A Resonant-Superlinear Elliptic Problem Revisited

( Dedicated to Klaus Schmitt on the occasion of his retirement )

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Abstract

We consider the resonant-superlinear elliptic problem \(-\Delta u = \lambda_1 u + (u^+)^q + f(x)\), with Dirichlet boundary conditions on a bounded regular domain of \(\mathbb{R}^N\). We assume that \(1 < q < \frac{N+1}{N-1}\), \(f \in L^s(\Omega)\) with \(s > N\) satisfies \(\int_\Omega f \varphi_1 < 0\) and \((\lambda_1, \varphi_1)\) is the first eigenpair of \(-\Delta\) on \(H_0^1(\Omega)\). We apply a non-well ordered lower and upper solution result on a family of modified problems and obtain a sequence of localized solutions of these modified problems. Thanks to this localization and a precise bootstrap argument we are able to prove that for large modification, these solutions are, in fact, solutions of our initial problem.

The problem was already considered in [2] by a totally different approach.

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1 Introduction

In this paper we consider the problem

$$-\Delta u = \lambda_1 u + (u^+)q + f(x) \quad \text{in} \quad \Omega,$$
$$u = 0 \quad \text{on} \quad \partial\Omega,$$

(1.1)

where $\Omega \subset \mathbb{R}^N$ is a bounded regular domain ($N \geq 3$), $u^+ := \max\{u, 0\}$ (we will use also the notation $u^- := \max\{-u, 0\}$), $\lambda_1$ is the first eigenvalue of $-\Delta$ on $H^1_0(\Omega)$, $f \in L^s(\Omega)$ with $s > N$ and $q > 1$. We will denote by $\varphi_1$ the positive eigenfunction corresponding to $\lambda_1$ with $\|\varphi_1\|_{H^1_0}^2 = 1$.

In a recent paper [2], M. Cuesta, D.G. de Figueiredo and P.N. Srikanth prove the following result.

**Theorem 1.1** Assume that $N \geq 3$, $f \in L^s(\Omega)$ with $s > N$ is such that

$$\int_\Omega f \varphi_1 \leq 0,$$

and $q$ satisfies

$$1 < q < \frac{N + 1}{N - 1}.$$

Then the problem (1.1) possesses at least one solution.

It is easy to observe that the condition $\int_\Omega f \varphi_1 \leq 0$ is necessary to have a solution. For what concerns the restriction on the growth of the superlinear term, to our knowledge, there had not been any improvement of this condition in this situation. Observe that we cannot reduce ourselves to the search of positive solutions or at least to solutions above a fixed function.

The proof of [2] uses Hardy-Sobolev type inequalities and argument as in Brezis-Turner famous paper [1]. As this approach uses deeply the selfadjointness of $-\Delta$ and the orthogonal decomposition of a function $u \in H^1_0(\Omega)$ in the form $u = a\varphi_1 + w$, it is not clear how to extend this approach for example to the $p$-Laplacian problem. On the other hand, it is easy to observe that the problem (1.1) possesses non well ordered lower and upper solutions. In this work, we apply then the result of C. De Coster [3] and prove the same result by a completely different approach which is more promising, for what concerns a possible extension to the $p$-Laplacian problem, as it does not use the orthogonal decomposition of the solution.

In [3], in case we have $\alpha$ and $\beta$ lower and upper solutions of (1.1) with $\alpha \nless \beta$, the second author defines

$$c := \inf_{\gamma \in \Gamma} \max_{s \in T_\gamma} \Phi(\gamma(s))$$

with

$$\Gamma = \{ \gamma \in C([0, 1], C^0_0(\Omega)) \mid \gamma(0) \in C^\alpha, \ \gamma(1) \in C_\beta \},$$

$$T_\gamma = \{ s \in [0, 1] \mid \gamma(s) \in C^0_0(\Omega) \setminus (C^\alpha \cup C_\beta) \},$$

(1.2)

where

$$C_\alpha = \{ u \in C^0_0(\Omega) \mid u \gg \alpha \},$$

$$C^\alpha = \{ u \in C^0_0(\Omega) \mid u \ll \beta \},$$

$$C_\beta = \{ u \in C^0_0(\Omega) \mid u \ll \beta \},$$

$$C^\beta = \{ u \in C^0_0(\Omega) \mid u \gg \alpha \},$$

$\gg$ and $\ll$ denote respectively the lower and upper order relations.
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(see Definition 2.1 for the definition of $u \ll v$) and

$$\Phi : H^1_0(\Omega) \to \mathbb{R}, u \mapsto \int_{\Omega} \left( \frac{\|\nabla u(x)\|^2}{2} - \lambda_1 \frac{|u|^2}{q+1} - fu \right) dx.$$ 

Then, in [3, Theorem 3.7], it is proved that, if $c \in \mathbb{R}$ and $\Phi$ satisfies the Palais-Smale condition, there exists $u \in C^1_0(\Omega) \setminus (C^\beta \cup C_\alpha)$ solution of (1.1) such that $\Phi(u) = c$.

In our case, we are unable to prove that $c \in \mathbb{R}$. To solve this problem, as in [4], we consider a family of modified problems (2.1) for which we prove the existence of a solution $u_r \in C^1_0(\Omega) \setminus (C^\beta \cup C_\alpha)$. Then the main difficulty is to come back to the original problem (1.1). This can be done using the localization information on the solution $u_r$ and a precise bootstrap argument.

2 Lower, upper solutions and modified problems

In this paper, we use the following notations.

**Definition 2.1** Given functions $u, v : \overline{\Omega} \to \mathbb{R}$, we write
(i) $u \leq v$ if for all $x \in \Omega, u(x) \leq v(x)$;
(ii) $u < v$ if $u \leq v$ and $u \neq v$;
(iii) $u \ll v$ if, for all $x \in \Omega, u(x) < v(x)$ and, for all $x \in \partial \Omega$, either $u(x) < v(x)$ or $\frac{\partial u}{\partial \eta}(x) > \frac{\partial v}{\partial \eta}(x)$. Here $\eta$ denotes the exterior normal direction with respect to $\partial \Omega$.

Recall the following notions of lower and upper solutions.

**Definition 2.2** A function $\alpha \in W^{2,p}(\Omega)$ (with $p > N$) is a lower solution of (1.1) if
(i) for a.e. $x \in \Omega, -\Delta \alpha(x) \leq \lambda_1 \alpha(x) + (\alpha^+(x))^q + f(x)$;
(ii) $\alpha \leq 0$ on $\partial \Omega$.

Similarly, an upper solution of (1.1) is defined by reversing the above inequalities.

**Lemma 2.1** Under the assumptions of Theorem 1.1, the problem (1.1) has an upper solution $\beta \ll 0$.

**Proof.** Let $w$ be the solution of

$$-\Delta w = \lambda_1 w + f(x) - \lambda_1 \int_{\Omega} f \varphi_1(x), \quad \text{in} \quad \Omega, \quad w = 0, \quad \text{on} \quad \partial \Omega.$$ 

We can choose $k > 0$ such that $\beta = -k \varphi_1 + w \ll 0$. Then we have

$$-\Delta \beta = \lambda_1 \beta + f(x) - \lambda_1 \int_{\Omega} f \varphi_1(x) \geq \lambda_1 \beta + (\beta^+)^q + f(x), \quad \text{in} \quad \Omega, \quad \beta = 0, \quad \text{on} \quad \partial \Omega,$$

which means that $\beta$ is an upper solution of (1.1).

**Lemma 2.2** Under the assumptions of Theorem 1.1, the problem (1.1) has a lower solution $\alpha \gg 0$. 


Proof: Let $\epsilon > 0$ such that $\lambda_1 + \epsilon \notin \sigma(-\Delta)$. Observe that, for all $u \geq 0$,
\[ w^\theta \geq \epsilon u - \epsilon \frac{\epsilon}{q} \int_1^q \left( q - \frac{1}{q} \right), \]
Define $w$ as the solution of
\[ -\Delta w = (\lambda_1 + \epsilon)w + f(x) - \epsilon \frac{\epsilon}{q} \int_1^q \left( q - \frac{1}{q} \right), \quad \text{in} \quad \Omega, \]
\[ w = 0, \quad \text{on} \quad \partial \Omega. \]
Choose $k > 0$ such that $\alpha = k\varphi_1 + w \gg 0$. Then we have
\[ -\Delta \alpha = \lambda_1 \alpha + f(x) - k\varphi_1 + \epsilon\alpha - \epsilon \frac{\epsilon}{q} \int_1^q \left( q - \frac{1}{q} \right) \leq \lambda_1 \alpha + f(x) + (\alpha^\theta)\theta, \]
in $\Omega$ and $\alpha = 0$ on $\partial \Omega$, which means that $\alpha$ is a lower solution of (1.1).

We have proved so far that problem (1.1) possesses a pair of non ordered lower and upper solutions. In order to be able to apply [3, Theorem 3.7], we are going to modify our problem in the following way.

For $r > 1 - \min \beta$, consider the modified problem
\[ -\Delta u = (\lambda_1 - \frac{1}{r}h_1(u))u + (u^\theta)\theta + f(x) =: g_r(x, u), \quad \text{in} \quad \Omega, \]
\[ u = 0, \quad \text{on} \quad \partial \Omega, \]
where
\[ h_1(u) = \begin{cases} 0, & \text{if } u > -r, \\ -(u + r), & \text{if } u \in [-r - 1, -r], \\ 1, & \text{if } u < -r - 1. \end{cases} \]
As $r > 1 - \min \beta$, the lower and upper solutions $\alpha$ and $\beta$ given by Lemmas 2.1 and 2.2 are still lower and upper solutions of (2.1).

Lemma 2.3 The modified problem (2.1) has a lower solution $\alpha_r \ll \beta$.

Proof. Observe first that, for all $u \leq 0$,
\[ (\lambda_1 - \frac{1}{r}h_1(u))u \geq \begin{cases} (\lambda_1 - \frac{1}{r})u - \frac{1}{r}(h_1(u) - 1)u, \\ (\lambda_1 - \frac{1}{r})u + \frac{1}{r}(h_1(u) - 1)(r + 1) \\ (\lambda_1 - \frac{1}{r})u - \frac{1}{r}(r + 1). \end{cases} \]
Let $w_r$ be the solution of
\[ -\Delta w = (\lambda_1 - \frac{1}{r})w + f(x) - \frac{r+1}{r}, \quad \text{in} \quad \Omega, \]
\[ w = 0, \quad \text{on} \quad \partial \Omega, \]
and $k_r > 0$ be large enough so that $\alpha_r = w_r - k_r\varphi_1 \ll \beta$, for example $k_r = \sup \left\{ \frac{w_r(x) - f(x)}{\varphi_1(x)} \mid x \in \Omega \right\} + 1$. It is then easy to observe that
\[ -\Delta \alpha_r = (\lambda_1 - \frac{1}{r})\alpha_r - \frac{r+1}{r}k_r\varphi_1 + f(x) \leq (\lambda_1 - \frac{1}{r}h_1(\alpha_r))\alpha_r + f(x) \]
\[ = (\lambda_1 - \frac{1}{r}h_1(\alpha_r))\alpha_r + (\alpha_r^\theta)\theta + f(x), \quad \text{in} \quad \Omega, \]
\[ \alpha_r = 0, \quad \text{on} \quad \partial \Omega, \]
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We use the lower solution \( \alpha_r \) to truncate the previous nonlinearity. Let us consider, for every \( r > 1 - \min \beta \), the (second) modified problem

\[
\begin{align*}
-\Delta u &= g_r(x, \gamma_r(x, u)), & \quad & \text{in } \Omega, \\
\quad u &= 0, & \quad & \text{on } \partial \Omega,
\end{align*}
\]

where

\[
\gamma_r(x, u) =
\begin{cases}
  u, & \text{if } u \geq \alpha_r(x), \\
  \alpha_r(x), & \text{if } u < \alpha_r(x).
\end{cases}
\]

Applying the maximum principle, it is easy to verify that every solution \( u \) of (2.2) satisfies \( u \geq \alpha_r \).

3 Proof of theorem 1.1

In this section we want to apply [3, Theorem 3.7] to problem (2.2). Let us define the \( C^1 \)-functional \( \Phi_r : H^1_0(\Omega) \to \mathbb{R} \) corresponding to (2.2):

\[
\Phi_r(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \int_\Omega G_r(x, u) \, dx
\]

where \( G_r(x, s) := \int_0^s g_r(x, \gamma_r(x, \rho)) \, d\rho \). Trivially, if \( r > 2/\lambda_1 \), the function

\[
\quad u \mapsto g_r(x, \gamma_r(x, u)) + u
\]

is increasing in \( u \) and locally Lipschitz in \( u \) uniformly in \( x \). This means that, for all \( r > 0 \), the assumption (H) of [3] is satisfied with \( m = 1 \).

Lemma 3.1 Let \( u_n \in H^1_0(\Omega) \) be a sequence of functions satisfying

\[
\int_\Omega (|\nabla u_n|^2 - \lambda_1 |u_n|^2) = o_{\infty}(\|u_n\|_{H^1_0}^2).
\]

Then either \( \|u_n\|_{H^1_0} \) is bounded or, up to a subsequence, \( (\|u_n\|_{H^1_0})_n \) converges strongly in \( H^1_0(\Omega) \) to \( \pm \varphi_1 \).

Proof. If \( \|u_n\|_{H^1_0} \to +\infty \), let us consider \( w_n := \frac{u_n}{\|u_n\|_{H^1_0}} \). As the sequence \( (w_n)_n \) is bounded, we have \( (w_n)_k \), a subsequence of \( (w_n)_n \), and \( w \in H^1_0(\Omega) \) such that \( w_n \to w \) weakly in \( H^1_0(\Omega) \), strongly in \( L^2(\Omega) \) and a.e. Using (3.1), we find

\[
0 \leq \int_\Omega (|\nabla w|^2 - \lambda_1 |w|^2) \leq \lim_{k \to \infty} \int_\Omega (|\nabla w_n|^2 - \lambda_1 |w_n|^2) = 0,
\]

from which we deduce \( w = c \varphi_1 \) for some constant \( c \in \mathbb{R} \). Moreover we have \( \int_\Omega |\nabla w_n|^2 = 1 \), which implies

\[
1 = \lim_{k \to \infty} \int_\Omega |\nabla w_n|^2 = \lambda_1 \lim_{k \to \infty} \int_\Omega |w_n|^2 = \lambda_1 \int_\Omega |w|^2
\]
and $\int_{\Omega} |\nabla w|^2 = \lambda_1 \int_{\Omega} |w|^2 = 1$. As $w_{n_k} \to w$ weakly in $H^1_0(\Omega)$ and $\|w_{n_k}\|_{H^1_0} \to \|w\|_{H^1_0}$, we then deduce that $w_{n_k} \to w$ strongly in $H^1_0(\Omega)$. By the normalization of $\varphi_1$, we have $c = \pm 1$.

**Proposition 3.1** Under the assumptions of Theorem 1.1, for all $r > r_0 := \max\{1 - \min \beta, 2/\lambda_1\}$, we have a solution $u_r$ of (2.1) with $u_r \neq \beta$, $u_r \geq \alpha_r$, $u_r \geq \alpha_r$ and $\Phi_r(u_r) = c_r$ where

$$c_r = \inf_{\gamma \in S} \max_{s \in T_r} \Phi_r(\gamma(s)),$$

and $\Gamma$ and $T_r$ are given by (1.2).

Moreover, there exists $d > 0$ such that, for all $r > r_0$, we have $c_r \leq d$.

**Proof. Step 1.** For all $r > r_0$, we have $c_r \in \mathbb{R}$ and there exists $d > 0$ such that, for all $r > r_0$, we have $c_r \leq d$. For all $\gamma \in \Gamma$, let $s_0 \in T_r$ such that $u_0 = \gamma(s_0) \in \partial \mathcal{C}$. Observe that $u_0 \ll 0$ and

$$\Phi_r(u_0) = \frac{1}{2} \int_{\Omega} |\nabla u_0|^2 \, dx - \int_{\Omega} \overline{G}(x, u_0) \, dx \geq \frac{1}{2} \|u_0\|_{H^1_0}^2 - c_r \|u_0\|_{H^1_0} - d_r,$$

with $c_r$, $d_r$ depending on $r$ and $\alpha_r$, but not on $u_0$. This implies the existence of $m_r \in \mathbb{R}$ such that, for all $\gamma \in \Gamma$,

$$\max_{s \in T_r} \Phi_r(\gamma(s)) \geq m_r,$$

and $c_r \in \mathbb{R}$.

Moreover, there exists a path $\gamma_0 \in \Gamma$ such that, for all $t \in [0, 1]$ and all $x \in \Omega$, $\gamma_0(t)(x) \geq \max\{-r, \alpha_r(x)\}$. Hence, for all $r > r_0$, we have

$$c_r \leq d := \max_{\gamma \in \mathcal{C}(\gamma_0(0), \mathcal{C}_n)} \Phi(\gamma(t)),$$

as, for $u \geq \max\{-r, \alpha_r(x)\}$, by definition, $\Phi_r(u) = \Phi(u)$.

**Step 2.** $\Phi_r$ satisfies the Palais-Smale condition.

Let $(u_{n_k})_k$ be a (PS) sequence for $\Phi_r$, i.e. there exists $M > 0$ and $\epsilon_n \to 0$ such that $|\Phi_r(u_{n_k})| \leq M$ and, for all $v \in H^1_0(\Omega)$, $|\Phi_r(u_{n_k}) v| \leq \epsilon_n \|v\|_{H^1_0}$. Using this last condition with $v = u_{n_k}$ we obtain

$$\epsilon_n \|u_{n_k}\|_{H^1_0}^2 \geq \|\Phi_r(u_{n_k}), u_{n_k}\| = \|\Phi_r(u_{n_k})\| \geq \int_{\Omega} |\nabla u_{n_k}|^2 - (\lambda_1 - \frac{1}{r}) \int_{\Omega} \left(\frac{1}{r} - 1ight) \int_{\Omega} (1 - h_r(x, u_{n_k}))(u_{n_k})^2 + \int_{\Omega} f u_{n_k}^2$$

$$\geq \int_{\Omega} |\nabla u_{n_k}|^2 - (\lambda_1 - \frac{1}{r}) \int_{\Omega} \left(\frac{1}{r} - 1\right) \int_{\Omega} (1 - h_r(x, u_{n_k}))(u_{n_k})^2 + \int_{\Omega} f u_{n_k}^2$$

from which we deduce that $\|u_{n_k}\|_{H^1_0}$ is bounded. Moreover we have

$$M + \epsilon_n \|u_{n_k}\|_{H^1_0} \geq \Phi_r(u_{n_k}) = \frac{1}{q + 1} \Phi_r(u_{n_k}) \geq \frac{1}{2} - \frac{1}{q + 1} \left(\int_{\Omega} |\nabla u_{n_k}|^2 \right)^2$$

$$\geq \frac{1}{2} \left(1 - \frac{1}{q + 1}\right) \int_{\Omega} \left(\frac{1}{r} - 1\right) \int_{\Omega} (1 - h_r(x, u_{n_k}))(u_{n_k})^2 + \int_{\Omega} f u_{n_k}^2 + \int_{\Omega} f u_{n_k}^2 + \frac{1}{2} \int_{\Omega} \alpha_r u_{n_k}.$$
As $\|u_n^r\|_{H^1_0}$ is bounded, we can write

$$\int_{\Omega} |\nabla u_n^r|^2 - \lambda_1 \int_{\Omega} (u_n^r)^2 = o_{\infty}(\|u_n^r\|_{H^1_0}^2),$$

and, using Lemma 3.1, either $\|u_n^r\|_{H^1_0}$ is bounded or $\frac{u_n^r}{\|u_n^r\|_{H^1_0}} \to \varphi_1$ strongly in $H^1_0(\Omega)$.

Let us prove by contradiction that the second situation cannot occur. By the (PS) condition with $v = \varphi_1$, we have

$$\epsilon_0 \|\varphi_1\|_{H^1_0} \geq -\langle \Phi_r(u_0), \varphi_1 \rangle \geq \int_{\Omega} (u_n^r)^2 \varphi_1 + \int_{\Omega} f \varphi_1 - \frac{1}{r} \int_{\Omega} h_r(x, u_n) \gamma_r(x, u_n) \varphi_1.$$

In case $\frac{u_n^r}{\|u_n^r\|_{H^1_0}} \to \varphi_1$ strongly in $H^1_0(\Omega)$, dividing by $\|u_n^r\|_{H^1_0}^2$ and passing to the limit we obtain the contradiction

$$\int_{\Omega} \varphi_1^{q+1} = 0.$$

Hence $\|u_n^r\|_{H^1_0}$ is bounded which proves the (PS) condition.

**Conclusion.** By the above claims, we can apply [3, Theorem 3.7] and, for all $r > r_0$, there exists a solution $u_r$ of (2.2) with $u_r \not\equiv \beta, u_r \not\equiv \alpha$ and $\Phi_r(u_r) = c_r$. Hence by construction, for all $r > r_0$, we have a solution $u_r$ of (2.1) with $u_r \not\equiv \beta, u_r \not\equiv \alpha, u_r \geq \alpha_r$ and $\Phi_r(u_r) = c_r$.

In order to conclude the proof of the result, we need some estimates. Let us denote by $\Phi_r$ the functional corresponding to (2.1) and $u_r$ the solution given by Proposition 3.1.

**Lemma 3.2** For all $r > r_0$, let $u_r$ be the solution of (2.1) given by Proposition 3.1. Then either $\|u_r^r\|_{H^1_0} \to \infty$ or $\|u_r\|_{C}$ is bounded if $r \to +\infty$.

In case $\lim_{r \to +\infty} \|u_r^r\|_{H^1_0} = 0$, the sequence $\frac{u_r^r}{\|u_r^r\|_{H^1_0}}$ converges strongly in $H^1_0(\Omega)$ to $\varphi_1$ when $r \to +\infty$.

**Proof.** As $u_r$ is a solution of (2.1), we have

$$0 = \|\Phi_r'(u_r), u_r^r\| \geq \int_{\Omega} |\nabla u_r^r|^2 - \lambda_1 \int_{\Omega} (u_r^r)^2 + \int_{\Omega} f u_r^r,$$

from which we deduce that

$$\int_{\Omega} |\nabla u_r^r|^2 - \lambda_1 \int_{\Omega} (u_r^r)^2 \leq -\int_{\Omega} f u_r^r. \quad (3.2)$$

By (3.2) and Lemma 3.1, either $\|u_r^r\|_{H^1_0}$ is bounded or $\frac{u_r^r}{\|u_r^r\|_{H^1_0}} \to \varphi_1$ strongly in $H^1_0(\Omega)$. It remains then to prove that, in case $\|u_r^r\|_{H^1_0}$ is bounded, $\|u_r\|_{C}$ is also bounded.

Recall the existence of $d > 0$ such that, for all $r > r_0, c_r \leq d$. As $u_r$ is solution of (2.1), we have in particular

$$\int_{\Omega} |\nabla u_r^r|^2 - \lambda_1 \int_{\Omega} (u_r^r)^2 - \int_{\Omega} (u_r^r)^{q+1} - \int_{\Omega} f u_r^r = \langle \Phi_r'(u_r), u_r^r \rangle = 0. \quad (3.3)$$
Lemma 3.3

For all \( r \) given by Proposition 3.1. If \( \lim_{r \to \infty} ||u_r^+||_{H^1_0} = \infty \) then there exists \( C > 0 \) such that, for \( r > r_0 \), we have

\[
||u_r^+||_{H^1_0} \leq C ||u_r||_{H^1_0}^{1/2}.
\]

Proof: As in the previous lemma, we deduce (3.4) and (3.3). Consider the function \( w_r = \frac{u_r^+}{||u_r^+||_{H^1_0}} \). By Lemma 3.2, we have \( \frac{w_r}{||w_r||_{H^1_0}} \to \varphi_1 \) and hence, from (3.4),

\[
\int_{\Omega} ||w_r||^2 - \lambda_1 \int_{\Omega} w_r^2 = o_o(||w_r||^2_{H^1_0}).
\]
By Lemma 3.1, either $\frac{w_r}{|w_r|_{H^1_0}} \to \varphi_1$ in $H^1_0(\Omega)$ or $|w_r|_{H^1_0} \leq C$. If $\frac{w_r}{|w_r|_{H^1_0}} = \frac{u_r}{|u_r|_{H^1_0}} \to \varphi_1$, dividing (3.3) by $|u_r|^{q+1}_{H^1_0}$, we obtain the contradiction

$$\int_{\Omega} \left( \frac{u_r}{|u_r|_{H^1_0}} \right)^{q+1} \to 0 \quad \text{and} \quad \int_{\Omega} \left( \frac{u_r}{|u_r|_{H^1_0}} \right)^{q+1} \to \int_{\Omega} \varphi_1^{q+1}.$$ 

This implies that the first situation cannot occur. Hence, there exists $C$ such that, for all $r > r_0$, we have $|w_r|_{H^1_0} \leq C$.

**Remark 3.1** Observe that this implies also that, if $\lim_{r \to +\infty} |u_r|_{H^1_0} = \infty$, then

$$\lim_{r \to +\infty} |u_r|_{H^1_0} = 1.$$

**Lemma 3.4** For all $r > r_0$, let $u_r$ be the solution of (2.1) given by Proposition 3.1. If $\lim_{r \to +\infty} |u_r|_{H^1_0} = \infty$ then, there exists $C > 0$ such that, for $r$ large, $\int_{\Omega} (u_r^*)^{q+1} \leq C |u_r|_{H^1_0}$.

**Proof.** Recall the existence of $d > 0$ such that, for all $r > r_0$, $c_r \leq d$. As $u_r$ is a solution of (2.1), we have

$$d \geq \Phi_r(u_r) - \frac{1}{2} \langle \Phi'_r(u_r), u_r^* \rangle \\
\geq \frac{1}{2} \int_{\Omega} |\nabla u_r|^2 - \frac{3}{2} \int_{\Omega} |u_r|^2 + (\frac{1}{2} - \frac{1}{q+1}) \int_{\Omega} (u_r^*)^{q+1} = \frac{1}{2} \int_{\Omega} u_r f + \int_{\Omega} u_r f \\
\geq \left( \frac{1}{2} - \frac{1}{q+1} \right) \int_{\Omega} (u_r^*)^{q+1} - \frac{1}{2} \int_{\Omega} u_r f + \int_{\Omega} u_r f.$$

Dividing by $|u_r|_{H^1_0}$ and using Lemmas 3.2 and 3.3, we obtain

$$\limsup_{r \to +\infty} \int_{\Omega} \frac{(u_r^*)^{q+1}}{|u_r|_{H^1_0}} \leq - \frac{2(q+1)}{q-1} \int_{\Omega} f \varphi_1,$$

which proves the estimate.

**Lemma 3.5** For all $r > r_0$, let $u_r$ be the solution of (2.1) given by Proposition 3.1. If $\lim_{r \to +\infty} |u_r|_{H^1_0} = \infty$ then there exists $C > 0$ such that, for all $r > r_0$,

$$|u_r| \leq C r \quad \text{and} \quad |u_r|_{H^1_0} \leq C r.$$

**Hence, there exists $C_1 > 0$ such that, for all $r > r_0$,$$
\left| \frac{1}{r} \int_{\Omega} h_r(u_r) u_r \right| \leq C_1.$$
Proof. Recall that, by construction \( u_r \geq \alpha_r \), where \( \alpha_r = w_r - k_r \varphi_1 \) with \( w_r \) solution of

\[
-\Delta w = (\lambda_1 - \frac{1}{r})w + f(x) - \frac{r + 1}{r}, \quad \text{in} \quad \Omega, \\
w = 0, \quad \text{on} \quad \partial\Omega,
\]

and \( k_r = \sup \{ \frac{w_r(x) - \beta(x)}{\varphi_1(x)} \mid x \in \Omega \} + 1 \).

Observe first that, as

\[
\int_\Omega |\nabla w_r|^2 - (\lambda_1 - \frac{1}{r}) \int_\Omega |w_r|^2 \geq \frac{1}{r} \int_\Omega |\nabla w_r|^2,
\]

we deduce the existence of a constant \( C_1 \) such that, for all \( r > r_0 \),

\[
\|w_r\|_{H^1_0}^2 \leq r \lambda_1 \int_\Omega (f - \frac{r + 1}{r}w_r) \leq r \lambda_1 \|f\| + 2\|\varphi_1\| \|w_r\|_{H^1_0},
\]

and hence \( \|w_r\|_{H^1_0}/r \) is bounded.

Moreover \( w_r \) satisfies

\[
-\Delta (\frac{w_r}{r}) - (\lambda_1 - 1)\frac{w_r}{r} = (1 - \frac{1}{r})\frac{w_r}{r} + \frac{f(x) - \varphi_1}{r}, \quad \text{in} \quad \Omega, \\
\frac{w_r}{r} = 0, \quad \text{on} \quad \partial\Omega.
\]

As now the right hand side of this equation is bounded, we conclude, by a bootstrap argument, the existence of a constant \( C_1 \) such that, for all \( r > r_0 \),

\[
\|w_r\|_{C^1} \leq C_1 r.
\]

Hence we have also the existence of \( C_2 > 0 \) such that, for all \( r > r_0 \),

\[
k_r = \sup \{ \frac{w_r(x) - \beta(x)}{\varphi_1(x)} \mid x \in \Omega \} + 1 < C_2 r.
\]

This allows to conclude the existence of \( C_3 \) such that, for all \( r > r_0 \),

\[
\|u_r\|_{C^1} \leq \|\alpha_r\|_{C^1} \leq C_3 r.
\]

Now multiplying (2.1) by \( \frac{w_r}{r^2} \) and integrating we deduce that

\[
\int_\Omega \frac{|\nabla u_r|^2}{r^2} = \int_\Omega (\lambda_1 - h_r(u_r)) \frac{(u_r)^2}{r^2} - \int_\Omega \frac{f u_r^2}{r^2}.
\]

As the right hand side of this equation is bounded, we deduce the existence of a constant \( C > 0 \) such that, for all \( r > r_0 \), \( \|u_r\|_{H^1_0} \leq Cr \). It follows from Lemma 3.3 that

\[
\|u_r\|_{H^1_0} \leq Cr^{1/2},
\]

which concludes the proof.
Proof of Theorem 1.1. Recall that by Proposition 3.1, for all \( r > r_0 \), we have a solution \( u_r \) of (2.1) with \( u_r \not\in \beta, u_r \not\in \alpha, u_r \geq \alpha \), and \( \Phi_r(u_r) = c_r \). In order to conclude the proof of the existence of a solution of problem (1.1), we wish to prove the following result:

Claim: There exists \( K > r_0 \) such that, for all \( r > K \), every solution \( u_r \) of (2.1) with \( u_r \not\in \beta, u_r \not\in \alpha \), \( u_r \geq \alpha \), and \( \Phi_r(u_r) = c_r \) satisfies \( u_r > -K \).

Assume by contradiction the existence of a sequence \((r_n)_n\) with \( \lim r_n = +\infty \) and a sequence of solutions \((u_n)_n\) of (2.1) with \( u_n \not\in \beta, u_n \not\in \alpha, u_n \geq \alpha_n \) and \( \Phi_n(u_n) = c_n \) which satisfies \( \min u_n \leq -n \).

Here and below we simplify the notation by writing \( r \), \( s \), \( \Phi \), etc.

In the course of this proof we use the following notations: \( a \leq b \) means the existence of a positive constant \( C \), which is independent of the quantities \( a \) and \( b \) under consideration such that \( a \leq C b \).

Let us write \( u_n = a_n \varphi_1 + w_n \) with \( \int_{\Omega} w_n \varphi_1 = 0 \) and \( a_n = \frac{\int_{\Omega} u_n \varphi_1}{\int_{\Omega} \varphi_1^2} \). Recall that by Lemmas 3.2 and 3.3, as \( \lim_{n \to \infty} ||u_n||_{\mathcal{H}_{0}^1} = \infty \), we have also \( \lim_{n \to \infty} ||u_n||_{\mathcal{H}_{0}^1} = \infty \) and \( \lim_{n \to \infty} \frac{u_n}{||u_n||_{\mathcal{H}_{0}^1}} = -\varphi_1 \) strongly in \( \mathcal{H}_{0}^1(\Omega) \).

Hence dividing \( a_n \) by \( ||u_n||_{\mathcal{H}_{0}^1} \) we obtain

\[
\lim_{n \to \infty} \frac{a_n}{||u_n||_{\mathcal{H}_{0}^1}} = \lim_{n \to \infty} \frac{1}{\int_{\Omega} \varphi_1^2} \int_{\Omega} \frac{u_n}{||u_n||_{\mathcal{H}_{0}^1}} \varphi_1 = -1,
\]

and also

\[
\lim_{n \to \infty} \frac{w_n}{||a_n||} = \lim_{n \to \infty} \frac{u_n}{||u_n||_{\mathcal{H}_{0}^1}} + \varphi_1 = 0 \quad \text{in} \ \mathcal{H}_{0}^1(\Omega).
\]

We would have the claim if we prove \( \frac{w_n}{||a_n||} \to 0 \) in \( \mathcal{C}_0^2(\Omega) \), as in that case, for \( n \) large enough, we obtain

\[
u_n = |a_n|(-\varphi_1 + \frac{w_n}{||a_n||}) \leq -\frac{|a_n|}{2} \varphi_1 \ll \beta,
\]

which contradicts the localization \( u_n \not\in \beta \).

Observe that \( w_n \) satisfies

\[
-\Delta w_n - \lambda_1 w_n = h_n(u_n)u_n^\ast + (u_n^\ast)^q + f(x), \quad \text{in} \ \Omega,
\]

\[
w_n = 0, \quad \text{on} \ \partial \Omega,
\]

\[
\int_{\Omega} w_n \varphi_1 = 0.
\]

By the regularity theory for linear equation, for every \( s > 0 \) such that \( h_n(u_n)u_n^\ast + (u_n^\ast)^q + f \in L^s(\Omega) \), we have \( w_n \in W^{2,s}(\Omega) \) and, by Lemma 3.5, we obtain

\[
||w_n||_{W^{2,s}} \leq ||h_n(u_n)u_n^\ast + (u_n^\ast)^q + f||_{L^s} \leq ||(u_n^\ast)^q||_{L^s} + 1.
\]

By Lemma 3.3 and Lemma 3.5, we obtain for \( s_1 = \frac{2N}{N-2} \),

\[
||(u_n^\ast)^q||_{L^{s_1}} \leq |a_n|_{s_1},
\]

and by Lemma 3.4 we have

\[
||(u_n^\ast)^q||_{L^{s_1}} \leq |a_n|_{s_1}. \quad (3.7)
\]

Observe also that as \( \frac{a_n}{||u_n||_{\mathcal{H}_{0}^1}} \to -1 \), for \( n \) large enough, \( u_n^\ast \leq w_n^\ast \).
Step 1: $N = 3$. By (3.5) and (3.6), we have
$$\|w_n\|_{W^{2, \frac{q}{s_1}}} \leq |a_n|^{\frac{q}{2}} + 1.$$ 
As $\frac{q}{N} > N$, we know that $W^{2, \frac{q}{N}}(\Omega)$ is compactly embedded in $C^1_0(\overline{\Omega})$ and hence
$$\|w_n\|_{C^1_0} \leq |a_n|^{\frac{q}{2}} + 1.$$ 
Observing that $\frac{q}{N} < \frac{1}{2}$, we conclude in this case that $w_n|_{a_n} \to 0$ in $C^1_0(\Omega)$.

Step 2: $N \geq 4$ and $q^2 \geq q + 1$. By (3.7) and (3.5), we have $w_n \in W^{2, \frac{q}{s_2}}(\Omega)$ with
$$\|w_n\|_{W^{2, \frac{q}{s_2}}} \leq |a_n|^{\frac{q}{2}} + 1.$$ 
Moreover, observe that, as $q > 1$, we have $\frac{q}{s_2} > \frac{q}{q+1} = \frac{2}{N} > 0$. By the continuous injection properties of Sobolev spaces, we obtain $w_n \in L^{s_2}(\Omega)$ for
$$\frac{1}{s_2} = \frac{q}{q+1} - \frac{2}{N},$$ 
with
$$\|w_n\|_{L^{s_2}} \leq |a_n|^{\frac{q}{s_2}} + 1. \quad (3.8)$$ 
This implies that
$$\|(w_n)^q\|_{L^{s_2}} \leq |a_n|^{\frac{q}{s_2}} + 1. \quad (3.9)$$ 
Observe that
$$\frac{q}{q+1} < \frac{q^2}{q+1} \quad \text{and} \quad \frac{s_2}{q} > \frac{q+1}{q}.$$ 
This means that (3.7) is better for what concerns the exponent but (3.9) is better for what concerns the regularity. The idea now is to make a bootstrap combining these two estimates in order to gain regularity but keeping an exponent less than 1.

Recall that, until this point, we did not use the condition $q^2 \geq q + 1$. Using $u_n^+ = w_n^+$, we have
$$\int_{\Omega} (u_n^+)^{qs_2} \leq \int_{\Omega} (u_n^+)^{\alpha q + \beta}(w_n^+)^{qs_2} \leq \|u_n^+\|_{L^{qs_2}} \|w_n^+\|_{L^{qs_2}} \leq |a_n|^{\frac{q}{s_2}} + |a_n|^{\frac{q}{s_2}} \quad (3.10)$$
with $\frac{1}{2}(\frac{\alpha q}{q+1} + \frac{\beta q}{q+1}) \leq 1$ in case $\alpha, \beta, s$ are chosen in such a way that
$$0 \leq \alpha(\frac{s}{q_1}), \quad 0 \leq \beta(\frac{s}{q_2}), \quad \frac{\alpha q}{q+1} + \frac{\beta q}{q+1} \leq 1, \quad \frac{\alpha q}{q+1} + \frac{\beta q}{q+1} \leq s.$$
The conditions in parenthesis can be deduced from the third one in case \( \alpha \geq 0 \) and \( \beta \geq 0 \).

As \( q^2 > q + 1 \), we see that the optimal \( s \) (i.e. the largest one) is given by

\[
s = (q - 1) \frac{1}{q^2 - 1 + \frac{1}{s^2}} = \frac{N(q - 1)}{Nq - N - 2}.
\]

This corresponds to

\[
\alpha = \frac{N(q^2 - q - 1)}{q(Nq - N - 2)} \quad \text{and} \quad \beta = \frac{N}{q(Nq - N - 2)}.
\]

We deduce then from (3.5) that

\[
\|w_n\|_{L^s} \lesssim |a_n| + 1.
\]

The condition \( q < \frac{N+1}{N-2} \) implies \( \frac{N(q-1)}{Nq-N-2} > N \) and, by the continuous injection properties of Sobolev spaces, we have, for some \( \alpha > 0 \), \( w_n \in C^{1,\alpha}_0(\Omega) \) with

\[
\|w_n\|_{C^{1,\alpha}} \lesssim |a_n| + 1.
\]

Hence, every subsequence \( (w_{n_k}) \) of \( (w_n) \) has a subsubsequence \( (w_{n_{k,j}}) \) such that \( w_{n_{k,j}} \|_{C^{1,\alpha}} \rightarrow w \) strongly in \( C^{1,\alpha}_0(\Omega) \) by the compact embedding of \( C^{1,\alpha}_0(\Omega) \) into \( C^{1,\alpha}_0(\Omega) \). We deduce from \( w_{n_{k,j}} \|_{L^s} \rightarrow 0 \) in \( H^1_0(\Omega) \), that \( w = 0 \). As this is true for all subsequence, we obtain \( w_n \|_{C^{1,\alpha}} \rightarrow 0 \) in \( C^{1,\alpha}_0(\Omega) \) and the proof is complete in this case.

**Step 3:** \( N \geq 4 \), \( q^2 < q + 1 \) and \( s^2 > qN \). By (3.8) and (3.5), we have \( \|w_n\|_{L^s} \lesssim \|u_n\|_{L^s} \lesssim |a_n| + 1 \) and hence, by (3.9), we have \( w_n \in W^{2,\frac{s^2}{q}}(\Omega) \) with

\[
\|w_n\|_{W^{2,\frac{s^2}{q}}} \lesssim |a_n|^\frac{s^2}{q} + 1. \tag{3.11}
\]

As \( \frac{s^2}{q} > N \), we obtain

\[
\|w_n\|_{C^{1,\alpha}} \lesssim |a_n|^\frac{s^2}{q} + 1,
\]

with \( \frac{s^2}{q} > 1 \) and the conclusion follows.

**Step 4:** \( N \geq 4 \), \( q^2 < q + 1 \) and \( \frac{q}{q^2} > \frac{q}{s^2} \). Working as in Step 3, we have (3.11) and as \( \frac{q}{q^2} > \frac{q}{s^2} \), we obtain

\[
\|w_n\|_{C^{1,\alpha}} \lesssim |a_n|^\frac{s^2}{q} + 1.
\]

Let us try to minimize the exponent keeping the regularity i.e. we want to make the same kind of bootstrap estimate as in (3.10) with \( \alpha, \beta \) and \( s \) such that

\[
s = \alpha + \beta,
\]
\[
0 \leq \alpha, \quad 0 \leq \beta,
\]
\[
\frac{q\alpha}{q+1} + \frac{q\beta}{s^2} \leq 1,
\]
\[
\frac{q\alpha}{q+1} + \frac{q\beta}{s^2} = \eta s,
\]
with \( s > \frac{N}{2} \) and \( \eta \) minimum. We easily see that we obtain
\[
\|w_n\|_{C^s} \lesssim |a_n|^\eta + 1,
\]
with
\[
\eta > \frac{2}{N + 2 - (N - 2)q} = \frac{2}{N} \left( \frac{qN}{2q} - 1 + q \left( 1 - \frac{qN}{2q+1} \right) \right).
\]
Now we use this information to improve the regularity. To this aim we compute
\[
\int_{\Omega} (u + n)^{qs} \leq \int_{\Omega} (u + n)^{q\alpha} (w + n)^{q\beta} \lesssim \|u + n\|^L^q \|w + n\|^C^q \lesssim |a_n|^\eta + 1 \|a_n\|^\eta + \eta^1,
\]
with \( \frac{1}{s} (\frac{q}{q+1} + \eta q^\beta) \leq 1 \) in case \( \alpha, \beta, s \) are chosen in such a way that
\[
s = \alpha + \beta, \quad 0 \leq \alpha \leq \frac{q+1}{q}, \quad 0 \leq \beta, \quad \frac{q}{q+1} + \eta q^\beta \leq s.
\]
Recall that we are looking for the larger \( s \) such that this system has a solution.

If \( q < \frac{N+1}{N+1} \) i.e. \( \frac{2q}{N+1} \leq \eta q < 1 \), we can choose \( \eta \) such that \( \frac{q}{N+1} < \eta q < 1 \). In that case we have for \( s > N \), \( \alpha = \frac{q+1}{q} \) and \( \beta = s - \frac{q+1}{q} \), the existence of \( 0 < \epsilon < \eta q < 1 \) such that
\[
\|w_n\|_{W^{1,s}} \lesssim |a_n|^{\epsilon} + 1,
\]
and hence
\[
\|w_n\|_{C^s} \lesssim |a_n|^\epsilon + 1,
\]
which allows to conclude that \( \frac{w_n}{|a_n|} \to 0 \) in \( C^1(I) \).

If \( q \geq \frac{N+2}{N} \) i.e. \( \eta q > \frac{2q}{N+2} \), the optimal is obtained for \( \alpha = \frac{q+1}{q} \) and \( \beta = s - \frac{q+1}{q} \), and \( s = \frac{q+1}{q} + \beta, \quad 1 + \eta q^\beta = s \).

This gives
\[
\beta = \frac{1}{q(\eta q - 1)} \left( N + 2 - (N - 2q) \frac{q}{q(N - (N + 2))} \right) \quad \text{and} \quad s = 1 + \frac{\eta}{\eta q - 1} < \frac{N(q - 1)}{Nq - (N + 2)}.
\]
The condition \( q < \frac{N+1}{N+1} \) implies again \( \frac{N(q-1)}{Nq - (N + 2)} > N \). Hence, choosing \( \eta \) close enough to \( \frac{2q}{N+1} \), we have \( s > N \), which implies that, for some \( \alpha > 0 \), \( w_n \in C^\alpha(I) \) with
\[
\|w_n\|_{C^\alpha} \lesssim |a_n| + 1.
\]
This allows to conclude in that case as in Step 2.
Step 5: \( N \geq 4, \frac{q}{q+1} < \frac{q}{2} \) and \( q^2 < q + 1 \leq q^3 \). If \( \frac{q}{q+1} \leq \frac{q}{2} \) then, define \( s_3 > 0 \) by

\[
\frac{1}{s_3} = \frac{q}{s_2} - \frac{2}{N}, \quad \text{if} \quad \frac{q}{s_2} - \frac{2}{N} > 0, \\
\quad s_3 > Nq, \quad \text{if} \quad \frac{q}{s_2} - \frac{2}{N} = 0.
\]

As \( q < \frac{N+1}{N-1} \), we easily see that \( s_3 > s_2 \) and

\[
\frac{1}{s_3} < \frac{1}{N}(q^2 - 1) - \left( \frac{q^3}{q+1} - 1 \right). \tag{3.12}
\]

Moreover we have by (3.11) and Sobolev inequalities

\[
\|w_n\|_{L^q} \lesssim |a_n|^{\frac{q^2}{q+1}} + 1,
\]

and hence

\[
\|(w_n^\#)\|_{L^q} \lesssim |a_n|^{\frac{q^3}{q+1}} + 1.
\]

As in Step 2, in case \( q^3 \geq q + 1 > q^2 \), we obtain \( (u_n^\#)^\# \in L^q(\Omega) \) with

\[
\|(u_n^\#)^\#\|_{L^q} \lesssim |a_n| + 1
\]

where

\[
s = (q^2 - 1) \frac{1}{q+1} \frac{q^3}{q+1} - 1 + \frac{1}{s_3}.
\]

This corresponds to

\[
\alpha = (q^3 - 1) \frac{1}{q+1} \frac{q^3}{q+1} - 1 + \frac{1}{s_3} \quad \text{and} \quad \beta = \frac{1}{q+1} \frac{q^3}{q+1} - 1 + \frac{1}{s_3}.
\]

By (3.12) we have \( s > N \). This implies that, for some \( \alpha > 0 \), \( w_n \in C_0^{1,\alpha}(\Omega) \) with

\[
\|w_n\|_{C_0^{1,\alpha}} \lesssim |a_n| + 1.
\]

We conclude in that case as in Step 2.

Step 6. We now proceed by recurrence. Let us define the sequence \((s_j)_{j \geq 3}\) by

\[
\frac{1}{s_j} = \frac{q}{s_{j-1}} - \frac{2}{N}. \tag{3.13}
\]

As we have proved in Step 5 that \( \frac{1}{s_3} < \frac{1}{s_2} \), it is easy to prove, by recurrence, that the sequence \((s_j)_{j \geq 3}\) is increasing. Moreover

\[
\lim_{j \to \infty} s_j = +\infty,
\]

as otherwise, if \( s_j \to l \), we can easily prove that \( l = \frac{N(q-1)}{2} < s_2 \) which contradicts the fact that \((s_j)\) is increasing. This implies the existence of \( J \) such that \( s_J > Nq \).
Now observe that, by bootstrap, for all \( j \in \{2, \ldots, J \}, \)

\[
\|w_n\|_{\psi^{-1}} \leq |a_n|^{\frac{q_j}{q + 1}} + 1. \tag{3.14}
\]

Let us prove that the result is valid if \( N \geq 4, q \) is such that \( q^j < q + 1 \) and, either \( \frac{s_j}{q} > \frac{N}{2} \) or \( q + 1 \leq q^j + 1 \).

Steps 3-4-5 concern the initial step \( j = 2 \). There, we have proved that the result is valid if \( N \geq 4, q^2 < q + 1 \) and, either \( \frac{s_2}{q} > \frac{N}{2} \) or \( q + 1 \leq q^2 \). Moreover, we have proved that, in case \( q + 1 \leq q^2 \), we have

\[
\frac{q^2 - 1}{q + 1} \geq N.
\]

So assume that the result is valid if \( N \geq 4, q^{j-1} < q + 1 \) and, either \( \frac{s_{j-1}}{q} > \frac{N}{2} \) or \( q + 1 \leq q^{j-1} \), and that, in case \( q + 1 \leq q^{j-1} \) we have

\[
\frac{q^{j-1} - 1}{q + 1} \geq N.
\]

Let us prove that the result is valid if \( N \geq 4, q^j < q + 1 \) and, either \( \frac{s_j}{q} > \frac{N}{2} \) or \( q + 1 \leq q^{j+1} \), and that, in case \( q + 1 \leq q^{j+1} \) we have

\[
\frac{q^j - 1}{q + 1} \geq N.
\]

**Case** \( \frac{s_j}{q} > N \). In that case, the result is proved as in Step 1 as, by (3.14), we have

\[
\|w_n\|_{C_0^1} \leq |a_n|^{\frac{q_j}{q + 1}} + 1 \quad \text{with} \quad \frac{q_j}{q + 1} < 1.
\]

**Case** \( \frac{s_j}{q} > \frac{N}{2} \). We obtain, by (3.14),

\[
\|w_n\|_{C} \leq |a_n|^{\frac{q_j}{q + 1}} + 1.
\]

As in Step 4, let us try now to minimize the exponent keeping the regularity. This means that we are looking for \( \alpha, \beta \) and \( s \) such that

\[
s = \alpha + \beta, \quad 0 \leq \alpha, \quad 0 \leq \beta, \quad \frac{\alpha q}{q + 1} + \frac{\beta q}{s_j} \leq 1, \quad \frac{\alpha q}{q + 1} + \frac{\beta q}{s_j} = \eta s,
\]

with \( s > \frac{N}{2} \) and \( \eta \) minimum.

By recurrence, we easily prove that

\[
\frac{2}{N} \left( \frac{\frac{qN}{s_j} - 1}{s_j} - 1 \right) = \frac{2}{N} \left( (\frac{2N}{s_j} - 1) + q^{j}(1 - \frac{\frac{qN}{s_j} - 1}{s_j}) \right) = \frac{2}{N + 2 - (N - 2)q}.
\]
Hence, as in Step 4, we obtain

$$\|w_n\|_C \leq |a_n|^\eta + 1 \quad \text{with} \quad \eta > \frac{2}{N + 2 - (N - 2)q}.$$ 

and we conclude as in Step 4.

**Case** $\frac{q}{q} < \frac{N}{2}$ **and** $q^j < q + 1 \leq q^{j+1}$. In that case, we prove as in Step 2 that

$$\|(a_n^j)^h\|_{L^s} \leq |a_n| + 1,$$

with

$$s = \frac{q^j -1}{q^{j+1}-1 + \frac{1}{s_{j+1}}}.$$ 

This corresponds to

$$\alpha = \left(\frac{q^{j+1}}{q+1} - 1\right) \frac{1}{q^{j+1} \left(\frac{q^{j+1}}{q+1} - 1 + \frac{1}{s_{j+1}}\right)} \quad \text{and} \quad \beta = \frac{1}{q + 1} \frac{1}{q \left(\frac{q^{j+1}}{q+1} - 1 + \frac{1}{s_{j+1}}\right)}.$$ 

It remains to prove that $s > N$ to conclude.

Assume that

$$\frac{q^{j+1} - 1}{q^{j+1} - 1 + \frac{1}{s_j}} > N,$$

and let us prove that

$$\frac{q^j - 1}{q^{j+1} - 1 + \frac{1}{s_{j+1}}} > N.$$ 

In fact, we have, by the recurrence assumption and as $q < \frac{N+1}{N+1}$,

$$N \left(\frac{q^{j+1}}{q+1} - 1 + \frac{1}{s_{j+1}}\right) = N \left(\frac{q^{j+1}}{q+1} - 1 + \frac{1}{s_j} - \frac{2}{N}\right)$$

$$= qN \left(\frac{q^j}{q+1} - 1 + \frac{1}{s_j}\right) + qN - N - 2$$

$$< q(q^{j-1} - 1) + qN - N - 2$$

$$= q^j - 1 + (N - 1)q - (N + 1)$$

$$< q^j - 1.$$ 

which proves the recurrence.

Hence in case $\frac{q}{q} < \frac{N}{2}$ **and** $q^j < q + 1 \leq q^{j+1}$, we have

$$\|w_n\|_{W^{2,s}} \leq |a_n| + 1 \quad \text{with} \quad s > N,$$

and we conclude as in Step 2.
Conclusion. If $N = 3$, the Claim is obtained by Step 1. So we assume $N \geq 4$.

Let us choose $j \in \{1, \ldots, J\}$ in the following way

\[ j = 1, \quad \text{if } q + 1 \leq q^2, \]
\[ j = J, \quad \text{if } q + 1 > q^J, \]
\[ j = k \in \{2, \ldots, J - 1\}, \quad \text{if } q^k < q + 1 \leq q^{k+1}. \]

The case $j = 1$ is considered in Step 2.

In case $j = J$, we have by construction

\[ \|w_n\|_{W^1_2} \lesssim |a_n|^\frac{q}{q+1} + 1, \]

with $\frac{q}{q+1} > N$ and $\frac{q}{q+1} < 1$. Hence we obtain

\[ \|w_n\|_{C_0^1} \lesssim |a_n|^\frac{q'}{q+1} + 1, \quad \text{with } \frac{q'}{q+1} < 1, \]

which allows to conclude that $\frac{\partial a_n}{\partial x_i} \to 0$ in $C_0^1(\Omega)$.

In case $j \in \{2, \ldots, J - 1\}$ the Claim is proved by Step 3-4-5 (initial case $j = 2$) or Step 6 (recurrence case). Hence the Claim is proved in every case.

By the Claim, we see that for $r > K$, the solution $u_r$ of (2.1) with $u_r \not\geq \beta$, $u_r \not\leq \alpha$, $u_r \geq \alpha_r$ and $\Phi_r(u_r) = c_r$ given by Proposition 3.1 satisfies $u_r > -K > -r$. This implies that, for this value of $r$, $u_r$ is a solution of (1.1) which concludes the proof.

References