# Weighted eigenvalue problems for quasilinear elliptic operators with mixed Robin-Dirichlet boundary conditions

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**Abstract**

We investigate the existence of principal eigenvalues type problems with weights for the quasilinear operator $-\Delta_p + V\psi_p$ with mixed weighted Robin-Dirichlet boundary conditions in a bounded regular domain. We also give some results on the existence of non principal eigenvalues.

*Keywords:* $p$-laplacian, indefinite weights, indefinite potential, principal eigenvalues Robin boundary conditions.
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## 1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary $\partial\Omega$ and let $\nu$ be its outer normal defined everywhere. Let $V$ be a bounded function defined in $\Omega$ and $\sigma$ a smooth function defined on $\partial\Omega$. It was pointed out in [8], [9] that the Fourier analysis for parabolic problems with dynamic boundary conditions leads, through a separation of variables, to the following eigenvalue problem with Robin type boundary conditions

$$(L) \qquad \begin{cases} -\Delta u + V(x)u = \lambda u & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = \lambda\sigma(x)u & \text{on} \quad \partial\Omega. \end{cases}$$

The complete analysis of this eigenvalue problem when $V \geq 0$ has been done in [8] in the case $\sigma = cst$, and in [9] in the case $\sigma \neq cst$ (for an slighter general operator than the laplacian). As a common feature, it appears that problem $(L)$ possesses an infinite sequence of positive eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ if $\sigma^+$ (the positive part of $\sigma$) is $\not\equiv 0$, and an infinite sequence of negative eigenvalues $\{\lambda_{-n}\}_{n\in\mathbb{N}}$ if $\sigma^- \not\equiv 0$ and $N \geq 2$. Moreover, $\lambda_{\pm 1}$ are both principal eigenvalue (i.e., an eigenvalue whose eigenfunctions are sign-constant) and simple (i.e the associated eigenfunctions are each a constant multiple of one another).

It is also well known that the spectra of the $-\Delta + V$, in the case $V^- \not\equiv 0$, could present features different to those of the spectra in the case $V \geq 0$. Problem $(L)$ with $V$ indefinite and Dirichlet boundary conditions has been extensively studied for instance by Allegretto-Mingarelli [4], Fleckinger-Hernandez-de Thelin [15] and Lopez-Gomez [20] among others.

Our intention in this paper is to initiate the *study of the spectrum* of the more general problem

$$(P) \qquad \begin{cases} -\Delta_p u + V(x)|u|^{p-2}u = \lambda m(x)|u|^{p-2}u & \text{in} \quad \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda\sigma|u|^{p-2}u & \text{on} \quad \Gamma_1, \\ u = 0 & \text{on} \quad \Gamma_2, \end{cases}$$

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with $V, m$ and $\sigma$ indefinite. Here $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$, denotes the $p$-Laplacian operator for $p > 1$. We will assume that $\Omega$ is a bounded smooth domain and that $\partial\Omega$ splits up in two sets $\Gamma_1$ and $\Gamma_2$ which are connected and closed $(n-1)-$manifolds.

The existence of principal eigenvalues for the quasilinear equation in problem $(P)$ with Dirichlet boundary condition and $V, m$ indefinite has been treated by Binding-Huang [10, 11] and [13]. It appears that sometimes there are not principal eigenvalues, a phenomenon that depends, loosely speaking, in how big the negative part of $V$ with respect to the negative part of $m$ is.

Problem $(P)$ for $p = 2$ with $V \equiv 0$, $\Gamma_2 = \emptyset$ and $m$ indefinite has already been considered by Afrouzi-Brown [2] when $\sigma = cst$, and later by K. Umezu [23] for indefinite $\sigma$. This last author proved that, besides the trivial eigenvalue $\lambda = 0$, problem $(P)$ possesses a *unique* positive principal eigenvalue if and only if

$$\oint_{\partial\Omega} \sigma \, d\rho + \int_{\Omega} m \, dx < 0.$$

Here $d\rho$ stands for the surface element of $\partial\Omega$.

In the case $V \not\equiv 0$ and possibly indefinite, the situation is much different since the energy functional $E_V(u) \stackrel{\text{def}}{=} \int_\Omega (|\nabla u|^p + V|u|^p)dx$ is indefinite. One approach to find principal eigenvalues that has been used by many authors is to define a new eigenvalue problem for each fixed $\lambda$ and to construct "an eigenvalue curve" as $\lambda$ varies. We apply this approach in section 3 and we give in section 4 a necessary and sufficient condition for the existence of eigenvalues in terms of the infimum of $E_V$ over the set of functions $\mathcal{G}$ satisfying $\int_\Omega |u|^p \, dx = 1$ and

$$\int_{\Omega} m|u|^p \, dx + \oint_{\Gamma_1} |u|^p \sigma \, d\rho = 0.$$

Indeed, this set $\mathcal{G}$ was already considered by [10], [11] and [13] for the quasilinear equation of problem $(P)$ with Dirichlet, Newmann or mixed boundary conditions. In fact, one cannot exclude that some eigenfunctions belong to $\mathcal{G}$ and, in that case, the corresponding energy levels are called in [11] "ghost states". Ghost states are interesting because they have the property of *losing of compactness*, see the discussion in section 8 and remark 8.6.

This paper is organized as follows. We construct the eigencurve associated to problem $(P)$ in section 3. The existence of principal eigenvalues is studied in section 4. A sufficient condition for the existence of principal eigenvalues is presented in section 5, where we also discuss the necessity of such condition for the coerciveness of the related functional. In section 6 we prove isolation and simplicity of principal eigenvalues of problem $(P)$. In section 7 we investigate the coerciveness of the restricted functional and in section 8 we prove the existence of two unbounded sequences of eigenvalues in the case where either $m^\pm \not\equiv 0$ or $\sigma^\pm \not\equiv 0$. We also exhibit an example of problem $(P)$ in dimension 1, with a changing-sign $\sigma$, that fails to have a double sequence of eigenvalues.

Notice that the Dirichlet-Newmann eigenvalue problem associated to $(P)$, that is

$$\begin{cases} -\Delta_p u + V(x)|u|^{p-2}u = \lambda m(x)|u|^{p-2}u & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \Gamma_1, \\ u = 0 & \text{on} \quad \Gamma_2, \end{cases}$$

corresponds to the case $\sigma \equiv 0$. As a by-product of our main result in Theorem 4.1 we will show that there exists always a unique principal eigenvalue for this problem.

## 2. Main assumptions and useful inequalities

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $C^{2,\alpha}$ for some $0 < \alpha < 1$. The Lebesgue measure of a measurable set $A$ of $\mathbb{R}^N$ will be denoted by $\lambda_N(A)$. We denote by $\rho$ the restriction to $\partial\Omega$ of

the $(N-1)$-Hausdorff measure, which coincides with the usual Lebesgue surface measure as $\partial\Omega$ is regular enough. We denote by $\nu = \nu(x)$ its outer normal at $x$, defined for all $x \in \partial\Omega$. We will assume that $\partial\Omega$ splits up in two sets $\Gamma_1$ and $\Gamma_2$ which are connected and closed $(n-1)-$manifolds. We allow $\Gamma_2$ to be the empty set.

Throughout this paper we always assume that the weights $V, m \in L^\infty(\Omega)$ and $\sigma \in C^{0,r}(\Gamma_1)$ for some $0 < r < 1$, are possibly indefinite. We will always assume that either $m$ or $\sigma$ is not equivalent to 0. We will denote

$$\Omega^+ \stackrel{\text{def}}{=} \{x \in \Omega \mid m(x) > 0\},\ \Omega^- \stackrel{\text{def}}{=} \{x \in \Omega \mid m(x) < 0\},\ \Omega^0 \stackrel{\text{def}}{=} \{x \in \Omega \mid m(x) = 0\}.$$

Similarly, we will denote $\Gamma_1^+, \Gamma_1^-$ and $\Gamma_1^0$ the positive, negative and null set of the weight $\sigma$ in $\Gamma_1$. We denote $W^{1,p}(\Omega)$ the classical Sobolev space endowed with the classical norm

$$\|u\| := \left( \int_\Omega |\nabla u|^p \, dx + \int_\Omega |u|^p \, dx \right)^{1/p}.$$

The Lebesgue norm of $L^q(\Omega)$ will be denoted by $\|\cdot\|_q$, and the Lebesgue norm of $L^q(\partial\Omega, \rho)$ by $\|\cdot\|_{q,\partial\Omega}$, for any $q \in [1, +\infty]$.
The conjugate of any $r \in [1, +\infty]$ will be denoted by $r'$, the critical Sobolev exponent for the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ will be denoted by $p^* := \frac{pN}{N-p}$ if $1 < p < N$, $p^* = +\infty$ otherwise.
The *trace* operator will be denoted by $\gamma$, that is

$$\gamma : W^{1,p}(\Omega) \to W^{1-\frac{1}{p},p}(\partial\Omega, \rho).$$

Recall that there is a continuous boundary trace embedding $W^{1,p}(\Omega) \to L^q(\partial\Omega, \rho)$ for every $q \in [1, p_*]$ and that those embeddings are compact for $q \in [1, p_*[$. Here we denote by $p_* = \frac{p(N-1)}{(N-p)^+}$ the critical exponent for the above trace embedding. For the properties of $\gamma$ (especially the surjectivity) we refer to [1].
We put $W \stackrel{\text{def}}{=} \{u \in W^{1,p}(\Omega) \mid \gamma(u) = 0 \text{ on } \Gamma_2\}$. One can show that

$$\|u\|_W := \begin{cases} \int_\Omega |\nabla u|^p \, dx + \int_{\partial\Omega} |u|^p d\rho & \text{if } \Gamma_2 = \emptyset; \\ \int_\Omega |\nabla u|^p \, dx & \text{if } \Gamma_2 \neq \emptyset, \end{cases}$$

is a norm on $W$ equivalent to $\|\cdot\|$.

Let us recall Picone's identity [3]:

**Lemma 2.1 ([3]).** *Let $w \geq 0, v > 0$ be two continuous functions in $\Omega$ differentiable a.e. Denote*

$$L(w,v) = |\nabla w|^p + (p-1)\tfrac{w^p}{v^p}|\nabla v|^p - p\tfrac{w^{p-1}}{v^{p-1}}|\nabla v|^{p-2}\nabla v \nabla w\,,$$

$$R(w,v) = |\nabla w|^p - |\nabla v|^{p-2}\nabla(\tfrac{w^p}{v^{p-1}})\nabla v.$$

*Then* **(i)** *$L(w,v) = R(w,v)$,* **(ii)** *$L(w,v) \geq 0$ a.e. and* **(iii)** *Assume that $\frac{w}{v} \in W^{1,1}_{loc}(\Omega)$. Then $L(w,v) = 0$ a.e. in $\Omega$ if and only if $w = kv$ for some $k \in \mathbb{R}$.*

The following property can be found for instance in [19]:

**Lemma 2.2 ([19]).** *The operator $-\Delta_p : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ satisfies the so called $S^+$ property: for all sequence $u_n \in W^{1,p}(\Omega)$ such that $u_n \hookrightarrow u_0$ weakly in $W^{1,p}(\Omega)$ and*

$$\lim_{n\to+\infty} \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u_0) dx = 0,$$

*it holds $\|\nabla u_n - \nabla u_0\|_p \to 0$.*

## 3. An eigenvalue curve associated to problem $(P)$

It is well established that, in order to prove the existence of principal eigenvalues of $(P)$, one fixes $\lambda$ and embeds the problem into the new *eigenvalue problem of parameter* $\mu$ :

$$\begin{cases} -\Delta_p u + V(x)|u|^{p-2}u = \lambda m(x)|u|^{p-2}u + \mu|u|^{p-2}u & \text{in} \quad \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda\sigma|u|^{p-2}u & \text{on} \quad \Gamma_1, \\ u = 0 & \text{on} \quad \Gamma_2. \end{cases} \tag{3.1}$$

A value $\mu \in \mathbb{R}$ is called an *eigenvalue* for problem (3.1) if and only if there exists $u \in W$, $u \not\equiv 0$, satisfying the equation $(P)$ in the weak sense, i.e., $\forall w \in W$

$$\int_\Omega (|\nabla u|^{p-2}\nabla u \cdot \nabla w + (V - \lambda m)|u|^{p-2}uw)\, dx = \mu \int_\Omega |u|^{p-2}uw\, dx + \lambda \oint_{\Gamma_1} \sigma|u|^{p-2}uw\, d\rho.$$

The function $u \in W$ is called an *eigenfunction*. An eigenvalue is called *principal* if it possesses an eigenfunction $u > 0$ a.e. in $\Omega$.

**Remark 3.1.** *We recall that, by the regularity results of [14] and [18, Theorem 2], bounded weak solutions of* (3.1) *are of class* $C^{1,\beta}(\overline{\Omega})$ *for some* $0 < \beta < 1$. *We use here that* $\Omega$ *is of class* $C^{2,\alpha}$ *as well as that* $\sigma \in C^{0,r}(\partial\Omega)$ *for some* $0 < \alpha, r < 1$. *That all the solutions of* (3.1) *belongs to* $L^\infty(\Omega) \cap L^\infty(\partial\Omega, \rho)$ *has been proved, for instance, in [17] in the case* $m = 0$. *See Theorem 9.1 in the appendix for a more general result.*

We are going to consider the smaller eigenvalue $\mu \in \mathbb{R}$ of problem (3.1). In order to do so, we define the energy functional

$$J_\lambda : W \to \mathbb{R}; \quad J_\lambda(u) \stackrel{\text{def}}{=} E_V(u) - \lambda I(u),$$

where

$$E_V(u) \stackrel{\text{def}}{=} \int_\Omega \left(|\nabla u|^p + V|u|^p\right) dx,$$

and

$$I(u) \stackrel{\text{def}}{=} \int_\Omega m\, |u|^p\, dx + \oint_{\Gamma_1} \sigma\, |u|^p\, d\rho.$$

Let us also consider the manifold

$$S \stackrel{\text{def}}{=} \{u \in W \mid \int_\Omega |u|^p\, dx = 1\}.$$

**Proposition 3.2.** *The value*

$$\mu_1(\lambda) \stackrel{\text{def}}{=} \inf\{J_\lambda(u) \mid u \in S\} \in \mathbb{R}$$

*is the smaller eigenvalue of* (3.1). *Moreover* $\mu_1(\lambda)$ *is principal, simple and it is the unique principal eigenvalue associated to* (3.1).
*If we denote by* $\varphi_\lambda \in S$ *the (unique) positive eigenfunction for problem* (3.1), *it holds that* $\varphi_\lambda \in C^{1,\beta}(\overline{\Omega})$, $\varphi_\lambda > 0$ *in* $\Omega \cup \Gamma_1$.

*Proof.* The proof is quite standard but we include it for the sake of completeness. One uses the following simple estimate that can be proved by arguing by contradiction:
$\forall \epsilon > 0 \quad \exists c(\epsilon) > 0$ *such that*

$$\oint_{\partial\Omega} |u|^p\, d\rho \le \epsilon \int_\Omega |\nabla u|^p\, dx + c(\epsilon) \int_\Omega |u|^p\, dx \quad \forall u \in W^{1,p}(\Omega). \tag{3.2}$$

Thus we have, for all $u \in S$ and all $\lambda \in \mathbb{R}$,

$$J_\lambda(u) \geq (1 - |\lambda|\epsilon \, \|\sigma\|_{\infty,\Gamma_1}) \int_\Omega |\nabla u|^p \, dx - \|V\|_\infty - |\lambda|(\|m\|_\infty + c(\epsilon) \, \|\sigma\|_{\infty,\Gamma_1}). \tag{3.3}$$

Then, by choosing any $0 < \epsilon < (|\lambda|\|\sigma\|_{\infty,\Gamma_1})^{-1}$ in (3.3), it comes that $J_\lambda$ is bounded from below on $S$. Since $J_\lambda$ is sequentially weakly lower semi-continuous, it follows that $\mu_1(\lambda)$ is achieved at some $u \in S$. By Lagrange multipliers rule, one concludes that $\mu_1(\lambda)$ is an eigenvalue for (3.1) and $u$ is an associated eigenfunction. Since $J_\lambda(u) = J_\lambda(|u|)$ it follows that $|u|$ is also an eigenfunction for $\mu_1(\lambda)$. By the regularity results already quoted in Remark 3.1, $u \in C^{1,\beta}(\overline{\Omega})$ and, by the well known Strong Maximum Principle of [24], we conclude that $|u| > 0$ in $\Omega$, so either $u > 0$ in $\Omega$ or $u < 0$ in $\Omega$, that is, $\mu_1(\lambda)$ is principal. Moreover $u > 0$ on $\Gamma_1$ otherwise, by the Hopf maximum principle of [24], one will infer that $\frac{\partial u}{\partial \nu} < 0$ on $\Gamma_1$, in contradiction with the boundary condition of (3.1).

To prove that $\mu_1(\lambda)$ is simple, assume that $w, v \in W$ are two different eigenfunctions of (3.1) for $\mu_1$. Thus, we can then assume that $w > 0$ and $v > 0$ in $\Omega$. Next, one uses Picone's identity as follows. Choose for any $\eta > 0$ the function $\frac{w^p}{(v+\eta)^{p-1}}$ as a test function in the equation (3.1) satisfied by $v$ and $w$ as a test function in the equation (3.1) satisfied by $w$ to get

$$0 \leq \int_\Omega R(w, v+\eta) \, dx = \int_\Omega |\nabla w|^p \, dx - \int_\Omega |\nabla v|^{p-2} \nabla v \cdot \nabla \left(\frac{w^p}{(v+\eta)^{p-1}}\right) dx =$$

$$\int_\Omega |\nabla w|^p \, dx - \int_\Omega (V - \lambda m - \mu_1(\lambda)) w^p \frac{v^p}{(v+\eta)^{p-1}} \, dx - \lambda \oint_{\Gamma_1} \sigma w^p \frac{v^p}{(v+\eta)^{p-1}} \, d\rho.$$

By letting $\eta \to 0$ it comes that $R(u, v) = 0$ and the conclusion follows. □

The curve $\mu_1 : \mathbb{R} \to \mathbb{R}$ is known as *eigencurve* associated to problem $(P)$. This notion was first used by [10, 11], for Newmann or Dirichlet boundary problems, as well as the following properties that we will prove here for the weighted Robin-Dirichlet boundary conditions.

**Proposition 3.3.** *Assume that $m$ or $\sigma$ are $\not\equiv 0$. The following properties hold :*

1. *$\mu_1$ is concave, differentiable and*

$$\mu_1'(\lambda) = -\left(\int_\Omega m \, \varphi_\lambda^p \, dx + \oint_{\Gamma_1} \sigma \varphi_\lambda^p \, d\rho\right) \tag{3.4}$$

.

2. *If $m^+ \not\equiv 0$ in $\Omega$ or $\sigma^+ \not\equiv 0$ on $\Gamma_1$ then $\lim_{\lambda \to +\infty} \mu_1(\lambda) = -\infty$.*
3. *If $m^- \not\equiv 0$ in $\Omega$ or $\sigma^- \not\equiv 0$ on $\Gamma_1$ then $\lim_{\lambda \to -\infty} \mu_1(\lambda) = -\infty$.*
4. *If $m \geq 0$ in $\Omega$ and $\sigma \geq 0$ on $\Gamma_1$ (resp. $m \leq 0$ in $\Omega$ and $\sigma \leq 0$ on $\Gamma_1$) then $\mu_1$ is strictly decreasing (resp. strictly increasing).*
5. *Let us denote*

$$\alpha \stackrel{\text{def}}{=} \inf\{E_V(u) \mid u \in S, \int_\Omega m \, |u|^p \, dx + \oint_{\Gamma_1} \sigma \, |u|^p d\rho = 0\}. \tag{3.5}$$

*Then $\alpha = \sup_{\lambda \in \mathbb{R}} \mu_1(\lambda)$.*

*Proof.* (1) For a fixed $u \in W$ the mapping $\lambda \mapsto J_\lambda(u)$ is concave and then the infimum over $S$, that is $\mu_1(\lambda)$, is also concave and therefore continuous. Now let $\lambda_n \to \lambda$ and $\varphi_n, \varphi_\lambda$ be the $L^p$-normalized positive eigenfunctions related to $\lambda_n, \lambda$ respectively. If we apply (3.3) with $\lambda = \lambda_n$ and $u = \varphi_n$ we have, after choosing $\epsilon$ small enough, that

$$\|\nabla \varphi_n\|_p^p \leq C_1$$

for some $C_1 > 0$. So we conclude that the sequence $\varphi_n$ is bounded in $W$. Hence there exists $\varphi_0$ such that, up to a subsequence, $\varphi_n \rightharpoonup \varphi_0$ in $W$, strongly in $L^p(\Omega)$ and in $L^p(\partial\Omega, \rho)$. Then $\|\varphi_0\|_p = 1$ and, from

$$\mu(\lambda) = \lim_{n\to+\infty} \mu(\lambda_n) \geq E_{V-\lambda m}(\varphi_0) - \lambda \oint_{\Gamma_1} \sigma|\varphi_0|^p \, d\rho \geq \mu(\lambda),$$

we infer that $\varphi_0 = \varphi_\lambda$. Furthermore

$$\begin{aligned}
\mu(\lambda_n) &= E_{V-\lambda_n m}(\varphi_n) - \lambda_n \oint_{\Gamma_1} \sigma|\varphi_n|^p \, d\rho \\
&= E_{V-\lambda m}(\varphi_n) + (\lambda - \lambda_n) \int_\Omega m|\varphi_n|^p \, dx - \lambda_n \oint_{\Gamma_1} \sigma|\varphi_n|^p \, d\rho \\
&\geq \mu(\lambda) + (\lambda - \lambda_n) \int_\Omega m|\varphi_n|^p dx + (\lambda - \lambda_n) \oint_{\Gamma_1} \sigma|\varphi_n|^p \, d\rho
\end{aligned}$$

and replacing $\lambda$ (resp. $\varphi$) by $\lambda_n$ (resp. $\varphi_n$) in this inequality we have, for $\lambda_n > \lambda$,

$$-\int_\Omega m\,|\varphi_n|^p \, dx - \oint_{\Gamma_1} \sigma|\varphi_n|^p \, d\rho \leq \frac{\mu(\lambda_n) - \mu(\lambda)}{\lambda_n - \lambda} \leq -\int_\Omega m\,|\varphi_\lambda|^p \, dx - \oint_{\Gamma_1} \sigma|\varphi_\lambda|^p \, d\rho.$$

Passing to the limit we get (3.4). A similar argument holds if $\lambda_n < \lambda$.
(2) If $m^+ \not\equiv 0$ then it is easy to see that there exists $u_0 \in W_0^{1,p}(\Omega) \cap S$ such that $\int_\Omega m|u_0|^p \, dx > 0$. Indeed, one can choose $u_0$ as the regularization of the characteristic function of any small ball $B$ strictly contained in $\Omega$ for which $m^+ \not\equiv 0$ in $B$. Hence

$$\mu_1(\lambda) \leq E_V(u_0) - \lambda \int_\Omega m|u_0|^p \, dx \to -\infty \quad \text{as } \lambda \to +\infty.$$

If $m^+ \equiv 0$ in $\Omega$ and $\sigma^+ \not\equiv 0$ on $\Gamma_1$ we use Lemma 3.4 below to get the existence of $u_0 \in S$ such that $\int_\Omega m|u_0|^p \, dx + \oint_{\Gamma_1} \sigma|u_0|^p \, d\rho = 1$ and the conclusion follows.
(3) Similar to the previous case.
(4) If $m \geq 0$ in $\Omega$ and $\sigma \geq 0$ on $\Gamma_1$, the result is clear from the fact that

$$\int_\Omega m|\varphi_\lambda|^p \, dx + \oint_{\Gamma_1} \sigma|\varphi_\lambda|^p \, d\rho > 0$$

for any $\lambda \in \mathbb{R}$. Indeed, if $\lambda_1 < \lambda_2$ then

$$\mu(\lambda_1) = E_V(\varphi_{\lambda_1}) - \lambda_1 \Big( \int_\Omega m\,|\varphi_{\lambda_1}|^p \, dx + \oint_{\Gamma_1} \sigma|\varphi_{\lambda_1}|^p \, d\rho \Big)$$

$$> E_V(\varphi_{\lambda_1}) - \lambda_2 \Big( \int_\Omega m\,|\varphi_{\lambda_1}|^p \, dx + \oint_{\Gamma_1} \sigma|\varphi_{\lambda_1}|^p \, d\rho \Big) \geq \mu(\lambda_2).$$

(5) Let us prove that $\sup_{\lambda\in\mathbb{R}} \mu(\lambda) = \alpha$. Notice that trivially

$$\alpha \geq \mu(\lambda)$$

for all $\lambda \in \mathbb{R}$. We distinguish the following two complementary cases :
(a1) Either $m \geq 0$ and $\sigma \geq 0$ or $m \leq 0$ and $\sigma \leq 0$. In this first alternative we know that $\mu(\lambda)$ is strictly decreasing so then

$$\sup \mu(\lambda) = \lim_{\lambda \to -\infty} \mu(\lambda). \tag{3.6}$$

Let $\lambda_n \to -\infty$ when $n \to \infty$ and let $\varphi_n$ be the associated $L^p$-normalized eigenfunction. We have

$$\mu(\lambda_n) = E_{V-\lambda_n m}(\varphi_n) - \lambda_n \oint_{\Gamma_1} \sigma |\varphi_n|^p \, d\rho \geq E_V(\varphi_n) \geq \|\varphi_n\|^p - 1 - \|V\|_\infty$$

for all $\lambda_n \leq 0$. Thus the sequence $\varphi_n$ is bounded in $W$ so there exists $\varphi \in W$ such that, up to a subsequence, $\varphi_n \rightharpoonup \varphi$ in $W$, strongly in $L^p(\Omega) \cap L^p(\partial\Omega, \rho)$. Thus $\|\varphi\|_p = 1$ and

$$\alpha \geq \lim_{n\to\infty} \mu(\lambda_n) \geq E_V(\varphi) - \lim_{n\to\infty} \lambda_n \Big( \int_\Omega m \, |\varphi_n|^p \, dx + \oint_{\Gamma_1} \sigma |\varphi_n|^p \, d\rho \Big). \tag{3.7}$$

If $m > 0$ a.e on $\Omega$ then, from (3.7), we have

$$\lim_{\lambda \to -\infty} \mu(\lambda) = +\infty = \alpha.$$

If $\lambda_N(\{x \in \Omega \mid m(x) = 0\}) > 0$ then from Proposition 3.5(iii) $\alpha < +\infty$ and therefore $\mu$ is bounded from above. We conclude from (3.7) that $\int_\Omega m|\varphi|^p \, dx + \oint_{\Gamma_1} \sigma |\varphi|^p \, d\rho = 0$ and then $\varphi$ is admissible in the definition of $\alpha$. Again from (3.7), we get

$$\alpha \geq \lim_{n\to\infty} \mu(\lambda_n) \geq E_V(\varphi) \geq \alpha$$

and the result follows from (3.6). If $m \leq 0$ and $\sigma \leq 0$ we argue similarly.

(a2) Either $m^+ \not\equiv 0$ and $\sigma^- \not\equiv 0$ or $m^- \not\equiv 0$ and $\sigma^+ \not\equiv 0$. In both cases $\alpha < +\infty$ from Proposition 3.5 below and it follows from (1)-(3) that $\mu$ is bounded from above. Then $\sup_{\lambda \in \mathbb{R}} \mu(\lambda)$ is achieved at some $\lambda_0$ that satisfies $0 = \mu'(\lambda_0) = -\int_\Omega m \, \varphi_{\lambda_0}^p \, dx - \oint_{\Gamma_1} \sigma \varphi_{\lambda_0}^p \, d\rho$. We conclude that $\varphi_{\lambda_0}$ is admissible in the definition of $\alpha$ and hence

$$\alpha \leq E_V(\varphi_{\lambda_0}) = \mu(\lambda_0) = \sup_{\lambda \in \mathbb{R}} \mu(\lambda).$$

□

**Lemma 3.4.** *If $\sigma^- \not\equiv 0$ or $m^- \not\equiv 0$ then there exists $u_0 \in W$ such that $supp\, u_0 \subset \Omega^-$, $supp\, \gamma(u_0) \subset \Gamma_1^-$ and*

$$\int_\Omega m \, |u_0|^p \, dx + \oint_{\Gamma_1} \sigma \, |u_0|^p d\rho < 0. \tag{3.8}$$

*Proof.* If $m^- \not\equiv 0$ one can take $u_0 \in W_0^{1,p}(\Omega)$ such that $\int_\Omega m|u_0|^p \, dx < 0$ and $supp\, u_0 \subset \Omega^-$. If $m^- \equiv 0$ pick any $0 < \psi \in C^\infty(\Gamma_1)$ with $supp\, \psi \subset \Gamma_1^-$. Put $a := \oint_{\Gamma_1} \sigma \psi^p \, d\rho < 0$ and take a function $0 \leq v \in W^{1,\infty}(\Omega)$ such that $\gamma(v) = \psi$. Consider for $\epsilon > 0$ small a cut-function $\xi_\epsilon \in C^\infty(\mathbb{R}^N)$ such that

$$\xi_\epsilon \equiv 1 \text{ in } \Omega_\epsilon, \quad \xi_\epsilon \equiv 0 \text{ in } \mathbb{R}^N \setminus \Omega_{\epsilon/2},$$

where

$$\Omega_\epsilon := \{x \in \Omega \mid dist\,(x, \Gamma_1) > \epsilon\}.$$

Thus $u_0 := (1 - \xi_\epsilon)v \in W$ and $\gamma(u_0) = \psi$. Choose $\epsilon > 0$ small enough to have

$$\lambda_N(\Omega \setminus \Omega_\epsilon) < \frac{-a}{2\|m\|_\infty \|v\|_\infty^p}.$$

Hence

$$\int_\Omega |m||u_0|^p\,dx \le \|m\|_\infty \|v\|_\infty^p \lambda_N(\Omega\setminus\Omega_\epsilon) \le -\frac{1}{2}\oint_{\Gamma_1}\sigma|u_0|^p\,d\rho$$

and therefore

$$\int_\Omega m\,|u_0|^p\,dx + \oint_{\Gamma_1}\sigma\,|u_0|^p d\rho < 0.$$

□

Let us state in which cases the value $\alpha$ is finite.

**Proposition 3.5.** *Let us define the set $\mathcal{G}$ as*

$$\mathcal{G} \stackrel{\text{def}}{=} \{u\in S \mid \int_\Omega m|u|^p\,dx + \oint_{\Gamma_1}\sigma|u|^p\,d\rho = 0\}. \tag{3.9}$$

*Then $\mathcal{G}\neq\emptyset$ if and only if either*
*(i) $m^\pm \not\equiv 0$ or (ii) $m^+\not\equiv 0$ and $\sigma^-\not\equiv 0$ or $m^-\not\equiv 0$ and $\sigma^+\not\equiv 0$ or (iii) $\lambda_N(\{x\in\Omega \mid m(x)=0\})>0$.*
*In all these cases the value $\alpha$ defined in (3.5) is achieved at some $0\le\xi_0\in W$.*

*Proof.* (i) If $m^\pm\not\equiv 0$ one can always find function on $W_0^{1,p}(\Omega)$ belonging to $\mathcal{G}$ by taking, for instance, $u=u_1-u_2$ with $u_1$ a positive function with support on $\Omega^+$, $u_2$ a positive function with support on $\Omega^-$ and, after rescaling, then make $u\in\mathcal{G}$.
(ii) If, say, $m^+\not\equiv 0$ and $\sigma^+\not\equiv 0$, we pick any $0<\psi\in C^\infty(\Gamma_1)$ such that $a:=\oint_{\Gamma_1}\sigma\psi^p\,d\rho<0$. Let $v\in W^{1,\infty}(\Omega)$ such that $\gamma(v)=\psi$ and $u_0=(1-\xi_\epsilon)v$ with $\xi_\epsilon$, as in Lemma 3.4 above. We now choose $\epsilon>0$ small enough to guarantee that $\lambda_N(\Omega_\epsilon\cap\Omega^+)\neq 0$. Let us take another function $0\le w\in W_0^{1,p}(\Omega_\epsilon)$ such that

$$\int_{\Omega_\epsilon} mw^p\,dx = -a.$$

Hence, as $u_0$ and $w$ have disjoint supports,

$$\int_\Omega m|w+u_0|^p dx + \oint_{\Gamma_1}\sigma\,|u_0+w|^p d\rho = 0.$$

(iii) If $\lambda_N(\Omega^0)>0$ we can always find a function $0\not\equiv u\in W_0^{1,p}(\Omega)$ satisfying $\int_\Omega m|u|^p\,dx=0$, so, after normalisation, $u$ is admissible in the definition of $\alpha$.

Finally notice that any minimizing sequence of $\alpha$ is bounded by the estimate

$$E_V(u)\ge\int_\Omega|\nabla u|^p\,dx - \|V\|_\infty$$

valid for any $u\in S$. Then, by standard arguments, one can show that $\alpha$ is achieved at some $\xi_0\ge 0$. □

## 4. Existence of principal eigenvalues of $(P)$

As a consequence of Proposition 3.3 we prove our main result of this section. Recall that the value $\alpha$ have been defined in (3.5) and the set $\mathcal{G}$ in (3.9).

**Theorem 4.1.** 1. (a) *If $m\ge 0$ in $\Omega$ and $\sigma\ge 0$ on $\Gamma_1$ then there exists a principal eigenvalue of $(P)$ if and only if $\alpha>0$. The principal eigenvalue is unique and will be denoted by $\lambda_1$. It is characterized by*

$$\lambda_1 = \min_{\mathcal{M}^+} E_V, \tag{4.1}$$

*where $\mathcal{M}^+ := \{u\in W \mid I(u)=1\}\neq\emptyset$.*

(b) *If $m \leq 0$ in $\Omega$ and $\sigma \leq 0$ on $\Gamma_1$ then there exists a principal eigenvalue of $(P)$ if and only if $\alpha > 0$. The principal eigenvalue is unique and will be denoted by $\lambda_{-1}$. It is characterized by*

$$\lambda_{-1} = -\min_{\mathcal{M}^-} E_V, \tag{4.2}$$

*where $\mathcal{M}^- := \{u \in W \mid I(u) = -1\} \neq \emptyset$.*

2. *If either $m$ and $\sigma$ are definite but with opposite sign or one of them is indefinite then there exists a principal eigenvalue of $(P)$ if and only if $\alpha \geq 0$. More precisely :*

   (a) *if $\alpha > 0$ then $(P)$ admits exactly two principal eigenvalues $\lambda_{-1} < \lambda_1$, with $\lambda_1$ characterized as in (4.1) and $\lambda_{-1}$ characterized as in (4.2).*

   (b) *If $\alpha = 0$ then $(P)$ has a unique principal eigenvalue $\lambda_*$ given by*

$$\lambda_* = \inf_{\mathcal{M}^+} E_V = -\inf_{\mathcal{M}^-} E_V.$$

   *These infima are not achieved. Moreover a function $u \in \mathcal{G}$ satisfies $E_V(u) = \alpha$ if and only if $u \in S$ is an eigenfunction associated to $\lambda_*$.*

*Whenever they exist, one has $\lambda_{-1} \leq \lambda_1$.*

By the regularity results mentioned earlier, the eigenfunctions associated to principal eigenvalues belong to $C^{1,\beta}(\overline{\Omega})$ and are $> 0$ in $\Omega \cup \Gamma_1$.

*Proof.* We only prove (2)(b) since the proof of the other statements can be carried with minor changes from the proof of [13, Theorem 7]. If $\alpha = 0$ then $\mu(\lambda_0) = 0$ at some $\lambda_0$, providing a principal eigenvalue of $(P)$. By Lemma 7.1 below this point $\lambda_0$ is unique. Moreover $\mu'(\lambda_0) = -\int_\Omega m|\varphi_{\lambda_0}|^p\,dx - \oint_{\Gamma_1} \sigma|\varphi_{\lambda_0}|^p\,d\rho = 0$. Let us prove that $\lambda_0 = \inf_{\mathcal{M}^+} E_V$. Take $u \in \mathcal{M}^+$ and assume that $u \geq 0$ by replacing $u$ by $|u|$ if necessary. Picone's identity (c.f. Lemma 2.1) applied to $u_T \stackrel{\text{def}}{=} \min\{u, T\}$ and $\varphi_{\lambda_0}$ yields (notice that $\frac{u_T^p}{\varphi_{\lambda_0}^{p-1}} \in W^{1,1}_{loc}(\Omega)$ )

$$\begin{aligned}
0 \quad &\leq \quad \int_\Omega L(u_T, \varphi_{\lambda_0})\,dx = \int_\Omega R(u_T, \varphi_{\lambda_0})\,dx \\
&= \quad \int_\Omega \left[ |\nabla u_T|^p - |\nabla \varphi_{\lambda_0}|^{p-2} \nabla \left( \frac{u_T^p}{\varphi_{\lambda_0}^{p-1}} \right) \nabla \varphi_{\lambda_0} \right] dx \\
&= \quad \int_\Omega |\nabla u_T|^p\,dx + \int_\Omega V\,u_T^p\,dx \\
&\qquad - \lambda_0 \int_\Omega m\,u_T^p\,dx - \lambda_0 \oint_{\Gamma_1} \sigma\,u_T^p\,d\rho
\end{aligned}$$

Now we let $T \to \infty$ to get $E_V(u) \geq \lambda_0$. Consider the sequence $u_n \in \mathcal{M}^+$ given by

$$u_n = \frac{\varphi_{\lambda_0} + \frac{\psi}{n}}{I(\varphi_{\lambda_0} + \frac{\psi}{n})^{\frac{1}{p}}}, \tag{4.3}$$

for some fixed $0 \leq \psi \in C^\infty(\overline{\Omega})$, $\psi = 0$ on $\Gamma_2$, such that $I(\psi) > 0$, and $< I'(\varphi_{\lambda_0}), \psi > > 0$. The existence of such a function $\psi$ can be proved using arguments similar to those in Lemma 3.4. One can easily prove that $I(\varphi_{\lambda_0} + \frac{\psi}{n}) > 0$ for $n$ large enough. Moreover, for such $n$ we can find $0 < t_n, s_n < \frac{1}{n}$ such that

$$E_V(\varphi_{\lambda_0} + \frac{\psi}{n}) = \frac{1}{n} \langle E_V'(\varphi_{\lambda_0} + t_n\psi), \psi \rangle$$

and

$$I(\varphi_{\lambda_0} + \frac{\psi}{n}) =< I'(\varphi_{\lambda_0} + s_n\psi), \psi >$$

Hence $E_V(u_n) \to \lambda_0$. Since $\varphi_{\lambda_0}$ satisfies $I(\varphi_{\lambda_0}) = 0$ it is clear that the $\inf_{\mathcal{M}^+} E_V$ is not achieved. Finally for any $L^p$-normalized function $u$ satisfying $E_V(u) = I(u) = 0$ one has

$$\sup_{\lambda\in\mathbb{R}} \mu(\lambda) = 0 = E_V(u) = E_{V-\lambda_0 m}(u) - \lambda_0 \oint_{\Gamma_1} \sigma|u|^p \, d\rho \geq \mu(\lambda_0) = 0,$$

so $u$ achieves $\mu(\lambda_0)$. Thus $u = c\varphi_{\lambda_0}$ for some constant $c$ and the result follows with $\lambda_* := \lambda_0$. □

## 5. A sufficient condition for the existence of principal eigenvalues

Let us consider the following eigenvalue problem with mixed Dirichlet-Newmann boundary conditions:

$$(N) \qquad \begin{cases} -\Delta_p u + V(x)|u|^{p-2}u = \lambda|u|^{p-2}u & \text{in} \quad \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \Gamma_1, \\ u = 0 & \text{on} \quad \Gamma_2, \end{cases}$$

which correspond to the case $\sigma = 0$ and $m \equiv 1$ in problem $(P)$. Denote

$$\lambda_1^{\mathcal{N},\mathcal{D}}(V) = \inf\{E_V(u) \mid u \in W, \int_\Omega |u|^p \, dx = 1\}.$$

Noticing that that $\mu(0) = \lambda_1^{\mathcal{N},D}(V)$ we have the following trivial consequence of Theorem 4.1:

**Proposition 5.1.** *Assume that $\alpha \geq 0$.*
*(1) If $\lambda_1^{\mathcal{N},D}(V) > 0$ then problem $(P)$ possesses exactly two principal eigenvalues of different sign except in the cases* **(i)** *$m \geq 0$ in $\Omega$ and $\sigma \geq 0$ on $\Gamma_1$ where there is exactly one principal eigenvalue, which is positive, or in the case* **(ii)** *$m \leq 0$ in $\Omega$ and $\sigma \leq 0$ on $\Gamma_1$ where there is exactly one principal eigenvalue, which is negative.*
*(2) If $\lambda_1^{\mathcal{N},D}(V) = 0$ then problem $(P)$ possesses a unique nontrivial eigenvalue (which is positive) if and only if $m^+ \not\equiv 0$ in $\Omega$ or $\sigma^+ \not\equiv 0$ on $\Gamma_1$ and*

$$d := \int_\Omega m|\varphi_0|^p \, dx + \oint_{\Gamma_1} \sigma|\varphi_0|^p \, d\rho < 0,$$

*where $\varphi_0$ is the positive eigenfunction associated to $\lambda_1^{\mathcal{N},D}(V)$ of $L^p$-norm equal to 1.*
*(3) If $\lambda_1^{\mathcal{N},D}(V) = 0$ then problem $(P)$ possesses a unique nontrivial eigenvalue (which is negative) if and only if $m^- \not\equiv 0$ in $\Omega$ or $\sigma^- \not\equiv 0$ on $\Gamma_1$ and $d > 0$.*
*(4) If $\lambda_1^{\mathcal{N},D}(V) = 0$ and $d = 0$ then $\lambda = 0$ is the unique principal eigenvalue of problem $(P)$.*

*Proof.* One uses that

$$\mu'(0) = -\int_\Omega m|\varphi_0|^p \, dx - \oint_{\Gamma_1} \sigma|\varphi_0|^p \, d\rho = -d.$$

(1) In this case $\alpha = \sup_{\lambda\in\mathbb{R}} \mu(\lambda) \geq \mu(0) = \lambda_1^{\mathcal{N},D}(V) > 0$ and the situations **(i)** and **(ii)** correspond to those of case (1) of Theorem 4.1.
(2) In this case we have $\lambda_{-1} = \mu(0) = \lambda_1^{\mathcal{N},D}(V) = 0$. Since $\mu'(0) = -d > 0$, by the concavity of the curve $\lambda \to \mu(\lambda)$ it must be $\alpha > 0$ and the result follows from the case (2)(a) of Theorem 4.1.
(3) Similar to case (2).
(4) If $\mu(0) = \lambda_1^{\mathcal{N},D}(V) = 0$ and $d = 0$ then $\alpha = 0$ and therefore $\lambda = 0$ is the unique principal eigenvalue of problem $(P)$. □

**Remark 5.2.** *When $V \equiv 0$ and $\Gamma_2 = \emptyset$ then $\lambda_1^{\mathcal{N},D}(V) = 0 = \mu(0)$ and we can choose $\varphi_0 = cst$. Our results (2) and (3) in Proposition 5.1 generalize for the p-laplacian operator the result of [23].*

## 6. On the coerciveness of the restricted functional

Let us show in this section that $\alpha > 0$ is a sufficient condition for the coerciveness of $E_V$ under the constrain $\mathcal{M}^{\pm}$.

**Proposition 6.1.** *If $\alpha > 0$ then, for any $M \in \mathbb{R}$, the set*

$$\{u \in \mathcal{M}^+ \mid E_V(u) \leq M\}$$

*is bounded. A similar result can be stated for $-E_V$ on $\mathcal{M}^-$.*

*Proof.* Assume by contradiction that for some $M \in \mathbb{R}$ there is sequence $(u_n)$ unbounded in $\mathcal{M}^+$ and satisfying $E_V(u_n) \leq M$. Then, there exits a subsequence of $v_n = \frac{u_n}{\|u_n\|}$, still denoted as $v_n$, that converges to some $v_0$ weakly in $W$ and strongly in $L^p(\Omega) \cap L^p(\partial\Omega, \rho)$. Since $I(v_n) = \dfrac{1}{\|u_n\|^p} \to 0$, we deduce that $I(v_0) = 0$ and

$$\int_\Omega |\nabla v_0|^p dx + \int_\Omega V|v_0|^p dx \leq \liminf_{n\to+\infty} E_V(v_n) = \liminf_{n\to+\infty} \frac{E_V(u_n)}{\|u_n\|^p} = 0. \tag{6.1}$$

We claim that $v_0 \neq 0$, otherwise

$$0 \leq \liminf_{n\to+\infty} \int_\Omega |\nabla v_n|^p \, dx = \liminf_{n\to+\infty} (E_v(v_n) - \int_\Omega V \, |v_n|^p \, dx) \leq 0$$

and consequently $v_n \to 0$ strongly in $W$, which is impossible since $\|v_n\| = 1$. Hence $\frac{v_0}{\|v_0\|_p}$ is an admissible function in the definition of $\alpha$ and consequently

$$\alpha \leq E_V(\frac{v_0}{\|v_0\|_p}) \leq 0,$$

gives a contradiction with the assumption $\alpha > 0$. □

Next let us justify that $\alpha > 0$ is "almost" a necessary for $E_V$ to be coercive on $\mathcal{M}^{\pm}$. The case $\Gamma_1 = \emptyset$ was already considered in [13].

**Proposition 6.2.** *Assume that $\sigma^+ \not\equiv 0$ (resp. $\sigma^- \not\equiv 0$) . If $\alpha < 0$ then $E_V$ (resp. $-E_V$) is unbounded from below on $M^+$ (resp. on $M^-$).*

*Proof.* Let $0 \leq u_0 \in \mathcal{G}$ realize $\alpha$. We distinguish four cases:
(a) If $m^+ \not\equiv 0$, we pick $0 \leq w \in W_0^{1,p}(\Omega)$ such that $0 \not\equiv w$ and $supp\, w \subset \Omega^+$. Thus, for each $n$, there is $0 < s_n < \frac{1}{n}$ such that

$$I(u_0 + \frac{w}{n}) = \frac{1}{n} \int_{\Omega^+} m(u_0 + s_n w)^{p-2}(u_0 + s_n w) w \, dx + \oint_{\Gamma_1} \sigma u_0^p \, d\rho > 0.$$

Hence

$$\frac{E_V(u_0 + \frac{w}{n})}{I(u_0 + \frac{w}{n})} \to -\infty.$$

(b) If $m \leq 0$ and $mu_0 \not\equiv 0$, we pick a function $0 \leq w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that $0 \not\equiv w \leq u_0$.

$$I(u_0 - \frac{w}{n}) = \int_\Omega m|u_0 - \frac{w}{n}|^p \, dx + \oint_{\Gamma_1} \sigma u_0^p \, d\rho > I(u_0) = 0$$

Then

$$\frac{E_V(u_0 - \frac{w}{n})}{I(u_0 - \frac{w}{n})} \to -\infty.$$

(c) If $m \leq 0$, $mu_0 \equiv 0$ and the adherence of $\Omega^0$ intersects $\Gamma_1^+$ we take a function $0 \leq w \in W$ with support in $\Omega^0$ and such that $supp\, \gamma(w) \subset \Gamma_1^+$ (we construct $w$ as in Lemma 3.4). Then

$$I(u_0 + \frac{w}{n}) = \frac{1}{n^p} I(w) > 0, \quad \frac{E_V(u_0 + \frac{w}{n})}{I(u_0 + \frac{w}{n})} \to -\infty. \tag{6.2}$$

(d) If $m \leq 0$, $mu_0 \equiv 0$ and we are not in the previous cases, then we must have $\overline{\Omega^0} \subset \Gamma_1^0$ because $I(u_0) = \oint_{\Gamma_1} \sigma u_0^p \, d\rho = 0$. In this case we take a function $0 \leq w \in W$ from Lemma 3.4 such that $\int_{\Gamma_1} \sigma w^p d\rho > 0$ and having support in $\Omega^-$ and also $supp\, \gamma(w) \subset \Gamma_1^+$. Then we also have (6.2). □

If $\alpha = 0$ we have a less general result:

**Proposition 6.3.** *Assume that $\sigma^+ \not\equiv 0$. If $\alpha = 0$ and $u_n \geq 0$ is a sequence in $\mathcal{M}^+$ such that $\lambda_N(\{x \in \Omega \mid u_n(x) > 0\}) \to 0$ and $(E_V(u_n))$ is bounded then $(u_n)$ is bounded in $W$. A similar result can be stated in the case $\sigma^- \not\equiv 0$ for $-E_V$ over $\mathcal{M}^-$.*

*Proof.* Assume by contradiction that the sequence $(u_n)_n$ is unbounded. Consider $v_n = \frac{u_n}{\|u_n\|}$, which converges weakly to some $v_0 \in W$ and strongly in $L^p(\Omega) \cap L^p(\partial\Omega, \rho)$. Thus, passing to the limit, we get $I(v_0) = 0$ and

$$\lim_{n \to +\infty} E_V(v_n) = \lim_{n \to +\infty} \frac{E_V(u_n)}{||u_n||^p} = 0.$$

Moreover

$$1 + \int_\Omega V \, |v_0|^p \, dx = 1 + \lim_{n \to +\infty} \int_\Omega V \, |v_n|^p \, dx = \lim_{n \to +\infty} E_V(v_n) = 0.$$

We deduce in particular that $v_0 \not\equiv 0$. Hence $\frac{v_0}{\|v_0\|_p}$ is admissible in the definition of $\alpha$. By Proposition 4.1 (2)(b) $v_0 > 0$ on $\Omega$, in contradiction with

$$\lambda_N(\{x \in \Omega \mid v_n(x) > 0\}) = \lambda_N(\{x \in \Omega \mid u_n(x) > 0\}) \to 0.$$

□

It is not difficult to construct examples of $V$ and $m$ such that $\alpha = \alpha(V, m, \sigma) = 0$ by adapting some ideas of [13].

**Example 6.3.1.** *Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ and $B_0 \subset \Omega$ be an open subset such that $\|\psi_1\|_{2,B_0}^2 = \frac{1}{2}$, where $\psi_1$ is an eigenfunction associated to the first eigenvalue of*

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & in \quad \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = \lambda \sigma |u|^{p-2} u & on \quad \Gamma_1, \\ u = 0 & on \quad \Gamma_2, \end{cases}$$

*satisfying $\int_\Omega \psi_1^p \, dx + \oint_{\Gamma_1} \sigma \psi_1^p \, d\rho = 1$. Thus, $\lambda_1 = \lambda_1(V \equiv 0, m \equiv 1, \sigma)$ as defined in Theorem 4.1. Set*

$$m = \begin{cases} a & in \quad \Omega \backslash B_0, \\ b & in \quad B_0 \end{cases} \tag{6.3}$$

*for some $a, b$ to be determined later. We have $\int_\Omega m\,\psi_1^p\,dx + \oint_{\Gamma_1} \sigma\psi_1^p\,d\rho = 0$ by choosing $\frac{a+b}{2} - 1 = -(\int_\Omega \psi_1^p)^{-1}$. Let*

$$V = \begin{cases} \lambda_1(a-1), & in \quad \Omega\backslash B_0 \\ \lambda_1(b-1), & in \quad B_0. \end{cases} \tag{6.4}$$

*If $u$ is such that $\int_\Omega m|u|^p\,dx + \oint_{\Gamma_1} \sigma|u|^p\,d\rho = 0$ then*

$$E_V(u) = \int_\Omega |\nabla u|^p\,dx + \lambda_1(b-1)\|u\|_{p,B_0}^p + \lambda_1(a-1)\|u\|_{p,\Omega\backslash B_0}^p$$

$$= \int_\Omega |\nabla u|^p\,dx - \lambda_1(\int_\Omega |u|^p\,dx + \oint_{\Gamma_1} \sigma|u|^p\,d\rho) \geq 0$$

*Moreover, $E_V(u) = 0$ holds precisely for multiples of $\psi_1$. Therefore $\alpha(V, m, \sigma) = 0$.*

**Example 6.3.2.** *We give another example where $\alpha(V, m, \sigma) = 0$ in the case $m \equiv 0$. Let us assume here that $\Gamma_1$ is not a single point if $N = 1$. For a given $V \in L^\infty(\Omega)$, let us take $\Gamma_0 \subset \Gamma_1$ a subset such that $\|\psi_0\|_{p,\Gamma_0}^p = \frac{1}{2}$, where $\psi_0$ is an eigenfunction associated to the first eigenvalue of problem $(N)$*

$$\begin{cases} -\Delta_p u = \lambda|u|^{p-2}u & in \quad \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & on \quad \Gamma_1, \\ u = 0 & on \quad \Gamma_2, \end{cases}$$

*satisfying $\oint_{\Gamma_1} \psi_0^p\,d\rho = 1$. Thus, $\lambda_1 = \lambda_1^{\mathcal{N},\mathcal{D}}(0)$. Set*

$$\sigma = \begin{cases} 1 & in \quad \Gamma_1\backslash\Gamma_0, \\ -1 & in \quad \Gamma_0, \end{cases} \tag{6.5}$$

*then we have $\oint_{\Gamma_1} \sigma\psi_0^p\,d\rho = 0$. Set $V := -\lambda_1^{\mathcal{N},\mathcal{D}}(0)$. If $u$ is such that $\oint_{\Gamma_1} \sigma|u|^p\,d\rho = 0$ then*

$$E_V(u) = \int_\Omega |\nabla u|^p\,dx + \int_\Omega V|u|^P\,dx \geq 0$$

*and $E_V(u) = 0$ holds precisely for multiples of $\psi_0$. Therefore $\alpha(V, 0, \sigma) = 0$.*

## 7. Simplicity and isolation of the principal eigenvalues

Here below we prove two properties of the principal eigenvalues of problem $(P)$.

**Proposition 7.1** (*Simplicity*). *Assume that $u, v \in W$ are two eigenfunctions of Problem $(P)$ associated respectively to $\lambda$ and $\beta$. Assume also that $u > 0$ and $v > 0$ in $\Omega$. If $\beta \geq \lambda$ (resp. $\beta \leq \lambda$) when $I(u) \geq 0$ (resp. when $I(u) \leq 0$) then $u = cv$ for some $c > 0$ and $\lambda = \beta$.*
*In particular, if $\alpha \geq 0$, the principal eigenvalues $\lambda_1$ and $\lambda_{-1}$ are simple. There are no other principal eigenvalues.*

*Proof.* We apply Picone's Identity of Lemma 2.1 to $u$ and $v+\eta$. After integration and letting $\eta \to 0$ we find

$$\begin{aligned} 0 &\leq \int_\Omega L(u,v)\,dx = \int_\Omega R(u,v)\,dx \\ &= \int_\Omega \left[ |\nabla u|^p - |\nabla v|^{p-2}\nabla\left(\frac{u^p}{v^{p-1}}\right)\nabla v\right] dx \\ &= \lambda(\int_\Omega m\,|u|^p\,dx + \oint_{\Gamma_1} \sigma\,u^p\,d\rho) - \int_\Omega V\,u^p\,dx + \\ &\quad \int_\Omega (V-\beta m)u^p\,dx - \beta \oint_{\Gamma_1} \sigma\,u^p\,d\rho \\ &= (\lambda-\beta)(\int_\Omega m\,u^p\,dx + \oint_{\Gamma_1} \sigma\,u^p\,d\rho). \end{aligned}$$

If $\int_\Omega m\,u^p\,dx + \oint_{\Gamma_1} \sigma\,u^p\,d\rho > 0$ (resp. $\int_\Omega m\,u^p\,dx + \oint_{\Gamma_1} \sigma\,u^p\,d\rho < 0$) and $\beta \geq \lambda$ (resp. $\beta \leq \lambda$) then $L(u,v)=0$. Hence $u = cv$ for some $c>0$ and therefore $\alpha=\beta$. If $\int_\Omega m\,u^p\,dx + \oint_{\Gamma_1} \sigma\,u^p\,d\rho = 0$ then $L(u,v)=0$ and we conclude again that $u=cv$ and $\alpha=\beta$. □

**Proposition 7.2** (*Isolation*). *Assume that $\alpha \geq 0$. Then the principal eigenvalues are isolated in the spectrum of $(P)$.*

*Proof.* We only prove that $\lambda_1$ is isolated, an analogous proof can be given for $\lambda_{-1}$. We can use similar arguments to those of [12]. First one gets an a-priori estimate of the measure of any nodal set $\mathcal{N}$ of a non-principal eigenfunction $u$ associated to $\lambda$ by using Sobolev and trace embeddings. We recall that a *nodal domain of* $u$ is a connected component of $\Omega \setminus \{x \in \Omega \mid u(x) = 0\}$. This estimate will read as follows

$$\lambda_N(\mathcal{N})^{\frac{p^*-p}{p^*}} + \rho\,(\overline{\mathcal{N}} \cap \Gamma_1)^{\frac{p_*-p}{p_*}} \geq (|\lambda|c_1 + c_2)^{-1} \tag{7.1}$$

for some positive constants $c_1, c_2$ depending only upon $m, \sigma, V$ and $\Omega$. We explain briefly how to prove (7.1). First observe that, since $u \in W \cap C(\overline{\Omega})$, then $u_{|\mathcal{N}} \in W_{\mathcal{N}}$. Hence the function $w$ defined as $w(x) = u(x)$ if $x \in \mathcal{N}$ and $w(x) = 0$ if $x \in \Omega \setminus \mathcal{N}$ belongs to $W$. Assume that $1 < p < N$. Using $w$ as a test function in the weak equation satisfied by $u$ we find

$$\int_{\mathcal{N}} |\nabla v|^p\,dx + \oint_{\overline{\mathcal{N}}\cap\Gamma_1} |u|^p\,d\rho \leq C\big(\|u\|^p_{p^*,\mathcal{N}}\,\lambda_N(\mathcal{N})^{\frac{p^*-p}{p^*}} + \|u\|^p_{p_*,\overline{\mathcal{N}}\cap\Gamma_1}\,\rho(\overline{\mathcal{N}}\cap\Gamma_1)^{\frac{p_*-p}{p_*}}\big)$$

by Hölder inequality, with $C = (|\lambda|(\|m\|_\infty + \|\sigma\|_\infty) + \|V\|_\infty + 1)$. On the other hand, using Sobolev's and trace embeddings we have that

$$\int_{\mathcal{N}} |\nabla v|^p\,dx + \oint_{\overline{\mathcal{N}}\cap\Gamma_1} |u|^p\,d\rho \geq D\big(\|v\|^p_{p^*,\mathcal{N}} + \|u\|^p_{p_*,\overline{\mathcal{N}}\cap\Gamma_1}\big)$$

for some new constant $D = D(N,p,\Omega)$ and the result follows. In the case $p \geq N$ we will proceed similarly.

Assume now by contradiction that there exists $(\lambda_n, u_n)$ a sequence of eigenvalues and eigenfunctions such that $\lambda_n \searrow \lambda_1$.
(a) Assume $\alpha > 0$. If $I(u_n) = 0$ then $0 < \alpha \leq E_V(u_n) = \lambda_n I(u_n) = 0$, a contradiction. Since $\lambda_n > \lambda_1$ it follows that $I(u_n) > 0$, otherwise

$$-\lambda_{-1} \leq -\frac{E_V(u_n)}{I(u_n)} = -\lambda_n < -\lambda_1,$$

a contradiction. Thus $v_n := \frac{u_n}{I(u_n)^{1/p}} \in \mathcal{M}^+$ is such that $E_V(v_n) = \lambda_n \searrow \lambda_1$. By Proposition 6.1 the sequence $v_n$ is bounded in $W$ so there exists $v_0 \in W$ such that $v_n \rightharpoonup u_0$ weakly in $W$ and strongly in $L^p(\Omega) \cap L^p(\partial\Omega, \rho)$. Using the fact that $\lambda_1 \leq E_V(v_0) \leq \lim_{n\to+\infty} E_v(v_n) = \lambda_1$, we conclude that $u_0 = \pm\varphi_1$. Remember that, by Proposition 3.2, $\varphi_1 > 0$ in $\Omega$ and $\varphi_1 > 0$ on $\Gamma_1$. In the case $v_0 = \varphi_1$, let $\Omega_n^-$ be a negative nodal domain of $v_n$. From the convergence in measure of $v_n$ towards $v_0$ we conclude that $\lambda_N(\Omega_n^-) + \rho\,(\overline{\Omega_n^-} \cap \Gamma_1) \to 0$ as $n \to +\infty$, in contradiction with (7.1). In the case $v_0 = -\varphi_1$ we will argue similarly.
(b) Assume $\alpha = 0$. If $I(u_n) = 0$ then, by the result of Theorem 4.1(2)(b), $u_n$ will be a multiple of $\varphi_{\lambda_*}$ which is impossible as $\lambda_n \neq \lambda_*$. Arguing as in the previous case we have that $I(u_n) > 0$. Let us prove that the sequence $v_n = \frac{u_n}{I(u_n)^{1/p}} \in \mathcal{M}^+$ is bounded in $W$. If not, take $w_n = \frac{v_n}{\|v_n\|}$ and $w_0$ a weak limit of a subsequence converging strongly in $L^p(\Omega) \cap L^p(\partial\Omega, \rho)$. Then $E_V(w_0) = I(w_0) = 0$. Moreover, since

$$\int_\Omega |\nabla w_0|^p\, dx \leq \liminf_{n\to+\infty}(E_V(w_n) - \int_\Omega V|w_n|^p\, dx) = 0,$$

then $w_0 \neq 0$ otherwise $w_n \to w_0 = 0$ strongly in $W$, which will contradict that $\|w_n\| = 1$. So $\frac{w_0}{\|w_0\|_p} \in \mathcal{G}$ is a function where the value $\alpha = 0$ is achieved. By Theorem 4.1 (2)(b), $w_0$ is an eigenfunction for the principal eigenvalue $\lambda_*$. Hence we will reach a contradiction using, as before, the estimate of the measure of the nodal domains of $w_n$. □

## 8. Existence of non principal eigenvalues

Our aim is two prove the existence of a sequence of eigenvalues for problem $(P)$. In some cases we can even establish the existence of *two sequences* of eigenvalues, one converging to $+\infty$ and the other to $-\infty$.

For simplicity, we will assume that either $\sigma^+ \not\equiv 0$ or $m^+ \not\equiv 0$ and we will prove the existence of a sequence of eigenvalues going to $+\infty$ by constructing a sequence of critical values of $E_V$ restricted to $\mathcal{M}^+$ via the Ljusternik-Schnirelmann critical theory on $C^1$ manifolds (see [21] or [22]). In order to do so, we define for any $k \in \mathbb{N}^*$,

$$\mathcal{A}_k \stackrel{\text{def}}{=} \{K \subset \mathcal{M}^+ \mid K \text{ symmetric, compact and } i(K) = k\},$$

where $i(K)$ denotes the Krasnoselski's genus of $K$ on $W^{1,p}(\Omega)$. Let us first investigate when $\mathcal{A}_k$ is a non empty set.

**Lemma 8.1.** *Assume that either $\sigma^+ \not\equiv 0$ or $m^+ \not\equiv 0$. In the case $m^+ \equiv 0$ assume furthermore that $N > 1$. Then $\mathcal{A}_k \neq \emptyset$ for all $k \in \mathbb{N}^*$.*

*Proof.* Let $k \in \mathbb{N}^*$ be fixed. If $m^+ \not\equiv 0$, we can construct functions $e_i \in W_0^{1,p}(\Omega)$, $i = 1, \cdots, k$, such that for $i \neq j$, $supp\, e_i \cap supp\, e_j = \emptyset$ and $\int_\Omega m\,|e_i|^p\, dx > 0$ by regularizing, for instance, characteristic functions on small disjoint balls of $\Omega$ that have non empty intersection with $\Omega^+$. Then

$$F := \{\sum_{i=1}^k \alpha_i e_i \mid \alpha_i \in \mathbb{R}\} \cap \mathcal{M}^+ \in \mathcal{A}_k.$$

If $m^+ \equiv 0$ we show how to construct functions $e_i \in W$ with $supp\, e_i \cap supp\, e_j = \emptyset$ and $supp\, \gamma(e_i) \cap supp\, \gamma(e_j) = \emptyset$, if $i \neq j$. To be more precise, pick $k$ disjoint balls $B_i$ of $\Omega$ such that $S_i := \overline{B_i} \cap \Gamma_1 \neq \emptyset$ and such that $\rho(S_i \cap \{x \in \Gamma_1 \mid \sigma(x) > 0\}) > 0$ for all $i = 1, \cdots k$. Notice that, in order to assure the existence of such disjoint supports on $\Gamma_1$, we need to assure that the boundary set $\Gamma_1$ is not a single point, and therefore we exclude the case of dimension $N = 1$. Then take $\psi_i \in C^\infty(\Gamma_1)$ a non negative function such that $a_i := \oint_{\Gamma_1} \sigma|\psi_i|^p\, d\rho > 0$, by regularizing the characteristic

function of $S_i \cap \{x \in \Gamma_1 \mid \sigma(x) > 0\}$. Let $v_i \in C^\infty(\Omega)$ such that $\gamma(v_i) = \psi_i$. Then multiply $v_i$ by a regularization of the characteristic function of a small $B_i^0 \subset B_i$ chosen in such a way that

$$\lambda_N(B_i^0) < \frac{a_i}{2\|m\|_\infty \|v_i\|_\infty^p},$$

to get a function $e_i \in W$ with trace $\gamma(e_i) = \psi_i$, $supp\, e_i \cap supp\, e_j = \emptyset$, if $i \neq j$, and such that $\int_\Omega m|e_i|^p\, dx + \oint_{\Gamma_1} \sigma|\psi_i|^p\, d\rho > 0$. □

**Theorem 8.2.** *Let us assume $\alpha > 0$ and that either $\sigma^+ \not\equiv 0$ or $m^+ \not\equiv 0$. In the case $m^+ \equiv 0$ assume that $N > 1$. Define, for any $k \in \mathbb{N}^*$,*

$$\lambda_k \stackrel{\text{def}}{=} \inf_{K \in \mathcal{A}_k} \max_{u \in K} E_V(u). \tag{8.1}$$

*Then $(\lambda_k)_k$ is a nondecreasing sequence of eigenvalues of the problem $(P)$ such that $\lim_{k\to+\infty} \lambda_k = +\infty$. Moreover*

$$\lambda_2 = \inf\{\lambda \mid \lambda > \lambda_1 \text{ and } \lambda \text{ is an eigenvalue of } (P)\}. \tag{8.2}$$

*Proof.* In order to apply the Ljusternik-Schnirelmann critical theory on $C^1$ manifolds it suffices to prove that the restriction of $E_V$ to $\mathcal{M}^+$ satisfies the Palais-Smale condition at the level $\lambda_k$. This will imply(c.f. [21, Theorem 3.54] or [22]) both that $\lambda_k$ is an eigenvalue associated to Problem $(P)$, and also $\lim_{k\to+\infty} \lambda_k = +\infty$.

Let $(u_n)$ be a Palais-Smale sequence in $\mathcal{M}^+$ for $E_V$, i.e.

$$\begin{aligned} &\text{(PS1)} \quad E_V(u_n) \longrightarrow c \\ &\text{(PS2)} \quad |\langle E_V'(u_n), \xi\rangle| \leq \varepsilon_n \|\xi\| \quad \text{for all } \xi \in T_{u_n}\mathcal{M}^+. \end{aligned}$$

For any $w \in W$, it is clear that

$$a_n(w) = w - \left[\int_\Omega m|u_n|^{p-2}u_n w + \oint_{\Gamma_1} \sigma|u_n|^{p-2}u_n w\right] u_n \in T_{u_n}\mathcal{M}^+.$$

Hence, taking $\xi = a_n(w)$ in (PS2), we get

$$|\langle E_V'(u_n), w\rangle - A_n E_V(u_n)| \leq \varepsilon_n \|w - A_n u_n\|, \tag{8.3}$$

where

$$A_n := \int_\Omega m|u_n|^{p-2}u_n w + \oint_{\Gamma_1} \sigma|u_n|^{p-2}u_n w.$$

By Proposition 6.1, the sequence $u_n$ is bounded and therefore, there exits a subsequence still denoted $u_n$, such that $u_n \to u_0$ weakly in $W$ and strongly in $L^p(\Omega) \cap L^p(\partial\Omega, \rho)$ for some $u_0$. Choosing $v = u_n - u_0$ in (8.3) and passing to the limit we obtain

$$\lim_{n\to+\infty} \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u_0)dx = 0.$$

Applying Lemma 2.2 and Hölder inequality, one easily derives that $\nabla u_n \to \nabla u_0$ in $L^p(\Omega)$ and consequently $u_n \to u_0$ in $W^{1,p}(\Omega)$.

Let us prove the characterization of $\lambda_2$. Of course $\lambda_2 > \lambda_1$ otherwise one will have $i(K_{\lambda_1}) \geq 2$, where $K_{\lambda_1}$ is the set of eigenfunctions in $\mathcal{M}^+$ associated to $\lambda_1$. This is a contradiction because $\lambda_1$ is simple. Assume now by contradiction that there exists an eigenvalue $\lambda$ of problem

$(P)$ between $\lambda_1$ and $\lambda_2$ and let $v \in \mathcal{M}^+$ a corresponding eigenfunction. Since $v$ changes sign by Proposition 7.1, by multiplying the equation $(P)$ by $v^\pm$ if follows that $E_v(v^\pm) = \lambda I(v^\pm)$. It comes from this identity that $I(v^\pm) \neq 0$ otherwise $v^\pm \in \mathcal{G}$, and $E_V(v^\pm) = 0$ and therefore $\alpha \leq 0$, a contradiction. Also $I(v^\pm) > 0$ otherwise

$$-\lambda_{-1} \leq \frac{E_V(v^\pm)}{-I(v^\pm)} = -\lambda < -\lambda_1,$$

which is absurd. Consider then the set

$$A = \{av^+ + bv^- \mid a, b \in \mathbb{R}\} \cap \mathcal{M}^+.$$

Using the fact that the supports of $v^+$ and $v^-$ are disjoint, one can easily prove that $A$ is homomorphic to a sphere of $\mathbb{R}^2$ and therefore $i(A) = 2$. Hence

$$\lambda_2 \leq \max_{v \in A} E_V(v) = \lambda,$$

a contradiction. □

A trivial corollary of the Theorem 8.2 is

**Corollary 8.3.** *Assume that $\sigma^- \not\equiv 0$ or $m^- \not\equiv 0$. In the case $m^- \equiv 0$ assume that $N > 1$. Assume also $\alpha > 0$. Then there exists a nonincreasing sequence $\lambda_{-k}$ of eigenvalues of the problem $(P)$ defined as*

$$\lambda_{-k} \stackrel{\text{def}}{=} \inf_{K \in \mathcal{B}_k} \max_{u \in K} -E_V(u),$$

*where $\mathcal{B}_k \stackrel{\text{def}}{=} \{K \subset \mathcal{M}^- \mid K$ symmetric, compact and $i(K) = k\}$, satisfying $\lim_{k \to +\infty} \lambda_{-k} = -\infty$.*

**Remark 8.4.** *The characterization of $\lambda_2$ as the second eigenvalue on the right of $(P)$ was first proved, for the Dirichlet problem and $V \equiv 0$, by [5]. In [13] a second characterization of the second eigenvalue on the right was found for the Dirichlet problem with $V, m$ indefinite.*

In the case $\alpha = 0$, it is more delicate to construct non principal eigenvalues by variational methods since *the $(PS)$ fails at the level $\lambda_1 = \lambda_{-1} = \lambda_*$*. Indeed, notice that the eigenvalue $\lambda_*$ of Theorem 4.1 *is not achieved* neither at $\mathcal{M}^+$ nor $\mathcal{M}^-$, but at any function of $\mathcal{G}$. Although one can define the Ljusternik-Schnirelmann sequence $\{\lambda_k\}_k$ as above, one needs a compactness condition to prove that those values are critical values of $E_V$ restricted to the manifold $\mathcal{M}^+$. We will prove in the next lemma a weaker compactness condition for all levels greater than $\lambda_*$.

**Lemma 8.5.** *Assume that $\alpha = 0$. Then $E_V$ satisfies the Palais-Smale Condition of Cerami at level $c$ ($(PSC)_c$ for short) on $\mathcal{M}^+$ for any $c > \lambda_*$.*

*Proof.* Let us prove the $(PSC)_c$ condition for any $c > \lambda_*$. Let $u_n$ be a $(PSC)_c$ sequence in $\mathcal{M}^+$ for $E_V$, i.e., there exists $\varepsilon_n \to 0$ such that

$$\begin{aligned} &\text{(PSC1)} \quad E_V(u_n) \longrightarrow c \\ &\text{(PSC2)} \quad |\langle E_V'(u_n), \xi\rangle| \leq \frac{\varepsilon_n}{1 + \|u_n\|}\|\xi\| \quad \text{for all } \xi \in T_{u_n}\mathcal{M}^+. \end{aligned}$$

Let us assume by contradiction that $(u_n)$ is unbounded and set $v_n = \frac{u_n}{\|u_n\|}$. Up to a subsequence, there is some $v_0$ such that $v_n \rightharpoonup v_0$ in $W$ and $v_n \to v_0$ in $L^p(\Omega) \cap L^p(\partial\Omega, \rho)$. We choose $\xi = a_n(v_n - v_0)$ in $(PSC2)$ and divide it by $\|u_n\|^{p-1}$ to obtain

$$|\langle E_V'(v_n), v_n - v_0\rangle - B_n E_V(u_n)| \quad \leq \quad \frac{\varepsilon_n \|u_n\|}{1 + \|u_n\|} \left\| \frac{v_n - v_0}{\|u_n\|^p} - B_n v_n \right\|, \tag{8.4}$$

where

$$B_n = \int_\Omega m|v_n|^{p-2}v_n(v_n - v_0) + \oint_{\Gamma_1} \sigma|v_n|^{p-2}v_n(v_n - v_0).$$

By letting $n \to \infty$ and using the $(S^+)$ property of the $p$-laplacian (c.f. Lemma 2.2) we get that $v_n \to v_0$ in $W$ and in particular $v_0 \neq 0$. Moreover $E_V(v_0) = 0$ and $\int_\Omega m\,|v_0|^p\,dx + \oint_{\Gamma_1} \sigma\,|v_0|^p\,d\rho = 0$, then $v_0$ achieves $\alpha$. By Theorem 4.1(2)(b), $v_0$ has definite sign and is an eigenfunction of $(P)$. Furthermore, when choosing $\xi = a_n(w)$ by any $w \in W$ and letting $n \to \infty$ in (8.4), we find that $v_0$ is an eigenfunction associated to $c$, in contradiction with $c > \lambda_*$. Therefore the sequence $u_n$ is bounded and, again by the $(S^+)$ property $(u_n)_n$ is convergent in $W$ up to a subsequence. □

**Remark 8.6.** *It is not clear in our context if the 2nd $L-S$ infmax-value is strictly greater than $\lambda_*$. To our knowledge, the compactness condition (either $(PS)$ or $(PSC)$) is needed to prove both that the second $L-S$ infmax-value is greater than $\lambda_1$ and that it is a critical value of $E_V$ restricted to $\mathcal{M}^+$. We fail to give an answer to any of these questions.*

Next we construct a nonprincipal eigenvalue using the ideas of [13]. We refer the reader to this paper for complete details relative of the following theorem.

**Theorem 8.7.** *Assume that either $\sigma^+ \not\equiv 0$ or $m^+ \not\equiv 0$. In the case $m^+ \equiv 0$ assume that $N > 1$. Let us also assume that $\alpha = 0$.*
*(i) Let $\mathcal{L} := \{h \in C([0,1], \mathcal{M}^+) \mid h(0) \geq 0,\ h(1) \leq 0\}$. Then the value*

$$\mu \stackrel{\text{def}}{=} \inf_{h\in\mathcal{L}} \max_{u\in h([0,1])} E_V(u),$$

*satisfies $\mu > \lambda_*$.*
*(ii) For any $u_1 \in \mathcal{M}^+$, $u_1 \geq 0$, such that $E_V(u_1) < \mu$,*

$$\lambda_2 \stackrel{\text{def}}{=} \inf_{h\in\mathcal{H}} \max_{u\in h([0,1])} E_V(u),$$

*where $\mathcal{H} := \{h \in C([0,1], \mathcal{M}^+) \mid h(0) = u_1,\ h(1) = -u_1\}$, is an eigenvalue for problem $(P)$.*
*(iii) $\mu = \lambda_2$.*
*(iv) $\lambda_2$ is the first non principal eigenvalue of problem $(P)$ satisfying (8.2).*

*Proof.* (i) First notice that $\mathcal{L} \neq \emptyset$ as one can always pick two functions in $\mathcal{M}^+$ of definite sign, $v_1 \geq 0$, $v_2 \leq 0$, with disjoint supports and construct the path $\gamma_1(t) := \frac{t^{\frac{1}{p}}v_1 - (1-t)^{\frac{1}{p}}v_2}{I(t^{\frac{1}{p}}v_1 - (1-t)^{\frac{1}{p}}v_2)^{1/p}}$ for $t \in [0,1]$. Then $\gamma_1 \in \mathcal{L}$. Let us assume by contradiction that equality $\lambda_* = \mu$ holds. Then there exists $h_k \in \mathcal{L}$ such that $\max_{t\in[0,1]} E_V(h_k(t)) \to \lambda_*$ when $k \to \infty$. For every $k$ one can find $t_k \in [0,1]$ satisfying

$$I(h_k(t_k)^+) = I(h_k(t_k)^-) = \frac{1}{2}. \tag{8.5}$$

We set $u_k = h_k(t_k)$. Notice, from (8.5), that $2^{\frac{1}{p}}u_k^\pm \in \mathcal{M}^+$ so that $E_V(u_k^\pm) \geq \frac{1}{2}\lambda_*$. Hence

$$\frac{1}{2}\lambda_* \quad \leq \quad E_V(u_k^\pm) \quad = \quad E_V(u_k) \ - \ E_V(u_k^\mp) \quad \leq \quad \max_{t\in[0,1]} E_V(\gamma_k(t)) \ - \ \frac{\lambda_*}{2} \quad \to \quad \frac{\lambda_*}{2}$$

so that

$$\lim_{k\to+\infty} E_V(u_k^\pm) = \frac{\lambda_*}{2}. \tag{8.6}$$

Let us show that the sequence $(u_k)$ is bounded in $W$. Assume by contradiction that the sequence $(u_k)$ is unbounded and set $v_k = \frac{u_k}{\|u_k\|}$, which, up to a subsequence, converges weakly to some

$v_0 \in W$ and strongly in $L^p(\Omega) \cap L^p(\partial\Omega, \rho)$. From $I(u_k) = 1$ we infer that $I(v_0) = 0$. If $v_0 = 0$ then from

$$\int_\Omega |\nabla v_0|^p \, dx \leq \liminf_{k\to\infty} \left(\frac{E_V(u_k)}{\|u_k\|^p} - \int_\Omega V|v_k|^p \, dx\right) = 0$$

we deduce that $v_k \to v_0 = 0$ strongly in $W^{1,p}(\Omega)$, a contradiction with $\|v_k\| = 1$. Thus it must be $v_0 \neq 0$. From $(PSC1)$ we infer $E_V(v_0) \leq \liminf_{k\to\infty} \frac{E_V(u_k)}{\|u_k\|^p} = 0$, then $\frac{v_0}{\|v_0\|_p}$ realizes $\alpha$. By Theorem 4.1(2)(b), $v_0$ is a definite eigenfunction for $(P)$ associated to $\lambda_*$. If, say, $v_0 > 0$, then the sequence $u_k^-$ converges in measure to 0. By Proposition 6.3 and (8.6) we conclude that the sequence $u_k^-$ is bounded in $W$. Then, up to a subsequence, $u_k^-$ converges weakly to some $z_0$ that satisfies $E_V(z_0) = \frac{1}{2}\lambda_*$ and $I(z_0) = \frac{1}{2}$. Hence $2^{\frac{1}{p}} z_0 \in \mathcal{M}^+$ realizes $\lambda_*$. Thus $z_0 > 0$, a contradiction with the fact that $u_k^-$ converges to 0 in measure. Therefore the sequence $u_k$ is bounded. Since the sequence $u_k$ is bounded, passing to the limit for a weakly convergent subsequence, we prove that the value $\lambda_*$ is achieved at some point of $\mathcal{M}^+$, a contradiction. We have just proved that $\mu > \lambda_*$.
(ii) Now, let us pick a function $u_1$ (for instance, a function belonging to the sequence defined in (4.3)) such that $E_V(u_1) < \mu$. That $\lambda_2$ defined in the statement is an eigenvalue for problem $(P)$ is a consequence, for instance of the version of Mountain Pass Theorem on $C^1$-manifolds of of [7, Theorem 4.1]. Observe that: (1) by Lemma 8.5, $(PSC)_{\lambda_2}$ holds because $\lambda_* < \mu \leq \lambda_2$ and (2) the geometric condition

$$\max\{E_V(u_1), E_V(-u_1)\} < \mu \leq \lambda_2$$

also holds.
(iii) Let us prove that $\lambda_2 \leq \mu$. Let $\epsilon > 0$ be small enough and $h \in \mathcal{L}$ such that

$$\max_h E_V \leq \mu + \epsilon.$$

Put $u_0 = h(0)$. We claim that there exists a path $\tilde{h}$ in $\mathcal{M}^+$ from $u_0$ to $u_1$ such that $E_V$ stays below the level $\mu+\epsilon$ on $\tilde{h}$ and consequently $\lambda_2 \leq \mu+\epsilon$ and the conclusion follows (since a similar argument holds for $h(1)$ and -$u_1$). For that purpose, we consider $V+t$ for $t > 0$ small in order to have $\alpha(V+t) > 0$ and consequently the Palais-Smale condition will be satisfied everywhere. We also choose $t > 0$ small enough to have $\max\{E_{V+t}(u_0), E_{V+t}(u_1)\} < \mu(V) + \epsilon$. Let us consider the open set $\mathcal{O} \stackrel{\text{def}}{=} \{u \in \mathcal{M}^+ \mid E_{V+t}(u) < \mu(V) + \epsilon\}$. Observe that, if $t > 0$ is small enough, then $\mu(V) < \mu(V+t)$. It follows from Lemma 14 of [6] that $\mathcal{O}$ has at most two arcwise connected components (because $\varphi_1(V+t)$ and $-\varphi_1(V+t)$ are the only critical points of the restriction of $E_{V+t}$ to $\mathcal{M}^+$ in $\mathcal{O}$). If $u_1$ and $-u_1$ lie in the same component then it comes that $\lambda_2 \leq \mu(V) + \epsilon$. Otherwise $u_0$ can be connected by a path to either $u_1$ or $-u_1$. But since $E_{V+t}(u) = E_{V+t}(\pm|u|)$ for every $u \in \mathcal{M}^+$, we can always find a path from $u_0$ to $u_1$ by taking absolute value.
(iv) Finally, the fact that $\lambda_2 = \mu$ is the first nonprincipal eigenvalue is due to the following observation. If $\lambda > \lambda_*$ is an eigenvalue for $(P)$ and $u$ is a corresponding eigenfunction, then the path $h$ defined as $h(t) = \frac{tu^+ - (1-t)u^-}{I(tu^+ - (1-t)u^-)^{1/p}}$ is well defined (we leave the details to the reader) and belongs to $\mathcal{H}$. Moreover $E_V(h(t)) = \lambda$ for all $t \in [0,1]$ because $E_V(u^\pm) = \lambda I(u^\pm)$. Thus it comes from the definition of $\mu$ that $\mu \leq \lambda$. □

Here below we give an example of problem $(P)$ in dimension $N = 1$ with a weight $\sigma$ changing sign showing that one can have *only one* eigenvalue on the right of $\lambda_1$.

**Example 8.7.1.** *Let us consider the problem*

$$\begin{cases} -u'' + u = -\lambda u & \text{in } [0,1] \\ -\sigma_1 u'(0) = \lambda u(0), \\ \sigma_2 u'(1) = \lambda u(1) \end{cases}$$

*With $\sigma_1+\sigma_2=-1$ and $\sigma_1<-1$. Hence $\lambda_1=-1$ is a principal eigenvalue with $u(t)=t+\sigma_1$ as corresponding eigenfunction. Any solution of the differential equation for $\beta^2:=\lambda+1>0$ is of the form $u(x)=Ae^{\beta x}+Be^{-\beta x}$ and the boundary condition are satisfied if and only if*

$$e^{2\beta}=\frac{\sigma_1\sigma_2\beta^2-\beta(\beta^2-1)+(\beta^2-1)^2}{\sigma_1\sigma_2\beta^2+\beta(\beta^2-1)+(\beta^2-1)^2}:=h(\beta).$$

*A simple analysis of $h$ shows that $h(\beta)\le 1+2\beta$, and that equality holds if and only if $\beta=0$. Thus, $\lambda=\lambda_1=-1$ and there are not eigenvalues greater than $\lambda_1=-1$.*

## 9. Appendix

We want to give here a proof of the boundedness of the solutions of problem $(P)$. This type of result is well established for Dirichlet or Newmann boundary conditions and it has already been proved by several authors,see for instance [16]. However we didn't find any precise reference for the type of equations with the mixed boundary conditions that we are considering here. We include the proof of the following theorem for sake of completeness.

**Theorem 9.1.** *Let $\Omega$ be a bounded regular domain satisfying the main assumptions of the introduction. Let $f:\Omega\times\mathbb{R}\to\mathbb{R}$ and Let $g:\Gamma_1\times\mathbb{R}\to\mathbb{R}$ be two Carathéodory functions satisfying the growth assumption*

$$|f(x,t)|\le a|t|^{q-1}+b\quad\forall(x,t)\in\Omega\times\mathbb{R};\tag{9.1}$$

$$|g(x,t)|\le a|t|^{r-1}+b\quad\forall(x,t)\in\Gamma_1\times\mathbb{R},\tag{9.2}$$

*for some $a,b\in\mathbb{R}^+$, $1\le q\le p^*$ and $1\le r\le p_*$. Then there exists a constant $C=C(a,b,q,r,\Omega,\Gamma_1,\|u\|_{p^*},\|u\|_{p_*,\Gamma_1})$ such that any weak solution $u$ of*

$$\begin{cases}-\Delta_p u=f(x,u) \text{ in} & \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial\nu}=g(x,u) & \text{on}\quad\Gamma_1,\\ u=0 & \text{on}\quad\Gamma_2,\end{cases}\tag{9.3}$$

*satisfies*
*(i) if $q<p^*$ and $r<p_*$ then*

$$\|u\|_\infty+\|u\|_{\infty,\Gamma_1}\le C,$$

*(ii) if $q=p^*$ or $r=p_*$ then $u\in L^t(\Omega)\cap L^t(\partial\Omega,\rho)$ for all $t\in[1,+\infty[$ and there exists a constant $C_t$, depending on the previous parameters and on $t$, such that*

$$\|u\|_t+\|u\|_{t,\Gamma_1}\le C_t.$$

*Proof.* By Sobolev's embedding theorem, it suffices to consider the case $N\ge p$. We may assume that $u\ge 0$ otherwise we will consider test functions involving $u^+$ and $u^-$ to obtain the a-priori bounds.
(i) Let us assume that $1\le q<p^*$ and $1\le r<p_*$. For $M>0$ we define $u_M(x):=\min\{u(x),M\}$ and $\Phi(x):=u_M^{kp+1}(x)\in W\cap L^\infty(\Omega)\cap L^\infty(\partial\Omega,\rho)$ for $k>0$ to be determined later. Using $\Phi$ a test function in (9.3), one obtains

$$c_k^{-1}\|\nabla(u_M^{k+1})\|_p^p\le a[\int_\Omega u_M^{kp+q}\,dx+\oint_{\Gamma_1}u_M^{kp+r}\,d\rho]+b[\int_\Omega u_M^{kp+1}\,dx+\oint_{\Gamma_1}u_M^{kp+1}\,d\rho]$$

where $c_k := \frac{(k+1)^p}{kp+1}$. From now on $C$ will denote a generic constant independent of $k$, but depending on $p, q, r$, and $\Omega$. Since $c_k > 1$ for any $k > 0$, adding $\|u_M^{(k+1)}\|_{p,\Gamma_1}^p$ to each side, letting $M \to +\infty$ and using Young's inequality twice it comes

$$\|u^{(k+1)}\|_W \leq C c_k^{1/p} \Big( \int_\Omega u^{kp+q}\, dx + \oint_{\Gamma_1} u^{kp+r}\, d\rho + 1 \Big)^{1/p}.$$

Since our purpose is to make $k \to +\infty$ we will replace all the constants of the form $C^{1/k}$ that should appears in Holder's estimates by a generic constant $C$.

By Sobolev's embedding theorem, there exists $C_1$ such that

$$\|u^{k+1}\|_{p^*} + \|u^{k+1}\|_{p_*,\Gamma_1} \leq C_1 \|u^{k+1}\|_W$$

and hence

$$\|u\|_{p^*(k+1)} + \|u\|_{p_*(k+1),\Gamma_1} \leq C c_k^{1/p(k+1)} \Big( \int_\Omega u^{kp+q}\, dx + \oint_{\Gamma_1} u^{kp+r}\, d\rho + 1 \Big)^{1/p(k+1)}. \tag{9.4}$$

Put $k_1 = \min\{\frac{p^*-q}{p}, \frac{p_*-r}{p}\}$, $\xi_0 = \|u\|_{p^*} + \|u\|_{p_*}$ and $\xi_1 = \|u\|_{p^*(k_1+1)} + \|u\|_{p_*(k_1+1),\Gamma_1}$. Using Holder's inequality and that $\max\{k_1 p + q, k_1 p + r\} \leq p^*$ it comes from (9.4) that

$$\xi_1 \leq C c_{k_1}^{1/p(k_1+1)} (\xi_0 + 1)^{\frac{p^*}{p(k_1+1)}}.$$

Define successively

$$k_n = \min\{\frac{(k_{n-1}+1)p^* - q}{p}, \frac{(k_{n-1}+1)p_* - r}{p}\}, \quad c_n := c_{k_n}.$$

Observe that $k_n > 0$ for all $n \in \mathbb{N}$. Let us denote for simplicity $\xi_n := \|u\|_{p^*(k_n+1)} + \|u\|_{p_*(k_n+1),\Gamma_1}$. By induction we get from (9.4) that

$$\xi_n \leq C c_n^{1/p(k_n+1)} \big(\xi_{n-1} + 1\big)^{p^*/p(k_n+1)}. \tag{9.5}$$

Observe that $\lim_{n\to+\infty} k_n = +\infty$ and therefore $u \in L^r(\Omega) \cap L^r(\partial\Omega, \rho)$ for all $r > 1$. To obtain an uniform bound for $u$, we define a new sequence

$$q_0 := p_*, \quad q_{n+1} := p_*(\frac{q_n}{sp} + \frac{1}{p'})$$

where $s$ is any fixed number satisfying $1 < s < \frac{p_*}{p}$. Let us show the new estimate

$$\big(\|u\|_{q_{n+1}} + \|u\|_{q_{n+1},\Gamma_1}\big)^{q_{n+1}} \leq C(\frac{q_{n+1}^p}{q_n})^{p_*/p} \big(\|u\|_{q_n} + \|u\|_{q_n,\Gamma_1}\big)^{q_n p_*/sp} \tag{9.6}$$

Indeed, since we know that $u \in L^r(\Omega) \cap L^r(\partial\Omega, \rho)$ for all $r > 1$, we can use $u^{q_n/s}$ as a test function to get from the right hand side of the equation (9.3)

$$\begin{aligned} R &= \int_\Omega f(x,u) u^{q_n/s}\, dx + \oint_{\Gamma_1} g(x,u) u^{q_n/s}\, d\rho \\ &\leq \int_\Omega [a u^{q-1} + b] u^{q_n/s}\, dx + \oint_{\Gamma_1} [a u^{r-1} + b] u^{q_n/s}\, d\rho \\ &\leq D\big(\|u\|_{q_n} + \|u\|_{q_n,\Gamma_1}\big)^{q_n/s} \end{aligned}$$

with $D = \|au^{q-1}+b\|_{s'} + \|au^{r-1}+b\|_{s',\Gamma_1}$, and in the left hand side (the gradient term) of (3.8), after using Sobolev's embedding as above,

$$C_1 \frac{q_n}{s}(\frac{q_n}{sp}+\frac{1}{p'})^{-p}\big(\|u\|_{q_{n+1}} + \|u\|_{q_{n+1},\Gamma_1}\big)^{q_{n+1}\frac{p}{p_*}} \le L,$$

where

$$L = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla (u^{q_n/s})\, dx.$$

The equality $R = L$ gives the estimate (9.6). For simplicity we denote

$$\theta_n := q_n \ln(\|u\|_{q_n} + \|u\|_{q_n,\Gamma_1}), \quad B_n := \ln(C(\frac{q_{n+1}^p}{q_n})^{p_*}).$$

By induction we get from (9.6) that

$$\theta_{n+1} \le B_n + (p_*/sp)\theta_n.$$

Then

$$\theta_n \le (p_*/sp)^n\theta_0 + \sum_{i=1}^n (p_*/sp)^i B_{n-i}. \tag{9.7}$$

A simple estimate gives

$$d_0 \le \sum_{i=1}^n (p_*/sp)^i B_{n-i} \le d_1(p_*/sp)^n$$

for some $d_0, d_1 > 0$, and also we have $q_n \ge d_2(p_*/sp)^n$ for some $d_2 > 0$. Then from (9.7)

$$\frac{\theta_n}{q_n} \le \frac{(\theta_0+d_1)(p_*/sp)^n}{q_n} \le \frac{\theta_0+d_1}{d_2}$$

and hence

$$\|u\|_{q_n} + \|u\|_{q_n,\Gamma_1} \le e^{\frac{\theta_0+d_1}{d_2}}.$$

We conclude by letting $n \to +\infty$.
(ii) Assume for instance that $q = p^*$ and $r = p_*$. The proof in the case when one of these exponents is subcritical can be done using both the arguments of case (i) and those of the present case. The details are left to the reader. Assume as before that $u \ge 0$ and take, for any $t \ge p$, the test function $\Phi(x) = u_M^{tp+1}(x)$. After multiplying the equation (9.3) by $\Phi$, integrating by parts and letting $M \to +\infty$, one obtains, from the gradient term, in the left hand side,

$$L := \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\Phi\, dx = C_t \int_\Omega |\nabla u^{\frac{t}{p}}|^p\, dx \ge S_t\big(\|u\|^t_{\frac{tp^*}{p}} + \|u\|^t_{\frac{tp_*}{p},\Gamma_1}\big),$$

where we have used Sobolev's embedding to obtain the last inequality. From the right hand side we get

$$R := \int_\Omega f(x,u)\Phi\, dx + \int_{\Gamma_1} g(x,u)\Phi\, d\rho$$

$$\le a[\int_\Omega u^{p^*+t-p}\, dx + \oint_{\Gamma_1} u^{p_*+t-p}\, d\rho] + b[\int_\Omega u^{t-p+1}\, dx + \oint_{\Gamma_1} u^{t-p+1}\, d\rho].$$

To estimate the 1st and the 2nd integrals of $R$ we define, for all $m > 0$, the sets

$$\Omega_m := \{x \in \Omega \mid u(x) \ge m\}, \quad \Gamma_m := \{x \in \Gamma_1 \mid u(x) \ge m\}$$

and we obtain, using again Holder's inequality,

$$\int_\Omega u^{p^*+t-p}\,dx \le m^{p^*-p}\int_{\Omega\setminus\Omega_m} u^t\,dx + \int_{\Omega_m} u^{p^*+t-p}\,dx$$
$$\le m^{p^*-p}\|u\|_t^t + \|u\|_{\frac{tp^*}{p}}^t\,(\int_{\Omega_m} u^{p^*}\,dx)^{p/N};$$

and similarly

$$\oint_{\Gamma_1} u^{p_*+t-p}\,d\rho \le m^{p_*-p}\|u\|_{t,\Gamma_1}^t + \|u\|_{\frac{tp_*}{p},\Gamma_1}^t\left(\oint_{\Gamma_m} u^{p_*}\,d\rho\right)^{(p-1)/(N-1)}.$$

Choose $m>0$ such that $(\int_{\Omega_m} u^{p^*}\,dx)^{p/N} + (\oint_{\Gamma_m} u^{p_*}\,d\rho)^{(p-1)/(N-1)} < S_t/2$ and use that $R=L$ to get

$$S_t/2\big(\|u\|_{\frac{tp^*}{p}}^t + \|u\|_{\frac{tp_*}{p},\Gamma_1}^t\big) \le C_t\big(\|u\|_t^t + \|u\|_{t,\Gamma_1}^t + \int_\Omega u^{t-p+1}\,dx + \oint_{\Gamma_1} u^{t-p+1}\,d\rho\big)$$

and, after using Holder's and Young's inequalities for the 3rd and 4th integrals,

$$\frac{S_t}{4}\big(\|u\|_{\frac{tp^*}{p}}^t + \|u\|_{\frac{tp_*}{p},\Gamma_1}^t\big) \le C_t\big(\|u\|_t^t + \|u\|_{t,\Gamma_1}^t + 1)\big). \tag{9.8}$$

Finally, we define

$$t_0 := p, \quad t_n := p(\frac{p_*}{p})^n$$

to obtain from (9.8) successively that $u \in L^{\frac{t_n p^*}{p}}(\Omega) \cap L^{\frac{t_n p_*}{p}}(\partial\Omega,\rho)$. Since $t_n \to +\infty$, we get the conclusion. □

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