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Nonlinear Analysis

Nonresonance to the right of the first eigenvalue for the one-dimensional p-Laplacian[1]

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1. Introduction and statements

Let us consider the following quasilinear two-point boundary value problem:

$$\begin{aligned} &-(|u'|^{p-2}u')' = f(u) + e(t) \text{ in }]0,T[, \\ &u(0) = u(T) = 0, \end{aligned} \tag{1.1}$$

where $p \in]1,+\infty[, f:\mathbb{R}\to\mathbb{R}$ is continuous and $e \in L^\infty(0,T)$. Let us set $F(s) = \int_0^s f(\xi)\,\mathrm{d}\xi$ and denote by λ_1 and λ_2 the first and the second eigenvalue, respectively, of the homogeneous problem

$$\begin{aligned} &-(|u'|^{p-2}u')' = \lambda|u|^{p-2}u \text{ in }]0,T[, \\ &u(0) = u(T) = 0. \end{aligned}$$

We recall that (see [7]) $\lambda_1 = (\pi_p/T)^p$ and $\lambda_2 = (2\pi_p/T)^p$, where $\pi_p = 2(p-1)^{1/p} \int_0^1 \mathrm{d}s/(1-s^p)^{1/p}$.

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As far as one is concerned with the case where f interacts in some sense only with λ_1 and λ_2, it is known that either the Hammerstein-type condition (see e.g. [1])

$$\limsup_{s\to\pm\infty} \frac{pF(s)}{|s|^p} < \lambda_1, \tag{1.2}$$

or the Dolph-type condition (see e.g. [3])

$$\lambda_1 < \liminf_{s\to\pm\infty} \frac{f(s)}{|s|^{p-2}s} \le \liminf_{s\to\pm\infty} \frac{f(s)}{|s|^{p-2}s} < \lambda_2 \tag{1.3}$$

yield the solvability of Eq. (1.1) for any given function e. In the literature, this situation is usually referred to as nonresonance. On the other hand, in some recent papers [8, 9, 12], it was observed that the assumption

$$\liminf_{s\to\pm\infty} \frac{pF(s)}{|s|^p} < \lambda_1, \tag{1.4}$$

which is evidently much weaker than Eq. (1.2), is sufficient to obtain the same conclusion. It is therefore natural to ask whether a condition similar to Eq. (1.4), but placed to the right of λ_1, can be introduced. The aim of this paper is to show that the analogue of Eq. (1.4) to the right of λ_1, that is

$$\limsup_{s\to\pm\infty} \frac{pF(s)}{|s|^p} > \lambda_1, \tag{1.5}$$

yields the solvability of Eq. (1.1) for any e, when it is coupled with

$$\lim_{|s|\to+\infty} f(s)\operatorname{sgn}(s) = +\infty, \tag{1.6}$$

and the following control with respect to λ_2 is assumed

$$\limsup_{s\to\pm\infty} \frac{f(s)}{|s|^{p-2}s} \le \lambda_2 \quad \text{and} \quad \liminf_{s\to\pm\infty} \frac{pF(s)}{|s|^p} < \lambda_2. \tag{1.7}$$

Let us observe that the second condition in Eq. (1.7) is exactly in the same spirit of Eqs. (1.4) and (1.5), but while Eqs. (1.4) and (1.5) allow $f(s)/|s|^{p-2}s$ to cross λ_1, the same sort of crossing of λ_2 is prevented by the first condition in Eq. (1.7). It is worth noticing here that adapting an example given in [6] shows that condition (1.5) or, respectively, the second condition in Eq. (1.7), cannot be relaxed to

$$\limsup_{s\to\pm\infty} \frac{f(s)}{|s|^{p-2}s} > \lambda_1$$

even if Eq. (1.6) is assumed, or respectively, to

$$\liminf_{s\to\pm\infty} \frac{f(s)}{|s|^{p-2}s} < \lambda_2,$$

even if the first condition in Eq. (1.7) is assumed.

At this point it must be said that some results in this direction are already available in the literature when $p=2$, so that problem (1.1) reads

$$\begin{aligned} &-u'' = f(u) + e(t) \quad \text{in }]0,T[,\\ &u(0)=u(T)=0. \end{aligned} \tag{1.8}$$

In this particular case, it has been proved that the Dolph-type condition (1.3), with $p=2$, can be weakened in various ways, still keeping nonresonance. In [13] it was proved that, if one assumes the monotonicity condition

$$f(s) - \lambda_1 s \text{ is nondecreasing on } \mathbb{R}, \tag{1.9}$$

then the conditions

$$\lambda_1 < \liminf_{s\to\pm\infty} \frac{2F(s)}{s^2} \le \limsup_{s\to\pm\infty} \frac{2F(s)}{s^2} < \lambda_2$$

imply the solvability of Eq. (1.8) for any given e. On the other hand, in [15] it was shown that nonresonance for Eq. (1.8) occurs, under assumptions (1.6),

$$\lambda_1 < \liminf_{s\to-\infty} \frac{f(s)}{s} \le \liminf_{s\to-\infty} \frac{2F(s)}{s^2} < \lambda_2$$

and

$$\lambda_1 < \limsup_{s\to+\infty} \frac{2F(s)}{s^2} \le \limsup_{s\to+\infty} \frac{f(s)}{s} < \lambda_2.$$

Furthermore, in [16, 4] it was proved that the conditions

$$\lambda_1 \le \liminf_{s\to\pm\infty} \frac{f(s)}{s} \le \limsup_{s\to\pm\infty} \frac{f(s)}{s} \le \lambda_2$$

and

$$\lambda_1 < \limsup_{s\to\pm\infty} \frac{2F(s)}{s^2}, \qquad \liminf_{s\to\pm\infty} \frac{2F(s)}{s^2} < \lambda_2$$

still yield nonresonance. In these works condition (1.9) is not assumed anymore. Comparing these results with our one, it is clear that we improve the statements in [4, 16], for what concerns the condition with respect to λ_1, and we complement those in [13,15]. Furthermore, our result is valid for the quasilinear problem (1.1). It is worth noticing however that also the statements in [4, 15, 16] might be likely extended to this more general situation; whereas such an extension does not seem evident for [13].

Actually, we obtain a result where the restriction from above with respect to the second eigenvalue is replaced by a similar restriction with respect to the first nontrivial branch Σ of the Dancer–Fučik spectrum of the problem

$$\begin{aligned} &-(|u'|^{p-2}u')' = \mu|u|^{p-2}u^+ - \nu|u|^{p-2}u^- \quad \text{in }]0,T[,\\ &u(0)=u(T)=0. \end{aligned}$$

We recall that Σ can be represented as follows (see [7]):

$$\Sigma = \{(\mu, \nu) \mid 1/\mu^{1/p} + 1/\nu^{1/p} = T/\pi_p\}.$$

Namely, we have the following result, which appears new even in the case $p = 2$.

Theorem 1.1. *Let* $(\mu, \nu) \in \Sigma$ *be given and assume that*

(h_1)

$$\lim_{s \to \pm\infty} f(s)\,\mathrm{sgn}(s) = +\infty,$$

(h_2)

$$\limsup_{s \to \pm\infty} \frac{pF(s)}{|s|^p} > \lambda_1,$$

(h_3)

$$\limsup_{s \to +\infty} \frac{f(s)}{|s|^{p-2}s} \le \mu \quad \textit{and} \quad \limsup_{s \to -\infty} \frac{f(s)}{|s|^{p-2}s} \le \nu,$$

(h_4)

$$\liminf_{s \to +\infty} \frac{pF(s)}{|s|^p} < \mu \quad \textit{or} \quad \liminf_{s \to -\infty} \frac{pF(s)}{|s|^p} < \nu.$$

Then, for any given function e, problem (1.1) *has at least one solution.*

The proof of Theorem 1.1 basically relies on the use of topological degree methods, in combination with time-mapping estimates for constructing wrongly ordered lower and upper solutions. The proof is divided in three parts. In the first one, we prove that conditions (h_1) and (h_2) imply the existence of a sequence $(\alpha_n)_n$ of positive lower solutions, with $\max \alpha_n \to +\infty$, and of a sequence $(\beta_n)_n$ of negative upper solutions, with $\min \beta_n \to -\infty$. These α_n and β_n are generated by a shooting method applied to an associated autonomous equation, whose solutions are controlled estimating the time-map. The argument used here is modelled on a similar one recently introduced in [14]. In the second part of the proof, we fix a pair of lower and upper solutions, constructed in the preceding step and satisfying $\beta(t) < \alpha(t)$ in $]0, T[$, and we define a suitable homotopy. Then, we show that conditions (h_3) and (h_4) provide some estimates on the maximum and the minimum values of the solutions u of the parametrized problem, satisfying $\beta(t_0) < u(t_0) < \alpha(t_0)$ for some $t_0 \in]0, T[$. Here, some ideas are borrowed from [10, 11]. In the final part of the proof, by introducing a further (more standard) homotopy, we show that the degree of an associated operator, whose zeroes are precisely the solutions of problem (1.1), is equal to -1 on the open set $\mathscr{U} = \{u \in C^1([0, T]) | \min(u - \alpha) < 0 < \max(u - \beta), A < \min u, \max u < B, \|u'\|_\infty < C\}$, where the constants A, B and C are found in the previous step. In this part, an argument from [10] is adapted to the quasilinear setting.

From this discussion it should be clear that actually a slightly more general result than Theorem 1.1 is valid, in which instead of assuming (h_1) and (h_2) we directly suppose the existence of a pair of wrongly ordered lower and upper solutions. We also wish to observe that a different approach, based on suitable truncations and modifications of the equation, as recently introduced in [2], might be exploited to get a result in which no ordering between α and β is required. However, this would not lead to any improvement of Theorem 1.1.

2. Proof

In the sequel we use the following notation. For any given $p \in]1, +\infty[$, we set $\varphi_p(s) = |s|^{p-2}s$, for $s \in \mathbb{R}$. Moreover, for $q \in [1, +\infty]$, we denote by $\|\cdot\|_q$ the norm in $L^q(0, T)$ and by $\|\cdot\|_{C^1}$ the norm in $C^1([0, T])$.

2.1. Construction of wrongly ordered lower and upper solutions

By a lower solution of problem (1.1) we mean a function $\alpha \in C^1([0, T])$ such that $\varphi_p(\alpha')$ is absolutely continuous and

$$-(\varphi_p(\alpha'))' \le f(\alpha) + e(t) \quad \text{in }]0, T[,$$
$$\alpha(0) \le 0, \alpha(T) \le 0.$$

An upper solution is defined similarly by reversing all the above inequalities.

Lemma 2.1. (i) *Assume*

$$\lim_{s \to +\infty} f(s) = +\infty \quad \textit{and} \quad \lim_{s \to +\infty} \sup \frac{pF(s)}{s^p} > \lambda_1. \tag{2.1}$$

Then, there exists a sequence $(\alpha_n)_n$ *of lower solutions of problem* (1.1) *satisfying*

$$\begin{aligned} &\alpha_n(t) > 0 \quad \textit{in }]0, T[, \\ &\alpha_n(0) = \alpha(T) = 0, \\ &\max \alpha_n \to +\infty \quad \textit{as } n \to +\infty. \end{aligned} \tag{2.2}$$

(ii) *Assume*

$$\lim_{s \to -\infty} f(s) = -\infty \quad \textit{and} \quad \lim_{s \to -\infty} \sup \frac{pF(s)}{|s|^p} > \lambda_1.$$

Then, there exists a sequence $(\beta_n)_n$ *of upper solutions of problem* (1.1) *satisfying*

$$\begin{aligned} &\beta_n(t) < 0 \quad \textit{in }]0, T[, \\ &\beta_n(0) = \beta(T) = 0, \\ &\min \beta_n \to -\infty \quad \textit{as } n \to +\infty. \end{aligned}$$

Proof. We prove only the first statement, since the proof of the second one proceeds in a quite parallel way. Let us set $g(s)=f(s)-\|e\|_\infty$ and $G(s)=\int_0^s g(\xi)\,\mathrm{d}\xi$. It is easy to see that, by Eq. (2.1), there exists a constant $c_0>0$ such that

$$g(s)>0 \quad \text{for } s>c_0 \tag{2.3}$$

and there exist a constant $K>\lambda_1^{1/p}$ and a sequence $(d_n)_n$, with $d_n\to+\infty$, such that

$$p(G(d_n)-G(s))\geq K^p(d_n^p-s^p) \quad \text{for all } 0\leq s\leq d_n. \tag{2.4}$$

Moreover, let $\gamma>0$ be a constant such that

$$g(s)\geq -\gamma \quad \text{for all } s\geq 0 \tag{2.5}$$

and take a number $L>0$ so large that

$$\varphi_p(L)\geq\varphi_p\left(\frac{2Kc_0}{(K-\lambda_1^{1/p})T}\right)+\gamma T/2. \tag{2.6}$$

Without restriction we can also suppose that

$$d_n>c_0+LT/2 \quad \text{for all } n. \tag{2.7}$$

Let us consider the initial value problem

$$\begin{aligned} &-(\varphi_p(u'))'=g(u) \quad \text{in } [0,T/2[,\\ &u(0)=d_n,\\ &u'(0)=0. \end{aligned} \tag{2.8}$$

By a local solution of Eq. (2.8) we mean a function u defined on some interval $I\subset[0,T/2[$, which is of class C^1 on I, together with $\varphi_p(u')$, and satisfies the initial conditions. It is well-known that Eq. (2.8) admits local solutions which can be extended to a maximal interval of existence $[0,\omega[\subset[0,T/2[$.

Let u be a (maximal) solution of Eq. (2.8) and define

$$\sigma=\sup\{t\in[0,\omega[\,|\,u(s)>c_0 \quad \text{in } [0,t[\}.$$

From Eqs. (2.8) and (2.3), we immediately see that $\varphi_p(u')$ and, hence, u' are decreasing on $[0,\sigma[$. Therefore, we have

$$u'(t)<0 \quad \text{for } t\in[0,\sigma[. \tag{2.9}$$

Multiplying Eq. (2.8) by u' and integrating between 0 and t, with $t\in]0,\sigma[$, we obtain

$$\frac{p-1}{p}|u'(t)|^p=G(d_n)-G(u(t))$$

and hence, using Eq. (2.9) and integrating again between 0 and t, we derive

$$t=\left(\frac{p-1}{p}\right)^{1/p}\int_0^t \frac{-u'(\xi)}{(G(d_n)-G(u(\xi)))^{1/p}}\,\mathrm{d}\xi$$

$$=\left(\frac{p-1}{p}\right)^{1/p}\int_{u(t)}^{d_n} \frac{ds}{(G(d_n)-G(s))^{1/p}}\,\mathrm{d}s.$$

Now, using the definition of σ and exploiting Eq. (2.4), we get

$$\sigma\le\frac{1}{K}(p-1)^{1/p}\int_{c_0}^{d_n}\frac{ds}{(d_n^p-s^p)^{1/p}}\,\mathrm{d}s.$$

Changing variable and recalling that

$$(p-1)^{1/p}\int_0^1\frac{dv}{(1-v^p)^{1/p}}\,\mathrm{d}v=\frac{T}{2}\lambda_1^{1/p},$$

we obtain

$$\sigma\le\frac{\lambda_1^{1/p}}{K}\frac{T}{2}\left(<\frac{T}{2}\right). \tag{2.10}$$

Since, by Eq. (2.9), u can be continued to σ as a solution and $u(\sigma)\ge c_0$, we can conclude that $\sigma<\omega$ and, therefore,

$$u(\sigma)=c_0.$$

Now, we want to prove that

$$u'(\sigma)<-L. \tag{2.11}$$

Assume that this is not true. Since u' is decreasing on $[0,\sigma]$, it follows that $u'(t)\ge -L$, for $t\in[0,\sigma]$, and hence, by integration,

$$d_n-c_0=u(0)-u(\sigma)\le LT/2,$$

which is in contradiction with Eq. (2.7).

Let us set now

$$\tau=\sup\{t\in[\sigma,\omega[\,|\,u(s)>0 \text{ in } [\sigma,t[\}. \tag{2.12}$$

Of course, we have $\tau>\sigma$. Integrating Eq. (2.8) between σ and t, with $t\in]\sigma,\tau[$, and using Eqs. (2.5) and (2.11), we have

$$-\varphi_p(u'(t))-\varphi_p(L)\ge-\varphi_p(u'(t))+\varphi_p(u'(\sigma))=\int_\sigma^t g(u(\xi))\,\mathrm{d}\xi\ge-\gamma T/2$$

and hence, by Eq. (2.6),

$$u'(t)\le\varphi_p^{-1}(-\varphi_p(L)+\gamma T/2)\le-\frac{2Kc_0}{(K-\lambda_1^{1/p})T}(<0). \tag{2.13}$$

Therefore, we can continue u to τ as a solution and we have $u(\tau)\geq 0$. Suppose that $u(\tau)>0$. This implies that $\tau=\omega=T/2$. Integrating Eq. (2.13) between σ and $T/2$, we obtain

$$-c_0 < u(T/2) - u(\sigma) \leq -\frac{2Kc_0}{(K-\lambda_1^{1/p})T}\left(\frac{T}{2}-\sigma\right).$$

Hence, by Eq. (2.10), we get a contradiction. Accordingly, we can conclude that $u(\tau)=0$, for some $\tau\in\,]0,T/2]$, with $u(t)>0$ for $t\in[0,\tau[$.

Now, we are in a position to build a sequence $(\alpha_n)_n$ of lower solutions of problem (1.1), satisfying Eq. (2.2). For each n, let us set

$$\alpha_n(t)=\begin{cases} u_n(\tau_n-t) & \text{for } 0\leq t\leq \tau_n,\\ d_n & \text{for } \tau_n<t<T-\tau_n,\\ u_n(t-T+\tau_n) & \text{for } T-\tau_n\leq t\leq T,\end{cases}$$

where u_n is a fixed solution of Eq. (2.8) and τ_n is defined according to Eq. (2.12). It is plain that α_n is of class C^1 and $\varphi_p(\alpha_n')$ is absolutely continuous on $[0,T]$. Moreover, we have

$$-(\varphi_p(\alpha_n'))'=0\leq g(d_n)=g(\alpha_n(t))\leq f(\alpha_n(t))+e(t)\quad \text{for a.e. } t\in\,]\tau_n,T-\tau_n[$$

and

$$-(\varphi_p(\alpha_n'))'=g(\alpha_n(t))\leq f(\alpha_n(t))+e(t)\quad \text{for a.e. } t\in\,]0,\tau_n[\,\cup\,]T-\tau_n,T[.$$

Therefore, α_n is a lower solution of problem (1.1). This concludes the proof of Lemma 2.1. □

2.2. Definition of a homotopy and a priori estimates

Let us fix a pair α,β of lower and upper solutions of problem (1.1) with the property

$$\beta(t)<0<\alpha(t)\quad \text{in }]0,T[,$$
$$\alpha(0)=\alpha(T)=\beta(0)=\beta(T)=0.$$

Let us also fix $\eta>0$ and $a>0$ such that

$$\lambda_1<a<\min\{u,v\}. \tag{2.14}$$

A further condition will be assumed later on η (see Eq. (2.25) below). We define, for $t\in\,]0,T[$ and $s\in\mathbb{R}$,

$$k(t,s)=\begin{cases} f(\alpha(t))+a\varphi_p(s)+\eta & \text{if } s\geq\alpha(t),\\ f(\beta(t))+a\varphi_p(s)-\eta & \text{if } s\leq\beta(t),\\ \text{a linear interpolation} & \text{if } \beta(t)<s<\alpha(t).\end{cases}$$

Let us consider the homotopy

$$-(\varphi_p(u'))' = (1-\rho)(f(u)+e(t)) + \rho k(t,u) \quad \text{in }]0,T[, \qquad (\mathrm{P}_\rho)$$
$$u(0)=u(T)=0,$$

where $\rho \in [0,1]$. The following four technical claims will provide some a priori estimates on the solutions of (P_ρ).

Claim 2.1. Let $(u_n)_n$ be a sequence of solutions of (P_{ρ_n}), $\rho_n \in [0,1]$, satisfying $\|u_n\|_\infty \to +\infty$ and

$$\beta(t_n) < u_n(t_n) < \alpha(t_n) \quad \textit{for some } t_n \in]0,T[. \tag{2.15}$$

Then, possibly for a subsequence, $\rho_n \to 0$ and $u_n/\|u_n\|_\infty \to v$ in $C^1([0,T])$, where v is a nontrivial solution of

$$-(\varphi_p(v'))' = \mu\varphi_p(v^+) - \nu\varphi_p(v^-) \quad \textit{in }]0,T[,$$
$$v(0)=v(T)=0. \tag{2.16}$$

Moreover, there exists a constant $L>1$, independent of $(u_n)_n$, such that for all large n

$$\frac{1}{L} \le \frac{\max u_n}{-\min u_n} \le L. \tag{2.17}$$

Finally, one has

$$\int_{\{v>0\}} \left| \frac{f(u_n)}{\|u_n\|_\infty^{p-1}} - \mu\varphi_p(v^+) \right| \mathrm{d}t \to 0 \tag{2.18}$$

and

$$\int_{\{v<0\}} \left| \frac{f(u_n)}{\|u_n\|_\infty^{p-1}} - \nu\varphi_p(v^-) \right| \mathrm{d}t \to 0.$$

This statement is known when $p=2$ (cf. [11]), so that we only give a sketch of the proof in the present more general framework.

Proof of Claim 2.1. Write for simplicity

$$h(t,s,\rho) = (1-\rho)(f(s)+e(t)) + \rho k(t,s) \tag{2.19}$$

and $v_n = u_n/\|u_n\|_\infty$. Thus v_n is a solution of the following problem:

$$-(\varphi_p(v_n'))' = \frac{h(t,u_n,\rho_n)}{\|u_n\|_\infty^{p-1}} \quad \text{in }]0,T[, \tag{2.20}$$
$$v_n(0)=v_n(T)=0.$$

By conditions (h_1) and (h_3) on f and the definition of k, there exist constants A,B such that

$$|h(t,s,\rho)| \le A|s|^{p-1} + B \tag{2.21}$$

for a.e. $t\in[0,T]$, $s\in\mathbb{R}$ and $\rho\in[0,1]$. Hence, the right-hand side of Eq. (2.20) is uniformly bounded in $L^\infty(0,T)$ and therefore, by standard regularity results, there exist constants $C>0$ and $\gamma\in]0,1[$ such that $\|v_n\|_{C^{1,\gamma}}\le C$. By compactness we get the existence of a subsequence, still denoted by $(v_n)_n$, and of a function $v\in C^1([0,T])$, satisfying $\|v\|_\infty=1$ and $v(0)=v(T)=0$, such that $v_n\to v$ in $C^1([0,T])$. Possibly passing to a further subsequence, we can also assume that $\rho_n\to\rho_0$ for some $\rho_0\in[0,1]$. Then, using again (h_1), (h_3) and the definition of k, it is easy to see that v solves an equation of the form

$$\begin{aligned}-(\varphi_p(v'))' = {}&((1-\rho_0)M(t)+\rho_0 a)\varphi_p(v^+)\\ &-((1-\rho_0)N(t)+\rho_0 a)\varphi_p(v^-)\quad \text{in }]0,T[\end{aligned}$$

for some $M,N\in L^\infty(0,T)$ satisfying

$$0\le M(t)\le\mu,\qquad 0\le N(t)\le\nu\quad\text{a.e. in } [0,T]. \tag{2.22}$$

We claim that v changes sign. Assume by contradiction that, say, $v\ge 0$ in $[0,T]$. The strong maximum principle (see for instance [17]) implies that $v(t)>0$ in $]0,T[$ and $v'(0)>0>v'(T)$. Since $v_n\to v$ in $C^1([0,T])$, then, for some $d>0$ and all n sufficiently large, we have that $v_n(t)\ge dv(t)$ in $[0,T]$. On the other hand, there exists $c>0$ such that $cv(t)\ge\alpha(t)$ in $[0,T]$. Consequently, if n is large enough to have $d\|u_n\|_\infty\ge c$, we conclude that

$$u_n(t)\ge d\|u_n\|_\infty v(t)\ge cv(t)\ge\alpha(t)\quad\text{in }]0,T[,$$

which contradicts Eq. (2.15). Thus, v changes sign. Finally, a standard argument (see [11], for $p=2$) shows that Eqs. (2.14) and (2.22) imply that $(1-\rho_0)M(t)+\rho_0 a\equiv\mu$ a.e. on $\{v>0\}$ and $(1-\rho_0)N(t)+\rho_0 a\equiv\nu$ a.e. on $\{v<0\}$. Accordingly, $\rho_0=0$, $M(t)\equiv\mu$ a.e. on $\{v>0\}$, $N(t)\equiv\nu$ a.e. on $\{v<0\}$ and therefore v satisfies Eq. (2.16).

Moreover, property Eq. (2.17) is a consequence of the convergence of $\max u_n/-\min u_n$ to $\max v/-\min v$ and of the structure of the solution set of Eq. (2.16) (cf. [7]).

Finally, property (2.18) follows from the L^∞-weak* convergence of the right-hand side of Eq. (2.20) to that of Eq. (2.16) (cf. [11]). This concludes the proof of Claim 2.1. □

Claim 2.2. There exists a constant $C>0$ such that, for any solution u of (P_ρ), $\rho\in[0,1]$; satisfying $\|u\|_\infty\ge 1$, one has

$$\|u'\|_\infty\le C\|u\|_\infty.$$

The proof of Claim 2.2 is an immediate consequence of condition (2.21).

Claim 2.3. (i) Assume that the first condition in (h_4) holds, that is

$$\liminf_{s\to+\infty} \frac{pF(s)}{s^p} < \mu. \tag{2.23}$$

Then, there exists a sequence $(R_n)_n$, with $R_n \to +\infty$, such that, for any solution u of (P_ρ), $\rho \in [0,1]$, satisfying

$$\beta(t_0) < u(t_0) < \alpha(t_0) \quad \text{for some } t_0 \in]0,T[, \tag{2.24}$$

one has

$$\max u \neq R_n.$$

(ii) When the second condition in (h_4) holds, a similar statement is true for a sequence of negative real numbers $(S_n)_n$, with $S_n \to -\infty$ and $\min u$ replaced by $\max u$.

Proof. We prove only the first statement, because the proof of the second one is similar. From Eq. (2.23), we infer the existence of a sequence $(R_n)_n$, with $R_n \to +\infty$, such that, for some $\varepsilon > 0$,

$$pF(R_n) \leq (\mu - \varepsilon) R_n^p.$$

We show that the conclusion of the claim holds for a tail-end of $(R_n)_n$. Assume by contradiction that this is not true, so that we can find a subsequence, still denoted by $(R_n)_n$, and a sequence $(u_n)_n$ of solutions of (P_{ρ_n}) satisfying Eq. (2.15) and $\max u_n = R_n$. In particular, $\|u_n\|_\infty \to +\infty$ and the conclusions of Claim 2.1 hold. Thus, u_n changes sign for n large enough. Hence, there exist points $\gamma_n < \delta_n$ in $[0,T]$, with $u_n(\gamma_n) = 0$, $u_n(\delta_n) = R_n$ and $u_n > 0$ on $]\gamma_n, \delta_n[$. Using Eq. (2.17) we get, for n large,

$$\begin{aligned}
\frac{\varepsilon}{pL}\|u_n\|_\infty^p \leq \frac{\varepsilon}{p} R_n^p &\leq \frac{\mu}{p} R_n^p - F(R_n) \\
&= \frac{\mu}{p}(u_n(\delta_n)^p - u_n(\gamma_n)^p) - F(u_n(\delta_n)) + F(u_n(\gamma_n)) \\
&= \int_{\gamma_n}^{\delta_n} (\mu\varphi_p(u_n) - f(u_n)) u_n' \, \mathrm{d}s.
\end{aligned}$$

Using Claim 2.2 we obtain

$$\begin{aligned}
\frac{\varepsilon}{pL} \leq &\int_{\gamma_n}^{\delta_n} \left| \mu\varphi_p(v^+) - \frac{f(u_n)}{\|u_n\|_\infty^{p-1}} \right| \frac{|u_n'|}{\|u_n\|_\infty} \, \mathrm{d}s \\
&+ \int_{\gamma_n}^{\delta_n} \mu \left| \frac{\varphi_p(u_n)}{\|u_n\|_\infty^{p-1}} - \varphi_p(v^+) \right| \frac{|u_n'|}{\|u_n\|_\infty} \, \mathrm{d}s
\end{aligned}$$

$$\leq C\int_{\gamma_n}^{\delta_n}\left|\mu\varphi_p(v^+) - \frac{f(u_n)}{\|u_n\|_\infty^{p-1}}\right| \mathrm{d}s$$

$$+ C\mu\int_{\gamma_n}^{\delta_n}\left|\frac{\varphi_p(u_n)}{\|u_n\|_\infty^{p-1}} - \varphi_p(v^+)\right| \mathrm{d}s$$

Then, passing to the limit, using Eq. (2.18) and the fact that $\gamma_n \to \gamma$ and $\delta_n \to \delta$, where $]\gamma, \delta[= \{v > 0\}$, we get a contradiction. This concludes the proof of Claim 2.3. □

Claim 2.4. (i) Assume that the first condition in (h_4) holds. Then, for all n sufficiently large, every solution u of (P_ρ), $\rho \in [0, 1]$, satisfying Eq. (2.24) and $\max u \leq R_n$, is such that

$$\min u > -LR_n.$$

(ii) When the second condition in (h_4) holds, then, for all n sufficiently large, every solution u of (P_ρ), $\rho \in [0, 1]$, satisfying Eq. (2.24) and $\min u \geq S_n$, is such that

$$\max u < -LS_n.$$

The proof of Claim 2.4 follows, arguing by contradiction and using Claim 2.3 and conclusion (2.17) of Claim 2.1.

2.3. Computation of the degree

Assume that the first condition in (h_4) holds, the treatment of the other case being completely similar. Take n sufficiently large and put $R = R_n$. Let $C > 0$ be the constant of Claim 2.2. We define the open set

$$\mathcal{U} = \left\{ \begin{array}{l} u \in C^1([0, T]) \mid \min(u - \alpha) < 0 < \max(u - \beta), \\ -LR < \min u, \ \max u < R, \ \|u'\|_\infty < C(LR + 1) \end{array} \right\}.$$

At this stage we also assume that the constant $\eta > 0$ appearing in the definition of k satisfies

$$\begin{aligned} &-a\|\varphi_p(\alpha)\|_\infty - \|e\|_\infty + \eta > 1, \quad a\|\varphi_p(\beta)\|_\infty + \|e\|_\infty - \eta < -1, \\ &-\|f(\alpha)\|_\infty + \eta > 0, \quad -\|f(\beta)\|_\infty + \eta > 0. \end{aligned} \tag{2.25}$$

Lemma 2.2. *There is no solution u of (P_ρ), $\rho \in]0, 1]$, such that $u \in \partial\mathcal{U}$.*

Proof. Assume, by contradiction, that there exists a solution u of (P_ρ), $\rho \in]0, 1]$, with $u \in \partial\mathcal{U}$. Then, by Claims 2.2–2.4, condition (2.24) cannot hold. Hence, one of the following alternatives occurs:

(i) $u(t) \geq \alpha(t)$ for all $t \in [0, T]$ and $u(t_0) = \alpha(t_0)$ for some $t_0 \in]0, T[$;

(ii) $u(t) > \alpha(t)$ for all $t \in]0, T[$ and either $u'(0) = \alpha'(0)$ or $u'(T) = \alpha'(T)$;

(iii) $u(t) \le \beta(t)$ for all $t \in [0,T]$ and $u(t_0) = \beta(t_0)$ for some $t_0 \in]0,T[$;
(iv) $u(t) < \beta(t)$ for all $t \in]0,T[$ and either $u'(0) = \beta'(0)$ or $u'(T) = \beta'(T)$.
This implies that, anyhow, we can find a point $t_0 \in [0,T]$ such that $u(t_0) = \alpha(t_0)$ and $u'(t_0) = \alpha'(t_0)$, or $u(t_0) = \beta(t_0)$ and $u'(t_0) = \beta'(t_0)$. Let us consider the first case, with $t_0 \in [0,T[$. The remainder of the cases can be treated similarly. Since $u(t_0) = \alpha(t_0)$ for any given $\varepsilon > 0$ there exists, by continuity, an interval $[t_0, t_0 + \delta[\subset [0,T[$, where

$$f(u(t)) - f(\alpha(t)) > -\frac{\varepsilon}{2}.$$

Then, it follows, from (P_ρ) and the choice of η in Eq. (2.25), that

$$-((\varphi_p(u'))' - (\varphi_p(\alpha'))') \ge -(1-\rho)\frac{\varepsilon}{2} + \rho$$

in $[t_0, t_0 + \delta[$, and hence, taking ε small enough,

$$-((\varphi_p(u'))' - (\varphi_p(\alpha'))') \ge -\frac{\rho}{2}.$$

Integrating between t_0 and $t \in]t_0, t_0+\delta[$ and using $u'(t_0) = \alpha'(t_0)$, we get $u'(t) < \alpha'(t)$ in $[t_0, t_0+\delta[$. This contradicts the fact that $u(t) \ge \alpha(t)$ on $[0,T]$. The proof of Lemma 2.2 is concluded. □

Observe that for $\rho = 0$, (P_0) is precisely problem (1.1). Therefore, we will assume in the sequel that (P_0) has no solution on $\partial\mathscr{U}$, otherwise we are done. Let us introduce the following operators. Denote by $K : L^\infty(0,T) \to C^{1,\gamma}([0,T])$, with $\gamma = \min\{1, 1/(p-1)\}$, the operator which sends l on the unique solution of

$$-\varphi_p(u')' = l \quad \text{in }]0,T[,$$
$$u(0) = u(T) = 0.$$

It is known that K is continuous (see [7]). The compact embedding J of $C^{1,\gamma}([0,T])$ into $C^1([0,T])$ then implies that $J \circ K : L^\infty(0,T) \to C^1([0,T])$ is compact. Moreover, for each $\rho \in [0,1]$, denote by $H_\rho : C^1([0,T]) \to L^\infty(0,T)$ the operator defined by $H_\rho(u) = h(\cdot, u, p)$, where h is given in Eq. (2.19). Clearly, H_ρ is continuous and maps bounded sets into bounded sets. Thus, for each $\rho \in [0,1]$, the operator $T_\rho = J \circ K \circ H_\rho$ is completely continuous from $C^1([0,T])$ into itself and its fixed points are precisely the solutions of (P_ρ).

Lemma 2.3. *We have*

$$\deg(I - T_0, \mathscr{U}, 0) = -1.$$

Proof. From Lemma 2.2 and the above considerations, we have that $\deg(I - T_\rho, \mathscr{U}, 0)$ is well defined and independent of $\rho \in [0,1]$. Therefore,

$$\deg(I - T_0, \mathscr{U}, 0) = \deg(I - T_1, \mathscr{U}, 0). \tag{2.26}$$

Notice that for $\rho = 1$ problem (P_1) reduces to

$$-(\varphi_p(u'))' = k(t,u) \quad \text{in }]0,T[,$$
$$u(0) = u(T) = 0.$$

Claim 2.5. There is no solution u of (P_1) satisfying $u(t) \geq \alpha(t)$ in $[0,T]$, or $u(t) \leq \beta(t)$ in $[0,T]$.

Proof. If $u(t) \geq \alpha(t)$ in $[0,T]$, then (P_1) reduces to

$$-(\varphi_p(u'))' = f(\alpha(t)) + a\varphi_p(u) + \eta \quad \text{in }]0,T[,$$
$$u(0) = u(T) = 0.$$

By the choice of η in Eq. (2.25), we have that

$$-(\varphi_p(u'))' \geq a\varphi_p(u) \quad \text{in }]0,T[. \tag{2.27}$$

Moreover, since $\alpha(t) > 0$, then $u(t) > 0$ in $]0,T[$. Hence, from Eq. (2.27) we have that $(\varphi_p(u'))' < 0$ in $]0,T[$. Thus u' is decreasing in $[0,T]$. Let $t_0 \in]0,T[$ be the unique point where the maximum of u is attained. Pick up a $t \in]0,t_0[$. We have, after multiplying Eq. (2.27) by u' and integrating between t and t_0,

$$\frac{p-1}{p}(u'(t))^p \geq \frac{a}{p}(u(t_0)^p - u(t)^p),$$

that is

$$(p-1)^{1/p}\frac{u'(t)}{(u(t_0)^p - u(t)^p)^{1/p}} \geq a^{1/p}.$$

Integrating this inequality between 0 and t_0, we get, after a change of variable,

$$(p-1)^{1/p}\int_0^1 \frac{\mathrm{d}s}{(1-s^p)^{1/p}} = (p-1)^{1/p}\int_0^{t_0} \frac{u'(t)}{(u(t_0)^p - u(t)^p)^{1/p}}\,\mathrm{d}t \geq a^{1/p}t_0.$$

Recalling the definition of λ_1, we finally obtain

$$\frac{T}{2}\lambda_1^{1/p} \geq a^{1/p}t_0.$$

Proceeding similarly on $[t_0,T]$, we find

$$\frac{T}{2}\lambda_1^{1/p} \geq a^{1/p}(T - t_0).$$

Summing up these two last inequalities and using $a > \lambda_1$, we get a contradiction. This concludes the proof of Claim 2.5. $\square$

As a consequence of Claim 2.5 we derive

$$\deg(I - T_1, \mathscr{U}, 0) = \deg(I - T_1, \mathscr{V}, 0), \tag{2.28}$$

where

$$\mathscr{V} = \{u \in C^1([0,T]) \mid -LR < \min u,\ \max u < R,\ \|u'\|_\infty < C(LR+1)\}.$$

Finally, in order to compute the degree to the right-hand side of Eq. (2.28), we consider the homotopy

$$\begin{aligned} &-(\varphi_p(u'))' = \rho k(t,u) + (1-\rho)a\varphi_p(u) \quad \text{in }]0,T[,\\ &u(0) = u(T) = 0, \end{aligned} \tag{Q_ρ}$$

where $\rho \in [0,1]$. Let us denote, for $\rho \in [0,1]$, by $N_\rho : C^1([0,T]) \to L^\infty(0,T)$ the operator defined by $N_\rho(u) = \rho k(\cdot,u) + (1-\rho)a\varphi_p(u)$ and set $S_\rho = J \circ K \circ N_\rho$. Of course, it is $S_1 = T_1$.

Claim 2.6. There exists a constant $D > 0$ such that, for every solution u of (Q_ρ), $\rho \in [0,1]$, one has

$$\|u\|_{C^1} < D.$$

Proof. Assume by contradiction that, for a sequence $(u_n)_n$ of solutions of (Q_{ρ_n}), we have $\|u_n\|_{C^1} \to +\infty$. Setting $v_n = u_n/\|u_n\|_{C^1}$, we find a subsequence, still denoted by $(v_n)_n$, and a function $v \in C^1([0,T])$ such that $v_n \to v$ in $C^1([0,T])$ and v is a nontrivial solution of

$$\begin{aligned} &-(\varphi_p(v'))' = a\varphi_p(v) \quad \text{in }]0,T[,\\ &v(0) = v(T) = 0. \end{aligned}$$

This is clearly impossible because a is not an eigenvalue. This concludes the proof of Claim 2.6. □

Now, take n so large that $R = R_n > D$, with D defined in Claim 2.6. Using Eqs. (2.26) and (2.28) and adapting the degree computation of [3], we have that

$$\begin{aligned} \deg(I - T_0, \mathscr{U}, 0) &= \deg(I - S_1, \mathscr{V}, 0)\\ &= \deg(I - S_0, \mathscr{V}, 0) = \deg(I - S_0, \mathscr{W}, 0) = -1, \end{aligned}$$

where $\mathscr{W}$ is the open ball of center 0 and radius D in $C^1([0,T])$. This concludes the proof of Lemma 2.3. □

Finally, the existence of a solution $u \in \bar{\mathscr{U}}$ of problem (1.1) follows from Lemma 2.3. This concludes the proof of Theorem 1.1.

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