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# A VARIATIONAL APPROACH TO NONRESONANCE WITH RESPECT TO THE FUČIK SPECTRUM

MABEL CUESTA and JEAN-PIERRE GOSSEZ
Département de Mathématique, C.P.214, Université Libre de Bruxelles, 1050 Bruxelles, Belgium

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## 1. INTRODUCTION

THIS PAPER is concerned with the existence of solutions to the following periodic problem

$$\begin{cases} -u''(t) = f(t, u(t)) & \text{in } [0, 2\pi], \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \end{cases} \tag{1.1}$$

in the case where the nonlinearity $f(t, s)$ interferes in a certain sense with the associated Fučik spectrum. We recall that this spectrum is defined as the set of $(\lambda_+, \lambda_-)$ in $\mathbb{R}^2$ for which the positively homogeneous problem

$$\begin{cases} -u''(t) = \lambda_+ u^+(t) - \lambda_- u^-(t) & \text{in } [0, 2\pi], \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi) \end{cases} \tag{1.2}$$

has a nontrivial solution; it is made of the union of the two lines $(\mathbb{R}, 0)$ and $(0, \mathbb{R})$ with a sequence of curves $C_k$, $k = 1, 2, \ldots$, where

$$C_k = \left\{(\lambda_+, \lambda_-) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ ; k\left(\frac{1}{\sqrt{\lambda_+}} + \frac{1}{\sqrt{\lambda_-}}\right) = 2\right\}$$

(cf. [1, 2]).

Several works have been devoted to these questions of existence and we refer for instance to [1–5]. Typically, in the study of nonresonance, one considers the limits

$$\gamma_\pm(t) = \liminf_{s \to \pm\infty} \frac{f(t, s)}{s}, \qquad \Gamma_\pm(t) = \limsup_{s \to \pm\infty} \frac{f(t,s)}{s}$$

and one assumes either

$$\begin{cases} 0 \le \gamma_+(t) \le \Gamma_+(t) \le Q_+ & \text{a.e.}, \\ 0 \le \gamma_-(t) \le \Gamma_-(t) \le Q_- & \text{a.e.}, \end{cases} \tag{1.3}$$

for one point $(Q_+, Q_-)$ in the first branch $C_1$, or

$$\begin{cases} q_+ \le \gamma_+(t) \le \Gamma_+(t) \le Q_+ & \text{a.e.}, \\ q_- \le \gamma_-(t) \le \Gamma_-(t) \le Q_- & \text{a.e.}, \end{cases} \tag{1.4}$$

for two points $(q_+, q_-)$ and $(Q_+, Q_-)$ in two consecutive branches $C_{k-1}$ and $C_k$. Solvability of (1.1) is then deduced from some additional assumption which, in particular, prevents $f(t, s)$ to be of the form $\lambda_+ s^+ - \lambda_- s^- + h(t)$, with $(\lambda_+, \lambda_-)$ in the Fučik spectrum.

Most of the existence results which have been established in that direction and which really involve the Fučik spectrum have been proved by means of degree theory. It is our purpose in this paper to use a variational approach. Besides providing a new insight, we are able to weaken some of the usual assumptions, by replacing conditions on $f$ by conditions on its primitive. We deal in this paper with the case where the nonlinearity $f(t, s)$ lies asymptotically between 0 and the first branch $C_1$. Our main abstract tool is a slight variant of the saddle point theorem of Rabinowitz, where the decomposition of the surrounding space is made according to cones instead of subspaces. A version of the Wirtinger inequality, where positive and negative parts play nonsymmetric roles, is of particular importance in our arguments.

Our existence theorem is stated in Section 2 and proved in Sections 3 and 4. In Section 5 we use the technique of Section 4 to give a variational characterization of the first branch $C_1$ of the Fučik spectrum. The Dirichlet problem can also be treated along these lines and we make some brief comments on that in Section 6.

## 2. STATEMENT OF THE THEOREM

Let $f: [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$ be a function satisfying the usual $L^2$ Caratheodory conditions. Given a point $(Q_+, Q_-)$ in the first branch $C_1$ of the Fučik spectrum, we will assume that the limits $\gamma_\pm(t)$ and $\Gamma_\pm(t)$ of $f(t, s)/s$ satisfy (1.3), with some uniformity with respect to $t$. This means precisely the following:

(f) for any $\varepsilon > 0$ there exists $a_\varepsilon \in L^2(0, 2\pi)$ such that for a.e.t.,

$$-\varepsilon s - a_\varepsilon(t) \le f(t, s) \le (Q_+ + \varepsilon)s + a_\varepsilon(t) \qquad \text{for } s \ge 0,$$

$$-\varepsilon s + a_\varepsilon(t) \ge f(t, s) \ge (Q_- + \varepsilon)s - a_\varepsilon(t) \qquad \text{for } s \le 0.$$

Writing $F(t, s) = \int_0^s f(t, r)\, dr$, the limits

$$\delta_\pm(t) = \liminf_{s \to \pm\infty} \frac{2F(t, s)}{s^2}, \qquad \Delta_\pm(t) = \limsup_{s \to \pm\infty} \frac{2F(t,s)}{s^2}$$

then also satisfy

$$\begin{cases} 0 \le \delta_+(t) \le \Delta_+(t) \le Q_+ & \text{a.e.,} \\ 0 \le \delta_-(t) \le \Delta_-(t) \le Q_- & \text{a.e.} \end{cases} \tag{2.1}$$

We will assume the following strict inequalities in (2.1):

(F) $\delta_+(t)$ and $\delta_-(t) > 0$ on subsets of positive measure, and $\Delta_+(t) < Q_+$ and $\Delta_-(t) < Q_-$ on a (common) subset of positive measure.

THEOREM 2.1. Assume (f) and (F) with $(Q_+, Q_-) \in C_1$. Then problem (1.1) admits at least one solution $u$ in $H^2(0, 2\pi)$.

Theorem 2.1 improves some results in [4, 5] where strict conditions are imposed on $\gamma_\pm$, $\Gamma_\pm$ instead of $\delta_\pm$, $\Delta_\pm$.

*Remark* 2.2. For an autonomous nonlinearity $f(t, s) = f(s) + h(t)$, with $h \in L^\infty(0, 2\pi)$, the conclusion of theorem 2.1 also follows from the main result in [6]. The proof in [6] is not variational and the strict condition on the primitive is exploited there through its equivalence with a positive density condition.

*Remark* 2.3. When $Q_+ = Q_- = 1$, i.e. when the usual spectrum is considered, the conclusion of theorem 2.1 can be derived by the method of [7] (which is variational).

## 3. PALAIS-SMALE CONDITION

We will work in the space $H^1_{2\pi}$ of functions $u$ in the Sobolev space $H^1(0, 2\pi)$ such that $u(0) = u(2\pi)$. Its norm is defined by

$$\|u\| = (\|u\|^2_{L^2} + \|u'\|^2_{L^2})^{1/2}.$$

The functional associated with our problem is

$$\Phi(u) = \tfrac{1}{2} \int_0^{2\pi} u'(t)^2 - \int_0^{2\pi} F(t, u(t)).$$

Since our assumptions imply that $f$ has linear growth, this functional is of class $C^1$ on $H^1_{2\pi}$. Its critical points are precisely the (weak) solutions of (1.1).

The following lemma will be used repeatedly. Its second part is a variant of a result in [4]. Here and below, we identify $[0, 2\pi]$ with the unit circle $S^1$.

LEMMA 3.1. Let $(Q_+, Q_-) \in C_1$ and let $m_+(t)$, $m_-(t)$ be $L^\infty$ functions such that

$$\begin{cases} 0 \le m_+(t) \le Q_+ & \text{a.e.,} \\ 0 \le m_-(t) \le Q_- & \text{a.e.} \end{cases} \tag{3.1}$$

Then any solution $u$ of

$$\begin{cases} -u'' = m_+(t)u^+ - m_-(t)u^- & \text{in } [0, 2\pi], \\ u(0) = u(2\pi), \qquad u'(0) = u'(2\pi) \end{cases} \tag{3.2}$$

is either constant or is strictly positive on an open interval (in $S^1$) of length $\pi/\sqrt{Q_+}$ and strictly negative on the complementary open interval of length $\pi/\sqrt{Q_-}$. Moreover, if, in addition to (3.1), $m_+(t) > 0$ and $m_-(t) > 0$ on subsets of positive measure, $m_+(t) < Q_+$ and $m_-(t) < Q_-$ on a (common) subset of positive measure, then (3.2) has only the trivial solution $u \equiv 0$.

*Proof.* Let $u$ be a solution of (3.2). Either (i) $u$ is of constant sign or (ii) $u$ changes sign. In case (i) assume, for instance, $u \ge 0$ on $[0, 2\pi]$. Integrating (3.2), we get

$$0 = \int_0^{2\pi} m_+(t)u(t)$$

and consequently, using (3.1),

$$m_+(t)u(t) = 0 \qquad \text{a.e. in } [0, 2\pi]. \tag{3.3}$$

Multiplying now (3.2) by $u$ and integrating, we get

$$\int_0^{2\pi} u'(t)^2 = \int_0^{2\pi} m_+(t)u(t)^2$$

where the right-hand side vanishes by (3.3). Consequently, $u$ is constant. In case (ii) let $]\alpha, \beta[$ be an interval in $S^1$ where $u$ is strictly positive, with $u(\alpha) = u(\beta) = 0$. Multiplying (3.2) by $u$ and integrating over $[\alpha, \beta]$, we have

$$\int_\alpha^\beta u'(t)^2 = \int_\alpha^\beta m_+(t)u^+(t)^2 \le Q_+ \int_\alpha^\beta u(t)^2. \tag{3.4}$$

It then follows from the Poincaré inequality

$$\int_\alpha^\beta u'(t)^2 \ge \frac{\pi^2}{(\beta - \alpha)^2} \int_\alpha^\beta u(t)^2 \tag{3.5}$$

that

$$\beta - \alpha \ge \pi/\sqrt{Q_+}.$$

Similarly, if $]\gamma, \delta[$ is an interval in $S^1$ where $u$ is strictly negative, with $u(\gamma) = u(\delta) = 0$, then

$$\delta - \gamma \ge \pi/\sqrt{Q_-}.$$

Combining these two inequalities with the definition of $C_1$, we obtain

$$\beta - \alpha = \pi/\sqrt{Q_+}, \qquad \delta - \gamma = \pi/\sqrt{Q_-}. \tag{3.6}$$

This completes the proof of the first part of the lemma.

Let us now turn to the second part of the lemma. Assume, by contradiction, that (3.2) has a nontrivial solution $u$. First we claim that $u$ must change sign. Indeed if $u$ is, say, $\ge 0$ on $[0, 2\pi]$, then, by the first part of the lemma, $u$ is constant; replacing in (3.2) and using $m_+(t) > 0$ on a set of positive measure, we deduce that this constant is zero, a contradiction. Using now the notations of the proof of the first part of the lemma, we deduce from (3.4) to (3.6) that

$$\int_\alpha^\beta (Q_+ - m_+(t))u^+(t)^2 = 0.$$

Similarly,

$$\int_\gamma^\delta (Q_- - m_-(t))u^-(t)^2 = 0.$$

Combining these relations with (3.6) and the assumption on $m_+(t)$ and $m_-(t)$ leads to a contradiction. ■

We are now in a position to prove the following proposition.

PROPOSITION 3.2. Assume (f) and (F) with $(Q_+, Q_-) \in C_1$. Then the functional $\Phi$ satisfies the (P.S.) condition on $H^1_{2\pi}$.

*Proof.* Let $u_n \in H^1_{2\pi}$ be a (P.S.) sequence, i.e.

$$|\Phi(u_n)| \le C, \tag{3.7}$$

$$|\Phi'(u_n)(v)| = \left| \int_0^{2\pi} u_n'(t)v'(t) - \int_0^{2\pi} f(t, u_n(t))v(t) \right| \le \varepsilon_n \|v\| \qquad \text{for } v \in H^1_{2\pi}, \tag{3.8}$$

where $C$ is a constant and $\varepsilon_n \to 0$. It clearly suffices to show that $u_n$ remains bounded in $H^1_{2\pi}$. Assume, by contradiction, that for a subsequence, $\|u_n\| \to +\infty$ (here and below, we keep the same index to denote subsequences). Let $v_n = u_n/\|u_n\|$ and take a subsequence such that $v_n \to v_0$ weakly in $H^1_{2\pi}$ and $v_n \to v_0$ uniformly on $[0, 2\pi]$.

We first claim that $\|v_0\| = 1$. To prove this, we consider $f(t, u_n(t))/\|u_n\|$ which, by the linear growth of $f$, remains bounded in $L^2$. Thus, for a subsequence, $f(t, u_n(t))/\|u_n\|$ converges weakly in $L^2$ to some $f_0 \in L^2$. We now divide (3.8) by $\|u_n\|$ and go to the limit to get

$$\int_0^{2\pi} v_0'(t)v'(t) - \int_0^{2\pi} f_0(t)v(t) = 0 \tag{3.9}$$

for all $v \in H^1_{2\pi}$. In particular, for $v = v_0$,

$$\int_0^{2\pi} v_0'(t)^2 = \int_0^{2\pi} f_0(t)v_0(t). \tag{3.10}$$

Using $\|u_n\| = 1$, we now deduce from (3.8) with $v = v_n/\|v_n\|$ that

$$\left| 1 - \int_0^{2\pi} v_n(t)^2 - \int_0^{2\pi} \frac{f(t, u_n(t))}{\|u_n\|} v_n(t) \right| \le \frac{\varepsilon_n}{\|u_n\|},$$

which gives

$$1 - \int_0^{2\pi} v_0(t)^2 = \int_0^{2\pi} f_0(t)v_0(t)\,\mathrm{d}t.$$

Comparing with (3.10) then yields the claim $\|v_0\| = 1$.

To proceed further in the proof of proposition 3.2, we need more information on $f_0$. By standard arguments based on assumption $(f)$, $f_0$ can be written as

$$f_0(t) = m_+(t)v_0^+(t) - m_-(t)v_0^-(t)$$

where the $L^\infty$ functions $m_+$ and $m_-$ satisfy

$$\begin{cases} 0 \le m_+(t) \le Q_+ & \text{a.e.,} \\ 0 \le m_-(t) \le Q_- & \text{a.e.} \end{cases} \tag{3.11}$$

(cf. e.g. [5, 8]). Consequently, by (3.9), $v_0$ is a (nontrivial) solution of

$$\begin{cases} -v_0'' = m_+(t)v_0^+ - m_-(t)v_0^- & \text{in } [0, 2\pi], \\ v_0(0) = v_0(2\pi), \qquad v_0'(0) = v_0'(2\pi), \end{cases} \tag{3.12}$$

which is a problem like (3.2). Observe also that without loss of generality we can assume

$$\begin{cases} m_+(t) = Q_+/2 & \text{on } \{t;\, v_0(t) < 0\}, \\ m_-(t) = Q_-/2 & \text{on } \{t;\, v_0(t) > 0\}. \end{cases} \tag{3.13}$$

We now distinguish three cases: (i) $m_+(t) \equiv 0$ or $m_-(t) \equiv 0$; (ii) $m_+(t) > 0$ and $m_-(t) > 0$ on subsets of positive measure and $\mathrm{meas}\{t;\, m_+(t) < Q_+ \text{ and } m_-(t) < Q_-\} = 0$; (iii) $m_+(t) > 0$ and $m_-(t) > 0$ on subsets of positive measure and $m_+(t) < Q_+$ and $m_-(t) < Q_-$ on a (common) subset of positive measure.

*Case* (i). Suppose, for instance, $m_-(t) \equiv 0$. Then, by (3.13), $v_0(t) \le 0$ a.e., and lemma 3.1 implies $v_0 \equiv$ cst. Call this constant $d(<0)$. We now use (3.7). Dividing (3.7) by $\|u_n\|^2$ and using $\|u_n\| = 1$, we obtain at the limit

$$1 - \int_0^{2\pi} v_0(t)^2 - \lim_{n\to\infty} 2 \int_0^{2\pi} \frac{F(t, u_n(t))}{\|u_n\|^2} = 0. \tag{3.14}$$

Since $v_0 \equiv$ cst and $\|v_0\| = 1$, $1 - \int_0^{2\pi} v_0(t)^2 = 0$, consequently, the last term in (3.14) must vanish. It then follows from Fatou's lemma that

$$0 = \lim_{n\to\infty} 2 \int_0^{2\pi} \frac{F(t, u_n(t))}{\|u_n\|^2} \ge 2 \int_0^{2\pi} \liminf_{n\to\infty} \frac{F(t, u_n(t))}{\|u_n\|^2} \ge \int_0^{2\pi} \delta_-(t) d^2$$

(since $u_n(t) = v_n(t)\|u_n\| \to -\infty$ a.e.). By assumption (F), $\delta_-(t) \ge 0$ a.e. and $\delta_-(t) > 0$ on a set of positive measure, which implies $d = 0$, i.e. $v_0(t) \equiv 0$, a contradiction.

*Case* (ii). In this case it follows from (3.13) that $m_+(t) = Q_+$ a.e. on $\{t; v_0(t) > 0\}$. Similarly, $m_-(t) = Q_-$ a.e. on $\{t; v_0(t) < 0\}$, and the equation in (3.12) thus reads

$$-v_0'' = Q_+ v_0^+ - Q_- v_0^- \qquad \text{in } [0, 2\pi]. \tag{3.15}$$

Integrating (3.15) immediately shows that $v_0$ must change sign. Multiplying (3.15) by $v_0$ and integrating gives

$$\int_0^{2\pi} v_0'(t)^2 = Q_+ \int_0^{2\pi} v_0^+(t)^2 + Q_- \int_0^{2\pi} v_0^-(t)^2. \tag{3.16}$$

We now argue as in case (i), using (3.7), to derive (3.14). Since $\|v_0\| = 1$, the sum of the first two terms of (3.14) is $\int_0^{2\pi} v_0'(t)^2\, dt$, which is given by (3.16). Replacing in (3.14) we obtain, using Fatou's lemma,

$$\begin{aligned} Q_+ \int_0^{2\pi} v_0^+(t)^2 + Q_- \int_0^{2\pi} v_0^-(t)^2 &\le \int_0^{2\pi} \limsup_{n\to\infty} \frac{2F(t, u_n(t))}{\|u_n\|^2} \\ &\le \int_0^{2\pi} \Delta_+(t) v_0^+(t)^2 + \int_0^{2\pi} \Delta_-(t) v_0^-(t)^2 \end{aligned}$$

(since $u_n(t) = v_n(t)\|u_n\| \to +(-\infty)$ where $v_0$ is $>0(<0)$). By assumption (F), $\Delta_\pm(t) \le Q_\pm$ a.e., and consequently,

$$\int_0^{2\pi} (Q_+ - \Delta_+(t)) v_0^+(t)^2 = 0,$$

$$\int_0^{2\pi} (Q_- - \Delta_-(t)) v_0^-(t)^2 = 0.$$

Moreover, by assumption (F), $\Delta_+(t) < Q_+$ and $\Delta_-(t) < Q_-$ on a set of positive measure. Using the information and the first part of lemma 3.1, we derive a contradiction from the above relations.

*Case* (iii). Here the second part of lemma 3.1 immediately implies that the solution $v_0$ of (3.12) is $\equiv 0$, a contradiction. ■

## 4. SADDLE SHAPE AND LINKING PROPERTY

PROPOSITION 4.1. Assume (f) and (F) with $(Q_+, Q_-) \in C_1$. Then there exist a subspace $V \subset H^1_{2\pi}$ and a cone $M \subset H^1_{2\pi}$ such that

$$\Phi(u) \to -\infty \qquad \text{as } \|u\| \to +\infty,\ u \in V, \tag{4.1}$$

$$\Phi(u) \to +\infty \qquad \text{as } \|u\| \to +\infty,\ u \in M. \tag{4.2}$$

Moreover, defining for $R > 0$

$$\Gamma_R = \{h \in C(\bar{B}(0, R) \cap V; H^1_{2\pi});\ h(u) = u \text{ for all } u \in \partial B(0, R) \cap V\},$$

we have that, for any $h \in \Gamma_R$,

$$h(B(0, R) \cap V) \text{ meets } M. \tag{4.3}$$

Once this proposition is proved, the corollary below follows from a standard deformation technique, as in the proof of the saddle point theorem of Rabinowitz [9]. This corollary clearly implies theorem 2.1.

COROLLARY 4.2. For $R$ sufficiently large,

$$c = \inf_{h \in \Gamma_R} \sup_{u \in \bar{B}(0,R) \cap V} \Phi(h(u))$$

is a critical value of $\Phi$.

The following variant of Wirtinger inequality will be used in the proof of proposition 4.1 and in Section 5.

LEMMA 4.3. Fix $(\lambda_+, \lambda_-) \in C_1$. Then

$$\int_0^{2\pi} u'(t)^2 \geq \lambda_+ \int_0^{2\pi} u^+(t)^2 + \lambda_- \int_0^{2\pi} u^-(t)^2 \tag{4.4}$$

for all $u \in H^1_{2\pi}$ satisfying

$$\lambda_+ \int_0^{2\pi} u^+(t) - \lambda_- \int_0^{2\pi} u^-(t) = 0. \tag{4.5}$$

Moreover, if equality holds in (4.4) for one $u \in H^1_{2\pi}$ satisfying (4.5), then $u$ is a solution of

$$\begin{cases} -u''(t) = \lambda_+ u^+(t) - \lambda_- u^-(t) & \text{in } [0, 2\pi], \\ u(0) = u(2\pi), \qquad u'(0) = u'(2\pi). \end{cases} \tag{4.6}$$

The classical Wirtinger inequality (cf. e.g. [10]) corresponds to the case $\lambda_+ = \lambda_- = 1$.

*Remark* 4.4. Integrations easily show that conversely any solution $u$ of (4.6) verifies (4.5) and yields equality in (4.4).

The subdifferential of the convex functional

$$\varphi(u) = \int_0^{2\pi} u^+(t)$$

will be used in the proof of lemma 4.3. We recall that $u^* \in \partial\varphi(u)$ if, by definition,

$$\varphi(v) \geq \varphi(u) + \langle u^*, v - u \rangle \qquad \text{for all } v \in H^1_{2\pi},$$

where $\langle , \rangle$ denotes the pairing between $H^1_{2\pi}$ and its dual. Taking in particular $v = u - 1$, we have

$$\langle u^*, 1 \rangle \geq \varphi(u) - \varphi(u - 1) \geq 0, \tag{4.7}$$

an inequality which will be used later.

*Proof of lemma* 4.3. We will show that

$$\inf \left\{ \begin{array}{ll} \displaystyle\int_0^{2\pi} u'(t)^2;\ u \in H^1_{2\pi} & \text{with } \lambda_+ \displaystyle\int_0^{2\pi} u^+(t)^2 + \lambda_- \int_0^{2\pi} u^-(t)^2 = 1 \\ & \text{and } \lambda_+ \displaystyle\int_0^{2\pi} u^+(t) - \lambda_- \int_0^{2\pi} u^-(t) = 0 \end{array} \right\} = 1, \tag{4.8}$$

which clearly implies (4.4). Assuming, for instance, $\lambda_+ \geq \lambda_-$, it will be convenient to write the second constraint in (4.8) in the following convex form:

$$\lambda_- \int_0^{2\pi} u(t) + (\lambda_+ - \lambda_-) \int_0^{2\pi} u^+(t) = 0.$$

Call $\delta$ the value of the infimum in (4.8). It follows from remark 4.4 above that $\delta \leq 1$. To prove that $\delta = 1$, we first observe, by a simple compactness argument, that the infimum in (4.8) is $>0$ and achieved at some function $u (\not\equiv 0)$. We then apply the Lagrange multiplier rule, as given, e.g. in [11], to derive the existence of $\alpha \geq 0, \beta, \gamma$, not all zero, and $u^* \in \partial\varphi(u)$ such that

$$\alpha \int_0^{2\pi} u'(t)v'(t) + \beta \left[ \lambda_+ \int_0^{2\pi} u^+(t)v(t) - \lambda_- \int_0^{2\pi} u^-(t)v(t) \right] + \gamma \left[ \lambda_- \int_0^{2\pi} v(t) + (\lambda_+ - \lambda_-)\langle u^*, v \rangle \right] = 0 \tag{4.9}$$

for all $v \in H^1_{2\pi}$.

Putting $v \equiv 1$ in (4.9) and using the second constraint in (4.8), we get

$$\gamma[2\pi\lambda_- + (\lambda_+ - \lambda_-)\langle u^*, 1 \rangle] = 0.$$

Since, by (4.7), the bracket above is $>0$, we deduce $\gamma = 0$. We also observe that $\alpha \neq 0$ because otherwise $\alpha = 0$, $\beta \neq 0$, and a contradiction follows by taking $v = u$ in (4.9). Assuming, without loss of generality, $\alpha = 1$ and taking $v = u$ in (4.9), we obtain, using the first constraint in (4.8),

$$\int_0^{2\pi} u'(t)^2 + \beta = 0,$$

i.e. $-\beta = \delta$, the infimum value in (4.8). Rewriting (4.9) now shows that $u$ is a nontrivial (weak) solution of

$$\begin{cases} -u''(t) = \delta(\lambda_+ u^+(t) - \lambda_- u^-(t)) & \text{in } [0, 2\pi], \\ u(0) = u(2\pi), \qquad u'(0) = u'(2\pi). \end{cases} \tag{4.10}$$

This implies $\delta \geq 1$ since $(\lambda_+, \lambda_-) \in C_1$. We thus conclude $\delta = 1$.

We now turn to the second part of lemma 4.3. If $u \not\equiv 0$ satisfies (4.5) and yields equality in (4.4), then its normalization

$$w = u\left[\lambda_+ \int_0^{2\pi} u^+(t)^2 + \lambda_- \int_0^{2\pi} u^-(t)^2\right]^{-1/2}$$

realizes the infimum in (4.8). The arguments above then show that $w$ solves (4.10) with $\delta = 1$, i.e. (4.6). ∎

Before starting the proof of proposition 4.1, we observe that our assumptions (f) and (F) imply the following condition on $F(t, s)$:

($F_1$) there exist $\theta_\pm$ and $\Theta_\pm$ in $L^\infty(0, 2\pi)$, with

$$0 \leq \theta_+(t) \leq \Theta_+(t) \leq Q_+ \quad \text{a.e.},$$

$$0 \leq \theta_-(t) \leq \Theta_-(t) \leq Q_- \quad \text{a.e.},$$

$\theta_+(t) > 0$ and $\theta_-(t) > 0$ on subsets of positive measure, $\Theta_+(t) < Q_+$ and $\Theta_-(t) < Q_-$ on a (common) subset of positive measure, such that the following holds: for any $\varepsilon > 0$, there exists $b_\varepsilon \in L^1(0, 2\pi)$ such that for a.e.t.,

$$(\theta_+(t) - \varepsilon)\frac{s^2}{2} - b_\varepsilon(t) \leq F(t, s) \leq (\Theta_+(t) + \varepsilon)\frac{s^2}{2} + b_\varepsilon(t) \qquad \text{for } s \geq 0,$$

$$(\theta_-(t) - \varepsilon)\frac{s^2}{2} - b_\varepsilon(t) \leq F(t, s) \leq (\Theta_-(t) + \varepsilon)\frac{s^2}{2} + b_\varepsilon(t) \qquad \text{for } s \leq 0.$$

This can be easily verified as in remark 2.2 of [5], using Egorov's theorem. Actually, (f) and (F) are equivalent to (f) and ($F_1$).

*Proof of proposition* 4.1. Let $V$ be the subspace of constant functions. For $u$ in $V$ with, say, $u(t) \equiv c > 0$, we deduce from ($F_1$) that

$$\begin{aligned} \Phi(u) &= -\int_0^{2\pi} F(t, c)\,\mathrm{d}t \\ &\leq \frac{-\|u\|^2}{4\pi}\int_0^{2\pi} (\theta_+(t) - \varepsilon) + \int_0^{2\pi} b_\varepsilon(t). \end{aligned}$$

Choosing

$$\varepsilon < \frac{1}{2\pi}\int_0^{2\pi} \theta_+(t),$$

and arguing in a similar way for $c < 0$, we obtain (4.1).

We now introduce the cone

$$M = \left\{ u \in H^1_{2\pi};\ Q_+ \int_0^{2\pi} u^+(t) - Q_- \int_0^{2\pi} u^-(t) = 0 \right\}.$$

To prove that $\Phi$ is coercive on $M$, i.e. (4.2), we first observe that, by $(F_1)$,

$$2\Phi(u) \geq \int_0^{2\pi} u'(t)^2 - \int_0^{2\pi} (\Theta_+(t) + \varepsilon) u^+(t)^2 - \int_0^{2\pi} (\Theta_-(t) + \varepsilon) u^-(t)^2 - 4 \int_0^{2\pi} b_\varepsilon(t).$$

Consequently, (4.2) will follow by a suitable choice of $\varepsilon$ if we show that the functional $\Psi(u)$, defined by

$$\Psi(u) = \int_0^{2\pi} u'(t)^2 - \int_0^{2\pi} \Theta_+(t) u^+(t)^2 - \int_0^{2\pi} \Theta_-(t) u^-(t)^2,$$

satisfies

$$\Psi(u) \geq \eta \|u\|^2 \tag{4.11}$$

for some $\eta > 0$ and all $u \in M$.

We first observe that, for $u \in M$,

$$\Psi(u) \geq \int_0^{2\pi} (Q_+ - \Theta_+(t)) u^+(t)^2 + \int_0^{2\pi} (Q_- - \Theta_-(t)) u^-(t)^2 \geq 0. \tag{4.12}$$

Indeed, the first inequality follows from lemma 4.3, while the second follows from $(F_1)$.

We now start the proof of (4.11). Assume, by contradiction, that (4.11) does not hold. Then there exists a sequence $w_n \in M$ such that $\|w_n\| = 1$ and $\Psi(w_n) \to 0$. Going to a subsequence, we have $w_n \to w$ weakly in $H^1_{2\pi}$ and $w_n \to w$ uniformly on $[0, 2\pi]$. We first observe that $w \not\equiv 0$. Indeed, it follows from $\Psi(w_n) \to 0$ and $\|w_n\| = 1$ that

$$\begin{aligned} 0 &= \lim \left[ \int_0^{2\pi} w_n'(t)^2 - \int_0^{2\pi} \Theta_+(t) w_n^+(t)^2 - \int_0^{2\pi} \Theta_-(t) w_n^-(t)^2 \right] \\ &= 1 - \int_0^{2\pi} w(t)^2 - \int_0^{2\pi} \Theta_+(t) w^+(t)^2 - \int_0^{2\pi} \Theta_-(t) w^-(t)^2, \end{aligned}$$

which clearly implies $w \not\equiv 0$. We also have that $w \in M$ (since $M$ is weakly closed) and that

$$\Psi(w) = 0. \tag{4.13}$$

Indeed, by (4.12), $\Psi(w) \geq 0$ since $w \in M$; moreover, by weak lower semicontinuity,

$$\Psi(w) \leq \liminf \Psi(w_n) = 0,$$

and (4.13) follows. Combining (4.12) and (4.13) gives

$$\int_0^{2\pi} Q_+ w^+(t)^2 = \int_0^{2\pi} \Theta_+(t) w^+(t)^2, \tag{4.14}$$

$$\int_0^{2\pi} Q_- w^-(t)^2 = \int_0^{2\pi} \Theta_-(t) w^-(t)^2. \tag{4.15}$$

It now follows from (4.13) to (4.15) that $w$ yields equality in (4.4). Consequently, by the second part of lemma 4.3, $w$ solves

$$\begin{cases} -w'' = Q_+ w^+(t) - Q_- w^-(t) & \text{in } [0, 2\pi], \\ w(0) = w(2\pi), \qquad w'(0) = w'(2\pi). \end{cases} \tag{4.16}$$

The function $w$ is thus $>0$ on an open interval of length $\pi/\sqrt{Q_+}$ and $<0$ on the complementary interval of length $\pi/\sqrt{Q_-}$. A contradiction then follows from (4.14), (4.15) and ($F_1$). This completes the proof of (4.11) and so of (4.2).

To conclude the proof of proposition 4.1, it remains to verify the linking property (4.3). Let $h \in \Gamma_R$. Consider the functional

$$\chi(u) = Q_+ \int_0^{2\pi} u^+(t) - Q_- \int_0^{2\pi} u^-(t)$$

and the function $\chi(h(u))$. For $u(t) \equiv -R$, we have $\chi(h(-R)) < 0$, and for $u(t) \equiv +R$, we have $\chi(h(+R)) > 0$. By the intermediate value theorem, there exists $u \in B(0, R) \cap V$ such that $\chi(h(u)) = 0$, i.e. $h(u) \in M$. This proves (4.3). ■

*Remark* 4.5. The conclusion of theorem 2.1 holds if the strict condition in ($F_1$) "$\Theta_+(t) < Q_+$ and $\Theta_-(t) < Q_-$ on a set of positive measure" is replaced by "$\Theta_+(t) < Q_+$ on a set which intersects (with positive measure) any interval in $S^1$ of length $\pi/\sqrt{Q_+}$". A strict condition at $Q_-$ only could, of course, also be considered. The proof is similar to the above. The result obtained in this way improves [5, theorem 2.3].

*Remark* 4.6. Another cone could be used in proposition 4.1 instead of $M$:

$$M' = \left\{ u \in H^1_{2\pi};\ \int_0^{2\pi} u^+(t)^2 = \left(\frac{Q_-}{Q_+}\right)^{3/2} \int_0^{2\pi} u^-(t)^2 \right\}.$$

The proof of the coercivity of $\Phi$ over $M'$ is, however, rather different from the above. It involves the following variant of the Poincaré inequality:

$$\int_0^{2\pi} u^{+\prime}(t)^2 \geq \frac{\pi^2}{|u > 0|} \int_0^{2\pi} u^+(t)^2$$

for any $u \in H^1_{2\pi}$ which vanishes somewhere and is positive somewhere. One uses this inequality (and a similar one for $u^-$) to show that, if $u \in M'$, then

$$\int_0^{2\pi} u'(t)^2 \geq \left[ \frac{\pi^2}{a^2} + \frac{\pi^2}{b^2} \left(\frac{Q_+}{Q_-}\right)^{3/2} \right] \int_0^{2\pi} u^+(t)^2 \tag{4.17}$$

for some $(a, b) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+$ with $a \geq |u > 0|$, $b \geq |u < 0|$ and $a + b = 2\pi$. Studying the minimum value of the bracket in (4.17), one eventually reaches (4.12), and the argument can be pursued with little change.

*Remark* 4.7. Any solution of (4.16) with $(Q_+, Q_-) \in C_k$ for some $k$ belongs to $M'$, as was observed in [12].

## 5. VARIATIONAL CHARACTERIZATION OF $C_1$

In this section we use a set similar to the set $M$ to derive a variational characterization of $C_1$. Taking a line $y = rx$ in $\mathbb{R}^2$, we will obtain its intersection with $C_1$ through minimizations of the Dirichlet integral.

Consider for a fixed $r > 0$ the set

$$M_r = \left\{ u \in H^1_{2\pi};\ \int_0^{2\pi} u^+(t) = r \int_0^{2\pi} u^-(t) \right\}$$

and define $(\lambda_+, \lambda_-) \in \mathbb{R}^+ \times \mathbb{R}^+$ by

$$\lambda_+ = \inf\left\{ \int_0^{2\pi} u'(t)^2 \Big/ \left( \int_0^{2\pi} u^+(t)^2 + r \int_0^{2\pi} u^-(t)^2 \right); u \in M_r, u \not\equiv 0 \right\}, \tag{5.1}$$

$$\lambda_- = \inf\left\{ \int_0^{2\pi} u'(t)^2 \Big/ \left( r^{-1} \int_0^{2\pi} u^+(t)^2 + \int_0^{2\pi} u^-(t)^2 \right); u \in M_r, u \not\equiv 0 \right\}. \tag{5.2}$$

It is clear that $\lambda_- = r\lambda_+$ and that a function $u$ achieves the infimum in (5.1) if and only if $u$ achieves the infimum in (5.2).

PROPOSITION 5.1. The point $(\lambda_+, \lambda_-)$ belongs to $C_1$. Moreover, the infima in (5.1), (5.2) are achieved precisely by the nontrivial solutions of

$$\begin{cases} -u''(t) = \lambda_+ u^+(t) - \lambda_- u^-(t) & \text{in } [0, 2\pi], \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi). \end{cases} \tag{5.3}$$

*Proof.* We clearly have

$$\lambda_+ = \inf\left\{ \int_0^{2\pi} u'(t)^2; u \in M_r \text{ and } \int_0^{2\pi} u^+(t)^2 + r \int_0^{2\pi} u^-(t)^2 = 1 \right\}, \tag{5.4}$$

which is an infimum similar to the one occuring in the proof of lemma 4.3. We then proceed exactly as in the proof of that lemma to obtain that infimum (5.4) is achieved and that if a function $u$ achieves it, then $u$ solves

$$\begin{cases} -u''(t) = \lambda_+(u^+ - ru^-) & \text{in } [0, 2\pi], \\ u(0) = u(2\pi), \quad u'(0) = u'(2\pi). \end{cases}$$

It follows that $(\lambda_+, \lambda_+ r)$ belongs to $C_k$ for some $k \geq 1$. To conclude that $k = 1$, we observe that if $(\mu_+, \mu_-) \in C_l$ with $\mu_- = r\mu_+$, then any solution $v$ of

$$\begin{cases} -v''(t) = \mu_+ v^+(t) - \mu_- v^-(t) & \text{in } [0, 2\pi], \\ v(0) = v(2\pi), \quad v'(0) = v'(2\pi) \end{cases}$$

is, after normalization, admissible in infimum (5.4). One obtains in this way $\lambda_+ \leq \mu_+$ for any $\mu_+$ as above. This implies $(\lambda_+, \lambda_+ r) \in C_1$, i.e. $(\lambda_+, \lambda_-) \in C_1$.

In the process of the above argument, we have shown that any function which achieves the infimum (5.1) solves (5.3). The converse implication is easily verified (cf. remark 4.4). ■

*Remark* 5.2. While this paper was being completed, we learnt of a work in progress by Ramos [13] where formulas similar to (5.1) and (5.2) are considered. Other formulas for $C_1$ can be found in [12, 14]. A cone similar to the cone $M'$ of remark 4.6 could also be used.

## 6. DIRICHLET PROBLEM

The approach of Sections 2–4 can easily be adapted to deal with the Dirichlet problem:

$$\begin{cases} -u''(t) = f(t, u(t)) & \text{in } [0, \pi], \\ u(0) = u(\pi) = 0. \end{cases} \tag{6.1}$$

Given a point $(Q_+, Q_-)$ in the first branch of the associated Fučik spectrum (i.e. here $Q_+, Q_- > 0$ and $1/\sqrt{Q_+} + 1/\sqrt{Q_-} = 1$), one assumes

$$\lambda_1 \le \gamma_+(t) \le \Gamma_+(t) \le Q_+,$$

$$\lambda_1 \le \gamma_-(t) \le \Gamma_-(t) \le Q_-,$$

with some uniformity with respect to $t$. Here $\lambda_1$ denotes the principal eigenvalue of $-u'' = \lambda u$ in $[0, \pi]$, $u(0) = u(\pi) = 0$, and $\varphi_1$ will denote a corresponding positive eigenfunction. One also assumes $\delta_+(t)$ and $\delta_-(t) > \lambda_1$ on subsets of positive measure, and $\Delta_+(t) < Q_+$ and $\Delta_-(t) < Q_-$ on a (common) subset of positive measure. Problem (6.1) is then solvable.

The basic inequality is now the following lemma.

Lemma 6.1. Fix $(\lambda_+, \lambda_-)$ in the first branch of the Fučik spectrum. Then

$$\int_0^\pi u'(t)^2 \ge \lambda_+ \int_0^\pi u^+(t)^2 + \lambda_- \int_0^\pi u^-(t)^2 \tag{6.2}$$

for all $u \in H_0^1(0, \pi)$ satisfying

$$(\lambda_+ - \lambda_1) \int_0^\pi u^+(t)\varphi_1(t) - (\lambda_- - \lambda_1) \int_0^\pi u^-(t)\varphi_1(t) = 0. \tag{6.3}$$

Moreover, if equality holds in (6.2) for one function $u \in H_0^1(0, \pi)$ satisfying (6.3), then $u$ solves

$$\begin{cases} -u''(t) = \lambda_+ u^+(t) - \lambda_- u^-(t) & \text{in } [0, \pi], \\ u(0) = u(\pi) = 0. \end{cases}$$

The proof of lemma 6.1 is obtained by adapting that of lemma 4.3 to the study of the following infimum problem:

$$\inf\left\{\int_0^\pi u'(t)^2 - \lambda_1 u(t)^2;\ u \in H_0^1(0, \pi) \text{ with } (\lambda_+ - \lambda_1) \int_0^\pi u^+(t)^2 + (\lambda_- - \lambda_1) \int_0^\pi u^-(t)^2 = 1\right.$$

$$\left.\text{and } (\lambda_+ - \lambda_1) \int_0^\pi u^+(t)\varphi_1(t) - (\lambda_- - \lambda_1) \int_0^\pi u^-(t)\varphi_1(t) = 0\right\}.$$

A variational characterization of the first branch of the associated Fučik spectrum can also be derived, as in Section 5. Defining, for $r > 0$,

$$N_r = \left\{ u \in H_0^1(0, \pi);\ \int_0^\pi u^+(t)\varphi_1(t) - r\int_0^\pi u^-(t)\varphi_1(t) = 0 \right\},$$

$$\lambda_+ = \lambda_1 + \inf\left\{ \left( \int_0^\pi (u'(t)^2 - \lambda_1 u(t)^2) \right) \Bigg/ \left( \int_0^\pi u^+(t)^2 + r\int_0^\pi u^-(t)^2 \right);\ u \in N_r,\ u \not\equiv 0 \right\},$$

$$\lambda_- = \lambda_1 + \inf\left\{ \left( \int_0^\pi (u'(t)^2 - \lambda_1 u(t)^2) \right) \Bigg/ \left( r^{-1}\int_0^\pi u^+(t)^2 + \int_0^\pi u^-(t)^2 \right);\ u \in N_r,\ u \not\equiv 0 \right\},$$

one obtains that $(\lambda_+, \lambda_-)$ belongs to that first branch.

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