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ASYMMETRIC ELLIPTIC PROBLEMS WITH INDEFINITE WEIGHTS

M. ARIAS [a], J. CAMPOS [a], M. CUESTA [b], J.-P. GOSSEZ [c]

[a] *Departamento de Matemática Aplicada, Universidad de Granada, 18071 Granada, Spain*
[b] *LMPA, Université du Littoral, 50, rue F. Buisson, BP 699, 6228 Calais, France*
[c] *Département de Mathématique, C. P. 214, Université Libre de Bruxelles, 1050 Bruxelles, Belgium*

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ABSTRACT. – We prove the existence of a first nontrivial eigenvalue for an asymmetric elliptic problem with weights involving the laplacian (cf. (1.2) below) or more generally the p-laplacian (cf. (1.3) below). A first application is given to the description of the beginning of the Fučik spectrum with weights for these operators. Another application concerns the study of nonresonance for the problems (1.1) and (1.5) below. One feature of our nonresonance conditions is that they involve eigenvalues with weights instead of pointwise restrictions. © 2001 Éditions scientifiques et médicales Elsevier SAS

RÉSUMÉ. – Nous démontrons l'existence d'une première valeur propre non triviale pour un problème asymétrique avec poids faisant intervenir le laplacien (cf. (1.2) ci-dessous) ou plus généralement le p-laplacien (cf. (1.3) ci-dessous). Une première application consiste en la description du début du spectre de Fučik avec poids pour ces opérateurs. Une autre application concerne l'étude de la nonrésonance pour les problèmes (1.1) et (1.5) ci-dessous. Une caractéristique de nos conditions de nonrésonance est qu'elles font intervenir des valeurs propres avec poids, plutôt que des restrictions ponctuelles. © 2001 Éditions scientifiques et médicales Elsevier SAS

1. Introduction

This paper is partly motivated by the study of the semilinear elliptic problem

$$-\Delta u = f(x,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$. It is well known that the asymptotic behaviour of the quotients $f(x,s)/s$ and $2F(x,s)/s^2$ (where $F(x,s) = \int_0^s f(x,t)\,\mathrm{d}t$) as $s \to +\infty$ and $s \to -\infty$ plays an important role in the study of the solvability of (1.1). Usually pointwise conditions are imposed on the limits of these quotients (for instance they

E-mail addresses: marias@goliat.ugr.es (M. Arias), jcampos@goliat.ugr.es (J. Campos), cuesta@lmpa.univ-littoral.fr (M. Cuesta), gossez@ulb.ac.be (J.-P. Gossez).

are required to lie for a.e. $x \in \Omega$ between two consecutive eigenvalues of $-\Delta$). When looking at the linear case $-\Delta u = a(x)u + b(x)$, it seems however more natural to impose conditions which would involve the limits of the above quotients as weights of eigenvalues. This is the approach that we wish to follow here. This approach of course requires the preliminary study of weighted asymmetric eigenvalue problems of the form

$$-\Delta u = \lambda\left[m(x)u^+ - n(x)u^-\right] \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where $u^\pm := \max(\pm u, 0)$.

The study of (1.2) is classical when $m(x) \equiv n(x)$ and corresponds to the theory of linear eigenvalue problems with weight. Several works have been devoted in the last 20 years to the study of (1.2) (and of its relations with (1.1)) in the case where $m(x)$ and $n(x)$ are constant and different; this has lead in particular to the notion of Fučik spectrum and to the so-called problems of Ambrosetti–Prodi type. The situation where $m(x)$ and $n(x)$ are non constant and different was investigated recently in the ODE case $N = 1$ in [23,7,37,36] (for $m(x)$ and $n(x) > 0$) and [2] (for $m(x)$ and $n(x)$ indefinite).

It is our purpose in this paper to initiate the study of (1.2) and of its relations with the solvability of (1.1) in the general case: $N \geqslant 1$, $m(x)$ and $n(x)$ possibly non constant, different and indefinite. More generally we will consider the quasilinear eigenvalue problem

$$-\Delta_p u = \lambda\left[m(x)(u^+)^{p-1} - n(x)(u^-)^{p-1}\right] \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.3}$$

as well as the Δ_p analogue of (1.1) (cf. (1.5) below). Here $1 < p < \infty$ and $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-laplacian. We wish however to point out that all our results are new even in the semilinear case $p = 2$ (with only one exception, Theorem 45, as we will see later).

Solutions u of (1.3) which do not change sign clearly arise if (and only if) λ is one of the first eigenvalues $\lambda_1(m)$, $\lambda_1(n)$, $\lambda_{-1}(m)$, $\lambda_{-1}(n)$ of the p-laplacian with weight (cf. the end of this introduction for definitions). Section 2 is devoted to the proof of the existence of a solution u of (1.3) which changes sign. Our construction is based on the mountain pass theorem, more precisely on a version of that theorem on a C^1 manifold. In Section 3 we show that the eigenvalue λ constructed in Section 2 is in fact the first eigenvalue of (1.3) with a sign-changing eigenfunction. This is probably one of the main results of our paper and the technique introduced in its proof (which in particular involves the consideration of 3 different manifolds) will be used repeatedly later. This distinguished eigenvalue, denoted by $c(m,n)$, plays in our asymmetric problems a role analogous to that of the usual "second eigenvalue". Several properties of $c(m,n)$ as a function of the weights m, n are investigated in Section 4: continuity, (strict) monotonicity, homogeneity.

As a first application, in Section 5, we study the Fučik spectrum with weights. This is defined as the set Σ of those $(\alpha, \beta) \in \mathbb{R}^2$ such that

$$-\Delta_p u = \alpha m(x)(u^+)^{p-1} - \beta n(x)(u^-)^{p-1} \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \tag{1.4}$$

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has a nontrivial solution. We show in particular that if m and n both change sign in Ω, then each of the four quadrants in the (α,β) plane contains a first (nontrivial) curve of Σ, which is hyperbolic like and has a variational characterization. We also study the asymptotic behaviour of these first curves. It turns out for instance that the first curve in $\mathbb{R}^+\times\mathbb{R}^+$ is asymptotic to the vertical line $\lambda_1(m)\times\mathbb{R}$ if $p\leqslant N$, or if $p>N$ and the support of n^+ intersects $\partial\Omega$, but it is not asymptotic to that line if $p>N$ and the support of n^+ is compact in Ω. A similar result of course holds with respect to the horizontal line $\mathbb{R}\times\lambda_1(n)$, which involves the support of m^+, and in the other quadrants.

The last two Sections 6 and 7 are mainly concerned with the study of nonresonance for the problem

$$-\Delta_p u = f(x,u) \quad \text{in } \Omega, \qquad u=0 \quad \text{on } \partial\Omega. \tag{1.5}$$

As mentionned earlier, we replace the usual pointwise conditions on the limits of $f(x,s)/|s|^{p-2}s$ and $pF(x,s)/|s|^p$ by conditions involving some eigenvalues having these limits as weights. This approach based on "weights" allows us to improve several results concerning (1.5), as will be indicated in details later. Section 6 is devoted to problems of the type "between the first two eigenvalues". We exploit here our results of Sections 2–4 relative to the distinguished eigenvalue $c(m,n)$ of (1.3). Section 7, which is independent from the previous sections, deals briefly with problems of the type "below the first eigenvalue". It contains an extension of the classical result of Hammerstein where we also impose conditions on eigenvalues with weights. Examples of non unicity are also investigated in Sections 6 and in 7, where nonconstant weights play a central role.

To conclude this introduction, let us briefly recall some properties of the spectrum of $-\Delta_p$ with weight to be used later. References are [3,34] (for a bounded weight), [39,1,13] (for an unbounded weight). Let Ω be a bounded domain in $\mathbb{R}^N$ and let $m\in L^r(\Omega)$ where $r>N/p$ if $p\leqslant N$ and $r=1$ if $p>N$. We also assume $m^+\not\equiv 0$. The eigenvalue problem under consideration here is

$$-\Delta_p u = \lambda m(x)|u|^{p-2}u \quad \text{in } \Omega, \qquad u=0 \quad \text{on } \partial\Omega. \tag{1.6}$$

The first positive eigenvalue $\lambda_1(m)$ of (1.6) is defined as

$$\lambda_1(m)=\lambda_1(m,\Omega):=\min\left\{\int_\Omega |\nabla u|^p\colon\ u\in W_0^{1,p}(\Omega) \text{ and } \int_\Omega m|u|^p=1\right\}.$$

It is known that $\lambda_1(m)$ is >0, simple, and admits an eigenfunction $\varphi_m=\varphi_{m,\Omega}\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)\cap C(\Omega)$, with φ_m satisfying $\varphi_m(x)>0$ in Ω and $\int_\Omega m(\varphi_m)^p=1$. Moreover $\lambda_1(m)$ is isolated in the spectrum, which allows to define the second positive eigenvalue $\lambda_2(m)$ as

$$\lambda_2(m):=\min\{\lambda\in\mathbb{R}\colon\ \lambda \text{ eigenvalue and } \lambda>\lambda_1(m)\}.$$

It is also known that any eigenfunction associated to a positive eigenvalue different from $\lambda_1(m)$ changes sign and that $\lambda_1(m)$ is strictly monotone decreasing with respect to m

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(i.e. $m \underset{\not\equiv}{\leqslant} \hat{m}$ implies $\lambda_1(m) > \lambda_1(\hat{m})$). In the case $m^- \not\equiv 0$, the first and second negative eigenvalues are obtained by reversing the sign of the weight: $\lambda_{-1}(m) = -\lambda_1(-m)$ and $\lambda_{-2}(m) = -\lambda_2(-m)$.

The main results of this paper were announced in [9].

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2. Construction of a nontrivial eigenvalue

In this section and in the following two we consider the eigenvalue problem (1.3) on a bounded domain $\Omega \subset \mathbb{R}^N$. It will always be assumed that $m, n \in L^r(\Omega)$ with r as in the introduction, i.e. $r > N/p$ if $p \leqslant N$ and $r = 1$ if $p > N$. We also assume, unless otherwise stated,

$$m^+ \quad \text{and} \quad n^+ \not\equiv 0 \quad \text{in } \Omega. \tag{2.1}$$

We look for eigenvalues λ of (1.3) with $\lambda > 0$.

Clearly (1.3) with $\lambda > 0$ has a nontrivial solution u which does not change sign if and only if $\lambda = \lambda_1(m)$ or $\lambda = \lambda_1(n)$. Moreover, multiplying by u^+ or u^-, one easily sees that if (1.3) with $\lambda > 0$ has a solution which changes sign, then $\lambda > \max\{\lambda_1(m), \lambda_1(n)\}$. Proving the existence of such a solution which changes sign is our purpose in this section.

Remark 1. – It is easily seen that (2.1) is a necessary condition for (1.3) with $\lambda > 0$ to have a solution which changes sign. Observe also that if, instead of (2.1), we have m^- and $n^- \not\equiv 0$, then, by reversing the sign of the weights, our approach will lead to negative eigenvalues of (1.3). So, if m and n both change sign, we will obtain positive as well as negative eigenvalues of (1.3).

We will use a variational approach and consider the functionals

$$A(u) := \int_\Omega |\nabla u|^p,$$

$$B_{m,n}(u) := \int_\Omega \bigl(m(u^+)^p + n(u^-)^p\bigr),$$

which are C^1 functionals on $W_0^{1,p}(\Omega)$. We are interested in the critical points of the restriction $\tilde{A}$ of A to the manifold

$$M_{m,n} := \bigl\{u \in W_0^{1,p}(\Omega):\ B_{m,n}(u) = 1\bigr\}.$$

Note that 1 is a regular value of $B_{m,n}$. Note also that φ_m and $-\varphi_n \in M_{m,n}$, and that $M_{m,n}$ contains functions which change sign. In fact, a standard argument of regularization shows that, under (2.1), there exists $u \in C_c^\infty(\Omega)$ such that $\int_\Omega m(u^+)^p > 0$ and $\int_\Omega n(u^-)^p > 0$, and consequently $u/(B_{m,n}(u))^{1/p}$ belongs to $M_{m,n}$.

By Lagrange's multiplier rule, $u \in M_{m,n}$ is a critical point of $\tilde{A}$ if and only if there exists $\lambda \in \mathbb{R}$ such that $A'(u) = \lambda B'_{m,n}(u)$, i.e.

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v = \lambda \int_\Omega \big(m(u^+)^{p-1} - n(u^-)^{p-1}\big)v \tag{2.2}$$

for all $v \in W_0^{1,p}(\Omega)$. This means that (1.3) holds in the weak sense. Moreover, taking $v = u$ in (2.2), one sees that the Lagrange multiplier λ is equal to the critical value $\tilde{A}(u)$. Our eigenvalue problem (1.3) is thus transformed into the problem of looking for critical points and critical values of $\tilde{A}$.

A first critical point of $\tilde{A}$ comes from global minimization. Indeed

$$\tilde{A}(u) \geqslant \lambda_1(m)\left[\int_\Omega m(u^+)^p\right]^+ + \lambda_1(n)\left[\int_\Omega n(u^-)^p\right]^+ \geqslant \min\{\lambda_1(m), \lambda_1(n)\}$$

for all $u \in M_{m,n}$, and one has $\tilde{A}(u) = \min\{\lambda_1(m), \lambda_1(n)\}$ for either $u = \varphi_m$ or $u = -\varphi_n$. Consequently either φ_m or $-\varphi_n$ is a global minimum of $\tilde{A}$ and so a critical point of $\tilde{A}$. The other one is also a critical point as follows from

PROPOSITION 2. – *φ_m and $-\varphi_n$ are strict local minima of $\tilde{A}$, with corresponding critical values $\lambda_1(m)$ and $\lambda_1(n)$.*

Proof. – Let us consider φ_m (similar argument for $-\varphi_n$). Assume by contradiction the existence of a sequence $u_k \in M_{m,n}$ with $u_k \neq \varphi_m$, $u_k \to \varphi_m$ in $W_0^{1,p}(\Omega)$ and $\tilde{A}(u_k) \leqslant \lambda_1(m)$. We first observe that u_k changes sign for k sufficiently large. Indeed, since $u_k \to \varphi_m$, u_k must be > 0 somewhere. If $u_k \geqslant 0$ in Ω, then

$$\tilde{A}(u_k) = \int_\Omega |\nabla u_k|^p > \lambda_1(m) \int_\Omega m|u_k|^p = \lambda_1(m)$$

since $u_k \neq \pm\varphi_m$, but this contradicts $\tilde{A}(u_k) \leqslant \lambda_1(m)$. So u_k changes sign for k sufficiently large. Now we have

$$\begin{aligned}
\lambda_1(m)\int_\Omega \big(m(u_k^+)^p + n(u_k^-)^p\big) = \lambda_1(m) &\geqslant \int_\Omega |\nabla u_k|^p \\
&= \int_\Omega \big(|\nabla u_k^+|^p + |\nabla u_k^-|^p\big) \\
&\geqslant \lambda_1(m)\int_\Omega m(u_k^+)^p + \int_\Omega |\nabla u_k^-|^p
\end{aligned}$$

and consequently

$$\lambda_1(m)\int_\Omega n\big(u_k^-\big)^p \geqslant \int_\Omega |\nabla u_k^-|^p.$$

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Since $u_k \to \varphi_m$, $|u_k^- > 0| \to 0$, where $|u_k^- > 0|$ denotes the measure of the set where u_k^- is > 0. The desired contradiction then follows from Lemma 2.3 below. $\square$

LEMMA 3. – *Let $v_k \in W_0^{1,p}(\Omega)$ with $v_k \geqslant 0$ and $|v_k > 0| \to 0$. Let $n_k \to n$ in $L^r(\Omega)$.*
$\not\equiv$
Then

$$\int_\Omega n_k(v_k)^p \Big/ \int_\Omega |\nabla v_k|^p \to 0.$$

Proof. – It is easily adapted from [15] which deals with the case $n_k \equiv 1$. Write $z_k := v_k / (\int_\Omega |\nabla v_k|^p)^{1/p}$. Clearly, for a subsequence, $z_k \rightharpoonup z$ in $W_0^{1,p}(\Omega)$ and $z_k \to z$ in $L^{r'p}(\Omega)$, with $z \geqslant 0$. If $z \equiv 0$, then

$$\int_\Omega n_k(v_k)^p \Big/ \int_\Omega |\nabla v_k|^p = \int_\Omega n_k z_k^p \to 0$$

and the lemma is proved. If $z \not\equiv 0$, then, for some $\varepsilon > 0$, $\eta := |z > \varepsilon| > 0$. We deduce that $|z_k > \varepsilon/2| > \eta/2$ for k sufficiently large, which contradicts the assumption that $|v_k > 0| \to 0$. $\square$

To get a third critical point, we will use a version of the mountain pass theorem on a C^1 manifold, which we now recall.

Let E be a real Banach space and let

$$M := \{u \in E \colon g(u) = 1\}, \tag{2.3}$$

where $g \in C^1(E, \mathbb{R})$ and 1 is a regular value of g. Let $f \in C^1(E, \mathbb{R})$ and consider the restriction $\tilde{f}$ of f to M. The differential of $\tilde{f}$ at $u \in M$ has a norm which will be denoted by $\|\tilde{f}'(u)\|_*$ and which is given by the norm of the restriction of $f'(u)$ to the tangent space

$$T_u(M) := \{v \in E \colon \langle g'(u), v\rangle = 0\};$$

here $\langle\ ,\ \rangle$ denotes the duality pairing between E^* and E. We recall that $\tilde{f}$ is said to satisfy the (PS) condition on M if, for any sequence $u_k \in M$ such that $\tilde{f}(u_k)$ is bounded and $\|\tilde{f}'(u_k)\|_* \to 0$, one has that u_k admits a converging subsequence.

PROPOSITION 4. – *Let $u, v \in M$ with $u \neq v$ and suppose that*

$$H := \{h \in C([-1, +1], M);\ h(-1) = u \text{ and } h(1) = v\} \tag{2.4}$$

is nonempty. Assume also that

$$c := \inf_{h \in H} \max_{w \in h[-1,+1]} f(w) > \max\{f(u), f(v)\} \tag{2.5}$$

and that $\tilde{f}$ satisfies the (PS) condition on M. Then c is a critical value of $\tilde{f}$.

Proposition 4 follows by applying Theorem 3.2 of [28] to a component of M. Note that the proof of Theorem 3.2 in [28] involves a deformation lemma on a C^1 Finsler

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manifold (cf. Lemma 3.7 in [28]). A direct proof of Proposition 4, which uses only Ekeland's variational principle, can be found in [14] in the case where E is uniformly convex. Note that the geometric assumption in [14] is weaker than (2.5): it reads

$$\max_{w\in h[-1,+1]} f(w) > \max\{f(u), f(v)\} \tag{2.6}$$

for any $h \in H$.

We will apply Proposition 4 with $E = W_0^{1,p}(\Omega)$, $f = A$ and $g = B_{m,n}$. First two preliminary results. The first one concerns the (PS) condition while the second one describes the geometry of $\tilde{A}$ near the strict local minima φ_m and $-\varphi_n$.

LEMMA 5. – *$\tilde{A}$ satisfies the (PS) condition on $M_{m,n}$.*

Proof. – Let $u_k \in M_{m,n}$ be a (PS) sequence for $\tilde{A}$. So $\int_\Omega |\nabla u_k|^p$ remains bounded and, for some $\varepsilon_k \to 0$,

$$\left|\int_\Omega |\nabla u_k|^{p-2}\nabla u_k \nabla v\right| \leqslant \varepsilon_k \|v\|_{1,p} \tag{2.7}$$

for all $v \in T_{u_k}(M_{m,n})$, where

$$T_u(M_{m,n}) = \left\{v \in W_0^{1,p}(\Omega)\colon \int_\Omega \left(m(u^+)^{p-1} - n(u^-)^{p-1}\right)v = 0\right\}$$

and $\| \quad \|_{1,p}$ denotes the $W_0^{1,p}(\Omega)$ norm. Clearly, for a subsequence, $u_k \to u$ weakly in $W_0^{1,p}(\Omega)$ and strongly in $L^{r'p}(\Omega)$, where r' denotes the Hölder conjugate of r. Now, given $w \in W_0^{1,p}(\Omega)$, one has $w - a_k(w)u_k \in T_{u_k}(M_{m,n})$ for $a_k(w) := \int_\Omega (m(u_k^+)^{p-1} - n(u_k^-)^{p-1})w$. Putting $v = (u_k - u) - a_k(u_k - u)u_k$ in (2.7) and observing that $a_k(u_k - u) \to 0$, one deduces that

$$\int_\Omega |\nabla u_k|^{p-2}\nabla u_k \nabla(u_k - u) \to 0$$

and consequently

$$\int_\Omega \left(|\nabla u_k|^{p-2}\nabla u_k - |\nabla u|^{p-2}\nabla u\right)(\nabla u_k - \nabla u) \to 0.$$

Using then the inequality

$$|\xi - \eta|^p \leqslant c\left[\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right)(\xi - \eta)\right]^{s/2}\left[|\xi|^p + |\eta|^p\right]^{1-s/2}$$

where $\xi, \eta \in \mathbb{R}^N$, $c = c(p) > 0$ and $s = 2$ if $p \geqslant 2$, $s = p$ if $1 < p < 2$ (cf. e.g. [34]), one easily obtains that $u_k \to u$ in $W_0^{1,p}(\Omega)$. $\square$

The second lemma can be stated in the more general framework of the manifold (2.3).

LEMMA 6. – *Let E, g, M, f and $\tilde f$ be as considered previously (cf. (2.3)). Let u_0 be a strict local minimum of $\tilde f$, i.e., for some $\varepsilon_0 > 0$,*

$$\tilde f(u_0) < \tilde f(u) \tag{2.8}$$

for all $u \in M$ with $u \neq u_0$ and $\|u - u_0\|_E < \varepsilon_0$. Assume that $\tilde f$ satisfies the (PS) condition on M (in fact the (PS) condition at level $\tilde f(u_0)$ suffices). Then, for any $0 < \varepsilon < \varepsilon_0$,

$$\tilde f(u_0) < \inf\bigl\{\tilde f(u)\colon\ u \in M \text{ and } \|u - u_0\|_E = \varepsilon\bigr\}. \tag{2.9}$$

Proof. – It is partly adapted from [17] where a similar situation without constraint is considered. Assume by contradiction the existence for some ε with $0 < \varepsilon < \varepsilon_0$ of a sequence $u_k \in M$ with $\|u_k - u_0\|_E = \varepsilon$ and, say, $\tilde f(u_k) \leqslant \tilde f(u_0) + 1/2k^2$. Call

$$C := \bigl\{u \in M\colon\ \varepsilon - \delta \leqslant \|u - u_0\|_E \leqslant \varepsilon + \delta\bigr\}$$

where $\delta > 0$ is chosen so that $0 < \varepsilon - \delta$ and $\varepsilon + \delta < \varepsilon_0$. Clearly $\inf\{\tilde f(u)\colon\ u \in C\} = \tilde f(u_0)$.

We apply for each k Ekeland's variational principle (cf. e.g. [17]) to the functional $\tilde f$ on C to get the existence of $v_k \in C$ such that

$$\tilde f(v_k) \leqslant \tilde f(u_k), \tag{2.10}$$

$$\|v_k - u_k\|_E \leqslant 1/k, \tag{2.11}$$

$$\tilde f(v_k) \leqslant \tilde f(u) + \frac{1}{k}\|u - v_k\|_E \quad \forall u \in C. \tag{2.12}$$

Our purpose is to show that v_k is a (PS) sequence for $\tilde f$, i.e. that $\tilde f(v_k)$ is bounded (which is clear by (2.10)) and that $\|\tilde f'(v_k)\|_* \to 0$. Once this is proved, we get that, for a subsequence, $v_k \to v$ in E. Clearly $v \in C$ and satisfies $\|v - u_0\|_E = \varepsilon$ and $\tilde f(v) = \tilde f(u_0)$, which contradicts (2.8).

To prove that $\|\tilde f'(v_k)\|_* \to 0$, we fix k with $1/k < \delta$, take $w \in T_{v_k}(M)$ and consider a C^1 path $\gamma :]-\eta, +\eta[\to M$ such that $\gamma(0) = v_k$ and $\gamma'(0) = w$ (cf. e.g. [40], vol. 3, Th. 43 C). For $|t|$ sufficiently small, $\gamma(t) \in C$. Indeed

$$\lim_{t\to 0}\bigl\|\gamma(t) - u_0\bigr\|_E = \|v_k - u_0\|_E \tag{2.13}$$

and it is easily seen, using (2.11), $1/k < \delta$ and $\|u_k - u_0\| = \varepsilon$, that the right-hand side of (2.13) is $< \varepsilon + \delta$ and $> \varepsilon - \delta$. So we can take $u = \gamma(t)$ in (2.12). This gives, for $t > 0$,

$$\frac{\tilde f(v_k) - \tilde f(\gamma(t))}{t} \leqslant \frac{1}{k}\left\|\frac{\gamma(t) - v_k}{t}\right\|_E$$

and so, going to the limit as $t \to 0$, we get

$$-\bigl\langle f'(v_k), w\bigr\rangle \leqslant \frac{1}{k}\|w\|_E.$$

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Consequently, since w is arbitrary in $T_{v_k}(M)$, $\|\tilde{f}'(v_k)\|_* \leqslant 1/k$. □

We are now in a position to apply the mountain pass theorem of Proposition 4.

THEOREM 7. – *Consider*

$$\Gamma := \{\gamma \in C([-1,+1], M_{m,n})\colon\ \gamma(-1) = \varphi_m \text{ and } \gamma(+1) = -\varphi_n\}.$$

Then

$$c(m,n) := \inf_{\gamma\in\Gamma} \max_{u\in\gamma[-1,+1]} \tilde{A}(u) \tag{2.14}$$

is a critical value of $\tilde{A}$, with $c(m,n) > \max\{\lambda_1(m), \lambda_1(n)\}$.

Proof. – The (PS) condition and the geometric assumption (2.5) are satisfied by the previous two lemmas. It remains to verify that Γ is nonempty. Clearly it suffices to construct a path γ in $W_0^{1,p}(\Omega)$ from φ_m to $-\varphi_n$ such that $B_{m,n}(\gamma(t)) > 0$ for all t. We start with a function $u \in W_0^{1,p}(\Omega)$ such that $\int_\Omega m(u^+)^p > 0$ and $\int_\Omega n(u^-)^p > 0$. As already observed the existence of such a function follows from (2.1). We first go from u to u^+ by convex combination: $tu + (1-t)u^+$, $t \in [0,1]$. Then we go on from u^+ to φ_m through the path

$$\left[t(u^+)^p + (1-t)(\varphi_m)^p\right]^{1/p}, \qquad t \in [0,1]. \tag{2.15}$$

(It is an interesting exercise to verify that if $v, w \in W_0^{1,p}(\Omega)$ with $v, w \geqslant 0$, then $(v^p + w^p)^{1/p} \in W_0^{1,p}(\Omega)$). Using the fact that $\int_\Omega m(\varphi_m)^p > 0$, one easily verifies that the path γ from u to φ_m constructed in the above way satisfies $B_{m,n}(\gamma(t)) > 0$ for all t. One goes in a similar way from u to $-\varphi_n$, and the conclusion follows. □

Remark 8. – When the weights m and n are $\geqslant 0$ in Ω, one can use in the above proof the standard convex combination $tu^+ + (1-t)\varphi_m$ instead of (2.15).

Remark 9. – Lemma 6 is in fact not needed to deduce that $c(m,n)$ is a critical value of $\tilde{A}$ (by using the version of Proposition 4 given in [14], which requires only (2.6)). Lemma 6 is however needed to deduce that $c(m,n)$ is $> \max\{\lambda_1(m), \lambda_1(n)\}$.

Remark 10. – It may happen that (1.3) does not admit any other positive eigenvalue beside $\lambda_1(m)$, $\lambda_1(n)$ and $c(m,n)$. In fact, when $N = 1$ and $p = 2$, given an odd integer k, there exist continuous weights satisfying (2.1) such that (1.3) admits exactly k eigenvalues $> \max\{\lambda_1(m), \lambda_1(n)\}$ (cf. [2]). On the other hand, again for $N = 1$ and $p = 2$, if m and n are continuous weights with the product $m^+n^+ \not\equiv 0$, then the positive eigenvalues of (1.3) constitue a sequence going to $+\infty$ (cf. [2]). We observe that for $N \geqslant 2$, the existence of further positive eigenvalues for (1.3) beside $\lambda_1(m)$, $\lambda_1(n)$ and $c(m,n)$ is an open question, even when $p = 2$ and the weights are constant but different.

3. First nontrivial eigenvalue

We have seen at the beginning of Section 2 that $\min\{\lambda_1(m), \lambda_1(n)\}$ and $\max\{\lambda_1(m), \lambda_1(n)\}$ are the first two positive eigenvalues of (1.3). The present section is devoted to the proof that the eigenvalue $c(m,n)$ constructed in (2.14) is the next eigenvalue of (1.3).

THEOREM 11. – *Problem* (1.3) *does not admit any eigenvalue in the open interval* $]\max\{\lambda_1(m), \lambda_1(n)\}, c(m,n)[$.

In particular, for $m = n$, we obtain the following variational characterization of the second eigenvalue $\lambda_2(m)$ of the p-laplacian with weight:

COROLLARY 12. – *One has*

$$\lambda_2(m) = \inf_{\gamma \in \Gamma_0} \max_{u \in \gamma[-1,+1]} \int_\Omega |\nabla u|^p, \tag{3.1}$$

where Γ_0 *is the family of all paths in* $M_{m,m} = \{u \in W_0^{1,p}(\Omega)\colon \int_\Omega m|u|^p = 1\}$ *going from* φ_m *to* $-\varphi_m$.

Remark 13. – Slightly different variational characterizations of $\lambda_2(m)$ have been obtained recently: [6] (bounded weight) considers a minimax procedure (P1) over all compact symmetric sets of genus $\geqslant 2$ in $M_{m,m}$, [4] (bounded weight) and [27,24] (no weight) consider a minimax procedure (P2) over all images of odd mappings from S^1 into $M_{m,m}$. These minimax procedures clearly involve more sets than (3.1). In fact $c(m,m)$ is $\geqslant$ the minimax value in (P2), which itself is $\geqslant$ the minimax value in (P1). Since the latter is $> \lambda_1(m)$ (by the Ljusternik–Schnirelman multiplicity theorem and the simplicity of $\lambda_1(m)$), we see that Theorem 3.1 implies the equality of the minimax values in (P1) and (P2) with $c(m,m)$. The mountain pass characterization (3.1) of $\lambda_2(m)$ was first derived in [15] for $m \equiv 1$.

The following lemma will be used in the proof of Theorem 3.1. It guarantees the existence of a critical point in any component of any sublevel set. As for Lemma 2.6, it can be stated in the general framework of the manifold (2.3).

LEMMA 14. – *Let* E, g, M, f *and* $\tilde f$ *be as considered previously* (*cf.* (2.3)). *Assume that* $\tilde f$ *is bounded from below on* M *and satisfies the* (PS) *condition on* M. *Let* $r \in \mathbb{R}$ *and consider*

$$\mathcal{O} := \{u \in M\colon\ \tilde f(u) < r\}.$$

Then any (*nonempty*) *component* $\mathcal{O}_1$ *of* $\mathcal{O}$ *contains a critical point of* $\tilde f$.

Proof. – It is partly adapted from [15]. Consider $d := \inf\{\tilde f(u)\colon\ u \in \bar{\mathcal{O}}_1\}$, where $\bar{\mathcal{O}}_1$ denotes the closure of $\mathcal{O}_1$. We will show that this infimum is achieved at some $u_0 \in \bar{\mathcal{O}}_1$. Let us accept this for a moment. Clearly $\tilde f(u_0) = d < r$ and so $u_0 \in \mathcal{O}$. Moreover $u_0 \in \mathcal{O}_1$ because $\mathcal{O}$ is locally arcwise connected. Consequently u_0 is a critical point of $\tilde f$.

To show that the infimum d above is achieved, let $u_k \in \mathcal{O}_1$ be a minimizing sequence with, say, $\tilde f(u_k) \leqslant d + 1/2k^2$. For each k, we apply Ekeland's variational principle to the functional $\tilde f$ on $\bar{\mathcal{O}}_1$ to get $v_k \in \bar{\mathcal{O}}_1$ such that

$$\tilde f(v_k) \leqslant \tilde f(u_k), \tag{3.2}$$

$$\|v_k - u_k\| \leqslant 1/k, \tag{3.3}$$

$$\tilde f(v_k) \leqslant \tilde f(u) + \frac{1}{k}\|u - v_k\|_E \quad \forall u \in \bar{\mathcal{O}}_1. \tag{3.4}$$

Our purpose is to show that v_k is a (PS) sequence for $\tilde f$ in S, i.e. that $\tilde f(v_k)$ is bounded (which follows from (3.2)) and that $\|\tilde f'(v_k)\|_* \to 0$. Once this is proved, we deduce from (3.3) that u_k admits a convergent subsequence, and consequently the infimum d is achieved.

To prove that $\|\tilde f'(v_k)\|_* \to 0$, we fix k, take $w \in T_{v_k}(M)$ and consider a C^1 path $\gamma :]-\eta, +\eta[\to M$ such that $\gamma(0) = v_k$ and $\gamma'(0) = w$ as in the proof of Lemma 6. We first observe that $v_k \in \mathcal{O}_1$ for k sufficiently large. Indeed, otherwise, $v_k \in \partial\mathcal{O}_1$ and consequently, since $\mathcal{O}$ is locally arcwise connected, $v_k \notin \mathcal{O}$, which implies $\tilde f(v_k) = r$; but this is impossible since, by (3.2),

$$\tilde f(v_k) \leqslant \tilde f(u_k) \leqslant d + \frac{1}{2k^2} < r$$

for k sufficiently large. So $\gamma(t) \in \mathcal{O}_1$ for t sufficiently small and we can take $u = \gamma(t)$ in (3.4). The argument now is identical to the one at the end of the proof of Lemma 6. It yields $\|\tilde f'(v_k)\|_* \leqslant 1/k$. □

We are now ready to start the

Proof of Theorem 3.1. – Assume by contradiction the existence of an eigenvalue λ of (1.3) with $\max\{\lambda_1(m), \lambda_1(n)\} < \lambda < c(m,n)$. We will construct a path in Γ on which $\tilde A$ remains $\leqslant \lambda$, which yields a contradiction with the definition (2.14) of $c(m,n)$.

Let $u \in M_{m,n}$ be a critical point of $\tilde A$ at level λ. So u satisfies

$$-\Delta_p u = \lambda\big[m(u^+)^{p-1} - n(u^-)^{p-1}\big] \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{3.5}$$

and we know that u changes sign. This implies

$$0 < \int_\Omega |\nabla u^+|^p = \lambda \int_\Omega m(u^+)^p \quad \text{and} \quad 0 < \int_\Omega |\nabla u^-|^p = \lambda \int_\Omega n(u^-)^p. \tag{3.6}$$

The desired path will be constructed in several steps, using u as starting point.

First we go from u to $v := u^+/B_{m,n}(u^+)^{1/p}$ by some sort of convex combination:

$$\gamma_1(t) := \frac{tu + (1-t)u^+}{B_{m,n}(tu + (1-t)u^+)^{1/p}}, \quad t \in [0,1]. \tag{3.7}$$

An easy calculation based on (3.6) shows that $\gamma_1(t)$ is well-defined, belongs to $M_{m,n}$ and satisfies $\tilde A(\gamma_1(t)) = \lambda\ \forall t \in [0,1]$. In a similar way we go from u to $-u^-/B_{m,n}(-u^-)^{1/p}$ in $M_{m,n}$ by staying at level λ. We will now describe the construction of a path in $M_{m,n}$ from v to φ_m which stays at levels $\leqslant \lambda$. A similar construction would yield a path in $M_{m,n}$ from $-u^-/B_{m,n}(-u^-)^{1/p}$ to $-\varphi_n$ which stays at levels $\leqslant \lambda$. Putting everything together, we get the desired path from φ_m to $-\varphi_n$.

To construct the path from v to φ_m, we first consider another manifold: $M_{m,m}$. Clearly $v \in M_{m,m}$. The critical points of the restriction of A to $M_{m,m}$ are the normalized eigenfunctions of $-\Delta_p$ for the weight m. Since v does not change sign and vanishes on a set of positive measure, v is not a critical point of this restriction of A to $M_{m,m}$. Consequently there exists a C^1 path $\nu:]-\varepsilon, +\varepsilon[\to M_{m,m}$ with $\nu(0) = v$ and $\frac{\mathrm{d}}{\mathrm{d}t}A(\nu(t))|_{t=0} \neq 0$. Following a little portion of this path ν in the positive or negative direction (call ν_1 that portion), we move from v to a point w by a path in $M_{m,m}$ which, with the exception of its starting point v where $A(v) = \lambda$, lies at levels $< \lambda$. The path $\gamma_2(t) := |\nu_1(t)|$ then lies in $M_{m,n}$ (because it lies in $M_{m,m}$ and is made of nonnegative functions), goes from v to $v_1 := |w|$ and remains, with the exception of its starting point v where $A(v) = \lambda$, at levels $< \lambda$ (since $A(|\nu_1(t)|) = A(\nu_1(t))$).

To go on from v_1 to φ_m, we first use Lemma 15 below to construct a weight $\hat{n} \in L^r(\Omega)$ such that $(\hat{n})^+ \not\equiv 0$, $\lambda_1(\hat{n}) > \lambda$ and $\hat{n} \leqslant m$ in Ω. It suffices to take $\hat{n} = m$ where $m \leqslant 0$ and $\hat{n} = \varepsilon m$ where $m > 0$, with $\varepsilon > 0$ sufficiently small. We then consider the manifold $M_{m,\hat{n}}$ and the sublevel set

$$\mathcal{O} := \{u \in M_{m,\hat{n}}:\ A(u) < \lambda\}.$$

Clearly v_1 and $\varphi_m \in \mathcal{O}$ (because they belong to $M_{m,m}$, are $\geqslant 0$ and have the right levels). Moreover the only critical point in $\mathcal{O}$ of the restriction $\hat{A}$ of A to $M_{m,\hat{n}}$ is φ_m (because the first two critical levels $\lambda_1(m)$ and $\lambda_1(\hat{n})$ of $\hat{A}$ verify $\lambda_1(m) < \lambda < \lambda_1(\hat{n})$ by the choice of $\hat{n}$). Applying Lemma 14 to the component of $\mathcal{O}$ which contains v_1 and using the fact that any open connected subset of a manifold is arcwise connected, we get a path γ_3 in $\mathcal{O}$ from v_1 to φ_m. We then consider the path

$$\gamma_4(t) := \frac{|\gamma_3(t)|}{(\int_\Omega m|\gamma_3(t)|^p)^{1/p}}.$$

By the choice of $\hat{n}$, one has

$$1 = \int_\Omega \left(m\left(\gamma_3(t)^+\right)^p + \hat{n}\left(\gamma_3(t)^-\right)^p\right) \leqslant \int_\Omega \left(m\left(\gamma_3(t)^+\right)^p + m\left(\gamma_3(t)^-\right)^p\right) = \int_\Omega m\left|\gamma_3(t)\right|^p, \tag{3.8}$$

and consequently $\gamma_4(t)$ is well-defined. Moreover γ_4 goes from v_1 to φ_m and belongs to our original manifold $M_{m,n}$. Finally

$$A\left(\gamma_4(t)\right) = \int_\Omega \left|\nabla\gamma_4(t)\right|^p = \frac{\int_\Omega |\nabla\gamma_3(t)|^p}{\int_\Omega m|\gamma_3(t)|^p} < \lambda$$

since $\gamma_3(t) \in \mathcal{O}$ and, by (3.8), $\int_\Omega m|\gamma_3(t)|^p \geqslant 1$. The path γ_4 thus allows us to move from v_1 to φ_m in $M_{m,n}$ by staying at levels $< \lambda$. □

LEMMA 15. – *If $m_k \in L^r(\Omega)$ with $m_k^+ \not\equiv 0$ and if $m_k^+ \to 0$ in $L^r(\Omega)$, then $\lambda_1(m_k) \to +\infty$.*

Proof. – A consequence of the following calculation where we use the Sobolev inequality (and have dropped the index k):

$$\frac{1}{\lambda_1(m)} = \frac{\int_\Omega m|\varphi_m|^p}{\int_\Omega |\nabla\varphi_m|^p} \leqslant \frac{\int_\Omega m^+|\varphi_m|^p}{\int_\Omega |\nabla\varphi_m|^p} \leqslant C\left(\int_\Omega (m^+)^r\right)^{1/r}$$

where $C = C(\Omega, N, p, r)$. □

Remark 16. – If we reproduce the proof of Theorem 11 starting from $c(m,n)$ instead of λ, we conclude that the infimum in the minimax formula (2.14) is a minimum.

Remark 17. – Let us observe for later reference that the last step in the proof of Theorem 11 shows the following: given $u \in M_{m,n}$ with $u \geqslant 0$ and $A(u) < \mu$ for some μ, there exists a path in $M_{m,n}$ from u to φ_m, which is made of nonnegative functions and which remains at levels $< \mu$. Note that it is the introduction of the manifolds $M_{m,m}$ and $M_{m,\hat{n}}$ in the proof of Theorem 11 which allows us to keep control of the sign of the functions constituing the paths.

Remark 18. – Let us also observe for later reference that the proof of Theorem 11 shows the following: given $u \in W_0^{1,p}(\Omega)$ with $\int_\Omega m(u^+)^p > 0$, $\int_\Omega n(u^-)^p > 0$, and such that

$$A\big(u^+/B_{m,n}(u^+)^{1/p}\big) \leqslant \mu \quad \text{and} \quad A\big(-u^-/B_{m,n}(-u^-)^{1/p}\big) \leqslant \mu$$

for some μ, then there exists a path in $M_{m,n}$ which goes from φ_m to $-\varphi_n$, contains $u/B_{m,n}(u)^{1/p}$ and remains at levels $\leqslant \mu$. In particular $c(m,n) \leqslant \mu$.

COROLLARY 19. – *$c(m,n)$ is the minimum of the positive eigenvalues of* (1.3) *associated to eigenfunctions which change sign.*

COROLLARY 20. – *The eigenvalues* $\min\{\lambda_1(m), \lambda_1(n)\}$ *and* $\max\{\lambda_1(m), \lambda_1(n)\}$ *are isolated in the spectrum of* (1.3).

4. Some properties of the first nontrivial eigenvalue

In this section we study the dependence of the first nontrivial eigenvalue $c(m,n)$ of (1.3) with respect to the weights m, n. Continuity and monotonicity will be considered, as well as some homogeneity properties. The weights in this section will always be assumed to belong to $L^r(\Omega)$ and to satisfy (2.1).

We start by modifying a little bit the variational characterization (2.14) of $c(m,n)$ in order to allow a larger family of paths, which in addition depends a little less on the weights.

PROPOSITION 21. – *One has*

$$c(m,n) = \inf_{\gamma\in\Gamma_1} \max_{u\in\gamma[-1,+1]} A(u) \tag{4.1}$$

where

$$\Gamma_1 := \big\{\gamma \in C([-1,+1], M_{m,n})\colon\ \gamma(-1) \geqslant 0 \text{ and } \gamma(1) \leqslant 0\big\}.$$

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Proof. – Let us call d the right-hand side of (4.1). Clearly $d \leqslant c(m,n)$. Assume by contradiction $d < c(m,n)$. Take μ with $d < \mu < c(m,n)$ and choose a path $\gamma \in \Gamma_1$ which remains at levels $< \mu$. We will construct a path in Γ which also remains at levels $< \mu$. This will contradict the definition (2.14) of $c(m,n)$. To construct this path we first go from φ_m to $\gamma(-1)$ by using Remark 17, then we follow γ from $\gamma(-1)$ to $\gamma(+1)$, and finally we go from $\gamma(+1)$ to $-\varphi_n$ by a construction analogous to that of Remark 17. □

The continuous and monotone dependence of $c(m,n)$ are easy consequences of this proposition.

PROPOSITION 22. – *If* $(m_k,n_k) \to (m_0,n_0)$ *in* $L^r(\Omega) \times L^r(\Omega)$*, then* $c(m_k,n_k) \to c(m_0,n_0)$.

Proof. – We first prove the upper semicontinuity. Let $\varepsilon > 0$ and take $\gamma \in \Gamma$ such that

$$\max_t A\big(\gamma(t)\big) < c(m_0,n_0) + \varepsilon.$$

Since $B_{m,n}(\gamma(t))$ is continuous in its 3 arguments (m,n,t), we deduce that, for k sufficiently large,

$$\max_t A\big(\gamma(t)/B_{m_k,n_k}\big(\gamma(t)\big)^{1/p}\big) < c(m_0,n_0) + \varepsilon. \tag{4.2}$$

By Proposition 21, $c(m_k,n_k)$ is $\leqslant$ than the left-hand side of (4.2) and consequently

$$\limsup c(m_k,n_k) \leqslant c(m_0,n_0) + \varepsilon.$$

Since ε is arbitrary, the upper semicontinuity follows.

To prove the lower semicontinuity, suppose by contradiction that, for a subsequence, $c(m_k,n_k) \to c_0$ with $c_0 < c(m_0,n_0)$. Let $u_k \in M_{m_k,n_k}$ be a solution of (1.3) for $\lambda = c(m_k,n_k)$ and for the weights m_k, n_k. For a further subsequence, $u_k \to u_0$ weakly in $W_0^{1,p}(\Omega)$ and strongly in $L^{r'p}(\Omega)$; moreover $u_0 \in M_{m_0,n_0}$ and u_0 is a solution of (1.3) for $\lambda = c_0$ and for the weights m_0, n_0. Since $c_0 < c(m_0,n_0)$, Theorem 11 implies that either $c_0 = \lambda_1(m_0)$ and $u_0 = \varphi_{m_0}$, or $c_0 = \lambda_1(n_0)$ and $u_0 = -\varphi_{n_0}$. Consider the first case (similar argument in the other case). In that case $|u_k^-| \to 0$ and Lemma 3 applies to give

$$\int_\Omega n_k\big(u_k^-\big)^p \Big/ \int_\Omega \big|\nabla u_k^-\big|^p \to 0. \tag{4.3}$$

But multiplying by u_k^- the equation satisfied by u_k, one gets that the expression in (4.3) is equal to $1/c(m_k,n_k)$, which goes to $1/c_0 \neq 0$, a contradiction. □

PROPOSITION 23. – *If* $m \leqslant \hat{m}$ *and* $n \leqslant \hat{n}$*, then* $c(m,n) \geqslant c(\hat{m},\hat{n})$.

Proof. – If γ is a path admissible in formula (2.14) for $c(m,n)$, then $\int_\Omega (\hat{m}(\gamma(t)^+)^p + \hat{n}(\gamma(t)^-)^p) \geqslant 1$ and consequently

$$\hat{\gamma}(t) := \gamma(t)/B_{\hat{m},\hat{n}}\big(\gamma(t)\big)^{1/p}$$

is well-defined and is a path admissible in formula (4.1) for $c(\hat{m},\hat{n})$. Moreover $A(\hat{\gamma}(t)) \leqslant A(\gamma(t))$, and the conclusions follows. □

The monotonicity provided by Proposition 23 is generally not strict as is seen from the following

Example 24. – We start with two weights $\hat{m},\hat{n}$ satisfying (2.1) and let u be an eigenfunction associated to $c(\hat{m},\hat{n})$. We then construct m,n with

$$m \equiv \hat{m} \text{ in } \{u>0\}, \quad n \equiv \hat{n} \text{ in } \{u<0\}, \quad m \underset{\not\equiv}{\leqslant} \hat{m} \text{ in } \Omega \quad \text{and} \quad n \underset{\not\equiv}{\leqslant} \hat{n} \text{ in } \Omega.$$

Note that m,n still satisfy (2.1). By Proposition 23, $c(m,n) \geqslant c(\hat{m},\hat{n})$. But

$$-\Delta_p u = c(\hat{m},\hat{n})\big[\hat{m}(u^+)^{p-1} - \hat{n}(u^-)^{p-1}\big] = c(\hat{m},\hat{n})\big[m(u^+)^{p-1} - n(u^-)^{p-1}\big]$$

which implies, by Theorem 3.1, $c(m,n) \leqslant c(\hat{m},\hat{n})$. Consequently $c(m,n) = c(\hat{m}.\hat{n})$.

PROPOSITION 25. – *If $m \leqslant \hat{m}$, $n \leqslant \hat{n}$ and if*

$$\int_\Omega (\hat{m}-m)(u^+)^p + \int_\Omega (\hat{n}-n)(u^-)^p > 0 \tag{4.4}$$

for at least one eigenfunction u associated to $c(m,n)$, then $c(m,n) > c(\hat{m},\hat{n})$.

Proof. – Let us consider the case where the first integral in (4.4) is >0 (similar argument if the second integral is >0). So

$$\int_\Omega m(u^+)^p < \int_\Omega \hat{m}(u^+)^p \quad \text{and} \quad \int_\Omega n(u^-)^p \leqslant \int_\Omega \hat{n}(u^-)^p. \tag{4.5}$$

We start by considering the path $\gamma \in \Gamma$ constructed from the eigenfunction u of assumption (4.4) as in the proof of Theorem 11. With the notations of that proof, γ is made of a first part from u to φ_m consisting of γ_1 followed by γ_2 followed by γ_4, and a similar second part from u to $-\varphi_n$. Note that γ_1 lies at level $c(m,n)$ while γ_2 and γ_4 lie at levels $<c(m,n)$ with the exception of the starting point of γ_2 whose level is $c(m,n)$. We then take the normalization of γ for the weights $\hat{m},\hat{n}$:

$$\hat{\gamma}(t) := \gamma(t)/B_{\hat{m},\hat{n}}\big(\gamma(t)\big)^{1/p}.$$

Since $B_{\hat{m},\hat{n}}(\gamma(t)) \geqslant 1$, $\hat{\gamma}(t)$ is well defined and clearly $\hat{\gamma}(t) \in M_{\hat{m},\hat{n}}$. To estimate the levels of A along $\hat{\gamma}(t)$, we distinguish two cases in relation with the second inequality in (4.5): either $\int_\Omega n(u^-)^p < \int_\Omega \hat{n}(u^-)^p$, or $\int_\Omega n(u^-)^p = \int_\Omega \hat{n}(u^-)^p$. In the first case a direct calculation shows that $A(\hat{\gamma}(t)) < c(m,n)$ for all t. This clearly implies the conclusion $c(\hat{m},\hat{n}) < c(m,n)$. In the second case the same calculation shows that $A(\hat{\gamma}(t)) < c(m,n)$ for all t except at the point $v := -u^-/B_{\hat{m},\hat{n}}(-u^-)^{1/p}$ where $A(v) = c(m,n)$. The path $\hat{\gamma}$ goes from $\varphi_m/B_{\hat{m},\hat{n}}(\varphi_m)^{1/p}$ to $-\varphi_n/B_{\hat{m},\hat{n}}(-\varphi_n)^{1/p}$, which both lie at levels $<c(m,n)$. We then apply Remark 17 to extend $\hat{\gamma}$ into a path $\hat{\hat{\gamma}}$ which goes from $\varphi_{\hat{m}}$ to $-\varphi_{\hat{n}}$ and

which remains at levels $< c(m,n)$ with the exception of v where the level is $c(m,n)$. Assume now by contradiction that $c(\hat m,\hat n) = c(m,n)$. Then we can apply Lemma 26 below to the restriction $\hat A$ of A to the manifold $M_{\hat m,\hat n}$ to conclude that $\hat{\hat\gamma}$ contains a critical point of $\hat A$ at level $c(\hat m,\hat n)$. Consequently v must be this critical point. But this is impossible since v does not change sign. □

The lemma below guarantees that in a mountain pass situation, any minimizing path contains a critical point at the mountain pass level. It is stated in the general setting of the manifold (2.3).

LEMMA 26. – *Let $E, g, M, f, \tilde f$ be as in* (2.3). *Let $u, v \in M$ with $u \neq v$ and assume that H defined in* (2.4) *is nonempty and that* (2.5) *holds. Suppose that $h \in H$ is such that*

$$\max_{u\in h[-1,+1]} \tilde f(u) = c,$$

where c is defined in (2.5). *Then there exists $u \in h[-1,+1]$ with $\tilde f(u) = c$ and which is a critical point of $\tilde f$.*

Proof. – Assume by contradiction that $C := \{h(t)\colon\ t \in [-1,+1]$ and $\tilde f(h(t)) = c\}$ does not contain any critical point of $\tilde f$. We apply the deformation lemma of [28] (Lemma 3.7) to our functional $\tilde f$ on the component of M which contains u, v. This yields another path $l \in H$ such that $\tilde f(l(t)) \leqslant \tilde f(h(t))$, with strict inequality on C. Thus $\tilde f(l(t)) < c$ for all t, which contradicts the definition (2.5) of c. □

Remark 27. – A direct proof of a version of Lemma 26 can be found in [14], which uses only Ekeland's variational principle. In this version E is uniformly convex and assumption (2.5) is weakened into (2.6). Note also that Lemma 26 is not needed in the above proof of Proposition 25 if the two integrals in assumption (4.4) are > 0.

Remark 28. – A different proof of Proposition 25 can be given, which does not use Lemma 26. It goes roughly as follows. Assuming (4.5), one picks a positive nodal domain Ω^+ of u and a negative nodal domain Ω^- of u such that $\int_{\Omega^+} m(u^+)^p < \int_{\Omega^+} \hat m(u^+)^p$ and $\int_{\Omega^-} n(u^-)^p \leqslant \int_{\Omega^-} \hat n(u^-)^p$. Taking $u|_{\Omega^+}$ and $u|_{\Omega^-}$ as testing functions in the equation satisfied by u, one then deduces that $\lambda_1(\hat m,\Omega^+) < c(m,n)$ and $\lambda_1(\hat n,\Omega^-) \leqslant c(m,n)$. The idea now is to argue as in [15] to increase Ω^+ and to decrease Ω^- so as to get two new disjoint open sets in Ω, Ω_1 and Ω_2, such that $\lambda_1(\hat m,\Omega_1) < c(m,n)$ and $\lambda_1(\hat n,\Omega_2) < c(m,n)$. Putting $v := \varphi_{\hat m,\Omega_1} - \varphi_{\hat n,\Omega_2}$ and $w := v/B_{\hat m,\hat n}(v)^{1/p}$, one then uses Remark 18 to construct from w a path in $M_{\hat m,\hat n}$ which goes from $\varphi_{\hat m}$ to $-\varphi_{\hat n}$ and which remains at levels $< c(m,n)$. This implies the conclusion $c(\hat m,\hat n) < c(m,n)$.

COROLLARY 29. – *If $m \leqslant \hat m$ and $n \leqslant \hat n$ with either $m < \hat m$ on $\{m > 0\}$ or $n < \hat n$ on $\{n > 0\}$, then $c(m,n) > c(\hat m,\hat n)$.*

Proof. – Let u be an eigenfunction associated to $c(m,n)$. Then $\int_\Omega m(u^+)^p > 0$ and $\int_\Omega n(u^-)^p > 0$, and consequently $\{m > 0\} \cap \{u > 0\}$ has positive measure, as well as $\{n > 0\} \cap \{u < 0\}$. It then follows easily that the assumptions of Corollary 29 imply (4.4). □

For $m \equiv n$ (and m bounded), the result of Corollary 29, which then reads $\lambda_2(m) > \lambda_2(\hat m)$, was obtained in [6].

Remark 30. – Let us agree to say that the unique continuation property (UCP) holds for $-\Delta_p$ if, for any $a \in L^r(\Omega)$, any nontrivial solution $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ of an equation like $-\Delta_p u = a(x)|u|^{p-2}u$ in Ω does not vanish on a set of positive measure. If (UCP) holds then $m \leqslant \hat{m}$, $n \leqslant \hat{n}$ together with the local condition

$$\left|\left\{x \in \Omega\colon\ m(x) < \hat{m}(x) \text{ and } n(x) < \hat{n}(x)\right\}\right| > 0$$

imply (4.4) and consequently $c(m,n) > c(\hat{m},\hat{n})$. (UCP) holds when $p = 2$ (cf. [33,18, 30]) or $N = 1$ (cf. e.g. [36]), but it is an open question whether it holds when $p \neq 2$ and $N \geqslant 2$, even for $a(x)$ constant.

To conclude this section, let us observe that definition (2.14) clearly implies that $c(m,n)$ is homogeneous of degree -1:

$$c(sm,sn) = \frac{1}{s}c(m,n) \quad \text{for } s > 0, \tag{4.6}$$

Some sort of separate sub-homogeneity also holds, which will be useful later:

PROPOSITION 31. – *If* $0 < s < \hat{s}$, *then*

$$c(sm,n) > c(\hat{s}m,n) \quad \textit{and} \quad c(m,sn) > c(m,\hat{s}n). \tag{4.7}$$

Proof. – We will deal with the first inequality in (4.7) (similar argument for the second one). Let u be an eigenfunction in $M_{sm,n}$ associated to $c(sm,n)$ and let γ be the path in $M_{sm,n}$ from φ_{sm} to $-\varphi_n$ constructed from u as in the proof of Theorem 11. The path $\hat{\gamma}(t) = (\frac{s}{\hat{s}})^{1/p}\gamma(t)^+ - \gamma(t)^-$ is then admissible in definition (2.14) of $c(\hat{s}m,n)$ and we have

$$A(\hat{\gamma}(t)) = \frac{s}{\hat{s}}\int_\Omega |\nabla\gamma(t)^+|^p + \int_\Omega |\nabla\gamma(t)^-|^p \leqslant A(\gamma(t)),$$

with strict inequality if $\gamma(t)^+ \not\equiv 0$. So the path $\hat{\gamma}$ goes in $M_{\hat{s}m,n}$ from $\varphi_{\hat{s}m}$ to $-\varphi_n$ and remains at levels $< c(sm,n)$ except at the point $v := -u^-/B_{sm,n}(-u^-)^{1/p}$ where the level is $c(sm,n)$. It follows that $c(\hat{s}m,n) \leqslant c(sm,n)$. Assume now by contradiction that $c(\hat{s}m,n) = c(sm,n)$. We can then apply Lemma 26 (or its version referred to in Remark 27) to the path $\hat{\gamma}$ in the manifold $M_{\hat{s}m,n}$ to conclude that v must be a critical point of the restriction of A to $M_{\hat{s}m,n}$ at level $c(\hat{s}m,n)$. But this is impossible since v does not change sign. □

Remark 32. – If $m \geqslant 0$ in Ω, then, the first inequality in (4.7) follows directly from Corollary 4.9. In general however this inequality should not be looked at as a property of monotonicity since, when m changes sign in Ω, sm and $\hat{s}m$ are not comparable. This last observation can also be made for the classical formulas $\lambda_1(sm) > \lambda_1(\hat{s}m)$ and $\lambda_2(sm) > \lambda_2(\hat{s}m)$ where $0 < s < \hat{s}$.

5. Fučik spectrum with weights

Let $m, n \in L^r(\Omega)$ with r as before. Unless otherwise stated, we also assume (2.1). The Fučik spectrum is thus defined as the set $\Sigma = \Sigma(m, n)$ of those $(\alpha, \beta) \in \mathbb{R}^2$ such that (1.4) has a nontrivial solution.

Σ clearly contains the lines $\lambda_1(m) \times \mathbb{R}$ and $\mathbb{R} \times \lambda_1(n)$, and also, if $m^- \not\equiv 0$ (resp. $n^- \not\equiv 0$), $\lambda_{-1}(m) \times \mathbb{R}$ (resp. $\mathbb{R} \times \lambda_{-1}(n)$). These lines are in fact exactly made of those $(\alpha, \beta) \in \Sigma$ for which (1.4) admits a solution which does not change sign. It will be convenient to denote by $\Sigma^* = \Sigma^*(m, n)$ the set Σ without these $2, 3$ or 4 trivial lines. From the properties of the first eigenvalue recalled in the introduction also follows that if $(\alpha, \beta) \in \Sigma^*$ with $\alpha \geqslant 0$ and $\beta \geqslant 0$ (resp. $\alpha \leqslant 0$ and $\beta \leqslant 0$, $\alpha \geqslant 0$ and $\beta \leqslant 0$, $\alpha \leqslant 0$ and $\beta \geqslant 0$), then $\alpha > \lambda_1(m)$ and $\beta > \lambda_1(n)$ (resp. $\alpha < \lambda_{-1}(m)$ and $\beta < \lambda_{-1}(n)$, $\alpha > \lambda_1(m)$ and $\beta < \lambda_{-1}(n)$, $\alpha < \lambda_{-1}(m)$ and $\beta > \lambda_1(n)$).

We will start by looking at the part of Σ^* which lies in $\mathbb{R}^+ \times \mathbb{R}^+$. The case of the other quadrants will be considered briefly at the end of the section.

THEOREM 33. – *For any $s > 0$, the line $\beta = s\alpha$ in the (α, β) plane intersects $\Sigma^* \cap (\mathbb{R}^+ \times \mathbb{R}^+)$. Moreover the first point in this intersection is given by $\alpha(s) = c(m, sn)$, $\beta(s) = s\alpha(s)$, where $c(\cdot, \cdot)$ is defined by* (2.14)*.*

Proof. – The results of Sections 2 and 3 clearly imply that if $(\alpha, \beta) \in \mathbb{R}^+ \times \mathbb{R}^+$, then (α, β) belongs to Σ^* and is such that no element of Σ^* belongs to the segment $[(0, 0), (\alpha, \beta)[$ if and only if $c(\alpha m, \beta n) = 1$. Since, by (4.6), $c(\alpha m, \alpha s n) = c(m, sn)/\alpha$ for $\alpha > 0$, the conclusion follows. □

Letting $s > 0$ vary, we get in this way a first curve $\mathcal{C} := \{(\alpha(s), \beta(s)) \colon s > 0\}$ in $\Sigma^* \cap (\mathbb{R}^+ \times \mathbb{R}^+)$. Here are some properties of this curve.

PROPOSITION 34. – *The functions $\alpha(s)$ and $\beta(s)$ in Theorem* 33 *are continuous. Moreover $\alpha(s)$ is strictly decreasing and $\beta(s)$ is strictly increasing. One also has that $\alpha(s) \to +\infty$ if $s \to 0$ and $\beta(s) \to +\infty$ if $s \to +\infty$.*

Proof. – Continuity follows from Proposition 22. The monotonicity of $\alpha(s)$ follows from Proposition 31. The monotonicity of $\beta(s)$ also follows from Proposition 4.11 since $\beta(s) = c(m/s, n)$. Finally let us assume by contradiction that $\alpha(s)$ remains bounded as $s \to 0$. Then $\beta(s) = s\alpha(s) \to 0$, which is impossible since $\beta(s) > \lambda_1(n)$ for all $s > 0$. Similar argument for the behaviour of $\beta(s)$ as $s \to +\infty$. □

The curve $\mathcal{C}$ is thus an hyperbolic like curve in $\mathbb{R}^+ \times \mathbb{R}^+$, with asymptotes $\alpha_\infty \times \mathbb{R}$ and $\mathbb{R} \times \beta_\infty$, where $\alpha_\infty := \lim_{s \to \infty} \alpha(s)$ and $\beta_\infty := \lim_{s \to o} \beta(s)$. Note that Proposition 34 implies that the lines $\lambda_1(m) \times \mathbb{R}$ and $\mathbb{R} \times \lambda_1(n)$ are isolated in $\Sigma \cap (\mathbb{R}^+ \times \mathbb{R}^+)$, in the sense that there does not exists $(\alpha_k, \beta_k) \in \Sigma^* \cap (\mathbb{R}^+ \times \mathbb{R}^+)$ such that $\alpha_k \to \alpha_0$ and $\beta_k \to \beta_0$ with (α_0, β_0) in one of these two lines. Note also that Proposition 34 implies that the curve $\mathcal{C}$ above coincides with the curve constructed in [19] when $p = 2$ and $m \equiv n \equiv 1$ and in [15] when $m \equiv n \equiv 1$.

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We now investigate the asymptotic behaviour of this first curve, i.e. the values of α_∞ and β_∞. Let us define

$$\bar{\alpha} := \inf\left\{\int_\Omega |\nabla u^+|^p \colon\ u \in W_0^{1,p}(\Omega),\ \int_\Omega m(u^+)^p = 1 \text{ and } \int_\Omega n(u^-)^p > 0\right\}, \tag{5.1}$$

$$\bar{\beta} := \inf\left\{\int_\Omega |\nabla u^-|^p \colon\ u \in W_0^{1,p}(\Omega),\ \int_\Omega n(u^-)^p = 1 \text{ and } \int_\Omega m(u^+)^p > 0\right\}. \tag{5.2}$$

Clearly $\bar{\alpha} \geqslant \lambda_1(m)$ and $\bar{\beta} \geqslant \lambda_1(n)$. Let us also recall that the support of a measurable function $u(x)$ in Ω is defined as supp $u := \bar{\Omega} \setminus \mathcal{O}$, where $\mathcal{O}$ is the largest open set in Ω such that $u = 0$ a.e. in $\mathcal{O}$.

PROPOSITION 35. – *The asymptotic values α_∞ and β_∞ are equal to $\bar{\alpha}$ and $\bar{\beta}$ respectively. Moreover if $p \leqslant N$, then $\bar{\alpha} = \lambda_1(m)$ and $\bar{\beta} = \lambda_1(n)$. If $p > N$, then* (i) *$\bar{\alpha} = \lambda_1(m)$ if supp n^+ intersects $\partial\Omega$ but $\bar{\alpha} > \lambda_1(m)$ if supp n^+ is compact in Ω,* (ii) *$\bar{\beta} = \lambda_1(n)$ if supp m^+ intersects $\partial\Omega$ but $\bar{\beta} > \lambda_1(n)$ if supp m^+ is compact in Ω.*

So, when $p \leqslant N$, whatever the weights (satisfying (2.1)), the first curve $\mathcal{C}$ is asymptotic to the trivial lines $\lambda_1(m) \times \mathbb{R}$ and $\mathbb{R} \times \lambda_1(n)$. On the contrary, when $p > N$, the asymptotic behaviour of $\mathcal{C}$ depends on the supports of m^+ and n^+. Note that the influence of the supports of the weights in the asymptotic behaviour of the first curve was already observed in the semilinear ODE case $N = 1$, $p = 2$, in [2] by using the shooting method. Note also that the present distinction between the cases $p \leqslant N$ and $p > N$ is of the same nature as that observed in [10] in the study of the antimaximum principle and of the Fučik spectrum for the Neumann p-laplacian without weight.

Proof of Proposition 35. – We first show that $\alpha_\infty = \bar{\alpha}$ (similar proof for β_∞). Let $(\alpha, \beta) \in \mathcal{C}$ and let u be a corresponding nontrivial solution of (1.4). Then

$$\alpha \int_\Omega m(u^+)^p = \int_\Omega |\nabla u^+|^p > 0, \qquad \beta \int_\Omega n(u^-)^p = \int_\Omega |\nabla u^-|^p > 0.$$

Consequently $\alpha \geqslant \bar{\alpha}$, which implies $\alpha_\infty \geqslant \bar{\alpha}$. Assume now by contradiction that $\alpha_\infty > \bar{\alpha}$. Then there exists $u \in W_0^{1,p}(\Omega)$ with $\int_\Omega m(u^+)^p = 1$, $\int_\Omega n(u^-)^p > 0$ and $\bar{\alpha} \leqslant \int_\Omega |\nabla u^+|^p < \alpha_\infty$. Since $\alpha_\infty \leqslant \alpha(s) = c(m, sn)$ $\forall s > 0$, we have, for this u,

$$\int_\Omega |\nabla u^+|^p < c(m, sn) \quad \forall s > 0. \tag{5.3}$$

We then choose $s > 0$ such that

$$\int_\Omega |\nabla u^-|^p \Big/ \int_\Omega sn(u^-)^p = \int_\Omega |\nabla u^+|^p$$

and apply Remark 18 for the weights m and sn and for $\mu = \int_\Omega |\nabla u^+|^p$. This yields $c(m, sn) \leqslant \mu$, which contradicts (5.3).

We now consider the case $p \leqslant N$ and show that $\bar{\alpha} = \lambda_1(m)$. This will clearly follow if we prove the existence of functions which are admissible in (5.1) and converge to φ_m. The construction of such functions is inspired from [19,10,29]. It consists in starting from φ_m and "digging a little hole" in order to have room to introduce a suitable negative part. To do so we first consider the following functions on $\mathbb{R}^N$: for $p < N$

$$a_k(x) = \begin{cases} 1 & \text{if } |x| \geqslant 1/k, \\ 2k|x| - 1 & \text{if } 1/2k < |x| < 1/k, \\ 0 & \text{if } |x| \leqslant 1/2k, \end{cases}$$

while for $p = N$,

$$a_k(x) = \begin{cases} 1 - 2/k & \text{if } |x| \geqslant 1/k, \\ |x|^{\delta_k} - 1/k & \text{if } (1/k)^{1/\delta_k} < |x| < 1/k, \\ 0 & \text{if } |x| \leqslant (1/k)^{1/\delta_k}, \end{cases}$$

where $\delta_k \in]0,1[$ is chosen so that $(1/k)^{\delta_k} = 1 - 1/k$. A simple calculation shows that a_k converges to the constant function 1 in $W^{1,p}_{\text{loc}}(\mathbb{R}^N)$ as $k \to \infty$; it is here that the assumption $p \leqslant N$ enters. It follows that for any given $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $x_0 \in \Omega$, the function $u(x)a_k(x - x_0)$ converges to u in $W_0^{1,p}(\Omega)$ as $k \to \infty$ and vanishes in a neighborhood of x_0. We apply this construction to φ_m, taking for x_0 a point in $(\operatorname{supp} n^+) \cap \Omega$. This yields a function $v_k(x) := \varphi_m(x)a_k(x - x_0)$ which is $\geqslant 0$ and vanishes on, say, the ball $B(x_0, \varepsilon_k) \subset \Omega$, $\varepsilon_k > 0$. Regularizing the characteristic function of $(\operatorname{supp} n^+) \cap B(x_0, \varepsilon_k/2)$, one gets a function $w_k \in C_c^\infty(B(x_0, \varepsilon_k))$ with $w_k \geqslant 0$ and $\int_\Omega n(w_k)^p > 0$. It follows that the function

$$u_k := v_k - w_k/k\|w_k\|_{1,p}$$

converges to φ_m in $W_0^{1,p}(\Omega)$ and, after normalization, is admissible in definition (5.1) of $\bar{\alpha}$. We conclude in this way that $\bar{\alpha} \leqslant \lambda_1(m)$ and consequently $\bar{\alpha} = \lambda_1(m)$.

We now consider the case where $p > N$ and the support of n^+ intersects $\partial\Omega$. We will show that here again $\bar{\alpha} = \lambda_1(m)$. The idea is as before to start with φ_m and to introduce a suitable negative part which however will now be located near $\partial\Omega$. Let us define

$$\Omega_\varepsilon := \{x \in \Omega\colon \operatorname{dist}(x, \Omega^c) > \varepsilon\}$$

and consider the corresponding first eigenvalue $\lambda_1(m, \Omega_\varepsilon)$ as well as its associated positive normalized eigenfunction $\varphi_m(\Omega_\varepsilon)$. Note that these are well defined for $\varepsilon > 0$ sufficiently small since $m^+ \not\equiv 0$ in Ω (cf. (2.1)). Moreover the argument of Lemma 5 in [19] immediately extends to the present situation to show that as $\varepsilon \to 0$, $\lambda_1(m, \Omega_\varepsilon) \to$

$\lambda_1(m,\Omega)$ and $\varphi_m(\Omega_\varepsilon) \to \varphi_m(\Omega)$ in $W_0^{1,p}(\Omega)$ (here $\varphi_m(\Omega_\varepsilon)$ is as usual extended by zero outside Ω_ε). We now use the assumption on the support of n^+ to deduce that for any $\varepsilon > 0$, $n^+ \not\equiv 0$ on $\Omega \setminus \bar{\Omega}_\varepsilon$. This allows us by a regularization procedure as before to construct $w_\varepsilon \in C_c^\infty(\Omega \setminus \bar{\Omega}_\varepsilon)$ with $w_\varepsilon \geqslant 0$ and $\int_\Omega n(w_\varepsilon)^p > 0$. It follows that the function

$$u_k := \varphi_m(\Omega_\varepsilon) - \varepsilon w_\varepsilon / \|w_\varepsilon\|_{1,p}$$

converges to $\varphi_m(\Omega)$ in $W_0^{1,p}(\Omega)$ and is admissible in definition (5.1) of $\bar{\alpha}$. We conclude in this way that $\bar{\alpha} \leqslant \lambda_1(m)$ and consequently $\bar{\alpha} = \lambda_1(m)$. Note that $p > N$ has not been used in the preceding argument.

We finally consider the case where $p > N$ and the support of n^+ is compact in Ω. We will show that $\bar{\alpha} > \lambda_1(m)$. Assume by contradiction $\bar{\alpha} = \lambda_1(m)$ and let u_k be a minimizing sequence in definition (5.1) of $\bar{\alpha}$. For a subsequence, u_k^+ converges weakly in $W_0^{1,p}(\Omega)$ and strongly in $C(\bar{\Omega})$ to a function $v \in W_0^{1,p}(\Omega)$ which is $\geqslant 0$ and satisfies $\int_\Omega |\nabla v|^p \leqslant \lambda_1(m)$ and $\int_\Omega m v^p = 1$. Consequently $v = \varphi_m$. Since $\varphi_m \geqslant$ some $\varepsilon > 0$ on the compact set supp n^+, we deduce that $u_k^+ \geqslant \varepsilon/2$ on supp n^+ for k sufficiently large. Consequently, for those k, $u_k^- = 0$ on supp n^+, which implies $\int_\Omega n(u_k^-)^p \leqslant 0$, a contradiction with the fact that u_k is admissible in definition (5.1) of $\bar{\alpha}$.

The properties of $\bar{\beta}$ are of course proved in a similar way. □

We now briefly indicate another variational characterization of $\bar{\alpha}$ in the case $p > N$. A similar result of course holds for $\bar{\beta}$. Recall that $W_0^{1,p}(\Omega) \subset C(\bar{\Omega})$ in the case under consideration.

PROPOSITION 36. – *Suppose $p > N$. Then*

$$\bar{\alpha} = \inf\left\{ \int_\Omega |\nabla u|^p \colon u \in W_0^{1,p}(\Omega),\ \int_\Omega m|u|^p = 1 \text{ and } u \text{ vanishes} \right.$$
$$\left. \text{somewhere on } \operatorname{supp} n^+ \right\}. \tag{5.4}$$

The infimum in (5.4) *is achieved. Moreover if u is a minimizer in* (5.4)*, then u does not change sign in Ω and u vanishes in at most one point in* $(\operatorname{supp} n^+) \cap \Omega$.

Proof. – Let us call $\bar{\bar{\alpha}}$ the right-hand side of (5.4). We distinguish two cases: (i) $\operatorname{supp} n^+$ intersects $\partial\Omega$, (ii) $\operatorname{supp} n^+$ is compact in Ω. In case (i), any $u \in W_0^{1,p}(\Omega)$ vanishes somewhere on $\operatorname{supp} n^+$ and consequently $\bar{\bar{\alpha}} = \lambda_1(m)$. The conclusions of Proposition 36 then follow easily, using Proposition 35. So from now on in the proof of Proposition 36, we suppose that we are in case (ii).

We first show that $\bar{\alpha} \leqslant \bar{\bar{\alpha}}$. Clearly one can restrict oneself to nonnegative functions in the definition of $\bar{\bar{\alpha}}$. Let $u \geqslant 0$ be admissible in the definition of $\bar{\bar{\alpha}}$. Since $u(x_0) = 0$ for some $x_0 \in \operatorname{supp} n^+ \subset \Omega$, $(u - \frac{1}{k})^+$, which converges to u in $W_0^{1,p}(\Omega)$ as $k \to \infty$, vanishes on a neighbourhood of x_0. We can then construct in this neighbourhood a small negative part so as to satisfy the last constraint in definition (5.1) of $\bar{\alpha}$. The construction here goes by regularization and is identical to that in the proof of Proposition 35. Arguing in this way one gets $\bar{\alpha} \leqslant \bar{\bar{\alpha}}$. We now show that $\bar{\bar{\alpha}} \leqslant \bar{\alpha}$. Let u be admissible in definition (5.1) of $\bar{\alpha}$. Then u is $\leqslant 0$ somewhere on $\operatorname{supp} n^+$ and consequently u^+ vanishes somewhere on supp n^+. The conclusion $\bar{\bar{\alpha}} \leqslant \bar{\alpha}$ then follows by considering u^+ in the definition of $\bar{\bar{\alpha}}$.

The infimum in (5.4) is clearly achieved. Let u be a minimizer and assume by contradiction that u vanishes in at least two points x_1 and x_2 in $(\operatorname{supp} n^+) \cap \Omega$. Then $v = |u|$ is also a minimizer which vanishes at x_1 and x_2. Arguing as in the proof of Lemma 3.1 in [10], one first observes that, for each $i = 1, 2$, v is a minimizer for

$$\bar{\alpha} = \inf\left\{ \int_\Omega |\nabla w|^p \colon\ w \in F_i \text{ and } \int_\Omega m|w|^p = 1 \right\},$$

where F_i is the subspace of $W_0^{1,p}(\Omega)$ made of the functions vanishing at x_i. Applying Lagrange's multiplier rule in each F_i and using the fact that any function in $W_0^{1,p}(\Omega)$ can be written as the sum of a function in F_1 and a function in F_2, one gets that v satisfies

$$\int_\Omega |\nabla v|^{p-2} \nabla v \, \nabla w = \bar{\alpha} \int_\Omega m|v|^{p-2} v \, w \quad \forall w \in W_0^{1,p}(\Omega),$$

i.e. that v is an eigenfunction associated to $\bar{\alpha}$. Since we are in case (ii), $\bar{\alpha} > \lambda_1(m)$ by Proposition 35, and consequently v must change sign, which is impossible since $v = |u|$.

It remains to see that every minimizer u in (5.4) does not change sign in Ω. One starts by verifying, as in the proof of Lemma 3.1 in [10], that if $u^+ \not\equiv 0$ in Ω, then $\int_\Omega m(u^+)^p > 0$ and $u^+/(\int_\Omega m(u^+)^p)^{1/p}$ is again a minimizer in (5.4). So, by what has already been proved, u^+ vanishes at at most one point in $(\operatorname{supp} n^+) \cap \Omega = \operatorname{supp} n^+$, and consequently $u \geqslant 0$ on $\operatorname{supp} n^+$. Now, if we also have $u^- \not\equiv 0$ in Ω, then the same argument applies to $v = -u$ and yields $u \leqslant 0$ on supp n^+. But then $u \equiv 0$ on supp n^+, which contradicts the fact that u vanishes in at most one point of $(\operatorname{supp} n^+) \cap \Omega = \operatorname{supp} n^+$. □

To conclude this section we consider the distribution of Σ^* in the other quadrants of $\mathbb{R} \times \mathbb{R}$. From now on we do not assume anymore below that m, n satisfy (2.1).

PROPOSITION 37. – *$\Sigma^*(m,n)$ intersects $\mathbb{R}^+ \times \mathbb{R}^+$ (resp. $\mathbb{R}^- \times \mathbb{R}^-$, $\mathbb{R}^+ \times \mathbb{R}^-$, $\mathbb{R}^- \times \mathbb{R}^+$) if and only if m^+ and $n^+ \not\equiv 0$, (resp. m^- and $n^- \not\equiv 0$, m^+ and $n^- \not\equiv 0$, m^- and $n^+ \not\equiv 0$).*

Proof. – The necessary conditions follow from the fact that, if $(\alpha, \beta) \in \Sigma^*$, then, for u a corresponding solution of (1.4),

$$0 < \int_\Omega |\nabla u^+|^p = \alpha \int_\Omega m(u^+)^p \quad \text{and} \quad 0 < \int_\Omega |\nabla u^-|^p = \beta \int_\Omega n(u^-)^p.$$

To prove the sufficient conditions, let us consider for instance $\mathbb{R}^+ \times \mathbb{R}^-$ (similar arguments in the other quadrants). We have that $(\alpha, \beta) \in \Sigma^*(m,n) \cap (\mathbb{R}^+ \times \mathbb{R}^-)$ if and only if $(\alpha, -\beta) \in \Sigma^*(m, -n) \cap (\mathbb{R}^+ \times \mathbb{R}^+)$. The assumption m^+ and $n^- \not\equiv 0$ means that the two weights $m, -n$ satisfy m^+ and $(-n)^+ \not\equiv 0$, i.e. (2.1). Consequently

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Theorem 5.1 implies that $\Sigma^*(m,-n) \cap (\mathbb{R}^+ \times \mathbb{R}^+)$ is nonempty, and consequently $\Sigma^*(m,n) \cap \mathbb{R}^+ \times \mathbb{R}^-$ is nonempty. □

COROLLARY 38. – *If m and n both change sign in Ω, then each of the four quadrants in the (α,β) plane contains a first curve of Σ^*.*

In the semilinear ODE case $N=1$, $p=2$, the results of Proposition 37 and Corollary 38 were derived recently in [2] by using the shooting method.

The result of Proposition 35 on the asymptotic behaviour of the first curve in $\mathbb{R}^+ \times \mathbb{R}^+$ of course extends to the other quadrants. For instance we have

COROLLARY 39. – *Suppose m^+ and $n^- \not\equiv 0$, and let $\mathcal{C}^{+,-}$ be the first curve of $\Sigma^*(m,n)$ in $\mathbb{R}^+ \times \mathbb{R}^-$. If $p \leqslant N$, then $\mathcal{C}^{+,-}$ is asymptotic to the lines $\lambda_1(m) \times \mathbb{R}$ and $\mathbb{R} \times \lambda_{-1}(n)$. If $p > N$, then* (i) *$\mathcal{C}^{+,-}$ is asymptotic to the line $\lambda_1(m) \times \mathbb{R}$ if* $\operatorname{supp} n^-$ *intersects $\partial\Omega$ but is not asymptotic to that line if* $\operatorname{supp} n^-$ *is compact in Ω,* (ii) *$\mathcal{C}^{+,-}$ is asymptotic to the line $\mathbb{R} \times \lambda_{-1}(n)$ if* $\operatorname{supp} m^+$ *intersects $\partial\Omega$ but is not asymptotic to that line if* $\operatorname{supp} m^+$ *is compact in Ω.*

Proof. – Observe, as in the proof of Proposition 37, that the first curve to $\Sigma^*(m,n)$ in $\mathbb{R}^+ \times \mathbb{R}^-$ is symmetric to the first curve of $\Sigma^*(m,-n)$ in $\mathbb{R}^+ \times \mathbb{R}^+$. Applying Proposition 35 to the latter then yields the conclusion. □

6. Nonresonance of the type "between the first two eigenvalues"

In this section, we study the solvability of the Dirichlet problem (1.5) under assumptions on the asymptotic behaviour of the quotients $f(x,s)/|s|^{p-2}s$ and $pF(x,s)/|s|^p$ which generalize the classical condition that for a.e. $x \in \Omega$, the limits at infinity of these quotients lie between the first two eigenvalues. Existence, unicity, as well as examples of nonunicity will be considered.

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying the growth condition

$$\left|f(x,s)\right| \leqslant a(x)\,|s|^{p-1} + b(x) \tag{6.1}$$

for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$. Here $a \in L^r(\Omega)$ and $b \in L^{p'}(\Omega)$, where r is as before (i.e. $r > N/p$ if $p \leqslant N$ and $r = 1$ if $p > N$) and p' is the Hölder conjugate. We assume that the L^r functions $\gamma_\pm$ and $\Gamma_\pm$ defined by

$$\gamma_\pm(x) := \liminf_{s\to\pm\infty} \frac{f(x,s)}{|s|^{p-2}s} \leqslant \limsup_{s\to\pm\infty} \frac{f(x,s)}{|s|^{p-2}s} := \Gamma_\pm(x) \tag{6.2}$$

have nontrivial positive parts and satisfy

$$\lambda_1(\gamma_+) \leqslant 1, \quad \lambda_1(\gamma_-) \leqslant 1, \quad c(\Gamma_+,\Gamma_-) \geqslant 1. \tag{f}$$

Here $c(\Gamma_+,\Gamma_-)$ is the eigenvalue of (1.3) considered in Sections 2–4. We also assume that the L^r functions $\delta_\pm$ and $\Delta_\pm$ defined by

$$\delta_\pm(x) := \liminf_{s\to\pm\infty} \frac{pF(x,s)}{|s|^p} \leqslant \limsup_{s\to\pm\infty} \frac{pF(x,s)}{|s|^p} := \Delta_\pm(x) \tag{6.3}$$

have nontrivial positive parts and satisfy

$$\lambda_1(\delta_+) < 1, \quad \lambda_1(\delta_-) < 1, \quad c(\Delta_+, \Delta_-) > 1. \tag{F_1}$$

Some uniformity with respect to x is also required in (6.3), which is made precise in (6.13) below. Note that one clearly has

$$\gamma_\pm(x) \leqslant \delta_\pm(x) \leqslant \Delta_\pm(x) \leqslant \Gamma_\pm(x) \text{ a.e. in } \Omega. \tag{6.4}$$

THEOREM 40. – *Assume* (6.1)*,* (f) *and* (F_1)*. Then problem* (1.5) *admits at least one solution u in* $W_0^{1,p}(\Omega)$.

The result of Theorem 40 is in the line of those in [12] ($p = 2$ and usual spectrum), [16] ($p = 2$, $N = 1$ and Fučik spectrum), [19] ($p = 2$ and Fučik spectrum), [15] ($1 < p < \infty$ and Fučik spectrum). The main difference comes from the fact that in all these works, the hypothesis takes the form of pointwise inequalities on the functions $\gamma_\pm, \Gamma_\pm, \delta_\pm, \Delta_\pm$. For instance in [19] it is assumed that for one point (α, β) in the first curve of the Fučik spectrum of $-\Delta_p$ (without weight), one has

$$\begin{cases} \lambda_1 \leqslant \gamma_+(x) \leqslant \Gamma_+(x) \leqslant \alpha, \quad \lambda_1 \leqslant \gamma_-(x) \leqslant \Gamma_-(x) \leqslant \beta \text{ a.e. in } \Omega, \\ \delta_+(x) > \lambda_1 \text{ and } \delta_-(x) > \lambda_1 \text{ on subsets of positive measure,} \\ \text{either } \Delta_+(x) < \alpha \text{ a.e. in } \Omega \text{ or } \Delta_-(x) < \beta \text{ a.e. in } \Omega. \end{cases} \tag{6.5}$$

Since (α, β) above belongs to the first curve, $c(\alpha, \beta) = 1$, and it follows from Propositions 23 and 25 that (6.5) implies (f) and (F_1). On the other hand using the continuity of $\lambda_1(.)$ and $c(\cdot,\cdot)$ (cf. Proposition 22), one easily constructs examples where (f) and (F_1) hold while the pointwise conditions (6.5) do not. Note in particular that the functions $\gamma_\pm, \Gamma_\pm, \delta_\pm, \Delta_\pm$ in Theorem 40 may change sign and be unbounded. Nonresonance conditions bearing as in Theorem 40 on eigenvalues with weight were already considered in [26,20,21]. In particular the result of Theorem 40 for $p = 2$ and under the stronger hypothesis

$$\lambda_1\big(\min\{\gamma_+, \gamma_-\}\big) \leqslant 1, \qquad \lambda_2\big(\max\{\Gamma_+, \Gamma_-\}\big) \geqslant 1,$$

$$\lambda_1\big(\min\{\delta_+, \delta_-\}\big) < 1, \qquad \lambda_2\big(\max\{\Delta_+, \Delta_-\}\big) > 1$$

was obtained in [21].

Proof of Theorem 40. – We consider the functional

$$\Phi(u) := \frac{1}{p}\int_\Omega |\nabla u|^p - \int_\Omega F(x, u). \tag{6.6}$$

Assumption (6.1) implies that Φ in a C^1 functional on $W_0^{1,p}(\Omega)$. Its critical points are exactly the solutions of (1.5).

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Claim 1. – *Φ satisfies the (PS) condition on $W_0^{1,p}(\Omega)$.*

Proof. – Let u_k be a (PS) sequence, i.e.

$$\left|\Phi(u_k)\right| \leqslant c, \tag{6.7}$$

$$\left|\langle\Phi'(u_k), w\rangle\right| \leqslant \varepsilon_k\|w\|_{1,p} \quad \forall w \in W_0^{1,p}(\Omega), \tag{6.8}$$

where c is a constant and $\varepsilon_k \to 0$. As usual, it suffices to prove that u_k remains bounded in $W_0^{1,p}(\Omega)$. Assume by contradiction that, for a subsequence, $\|u_k\|_{1,p} \to +\infty$. Write $v_k := u_k/\|u_k\|_{1,p}$. For a further subsequence, $v_k \rightharpoonup v_0$ in $W_0^{1,p}(\Omega)$ and a.e. in Ω, and also, using (6.1) (and the Dunford–Pettis theorem when $p > N$), $f(x,u_k)/\|u_k\|_{1,p}^{p-1} \rightharpoonup f_0(x)$ in $L^q(\Omega)$ for some q with $q > (p^*)'$ if $p \leqslant N$ and $q = 1$ if $p > N$ (here p^* denotes the critical Sobolev exponent). We first take $w = v_0 - v_k$ in (6.8) and divide by $\|u_k\|_{1,p}^{p-1}$ to deduce from (6.1) that

$$\int_\Omega |\nabla v_k|^{p-2}\nabla v_k\,\nabla(v_k - v_0) \to 0;$$

arguing as at the end of the proof of Lemma 2.5, one obtains that $v_k \to v_0$ in $W_0^{1,p}(\Omega)$. In particular $\|v_0\|_{1,p} = 1$. One also deduces in a similar manner from (6.8) that

$$\int_\Omega |\nabla v_0|^{p-2}\nabla v_0\nabla w = \int_\Omega f_0 w \quad \forall w \in W_0^{1,p}(\Omega). \tag{6.9}$$

Now, by standard arguments based on (6.2) (cf. e.g. [31]), the function $f_0(x)$ can be written as $\alpha(x)(v_0^+)^{p-1} - \beta(x)(v_0^-)^{p-1}$ for some L^r functions α, β satisfying

$$\gamma_+(x) \leqslant \alpha(x) \leqslant \Gamma_+(x), \quad \gamma_-(x) \leqslant \beta(x) \leqslant \Gamma_-(x) \quad \text{a.e. in } \Omega. \tag{6.10}$$

Since the values of $\alpha(x)$ (resp. $\beta(x)$) on $\{v_0 \leqslant 0\}$ (resp. $\{v_0 \geqslant 0\}$) are irrelevant in the above expression of $f_0(x)$ as $\alpha(x)(v_0^+)^{p-1} - \beta(x)(v_0^-)^{p-1}$, we can assume that

$$\alpha(x) = \Delta_+(x) \quad \text{on } \{v_0 \leqslant 0\}, \qquad \beta(x) = \Delta_-(x) \quad \text{on } \{v_0 \geqslant 0\}. \tag{6.11}$$

We now distinguish three cases: (i) $v_0 \geqslant 0$ a.e. in Ω, (ii) $v_0 \leqslant 0$ a.e. in Ω, (iii) v_0 changes sign in Ω. We will see that each case leads to a contradiction. □

In case (i), Eq. (6.9) implies $\lambda_1(\alpha) = 1$ and $v_0(x) > 0$ in Ω. It then follows from (6.10) and (f) that $\lambda_1(\gamma_+) = 1$ and also, by the strict monotonicity of λ_1 with respect to the weight, that $\alpha = \gamma_+$, a.e. in Ω. Dividing (6.7) by $\|u_k\|_{1,p}^p$ and going to the limit, using (6.3) and Fatou's lemma, one gets

$$\int_\Omega \alpha v_0^p = \int_\Omega |\nabla v_0|^p = \lim \int_\Omega \frac{pF(x,u_k)}{\|u_k\|_{1,p}^p} \geqslant \int_\Omega \delta_+ v_0^p.$$

Since $\alpha = \gamma_+ \leqslant \delta_+$ and $v_0 > 0$, we deduce $\alpha = \delta_+$. Consequently $\lambda_1(\delta_+) = 1$, which contradicts (F_1). Case (ii) can be treated similarly. In case (iii), (6.9) shows that v_0 is a solution of $-\Delta_p u = \alpha(u^+)^{p-1} - \beta(u^-)^{p-1}$ which changes sign, and consequently $c(\alpha,\beta) \leqslant 1$. Proposition 23 together with (6.10) and (f) then yield $c(\alpha,\beta) = c(\Gamma_+,\Gamma_-) = 1$. Dividing (6.7) by $\|u_k\|_{1,p}^p$ and going to the limit, using (6.3) and Fatou's lemma, one gets

$$\begin{aligned}\int_\Omega \big(\alpha\big(v_0^+\big)^p + \beta\big(v_0^-\big)^p\big) = \int_\Omega |\nabla v_0|^p = \lim \int_\Omega \frac{pF(x,u_k)}{\|u_k\|_{1,p}^p}&\\ \leqslant \int_\Omega \big(\Delta_+\big(v_0^+\big)^p + \Delta_-\big(v_0^-\big)^p\big)&\\ \leqslant \int_\Omega \big(\Gamma_+\big(v_0^+\big)^p + \Gamma_-\big(v_0^-\big)^p\big).& \qquad (6.12)\end{aligned}$$

In fact the first integral and the last integral in (6.12) are equal because otherwise, Proposition 25 yields $c(\alpha,\beta) > c(\Gamma_+,\Gamma_-)$, in contradiction with what we have just proved. So all the terms are equal in (6.12) and we deduce, using (6.4), that $\Delta_+ = \Gamma_+$ on $\{v_0 > 0\}$, $\Delta_- = \Gamma_-$ on $\{v_0 < 0\}$, and using (6.10), that $\alpha = \Gamma_+$ on $\{v_0 > 0\}$, $\beta = \Gamma_-$ on $\{v_0 < 0\}$. Combining with (6.11), we finally get $\alpha = \Delta_+$ and $\beta = \Delta_-$ a.e. in Ω. Consequently $c(\Delta_+,\Delta_-) = 1$, which contradicts (F_1). This concludes the proof of Claim 1.

We now turn to the study of the geometry of Φ and first look for directions along which Φ goes to $-\infty$.

Claim 2. – Let w_+ (resp. w_-) be the positive eigenfunction associated to $\lambda_1(\delta_+)$ (resp. $\lambda_1(\delta_-)$) and normalized by $\int_\Omega \delta_+ w_+^p = 1$ (resp. $\int_\Omega \delta_- w_-^p = 1$). Then $\Phi(Rw_+) \to -\infty$ and $\Phi(-Rw_-) \to -\infty$ as $R \to +\infty$.

Proof. – We will prove the assertion relative to $\Phi(Rw_+)$, the other one is proved similarly. Let us first recall the precise meaning of the fact that the limits (6.3) are uniform with respect to x: for any $\varepsilon > 0$ there exists $a_\varepsilon \in L^1(\Omega)$ such that

$$\begin{aligned}&\frac{1}{p}\delta_+(x)(s^+)^p + \frac{1}{p}\delta_-(x)(s^-)^p - \frac{\varepsilon}{p}|s|^p - a_\varepsilon(x)\\ &\quad\leqslant F(x,s) \leqslant \frac{1}{p}\Delta_+(x)(s^+)^p + \frac{1}{p}\Delta_-(x)(s^-)^p + \frac{\varepsilon}{p}|s|^p + a_\varepsilon(x)\end{aligned}\qquad (6.13)$$

for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$. This implies, for $R > 0$,

$$\begin{aligned}\Phi(Rw_+) &\leqslant \frac{R^p}{p}\int_\Omega \big(|\nabla w_+|^p - \delta_+ w_+^p + \varepsilon w_+^p\big) + \int_\Omega a_\varepsilon\\ &\leqslant \frac{R^p}{p}\left(1 - \frac{1}{\lambda_1(\delta_+)} + \frac{\varepsilon}{\lambda_1}\right)\int_\Omega |\nabla w_+|^p + \int_\Omega a_\varepsilon,\end{aligned}$$

where $\lambda_1 = \lambda_1$ (constant weight 1). Choosing $\varepsilon > 0$ such that $1 - \frac{1}{\lambda_1(\delta_+)} + \frac{\varepsilon}{\lambda_1} < 0$, which is possible by Assumption (F_1), we get that $\Phi(Rw_+) \to -\infty$ as $R \to +\infty$. $\square$

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Claim 3. – *There exists R_0 such that for all $R \geqslant R_0$ and for all $h \in H_R := \{h \in C([-1,+1], W_0^{1,p}(\Omega))\colon h(-1) = Rw_+$ and $h(1) = -Rw_-\}$, one has*

$$\max_{u \in h[-1,+1]} \Phi(u) > \max\{\Phi(Rw_+), \Phi(-Rw_-)\}. \tag{6.14}$$

Once this last claim is proved, we can pick $R \geqslant R_0$ and apply a version of the mountain pass theorem in a Banach space as given for instance in [17] to conclude that

$$\inf_{h \in H_R} \max_{u \in h[-1,+1]} \Phi(u) \tag{6.15}$$

is a critical value of Φ. Theorem 40 will then be proved.

Proof of Claim 3. – Since by assumption (F_1), $c(\Delta_+, \Delta_-) > 1$, we can pick $\varepsilon > 0$ with $\varepsilon < (1 - 1/c(\Delta_+, \Delta_-))\lambda_1$. We then take a_ε according to (6.13) and use Claim 2 to choose $R_0 > 0$ such that

$$-\int_\Omega a_\varepsilon > \max\{\Phi(Rw_+), \Phi(-Rw_-)\} \tag{6.16}$$

for all $R \geqslant R_0$. Take such a value R and let $h \in H_R$. To prove (6.14), we distinguish two cases: either (i) $B_{\Delta_+,\Delta_-}(h(t_0)) \leqslant 0$ for some $t_0 \in [-1,+1]$, or (ii) $B_{\Delta_+,\Delta_-}(h(t)) > 0$ for all $t \in [-1,+1]$. We recall here that B_{Δ_+,Δ_-} is the function which defines the manifold M_{Δ_+,Δ_-} (cf. Section 2). We also recall that by Proposition 21,

$$c(\Delta_+, \Delta_-) = \inf_{\gamma \in \Gamma_1} \max_{u \in \gamma[-1,+1]} \int_\Omega |\nabla u|^p \tag{6.17}$$

where $\Gamma_1 := \{\gamma \in C([-1,+1], M_{\Delta_+,\Delta_-})\colon \gamma(-1) \geqslant 0$ and $\gamma(+1) \leqslant 0\}$.

Case (i). We first use (6.13) to obtain

$$\Phi(u) \geqslant \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{1}{p}\int_\Omega \left(\Delta_+(u^+)^p + \Delta_-(u^-)^p\right) - \frac{\varepsilon}{p}\int_\Omega |u|^p - \int_\Omega a_\varepsilon.$$

This implies, since we are in case (i),

$$\begin{aligned}\max_{u \in h[-1,+1]} \Phi(u) \geqslant \Phi\big(h(t_0)\big) &\geqslant \frac{1}{p}\int_\Omega \left|\nabla h(t_0)\right|^p - \frac{\varepsilon}{p}\int_\Omega \left|h(t_0)\right|^p - \int_\Omega a_\varepsilon \\ &\geqslant \frac{1}{p}(\lambda_1 - \varepsilon)\int_\Omega \left|h(t_0)\right|^p - \int_\Omega a_\varepsilon.\end{aligned}$$

Now, by the choice of ε, one has $\varepsilon < \lambda_1$, and consequently, by (6.16),

$$\max_{u \in h[-1,+1]} \Phi(u) \geqslant -\int_\Omega a_\varepsilon > \max\{\Phi(Rw_+), \Phi(-Rw_-)\},$$

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which implies the inequality (6.14) of Claim 3.

Case (ii). In this case we can normalize the path $h(t)$ to get a path

$$\tilde{h}(t) := h(t)/B_{\Delta_+,\Delta_-}\big(h(t)\big)^{1/p}$$

on the manifold M_{Δ_+,Δ_-} which satisfies, by (6.17),

$$\max_{u\in\tilde{h}[-1,+1]} \int_\Omega |\nabla u|^p \geqslant c(\Delta_+, \Delta_-). \tag{6.18}$$

We now use (6.13) to get

$$\Phi(u) \geqslant \frac{1}{p}\left(1 - \frac{\varepsilon}{\lambda_1}\right)\int_\Omega |\nabla u|^p - \frac{1}{p}\int_\Omega \big(\Delta_+(u^+)^p + \Delta_-(u^-)^p\big) - \int_\Omega a_\varepsilon$$

which implies, by (6.18),

$$\max_{u\in h[-1,+1]}\left(p\Phi(u) + B_{\Delta_+,\Delta_-}(u) + p\int_\Omega a_\varepsilon\right)\Big/ B_{\Delta_+,\Delta_-}(u) \geqslant \left(1 - \frac{\varepsilon}{\lambda_1}\right)c(\Delta_+, \Delta_-).$$

Hence there exists $u_0 \in h[-1, +1]$ such that

$$p\Phi(u_0) \geqslant \left(\left(1 - \frac{\varepsilon}{\lambda_1}\right)c(\Delta_+, \Delta_-) - 1\right)B_{\Delta_+,\Delta_-}(u_0) - p\int_\Omega a_\varepsilon.$$

This yields, by the choice of ε,

$$\max_{u\in h[-1,+1]} \Phi(u) \geqslant \Phi(u_0) \geqslant -\int_\Omega a_\varepsilon,$$

and the inequality (6.14) follows by using (6.16). This concludes the proof of Claim 3 and also of Theorem 40. □

Remark 41. – For later reference, let us observe that among the critical points of Φ at level (6.15), there is at least one, say u_1, which is such that there exists a sequence h_k of paths in H_R with the property that $\max_{u\in h_k[-1,+1]} \Phi(u) \to \Phi(u_1)$ and $\mathrm{dist}(u_1, h_k[-1, +1]) \to 0$. This follows from the proof of the mountain pass theorem as given for instance in [11].

Theorem 6.1 yields in particular a solution to the semilinear problem

$$-\Delta u = m(x)u^+ - n(x)u^- + h \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{6.19}$$

where $m, n \in L^r(\Omega)$ with (2.1), $h \in L^2(\Omega)$, if we assume

$$\lambda_1(m) < 1, \quad \lambda_1(n) < 1, \quad c(m, n) > 1. \tag{6.20}$$

In the rest of this section, we will be interested in the unicity of the solution to (6.19) when (6.20) holds.

PROPOSITION 42. – *If* $\min\{m,n\}$ *has a nontrivial positive part and if we assume*

$$\lambda_1\big(\min\{m,n\}\big) < 1, \quad \lambda_2\big(\max\{m,n\}\big) > 1, \tag{6.21}$$

then (6.19) *has an unique solution.*

Proof. – Existence follows from Theorem 40 since $\lambda_2(\max\{m,n\}) = c(\max\{m,n\}, \max\{m,n\})$ and so, by monotonicity, (6.21) implies (6.20). Assume now that u_1 and u_2 are two solutions of (6.19) and put $v := u_1 - u_2$. Then v solves

$$-\Delta v = d(x)v \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega, \tag{6.22}$$

where

$$d(x) := \begin{cases} \dfrac{m(x)(u_1^+(x) - u_2^+(x)) - n(x)(u_1^-(x) - u_2^-(x))}{u_1(x) - u_2(x)} & \text{if } v(x) \neq 0, \\ \min\{m(x), n(x)\} & \text{if } v(x) = 0. \end{cases}$$

Since $d(x)$ verifies $\min\{m,n\} \leqslant d \leqslant \max\{m,n\}$, we have

$$\lambda_1(d) \leqslant \lambda_1\big(\min\{m,n\}\big) < 1 < \lambda_2\big(\max\{m,n\}\big) \leqslant \lambda_2(d).$$

This implies that 1 is not an eigenvalue of $-\Delta$ for the weight d and consequently, by (6.22), $v \equiv 0$. □

The following two propositions describe two situations where unicity fails in (6.19) although (6.20) holds. In Proposition 43 it is the first part of (6.21) which is violated, while in Proposition 44 it is the second part of (6.21) which is violated. Note that the example in Proposition 43 requires nonconstant weights.

PROPOSITION 43. – *Suppose* $\partial\Omega$ *of class* C^2. *Then there exist* $m, n \in C^\infty(\bar{\Omega})$ *with* $m, n > 0$ *on* $\bar{\Omega}$,

$$\lambda_1(m) < 1, \quad \lambda_1(n) < 1, \quad \lambda_1\big(\min\{m,n\}\big) > 1, \quad \lambda_2\big(\max\{m,n\}\big) > 1 \tag{6.23}$$

such that, for some $h \in L^2(\Omega)$, (6.19) *has at least two solutions.*

Proof. – We start with the following

Claim 1. – *There exist* $m, n \in C^\infty(\bar{\Omega})$ *with* $m, n > 0$ *on* $\bar{\Omega}$ *and satisfying* (6.23), *and* $u_0 \in H_0^1(\Omega) \cap H^2(\Omega) \cap C(\bar{\Omega})$ *such that* $|u_0 = 0| = 0$, $m \leqslant n$ *on* $\{u_0 > 0\}$ *and* $n \leqslant m$ *on* $\{u_0 < 0\}$.

Let us admit this claim for a moment. With m, n, u_0 as in Claim 1, we define $h := -\Delta u_0 - m u_0^+ + n u_0^- \in L^2(\Omega)$ and consider (6.19) for these particular m, n, h. Clearly, by construction, u_0 is a solution of (6.19). In fact one can say more:

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Claim 2. – u_0 is a strict local minimum of the associated functional

$$\Psi(u) := \frac{1}{2}\int_\Omega \left(|\nabla u|^2 - m(u^+)^2 - n(u^-)^2\right) - \int_\Omega hu$$

on $H_0^1(\Omega)$.

Let us also admit this claim for a moment. We will use it to see that u_0 is different from the solution u_1 of (6.19) provided by the mountain pass argument of the proof of Theorem 40, more precisely provided by Remark 41. We thus have

$$\Psi(u_1) = \inf_{h\in H_R}\max_{u\in H[-1,+1]}\Psi(u)$$

(cf. (6.15)), with in addition the existence of a sequence of paths $h_k \in H_R$ such that $\max_{u\in h_k[-1,+1]}\Psi(u) \to \Psi(u_1)$ and $\operatorname{dist}(u_1, h_k[-1,+1]) \to 0$ (cf. Remark 6.2). So for $\varepsilon > 0$ with $\varepsilon < \max\{\|u_1 - Rw_+\|_{1,2}, \|u_1 - (-Rw_-)\|_{1,2}\}$, where Rw_+ and $-Rw_-$ are the functions involved in the definition of H_R in (6.15), each path h_k with k sufficiently large intersects $\{u\colon \|u - u_1\|_{1,2} = \varepsilon\}$. On the other hand, since Ψ satisfies the (PS) condition on $H_0^1(\Omega)$, Claim 2 and Theorem 5.10 in [17] imply that for any $\varepsilon > 0$ sufficiently small,

$$\inf\left\{\Psi(u)\colon \|u - u_0\|_{1,2} = \varepsilon\right\} > \Psi(u_0). \tag{6.24}$$

Fix now $\varepsilon > 0$ so that the two properties above hold simultaneously and assume by contradiction that $u_0 = u_1$. Then, by (6.24),

$$\max_{u\in h_k[-1,+1]}\Psi(u) \geqslant \inf\left\{\Psi(u)\colon \|u - u_0\|_{1,2} = \varepsilon\right\} > \Psi(u_0).$$

This contradicts the fact that $\max_{u\in h_k[-1,+1]}\Psi(u) \to \Psi(u_1) = \Psi(u_0)$. Consequently $u_1 \neq u_0$ and the conclusion of Proposition 43 follows.

Proof of Claim 2. – The proof uses the following easily verified Taylor type identities: for $x, y \in \mathbb{R}$, one has

$$y^2 - x^2 = 2x(y-x) + (y-x)^2,$$

$$(y^+)^2 - (x^+)^2 = 2x^+(y-x) + S(x^+)(y-x)^2 + R_+(x,y),$$

$$(y^-)^2 - (x^-)^2 = -2x^-(y-x) + S(x^-)(y-x)^2 + R_-(x,y),$$

where $S(t)$ denotes the sign function ($S(t) = 1$ if $t > 0$, 0 if $t = 0$, -1 if $t < 0$) and where R_+ and R_- satisfy $R_+(x,y) = 0$ if $xy > 0$, $|R_+(x,y)| \leqslant y^2$ if $xy \leqslant 0$, $R_-(x,y) = 0$ if $xy > 0$, $|R_-(x,y)| \leqslant y^2$ if $xy \leqslant 0$.

Assume by contradiction that u_0 is not a strict local minimum of Ψ. Thus there exists a sequence u_k converging to u_0 in $H_0^1(\Omega)$ with $u_k \neq u_0$ and $\Psi(u_k) \leqslant \Psi(u_0)$ for all k. For a subsequence, $u_k \to u_0$ a.e. in Ω. Using the above identities and the fact that u_0 solves (6.19), we get

$$
\begin{aligned}
\Psi(u_k)-\Psi(u_0) &= \frac{1}{2}\int_\Omega \left(|\nabla u_k|^2-|\nabla u_0|^2\right)-\frac{1}{2}\int_\Omega m\left(\left(u_k^+\right)^2-\left(u_0^+\right)^2\right)\\
&\quad -\frac{1}{2}\int_\Omega n\left(\left(u_k^-\right)^2-\left(u_0^-\right)^2\right)-\int_\Omega h(u_k-u_0)\\
&= \frac{1}{2}\int_\Omega \left(|\nabla u_k-\nabla u_0|^2\right)-\frac{1}{2}\int_\Omega \left(mS\left(u_0^+\right)+nS\left(u_0^-\right)\right)(u_k-u_0)^2\\
&\quad -\frac{1}{2}\int_\Omega \left(mR_+(u_0,u_k)+nR_-(u_0,u_k)\right). \qquad (6.25)
\end{aligned}
$$

By the properties of m, n, u_0 in Claim 1, we have

$$
\int_\Omega \left(mS\left(u_0^+\right)+nS\left(u_0^-\right)\right)(u_k-u_0)^2=\int_\Omega \min\{m,n\}(u_k-u_0)^2,
$$

where the integral over $\{u_0=0\}$ has been neglected since $|u_0=0|=0$. We also have

$$
\begin{aligned}
\left|\int_\Omega mR_+(u_0,u_k)\right| &\leqslant \int_{u_0u_k\leqslant 0} mu_k^2\leqslant \int_{u_0u_k\leqslant 0} m(u_k-u_0)^2\\
&\leqslant c_k\|u_k-u_0\|^2_{L^{2q}(\Omega)}
\end{aligned}
$$

where q is chosen with $2<2q<2^*$ and $c_k=\|m\|_\infty|u_0u_k\leqslant 0|^{1/q'}$; since $|u_0=0|=0$ and $u_k\to u_0$ a.e., $|u_0u_k\leqslant 0|\to 0$ and consequently $c_k\to 0$. A similar estimate of course holds for $\int_\Omega nR_-(u_0,u_k)$. It now follows from (6.25) that

$$
\Psi(u_k)-\Psi(u_0)\geqslant \left[\frac{1}{2}\left(1-\frac{1}{\lambda_1(\min\{m,n\})}\right)-\tilde{c}_k\right]\int_\Omega |\nabla u_k-\nabla u_0|^2
$$

where $\tilde{c}_k\to 0$. Since $\lambda_1(\min\{m,n\})>1$ and $u_k\neq u_0$, we deduce $\Psi(u_k)>\Psi(u_0)$ for k sufficiently large, which contradicts the fact that $\Psi(u_k)\leqslant\Psi(u_0)$. This completes the proof of Claim 2.

Proof of Claim 1. – Take for u_0 an eigenfunction of $-\Delta$ on $H_0^1(\Omega)$ which changes sign. It is well known that $u_0\in H^2(\Omega)\cap C(\bar{\Omega})$ and that $|u_0=0|=0$ (the regularity of $\partial\Omega$ is used here). Take balls B_+, B_- with $\overline{B}_+\subset\{u_0>0\}$ and $\overline{B}_-\subset\{u_0<0\}$. We start with the constant weight λ_1. Increasing it a little bit on B_- (resp. B_+), we can get a $C^\infty(\bar{\Omega})$ weight $\hat{m}$ (resp. $\hat{n}$) with $\lambda_1(\hat{m})<1$ (resp. $\lambda_1(\hat{n})<1$), and we can also impose $\hat{m}$ and $\hat{n}<\lambda_2$. At this stage, $\min\{\hat{m},\hat{n}\}\equiv\lambda_1$ and so $\lambda_1(\min\{\hat{m},\hat{n}\})=1$. We then define $m=\hat{m}-\delta$ and $n=\hat{n}-\delta$ with $\delta>0$ so small that we still have $\lambda_1(m)<1$ and $\lambda_1(n)<1$. Now $\min\{m,n\}\equiv\lambda_1-\delta$ and consequently $\lambda_1(\min\{m,n\})>1$. We also have $\lambda_2(\max\{m,n\})\geqslant\lambda_2(l)>1$, where l denotes the constant weight $\lambda_2-\delta$. Finally it is clear from the construction that $m\leqslant n$ on $\{u_0>0\}$ and $n\leqslant m$ on $\{u_0<0\}$. This concludes the proof of Claim 1 and of Proposition 43. □

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The following proposition is proved in [8], where many more precise results on the number of solutions can be found.

PROPOSITION 44. – *Let* $\Omega =]0,1[$. *For any constant positive weights* m, n *with*

$$\lambda_1\big(\min\{m,n\}\big) < 1, \quad \lambda_2\big(\max\{m,n\}\big) < 1, \quad c(m,n) > 1,$$

(6.19) *with* $h(x) \equiv 1$ *has at least two solutions.*

Note that with respect to the Fučik spectrum (without weight) of $-\Delta$ on $H_0^1(\Omega)$, the point (m,n) in Proposition 6.5 lies strictly between the trivial horizontal-vertical lines through (λ_1, λ_1) and the first curve through (λ_2, λ_2), but lies outside the closed square having $[(\lambda_1,\lambda_1),(\lambda_2,\lambda_2)]$ as diagonal.

7. Nonresonance of the type "below the first eigenvalue"

In this section, which is independant from the previous ones, we go on with the study of the Dirichlet problem (1.5) but now under assumptions on the asymptotic behaviour of the quotient $pF(x,s)/|s|^p$ which generalize the classical conditions that for a.e. $x \in \Omega$, the limits at infinity of this quotient lie below the first eigenvalue. Existence, unicity, as well as an example of nonunicity will be considered.

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying the growth condition (6.1). Denoting as before by $F(x,s)$ a primitive of $f(x,s)$, we assume that the L^r functions $\Delta_\pm$ defined by

$$\limsup_{s \to \pm\infty} p \frac{F(x,s)}{|s|^p} := \Delta_\pm(x) \tag{7.1}$$

have nontrivial positive parts and satisfy

$$\lambda_1(\Delta_+) > 1, \quad \lambda_1(\Delta_-) > 1. \tag{F_2}$$

Some uniformity with respect to x is also required in (7.1), which here corresponds to the second inequality in (6.13).

THEOREM 45. – *Assume* (6.1) *and* (F_2). *Then* (1.5) *has at least one solution* u_1 *which minimizes*

$$\Phi(u) := \frac{1}{p} \int_\Omega |\nabla u|^p - \int_\Omega F(x,u)$$

on $W_0^{1,p}(\Omega)$.

The result of Theorem 45 goes in the line of those in [32,35,5]. In the latter work for instance, it is assumed that $\Delta_+(x) \leqslant \lambda_1$ and $\Delta_-(x) \leqslant \lambda_1$ a.e. in Ω, with strict inequality on subsets of positive measure. This clearly imply (F_2). As already mentioned in Section 6, nonresonance conditions bearing on eigenvalues with weight were considered in [26,20,21]. In particular the result of Theorem 45 for $p = 2$ was obtained in [20].

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Remark 46. – Consider the problem

$$-\Delta_p u = \lambda m(x)|u|^{p-2}u + h \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \tag{7.2}$$

where $\lambda \in \mathbb{R}$, $m \in L^r(\Omega)$ and $h \in L^{p'}(\Omega)$, and assume that m changes sign in Ω. It is then easily verified that Theorem 45 applies if and only if $\lambda_{-1}(m) < \lambda < \lambda_1(m)$. So in fact we are dealing in this section with nonresonnance of the type "between the first negative eigenvalue and the first positive eigenvalue".

Proof of Theorem 7.1. – We recall that the uniformity in (7.1) precisely means that for any $\varepsilon > 0$ there exists $a_\varepsilon \in L^1(\Omega)$ such that

$$pF(x,s) \leqslant \Delta_+(x)(s^+)^p + \Delta_-(x)(s^-)^p + \varepsilon|s|^p + a_\varepsilon(x) \tag{7.3}$$

for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$. One easily deduces from (7.3) that

$$\Phi(u) \geqslant \frac{1}{p}\left(1 - \frac{1}{\min\{\lambda_1(\Delta_+), \lambda_1(\Delta_-)\}} - \frac{\varepsilon}{\lambda_1}\right)\int_\Omega |\nabla u|^p - \int_\Omega a_\varepsilon.$$

Taking $\varepsilon > 0$ sufficiently small and using (F_2), we get that Φ is bounded below and coercive on $W_0^{1,p}(\Omega)$. On the other hand standard arguments based on (7.3) and Fatou's lemma imply that Φ is weakly lower semicontinuous on $W_0^{1,p}(\Omega)$. Consequently Φ achieves its minimum on $W_0^{1,p}(\Omega)$ at some u_1. Since (6.1) implies that Φ is C^1 on $W_0^{1,p}(\Omega)$, u_1 solves (1.5). □

Theorem 45 yields in particular a solution to the semilinear problem

$$-\Delta u = m(x)u^+ - n(x)u^- + h \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \tag{7.4}$$

where $m, n \in L^r(\Omega)$ with (2.1), $h \in L^2(\Omega)$, if we assume

$$\lambda_1(m) > 1, \quad \lambda_1(n) > 1. \tag{7.5}$$

In the rest of this section we will be interested in the unicity of the solution of (7.4) when (7.5) holds.

PROPOSITION 47. – *Assume*

$$\lambda_1\bigl(\max\{m,n\}\bigr) > 1. \tag{7.6}$$

Then (7.4) *admits an unique solution.*

Proof. – Existence follows from Theorem 45 since (7.6) clearly implies (7.5). Assume now that u_1 and u_2 are two solutions of (7.4) and put $v = u_1 - u_2$. Then v solves

$$-\Delta v = d(x)v \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega, \tag{7.7}$$

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where $d(x)$ is defined as in the proof of Proposition 42. If $d \leqslant 0$, then (7.7) clearly implies $v \equiv 0$. If $d^+ \not\equiv 0$, then, since $d \leqslant \max\{m,n\}$, we have $\lambda_1(d) \geqslant \lambda_1(\max\{m,n\}) > 1$. Consequently 1 is not an eigenvalue of $-\Delta$ for the weight d and thus (7.7) implies $v \equiv 0$. □

Remark 48. – A result analogous to that of Proposition 47 does not hold for the p-laplacian, even when $m \equiv n \equiv$ constant and $N = 1$ (cf. [22] and [25]).

The following proposition shows that unicity may fail in (7.4) under (7.5). Note that as in Proposition 43, the example in Proposition 49 below requires nonconstant weights.

PROPOSITION 49. – *Suppose $\partial\Omega$ of class C^2. Then there exist $m, n \in C^\infty(\bar{\Omega})$ with $m, n > 0$ in $\bar{\Omega}$,*

$$\lambda_1(m) > 1, \quad \lambda_1(n) > 1, \quad \lambda_1\big(\max\{m,n\}\big) < 1, \tag{7.8}$$

such that, for some $h \in L^2(\Omega)$, (7.4) has at least two solutions.

Proof. – We start with the following

Claim 1. – There exist $m, n \in C^\infty(\bar{\Omega})$ with $m, n > 0$ in $\bar{\Omega}$, (7.8), and $u_0 \in H_0^1(\Omega) \cap H^2(\Omega) \cap C(\Omega)$ such that $|u_0 = 0| = 0$, $m \geqslant n$ on $\{u_0 > 0\}$ and $n \geqslant m$ on $\{u_0 < 0\}$.

Let us admit this claim for a moment. With m, n, u_0 as in Claim 1, we define $h := -\Delta u_0 - m u_0^+ + n u_0^-$ and consider (7.4) for these m, n, h. Clearly, by construction, u_0 is a solution of (7.4). We will show that u_0 is not a global minimum for the associated functional

$$\Psi(u) := \frac{1}{2} \int_\Omega \big(|\nabla u|^2 - m(u^+)^2 - n(u^-)^2\big) - \int_\Omega hu,$$

which implies that u_0 is different from the solution of (7.4) provided by Theorem 45.

To prove that u_0 is not a global minimum of Ψ, we take an eigenfunction v associated to λ_1 $(\max\{m,n\})$ and let $g(t) := \Psi(u_0 + tv)$. Clearly $g \in C^1(\mathbb{R})$ and

$$g'(t) = \int_\Omega \big(\nabla(u_0 + tv)\nabla v - m(u_0 + tv)^+ v + n(u_0 + tv)^- v\big) - \int_\Omega hv.$$

We have $g'(0) = 0$ since u_0 is a solution of (7.4). Moreover, since $|u_0 = 0| = 0$, we can use Proposition 2.2 from [38] to get that $g''(0)$ exists and that $g''(0) = \int_\Omega |\nabla v|^2 - \int_{u_0>0} m v^2 - \int_{u_0<0} n v^2$. Consequently, by the properties of m, n, u_0 in Claim 1,

$$g''(0) = \int_\Omega |\nabla v|^2 - \int_\Omega \max\{m,n\} v^2 = \big[\lambda_1(\max\{m,n\}) - 1\big] \int_\Omega \max\{m,n\} v^2 < 0.$$

This implies $g(t) < g(0)$ for $|t| > 0$ small, and it follows that u_0 is not a local and a fortiori global minimum of Ψ.

Proof of Claim 1. – It is rather similar to that of Claim 1 from the proof of Proposition 43. Take u_0 an eigenfunction of $-\Delta$ on $H_0^1(\Omega)$ which changes sign. So $u_0 \in H^2(\Omega) \cap C(\Omega)$ and $|u_0 = 0| = 0$. Take balls B_+, B_- with $\bar{B}_+ \subset \{u_0 > 0\}$ and

$\bar{B}_- \subset \{u_0 < 0\}$. Starting with the constant weight λ_1 and decreasing it a little bit on B_- (resp. B_+), we can get a positive $C^\infty(\bar{\Omega})$ weight $\hat{m}$ (resp. $\hat{n}$) with $\lambda_1(\hat{m}) > 1$ (resp. $\lambda_1(\hat{n}) > 1$). Clearly $\max\{\hat{m}, \hat{n}\} \equiv \lambda_1$ and so $\lambda_1(\max\{\hat{m}, \hat{n}\}) = 1$. Now, for $\delta > 0$ sufficiently small, we get that $m = \hat{m} + \delta$ and $n = \hat{n} + \delta$ satisfy $\lambda_1(m) > 1$, $\lambda_1(n) > 1$ and $\lambda_1(\max\{m, n\}) < 1$. Moreover, from the construction, $m \geqslant n$ on $\{u_0 > 0\}$ and $n \geqslant m$ on $\{u_0 < 0\}$. This concludes the proof of Claim 1 and of Proposition 49. □

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