Φ_Y, i.e. a Poisson point process in $[0,1] \times \mathbb{R}$ with random intensity measure $\mu(\mathrm{d}t, \mathrm{d}x) = m_Y(\mathrm{d}t)\nu(\mathrm{d}x)$, where $m_Y(t) = m(Y(0,t))$.

2.2 Technical results

The proof of Proposition 2 is mainly based on Kallenberg's theorem (see e.g. Proposition 3.22 in [**?**]) and on two technical lemmas which are stated below.

Theorem 3 (Kallenberg)**.** *Suppose Φ is a simple point process on E and $\mathcal{I}$ is a basis of relatively compact open sets such that $\mathcal{I}$ is closed under finite unions and intersections and, for $I \in \mathcal{I}$,*

$$\mathbb{P}(\#\Phi \cap \partial I = 0) = 1,$$

where ∂I is the boundary of I. Let (Φ_n) be a sequence of point processes on E such that, for all $I \in \mathcal{I}$,

$$\lim_{n\to+\infty} \mathbb{E}\left(\#\Phi_n \cap I\right) = \mathbb{E}\left(\#\Phi \cap I\right)$$

and

$$\lim_{n\to+\infty} \mathbb{P}\left(\#\Phi_n \cap I = 0\right) = \mathbb{P}\left(\#\Phi \cap I = 0\right).$$

Then Φ_n converges weakly to Φ in distribution.

The following lemma is a direct adaptation of Lemma 1 in [5] and deals with the independence between the sequence $(\xi(S_n))_{n\geq 0}$ and the sequence $(\tau_k)_{k\geq 1}$.

Lemma 1. *For all measurable sets $B \subset \mathbb{N}_+$ and $A \subset \mathbb{R}$, we have*

$$\mathbb{P}\left(\tau_k \in B, \xi(S_{\tau_k}) \in A\right) = \mathbb{P}\left(\tau_k \in B\right)\mathbb{P}\left(\xi \in A\right).$$

The second lemma is an extension of [1]. More precisely, under the assumptions that the $\mathbf{D}(u_n)$ and $\mathbf{D}'(u_n)$ conditions hold for the sequence $(\xi(s))_{s\in\mathbb{Z}}$, we have shown in [1] that

$$\mathbb{P}\left(\bigcap_{k\geq 1:\frac{\tau_k}{n}\in(0,1]} \left\{\frac{\xi(S_{\tau_k}) - b_{m(n)}}{a_{m(n)}} \notin (x,\infty)\right\}\right) - \mathbb{E}\left(\exp\left(-\frac{R_n}{m(n)}\nu(x,\infty)\right)\right) \underset{n\to\infty}{\longrightarrow} 0$$

when (1) holds for any threshold $u_n = u_n(x)$, $x \in \mathbb{R}$. The following lemma deals with the case where the interval $(0,1]$ (resp. (x,∞)) is replaced by $(a,b]$ (resp. $A \subset \mathbb{R}$) in the above equation.

Lemma 2. *Let A be a Borel subset in $\mathbb{R}$ and let $0 \leq a < b \leq 1$. Under the same assumptions as Proposition 2, for almost all realization of $(S_n)_{n\geq 0}$, we have*

$$\lim_{n\to\infty} \mathbb{P}\left(\bigcap_{k\geq 1:\frac{\tau_k}{n}\in(a,b]} \left\{\frac{\xi(S_{\tau_k}) - b_{m(n)}}{a_{m(n)}} \notin A\right\}\right) - \mathbb{E}\left(\exp\left(-\frac{R_{\lfloor nb\rfloor} - R_{\lfloor na\rfloor}}{m(n)}\nu(A)\right)\right) = 0.$$

2.3 Proofs

Proof of Lemma 1. Since the random walk and the random scenery are independent, we have

$$\begin{aligned}
\mathbb{P}\left(\tau_k \in B, \xi(S_{\tau_k}) \in A\right) &= \sum_{m\in B} \mathbb{P}\left(\tau_k = m, \xi(S_m) \in A\right) \\
&= \sum_{m\in B}\sum_{s\in\mathbb{Z}} \mathbb{P}\left(\tau_k = m, S_m = s, \xi(s) \in A\right) \\
&= \sum_{m\in B}\sum_{s\in\mathbb{Z}} \mathbb{P}\left(\tau_k = m, S_m = s\right)\mathbb{P}\left(\xi(s) \in A\right) \\
&= \mathbb{P}\left(\tau_k \in B\right)\mathbb{P}\left(\xi \in A\right).
\end{aligned}$$

□

Proof of Lemma 2. The proof will be sketched since it relies on a simple adaptation of the proof of Theorem 1 in [1].

Let (k_n), (ℓ_n) be as in (3) and let

$$r_n = \left\lfloor \frac{n}{k_n - 1} \right\rfloor + 1, \tag{5}$$

for n large enough. Given a realization of $(S_n)_{n\geq 0}$, we write

$$\mathcal{S}_{(na,nb]} = \left\{ S_{\tau_k} : k \geq 1, \frac{\tau_k}{n} \in (a,b] \right\} \quad \text{and} \quad R_{\lfloor nb \rfloor} - R_{\lfloor na \rfloor} = \#\mathcal{S}_{(na,nb]}.$$

To capture the fact that $(\xi(s))_{s\in\mathbb{Z}}$ satisfies the condition $\mathbf{D}(u_n)$, we construct blocks and stripes as follows. Let

$$K_n = \left\lfloor \frac{R_{\lfloor nb \rfloor} - R_{\lfloor na \rfloor}}{r_n} \right\rfloor + 1.$$

We subdivide the set $\mathcal{S}_{(na,nb]}$ into subsets $B_i \subset \mathcal{S}_{(na,nb]}$, $1 \leq i \leq K_n$, referred to as *blocks*, in such a way that $\#B_i = r_n$ and $\max B_i < \min B_{i+1}$ for all $i \leq K_n - 1$. Notice that $K_n \leq k_n$ and $\#B_{K_n} = R_{\lfloor nb \rfloor} - R_{\lfloor na \rfloor} - (K_n - 1) \cdot r_n$ a.s.. For each $j \leq K_n$, we denote by L_j the family consisting of the ℓ_n largest terms of B_j (e.g. if $B_j = \{x_1, \ldots, x_{r_n}\}$, with $x_1 < \cdots < x_{r_n}$, $j \leq K_n - 1$, then $L_j = \{x_{r_n - \ell_n + 1}, \ldots, x_{r_n}\}$). When $j = K_n$, we take the convention $L_{K_n} = \emptyset$ if $\#B_{K_n} < \ell_n$. The set L_j is referred to as a *stripe*, and the union of the stripes is denoted by $\mathcal{L}_n = \bigcup_{j \leq K_n} L_j$. Proceeding in the same spirit as in the proofs of Lemmas 1 and 2 of [1], we can easily that for almost all realization of $(S_n)_{n\geq 0}$,

- $\mathbb{P}\left(\bigcap_{s\in\mathcal{S}_{(na,nb]}} \left\{\frac{\xi(s)-b_{m(n)}}{a_{m(n)}} \notin A\right\}\right) - \mathbb{P}\left(\bigcap_{s\in\mathcal{S}_{(na,nb]}\setminus\mathcal{L}_n} \left\{\frac{\xi(s)-b_{m(n)}}{a_{m(n)}} \notin A\right\}\right) \underset{n\to\infty}{\longrightarrow} 0;$
- $\mathbb{P}\left(\bigcap_{s\in\mathcal{S}_{(na,nb]}\setminus\mathcal{L}_n} \left\{\frac{\xi(s)-b_{m(n)}}{a_{m(n)}} \notin A\right\}\right) - \prod_{i\leq K_n} \mathbb{P}\left(\bigcap_{s\in B_i\setminus\mathcal{L}_n} \left\{\frac{\xi(s)-b_{m(n)}}{a_{m(n)}} \notin A\right\}\right) \underset{n\to\infty}{\longrightarrow} 0;$
- $\prod_{i\leq K_n} \mathbb{P}\left(\bigcap_{s\in B_i\setminus\mathcal{L}_n} \left\{\frac{\xi(s)-b_{m(n)}}{a_{m(n)}} \notin A\right\}\right) - \prod_{i\leq K_n} \mathbb{P}\left(\bigcap_{s\in B_i} \left\{\frac{\xi(s)-b_{m(n)}}{a_{m(n)}} \notin A\right\}\right) \underset{n\to\infty}{\longrightarrow} 0;$
- $\prod_{i\leq K_n} \mathbb{P}\left(\bigcap_{s\in B_i} \left\{\frac{\xi(s)-b_{m(n)}}{a_{m(n)}} \notin A\right\}\right) - \mathbb{E}\left(\exp\left(-\frac{R_{\lfloor nb \rfloor}-R_{\lfloor na \rfloor}}{m(n)}\nu(A)\right)\right) \underset{n\to\infty}{\longrightarrow} 0.$

The first and the third assertions come from the fact that the size of the stripes is negligible compared to the size of the blocks, i.e. $\ell_n = o(r_n)$. The second assertion is a consequence of the fact that the sequence $(\xi(s))_{s\in\mathbb{Z}}$ satisfies the $\mathbf{D}(u_n)$ condition and the last one is obtained by using the $\mathbf{D}(u_n)$ and $\mathbf{D}'(u_n)$ conditions. Lemma 2 follows directly from the four assertions. □

Proof of Proposition 2. According to Kallenberg's theorem, it is sufficient to show that

(i) $\lim_{n\to\infty} \mathbb{E}\left(\#\Phi_n \cap I\right) = m_{[0,1]} \otimes \nu(I)$,

(ii) $\lim_{n\to\infty} \mathbb{P}\left(\#\Phi_n \cap I = 0\right) = e^{-m_{[0,1]}\otimes\nu(I)}$,

for all set I of the form $I = (a,b] \times A$, where $0 \leq a < b \leq 1$ and where A is an open subset of E.

To deal with (i), we write

$$\begin{aligned}
\mathbb{E}\left(\#\Phi_n \cap I\right) &= \sum_{k\geq 1} \mathbb{P}\left(\left(\frac{\tau_k}{n}, \frac{\xi(S_{\tau_k}) - b_{\lfloor qn\rfloor}}{a_{\lfloor qn\rfloor}}\right) \in I\right) \\
&= \sum_{k\geq 1} \mathbb{P}\left(\frac{\tau_k}{n} \in (a,b]\right) \mathbb{P}\left(\frac{\xi - b_{\lfloor qn\rfloor}}{a_{\lfloor qn\rfloor}} \in A\right) \\
&= \sum_{k\geq 1} \mathbb{P}\left(\frac{\tau_k}{n} \in (a,b]\right) P_{\lfloor qn\rfloor}(A),
\end{aligned}$$

where the second line comes from Lemma 1. Using the fact that $\sum_{k\geq 1} \mathbf{1}_{\frac{\tau_k}{n}\in(a,b]} = R_{\lfloor nb\rfloor} - R_{\lfloor na\rfloor}$, we have

$$\begin{aligned}
\mathbb{E}\left(\#\Phi_n \cap I\right) &= \mathbb{E}\left(\sum_{k\geq 1} \mathbf{1}_{\frac{\tau_k}{n}\in(a,b]}\right) P_{\lfloor qn\rfloor}(A) \\
&= \mathbb{E}\left(R_{\lfloor nb\rfloor} - R_{\lfloor na\rfloor}\right) P_{\lfloor qn\rfloor}(A).
\end{aligned}$$

Moreover, according to Theorem 1 and to the Lebesgue's dominated convergence theorem, we know that $\mathbb{E}(R_{\lfloor nb\rfloor} - R_{\lfloor na\rfloor}) \underset{n\to\infty}{\sim} nq(b-a)$. This, together with (2) implies

$$\mathbb{E}\left(\#\Phi_n \cap I\right) \underset{n\to\infty}{\longrightarrow} (b-a) \times \nu(A) = m_{[0,1]} \otimes \nu(I).$$

To deal with (ii), we observe that

$$\mathbb{P}\left(\#\Phi_n \cap I = 0\right) = \mathbb{P}\left(\bigcap_{k\geq 1:\frac{\tau_k}{n}\in(a,b]} \left\{\frac{\xi(S_{\tau_k}) - b_{\lfloor qn\rfloor}}{a_{\lfloor qn\rfloor}} \notin A\right\}\right).$$

According to Lemma 2, Theorem 1 and the Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}
\mathbb{P}\left(\#\Phi_n \cap I = 0\right) &= \mathbb{E}\left(\exp\left(-\frac{R_{\lfloor nb\rfloor} - R_{\lfloor na\rfloor}}{\lfloor qn\rfloor}\nu(A)\right)\right) + o(1) \\
&\underset{n\to\infty}{\longrightarrow} \exp\left(-(b-a)\nu(A)\right).
\end{aligned}$$

This, together with the fact that $(b-a)\nu(A) = m_{[0,1]} \otimes \nu(I)$, concludes the proof of Proposition 2. □

3 Properties of $(\xi(S_n))_{n\geq 0}$

In this section, we give some properties of $(\xi(S_n))_{n\geq 0}$. More precisely, we show that the latter satisfies the $D(u_n)$ condition and an extension of the so-called $D^{(k)}(u_n)$ condition, but does not satisfy the $D'(u_n)$ condition.

3.1 Distributional mixing property

The following extends Proposition 2 in [5], which deals with the case where the $\xi(s)$'s are i.i.d., to sequences which only satisfy the $\mathbf{D}(u_n)$ and $\mathbf{D}'(u_n)$ conditions.

Proposition 4. *Let $\alpha < 1$. Assume that the sequence $(\xi(s))_{s\in\mathbb{Z}}$ satisfies the $\mathbf{D}(u_n)$ and $\mathbf{D}'(u_n)$ conditions for a threshold u_n such that $n\mathbb{P}(\xi > u_n) \underset{n\to\infty}{\longrightarrow} \tau$, with $\tau > 0$. Then $(\xi(S_n))_{n\geq 0}$ satisfies the $\mathbf{D}(u_n)$ condition.*

Proof of Proposition 4. We adapt several arguments of [5] in our context. Let $0 \leq i_1 < \cdots < i_p < j_1 < \cdots < j_{p'} \leq n$ be a family of integers, with $j_1 - i_p > \ell_n$ and $k_n\ell_n = o(n)$. To prove that $(\xi(S_n))_{n\geq 0}$ satisfies the $\mathbf{D}(u_n)$, we have to show that

$$|F'_{i_1,\ldots,i_p,j_1,\ldots,j_{p'}}(u_n) - F'_{i_1,\ldots,i_p}(u_n)F'_{j_1,\ldots,j_{p'}}(u_n)| \leq \tilde{\alpha}_{n,\ell_n},$$

for some sequence $(\tilde{\alpha}_{n,\ell})_{(n,\ell)\in\mathbb{N}^2}$ such that $k_n\tilde{\alpha}_{n,\ell_n} \underset{n\to\infty}{\longrightarrow} 0$, with

$$F'_{i_1,\ldots,i_p}(u_n) = \mathbb{P}\left(\xi(S_{i_1}) \leq u_n, \ldots, \xi(S_{i_p}) \leq u_n\right).$$

We will use below the following notation:

- $R_{i_1,\ldots,i_p,j_1,\ldots,j_{p'}} = \#\{S_{i_1}, \ldots, S_{i_p}, S_{j_1}, \ldots, S_{j_{p'}}\}$;
- $R_{i_1,\ldots,i_p} = \#\{S_{i_1}, \ldots, S_{i_p}\}$;
- $R_{j_1,\ldots,j_{p'}} = \#\{S_{j_1}, \ldots, S_{j_{p'}}\}$;
- $R^{i_1,\ldots,i_p}_{j_1,\ldots,j_{p'}} = \#\{S_{i_1}, \ldots, S_{i_p}\} \cap \{S_{j_1}, \ldots, S_{j_{p'}}\} = R_{i_1,\ldots,i_p} + R_{j_1,\ldots,j_{p'}} - R_{i_1,\ldots,i_p,j_1,\ldots,j_{p'}}$.

We have

$$\begin{aligned}
&|F'_{i_1,\ldots,i_p,j_1,\ldots,j_{p'}}(u_n) - F'_{i_1,\ldots,i_p}(u_n)F'_{j_1,\ldots,j_{p'}}(u_n)| \\
&\leq \left|F'_{i_1,\ldots,i_p,j_1,\ldots,j_{p'}}(u_n) - \mathbb{E}\left(\exp\left(-\frac{R_{i_1,\ldots,i_p,j_1,\ldots,j_{p'}}}{n}\tau\right)\right)\right| \\
&+ \left|\mathbb{E}\left(\exp\left(-\frac{R_{i_1,\ldots,i_p,j_1,\ldots,j_{p'}}}{n}\tau\right)\right) - \mathbb{E}\left(\exp\left(-\frac{R_{i_1,\ldots,i_p}+R_{j_1,\ldots,j_{p'}}}{n}\tau\right)\right)\right| \\
&\qquad + \left|\mathbb{E}\left(\exp\left(-\frac{R_{i_1,\ldots,i_p}+R_{j_1,\ldots,j_{p'}}}{n}\tau\right)\right) - F'_{i_1,\ldots,i_p}(u_n)F'_{j_1,\ldots,j_{p'}}(u_n)\right|. \qquad (6)
\end{aligned}$$

To deal with the first and the third terms of the right-hand side of (6), we will use the following lemma.

Lemma 3. *For almost all realization of* $(S_n)_{n\geq 0}$ *and for all* $0 \leq i_1 < i_2 < \cdots < i_p \leq n$,

$$\left| F'_{i_1,\ldots,i_p}(u_n) - \exp\left(-\frac{R_{i_1,\ldots,i_p}}{n}\tau\right)\right| \leq \varepsilon_n,$$

with $\varepsilon_n = \varepsilon_n^{(1)} + \varepsilon_n^{(2)}$, *where* $\varepsilon_n^{(1)}$ *and* $\varepsilon_n^{(2)}$ *are defined in* (7) *and* (9) *respectively.*

Proof of Lemma 3. Similarly to Lemma 2, the main idea is to adapt several arguments appearing in the proofs of Lemmas 1 and 2 in [1] in our context. Let (k_n) and (r_n) be as in (3) and (5). Given $1 \leq i_1 < i_2 < \cdots < i_p \leq n$, we subdivide the random set $\{S_{i_1}, \ldots, S_{i_p}\}$ into K_n blocks, with $K_n = \lfloor \frac{R_{i_1,\ldots,i_p}}{r_n} \rfloor + 1$, in the same spirit as we did in the proof of Lemma 2. More precisely, there exists a unique K_n-tuple of subsets $B_i \subset \mathcal{S}_n$, $i \leq K_n$, such that the following properties hold: $\bigcup_{j\leq K_n} B_j = \{S_{i_1}, \ldots, S_{i_p}\}$, $\#B_i = r_n$ and $\max B_i < \min B_{i+1}$ for all $i \leq K_n - 1$. In particular, we have $K_n \leq k_n$ and $\#B_{K_n} = R_n - (K_n - 1)\cdot r_n$ a.s.. Without loss of generality, we assume that $\#B_{K_n} = \#B_i = r_n$ for all $i \leq K_n - 1$, so that $R_{i_1,\ldots,i_p} = K_n r_n$. For each $j \leq K_n$, we also denote by L_j the family consisting of the ℓ_n largest terms of B_j and we let $\mathcal{L}_n = \bigcup_{j\leq K_n} L_j$. In the rest of the paper, we write $M_B = \max_{s\in B} \xi(s)$ for all subset $B \subset \mathbb{Z}$.

Adapting the proof of Lemma 1 in [1], we can show that the following inequalities hold for almost all realization of $(S_n)_{n\in\geq 0}$ and for n larger than some deterministic integer n_0:

$$\left|\mathbb{P}\left(M_{\{S_{i_1},\ldots,S_{i_p}\}} \leq u_n\right) - \mathbb{P}\left(M_{\{S_{i_1},\ldots,S_{i_p}\}\setminus\mathcal{L}_n} \leq u_n\right)\right| \leq k_n\ell_n\mathbb{P}\left(\xi > u_n\right);$$

$$\left|\mathbb{P}\left(M_{\{S_{i_1},\ldots,S_{i_p}\}\setminus\mathcal{L}_n} \leq u_n\right) - \prod_{j\leq K_n}\mathbb{P}\left(M_{B_j\setminus\mathcal{L}_n} \leq u_n\right)\right| \leq k_n\alpha_{n,\ell_n};$$

$$\left|\prod_{j\leq K_n}\mathbb{P}\left(M_{B_j\setminus\mathcal{L}_n} \leq u_n\right) - \prod_{j\leq K_n}\mathbb{P}\left(M_{B_j} \leq u_n\right)\right| \leq 2\frac{\tau k_n\ell_n}{n}.$$

Since $F'_{i_1,\ldots,i_p}(u_n) = \mathbb{P}\left(M_{\{S_{i_1},\ldots,S_{i_p}\}} \leq u_n\right)$ and $\mathbb{P}\left(\xi > u_n\right) \underset{n\to\infty}{\sim} \frac{\tau}{n}$, we get for almost all realization of $(S_n)_{n\geq 0}$,

$$\left|F'_{i_1,\ldots,i_p}(u_n) - \prod_{j\leq K_n}\mathbb{P}\left(M_{B_j} \leq u_n\right)\right| \leq \varepsilon_n^{(1)},$$

with

$$\varepsilon_n^{(1)} = c\cdot\left(\frac{k_n\ell_n}{n} + k_n\alpha_{n,\ell_n}\right). \tag{7}$$

Without loss of generality, we assume from now on that $\mathbb{P}\left(\xi > u_n\right) = \frac{\tau}{n}$. We show below that

$$\left|\prod_{j\leq K_n}\mathbb{P}\left(M_{B_j} \leq u_n\right) - \exp\left(-\frac{R_{i_1,\ldots,i_p}}{n}\tau\right)\right| \leq \varepsilon_n^{(2)}, \tag{8}$$

for some deterministic sequence $\varepsilon_n^{(2)} \underset{n\to\infty}{\longrightarrow} 0$. To do it, we adapt several arguments of Lemma 2 in [1]. First, we notice that for n large enough,

$$\begin{aligned}
&\prod_{j\leq K_n} \mathbb{P}\left(M_{B_j} \leq u_n\right) - \exp\left(-\frac{R_{i_1,\ldots,i_p}}{n}\tau\right) \\
&\qquad\qquad \geq \exp\left(K_n \log(1 - r_n\mathbb{P}\left(\xi > u_n\right))\right) - \exp\left(-\frac{R_{i_1,\ldots,i_p}}{n}\tau\right) \\
&\qquad\qquad \geq \exp\left(-K_n r_n \mathbb{P}\left(\xi > u_n\right) - K_n(r_n\mathbb{P}\left(\xi > u_n\right))^2\right) - \exp\left(-\frac{R_{i_1,\ldots,i_p}}{n}\tau\right),
\end{aligned}$$

where the last line comes from the facts that $\log(1-x) \geq -x - x^2$ for $|x|$ small enough and that $r_n\mathbb{P}\left(\xi > u_n\right) \underset{n\to\infty}{\longrightarrow} 0$. Because $K_n r_n = R_{i_1,\ldots,i_p}$ and $\mathbb{P}\left(\xi > u_n\right) = \frac{\tau}{n}$, we have

$$\begin{aligned}
&\prod_{j\leq K_n} \mathbb{P}\left(M_{B_j} \leq u_n\right) - \exp\left(-\frac{R_{i_1,\ldots,i_p}}{n}\tau\right) \\
&\qquad\qquad \geq \exp\left(-\frac{R_{i_1,\ldots,i_p}}{n}\tau\right)\left(\exp\left(-K_n(r_n\mathbb{P}\left(\xi > u_n\right))^2\right) - 1\right) \\
&\qquad\qquad \geq \exp(-k_n(r_n\mathbb{P}\left(\xi > u_n\right))^2) - 1,
\end{aligned}$$

where the last line comes from the fact that $K_n \leq k_n$ a.s.. Since $k_n r_n \underset{n\to\infty}{\sim} n$, we have

$$\prod_{j\leq K_n} \mathbb{P}\left(M_{B_j} \leq u_n\right) - \exp\left(-\frac{R_{i_1,\ldots,i_p}}{n}\tau\right) \geq c \cdot \frac{1}{k_n}.$$

Moreover, because $\prod_{j\leq K_n} \mathbb{P}\left(M_{B_j} \leq u_n\right) \leq \exp\left(-\sum_{j\leq K_n} \mathbb{P}\left(M_{B_j} > u_n\right)\right)$, it follows from the Bonferroni inequalities (see e.g. p110 in Feller [3]) that

$$\begin{aligned}
&\prod_{j\leq K_n} \mathbb{P}\left(M_{B_j} \leq u_n\right) \\
&\qquad \leq \exp\left(-(K_n - 1)r_n\mathbb{P}\left(\xi > u_n\right) + \sum_{j\leq K_n} \sum_{\alpha<\beta;\alpha,\beta\in B_j} \mathbb{P}\left(\xi(\alpha) > u_n, \xi(\beta) > u_n\right)\right).
\end{aligned}$$

Since $K_n r_n = R_{i_1,\ldots,i_p}$ and $\mathbb{P}\left(\xi > u_n\right) = \frac{\tau}{n}$, we have

$$\begin{aligned}
&\prod_{j\leq K_n} \mathbb{P}\left(M_{B_j} \leq u_n\right) - \exp\left(-\frac{R_{i_1,\ldots,i_p}}{n}\tau\right) = \exp\left(-\frac{R_{i_1,\ldots,i_p}}{n}\tau\right) \\
&\qquad \times \left(\exp\left(r_n\mathbb{P}\left(\xi > u_n\right) + \sum_{j\leq K_n} \sum_{\alpha<\beta;\alpha,\beta\in B_j} \mathbb{P}\left(\xi(\alpha) > u_n, \xi(\beta) > u_n\right)\right) - 1\right)
\end{aligned}$$

and therefore

$$\prod_{j\le K_n}\mathbb{P}\left(M_{B_j}\le u_n\right)-\exp\left(-\frac{R_{i_1,\dots,i_p}}{n}\tau\right)$$
$$\le \exp\left(r_n\mathbb{P}\left(\xi>u_n\right)+\sum_{j\le K_n}\sum_{\alpha<\beta;\alpha,\beta\in B_j}\mathbb{P}\left(\xi(\alpha)>u_n,\xi(\beta)>u_n\right)\right)-1.$$

Proceeding along the same lines as in the proof of Lemma 2 in [1], we can show that

$$\exp\left(r_n\mathbb{P}\left(\xi>u_n\right)+\sum_{j\le K_n}\sum_{\alpha<\beta;\alpha,\beta\in B_j}\mathbb{P}\left(\xi(\alpha)>u_n,\xi(\beta)>u_n\right)\right)-1$$
$$\le c\left(\frac{1}{k_n}+n\sum_{s=1}^{\lfloor n/k_n\rfloor}\mathbb{P}\left(\xi(0)>u_n,\xi(s)>u_n\right)\right).$$

This shows (8) with

$$\varepsilon_n^{(2)}=c\left(\frac{1}{k_n}+n\sum_{s=1}^{\lfloor n/k_n\rfloor}\mathbb{P}\left(\xi(0)>u_n,\xi(s)>u_n\right)\right). \tag{9}$$

and consequently concludes the proof of Lemma 3. □

According to (3), the fact that $(\xi(s))_{s\in\mathbb{Z}}$ satisfies the $\mathbf{D'}(u_n)$ condition and the fact that $k_n\alpha_{n,\ell_n}\underset{n\to\infty}{\longrightarrow}0$, we have $\varepsilon_n\underset{n\to\infty}{\longrightarrow}0$. It follows from Lemma 3 that the first and the third terms of the right-hand side of (6) converge to 0 as n goes to infinity. To deal with the second one, we write

$$\left|\exp\left(-\frac{R_{i_1,\dots,i_p,j_1,\dots,j_{p'}}}{n}\tau\right)-\exp\left(-\frac{R_{i_1,\dots,i_p}+R_{j_1,\dots,j_{p'}}}{n}\tau\right)\right|$$
$$=\exp\left(-\frac{R_{i_1,\dots,i_p}+R_{j_1,\dots,j_{p'}}}{n}\tau\right)\left(\exp\left(\frac{R_{i_1,\dots,i_p}^{j_1,\dots,j_{p'}}}{n}\tau\right)-1\right)$$
$$\le\exp\left(\frac{R_{1,\dots,i_p}^{i_p+\ell_n+1,\dots,n}}{n}\tau\right)-1,$$

where the last line comes from the fact that $j_1-i_p>\ell_n$. Since $\ell_n\ge 0$, we get

$$\sup\left|\exp\left(-\frac{R_{i_1,\dots,i_p,j_1,\dots,j_{p'}}}{n}\tau\right)-\exp\left(-\frac{R_{i_1,\dots,i_p}+R_{j_1,\dots,j_{p'}}}{n}\tau\right)\right|$$
$$\le\sup_{i\le n}\exp\left(\frac{R_{1,\dots,i}^{i+1,\dots,n}}{n}\tau\right)-1, \tag{10}$$

where the supremum in the left-hand side is taken over all integers $0 \leq i_1 < \cdots < i_p < j_1 < \cdots < j_{p'} \leq n$, with $j_1 - i_p > \ell_n$. Moreover, using the fact that $R^{i+1,\dots,n}_{1,\dots,i} = R_{1,\dots,i} + R_{i+1,\dots,n} - R_{1,\dots,n}$ and following [8], we have $\sup_{i\leq n} \frac{R^{i+1,\dots,n}_{1,\dots,i}}{n} \underset{n\to\infty}{\longrightarrow} 0$ a.s.. This, together with (10) and the Lebesgue's dominated convergence theorem implies

$$\sup \left| \mathbb{E}\left[\exp\left(-\frac{R_{i_1,\dots,i_p,j_1,\dots,j_{p'}}}{n}\tau\right)\right] - \mathbb{E}\left[\exp\left(-\frac{R_{i_1,\dots,i_p} + R_{j_1,\dots,j_{p'}}}{n}\tau\right)\right]\right| \underset{n\to\infty}{\longrightarrow} 0$$

and consequently concludes the proof of Proposition 4. □

3.2 The $D^{(k)}(u_n)$ as $k \to \infty$

In [?], the authors introduce a local mixing condition, referred to as the $D^{(k)}(u_n)$ condition, which allows to express the extremal index in terms of joint distribution. We recall the latter below.

Condition $\mathbf{D}^{(k)}(u_n)$ Let $(\xi(s))_{s\in\mathbb{Z}}$ be a sequence of random variables and let u_n be a threshold such that $n\mathbb{P}(\xi > u_n) \underset{n\to\infty}{\longrightarrow} \tau$, for some $\tau > 0$. In conjunction with the $D(u_n)$ condition, we say that the $D^{(k)}(u_n)$ condition, $k \geq 1$, holds if there exist two sequences of integers (k_n) and (ℓ_n) such that

$$k_n \to \infty, \qquad k_n\alpha_{n,\ell_n} \to 0,\ k_n\ell_n = o(n)$$

and

$$\lim_{n\to\infty} n\mathbb{P}\left(\xi(1) > u_n \geq M_{2,k},\ M_{k+1,r_n} > u_n\right) = 0, \tag{11}$$

where r_n is as in (5) and where $M_{i,j} = \max\{\xi(i), \xi(i+1), \dots, \xi(j)\}$ for all $i \leq j$, with the convention $M_{i,j} = -\infty$ if $i > j$. As mentioned in [?], Equation (11) is implied by the condition

$$\lim_{n\to\infty} n \sum_{s=k+1}^{r_n} \mathbb{P}\left(\xi(1) > u_n \geq M_{2,k},\ \xi(s) > u_n\right) = 0.$$

Observe that the last line is the $D'(u_n)$ condition if $k = 1$.

Roughly, the following proposition states that the sequence $(\xi(S_n))_{n\geq 0}$ satisfies the $D^{(k)}(u_n)$ condition as k goes to infinity.

Proposition 5. *Under the same assumptions as Proposition 4, we have*

$$\lim_{k\to\infty}\lim_{n\to\infty} n \sum_{j=k+1}^{r_n} \mathbb{P}\left(\xi(S_1) > u_n \geq M'_{2,k},\ \xi(S_j) > u_n\right) = 0,$$

where $M'_{i,j} = \max_{i\leq t\leq j} \xi(S_t)$ *if* $i \leq j$ *and* $M'_{i,j} = -\infty$ *if* $i > j$.

Proof of Proposition 5. For all $k \geq 1$, we have

$$
\begin{aligned}
& n \sum_{j=k+1}^{r_n} \mathbb{P}\left(\xi(S_1) > u_n \geq M'_{2,k},\ \xi(S_j) > u_n\right) \\
& = n \sum_{j=k+1}^{r_n} \mathbb{P}\left(\xi(S_1) > u_n \geq M'_{2,k},\ \xi(S_j) > u_n | S_j = S_1\right) \mathbb{P}\left(S_j = S_1\right) \\
& \qquad + n \sum_{j=k+1}^{r_n} \mathbb{P}\left(\xi(S_1) > u_n \geq M'_{2,k},\ \xi(S_j) > u_n | S_j \neq S_1\right) \mathbb{P}\left(S_j \neq S_1\right). \qquad (12)
\end{aligned}
$$

The first term of the right-hand side of (12) tends to zero as $k, n \to \infty$. Indeed,

$$
\begin{aligned}
& \mathbb{P}\left(\xi(S_1) > u_n \geq M'_{2,k},\ \xi(S_j) > u_n | S_j = S_1\right) \mathbb{P}\left(S_j = S_1\right) \\
& \qquad \leq \mathbb{P}\left(\xi(S_1) > u_n\right) \mathbb{P}\left(S_j = S_1\right).
\end{aligned}
$$

Moreover, because $(S_n)_{n\geq 1}$ is a transient random walk, we have $\sum_{j=2}^{\infty} \mathbb{P}\left(S_j = S_1\right) < \infty$, which implies

$$
\lim_{k\to\infty} \lim_{n\to\infty} n\mathbb{P}\left(\xi(S_1) > u_n\right) \sum_{j=k+1}^{r_n} \mathbb{P}\left(S_j = S_1\right) = 0,
$$

and therefore

$$
\lim_{k\to\infty} \lim_{n\to\infty} n \sum_{j=k+1}^{r_n} \mathbb{P}\left(\xi(S_1) > u_n \geq M'_{2,k},\ \xi(S_j) > u_n | S_j = S_1\right) \mathbb{P}\left(S_j = S_1\right) = 0.
$$

To prove that the second term of the right-hand side of (12) goes to 0, we write

$$
\begin{aligned}
& n \sum_{j=k+1}^{r_n} \mathbb{P}\left(\xi(S_1) > u_n \geq M'_{2,k},\ \xi(S_j) > u_n | S_j \neq S_1\right) \mathbb{P}\left(S_j \neq S_1\right) \\
& = n \sum_{j=k+1}^{r_n} \mathbb{P}\left(\xi(S_1) > u_n \geq M'_{2,k},\ \xi(S_j) > u_n | S_j \in B^*(S_1, r_n)\right) \mathbb{P}\left(S_j \in B^*(S_1, r_n)\right) \\
& + n \sum_{j=k+1}^{r_n} \mathbb{P}\left(\xi(S_1) > u_n \geq M'_{2,k},\ \xi(S_j) > u_n | S_j \notin B(S_1, r_n)\right) \mathbb{P}\left(S_j \notin B(S_1, r_n)\right), \qquad (13)
\end{aligned}
$$

where $B(S_1, r_n) := \{S \in \mathcal{S}_n : |S - S_1| \leq r_n\}$ and $B^*(S_1, r_n) = B(S_1, r_n) \setminus \{S_1\}$. We prove below that the last two terms in (13) converge to 0. For the first one, we write

$$
\begin{aligned}
& n \sum_{j=k+1}^{r_n} \mathbb{P}\left(\xi(S_1) > u_n \geq M'_{2,k},\ \xi(S_j) > u_n | S_j \in B^*(S_1, r_n)\right) \mathbb{P}\left(S_j \in B^*(S_1, r_n)\right) \\
& \qquad \leq n \sum_{j=2}^{r_n} \mathbb{P}\left(\xi(0) > u_n,\ \xi(S_j - S_1) > u_n | S_j \in B^*(S_1, r_n)\right).
\end{aligned}
$$

The last quantity converges to 0 as n goes to infinity since the sequence $(\xi(s))_{s\in\mathbb{Z}}$ satisfies the $\mathbf{D}'(u_n)$ condition. To deal with the second term of (13), we write

$$n\sum_{j=k+1}^{r_n}\mathbb{P}\left(\xi(S_1)>u_n\geq M'_{2,k},\ \xi(S_j)>u_n|S_j\notin B(S_1,r_n)\right)\mathbb{P}\left(S_j\notin B(S_1,r_n)\right)$$

$$\leq n\sum_{j=k+1}^{r_n}\mathbb{P}\left(\xi(S_1)>u_n,\ \xi(S_j)>u_n|S_j\notin B(S_1,r_n)\right)$$

$$\leq n\sum_{j=k+1}^{r_n}\mathbb{P}\left(\xi>u_n\right)^2+n\sum_{j=k+1}^{r_n}\left|\mathbb{P}\left(\xi(S_1)>u_n,\ \xi(S_j)>u_n|S_j\notin B(S_1,r_n)\right)-\mathbb{P}\left(\xi>u_n\right)^2\right|.$$

The first series tends to 0 as n goes to infinity because

$$n\sum_{j=k+1}^{r_n}\mathbb{P}\left(\xi>u_n\right)^2\leq nr_n\mathbb{P}\left(\xi>u_n\right)^2\underset{n\to\infty}{\sim}\tau^2\frac{r_n}{n},$$

and $r_n=o(n)$. To deal with the second series, we use the $\mathbf{D}(u_n)$ condition. This gives

$$n\sum_{j=k+1}^{r_n}|\mathbb{P}\left(\xi(S_1)>u_n,\xi(S_j)>u_n|S_j\notin B(S_1,r_n)\right)-\mathbb{P}\left(\xi>u_n\right)^2|\leq nr_n\alpha_{n,r_n}$$

$$\leq\frac{n^2}{k_n}\alpha_{n,r_n},$$

which converges to 0 as n goes to infinity according to (3). This concludes the proof of Proposition 5. □

3.3 The extremal index

Let (k_n) and (r_n) be as in (3) and (5). Let us denote by $R_n=\#\mathcal{S}_n$ and $K_n=\left\lfloor\frac{R_n}{r_n}\right\rfloor+1$. The following proposition deals with $M_{\mathcal{S}_n}$ under the $\mathbf{D}(u_n)$ condition.

Proposition 6. *Let $\alpha<1$. Assume that the sequence $(\xi(s))_{s\in\mathbb{Z}}$ satisfies the $\mathbf{D}(u_n)$ conditions for a threshold u_n such that $n\mathbb{P}\left(\xi>u_n\right)\underset{n\to\infty}{\longrightarrow}\tau$, with $\tau>0$. Then for almost all realization of $(S_n)_{n\geq0}$,*

$$\mathbb{P}\left(M_{\mathcal{S}_n}\leq u_n\right)-\exp\left(-\sum_{j=1}^{K_n}\sum_{i=1}^{r_n}\mathbb{P}\left(\xi(S_{((j-1)r_n+i)})>u_n\geq M'_{((j-1)r_n+i+1,\ jr_n)}\right)\right)\underset{n\to\infty}{\longrightarrow}0,$$

where

$$M'_{(i,j)}:=\begin{cases}\max\limits_{i\leq t\leq j}\xi(S_{(t)}), & i\leq j\\ -\infty, & i>j\end{cases}$$

and where $S_{(t)}$ is the t-th largest value of the $\xi(S_i)'s$, $i\leq n$.

A similar result was obtained by O'Brien (Theorem 2.1. in [12]). However, the above proposition is not a consequence of the latter. Proposition 6 remains true if the sequence $(\xi(s))_{s\in\mathbb{Z}}$ only satisfies the $D(u_n)$ condition (i.e. when $k_n\alpha_{n,\ell_n} \underset{n\to\infty}{\longrightarrow} 0$ instead of $\frac{n^2}{k_n}\alpha_{n,\ell_n} \underset{n\to\infty}{\longrightarrow} 0$). As a direct consequence of such a result, if for almost all realization of $(S_n)_{n\geq 0}$,

$$\frac{1}{n}\sum_{j\leq K_n}\sum_{i=1}^{r_n}\mathbb{P}\left(M'_{((j-1)r_n+i+1,\, jr_n)} \leq u_n | \xi(S_{((j-1)r_n+i)}) > u_n\right) \underset{n\to\infty}{\longrightarrow} \theta,$$

for some $\theta\in[0,1]$, then $\mathbb{P}\left(M_{\mathcal{S}_n}\leq u_n\right) \underset{n\to\infty}{\longrightarrow} e^{-\theta\tau}$. In this case, the term θ is referred to as the *extremal index* (see e.g. [10]) and can be interpreted as the reciprocal of the mean size of a cluster of exceedances. As stated in Theorem 1 in [1], when the sequence $(\xi(s))_{s\in\mathbb{Z}}$ satisfies the $\mathbf{D}(u_n)$ and $\mathbf{D}'(u_n)$ conditions, we have

$$\mathbb{P}\left(M_{\mathcal{S}_n}\leq u_n\right) \underset{n\to\infty}{\longrightarrow} e^{-q\tau}. \tag{14}$$

In other words, under these conditions, the extremal index θ exists and $\theta = q$.

Proof of Proposition 6. Let us write $\mathcal{S}_n = \{S_{(1)},\dots,S_{(R_n)}\}$ with $S_{(1)} < S_{(2)} < \cdots < S_{(R_n)}$, and partitition $\mathcal{S}_n$ into K_n blocks as in Lemma 2. Without loss of generality, assume that the last block has the same size as the others, so that $\frac{R_n}{K_n}$ is an integer. Let $B_j = \{S_{((j-1)r_n+1)},\dots,S_{(jr_n)}\}$ be the j-th block of size r_n. According to Lemma 1 in [1], for almost all realization of $(S_n)_{n\geq 0}$, we have

$$\mathbb{P}\left(M_{\mathcal{S}_n}\leq u_n\right) - \exp\left(\sum_{j\leq K_n}\log\left(1-\mathbb{P}\left(M_{B_j}>u_n\right)\right)\right) \underset{n\to\infty}{\longrightarrow} 0.$$

Moreover, because $|\log(1-x)+x|\leq Cx^2$ for $|x|$ small enough and because $\mathbb{P}\left(M_{B_j}>u_n\right)\leq r_n\mathbb{P}\left(\xi>u_n\right)$ converges to 0 as n goes to infinity, we have

$$\begin{aligned}\left|\sum_{j\leq K_n}\log\left(1-\mathbb{P}\left(M_{B_j}>u_n\right)\right)+\sum_{j\leq K_n}\mathbb{P}\left(M_{B_j}>u_n\right)\right| \\ &\leq \sum_{j\leq K_n}\left|\log\left(1-\mathbb{P}\left(M_{B_j}>u_n\right)\right)+\mathbb{P}\left(M_{B_j}>u_n\right)\right| \\ &\leq C\sum_{j\leq K_n}\mathbb{P}\left(M_{B_j}>u_n\right)^2 \\ &\leq Ck_nr_n^2\mathbb{P}\left(\xi>u_n\right)^2.\end{aligned}$$

The last term converges to 0 as n goes to infinity since $k_nr_n \underset{n\to\infty}{\sim} n$, $n\mathbb{P}\left(\xi>u_n\right) \underset{n\to\infty}{\longrightarrow} \tau$ and $r_n\mathbb{P}\left(\xi>u_n\right) \underset{n\to\infty}{\longrightarrow} 0$. This shows that for almost all realization of $(S_n)_{n\geq 0}$

$$\mathbb{P}\left(M_{\mathcal{S}_n}\leq u_n\right) - \exp\left(-\sum_{j\leq K_n}\mathbb{P}\left(M_{B_j}>u_n\right)\right) \underset{n\to\infty}{\longrightarrow} 0. \tag{15}$$

Besides, following the same lines as [12], we have

$$\begin{aligned}\mathbb{P}\left(M_{B_j} \leq u_n\right) &= 1 - \mathbb{P}\left(M_{B_j} > u_n\right)\\ &= 1 - \sum_{i=1}^{r_n} \mathbb{P}\left(\xi(S_{((j-1)r_n+i)}) > u_n \geq M'_{((j-1)r_n+i+1,\, jr_n)}\right)\end{aligned}$$

This together with (15) concludes the proof of Proposition 6. □

3.4 The $D'(u_n)$ condition

Recall that, in the classical literature (see e.g. (3.2.1) in [11]), the $D'(u_n)$ condition holds for the sequence (Z_n) if, in conjunction with the $D(u_n)$ condition,

$$\lim_{n\to\infty} n \sum_{i=2}^{[n/k_n]} \mathbb{P}\left(Z_1 > u_n, Z_i > u_n\right) = 0,$$

for some sequence of integers (k_n) such that $k_n \underset{n\to\infty}{\longrightarrow} \infty$, $k_n\alpha_{n,\ell_n} \underset{n\to\infty}{\longrightarrow} 0$ and $k_n\ell_n = o(n)$. The following result is an extension of Proposition 3 in [5]. However, we give a simpler proof which is based on [10].

Proposition 7. *Under the same assumptions as Proposition 4, the sequence $(\xi(S_n))_{n\geq 0}$ does not satisfy the $D'(u_n)$ condition.*

Proof of Proposition 7. On the opposite, if $(\xi(S_n))_{n\geq 0}$ satisfies the $D'(u_n)$ condition, then $\mathbb{P}\left(M_{\mathcal{S}_n} \leq u_n\right) \underset{n\to\infty}{\longrightarrow} e^{-\tau}$ according to Theorem 1.2 in [10]. This contradicts (14) since $q \neq 1$. □

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