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NONLINEAR EIGENVALUE PROBLEMS FOR DEGENERATE ELLIPTIC SYSTEMS

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Abstract. The following nonlinear eigenvalue problem for a pair of real parameters (λ, μ) is studied:

$$\begin{cases} -\Delta_p u = \lambda\, a(x)\, |u|^{\alpha_1} |v|^{\beta_1 - 1} v & \text{in } \Omega; \\ -\Delta_q v = \mu\, b(x)\, |v|^{\alpha_2} |u|^{\beta_2 - 1} u & \text{in } \Omega; \\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

Here, $p, q \in (1, \infty)$ are given numbers, Ω is a bounded domain in $\mathbb{R}^N$ with a C^2-boundary, $a, b \in L^\infty(\Omega)$ are given functions, both assumed to be strictly positive on compact subsets of Ω, and the coefficients α_i, β_i are nonnegative numbers satisfying either the conditions $\alpha_1 + \beta_1 = p - 1$ and $\alpha_2 + \beta_2 = q - 1$, or the condition

$$(p - 1 - \alpha_1)(q - 1 - \alpha_2) = \beta_1 \beta_2.$$

A *smooth curve* of pairs (λ, μ) in $(0, \infty) \times (0, \infty)$ is found for which the quasilinear elliptic system possesses a solution pair (u, v) consisting of nontrivial, nonnegative functions $u \in W_0^{1,p}(\Omega)$ and $v \in W_0^{1,q}(\Omega)$. Key roles in the proof are played by the strong comparison principle and a nonlinear Kreĭn-Rutman theorem obtained by the authors in earlier works. The main result is applied to some quasilinear elliptic systems related to the above system.

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1. Introduction

We consider the following system of quasilinear elliptic boundary-value problems:

$$\begin{cases} -\Delta_p u = \lambda\, a(x)\, |u|^{\alpha_1} |v|^{\beta_1 - 1} v \ \text{ in } \ \Omega; \\ -\Delta_q v = \mu\, b(x)\, |v|^{\alpha_2} |u|^{\beta_2 - 1} u \ \text{ in } \ \Omega; \\ \qquad u = v = 0 \qquad\qquad\qquad \text{on } \ \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ whose boundary $\partial\Omega$ is a C^2-manifold $\partial\Omega$ (which is not assumed to be connected), $x = (x_1, \dots, x_N)$ is a generic point in Ω, $p, q \in (1, \infty)$ are given numbers, $a, b \in L^\infty(\Omega)$ are given functions satisfying

$$a_0 \stackrel{\text{def}}{=} \operatorname*{ess\,inf}_{x\in\Omega} a(x) > 0 \quad \text{ and } \quad b_0 \stackrel{\text{def}}{=} \operatorname*{ess\,inf}_{x\in\Omega} b(x) > 0,$$

and α_i, β_i are constants with $\alpha_i \geq 0$ and $\beta_i > 0$ for $i = 1, 2$. The quasilinear elliptic operator $u \mapsto \Delta_p u \stackrel{\text{def}}{=} \operatorname{div}(|\nabla u|^{p-2} \nabla u)$, called the p-Laplacian, is defined for $u \in W_0^{1,p}(\Omega)$ with values $\Delta_p u \in W^{-1,p'}(\Omega)$, the dual space of $W_0^{1,p}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. We view System (1.1) as a *homogeneous nonlinear eigenvalue problem* for the unknown pair of parameters $(\lambda, \mu) \in \mathbb{R}_+^* \times \mathbb{R}_+^* = (0, \infty)^2$ associated with the unknown pair of nonnegative eigenfunctions $u \in W_0^{1,p}(\Omega)$ and $v \in W_0^{1,q}(\Omega)$. We will refer to such a couple (λ, μ) as a "principal eigenvalue" of system (1.1).

Notice that System (1.1) is neither variational nor of Hamiltonian type, in general, except for the cases when either $\frac{\partial f}{\partial v} \equiv \frac{\partial g}{\partial u}$ or $\frac{\partial f}{\partial u} \equiv \frac{\partial g}{\partial v}$ where we have denoted by $f(x, u, v)$ and $g(x, u, v)$, respectively, the right-hand side of the first and second equations in (1.1). Computing the partial derivatives

$$\begin{cases} \frac{\partial f}{\partial v} = \lambda\beta_1\, a(x)\, |u|^{\alpha_1} |v|^{\beta_1 - 1}, & \frac{\partial g}{\partial u} = \mu\beta_2\, b(x)\, |v|^{\alpha_2} |u|^{\beta_2 - 1}, \\ \frac{\partial f}{\partial u} = \lambda\alpha_1\, a(x)\, |u|^{\alpha_1 - 2} u |v|^{\beta_1 - 1} v, & \frac{\partial g}{\partial v} = \mu\alpha_2\, a(x)\, |v|^{\alpha_1 - 2} v |u|^{\beta_1 - 1} u, \end{cases}$$

we observe that the former case occurs if and only if

$$\alpha_1 = \beta_2 - 1,\ \alpha_2 = \beta_1 - 1, \ \text{ and } \ \lambda\beta_1 a(x) = \mu\beta_2 b(x) \ \text{ for a.e. } x \in \Omega,$$

whereas the latter case occurs if and only if either

$$\alpha_1 - 1 = \beta_2,\ \alpha_2 - 1 = \beta_1, \ \text{ and } \ \lambda\alpha_1 a(x) = \mu\alpha_2 b(x) \ \text{ for } x \in \Omega,$$

or $\alpha_1 = \alpha_2 = 0$. The special "superlinear" case when $\alpha_1 = \alpha_2 = 0$ and $\beta_1\beta_2 > (p - 1)(q - 1)$ was treated in Ph. Clément, R. F. Manásevich, and E. Mitidieri [2]. Systems with variational and Hamiltonian structures have been studied in D. G. de Figueiredo [6], for instance. Our method does not require any variational or Hamiltonian structure for System (1.1).

We wish to apply a simplified version of a Kreĭn-Rutman theorem for homogeneous nonlinear mappings due to P. Takáč [20, Theorem 3.5, page 1763]. Given any $f \in L^\infty(\Omega)$, we denote by $T_p(f) \equiv u \in W_0^{1,p}(\Omega)$ the unique weak solution of the boundary-value problem

$$-\Delta_p u = f(x) \ \text{ in } \Omega; \qquad u = 0 \ \text{ on } \partial\Omega. \tag{1.2}$$

It is well known [7, 15, 24] that $u \in C^{1,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$. We denote $X = [C_0^1(\overline{\Omega})]^2$, $X_+ = \{(f,g) \in X : f \geq 0 \text{ and } g \geq 0 \text{ in } \Omega\}$, and $\overset{\circ}{X}_+$ is the topological interior of X_+ in X. Finally define the map $S : X \to X$ by $S(u,v) \overset{\text{def}}{=} (\tilde{u}, \tilde{v})$ with

$$\tilde{u} = T_p(a|u|^{\alpha_1}|v|^{\beta_1 - 1}v) \quad \text{ and } \quad \tilde{v} = T_q(b|v|^{\alpha_2}|u|^{\beta_2 - 1}u)$$

for $(u,v) \in X$.

In Section 2, we treat the case when S is homogeneous whereas in Section 3, we deal with a slightly more general case. We prove in both cases the existence of a curve $\mathcal{C}_1$ of principal eigenvalues (λ, μ) for system (1.1). In Section 3 we turn to the nonhomogeneous problem

$$\begin{cases} -\Delta_p u = \lambda a(x)|u|^{\alpha_1}|v|^{\beta_1 - 1}v + f(x) & \text{in } \Omega; \\ -\Delta_q v = \mu b(x)|v|^{\alpha_2}|u|^{\beta_2 - 1}u + g(x) & \text{in } \Omega; \\ u = v = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

We discuss the solvability of (1.3) when either **(a)** $\lambda, \mu > 0$, (λ, μ) lies below or to the left of $\mathcal{C}_1$, and $f, g \geq 0$, or **(b)** (λ, μ) lies on $\mathcal{C}_1$ and $f, g \geq 0$, or **(c)** (λ, μ) lies above or to the right of $\mathcal{C}_1$, and $f, g \leq 0$ ("antimaximum principle"). We also briefly discuss the uniqueness of solutions in these cases.

Finally in the appendix we present first a new theorem on the strong comparison principle for the p-Laplacian and a simplified version of the Kreĭn-Rutman theorem for homogeneous nonlinear mappings of P. Takáč [20, Theorem 3.5, page 1763].

2. A curve of principal eigenvalues

2.1. **The case when S is homogeneous.** We consider $C_0^1(\overline{\Omega}) \overset{\text{def}}{=} \{f \in C^1(\overline{\Omega}) : f = 0 \text{ on } \partial\Omega\}$ and its positive cone $(C_0^1(\overline{\Omega}))_+ = \{f \in C_0^1(\overline{\Omega}) : f \geq 0 \text{ in } \Omega\}$ which is normal and has nonempty topological interior $(C_0^1(\overline{\Omega}))_+^\circ$

characterized by $v \in (C_0^1(\overline{\Omega}))_+^\circ$ if and only if $v \in C_0^1(\overline{\Omega})$ satisfies the strong maximum principle ([23, 25]):

$$v > 0 \text{ in } \Omega \quad \text{and} \quad \frac{\partial v}{\partial \nu} < 0 \text{ on } \partial\Omega. \tag{2.1}$$

We consider also the Cartesian product $X = [C_0^1(\overline{\Omega})]^2$, which is a strongly ordered Banach space endowed with the natural norm and ordering for pairs of functions $(f, g) \in X$. Its positive cone $X_+ = \{(f, g) \in X : f \geq 0 \text{ and } g \geq 0 \text{ in } \Omega\}$ is normal and has nonempty topological interior $\overset{\circ}{X}_+$, where $\overset{\circ}{X}_+ = [(C_0^1(\overline{\Omega}))_+^\circ]^2$.

We define the map $S : X \to X$ by $S(u, v) \overset{\text{def}}{=} (\tilde{u}, \tilde{v})$ with

$$\tilde{u} = T_p(a|u|^{\alpha_1}|v|^{\beta_1 - 1}v) \quad \text{and} \quad \tilde{v} = T_q(b|v|^{\alpha_2}|u|^{\beta_2 - 1}u)$$

for $(u, v) \in X$, where T_p and T_q have been defined in (1.2).

The following lemma describes the interaction between positive eigenvalues and the different homogeneities of S_1 and S_2, the two components of S.

Lemma 2.1. **(i)** *A couple $(u, v) \in X_+ \setminus \{0\}$ is a weak solution of (1.1) for some $(\lambda, \mu) \in (\mathbb{R}_+^*)^2$ if and only if $(u, v) \in \overset{\circ}{X}_+$ and $S(u, v) = (\lambda^{-1/(p-1)}u, \mu^{-1/(q-1)}v)$.*
(ii) *For all $\rho, \sigma \in \mathbb{R}+$ and for all $(u, v) \in X_+$ we have*

$$S(\rho u, \sigma v) = \left((\rho^{\alpha_1}\sigma^{\beta_1})^{1/(p-1)} S_1(u, v), (\rho^{\beta_2}\sigma^{\alpha_2})^{1/(q-1)} S_2(u, v) \right).$$

(iii) *If $(u, v) \in X_+$ solves (1.1) with (λ', μ') in place of (λ, μ), then for any $\rho, \sigma > 0$, the pair $(\rho u, \sigma v)$ solves (1.1) with (λ, μ) satisfying*

$$\lambda = \rho^{p-1-\alpha_1}\sigma^{-\beta_1}\lambda', \qquad \mu = \sigma^{q-1-\alpha_2}\rho^{-\beta_2}\mu'. \tag{2.2}$$

The proof is left to the reader.

Let us now look for principal eigenvalues of the map S via the Kreĭn-Rutman theorem, cf. Theorem A.2. We have the following.

Theorem 2.2. *Assume*

$$\alpha_1 + \beta_1 = p - 1 \quad \text{and} \quad \alpha_2 + \beta_2 = q - 1. \tag{2.3}$$

Then there exists $\Lambda > 0$ and a couple $(u_1, v_1) \in \overset{\circ}{X}_+$ such that system (1.1) possesses a positive weak solution $(u, v) \in X_+$ associated to some $(\lambda, \mu) \in (\mathbb{R}_+^*)^2$ if and only if

$$\lambda^{\frac{1}{\beta_1}}\mu^{\frac{1}{\beta_2}} = \Lambda. \tag{2.4}$$

Moreover, $(u, v) = c(u_1, v_1)$ for some constant $c > 0$.

Proof. It is easy to see from Lemma 2.1(ii) that S is homogeneous; i.e., $S(tu,tv) = tS(u,v)$ for every $t \in \mathbb{R}_+ \stackrel{\text{def}}{=} [0,\infty)$ if and only if (2.3) is satisfied. Furthermore $S : X_+ \to X_+$ is strongly monotone; that is, if $(u_i, v_i) \in X$, $i = 1, 2$, satisfy $0 \le u_1 \le u_2$ and $0 \le v_1 \le v_2$ in Ω with $u_1 \not\equiv u_2$ or $v_1 \not\equiv v_2$ in Ω, then $S(u_2, v_2) - S(u_1, v_1) \in \overset{\circ}{X}_+$. This result follows from the strong comparison principle established in [4] and [16]. Finally by a regularity result of de Thélin [22, Théorème 1, page 376] or Ladyzhenskaya and Ural'tseva [14, Théorème 7.1] (for the $L^\infty(\Omega)$ bound of u and v) and the results of Lieberman [15, Theorem 1, page 1203] and DiBenedetto [7] or Tolksdorf [24] for interior regularity it follows that the mapping $T_p : L^\infty(\Omega) \to C^{1,\beta}(\overline{\Omega})$ is continuous and bounded; that is, it maps bounded sets into bounded sets. It follows from Arzelá-Ascoli's theorem that $T_p : L^\infty(\Omega) \to C^{1,\beta'}(\overline{\Omega})$ is compact whenever $0 < \beta' < \beta$; that is, it maps bounded sets into sets with compact closure. By Theorem A.1 there exists a unique number $\Lambda_1 \in \mathbb{R}$ and $e = (u_1, v_1) \in \overset{\circ}{X}_+$ such that $S(u_1, v_1) = \Lambda_1(u_1, v_1)$. Hence, by Lemma 2.1**(i)**, the couple $(\lambda', \mu') \stackrel{\text{def}}{=} (\Lambda_1^{-(p-1)}, \Lambda_1^{-(q-1)})$ is a principal eigenvalue for system (1.1). Using this eigenvalue in Lemma 2.1**(iii)** and condition (2.3) it follows that, for any (λ, μ) satisfying

$$\left(\frac{\lambda}{\Lambda_1^{-(p-1)}}\right)^{1/\beta_1} \left(\frac{\mu}{\Lambda_1^{-(q-1)}}\right)^{1/\beta_2} = 1, \tag{2.5}$$

the couple $\rho = \left(\frac{\lambda}{\Lambda_1^{-(p-1)}}\right)^{1/\beta_1}$, $\sigma = 1$ solves (2.2). Therefore, (λ, μ) is a principal eigenvalue of (1.1). Conversely, assume that system (1.1) possesses a positive weak solution $(u, v) \in \overset{\circ}{X}_+$ associated to some $(\lambda, \mu) \in (\mathbb{R}_+^*)^2$ and choose $\rho, \sigma > 0$ such that $\left((\frac{\rho}{\sigma})^{\beta_1}\lambda\right)^{\frac{-1}{p-1}} = \left((\frac{\sigma}{\rho})^{\beta_2}\mu\right)^{\frac{-1}{q-1}} \stackrel{\text{def}}{=} \Lambda_0$. Then, by Lemma 2.1, parts **(ii)** and **(iii)**, $S(\rho u, \sigma v) = \Lambda_0(\rho u, \sigma v)$ and, therefore, by the uniqueness results of Theorem A.2, $\Lambda_0 = \Lambda_1$. It follows from the definition of Λ_0 above that (λ, μ) satisfies (2.5). The conclusion of the theorem is now obtained with

$$\Lambda \stackrel{\text{def}}{=} \Lambda_1^{-(p-1)/\beta_1 - (q-1)/\beta_2}. \tag{2.6}$$

Finally, the fact that $(u, v) = c(u_1, v_1)$ with some positive constant c follows from the uniqueness in Theorem A.2. □

Let us denote the set of all principal eigenvalues (λ, μ) of (1.1) by

$$\mathcal{C}_1 \stackrel{\text{def}}{=} \left\{(\lambda, \mu) \in (\mathbb{R}_+^*)^2 : \lambda^{\frac{1}{\beta_1}} \mu^{\frac{1}{\beta_2}} = \Lambda\right\}.$$

We will call it the *principal eigenvalue curve* of system (1.1).

2.2. **The general case.** Let us now consider $\lambda > 0$, $\mu > 0$, the mapping $S : X_+ \to X_+$ defined in Section 2 above, and its square iterate $S^2 \equiv S \circ S$. We wish to apply the Kreĭn-Rutman theorem to S^2. Notice that $S^2 : X_+ \to X_+$ is homogeneous if and only if the following equations are satisfied:

$$\alpha_1 = \alpha_2 = 0, \quad \beta_1\beta_2 = (p-1)(q-1). \tag{2.7}$$

Indeed, given $(u, v) \in X_+$, set $(\tilde{u}, \tilde{v}) = S(u, v)$ and $(\tilde{\tilde{u}}, \tilde{\tilde{v}}) = S(\tilde{u}, \tilde{v})$. For $t \in \mathbb{R}_+$ we have

$$(t^{\gamma_1}\tilde{u}, t^{\gamma_2}\tilde{v}) = S(tu, tv), \tag{2.8}$$

and, therefore,

$$(t^{\delta_1}\tilde{\tilde{u}}, t^{\delta_2}\tilde{\tilde{v}}) = S^2(tu, tv),$$

where $\delta_1 = \frac{\alpha_1\gamma_1+\beta_1\gamma_2}{p-1}$, $\delta_2 = \frac{\alpha_2\gamma_2+\beta_2\gamma_1}{q-1}$. Thus, S^2 is homogeneous if and only if $\delta_1 = \delta_2 = 1$. These equations are equivalent to

$$\alpha_1(\gamma_1 - \gamma_2) = (p-1)(1 - \gamma_1\gamma_2), \quad \beta_1(\gamma_1 - \gamma_2) = (p-1)(\gamma_1^2 - 1),$$
$$\alpha_2(\gamma_1 - \gamma_2) = (q-1)(\gamma_1\gamma_2 - 1), \quad \beta_2(\gamma_1 - \gamma_2) = (q-1)(1 - \gamma_2^2).$$

The case $\gamma_1 = \gamma_2$ forces $\gamma_1\gamma_2 = 1$ which yields $\gamma_1 = \gamma_2 = 1$. Consequently, α_i and β_i satisfy (2.3). Therefore, without loss of generality, from now on we may assume $\gamma_1 < \gamma_2$. Using $\alpha_1 \geq 0$ we obtain $1 - \gamma_1\gamma_2 \leq 0$, whereas $\alpha_2 \geq 0$ yields $\gamma_1\gamma_2 - 1 \leq 0$. It follows that $\gamma_1\gamma_2 = 1$ again. This forces also $\alpha_1 = \alpha_2 = 0$ and $\beta_1\beta_2 = (p-1)(q-1)$.

Since these conditions on α_i, β_i are very restrictive, one would prefer using a better method than the one we have just described. The idea is the following. Let us assume throughout this section that $\alpha_1 < p-1$ and $\alpha_2 < q-1$. We introduce a new mapping $T : X_+ \to X_+$ defined by $T(u, v) \stackrel{\text{def}}{=} (J_1(v), J_2(u))$ where, for $(u, v) \in X$, $J_1(v)$ is the unique (weak) solution $\tilde{u}$ of

$$\begin{cases} -\Delta_p \tilde{u} = a(x)|\tilde{u}|^{\alpha_1} v^{\beta_1} & \text{in } \Omega; \\ \tilde{u} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.9}$$

and $J_2(u)$ is the unique (weak) solution $\tilde{v}$ of

$$\begin{cases} -\Delta_q \tilde{v} = b(x)|\tilde{v}|^{\alpha_2} u^{\beta_2} & \text{in } \Omega; \\ \tilde{v} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.10}$$

The existence is obtained by classical minimization while uniqueness follows from a convexity argument ([11, Theorem 3, page 151]). The regularity results combined with the strong maximum principle already mentioned imply that the pair $(\tilde{u}, \tilde{v})$ belongs to $\overset{\circ}{X}_+$.

Let us to consider the mapping T^2. Notice that

$$T^2(u, v) = (J_1 \circ J_2(u), J_2 \circ J_1(v))$$

for any $(u, v) \in X$, so we can decouple T^2 and look for eigenvalues of each component. We have now that the condition (2.14) below for the homogeneity of $J_1 \circ J_2$ and $J_2 \circ J_1$ is less restrictive than the one found before for S^2. We will need the following result.

Lemma 2.3. *Let us denote $V = C_0^1(\overline{\Omega})$. Then the mapping $J_i : V_+ \to V_+$ is nondecreasing for $i = 1, 2$.*

Proof. We prove the result only for J_1. Let $v_1, v_2 \in V_+$, $0 \leq v_1 \leq v_2$, $v_1 \not\equiv 0$, and denote $m_i(x) = a(x)v_i^{\beta_1}$ and $u_i = J_1(v_i)$ for $i = 1, 2$. Since $m_1 \leq m_2$, it follows that u_2 is an upper solution for the following problem:

$$-\Delta_p u = m_1(x)|u|^{\alpha_1} \text{ in } \Omega; \quad u = 0 \text{ on } \partial\Omega. \tag{2.11}$$

Let us denote by φ the positive eigenfunction of the Dirichlet p-Laplacian with weight m_1; that is, there exists $\lambda_1 > 0$ such that

$$-\Delta_p \varphi = \lambda_1 m_1(x)\varphi^{p-1} \text{ in } \Omega; \quad \varphi = 0 \text{ on } \partial\Omega. \tag{2.12}$$

We can assume that $0 \leq \varphi \leq 1$ on Ω. It follows that, for any constant $c > 0$ sufficiently small, we have $\lambda_1(c\varphi)^{p-1} \leq (c\varphi)^{\alpha_1}$ in Ω, whence $c\varphi$ is a lower solution for problem (2.11). By choosing c even smaller if necessary we can assume $c\varphi < u_2$ in Ω. Let us define the sequence $\{z_n\}_{n=0}^{\infty}$ recursively by $z_0 = u_2$ and z_n is the unique solution of

$$-\Delta_p z_n = m_1(x)z_{n-1}^{\alpha_1} \text{ in } \Omega; \quad z_n = 0 \text{ on } \partial\Omega \tag{2.13}$$

in $W_0^{1,p}(\Omega)$, which is positive. Since obviously this sequence is bounded below and above by $c\varphi \leq z_n \leq u_2$ in Ω, using regularity and compactness results, we can prove that it converges in the norm of V to some function $u \in V$ which is a solution of problem (2.11) and satisfies $c\varphi \leq u \leq u_2$ in Ω. We refer the reader to D. H. Sattinger [18] for details on this monotone iteration method. Thus, by uniqueness, $u = u_1$ and the conclusion follows. $\square$

Now we give the analogue of Lemma 2.1 for the mappings $J_1 \circ J_2$ and $J_2 \circ J_1$.

Lemma 2.4. **(i)** *A couple $u, v \in V_+ \setminus \{0\}$ is a weak solution of (1.1) for some $(\lambda, \mu) \in (\mathbb{R}_+^*)^2$ if and only if $u, v \in \overset{\circ}{V}_+$ and $J_1(v) = \lambda^{\frac{-1}{p-1-\alpha_1}} u$, $J_2(u) = \mu^{\frac{-1}{q-1-\alpha_2}} v$.*

(ii) *For any* $\rho, \sigma > 0$ *we have*

$$(J_1 \circ J_2)(\rho u) = \rho^{\frac{\beta_1\beta_2}{(p-1-\alpha_1)(q-1-\alpha_2)}} (J_1 \circ J_2)(u),$$

$$(J_2 \circ J_1)(\sigma v) = \sigma^{\frac{\beta_1\beta_2}{(p-1-\alpha_1)(q-1-\alpha_2)}} (J_2 \circ J_1)(v).$$

The proof is left to the reader.
The following theorem holds.

Theorem 2.5. *Assume* $\alpha_1 < p-1$, $\alpha_2 < q-1$ *and*

$$\beta_1\beta_2 = (p-1-\alpha_1)(q-1-\alpha_2). \tag{2.14}$$

Then there exists $\Lambda' > 0$ *and a couple* $(u', v') \in \overset{\circ}{X}_+$ *such that system* (1.1) *possesses a positive weak solution* $(u, v) \in X_+$ *associated to some* $(\lambda, \mu) \in (\mathbb{R}_+^*)^2$ *if and only if*

$$\lambda^{\frac{1}{\sqrt{\beta_1(p-1-\alpha_1)}}} \mu^{\frac{1}{\sqrt{\beta_2(q-1-\alpha_2)}}} = \Lambda'. \tag{2.15}$$

Moreover, $(u, v) = (\rho u', \rho\mu^{1/\beta_2} v')$ *with some positive constant* ρ.

Proof. It follows from Lemma 2.4**(ii)** and condition (2.14) that the mappings $J_i \circ J_j : V_+ \to V_+$, $i, j \in \{1, 2\}$, $i \neq j$, are homogeneous. Moreover, both mappings are strongly monotone by the strong comparison principle in [4]. The regularity results quoted before imply that $J_1 \circ J_2$ and $J_2 \circ J_1$ maps bounded sets into sets with compact closure. By Theorem A.2, there exists a unique number $\Lambda_1 \in \mathbb{R}_+^*$ such that $(J_1 \circ J_2)(u') = \Lambda_1 u'$ holds for some $u' \in \overset{\circ}{V}_+$ which is unique up to a positive constant multiple. Similarly, there exists a unique number $\Theta_1 \in \mathbb{R}_+^*$ such that $(J_2 \circ J_1)(v') = \Theta_1 v'$ holds for some $v' \in \overset{\circ}{V}_+$. The $\frac{\beta_2}{q-1-\alpha_2}$-homogeneity of J_2 applied to $(J_1 \circ J_2)(u') = \Lambda_1 u'$ yields

$$(J_2 \circ J_1)(J_2(u')) = \Lambda_1^{\beta_2/(q-1-\alpha_2)} J_2(u').$$

Similarly, the $\frac{\beta_1}{p-1-\alpha_1}$-homogeneity of J_1 applied to $(J_2 \circ J_1)(v') = \Theta_1 v'$ yields

$$(J_1 \circ J_2)(J_1(v')) = \Theta_1^{\beta_1/(p-1-\alpha_1)} J_1(v').$$

The uniqueness of Θ_1 and v' yields

$$\Theta_1 = \Lambda_1^{\beta_2/(q-1-\alpha_2)} \quad \text{and} \quad v' = \theta J_2(u') \quad \text{for some } \theta \in (0, \infty).$$

Hence, we have also

$$\Lambda_1 = \Theta_1^{\beta_1/(p-1-\alpha_1)} \quad \text{and} \quad u' = \theta^{-\beta_1/(p-1-\alpha_1)} \Lambda_1^{-1} J_1(v').$$

By Lemma 2.4**(i)**, the pair $(u', J_2(u'))$ solves (1.1) with $\lambda = \Lambda_1^{1-p+\alpha_1}$ and $\mu = 1$. Similarly, $(J_1(v'), v')$ solves (1.1) with $\lambda = 1$ and $\mu = \Theta_1^{1-q+\alpha_2} = \Lambda_1^{-\beta_2}$. Thus, if λ, μ are positive real numbers satisfying

$$\mu^{-1/\beta_2}\lambda^{-1/(p-1-\alpha_1)} = \Lambda_1, \tag{2.16}$$

it is easy to see that the pair

$$\rho = \lambda^{1/(p-1-\alpha_1)}\Lambda_1, \quad \sigma = 1 \tag{2.17}$$

satisfies (2.2). Hence, (λ, μ) is a principal eigenvalue of (1.1). Conversely, if $u, v \in V_+$ solves (1.1) for some (λ, μ) then, by Lemma 2.4**(i)**, it follows that

$$(J_1 \circ J_2)(u) = \left(\mu^{-1/(q-1-\alpha_2)}\lambda^{-1/\beta_1}\right)^{\beta_1/(p-1-\alpha_1)} u = \mu^{-1/\beta_2}\lambda^{-1/(p-1-\alpha_1)}\, u.$$

By the uniqueness results of Theorem A.2, we have equation (2.16) and $u = \rho u'$ for some $\rho > 0$. Raising both sides of equation (2.16) to the power $-\sqrt{\beta_2/(q-1-\alpha_2)}$, we find that (λ, μ) satisfies (2.15) with $\Lambda' \stackrel{\text{def}}{=} \Lambda_1^{-\sqrt{\beta_2/(q-1-\alpha_2)}}$.

Finally, the result $v = \mu^{1/\beta_2}\rho v'$ follows from (2.2) and (2.17). □

Remark 2.6. One can easily prove that if we restrict ourselves to the case (2.3) then $\Lambda = \Lambda'$.

We will denote also in this case

$$\mathcal{C}_1 \stackrel{\text{def}}{=} \left\{(\lambda, \mu) \in (\mathbb{R}_+^*)^2 : \lambda^{1/\sqrt{\beta_1(p-1-\alpha_1)}}\mu^{1/\sqrt{\beta_2(q-1-\alpha_2)}} = \Lambda'\right\}. \tag{2.18}$$

Let us also denote by $(\varphi_\lambda, \varphi_\mu)$ the positive eigenfunction associated to $(\lambda, \mu) \in \mathcal{C}_1$ with $\|\varphi_\lambda\|_{L^\infty(\Omega)} = 1$.

The following proposition gives some properties of the principal eigenvalues from $\mathcal{C}_1$.

Proposition 2.7. *Assume that $\alpha_1 < p-1$, $\alpha_2 < q-1$, and (2.14) holds.*
(i) *Uniqueness. $(\lambda, \mu) \in \mathbb{R}_+ \times \mathbb{R}_+$ is a principal eigenvalue of (1.1) if and only if $(\lambda, \mu) \in \mathcal{C}_1$.*
(ii) *Simplicity in $\overset{\circ}{X}_+$. Let $(\lambda, \mu) \in \mathcal{C}_1$ and $(u, v), (u', v') \in \overset{\circ}{X}_+$ be a couple of eigenfunctions associated to (λ, μ). Then there exists $\rho > 0$ such that $u = \rho u'$ and $v = \rho\mu^{1/\beta_2}v'$.*
(iii) *Simplicity in X. Assume that $\alpha_1 = \alpha_2 = 0$ and let $(u, v) \in X$ be an eigenfunction associated to $(\lambda, \mu) \in \mathcal{C}_1$. Then either $(u, v) \in \overset{\circ}{X}_+$ or $(-u, -v) \in \overset{\circ}{X}_+$.*

Proof. The uniqueness and simplicity in $\overset{\circ}{X}_+$ of principal eigenvalues follow from the previous theorem and Theorem A.2, so we just prove **(iii)**. It is clear that there exists some $\gamma \in \mathbb{R}_+$ such that

$$-u \le \gamma\varphi_\lambda \quad \text{and} \quad -v \le \gamma^\omega \varphi_\mu,$$

where $\omega = \frac{\beta_2}{q-1}$; let $\overline{\gamma}$ be the minimum of such γ's. We assume by contradiction that $\overline{\gamma} > 0$. Then if we have also $-u \equiv \overline{\gamma}\varphi_\lambda$ in Ω, it follows from the second equation of system (1.1) that $\mu b(x)|-u|^{\beta_2-1}(-u) = \mu b(x)|\overline{\gamma}\varphi_\lambda|^{\beta_2-1}\overline{\gamma}\varphi_\lambda$ and, consequently, $-v = \overline{\gamma}^\omega \varphi_\mu$ and we are done. Thus, it remains to treat the case $-u \not\equiv \overline{\gamma}\varphi_\lambda$. Hence, we have

$$-\Delta_p(-u) = \lambda a(x)|-v|^{\beta_1-1}(-v) \le (\not\equiv) - \Delta_p(\overline{\gamma}\varphi_\lambda) \ \text{ in } \ \Omega,$$
$$-\Delta_q(-v) = \mu b(x)|-u|^{\beta_2-1}(-u) \le (\not\equiv) - \Delta_q(\overline{\gamma}^\omega\varphi_\mu) \ \text{ in } \ \Omega,$$

together with $-u = \overline{\gamma}\varphi_\lambda = -v = \overline{\gamma}^\omega\varphi_\mu = 0$ on $\partial\Omega$. It follows from the strong comparison principle (SCP) of Theorem A.1 that $-u \ll \overline{\gamma}\varphi_\lambda$, $-v \ll \overline{\gamma}^\omega\varphi_\mu$ (see (A.3) in the Appendix for the definition of the strong ordering "$\ll$" in $C^1(\overline{\Omega})$). Thus, we can find $0 < \varepsilon < 1$ such that $-u \le \varepsilon\overline{\gamma}\varphi_\lambda$ and $-v \le (\varepsilon\overline{\gamma})^\omega\varphi_\mu$, a contradiction with our definition of $\overline{\gamma}$. □

3. Systems of non-homogeneous equations

We turn to the following system of two non-homogeneous equations:

$$\begin{cases} -\Delta_p u = \lambda a(x)|u|^{\alpha_1}|v|^{\beta_1-1}v + f(x) & \text{in } \ \Omega; \\ -\Delta_q v = \mu b(x)|v|^{\alpha_2}|u|^{\beta_2-1}u + g(x) & \text{in } \ \Omega; \\ u = v = 0 \quad \text{on } \ \partial\Omega, \end{cases} \tag{3.1}$$

where $0 \le f, g \in L^\infty(\Omega)$ are given functions. Our aim is to study the solvability of system (3.1) in the following three cases: (a) $\lambda, \mu > 0$ and (λ, μ) below or to the left of the eigenvalue curve $\mathcal{C}_1$ (in Section (3.1)), (b) $(\lambda, \mu) \in \mathcal{C}_1$ (in Section (3.2)), and (c) $\lambda, \mu > 0$ and (λ, μ) above or to the right of, but close to, the eigenvalue curve $\mathcal{C}_1$ (in Section (3.3)). We recall that the eigenvalue curve $\mathcal{C}_1$ has been defined in (2.18) with $\Lambda' \overset{\text{def}}{=} \Lambda_1^{-\sqrt{\beta_2/(q-1-\alpha_2)}}$.

3.1. The case when (λ, μ) lies below or to the left of $\mathcal{C}_1$.

Let us now consider system (3.1) when (λ, μ) are in the first quadrant of $\mathbb{R}^2$ and below or to the left of the principal eigenvalue curve $\mathcal{C}_1$. Of course, in order to assure the existence of $\mathcal{C}_1$ we assume the more general condition (2.14) jointly with $\beta_1, \beta_2 > 0$, $0 \le \alpha_1 < p - 1$ and $0 \le \alpha_2 < q - 1$.

Theorem 3.1. *Let $f, g \in L^\infty(\Omega)$, $f \geq 0$, $g \geq 0$, and $\lambda > 0, \mu > 0$ be such that*

$$\lambda^{\frac{1}{\sqrt{\beta_1(p-1-\alpha_1)}}} \mu^{\frac{1}{\sqrt{\beta_2(q-1-\alpha_2)}}} < \Lambda'.$$

Then system (3.1) has a unique weak solution $(u, v) \in X_+$. If, moreover, $\alpha_1 = \alpha_2 = 0$ and $f + g \not\equiv 0$ then there exists a unique solution in X.

Proof. First observe that we can rule out the case when either $f \equiv 0$ and $\alpha_1 \neq 0$, or $g \equiv 0$ and $\alpha_2 \neq 0$, because in these cases either $(0, T_q(g))$ or $(T_p(f), 0)$ are solutions of (3.1). Otherwise consider $(u_0, v_0) = (0, 0)$ and define recursively for $n \in \mathbb{N}$: $u_n = T_p\left(a|u_{n-1}|^{\alpha_1}|v_{n-1}|^{\beta_1-1}v + f\right)$ and $v_n = T_q\left(b|v_{n-1}|^{\alpha_2}|u_{n-1}|^{\beta_2-1}v + g\right)$. It is enough to prove that both sequences, $\{u_n\}_{n=1}^\infty$ and $\{v_n\}_{n=1}^\infty$, are uniformly bounded in $L^\infty(\Omega)$. By the regularity results already quoted, these sequences will also be uniformly bounded in $C_0^{1,\alpha}(\overline{\Omega})$. First observe that

$$\begin{aligned} &0 \leq (\not\equiv) u_2 \leq \cdots \leq u_n \leq u_{n+1} \leq \ldots \quad \text{and} \\ &0 \leq (\not\equiv) v_2 \leq \cdots \leq v_n \leq v_{n+1} \leq \ldots \quad \text{pointwise a.e. in } \Omega. \end{aligned} \tag{3.2}$$

Hence, the functions $\hat{u}_n \stackrel{\text{def}}{=} u_n/\|u_{n+1}\|_\infty$ and $\hat{v}_n \stackrel{\text{def}}{=} v_n/\|v_{n+1}\|_\infty$ satisfy $0 \leq \hat{u}_n, \hat{v}_n \leq 1$ almost everywhere in Ω, $n \in \mathbb{N}$. Consequently, the right-hand sides of the following equations:

$$\begin{cases} -\Delta_p\left(\dfrac{u_n}{\|u_n\|_\infty^{\frac{\alpha_1}{p-1}}\|v_n\|_\infty^{\frac{\beta_1}{p-1}}}\right) = \lambda a(x)\hat{u}_{n-1}^{\alpha_1}\hat{v}_{n-1}^{\beta_1} + \dfrac{f(x)}{\|u_n\|_\infty^{\alpha_1}\|v_n\|_\infty^{\beta_1}} & \text{in } \Omega, \\ -\Delta_q\left(\dfrac{v_n}{\|v_n\|_\infty^{\frac{\alpha_2}{q-1}}\|u_n\|_\infty^{\frac{\beta_2}{q-1}}}\right) = \mu b(x)\hat{v}_{n-1}^{\alpha_2}\hat{u}_{n-1}^{\beta_2} + \dfrac{g(x)}{\|v_n\|_\infty^{\alpha_2}\|u_n\|_\infty^{\beta_2}} & \text{in } \Omega, \\ u_n = v_n = 0 \quad \text{on } \partial\Omega \end{cases} \tag{3.3}$$

are uniformly bounded in $L^\infty(\Omega)$. Employing the regularity result in $C^{1,\alpha}(\overline{\Omega})$, we obtain that both sequences on the left-hand sides above,

$$\frac{u_n}{\|u_n\|_\infty^{\frac{\alpha_1}{p-1}}\|v_n\|_\infty^{\frac{\beta_1}{p-1}}} \quad \text{and} \quad \frac{v_n}{\|v_n\|_\infty^{\frac{\alpha_2}{q-1}}\|u_n\|_\infty^{\frac{\beta_2}{q-1}}}, \tag{3.4}$$

are bounded in $C^{1,\alpha}(\overline{\Omega})$ and, in particular, also in $L^\infty(\Omega)$; that is, there is a constant $C > 0$ such that

$$\frac{\|u_n\|_\infty}{\|u_n\|_\infty^{\frac{\alpha_1}{p-1}}\|v_n\|_\infty^{\frac{\beta_1}{p-1}}} \leq C \quad \text{and} \quad \frac{\|v_n\|_\infty}{\|v_n\|_\infty^{\frac{\alpha_2}{q-1}}\|u_n\|_\infty^{\frac{\beta_2}{q-1}}} \leq C, \tag{3.5}$$

$n \in \mathbb{N}$. Thus, $\|u_n\|_\infty$ is uniformly bounded if and only if $\|v_n\|_\infty$ is uniformly bounded. Now assume that, by contradiction, $\|u_n\|_\infty \to +\infty$ and

$\|v_n\|_\infty \to +\infty$. We combine (3.4) with (3.5) to conclude that also the sequences $u_n/\|u_n\|_\infty$ and $v_n/\|v_n\|_\infty$ are uniformly bounded in $C^{1,\alpha}(\overline{\Omega})$. Hence, for a subsequence denoted again by (u_n, v_n) with $n \in \mathbb{N}$, there exist some functions $u_*, v_* \in C_0^1(\overline{\Omega})$, $u_*, v_* \geq 0$ in Ω, $\|u_*\|_\infty = \|v_*\|_\infty = 1$, such that

$$\frac{u_n}{\|u_n\|_\infty} \to u_* \quad \text{and} \quad \frac{v_n}{\|v_n\|_\infty} \to v_* \quad \text{in } C_0^1(\overline{\Omega}) \text{ as } n \to \infty.$$

Furthermore, there exist $\lambda_*, \mu_* \in \mathbb{R}$ such that also

$$s_n \stackrel{\text{def}}{=} \frac{\|v_n\|_\infty^{\beta_1}}{\|u_n\|_\infty^{p-1-\alpha_1}} \to \lambda_* \quad \text{and} \quad t_n \stackrel{\text{def}}{=} \frac{\|u_n\|_\infty^{\beta_2}}{\|v_n\|_\infty^{q-1-\alpha_2}} \to \mu_* \quad \text{as } n \to \infty.$$

Observe that $s_n^{\beta_2} t_n^{p-1-\alpha_1} = 1$ forces $\lambda_*^{\beta_2} \mu_*^{p-1-\alpha_1} = 1$. Passing to the limit in system (3.3) above, as $n \to \infty$, we find

$$\begin{cases} -\Delta_p u_* \leq \lambda\lambda_* a(x) u_*^{\alpha_1} v_*^{\beta_1} & \text{in } \Omega; \\ -\Delta_q v_* \leq \mu\mu_* b(x) v_*^{\alpha_2} u_*^{\beta_2} & \text{in } \Omega; \\ u_* = v_* = 0 \quad \text{on } \partial\Omega. \end{cases}$$

The inequalities are obtained with a help from inequalities (3.2). From the first inequality in the system above we deduce that $(\lambda\lambda_*)^{-1/(p-1-\alpha_1)} u_*$ is a lower solution of problem (2.9) and $(\mu\mu_*)^{-1/(q-1-\alpha_2)} v_*$ is a lower solution of problem (2.10). Using the uniqueness of positive solutions of these problems we deduce that $(\lambda\lambda_*)^{-1/p-1-\alpha_1} u_* \leq J_1(v_*)$ and $(\mu\mu_*)^{-1/q-1-\alpha_2} v_* \leq J_2(u_*)$ and, consequently, using also $\lambda_*^{\beta_2} \mu_*^{p-1-\alpha_1} = 1$, we get

$$\mu^{-1/\beta_2}\, \lambda^{-1/(p-1-\alpha_1)} u_* \leq (J_1 \circ J_2)(u_*).$$

From the uniqueness results of Theorem A.2 we infer $\mu^{-1/\beta_2}\, \lambda^{-1/(p-1-\alpha_1)} \leq \Lambda_1$. We recall that Λ_1 is the unique principal eigenvalue of $J_1 \circ J_2$ associated to some $u' \in \overset{\circ}{V}_+$. Thus, using the definition of Λ' in (2.6), we obtain a contradiction with the hypothesis $\lambda^{\frac{1}{\sqrt{\beta_1(p-1-\alpha_1)}}}\, \mu^{\frac{1}{\sqrt{\beta_2(q-1-\alpha_2)}}} < \Lambda'$. We have just proved that both sequences $\{u_n\}_{n=1}^\infty$ and $\{v_n\}_{n=1}^\infty$ are uniformly bounded in $C_0^{1,\alpha}(\Omega)$. Hence, there exists $(u, v) \in X_+$ such that $u_n \to u$ and $v_n \to v$ in $C_0^1(\Omega)$ and u, v solves (3.1).

Assume now by contradiction that $(\tilde{u}, \tilde{v}) \in X_+$ is a second solution of (3.1); we distinguish between two cases: (a) $u \not\equiv 0$, $v \not\equiv 0$ and (b) either $u \equiv 0$ or $v \equiv 0$.

In case (a) let us assume that for instance $f \not\equiv 0$ (a similar proof works if $g \not\equiv 0$). Since $-\Delta_p \tilde{u} \geq f = -\Delta_p u_1$, we have $\tilde{u} \geq u_1$ and, after an iteration process, $\tilde{u} \geq u_n$ for all $n \in \mathbb{N}$. Thus, $u \leq \tilde{u}$ and $v \leq \tilde{v}$. Observe that if for instance $u = \tilde{u}$ then it follows from the first equation of system (3.1) that $\lambda a(x) u^{\alpha_1} v^{\beta_1} + f = \lambda a(x) u^{\alpha_1} \tilde{v}^{\beta_1} + f$ and consequently $v = \tilde{v}$. Thus, let us assume by contradiction that $u \leq (\not\equiv) \tilde{u}$ and $v \leq (\not\equiv) \tilde{v}$. Denote $\omega \stackrel{\text{def}}{=} (p - 1 - \alpha_1)/\beta_1$, and consider the smallest $k \in \mathbb{R}$ such that $\tilde{u} \leq ku$ and $\tilde{v} \leq k^\omega v$. We have $\omega = \beta_2/(q - 1 - \alpha_2)$. Let us assume, by contradiction, that $k > 1$. From the first equation of (3.1) we have

$$-\Delta_p(ku) = \lambda a(x)(ku)^{\alpha_1}(k^\omega v)^{\beta_1} + k^{p-1} f \geq (\not\equiv) - \Delta_p \tilde{u} \ \text{ in } \ \Omega,$$

and $ku = \tilde{u} = 0$ on $\partial\Omega$. From the strong comparison principle of [4] we infer that $\tilde{u} \ll ku$; see (A.3) for the definition of "$\ll$". Taking advantage of the strict inequality in the second equation of system (3.1) we get, again by using the same SCP, that $\tilde{v} \ll k^\omega v$, in contradiction with our choice of k.

In case (b) if both $u \equiv 0$ and $v \equiv 0$ then $f = g \equiv 0$ and we can prove directly from the system that $\mu^{-1/\beta_2} \lambda^{-1/(p-1-\alpha_1)} \tilde{u} = (J_1 \circ J_2)(\tilde{u})$. Therefore, from Theorem A.2, it follows that $\mu^{-1/\beta_2} \lambda^{-1/(p-1-\alpha_1)} = \Lambda_1$, a contradiction with our hypothesis. In the case $u \equiv 0$ it is trivial that the couple $(0, T_q(g))$ is the unique solution of (3.1). A similar result holds if $v \equiv 0$.

Finally, to prove the uniqueness in X when $\alpha_1 = \alpha_2 = 0$, we argue as in Part **(iii)** of Proposition 2.7 to get the conclusion. We leave the details to the reader. $\square$

As a corollary of this theorem we have the following more general existence result.

Corollary 3.2. *Let us consider $f, g \in L^\infty(\Omega)$ and let $\lambda > 0$, $\mu > 0$ be such that*

$$\lambda^{\frac{1}{\sqrt{\beta_1(p-1-\alpha_1)}}} \mu^{\frac{1}{\sqrt{\beta_2(q-1-\alpha_2)}}} < \Lambda'.$$

Then system (3.1) *possesses at least one solution.*

Proof. Assume $|f| + |g| \not\equiv 0$, otherwise the conclusion is trivial. Let $(u_1, v_1) \in \overset{\circ}{X}_+$ be the (unique) solution of (3.1) for the functions $|f|$ and $|g|$ instead of f and g. Then trivially (u_1, v_1) is an upper solution of our problem. Similarly $(-u_1, -v_1)$ is a lower solution of our problem. We then apply degree theory to the mapping $T_{f,g} : X \mapsto X$ defined by

$$T_{f,g}(u, v) \stackrel{\text{def}}{=} \Big(T_p(a|u|^{\alpha_1}|v|^{\beta_1 - 1} v + f),\, T_q(b|v|^{\alpha_2}|u|^{\beta_2 - 1} u + g)\Big)$$

to get the conclusion (cf., for instance, [5]). $\square$

3.2. **The case when** $(\lambda, \mu) \in \mathcal{C}_1$**.** In this section we prove a nonexistence result for system (3.1) in the case when $(\lambda, \mu) \in \mathcal{C}_1$, $f \geq 0$, $g \geq 0$, and $f + g \not\equiv 0$.

Proposition 3.3. *Let $f, g \in L^\infty(\Omega)$, $f \geq 0, g \geq 0$, $f + g \not\equiv 0$, and $\lambda > 0$, $\mu > 0$ be such that*

$$\lambda^{\frac{1}{\sqrt{\beta_1(p-1-\alpha_1)}}} \mu^{\frac{1}{\sqrt{\beta_2(q-1-\alpha_2)}}} = \Lambda'.$$

Then system (3.1) *has no solution in X_+. If, moreover, $\alpha_1 = \alpha_2 = 0$ then there is no solution in X.*

Proof. Assume by contradiction that $(u, v) \in X_+$ is a solution of (3.1). Then,

$$\begin{cases} -\Delta_p u \geq \lambda a(x) u^{\alpha_1} v^{\beta_1} & \text{in } \Omega; \\ -\Delta_q v \geq \mu b(x) v^{\alpha_2} u^{\beta_2} & \text{in } \Omega; \\ u = v = 0 \quad \text{on } \partial\Omega, \end{cases}$$

and arguing as in the first part of the proof of Theorem 3.1 we conclude that $J_1 \circ J_2(u) \leq \Lambda_1 u$. By Theorem A.2, it follows that $u = \rho\varphi_\lambda$ for some $\rho > 0$ and similarly $v = \rho\mu^{1\beta_2}\varphi_\mu$. But this is impossible because of $f + g \not\equiv 0$.

In the case $\alpha_1 = \alpha_2 = 0$ let us assume, by contradiction, that there exists a solution $(u, v) \in X$ and let us consider the smallest $\gamma \in \mathbb{R}_+$ such that $-u \leq \gamma\varphi_\lambda$, $-v \leq \gamma^\omega \varphi_\mu$, where $\omega = \frac{\beta_2}{q-1}$. If $\gamma = 0$ then $(u, v) \in X_+$ and we argue as previously to arrive at a contradiction with $f + g \not\equiv 0$. Hence, we may assume $\gamma > 0$. If, for instance, $f \not\equiv 0$, then, from the first equation of system (3.1), we have

$$-\Delta_p(-u) = \lambda a(x)|-v|^{\beta_1 - 1}(-v) - f \leq (\not\equiv) - \Delta_p(\gamma\varphi_\lambda) \text{ in } \Omega.$$

Applying Theorem A.1, we conclude that $-u \ll \gamma\varphi_\lambda$. Then, using the second equation of (3.1), we get also that $-v \ll \gamma^\omega\varphi_\mu$ and thus a contradiction with the minimality of γ. $\square$

3.3. **An antimaximum principle for systems.** Here we treat the case when (λ, μ) lies above or to the right of, but close to, $\mathcal{C}_1$.

Let us recall the so-called "antimaximum principle" for a single equation (cf. for instance [9, 10]): Assume that Ω is a bounded domain in $\mathbb{R}^N$ with a C^2-boundary $\partial\Omega$. If $f \in L^\infty(\Omega)$, $f \geq 0$, $f \not\equiv 0$ in Ω, then there exists $\delta > 0$ such that, for each $\lambda \in (\lambda_1, \lambda_1 + \delta)$, every weak solution $u \in W_0^{1,p}(\Omega)$ of

$$-\Delta_p u = \lambda|u|^{p-2}u + f \text{ in } \Omega \quad u = 0 \text{ on } \partial\Omega$$

satisfies $-u \in \overset{\circ}{V}_+$; i.e., u is of class C^1 and satisfies $u \ll 0$ (cf. (A.3) for the definition of "$\ll$").

In this section we consider again system (3.1) with $\alpha_1 = \alpha_2 = 0$, $f \geq 0$ and $g \geq 0$. More precisely, we have the following result.

Theorem 3.4. *Let $\alpha_1 = \alpha_2 = 0$, $\beta_1\beta_2 = (p-1)(q-1)$, and $(\lambda_1, \mu_1) \in \mathcal{C}_1$. Consider two functions $0 \leq f, g \in L^\infty(\Omega)$ with $f + g \not\equiv 0$. Then there exists $\delta > 0$ such that, for all pairs $(\lambda, \mu) \in \mathbb{R}^2$ with $\lambda_1 < \lambda < \lambda_1 + \delta$ and $\mu_1 < \mu < \mu_1 + \delta$, every (weak) solution (u, v) of system* (3.1) *satisfies $u \ll 0$ and $v \ll 0$ in Ω.*

Proof. Assume that, by contradiction, there exist a sequence $\{(\lambda'_n, \mu'_n)\}_{n=1}^\infty \subset \mathbb{R}^2$, such that $\lambda'_n > \lambda_1$, $\mu'_n > \mu_1$, and $\lambda'_n \to \lambda_1$, $\mu'_n \to \mu_1$ as $n \to \infty$, and another sequence of solutions $(u_n, v_n) \in X$ of

$$\begin{cases} -\Delta_p u_n = \lambda'_n a(x)|v_n|^{\beta_1 - 1} v_n + f(x) \quad \text{in } \Omega; \\ -\Delta_q v_n = \mu'_n b(x)|u_n|^{\beta_2 - 1} u_n + g(x) \quad \text{in } \Omega; \\ u_n = v_n = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3.6}$$

with at least one of $-u_n$ or $-v_n$ not in $(C_0^1(\overline{\Omega}))_+^\circ$. We distinguish between two cases: (a) both $\|u_n\|_\infty$ and $\|v_n\|_\infty$ are bounded and (b) either $\|u_n\|_\infty$ or $\|v_n\|_\infty$ is unbounded.

In case (a) it follows that the sequences are bounded in $C_0^{1,\alpha}(\Omega)$, by regularity. Employing Arzelà-Ascoli's theorem, we may pass to the limit in $C_0^1(\Omega)$. Then the limit functions $u, v \in C_0^1(\Omega)$ satisfy

$$\begin{cases} -\Delta_p u = \lambda a(x)|v|^{\beta_1 - 1} v + f(x) \quad \text{in } \Omega; \\ -\Delta_q v = \mu b(x)|u|^{\beta_2 - 1} u + g(x) \quad \text{in } \Omega; \\ u = v = 0 \quad \text{on } \partial\Omega. \end{cases}$$

But this contradicts Proposition 3.3.

In case (b) we argue as in the proof of Theorem 3.1; see system (3.3). We observe that both sequences in (3.4) are bounded in $C^{1,\alpha}(\overline{\Omega})$ and, in particular, also in $L^\infty(\Omega)$; that is, there is a constant $C > 0$ such that both inequalities in (3.5) hold for every $n \in \mathbb{N}$. Thus, $\|u_n\|_\infty \to \infty$ if and only if $\|v_n\|_\infty \to \infty$ as $n \to \infty$. For a subsequence denoted again by (u_n, v_n) with $n \in \mathbb{N}$, there exist some functions $u_*, v_* \in C_0^1(\overline{\Omega})$, $\|u_*\|_\infty = \|v_*\|_\infty = 1$, such that

$$\frac{u_n}{\|u_n\|_\infty} \to u_* \quad \text{and} \quad \frac{v_n}{\|v_n\|_\infty} \to v_* \quad \text{in } C_0^1(\overline{\Omega}) \text{ as } n \to \infty. \tag{3.7}$$

Furthermore, there exist $\lambda_*, \mu_* \in \mathbb{R}$ such that also

$$s_n \stackrel{\text{def}}{=} \frac{\|v_n\|_\infty^{\beta_1}}{\|u_n\|_\infty^{p-1}} \to \lambda_* \quad \text{and} \quad t_n \stackrel{\text{def}}{=} \frac{\|u_n\|_\infty^{\beta_2}}{\|v_n\|_\infty^{q-1}} \to \mu_* \quad \text{as } n \to \infty,$$

thanks to $\alpha_1 = \alpha_2 = 0$. We have $\lambda_*^{\beta_2} \mu_*^{p-1} = 1$. Passing to the limit in system (3.6) above, as $n \to \infty$, for $u_n/\|u_n\|_\infty$ and $v_n/\|v_n\|_\infty$, we find

$$\begin{cases} -\Delta_p u_* = \lambda_1 \lambda_* a(x) |v|_*^{\beta_1 - 1} v_* & \text{in } \Omega; \\ -\Delta_q v_* = \mu_1 \mu_* b(x) |u|_*^{\beta_2 - 1} u_* & \text{in } \Omega; \\ u_* = v_* = 0 \quad \text{on } \partial\Omega. \end{cases}$$

This shows that $(\lambda_1 \lambda_*, \mu_1 \mu_*) \in \mathcal{C}_1$. By Proposition 2.7**(iii)**, we must have either $(u_*, v_*) \in \mathring{X}_+$ or $(-u_*, -v_*) \in \mathring{X}_+$.

If $(u_*, v_*) \in \mathring{X}_+$, we rewrite system (3.6) as follows:

$$\begin{cases} -\Delta_p u_n = \lambda_1 a(x) |v_n|^{\beta_1 - 1} v_n + (\lambda_n' - \lambda_1) a(x) |v_n|^{\beta_1 - 1} v_n + f(x) & \text{in } \Omega; \\ -\Delta_q v_n = \mu_1 b(x) |u_n|^{\beta_2 - 1} u_n + (\mu_n' - \mu_1) b(x) |u_n|^{\beta_2 - 1} u_n + g(x) & \text{in } \Omega; \\ u_n = v_n = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.8}$$

From the convergence in (3.7) we thus conclude that also $(u_n, v_n) \in \mathring{X}_+$ for all $n \geq n_0$ with $n_0 \in \mathbb{N}$ large enough. But then (3.8) contradicts the nonexistence result in Proposition 3.3. Thus, we must have $(-u_*, -v_*) \in \mathring{X}_+$. Using the convergence in (3.7) again, we now have $(-u_n, -v_n) \in \mathring{X}_+$ for all $n \geq n_0$ with $n_0 \in \mathbb{N}$ large enough. But this contradicts our hypothesis that for every $n \in \mathbb{N}$ we have $-u_n \notin (C_0^1(\overline{\Omega}))_+^\circ$ or $-v_n \notin (C_0^1(\overline{\Omega}))_+^\circ$.

The proposition is proved. □

APPENDIX A. APPENDIX

A.1. **Strong comparison principle.** In this paragraph we establish a version of the *strong comparison principle* (SCP, for brevity) for the p-Laplacian. The present version of the SCP is a modification of those in D. Arcoya and D. Ruiz [1, Proposition 2.6, page 853], M. Cuesta and P. Takáč [3, Theorem 1, page 81] and [4, Theorem 2.1, page 725], and M. Lucia and S. Prashanth [16, Theorem 1.3, pages 1006–1007]. Let f and g be two functions from $L^\infty(\Omega)$ satisfying $f \leq g$ almost everywhere in Ω. Assume that

$u, v \in W_0^{1,p}(\Omega)$ are any weak solutions to the following boundary value problems, respectively:

$$-\Delta_p u = f(x) \ \text{ in } \Omega; \qquad u = 0 \ \text{ on } \partial\Omega, \tag{A.1}$$

$$-\Delta_p v = g(x) \ \text{ in } \Omega; \qquad v = 0 \ \text{ on } \partial\Omega. \tag{A.2}$$

Then we have $u \leq v$ in Ω, by the *weak comparison principle* (WCP, for brevity) due to P. Tolksdorf [23, Lemma 3.1, page 800]. A classical version of the SCP would claim that if f and g satisfy also $f \not\equiv g$ in Ω then

$$u < v \ \text{ in } \Omega \quad \text{ and } \quad \frac{\partial u}{\partial \nu} > \frac{\partial v}{\partial \nu} \ \text{ on } \partial\Omega. \tag{A.3}$$

We abbreviate $u \ll v$ if (A.3) holds. It is still an open question if this claim holds without additional hypotheses. For $f \equiv 0$ and $u \equiv 0$ in Ω, this is the strong maximum principle due to P. Tolksdorf [23, Proposition 3.2.1 and 3.2.2, page 801] and J. L. Vázquez [25, Theorem 5, page 200].

As usual, $\nu \equiv \nu(x_0)$ denotes the exterior unit normal to $\partial\Omega$ at $x_0 \in \partial\Omega$. We recall that $u, v \in C^{1,\beta}(\overline{\Omega})$ by a regularity result due to E. DiBenedetto [7, Theorem 2, page 829] and P. Tolksdorf [24, Theorem 1, page 127] (interior regularity, shown independently), and to G. Lieberman [15, Theorem 1, page 1203] (regularity near the boundary).

Here, we verify the SCP (A.3) under the following additional hypotheses.

Theorem A.1. *Let $1 < p < \infty$. Assume that Ω is either a bounded domain in $\mathbb{R}^N$ whose boundary $\partial\Omega$ is a C^2-manifold if $N \geq 2$, or a bounded open interval in $\mathbb{R}^1$ if $N = 1$. Let $f, g \in L^\infty(\Omega)$ be such that $f \leq g$ and $f \not\equiv g$ in Ω and also $g \geq 0$ and $g \not\equiv 0$ in Ω. Then the SCP* (A.3) *is valid for any weak solutions $u, v \in W_0^{1,p}(\Omega)$ of* equations (A.1) *and* (A.2).

Proof. The case $N = 1$ is proved in M. Cuesta and P. Takáč [4, pages 731–732]. In the sequel we therefore assume $N \geq 2$.

First, the WCP with $0 \equiv f \leq g$ and $u \equiv 0$ in Ω guarantees $v \geq 0$ in Ω. Now we may apply the strong maximum principle ([23, 25]) to obtain (2.1). Take $\gamma > 0$ and $\delta > 0$ small enough, such that $|\nabla v(x)| \geq \gamma$ holds for every $x \in \Omega_\delta$, where

$$\Omega_\delta = \{x \in \Omega : \ d(x) < \delta\} \tag{A.4}$$

is the open δ-neighborhood in Ω of the boundary $\partial\Omega$. As usual, $d(x) \stackrel{\text{def}}{=} \operatorname{dist}(x, \partial\Omega)$ denotes the distance from a point $x \in \Omega$ to the boundary $\partial\Omega$. Set $\Omega'_\delta = \Omega \setminus \overline{\Omega_\delta}$. Since $\partial\Omega$ is assumed to be a compact manifold of class C^2, so is $\partial\Omega'_\delta$ provided $\delta > 0$ is small enough. Indeed, the last claim is a consequence of $d \in C^2(\overline{\Omega_\delta})$, by [12, Lemma 14.16, page 355] and its proof, where it is shown that $\overline{\Omega_\delta}$ is C^1-diffeomorphic to $\partial\Omega\times[0,\delta]$ with $x \mapsto (x,0)$ for all $x \in \partial\Omega$, and $\overline{\Omega'_\delta} = \Omega\setminus\Omega_\delta$ is C^1-diffeomorphic to $\overline{\Omega}$. Both diffeomorphisms are considered between manifolds with boundary of class C^2. Of course, they can be replaced by C^2-diffeomorphisms, by M. W. Hirsch [13, Theorem 3.5, page 57].

Second, set $w = v - u$; hence $0 \leq w \in C^{1,\beta}(\overline{\Omega})$ with $w = 0$ on $\partial\Omega$. Subtracting Equation (A.1) from (A.2), we find out that w satisfies the following linear elliptic inequality in the sense of distributions in Ω_δ:

$$\begin{aligned} -\operatorname{div}(\mathbf{A}(x)\nabla w) &\stackrel{\text{def}}{=} -\textstyle\sum_{i,j=1}^N \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial w}{\partial x_j}\right) \\ &= g - f \geq 0 \quad \text{ for } x \in \Omega_\delta. \end{aligned} \tag{A.5}$$

Here, the coefficients

$$a_{ij}(x) = \textstyle\int_0^1 \hat{a}_{ij}\left((1-s)\nabla u(x) + s\nabla v(x)\right)\,\mathrm{d}s \tag{A.6}$$

of the matrix $\mathbf{A}(x)$ belong to $C^\beta(\overline{\Omega_\delta})$ and form a uniformly elliptic operator in Ω_δ, where

$$\hat{a}_{ij}(\mathbf{z}) = |\mathbf{z}|^{p-2}\left(\delta_{ij} + (p-2)\,|\mathbf{z}|^{-2}\,z_i z_j\right)$$

is the Jacobian matrix of the mapping $\mathbf{z} \mapsto |\mathbf{z}|^{p-2}\mathbf{z}$ for $\mathbf{z} = (z_1,\dots,z_N) \in \mathbb{R}^N\setminus\{\mathbf{0}\}$, with the Kronecker symbol δ_{ij}. The uniform ellipticity is verified as follows, cf. P. Takáč [21, Appendix A, pages 233–235]:

The symmetric matrix with the entries $\hat{\hat{a}}_{ij}(\mathbf{z}) = \delta_{ij} + (p-2)\,|\mathbf{z}|^{-2}\,z_i z_j$ has only two eigenvalues, equal to 1 and $p-1$. The expression $|\mathbf{z}|^{p-2}$ for $\mathbf{z} = (1-s)\mathbf{a} + s\mathbf{b}$ with $\mathbf{a} = \nabla u(x)$ and $\mathbf{b} = \nabla v(x)$ is estimated by the inequalities

$$\begin{gathered} c_p\cdot\Big(\max_{0\leq s\leq 1}|\mathbf{a}+s\mathbf{b}|\Big)^{p-2} \leq \int_0^1 |\mathbf{a}+s\mathbf{b}|^{p-2}\,\mathrm{d}s \leq C_p\cdot\Big(\max_{0\leq s\leq 1}|\mathbf{a}+s\mathbf{b}|\Big)^{p-2} \\ \text{for all } \mathbf{a},\mathbf{b}\in\mathbb{R}^N \text{ with } |\mathbf{a}|+|\mathbf{b}|>0, \end{gathered} \tag{A.7}$$

where $0 < c_p \leq C_p < \infty$ are some constants and we have substituted $\mathbf{b}$ for the difference $\mathbf{b}-\mathbf{a}$ in $\mathbf{z} = \mathbf{a} + s(\mathbf{b}-\mathbf{a})$ to simplify our notation. The first inequality is trivial for $1 < p \leq 2$ (take $c_p = 1$), the second one for $2 \leq p < \infty$ (take $C_p = 1$); the remaining inequalities are proved in [21, Lemma A.1, page 233]. The desired uniform ellipticity now follows from our choice of Ω_δ; we have $|\nabla v| \geq \gamma = \text{const} > 0$ in Ω_δ.

Third, in every subdomain σ of Ω_δ with $\overline{\sigma}\cap\partial\Omega = \emptyset$, we apply the strong maximum principle from D. Gilbarg and N. S. Trudinger [12, Theorem 8.19, page 198] to the linear elliptic inequality (A.5) considered in σ. If $\partial\Omega$ is connected as in M. Cuesta and P. Takáč [3, 4], then so is Ω_δ and, consequently, we obtain either $w = v - u > 0$ in Ω_δ, or else $w = v - u \equiv 0$ in Ω_δ. If $\partial\Omega$ is not connected as in M. Lucia and S. Prashanth [16], Proposition 4.1 in [16, page 1009] guarantees that either $w \equiv 0$ in some connected component of Ω_δ, or else $w > 0$ in Ω_δ (cf. Cases A and B on page 1009).

If $u < v$ in Ω_δ then, for any fixed number $\eta \in (0, \delta)$, we can find a constant $c > 0$ such that $v \geq u + c$ holds on the boundary $\partial\Omega'_\eta = \partial\Omega_\eta \setminus \partial\Omega \subset \Omega_\delta$ of the domain $\Omega'_\eta = \Omega \setminus \overline{\Omega_\eta}$. We combine this boundary inequality with

$$-\Delta_p(u + c) = -\Delta_p u = f(x) \leq g(x) = -\Delta_p v \quad \text{ in } \Omega'_\eta$$

to conclude that $v \geq u + c$ holds throughout $\overline{\Omega'_\eta}$, by the WCP (P. Tolksdorf [23, Lemma 3.1, page 800]). We have thus obtained $u < v$ in $\Omega = \Omega_\delta \cup \overline{\Omega'_\eta}$ as desired. Furthermore, we can make use of the boundary point principle as shown in R. Finn and D. Gilbarg [8, Lemma 7, page 31] (for $N \geq 3$, see also [8, Remarks, page 35]) in order to deduce that $-\frac{\partial u}{\partial \nu}(x_0) < -\frac{\partial v}{\partial \nu}(x_0)$ holds at an arbitrary boundary point $x_0 \in \partial\Omega$.

Finally, assume $u \equiv v$ in Ω_δ. Next, we employ a version of the divergence theorem proved in M. Cuesta and P. Takáč [3, Lemma A.1, page 742]. For $\eta \in (0, \delta)$ small enough we apply the divergence theorem to equations (A.1) and (A.2) over the domain $\Omega'_\eta = \Omega \setminus \overline{\Omega_\eta}$. We thus obtain

$$-\int_{\partial\Omega'_\eta} |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nu(x)\, d\sigma(x) = \int_{\Omega'_\eta} f(x)\, \mathrm{d}x, \tag{A.8}$$

$$-\int_{\partial\Omega'_\eta} |\nabla v(x)|^{p-2} \nabla v(x) \cdot \nu(x)\, d\sigma(x) = \int_{\Omega'_\eta} g(x)\, \mathrm{d}x. \tag{A.9}$$

Since $u \equiv v$ in Ω_δ and $\partial\Omega'_\eta \subset \Omega_\delta$, the two surface integrals on the left-hand side in equations (A.8) and (A.9) are equal. Therefore, we have

$$\int_{\Omega'_\eta} f(x)\, \mathrm{d}x = \int_{\Omega'_\eta} g(x)\, \mathrm{d}x.$$

Combined with $f \leq g$ in Ω, this equality forces $f \equiv g$ in Ω'_η. From $u \equiv v$ in Ω_δ we obtain also $f \equiv g$ in Ω_δ. Thus, we arrive at $f \equiv g$ throughout $\Omega = \Omega_\delta \cup \Omega'_\eta$, a contradiction to our hypothesis $f \not\equiv g$ in Ω.

The proposition is proved. □

A.2. **Kreĭn-Rutman theorem for nonlinear mappings.** We need the following version of the Kreĭn-Rutman theorem for nonlinear homogeneous mappings which is essentially due to P. Takáč [20, Theorem 3.5, page 1763];

cf. also R. Nussbaum [17, Proposition 3.1, page 100]. As stated below, this version includes also some results from the proof of Theorem 3.5 in [20].

We assume that $E = (E, \leq)$ is a strongly ordered Banach space; i.e., E is a real Banach space endowed with an ordering "$\leq$" that is compatible with the norm topology on E and such that the positive cone $E_+ \stackrel{\text{def}}{=} \{x \in E : x \geq 0\}$ has nonempty interior (in E) denoted by $\overset{\circ}{E}_+$. For $x, y \in E$ we write $x \leq y$ (or equivalently $y \geq x$) if and only if $y - x \in E_+$. Similarly, we write $x < y$ (or equivalently $y > x$) in E if and only if $y - x \in E_+ \setminus \{0\}$, whereas $x \ll y$ (or equivalently $y \gg x$) in E if and only if $y - x \in \overset{\circ}{E}_+$. In particular, $\overset{\circ}{E}_+ = \{x \in E : x \gg 0\}$. For $a \leq b$ in E, the set $[a, b] \stackrel{\text{def}}{=} \{x \in E : a \leq x \leq b\}$ is called a closed order interval in E. (The reader is referred to the monograph by H. H. Schaefer [19] for details on ordered Banach spaces.)

A (nonlinear) self-mapping $T : E_+ \to E_+$ is called *homogeneous* if $T(sx) = sTx$ holds for all $x \in E_+$ and $s \in [0, 1]$. We say that $T : X \subset E \to E$ is *monotone* if $x \leq y$ in X implies $Tx \leq Ty$, and *strongly monotone* if $x < y$ in X implies $Tx \ll Ty$. Finally, a monotone mapping $T : X \subset E \to E$ is called *order-compact* if the set $T([a, b] \cap X) = \{Tx : x \in [a, b] \cap X\}$ has compact closure in E for each pair $a \leq b$ in E.

Theorem A.2. ([20, Theorem 3.5, page 1763]) *Let E be a strongly ordered Banach space and $T : E_+ \to E_+$ a continuous homogeneous mapping. Assume that T is strongly monotone and order-compact. Then there exist a number $\Lambda_1 > 0$ and $e \in \overset{\circ}{E}_+$ such that $Te = \Lambda_1 e$. Furthermore, if $\lambda \in [0, \Lambda_1]$ and $u \in E_+ \setminus \{0\}$ satisfy $Tu \leq \lambda u$, then $\lambda = \Lambda_1$ and $u = ce$ for some constant $c > 0$, and thus $Tu = \Lambda_1 u$. Finally, if $u \in E_+ \setminus \{0\}$ satisfies $Tu \geq \Lambda_1 u$, then again $u = ce$ with a constant $c > 0$.*

We wish to apply this theorem in the Banach space $E = C_0'(\overline{\Omega})$ (or in $E^2 = E \times E$) of all continuous functions $u : \overline{\Omega} \to \mathbb{R}$ with $u = 0$ on $\partial\Omega$, such that u possesses a continuous normal derivative $\partial u / \partial \nu : \partial\Omega \to \mathbb{R}$. Recall that $\nu \equiv \nu(x_0)$ denotes the exterior unit normal to $\partial\Omega$ at $x_0 \in \partial\Omega$. More precisely, $C_0'(\overline{\Omega})$ is defined to be the completion of the vector space

$$C_0^1(\overline{\Omega}) \stackrel{\text{def}}{=} \left\{u \in C^1(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\right\}$$

under the *order norm*

$$\|u\|' \stackrel{\text{def}}{=} \max_{\overline{\Omega}} |u(x)| + \max_{\partial\Omega} \left|\frac{\partial u}{\partial \nu}\right| \quad \text{for } u \in C_0^1(\overline{\Omega}). \tag{A.10}$$

The ordering " $\leq$ " on $C_0'(\overline{\Omega})$ is induced by the natural pointwise ordering of functions; that is, $u \leq v$ in E is defined by $u(x) \leq v(x)$ for all $x \in \Omega$. Thus,

$$C_0'(\overline{\Omega})_+ = \left\{ u \in C_0'(\overline{\Omega}) : \, u \geq 0 \ \text{ in } \ \Omega \right\}$$

is the positive cone in $C_0'(\overline{\Omega})$; the interior $\mathring{C}_0'(\overline{\Omega})$ consists of all functions $u \in C_0'(\overline{\Omega})$ that satisfy both inequalities of the strong maximum principle,

$$u > 0 \ \text{ in } \Omega \quad \text{ and } \quad \frac{\partial u}{\partial \nu} < 0 \ \text{ on } \partial\Omega. \tag{A.11}$$

The self-mapping $T : \, C_0'(\overline{\Omega}) \to C_0'(\overline{\Omega})$ may typically be of the form $Tu \stackrel{\text{def}}{=} v$, where $u \in C_0'(\overline{\Omega})$ is arbitrary and $v \in W_0^{1,p}(\Omega)$ is the unique weak solution to the Dirichlet boundary-value problem

$$-\Delta_p v = m(x)\, |u|^{p-2} u \ \text{ in } \Omega; \qquad v = 0 \ \text{ on } \partial\Omega. \tag{A.12}$$

Hence, $v \in C^{1,\beta}(\overline{\Omega}) \cap C_0'(\overline{\Omega})$, by regularity. The function $m \in L^\infty(\Omega)$ is a positive "weight;" i.e., it satisfies $m > 0$ almost everywhere in Ω. It is easy to see that this mapping satisfies all hypotheses of Theorem A.2 above, thanks to Theorem A.1 combined with the uniqueness and regularity of v; cf. [20], Example 5.2 (pages 1771–1773) and Corollary 5.3 (page 1773).

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