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On a resonant-superlinear elliptic problem

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Abstract. We start by discussing the solvability of the following superlinear problem

$$\begin{cases} -\Delta u = \lambda_1 u + (u^+)^p + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $1 < p < \frac{N+1}{N-1}$, $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain and f satisfies a one-sided Landesman-Lazer condition. We also consider systems of semilinear elliptic equations with nonlinearities of the above form, so exhibiting superlinearity as $u \to +\infty$ and resonance as $u \to -\infty$. A priori bounds for the solutions of the equation and the system are obtained by using Hardy-type inequalities . Existence of solutions is then obtained using topological degree arguments.

1. Introduction

Existence of solutions for semilinear elliptic Dirichlet problems

$$\begin{cases} -\Delta u = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with distinct behaviours of $\frac{g(x,s)}{s}$ as $s \to \pm\infty$ has been difficult to establish in the case when (i) $g(x,0) \neq 0$ (so there is no trivial solution), (ii) there is resonance in one direction and (iii) the problem is superlinear in the other. The problem seems to be particularly harder to deal if such a resonance is at the first eigenvalue of the Laplacian, in view of the fact that the corresponding first eigenfuction has a definite sign. It is a problem of this kind that it will be treated here, namely

$$\begin{cases} -\Delta u = \lambda_1 u + (u^+)^p + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

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where Ω is a smooth bounded domain in $\mathbb{R}^N, N \geq 3$, $f(x) \not\equiv 0$ is a function satisfying the following *condition (*)* :

$$f \in L^r(\Omega) \text{ for some } r > N$$

and $1 < p < \frac{N+1}{N-1}$. λ_1 denotes the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$.

Observe that the restriction on the power p is a bit disappointing!. One should aim to getting it for any $p < 2^* := \frac{N+2}{N-2}$, when considering subcritical problems. The critical case $p = \frac{N+2}{N-2}$ has been treated in [3]. On the subcritical case we refer to [4] [5], where the restriction on p is stronger than the one treated here, namely $p < \frac{N}{N-1}$.

Our first main result is the following :

Theorem 1.1. *Let $1 < p < \frac{N+1}{N-1}$ and let the function f satisfy both condition (*) and the hypothesis*

(H1) $\int_\Omega f\varphi_1 < 0$,

where φ_1 is the positive eigenfunction associated to λ_1 and normalized to have L^2-norm equal to 1 . Then problem (1.1) *possesses at least one solution in $W^{2,r}(\Omega) \cap H_0^1(\Omega)$.*

Since we will use topological arguments to prove Theorem 1.1, we shall need a-priori bounds on the eventual solutions of (1.1). This is the contents of the next result. Notice that from the regularity theory all weak solutions of (1.1) belong to $W^{2,r}(\Omega)$, and recall that $W^{2,r}(\Omega) \subset C^1(\overline{\Omega})$ because $r > N$.

Theorem 1.2. *Assume (H1), condition (*) and $1 < p < \frac{N+1}{N-1}$. Let $u \in H_0^1(\Omega)$ be a solution of equation* (1.1)*. Then there exists an increasing continuous function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$, depending only on p and Ω, such that $\rho(0) = 0$ and*

$$\|u\|_{C_0^1(\overline{\Omega})} \leq \rho(\|f\|_r),$$

where $\|f\|_r$ denotes the L^r-norm of f.

Let us remark that hypothesis (H1) is a necessary condition for both the existence of solutions *with nontrivial positive part* and the existence of a-priori bounds.

In the second part of this paper we deal with the solvability of the following system :

$$\begin{cases} -\Delta u = \lambda_1 u + (v^+)^p + f & \text{in } \Omega, \\ -\Delta v = \lambda_1 v + (u^+)^q + g & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where f, g satisfy condition (*), hypothesis (H1) and the exponents $p, q > 1$ are such that

(H2) $\quad \frac{1}{p+1} + \frac{N-1}{N+1}\frac{1}{q+1} > \frac{N-1}{N+1}$,

(H3) $\frac{1}{q+1} + \frac{N-1}{N+1}\frac{1}{p+1} > \frac{N-1}{N+1}$.

When the signs $>$ in the above inequalities are replaced by $=$ we obtain equations of hyperbolas in the (p, q) plane. These two hyperbolas where introduced by Clément-de Figueiredo-Mitidieri in [2] for the study of some class of superlinear systems that are non nonresonant at λ_1. Let us remark that hypothesies (H2) and (H3) imply that $p, q < \frac{N+3}{N-1}$, which again is less than $\frac{N+2}{N-2}$.

We prove in Sects. 4 and 5 the following result :

Theorem 1.3. *Assume that f and g satisfy condition (*) and (H1). Assume further that $p, q > 1$ satisfy (H2) and (H3). Then there exist at least one solution $(u, v) \in H_0^1(\Omega)^2$ of system* (1.2)*.*

In order to prove the above theorem, we use similar topological arguments as those used in the proof of Theorem 1.1. And in this case, the a-priori bounds in $C_0^1(\overline{\Omega})$norm for the solutions of (1.2) are established in the result below:

Theorem 1.4. *Assume that the hypotheses of Theorem 1.3 hold and let $(u, v) \in H_0^1(\Omega)^2$ be a solution of* (1.2)*. Then there exists an increasing continuous function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$, depending only on p, q and Ω, such that $\gamma(0) = 0$ and*

$$\|u\|_{C_0^1(\overline{\Omega})} + \|v\|_{C_0^1(\overline{\Omega})} \leq \gamma(\|f\|_r + \|g\|_r). \tag{1.3}$$

To our knowledge Theorem 1.3 is the first existence result in the literature for semilinear elliptic systems which are resonant in one direction and superlinear in the other. Theorem 1.3 is proved in Sect. 5.

From now on we will denote the natural norm of $H_0^1(\Omega)$ by $\|\cdot\|$, that is, $\|u\| = (\int_\Omega |\nabla u|^2\, dx)^{1/2}$. The space $C_0^1(\overline{\Omega})$ is defined as

$$C_0^1(\overline{\Omega}) = \left\{u \in C^1(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\right\},$$

which is a Banach space with the norm

$$\|u\|_{C_0^1(\overline{\Omega})} = \max_{x\in\overline{\Omega}} |u(x)| + \max_{x\in\overline{\Omega}} |\nabla u(x)|.$$

2. Proof of Theorem 1.2

Let us first prove the following lemma based on the Hardy-Sobolev inequality (c.f. for instance [1]).

Lemma 2.1. *Let $1 < p \leq \frac{N+2}{N-2}$. Then there exists a constant $C = C(p, \Omega)$ such that, for all $u, v \in H_0^1(\Omega)$ with $|u| \leq v$ a.e., it holds*

$$\int_\Omega |u|^p v\, dx \leq C(\int_\Omega |u|^p \varphi_1\, dx)^\alpha (\int_\Omega |\nabla v|^2\, dx)^{\delta/2}, \tag{2.1}$$

where

$$\alpha = 1 - \frac{N}{2 + 2N - (N-2)p} \in [0, 1),$$

$$\delta = 1 + \frac{Np}{2 + 2N - (N-2)p} \in (1, 2^*].$$

If moreover, $p < \frac{N+1}{N-1}$ then $\alpha \in (0, 1)$ and $\delta \in (1, 2)$.

Proof. By Hölder inequality,

$$\int_\Omega |u|^p v\,dx = \int_\Omega |u|^{p\alpha}\varphi_1^\alpha \frac{|u|^{p(1-\alpha)}v}{\varphi_1^\alpha}\,dx$$
$$\leq \left(\int_\Omega |u|^p\varphi_1 dx\right)^\alpha \left(\int_\Omega \frac{|v|^{p+\frac{1}{1-\alpha}}}{\varphi_1^{\frac{\alpha}{1-\alpha}}}dx\right)^{1-\alpha}$$

for any $0 < \alpha < 1$. Equation (2.1) is the result of applying Hardy-Sobolev inequality to the second integral on the left. Let us remember that Hardy-Sobolev inequality states that

$$\int_\Omega \frac{|v|^t}{\varphi_1^{\tau t}}\,dx \leq C(\int_\Omega |\nabla v|^2\,dx)^{t/2}$$

for any $0 \leq \tau \leq 1$ and $t > 1$ such that

$$\frac{1}{t} = \frac{1}{2} - \frac{1-\tau}{N}. \tag{2.2}$$

Thus it suffices to choose $0 < \alpha < 1$ such that (2.2) holds with $t = p + \frac{1}{1-\alpha}$ and $\tau = \frac{\alpha}{t(1-\alpha)}$. A simple computation gives the value α of the lemma. Notice that the power δ is exactly $t(1-\alpha)$. □

Throughout the rest of this section, we use the same letter C to denote distinct constants. In addition, we remark that all of them are independent of u and f.

Proof of Theorem 1.2. Let $u \in H_0^1(\Omega)$ be a weak solution of (1.1). Let us write $u = t\varphi_1 + u_1$ with $\int_\Omega u_1\varphi_1\,dx = 0$. By multiplying (1.1) by φ_1 we find

$$\int_\Omega (u^+)^p\varphi_1\,dx = -\int_\Omega f\varphi_1\,dx \leq C\|f\|_r \tag{2.3}$$

and therefore

$$t = \int_\Omega u^+\varphi_1\,dx - \int_\Omega u^-\varphi_1\,dx \leq C(\int_\Omega (u^+)^p\varphi_1\,dx)^{1/p} \leq C\|f\|_r^{1/p}. \tag{2.4}$$

We break the proof into two parts, according to $t < 0$ or $t \geq 0$.
Case 1. $t \geq 0$. From the previous inequality we obtain a bound on $|t|$. In order to get an estimate on u_1 we multiply (1.1) by u_1 to obtain

$$\int_\Omega |\nabla u_1|^2\,dx - \lambda_1\int_\Omega u_1^2\,dx = \int_\Omega (u^+)^p u_1\,dx + \int_\Omega f u_1\,dx$$

and using that $u_1 \in <\varphi_1>^\perp$ and the variational characterization of the second eigenvalue λ_2 of $(-\Delta, H_0^1(\Omega))$ we have

$$(1 - \frac{\lambda_1}{\lambda_2})\|u_1\|^2 \leq C\|f\|_r\|u_1\| + |\int_\Omega (u^+)^p u_1 dx|. \tag{2.5}$$

Next we use the fact that $u_1^+ \le u^+$ and that $u^+ \le t\varphi_1$ on $\{u_1 \le 0\}$ to obtain

$$|\int_\Omega (u^+)^p u_1| \le \int_\Omega (u^+)^p u_1^+ + \int_\Omega (u^+)^p u_1^- \le \int_\Omega (u^+)^{p+1} + t^p \int_\Omega \varphi_1^p u_1^-$$

and using (2.4) we get

$$|\int_\Omega (u^+)^p u_1\, dx| \le \int_\Omega (u^+)^{p+1}\, dx + C\|f\|_r \|u_1\|. \tag{2.6}$$

To estimate the integral in the right side of (2.6) we use (2.3) and Lemma 2.1 with $u = v = u^+$. Hence

$$\int_\Omega (u^+)^{p+1} \le C\|f\|_r^\alpha \|u\|^\delta \le C\|f\|_r^\alpha (\|f\|_r^{\delta/p} + \|u_1\|^\delta). \tag{2.7}$$

Replacing (2.6) and (2.7) in (2.5) we obtain

$$\|u_1\|^2 \le C\left(\|f\|_r^{\alpha+\delta/p} + \|f\|_r^\alpha \|u_1\|^\delta + \|f\|_r \|u_1\|\right)$$

Using Young inequality in the above estimate and the bound on t given in (2.4) we finally get

$$\|u\| \le C \max\{\|f\|_r^{1/p}, \|f\|_r^{\alpha/2-\delta}\}.$$

A bootstrap argument applied to (1.1) and the previous bound of the H_0^1-norm of u will give the claim of the theorem.

Case 2. $t < 0$. By Hopf's Maximum Principle the first eigenfunction of $(-\Delta, H_0^1(\Omega))$, $\varphi_1 > 0$ lies in the interior of the cone of positive functions in the space $C_0^1(\overline{\Omega})$. Then there exists $\epsilon > 0$ such that

$$v \in B_{C_0^1(\overline{\Omega})}(\varphi_1, \epsilon) \Rightarrow v > 0 \text{ in } \Omega \text{ and } \frac{\partial v}{\partial n} < 0 \text{ on } \partial\Omega,$$

where n denotes the exterior normal derivative at the boundary of Ω. Let ϵ_0 be the supremum of such ϵ's. Recall that our solution u of problem (1.1), as well as u_1, belong to $C_0^1(\overline{\Omega})$. Let us write $u = t(\varphi_1 + u_1/t)$,. Since $u^+ \not\equiv 0$ and $t < 0$ necessarily $-\frac{u_1}{t} \notin B_{C_0^1(\overline{\Omega})}(0, \epsilon_0)$ or equivalently

$$|t| \le \frac{1}{\epsilon_0} \|u_1\|_{C_0^1(\overline{\Omega})}. \tag{2.8}$$

Thus it is enough to find an a-priori bound of $\|u_1\|_{C_0^1(\overline{\Omega})}$. Similarly to the previous case we have also (2.5). But in this case $|\int_\Omega (u^+)^p u_1\, dx|$ can be estimated directly by using Lemma 2.1. Indeed, notice that we have $u^+ \le u_1$, since $t < 0$. Hence, by Lemma 2.1 and (2.3), we have

$$|\int_\Omega (u^+)^p u_1\, dx| \le C\|f\|_r^\alpha \|u_1\|^\delta$$

and replacing this estimate in (2.5) we find

$$\|u_1\|^2 \le C(\|f\|_r \|u_1\| + \|f\|_r^\alpha \|u_1\|^\delta).$$

By Young's inequality we deduce that

$$\|u_1\| \leq C(\|f\|_r + \|f\|_r^{\frac{\alpha}{2-\delta}}). \tag{2.9}$$

In order to get an estimate for $\|u_1\|_{C_0^1(\overline{\Omega})}$ from the estimate of $\|u_1\|$ we now use that u_1 solves the problem

$$\begin{cases} -\Delta u_1 = \lambda_1 u_1 + (u^+)^p + f & \text{in } \Omega \\ u_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $u^+ \leq u_1$, a bootstrap argument to this equation gives that

$$\|u_1\|_{C_0^1(\overline{\Omega})} \leq C(\|u_1\|^p + \|f\|_r)$$

and using (2.9) and (2.8) we finish the proof of the theorem. □

3. Proof of Theorem 1.1.

In this proof we use the proposition below. For that matter, let us introduce the following fixed point formulation of problem (1.1). Let $T_f : C_0^1(\overline{\Omega}) \to C_0^1(\overline{\Omega})$ be the map

$$T_f(u) = (-\Delta)^{-1}(\lambda_1 u + (u^+)^p + f).$$

T_f is a compact continuous map and $T_f(u) = u \Leftrightarrow u$ solves (1.1). In what follows $d(\cdot,\cdot,\cdot)$ denotes the Leray-Schauder degree.

Proposition 3.1. *There exist $\epsilon > 0$ and $R_0 > 0$ such that for all functions f satisfying condition (*) and hypothesis (H1), with $\|f\|_r < \epsilon$, and for which problem* (1.1) *possesses at least one solution, it follows that*

$$d(Id - T_f, B_{C_0^1(\overline{\Omega})}(0,R), 0) \neq 0,$$

for all $R \geq R_0$.

Proof. First we prove that there exists $\epsilon > 0$ such that for all $f \in L^r(\Omega)$ with $\|f\|_r \leq \epsilon$, any solution of problem (1.1) with f is non-degenerate with Morse index equal to 1. Let ρ be the function given by Theorem 1.2 and let $\epsilon < 1$ be such that $\rho(\epsilon) < (\frac{\lambda_2 - \lambda_1}{p})^{1/p-1}$. Let $f \in L^r(\Omega)$ be a function satisfying (H1), and be such that $\|f\|_r \leq \epsilon$. By Theorem 1.2 any solution u_0 of (1.1) satisfies

$$\|u\|_{C_0^1(\overline{\Omega})} < (\frac{\lambda_2 - \lambda_1}{p})^{\frac{1}{p-1}} =: R_0. \tag{3.1}$$

The linearized problem of (1.1) at u_0 is the following

$$\begin{cases} -\Delta v = [\lambda_1 + p(u_0^+)^{p-1}]v & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Denote by $\mu_1(a) < \mu_2(a) < \dots$ denote the eigenvalues of the following eigenvalue problem of weight a : $-\Delta v = \mu a(x) v$ in Ω; $v = 0$ on $\partial\Omega$. By (3.1) $\lambda_1 < a(x) := \lambda_1 + p(u_0^+(x))^{p-1} < \lambda_2$ a.e., so then $\mu_1(a) < 1 < \mu_2(a)$. Hence $v \equiv 0$ is the

unique solution of (3.2) and therefore u_0 is a non-degenerate solution of (1.1) of Morse index equal to 1.

By estimate (3.1) the degree above is well defined for all $R \geq R_0$. Moreover, since all possible solutions of $u = T_f(u)$ are non-degenerated, it follows that they are isolated and that there is only a finite number of them in $B_{C_0^1(\overline{\Omega})}(0,R)$. We recall that the index of each solution is equal to $(-1)^\beta$ where β is the Morse index. So $d(Id - T_f, B_{C_0^1(\overline{\Omega})}(0,R),0) = \sum'(-1) \neq 0$.

Remark 3.1. For the proof of Theorem 1.1 it will be enough to prove the existence of at least one f_1 for which we can prove that $d(Id - T_{f_1}, B_{C_0^1(\overline{\Omega})}(0,R),0) \neq 0$ for all $R > 0$ large enough.

Proof of Theorem 1.1. Choose $f_1 := -(t\varphi_1)^p$ with $0 < t < (\frac{\epsilon}{\|\varphi_1^p\|_r})^{1/p}$, where ϵ is given by Proposition 3.1. Notice that $u = t\varphi_1$ is a solution of problem (1.1) for $f_1 = -(t\varphi_1)^p$ and then Proposition 3.1 applies. Thus

$$d(Id - T_{f_1}, B_{C_0^1(\overline{\Omega})}(0,R),0) \neq 0$$

for R large enough. Consider the following homotopy

$$\begin{cases} -\Delta u = \lambda_1 u + (u^+)^p + (1-\tau)f + \tau f_1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.3}$$

where $0 \leq \tau \leq 1$. From the a-priori estimates of Theorem 1.2, all the solutions of problem (3.3) are uniformly bounded in $C_0^1(\overline{\Omega})$ by, say, $R_1 := \rho(\max\{\|f\|_r, \|f_1\|_r\})$. Hence, if $R > \max\{R_0, R_1\}$,

$$d(Id - T_f, B_{C_0^1(\overline{\Omega})}(0,R),0) = d(Id - T_{f_1}, B_{C_0^1(\overline{\Omega})}(0,R),0) \neq 0$$

and the conclusion of the theorem follows. □

4. Proof of Theorem 1.4

We follow here the approach of [2] which consists in getting estimates of the $W^{2,\frac{p+1}{p}}(\Omega)$-norm of u and the $W^{2,\frac{q+1}{q}}(\Omega)$-norm of v directly from the equations of (1.2). A bootstrap argument will allow us to obtain the a-priori bounds in $C_0^1(\overline{\Omega})$ in terms of $\|f\|_r$ and $\|g\|_r$.

The following version of the Hardy-Sobolev inequality will be used in the proof of Theorem 1.4, cf [2]. We denote the norm of $W^{2,m}(\Omega)$ by $\|\cdot\|_{2,m}$.

Lemma 4.1. *Let $u \in H_0^1(\Omega) \cap W^{2,m}(\Omega)$, $1 < m < \infty$. Then the following inequality holds*

$$\|\frac{u}{\varphi_1^\tau}\|_t \leq C\|u\|_{2,m}$$

where the constant C depends only on τ, m and N; $\tau \in [0,1]$ and $t > 1$ are such that

(i) $\frac{1}{t} = \frac{1}{m} - \frac{2-\tau}{N}$ *if* $m < \frac{N}{2}$,

(ii) $\frac{1}{t} > \frac{\tau}{N}$ *if* $m = \frac{N}{2}$,

(iii) $\frac{1}{t} = \tau(\frac{1}{m} - \frac{1}{N})$ *if* $\frac{N}{2} < m < N$,

(iv) $\forall t > 1,\ \forall \tau \in [0,1]$ *if* $m \geq N$.

Proof of Theorem 1.4. Let $(u, v) \in H_0^1(\Omega)^2$ be a solution of (1.2). First we prove that there exists an increasing continuous function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$, $\gamma(0) = 0$, depending only on p, q and Ω, such that

$$\|u\|_{2,\frac{p+1}{p}} + \|v\|_{2,\frac{q+1}{q}} \leq \gamma(\|f\|_r + \|g\|_r). \tag{4.1}$$

By multiplying both equations of the system by φ_1 we find

$$\begin{aligned} \int_\Omega (v^+)^p \varphi_1 dx &= -\int_\Omega f\varphi_1 dx \leq C\|f\|_r, \\ \int_\Omega (u^+)^q \varphi_1 dx &= -\int_\Omega g\varphi_1 dx \leq C\|g\|_r \end{aligned} \tag{4.2}$$

and then

$$\int_\Omega v^+ \varphi_1 \, dx \leq C(\|f\|_r)^{1/p}, \quad \int_\Omega u^+ \varphi_1 \, dx \leq C(\|g\|_r)^{1/q}.$$

Let us write $u = s\varphi_1 + u_1, v = t\varphi_1 + v_1$ where $\int_\Omega u_1\varphi_1 \, dx = \int_\Omega v_1\varphi_1 \, dx = 0$. It follows from the previous inequality that

$$\begin{cases} s = \int_\Omega u^+\varphi_1 \, dx - \int_\Omega u^-\varphi_1 \, dx \leq C\|f\|_r^{1/p}, \\ t = \int_\Omega v^+\varphi_1 \, dx - \int_\Omega v^-\varphi_1 \, dx \leq C\|g\|_r^{1/q}. \end{cases} \tag{4.3}$$

We break the proof in three parts, according to the signs of s and t.
Case 1. $s \geq 0$ and $t \geq 0$. From (4.3) we obtain a bound on s and a bound on t. Since u_1 satisfies the following equation

$$\begin{cases} -\Delta u_1 = \lambda_1 u_1 + (v^+)^p + f & \text{in } \Omega \\ u_1 = 0 & \text{on } \partial\Omega, \end{cases}$$

by taking the $L^{\frac{p+1}{p}}$-norm on both sides of the previous equation we have

$$\int_\Omega |\Delta u_1 + \lambda_1 u_1|^{\frac{p+1}{p}} \, dx \leq C\int_\Omega (v^+)^{p+1} \, dx + D\|f\|_r^{\frac{p+1}{p}}. \tag{4.4}$$

Let us write

$$\int_\Omega (v^+)^{p+1} = \int_\Omega (v^+)^{p\alpha} \varphi_1^\alpha \varphi_1^{-\alpha} (v^+)^{p(1-\alpha)+1} \, dx$$

for some $0 < \alpha < 1$ to be determined later. By Hölder inequality

$$\int_\Omega (v^+)^{p+1} \leq (\int_\Omega (v^+)^p \varphi_1 \, dx)^\alpha (\int_\Omega \frac{(v^+)^{p+\frac{1}{1-\alpha}}}{\varphi_1^{\frac{\alpha}{1-\alpha}}} \, dx)^{1-\alpha}. \tag{4.5}$$

Then we use (4.2) to the first integral on the right and Lemma 4.1 to the second one to find

$$\int_\Omega (v^+)^{p+1} \le C\|f\|_r^\alpha \|v\|_{2,\frac{q+1}{q}}^{p(1-\alpha)+1}. \tag{4.6}$$

In order to apply Lemma 4.1 to estimate $\int_\Omega \frac{(v^+)^{p+\frac{1}{1-\alpha}}}{\varphi_1^{\frac{\alpha}{1-\alpha}}}\,dx$ by $\|v\|_{2,\frac{q+1}{q}}$ we have chosen in all three cases below $m = \frac{q+1}{q}$, $t = p + \frac{1}{1-\alpha}$, $\tau t = \frac{\alpha}{1-\alpha}$ and a further condition on t according to :
(i) $(N \ge 4, q > 1)$ or $(N = 3$ and $q > 2)$. In this case $\frac{q+1}{q} < \frac{N}{2}$ so $\frac{1}{t} = \frac{q}{q+1} - \frac{2-\tau}{N}$. With all these conditions we get $\alpha = \frac{N(1-L-Lp)}{1+N-LNp}$ where $L := \frac{q}{q+1} - \frac{2}{N}$. Clearly $\alpha < 1$ and (H2) implies that $\alpha > 0$.

(ii) $N = 3$ and $q = 2$. Then $\frac{q+1}{q} = \frac{N}{2} = \frac{3}{2}$. Take $\frac{1}{t} > \frac{\tau}{3}$. With these three conditions one can choose any $0 < \alpha < \frac{3}{4}$.

(iii) $N = 3$ and $1 < q < 2$. In this case $N = 3 > \frac{q+1}{q} > \frac{3}{2} = \frac{N}{2}$. Then we choose $\frac{1}{t} = \tau(\frac{q}{q+1} - \frac{1}{3})$, which gives $\alpha = \frac{3q+3}{5q+2}$. Observe that $\frac{3}{4} < \alpha < \frac{6}{7}$.

Putting (4.4) and (4.6) together we get

$$\int_\Omega |\Delta u_1 + \lambda_1 u_1|^{\frac{p+1}{p}}\,dx \le C(\|f\|_r^\alpha \|v\|_{2,\frac{q+1}{q}}^{p(1-\alpha)+1} + \|f\|_r^{\frac{p+1}{p}}).$$

Now we use the fact that $-\Delta - \lambda_1 Id : W_*^{2,\frac{p+1}{p}}(\Omega) \to L_*^{\frac{p+1}{p}}(\Omega)$ is an isomorphism where $W_*^{2,\frac{p+1}{p}}(\Omega) := \{u \in W^{2,\frac{p+1}{p}}(\Omega) : \int_\Omega u\varphi_1\,dx = 0\}$ and $L_*^{\frac{p+1}{p}} := \{u \in L^{\frac{p+1}{p}}(\Omega) : \int_\Omega u\varphi_1\,dx = 0\}$. Thus we have

$$\|u_1\|_{2,\frac{p+1}{p}} \le C\|\Delta u_1 + \lambda_1 u_1\|_{\frac{p+1}{p}}$$

and from the previous inequality we obtain

$$\|u_1\|_{2,\frac{p+1}{p}}^{\frac{p+1}{p}} \le C(\|f\|_r^\alpha \|v\|_{2,\frac{q+1}{q}}^{p(1-\alpha)+1} + \|f\|_r^{\frac{p+1}{p}}). \tag{4.7}$$

Proceeding in a similar way with the second equation of the system (1.2) we obtain

$$\|v_1\|_{2,\frac{q+1}{q}}^{\frac{q+1}{q}} \le C(\|g\|_r^{\alpha'} \|u\|_{2,\frac{p+1}{p}}^{q(1-\alpha')+1} + \|g\|_r^{\frac{q+1}{q}}) \tag{4.8}$$

where α' is chosen with the same criteria as α by considering the corresponding cases (i), (ii) and (iii) with the role of p and q interchanged and the value $K := \frac{p}{p+1} - \frac{2}{N}$ instead of L.

From (4.7) and (4.8) and the estimates of of t and s in (4.3) we deduce

$$\begin{aligned}
\|u_1\|_{2,\frac{p+1}{p}}^{\frac{p+1}{p}} &\le C[\|f\|_r^\alpha(\|g\|_r^{\frac{p(1-\alpha)+1}{q}} + \|v_1\|_{2,\frac{q+1}{q}}^{p(1-\alpha)+1}) + \|f\|_r^{\frac{p+1}{p}}] \\
\|v_1\|_{2,\frac{q+1}{q}}^{\frac{q+1}{q}} &\le C[\|g\|_r^{\alpha'}(\|f\|_r^{\frac{q(1-\alpha')+1}{p}} + \|u_1\|_{2,\frac{p+1}{p}}^{q(1-\alpha')+1}) + \|g\|^{\frac{q+1}{q}}].
\end{aligned} \tag{4.9}$$

The conclusion (4.1) will follow from (4.9) if we prove that

$$(p(1-\alpha)+1)\frac{q}{q+1}<1 \text{ and } (q(1-\alpha')+1)\frac{p}{p+1}<1. \tag{4.10}$$

Let us check carefully the first inequality in (4.10) by distinguish the following cases. The proof of the second inequality is completely analogous.

(i) $(N\geq 4, q>1)$ or $(N=3, q>2)$. In this case (4.10) follows directly from (H3).

(ii) $N=3, q=2$. In this case (H2) and (H3) imply that $1<p<2$. We have

$$p(1-\alpha)+1<\frac{q+1}{q}=\frac{3}{2}\Leftrightarrow \alpha>1-\frac{1}{2p}.$$

Since $\alpha<\frac{3}{4}$, it is enough to have $\frac{3}{4}>1-\frac{1}{2p}$,which trivially holds for $p<2$.

(iii) $N=3$, $1<q<2$. Then

$$p(1-\alpha)+1<\frac{q+1}{q}\Leftrightarrow 1-\frac{1}{pq}<\frac{3q+3}{5q+2}.$$

Using that hypothesis $(H2)$ for $N=3$ is equivalent to $pq<q+2$, it will enough to have $1-\frac{1}{q+2}<\frac{3q+3}{5q+2}$, i.e., $q(q-1)<2$, which clearly holds for $1<q<2$.

Case 2. $t<0$ or $s<0$. Assume for instance that $t<0$. As in the proof of Theorem 1.2, let $\epsilon_0>0$ be such that $|t|\leq \frac{1}{\epsilon_0}\|v_1\|_{C_0^1(\overline{\Omega})}$. We proceed as in case 1 except for the following minor changes. In (4.5) we use that $v^+=t\varphi_1+v_1\leq v_1$ to get instead

$$\int_\Omega (v^+)^{p+1}\leq C(\int_\Omega (v^+)^p\varphi_1\,dx)^\alpha(\int_\Omega \frac{v_1^{p+\frac{1}{1-\alpha}}}{\varphi_1^{\frac{\alpha}{1-\alpha}}}\,dx)^{1-\alpha}.$$

Consequently we get

$$\int_\Omega (v^+)^{p+1}\leq C\|f\|_r^\alpha\|v_1\|^{p(1-\delta)+1}_{W^{2,\frac{q+1}{q}}}$$

instead of (4.6). With the same choice of α we will obtain finally

$$\|u_1\|^{\frac{p+1}{p}}_{W^{2,\frac{p+1}{p}}}\leq C(\|v_1\|^{p(1-\alpha)+1}_{W^{2,\frac{q+1}{q}}}\|f\|_r^\alpha+\|f\|_r^{\frac{p+1}{p}}).$$

We continue the proof as we have done after (4.9).

So the estimate (4.1) is proved. In order to complete the proof of Theorem 1.4 it is enough to find an a-priori bound of $\|u\|_{C_0^1(\overline{\Omega})}+\|v\|_{C_0^1(\overline{\Omega})}$. This can be done using a bootstrap argument to (1.2). We start the bootstrap process with $\|u\|_{1/K}, \|v\|_{1/L}\leq C$ where the constant C depends increasingly on $\|f\|_r, \|g\|_r$. Then we use the fact that

$$\frac{1}{L},\ \frac{1}{K}\geq\frac{2N}{N-2}$$

and the subcritical growth of the nonlinear terms. □

5. Proof of Theorem 1.3

We proceed as in the proof of Theorem 1.1. We start by proving that the solutions (u, v) of (1.2) are non-degenerated of index 1, provided f and g are"small" and satisfy (H1). The linearized problem of (1.2) at some solution (u_0, v_0) is the following

$$\begin{cases} -\Delta w = \lambda_1 w + p(v_0^+)^{p-1} z & \text{in } \Omega \\ -\Delta z = \lambda_1 z + q(u_0^+)^{q-1} w & \text{in } \Omega \\ w = z = 0 & \text{in } \partial\Omega \end{cases} \tag{5.1}$$

The non-degeneracy and the computation of the index will be a direct consequence of the following two lemmae.

Lemma 5.1. *There exists $\epsilon > 0$ such that for any $a, b \in L^\infty(\Omega)$, $c \in \mathbb{R}$ and $t \in [0,1]$ such that $a, b \geq 0$ a.e., $a \not\equiv 0, b \not\equiv 0$, $\|a\|_\infty < \epsilon, \|b\|_\infty < \epsilon$ and $0 < c < \epsilon$, the system*

$$\begin{cases} -\Delta w = \lambda_1 w + ta(x)z + (1-t)cz & \text{in } \Omega \\ -\Delta z = \lambda_1 z + tb(x)w + (1-t)cw & \text{in } \Omega \\ w = z = 0 & \text{on } \partial\Omega \end{cases} \tag{5.2}$$

possesses only the trivial solution $w = z = 0$.

Lemma 5.2. *Let $0 < c < \lambda_2 - \lambda_1$ be fixed and consider the following eigenvalue problem of parameter μ*

$$\begin{cases} -\Delta w = \mu(\lambda_1 w + cz) & \text{in } \Omega, \\ -\Delta z = \mu(\lambda_1 z + cw) & \text{in } \Omega, \\ w = z = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.3}$$

Then there is only one eigenvalue μ on the interval $[0,1]$. This eigenvalue is precisely $\mu_1 := \frac{\lambda_1}{\lambda_1 + c}$.

Proof of Lemma 5.1. Assume, by contradiction, that there exist sequences of functions $0 \not\equiv \leq a_n(x), b_n(x) \leq \frac{1}{n}$, and sequences of numbers $t_n \in [0,1]$, $c_n \in]0, \frac{1}{n}]$ such that (5.2) has nontrivial solution (w_n, z_n) for those corresponding coefficients. Consider the sequences

$$\tilde{w}_n = \frac{w_n}{||w_n||_2}, \tilde{z}_n = \frac{z_n}{\|z_n\|_2}$$

and the following two possibilities :

1) the sequence $\frac{\|z_n\|_2}{\|w_n\|_2}$ is bounded. In this case we have

$$\begin{cases} -\Delta \tilde{w}_n = \lambda_1 \tilde{w}_n + (t_n a_n(x) + (1-t_n)c_n)\frac{z_n}{||w_n||_2} & \text{in } \Omega, \\ \tilde{w}_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.4}$$

By testing the previous equation against $\tilde{w}_n$ we easily deduce that $\tilde{w}_n$ is bounded in $H_0^1(\Omega)$. Thus there exists $\tilde{w} \in H_0^1(\Omega)$ such that $\tilde{w}_n$ converges weakly in $H_0^1(\Omega)$

to $\tilde{w}$ and strongly in $L^2(\Omega)$, so then $\|\tilde{w}\|_2 = 1$. Passing to the limit in (5.4) we find that $\tilde{w} \not\equiv 0$ satisfies

$$\begin{cases} -\Delta\tilde{w} = \lambda_1\tilde{w} & \text{in } \Omega, \\ \tilde{w} = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence $\tilde{w} = \pm\varphi_1$ and $\tilde{w}_n \to \pm\varphi_1$ strongly in $H_0^1(\Omega)$ (because, from (5.4), $\|\tilde{w_n}\| \to \lambda_1 = \|\tilde{w}\|$). It also holds that $\tilde{w}_n \to \tilde{w}$ in $C_0^1(\overline{\Omega})$. This comes as a result of the regularity theory for elliptic equations applied to the equation

$$\begin{cases} -\Delta(\tilde{w}_n \pm \varphi_1) = \lambda_1(\tilde{w}_n \pm \varphi_1) + (t_n a_n(x) + (1-t_n)c_n)\frac{z_n}{\|w_n\|_2} & \text{in } \Omega, \\ \tilde{w}_n \pm \varphi_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

Hence $\tilde{w}_n$ (and then w_n) has definite sign for n large. On the other hand, by testing the second equation of (5.2) by φ_1 we deduce that $\int(t_n b_n(x) + (1-t_n)c_n)w_n\varphi_1 = 0$. This is a contradiction with the fact that w_n has a definite sign, that $0 < c_n$ and that $0 \le b_n \not\equiv 0$.

2) $\frac{\|z_n\|_2}{\|w_n\|_2} \to \infty$. In this case the sequence $\frac{\|w_n\|_2}{\|z_n\|_2}$ is bounded and we are in the same situation as in case 1), working now with the second equation of (5.2). □

Proof of Lemma 5.2. The procedure to find the eigenvalues of (5.3) is standard but we recall it here for sake of completeness. Let $\varphi_n, n \ge 1$ be a base of eigenfunctions of the Laplacian in $H_0^1(\Omega)$, $-\Delta\varphi_n = \lambda_n\varphi_n$, and write $w = \sum t_n\varphi_n, z = \sum s_n\varphi_n$. It follows then from (5.3) that a necessary and sufficient condition for a real μ to be an eigenvalue is that $\mu = \frac{\lambda_n}{c+\lambda_1}$ or $\mu = \frac{\lambda_n}{\lambda_1 - c}$ for some $n \ge 1$. Then the corresponding solution (up to a constant multiple) is $(w, z) = (\varphi_n, \varphi_n)$ (resp. $(w, z) = (\varphi_n, -\varphi_n)$). The conclusion of the lemma follows from the observation that if $0 < c < \lambda_2 - \lambda_1$ then

$$\frac{\lambda_1}{c+\lambda_1} < 1 < \frac{\lambda_2}{c+\lambda_1}.$$

□

Proof of Theorem 1.4. Let us introduce the following fixed point formulation of problem (1.2). Let $T_{(f,g)} : C_0^1(\overline{\Omega})^2 \to C_0^1(\overline{\Omega})^2$ be the map

$$T_{(f,g)}(u, v) = ((-\Delta)^{-1}(\lambda_1 u + (v^+)^p + f), (-\Delta)^{-1}(\lambda_1 v + (u^+)^q + g)).$$

$T_{(f,g)}$ is a compact continuous map and $T_{(f,g)}(u, v) = (u, v) \Leftrightarrow (u, v)$ solves (1.2).

Choose f_1, g_1 as $f_1 = -(\alpha\varphi_1)^p$ and $g_1 = -(\beta\varphi_1)^q$ with $\beta > 0, \alpha > 0$. Then $(u_0, v_0) = (\beta\varphi_1, \alpha\varphi_1)$ is a solution of problem (1.2) for (f_1, g_1). Consider further the following homotopy

$$\begin{cases} -\Delta u = \lambda_1 u + (v^+)^p + (1-\tau)f + \tau f_1 & \text{in } \Omega, \\ -\Delta v = \lambda_1 v + (u^+)^q + (1-\tau)g + \tau g_1 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega; \end{cases}$$

where $0 \leq \tau \leq 1$. From the a-priori estimates of Theorem 1.4, all the solutions of the previous homotopy are uniformly bounded in $C_0^1(\overline{\Omega})$ by some constant depending continuously on α and β. Hence for any $R > 0$ large enough,

$$d\left(Id - T_{(f,g)}, B_{C_0^1(\overline{\Omega})^2}(0,R), 0\right) = d\left(Id - T_{(f_1,g_1)}, B_{C_0^1(\overline{\Omega})^2}(0,R), 0\right).$$

Now we take α, β small enough such that Lemma 5.1 applies for $a = p((v)^+)^{p-1}, b = q((u)^+)^{q-1}$ for any (u,v) arbitrary solution of (1.2) with (f_1, g_1). We obtain, by considering $t = 1$ in (5.2), that (u,v) is non-degenerate. Moreover the index $Id - T_{(f_1,g_1)}$ at (u,v) can be computed through the homotopy given in (5.2), and it coincides with the index of the trivial solution of (5.2) for $t = 0$. Using Lemma 5.2 we deduce that this index is -1 and hence $d\left(Id - T_{(f_1,g_1)}, B_{C_0^1(\overline{\Omega})^2}(0,R), 0\right) = \sum'(-1) \neq 0$. □

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