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An Initial-Boundary-Value Problem for a Certain Density-Dependent Diffusion System

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1. Introduction

In this paper, we consider the quasilinear parabolic system

$$Q(a, \ldots, e, W, T)\quad \begin{aligned} &u_t = \Delta_x\{(c_1+d_1 v)u\} + (e_1 - a_1 v - b_1 u)u \\ &v_t = \Delta_x\{(c_2+d_2 u)v\} + (e_2 - a_2 u - b_2 v)v \quad \text{in } \bar{\Omega}\times[0, T) \\ &\frac{\partial u}{\partial \nu}(x, s) = 0 = \frac{\partial v}{\partial \nu}(x, s) \quad \text{on } \partial\Omega\times[0, T) \\ &u(x, 0) = W_1(x), \qquad v(x, 0) = W_2(x) \quad \text{on } \bar{\Omega}, \end{aligned}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ and outward unit normal ν.

Furthermore, T belongs to $(0, \infty]$; the coefficients $a = (a_1, a_2), \ldots, e = (e_1, e_2)$ are constants from $[0, \infty)^2$, with $c_1, c_2 > 0$, and the initial data $W = (W_1, W_2)$ consist of functions $W_1, W_2 \in C^0(\bar{\Omega})$ with $W_1, W_2 \geqq 0$.

By a solution of $Q(a, \ldots, e, W, T)$, we mean a pair (u, v) of continuous functions from $\bar{\Omega}\times[0, T)$ into $\mathbb{R}$ such that the derivatives appearing in $Q(a, \ldots, e, W, T)$ exist, and $Q(a, \ldots, e, W, T)$ is satisfied pointwise. A solution to $Q(a, \ldots, e, W, \infty)$ will be called global solution.

The system $Q(a, \ldots, e, W, T)$ arises in ecology as a model of two competing species with self- and cross-population pressures, where u, v denotes the population density of the two species (see [4]).

For the case of space dimension $N = 1$, Kim [5] proved local existence of solutions to $Q(a, \ldots, e, W, T)$; i.e. the existence of solutions for some $T \in (0, \infty)$, T possibly small. Moreover, Kim could show global solvability under the assumptions $N = 1$, $c_1 = c_2$.

In [7, 8] Mimura et al. announced a result on global solutions under the assumptions $N = 1$, $d_2 = 0$.

Amann [1] considers more general quasilinear parabolic systems. In order to apply his results to the system $Q(a, \ldots, e, W, T)$, one must either assume that $d_i = 0$ for at least one index $i \in \{1, 2\}$; then $Q(a, \ldots, e, W, T)$ is of the type "upper-block-triangle", as defined in [1]. Or one has to make sure that the initial data do not violate a certain ellipticity condition in [1]; this implies an assumption of the type

$$\min\{c_1, c_2\} - \max\{|d_1 W_1|_0, |d_2 W_2|_0\} > 0.$$

In either case, it follows from [1] that solutions to $Q(a, \ldots, e, W, T)$ exist *locally*. To obtain global existence results from [1], it must first be proved that $u(\cdot, t)$, $v(\cdot, t)$ are a-priori-bounded, uniformly in t, in the norm of the Sobolev-space $W_p^\sigma(\Omega)$, for some $p > n$ and $\sigma \in \left(1 + \frac{n}{p}, 2\right)$. Here we could only find a single a priori estimate for solutions (u, v) to $Q(a, \ldots, b, W, T)$, namely: $u, v \geqq 0$ (see Theorem 4.4).

In spite of that, we shall show the existence of a uniquely determined global solution, for a class of coefficients and initial data which may be characterized as follows:

Take $\delta \in (0, 1)$ and $m, M \in R$ with $0 < m \leqq M$. Then there is a number $E > 0$ such that $Q(a, \ldots, e, W, \infty)$ is solvable for $a_i \in [0, M]$, $b_i, c_i \in [m, M]$, $d_i \in [0, E]$, $e_i \in [0, mM/2]$, and for nonnegative initial data W_i with $|W_i|_{2,\delta} \leqq M$. This result is true for any space dimension N. The number E depends on Ω, δ, m, M and is smaller than m/M.

In order to prove this existence result, we consider a certain semilinear parabolic equation. With the help of Schauder-type estimates for linear parabolic equations of second order, we derive a priori bounds for this semilinear equation.

Then, using Schauder's fixed point theorem, we arrive at solutions to the system $Q(a, \ldots, e, W, T)$, for any $T \in (0, \infty)$. Now a uniqueness argument yields solutions to $Q(a, \ldots, e, W, \infty)$.

2. Notations

Let n be an integer. For $x \in \mathbb{R}^n$, $\sigma > 0$, set $K_\sigma(x) := \{y \in \mathbb{R}^n : |y - x| < \sigma\}$, where $|\ |$ denotes the Euclidean norm in $\mathbb{R}^n$. For $A \subset \mathbb{R}^n$, $\bar{A}$ denotes the closure of A and ∂A the boundary of A, in the usual topology of $\mathbb{R}^n$. If f is a function from A into $\mathbb{R}$, then $|f|_0$ means the supremum norm of f. Furthermore, we set for $\alpha \in (0, 1)$:

$$|f|_{0,\alpha} := |f|_0 + \sup\left\{\frac{|f(x) - f(x')|}{|x - x'|^\alpha} : x, x' \in A,\ x \neq x'\right\}.$$

Let B be an open subset of $\mathbb{R}^n$, f a function from B into $\mathbb{R}$, and $k \in \{1, \ldots, n\}$. If $\frac{\partial}{\partial x_k} f(x)$ exists for any $x \in B$, then $D_k f$ denotes the corresponding derivative, as a function from B into $\mathbb{R}$. For $i \in \mathbb{N}$ and $k_1, \ldots, k_i \in \{1, \ldots, n\}$, $D_{k_1} \ldots D_{k_i} f$ is defined by iteration. If g is a function from $\bar{B}$ into $\mathbb{R}$ such that $D_{k_1} \ldots D_{k_i} (g|B)$ exists and can be extended to $\bar{B}$ continuously, then we denote this extension by $D_{k_1} \ldots D_{k_i} g$. The function spaces $C^m(B)$, $C^m(\bar{B})$ $(m \in \mathbb{N} \cup \{0\} \cup \{\infty\})$ are defined in the usual way. If $f\in C^2(\bar{B})$, $\alpha\in(0,1)$, we set

$$|f|_2:=|f|_0+\sum_{k=1}^{N}|D_kf|_0+\sum_{k,m=1}^{N}|D_kD_mf|_0,$$

$$|f|_{2,\alpha}:=|f|_{0,\alpha}+\sum_{k=1}^{N}|D_kf|_{0,\alpha}+\sum_{k,m=1}^{N}|D_kD_mf|_{0,\alpha}.$$

$C_{2,\alpha}(\bar{B})$ denotes the set $\{f\in C^2(\bar{B}): |f|_{2,\alpha}<\infty\}$.

In the following, we take the integer N as fixed. If $E\subset\mathbb{R}^{N+1}$, $\alpha\in(0,1)$, and u a function which takes E into $\mathbb{R}$, we define

$$|u|_\alpha:=|u|_0+\sup\left\{\frac{|u(x,s)-u(x',s')|}{(|x-x'|^2+|s-s'|)^{\alpha/2}}: (x,s),(x',s')\in E, (x,s)\neq(x',s')\right\},$$

$$C_\alpha(E):=\{u: E\to\mathbb{R}: |u|_\alpha<\infty\}.$$

Let U be an open subset of $\mathbb{R}^{N+1}$. If $v: U\to\mathbb{R}$ or $v: \bar{U}\to\mathbb{R}$ is a function such that $D_1v,\dots,D_Nv$ exist, we set $\nabla v:=(D_1v,\dots,D_Nv)$. Furthermore, $\Delta v:=\sum_{i=1}^{N}D_iD_iv$, provided the derivatives D_iD_iv exist for $1\leqq i\leqq N$. Note that the derivative $D_{N+1}v$ does not enter into the definition of ∇v, Δv. $C^{2,1}(U)$ denotes the set of functions v from U into $\mathbb{R}$ such that D_kv, D_mD_kv $(1\leqq k,m\leqq N)$, and $D_{N+1}v$ exist and are continuous. The set $C^{2,1}(\bar{U})$ consists of all functions $u: \bar{U}\to R$ such that $u|U\in C^{2,1}(U)$, and the derivatives $D_k(u|U)$, $D_mD_k(u|U)$ $(1\leqq k,m\leqq N)$, $D_{N+1}(u|U)$ may be extended to $\bar{U}$ continuously.

Take $\alpha\in(0,1)$. If $v\in C^{2,1}(U)$ or $v\in C^{2,1}(\bar{U})$, we define

$$|v|_{2+\alpha}:=|v|_\alpha+\sum_{i=1}^{N}|D_iv|_\alpha+\sum_{i,j=1}^{N}|D_iD_jv|_\alpha+|D_{N+1}v|_\alpha.$$

Finally we set

$$C_{2+\alpha}(\bar{U}):=\{u\in C^0(\bar{U}): u|U\in C^{2,1}(U), |u|U|_{2+\alpha}<\infty\}.$$

Note that $C_{2+\alpha}(\bar{U})\subset C^{2,1}(\bar{U})$. Furthermore, $C_{2+\alpha}(\bar{U})$ with the norm $|\ |_{2+\alpha}$ is a Banach space. For $\alpha,\beta\in(0,1)$ with $\alpha<\beta$, and for U bounded, the identity map $u\to u$ from $C_{2+\beta}(\bar{U})$ into $C_{2+\alpha}(\bar{U})$ is compact (see [2] Theorem 7.1).

In the following, we take $\Omega\subset\mathbb{R}^N$ as fixed. We assume that Ω is a bounded domain with C^3-boundary; ν denotes the outward unit normal to Ω. We set $Z_T:=\Omega\times(0,T)$ for $T\in(0,\infty]$. Furthermore, $S_T:=\partial\Omega\times[0,T]$ for $T\in(0,\infty)$, $S_\infty:=\partial\Omega\times[0,\infty)$.

For $f\in C^1(\bar{\Omega})$, $x\in\partial\Omega$, we set

$$\frac{\partial f}{\partial\nu}(x):=\sum_{i=1}^{N}\nu_i(x)D_if(x).$$

3. Auxiliary Results

Theorem 3.1. *There is a function* K_Ω: $(0, 1) \times (0, \infty)^2 \to (0, \infty)$ *with the following property*

(1) *Let* α *belong to* $(0, 1)$, $K_1 \geqq K_2 > 0$, $T \in [1, \infty)$, $a, b_i, c, f \in C_\alpha(\bar{Z}_T)$, $\phi \in C_{2,\alpha}(\bar{\Omega})$, $u \in C^{2,1}(\bar{Z}_T)$, with $|a|_\alpha \leqq K_1$, $|b_i|_0 \leqq K_1$ $(1 \leqq i \leqq N)$, $|c|_0 \leqq K_1$, $a(x, s) \geqq K_2$ $((x, s) \in \bar{Z}_T)$,

$$a \Delta u + \sum_{i,j=1}^{N} b_i D_i u + cu - D_{N+1} u = f,$$

$$\frac{\partial u}{\partial \nu}(x, s) = 0 \qquad ((x, s) \in S_T), \qquad u(x, 0) = \phi(x) \qquad (x \in \bar{\Omega}).$$

Then

$$|u|_\alpha \leqq K_\Omega(\alpha, K_1, K_2)(|f|_0 + |\phi|_2 + |u|_0).$$

Furthermore, there is a function L_Ω: $(0, 1) \times (0, \infty)^2 \to (0, \infty)$ *with this property:*

(2) *Let* $\alpha, K_1, K_2, T, a, b_i, c, f, \phi, u$ *be given as in* (1), *but with the assumption* $|b_i|_0, |c|_0 \leqq K_1$ *replaced by the stronger condition* $|b_i|_\alpha, |c|_\alpha \leqq K_1$.

Then

$$|u|_{2+\alpha} \leqq L_\Omega(\alpha, K_1, K_2)(|f|_\alpha + |\phi|_{2,\alpha} + |u|_0).$$

Remark. Note that in Theorem 3.1 (1), the factor $K_\Omega(\alpha, K_1, K_2)$ depends on an upper bound for $|b_i|_0$ and $|c|_0$, but not on an upper bound for $|b_i|_\alpha$ and $|c|_\alpha$, even though we assume $b_i, c \in C_\alpha(\bar{Z}_T)$. (This condition could be somewhat relaxed in (1), but this is not necessary in the context of this paper.)

It is essential that the factors $K_\Omega(\alpha, K_1, K_2)$, $L_\Omega(\alpha, K_1, K_2)$ do not depend on T. This is made possible by including the term $|u|_0$ on the right side of the corresponding inequalities.

It is known from [2] Theorem 7.4′ that under the assumptions of Theorem 3.1 (1), the following inequality holds:

$$|u|_\alpha \leqq c(|f|_0 + |\phi|_2), \qquad \text{with } c \text{ dependent on } \alpha, K_1, K_2, T.$$

Similarly, assuming the conditions in Theorem 3.1 (2), one may conclude from [6] Theorem IV.5.3:

$$|u|_{2+\alpha} \leqq \tilde{c}(|f|_\alpha + |\phi|_{2,\alpha}), \qquad \tilde{c} \text{ dependent on } \alpha, K_1, K_2, T.$$

In the proof below, we shall reduce Theorem 3.1 to the just mentioned results in [2] and [6], via a partition of unity on $(0, \infty)$. This method was suggested by Prof. von Wahl, who used it in a similar context (see [11], p. 68).

Since [2] Theorem 7.4′ is not a good reference – it is given in the form of a problem – we shall summarize the proof of this theorem at the end of this paper (see Appendix, Theorem A).

We note that the condition $T \geqq 1$ in Theorem 3.1 could have been replaced by $T \geqq T_0$, for any fixed $T_0 > 0$. Possibly it would even suffice to assume $T > 0$.

But since Theorem 3.1 will be applied only for large T, there was no need to look for the best possible lower bound on T.

Proof of Theorem 3.1. As a special case of [6] Theorem IV.5.3, we have this result:

There is a function $L_{\Omega,1}: (0,1)\times(0,\infty)^2 \to (0,\infty)$ such that the estimate

(3) $|U|_{2+\alpha} \leqq L_{\Omega,1}(\alpha, K_1, K_2)(|\Psi|_{2,\alpha} + |F|_\alpha)$

is valid for $\alpha\in(0,1)$, $0<K_2\leqq K_1$, A, B_i, C, $F\in C_\alpha(\bar{Z}_1)$, $\Psi\in C_{2,\alpha}(\bar{\Omega})$, $U\in C^{2,1}(\bar{Z}_1)$ with

$$|A|_\alpha,\ |B_i|_\alpha,\ |C|_\alpha \leqq K_1 \qquad (1\leqq i,j\leqq N),$$

$$A(x,s)\geqq K_2 \qquad ((x,s)\in\bar{Z}_T),$$

$$A\Delta U + \sum_{i=1}^{N} B_i D_i U + CU - D_{N+1}U = F,$$

$$\frac{\partial U}{\partial \nu}(x,s) = 0 \qquad \text{for } (x,s)\in S_T,$$

$$U(x,0) = \Psi(x) \qquad \text{for } x\in\bar{\Omega}.$$

Using (3) and (1), we may now show (2):

Choose a function $\zeta\in C^2(\mathbb{R})$ with $\zeta(t)=1$ for $t\geqq 1/2$, $\zeta(t)=0$ for $t\leqq 0$. Define

$$M := |\varphi'|_0 + \sup\left\{\frac{|\varphi'(t)-\varphi'(s)|}{|t-s|^{\alpha/2}} : t,s\in\mathbb{R},\ t\neq s\right\}.$$

Let α, K_1, K_2, T, a, b_i, c, f, ϕ, u be given as in (2). Then we conclude by (3):

(4) $|u|\bar{Z}_1|_{2+\alpha} \leqq L_{\Omega,1}(\alpha, K_1, K_2)(|\phi|_{2,\alpha} + |f|_a)$.

Let $\tilde{T}\in[1,T]$. For $(x,s)\in\bar{Z}_1$, $1\leqq i\leqq N$, we set

$$\begin{aligned}
&A(x,s) := a(x, s+\tilde{T}-1), \qquad B_i(x,s) := b_i(x, s+\tilde{T}-1),\\
&C(x,s) := c(x, s+\tilde{T}-1), \qquad \Psi := 0,\\
&U(x,s) := u(x, s+\tilde{T}-1)\,\zeta(s),\\
&F(x,s) := f(x, s+\tilde{T}-1) - \zeta'(s)\,u(x, s+\tilde{T}-1).
\end{aligned}$$

Then the assumptions in (3) are satisfied. Hence

$$|U|_{2+\alpha} \leqq L_{\Omega,1}(\alpha, K_1, K_2)\,|F|_\alpha.$$

Observing that

$$|u|\bar{\Omega}x[\tilde{T}-\tfrac{1}{2}, \tilde{T}]|_{2+\alpha} \leqq |U|_{2+\alpha}, \qquad |F|_\alpha \leqq |f|_\alpha + M|u|_\alpha,$$

it follows:

(5) $|u|\bar{\Omega}x[\tilde{T}-\tfrac{1}{2}, \tilde{T}]|_{2+\alpha} \leqq L_{\Omega,1}(\alpha, K_1, K_2)(|f|_\alpha + M|u|_\alpha)$.

Now using (1), we obtain from (5):

(6) $|u|\bar{\Omega}x[\tilde{T}-\tfrac{1}{2},\tilde{T}]|_{2+\alpha}\leqq\Gamma(|\phi|_{2,\alpha}+|u|_0+|f|_\alpha)$,

where $\Gamma:=L_{\Omega,1}(\alpha,K_1,K_2)(1+MK_\Omega(\alpha,K_1,K_2))$.

Note that $\Gamma\geqq L_{\Omega,1}(\alpha,K_1,K_2)$. (6) holds for any $\tilde{T}\in[1,T]$.

Let $\sigma\in\mathbb{N}_0^N$, $m\in\mathbb{N}_0$ with $\sigma_1+\ldots+\sigma_N+2m\leqq 2$, and $(y,s)\in\bar{Z}_T$. From (4) in the case $s\leqq 1$, and from (6) (with $\tilde{T}:=s$) in the case $s>1$, we get:

(7) $|D_s^m D_y^\sigma u(y,s)|\leqq\Gamma(|\phi|_{2,\alpha}+|u|_0+|f|_\alpha)$,

where the meaning of $D_s^m D_y^\sigma u(y,s)$ is obvious.

Next, let $(x,t),(x',t')\in\bar{Z}_T$, with $(x,t)\neq(x',t')$. We abbreviate:

$$D^{\sigma,m}H(x,t,x',t'):=\frac{|D_t^m D_x^\sigma u(x,t)-D_{t'}^m D_{x'}^\sigma u(x',t')|}{(|x-x'|^2+|t-t'|)^{\alpha/2}}.$$

Without loss of generality, we may assume $t'\leqq t$.

In the case $t\leqq 1$, we have from (4):

$$D^{\sigma,m}H(x,t,x',t')\leqq L_{\Omega,1}(\alpha,K_1,K_2)(|\phi|_{2,\alpha}+|f|_\alpha).$$

In the case $t>1$, $t-t'\geqq 1/2$, we may conclude with (7):

$$D^{\sigma,m}H(x,t,x',t')\leqq(|D_t^m D_x^\sigma u(x,t)|+|D_{t'}^m D_{x'}^\sigma u(x',t')|)2^{\alpha/2}\leqq 4\Gamma(|\phi|_{2,\alpha}+|u|_0+|f|_\alpha).$$

Finally, in the case $t>1$, $t-t'<\tfrac{1}{2}$, it follows from (6), with $\tilde{T}:=t$:

$$D^{\sigma,m}H(x,t,x',t')\leqq\Gamma(|\phi|_{2,\alpha}+|u|_0+|f|_\alpha).$$

Therefore, we have in any case:

(8) $D^{\sigma,m}H(x,t,x',t')\leqq 4\Gamma(|\phi|_{2,\alpha}+|u|_0+|f|_\alpha)$.

From (7), (8) and the definition of the norm $|\ |_{2+\alpha}$, it follows:

$$|u|_{2+\alpha}\leqq(2+N+N^2)\,5\Gamma(|\phi|_{2,\alpha}+|u|_0+|f|_\alpha).$$

Thus we may set $L_\Omega(\alpha,K_1,K_2):=(2+N+N^2)\,5\Gamma$.

For the proof of (1), we use [2] Theorem 7.4′, as formulated in Theorem A of the Appendix. From this theorem, it follows:

(9) There is a function $K_{\Omega,1}:(0,1)\times(0,\infty)^2\to(0,\infty)$ such that

$$|U|_\alpha\leqq K_{\Omega,1}(\alpha,K_1,K_2)(|\Psi|_2+|F|_0)$$

for α, K_1, K_2, A, B_i, C, F, Ψ, U as in (3), but with the estimate $|B_i|_\alpha, |C|_\alpha\leqq K_1$ replaced by the weaker condition $|B_i|_0, |C|_0\leqq K_1$ $(i\in\{1,\ldots,N\})$.

Now, with α, K_1, K_2, T, a, b_i, c, f, ϕ, u is in (1), we may conclude from (9):

(10) $|u|\bar{Z}_1|_\alpha\leqq K_{\Omega,1}(\alpha,K_1,K_2)(|\phi|_2+|f|_0)$;

(11) $|u|\bar{\Omega}x[\tilde{T}-\tfrac{1}{2},\tilde{T}]|_\alpha\leqq K_{\Omega,1}(\alpha,K_1,K_2)(|f|_0+|\zeta'|_0|u|_0)$ for $\tilde{T}\in[1,T]$.

Proceeding in a similar way as in the proof of (2), we obtain from (10) and (11):

$$|u|_\alpha \leqq 5K_{\Omega,1}(\alpha, K_1, K_2)(1+|\zeta'|_0)(|\phi|_2+|u|_0+|f|_0).$$

Hence, (1) is valid for

$$K_\Omega(\alpha, K_1, K_2) := 5K_{\Omega,1}(\alpha, K_1, K_2)(1+|\zeta'|_0).$$

Theorem 3.2. *Let T, m be positive reals;* $a_{ij}, b_i, c \in C^0(\bar{Z}_T)$ *with* $a_{ij}=a_{ji}$ $(1 \leqq i, j \leqq N)$,

(1) $\sum_{i,j=1}^{N} a_{ij}(x,s)\xi_i\xi_j \geqq m|\xi|^2 \quad ((x,s)\in \bar{Z}_T,\ \zeta\in \mathbb{R}^N).$

Let $u\in C^{2,1}(\bar{Z}_T)$ *with* max $u>0$ (*or* min $u<0$ *respectively*), *and*

$$\sum_{i,j=1}^{N} a_{ij}D_iD_ju+\sum_{i=1}^{N} b_iD_iu+cu-D_{N+1}u \geqq 0 \quad (\leqq 0).$$

Assume there are elements $\overset{0}{x}\in\partial\Omega$, $t_0\in(0, T]$ *such that* $u(\overset{0}{x}, t_0)=\max u$ $(u(\overset{0}{x}, t_0)=\min u)$,

(2) *and* $u(x,s)<\max u$ $(u(x,s)>\min u)$ *for* $(x,s)\in(\bar{\Omega}\times[0,t_0))\cup(\Omega\times\{t_0\})$.

(3) *Furthermore, assume there are elements* $\sigma>0$, $s_0\in[0,t_0)$ *with* $c(x,s)\leqq 0$ *for* $(x,s)\in\overline{\Omega\cap K_\sigma(\overset{0}{x})}\times[s_0,t_0]$.

Then it follows: $\frac{\partial u}{\partial \nu}(\overset{0}{x}, t_0)>0$ $\left(\frac{\partial u}{\partial \nu}(\overset{0}{x}, t_0)<0\right)$.

Proof (compare [10], p. 89). It suffices to prove Theorem 3.2 for the case which relates to a maximum of *u*.

Since Ω has a C^3-boundary, there is a number $\varepsilon>0$ such that

(4) $K_\varepsilon(\overset{0}{x}-\varepsilon\nu(\overset{0}{x}))\subset\Omega, \quad \overline{K_\varepsilon(\overset{0}{x}-\varepsilon\nu(\overset{0}{x}))}\cap\partial\Omega=\{\overset{0}{x}\}.$

(Here it would have been sufficient to assume Ω C^2-bounded.) Without boss of generality we may suppose $\sigma<\varepsilon/2$. Define

$$\overset{1}{x} := \overset{0}{x}-\varepsilon\nu(\overset{0}{x}), \qquad D:=(K_\varepsilon(\overset{1}{x})\cap K_\sigma(\overset{0}{x}))\times(s_0,t_0),$$

$$\Gamma_1 := \overline{K_\varepsilon(\overset{1}{x})}\cap\partial K_\sigma(\overset{0}{x}), \qquad \Gamma_2 := \partial K_\varepsilon(\overset{1}{x})\cap K_\sigma(\overset{0}{x}).$$

We have $\Gamma_1\subset\Omega$, as follows from (4). Hence, because of (2), there is a number $\delta>0$ such that

(5) $u|\Gamma_1\times[s_0,t_0]\leqq\max u-\delta,\ u(x,s_0)\leqq\max u-\delta \quad$ for $x\in\bar{\Omega}$.

Furthermore we note

(6) $u|\Gamma_2\setminus\{\overset{0}{x}\}\times[s_0, t_0]<\max u.$

Set $B:=\max\{|a_{11}|_0, \ldots, |a_{NN}|_0, |b_1|_0, \ldots, |b_N|_0\},$

$\alpha:=4NB(1+\varepsilon)/(m\varepsilon^2),$

$h(x, s):=\exp(-\alpha|\overset{1}{x}-x|^2)-\exp(-\alpha\varepsilon^2)$ for $(x, s)\in\bar{D},$

$\kappa:=\delta/(2|h|_0),$

$v(x, s):=u(x, s)+\kappa h(x, s)$ for $(x, s)\in\bar{D}.$

Then we find:

(7) $$\sum_{i,j=1}^{N} a_{ij}(x, s)D_iD_jv(x, s)+\sum_{i=1}^{N} b_i(x, s)D_iv(x, s)+c(x, s)v(x, s)$$
$$-D_{N+1}v(x, s)>0 \quad \text{for } (x, s)\in D,$$

(8) $v(\overset{0}{x}, t_0)=u(\overset{0}{x}, t_0)=\max u>0,$

(9) $$v(x, s)<\max u \quad \text{for } (x, s)\in(\Gamma_2\setminus\{\overset{0}{x}\}\times[s_0, t_0])$$
$$\cup(\Gamma_1\times[s_0, t_0])\cup(\overline{K_\varepsilon(\overset{1}{x})\cap K_\sigma(\overset{0}{x})}\times\{s_0\}),$$

where (9) is implied by (5), (6). To derive (7), we used (1) and the assumption $\sigma<\varepsilon/2$.

Assume that v takes its maximum at a point $(x', s')\in K_\varepsilon(\overset{1}{x})\cap(K_\sigma(\overset{0}{x})\times[s_0, t_0]$. Then $v(x', s')\geqq v(\overset{0}{x}, t_0)$; therefore from (8):

$$v(x', s')\geqq\max u>0.$$

Because of (3), (7), and $\max v>0$, we may apply the maximum principle for linear parabolic equations (for example, see [2] Theorem 2.4'); it follows:

$$v(x, s)=\max v \quad \text{for } (x, s)\in K_\varepsilon(\overset{1}{x})\cap K_\sigma(\overset{0}{x})\times[s_0, s'].$$

Since $\max v\geqq\max u$, we have a contradiction to (9). Thus

$$v(x, s)<\max v \quad \text{for } (x, s)\in K_\varepsilon(\overset{1}{x})\cap K_\sigma(\overset{0}{x})\times[s_0, t_0].$$

Hence we may conclude with (8), (9):

$$\max v=v(\overset{0}{x}, t_0).$$

Therefore $\frac{\partial v}{\partial \nu}(\overset{0}{x}, t_0)\geqq 0$. Since $\frac{\partial h}{\partial \nu}(\overset{0}{x}, t_0)<0$, this means: $\frac{\partial u}{\partial \nu}(\overset{0}{x}, t_0)>0$.

Corollary 3.1. *Let T, $m\in(0,\infty)$, a_{ij}, b_i, $c\in C^0(\bar{Z}_T)$ with $a_{ij}=a_{ji}$ $(1\leqq i,j\leqq N)$,*

$$\sum_{i,j=1}^{N} a_{ij}(x,s)\xi_i\xi_j\geqq m|\xi|^2 \qquad ((x,s)\in\bar{Z}_T,\ \xi\in\mathbb{R}^N).$$

Assume further that $c\leqq 0$ throughout $\bar{Z}_T$.
Let $u\in C^{2,1}(\bar{Z}_T)$ satisfy

$$\sum_{i,j=1}^{N} a_{ij}D_iD_ju+\sum_{i=1}^{N} b_iD_iu+cu-D_{N+1}u\geqq 0 \qquad (\text{or } \leqq 0 \text{ respectively}),$$

(1) $\dfrac{\partial u}{\partial \nu}(x,t)=0$ *for* $(x,t)\in S_T$.

Then $\max u\leqq\max(\{0\}\cup\{u(x,0)\colon x\in\bar{\Omega}\})$
$(\min u\geqq\min(\{0\}\cup\{u(x,0)\colon x\in\bar{\Omega}\}))$.

Proof. It is sufficient to prove that part of Corollary 3.1 which relates to a maximum of u.

Assume that $\max u>\max(\{0\}\cup\{u(x,0)\colon x\in\bar{\Omega}\})$.
Take $s_0:=\inf\{r\in[0,T]\colon u(x,r)=\max u$ for some $x\in\bar{\Omega}\}$.
Then $s_0>0$, and $u(x,s)<\max u$ for $(x,s)\in\bar{\Omega}\times[0,s_0)$.
We choose $\overset{0}{x}\in\bar{\Omega}$ with $\max u=u(\overset{0}{x},s_0)$.

Consider the case $u(x,s_0)<\max u$ for all $x\in\Omega$.

Then $\overset{0}{x}\in\partial\Omega$, and we may conclude from Theorem 3.2: $\dfrac{\partial u}{\partial\nu}(\overset{0}{x},s_0)>0$, a contradiction to (1).

On the other hand, if $u(x',s_0)=\max u$ for some $x'\in\Omega$, it follows from the maximum principle for linear parabolic equations (see [2] Theorem 2.4′):

$$u(x,s)=\max u \qquad \text{for } (x,s)\in\Omega\times[0,s_0].$$

But this equation contradicts the assumption at the beginning the proof.
Thus the estimate

$$\max u\leqq\max(\{0\}\cup\{u(x,0)\colon x\in\bar{\Omega}\})$$

must be true.

Corollary 3.2. *Let T, m, a_{ij}, b_i, c, u $(1\leqq i,j\leqq N)$ be given as in Corollary 3.1, but without the condition $c\leqq 0$. Then*

$$\max u\leqq\max(\{0\}\cup\{u(x,0)e^{|c|_0T}\colon x\in\bar{\Omega}\}),$$
$$(\min u\geqq\min(\{0\}\cup\{u(x,0)e^{|c|_0T}\colon x\in\bar{\Omega}\})).$$

Proof. Apply Corollary 3.1, with c replaced by $c-|c|_0$, u replaced by $u(x,s)e^{-|c|_0 s}((x,s)\in\bar{Z}_T)$.

4. Existence and Uniqueness of Solutions to $Q(a, \ldots, e, W, \infty)$

Theorem 4.1. *Let* $\delta \in (0, 1)$, m, $M \in \mathbb{R}$ *with* $0 < m \leqq M$. *Take*

$$P := P(\delta, m, M) := 2K_\Omega(\delta, M + mM + 2M^2, m),$$

$$R := R(\delta, m, M) := \min\left\{\frac{m}{M}, \frac{m}{2L_\Omega(\delta, M + mM, m)(M^2(1+2P)+1)}\right\}$$

where the functions K_Ω, L_Ω *were introduced in Theorem 3.1.*

Take

$$a_0 \in \left[0, \frac{M^2}{R}\right], \quad b_0 \in \left[\frac{mM}{R}, \frac{M^2}{R}\right], \quad c_0 \in [m, M], \quad d_0 \in \left[0, \frac{M}{2}\right],$$

$$e_0 \in \left[0, \frac{mM}{2}\right], \quad \Psi \in C_{2,\delta}(\bar{\Omega}) \text{ with } \frac{\partial \Psi}{\partial \nu}(x) = 0 \text{ for } x \in \partial\Omega;$$

furthermore $\Psi \geqq 0$ *and* $|\Psi|_{2,\delta} \leqq R$.

Let $T \in [1, \infty)$, $g \in C_{2+\delta}(\bar{Z}_T)$ *with* $0 \leqq g \leqq R$, $|g|_\delta \leqq PR$, $|g|_{2+\delta} \leqq m$, $w \in C^{2,1}(\bar{Z}_T)$ *with*

$$S(a_0, \ldots, e_0, g, \Psi, T) \quad \begin{aligned} &(c_0 + d_0 g)\Delta w + 2d_0 \nabla g \nabla w + (d_0 \Delta g + e_0 - a_0 g - b_0 w)w \\ &- D_{N+1} w = 0, \\ &w(x, 0) = \Psi(x) \qquad (x \in \bar{\Omega}), \\ &\frac{\partial w}{\partial \nu}(x, s) = 0 \qquad ((x, s) \in S_T). \end{aligned}$$

Then it follows: $0 \leqq w \leqq R$.

Remark. Some of the assumptions in Theorem 4.1 are necessary only for the proof of Theorem 4.2. But it is more convenient to list all these assumptions at one place.

Theorems 4.1 and 4.2 show that under suitable conditions on $a_0, \ldots, e_0, g, \Psi$, the function g and the solution w to $S(a_0, \ldots, e_0, g, \Psi, T)$ both belong to a class of functions V, which is given by

$$V := \{h \in C^{2,1}(\bar{Z}_T) : 0 \leqq h \leqq R, |h|_\delta \leqq PR, |h|_{2+\delta} \leqq m\}.$$

Later on, this result will be used for a fixed point argument.

Note that $R = R(\delta, m, M) \to 0$ for $M \to \infty$. Therefore, the conditions $a_0 \in \left[0, \frac{M^2}{R}\right]$, $b_0 \in \left[\frac{mM}{R}, \frac{M^2}{R}\right]$ imply that a_0, b_0 may be large. We did not consider the behavior of the functions K_Ω, L_Ω appearing in the definitions of P, R. Thus R may be small, and so the assumptions $|\Psi|_{2,\delta} \leqq R$, $0 \leqq g \leqq R$ imply a smallness condition for $|\Psi|_{2,\delta}$, $|g|_0$.

Proof of Theorem 4.1. We have

$$T < \infty, \quad g \in C^{2,1}(\bar{Z}_T) \text{ with } g \geqq 0, \quad w \in C^0(\bar{Z}_T), \quad c_0 > 0, \quad d_0 \geqq 0.$$

Therefore

$$c_0+d_0 g,\ 2d_0 D_i g,\ d_0 \Delta g+e_0-a_0 g-b_0 w \in C^0(\bar{Z}_T) \qquad (1 \leqq i \leqq N),$$

$$c_0+d_0 g \geqq c_0>0.$$

Moreover, we note that

$$w(x, 0)=\Psi(x) \geqq 0 \qquad \text{for } x \in \bar{\Omega}.$$

Thus it follows from Corollary 3.2: $w \geqq 0$.

Now assume that $\max w>R$. Then $\Psi \leqq R<\max w$. Thus

$$t_1:=\sup\{r \in[0, T]: w(x, s)<\max w \text{ for } (x, s) \in \bar{Z}_r\}$$

is well defined, with $t_1>0$, and there is some $\overset{0}{x} \in \bar{\Omega}$ with $w(\overset{0}{x}, t_1)=\max w$.

A short computation shows:

(1) $(d_0 \Delta g+e_0-a_0 g-b_0 w)(\overset{0}{x}, t_1)<0$;

here we used $a_0 \geqq 0$, $g \geqq 0$, $|g|_{2+\delta} \leqq m$, $e_0 \leqq mM/2$, $d_0 \leqq M/2$, $b_0 \geqq mM/R$, $w(\overset{0}{x}, t_1)>R$.

Consider the case $w(x, t_1)<\max w$ for $x \in \Omega$. Then $\overset{0}{x} \in \partial \Omega$, and inequality (1) implies the existence of some $\sigma>0$, $\tilde{t} \in[0, t_1)$ such that

$$(d_0 \Delta g+e_0-a_0 g-b_0 w)(x, s)<0 \qquad \text{for } (x, s) \in \overline{\Omega \cap K_\sigma(\overset{0}{x})} \times[\tilde{t}, t_1].$$

Now we may conclude from Theorem 3.2: $\frac{\partial w}{\partial \nu}(\overset{0}{x}, t_1)>0$, a contradiction to the properties of w.

Next, consider the case that $w(x', t_1)=\max w$ for some $x' \in \Omega$. Because of (1), we may choose $\varepsilon>0$, $\tilde{t} \in[0, t_1)$ with $\overline{K_\varepsilon(x')} \subset \Omega$, and

$$(d_0 \Delta g+e_0-a_0 g-b_0 w) \mid \overline{K_\varepsilon(x')} \times[\tilde{t}, t_1]<0.$$

Then, from the maximum principle for linear parabolic equations:

$$w(x, s)=\max w \qquad \text{for } (x, s) \in \overline{K_\varepsilon(x')} \times[\tilde{t}, t_1].$$

This is a contradiction to the definition of t_1.

Theorem 4.2. *Let δ, m, M, a_0, b_0, c_0, d_0, e_0, Ψ, T, g be given as in Theorem 4.1. Then there is a uniquely determined function $w \in C^{2,1}(\bar{Z}_T)$, which solves $S(a_0, \ldots, e_o, g, \Psi, T)$.*

For this solution w, the following estimates hold:

$$|w|_\delta \leqq PR, \qquad |w|_{2+\delta} \leqq m,$$

where $P \equiv P(\delta, m, M)$, $R \equiv R(\delta, m, M)$ were introduced in Theorem 4.1.

Proof. The existence of a solution $w \in C_{2+\delta}(\bar{Z}_T)$ to $S(a_0, \ldots, e_0, g, \Psi, T)$ is well known. For example, one may consider the initial-boundary-value problem

(1) $(c_0+d_0 g)\Delta w+2d_0 \nabla g \nabla w+(d_0 \Delta g+e_0-a_0 g)w-D_{N+1}w$
$=b_0 \min\{w^2,(R+1)^2\}$ on $\bar{Z}_T$,

$$\frac{\partial w}{\partial \nu}(x,s)=0 \quad \text{for } (x,s)\in S_T, \qquad w(x,0)=\Psi(x) \quad \text{for } x\in\bar{\Omega}.$$

Problem (1) has a solution $w\in C_{2+\delta}(\bar{Z}_T)$, as may be shown by standard arguments (see Appendix, Theorem B).

Having found a solution $w\in C_{2+\delta}(\bar{Z}_T)$ to (1), it follows from Theorem 4.1 that w is also a solution to $S(a_0, \dots, e_0, g, \Psi, T)$.

Uniqueness of this solution in the class $C^{2,1}(\bar{Z}_T)$ is implied by Corollary 3.2.

From Theorem 4.1 we know: $0\leqq w\leqq R$.

Furthermore, we note:

$$|c_0+d_0 g|_\delta,\ |2d_0 D_i g|_0,\ |d_0\Delta g+e_0-a_0 g-b_0 w|_0\leqq M+mM+2M^2 \qquad (1\leqq i\leqq N),$$

(2) $c_0+d_0 g\geqq m$;

here we used the assumptions on $a_0, \dots, e_0$, g and the estimate $|w|_0\leqq R$. Recalling that w solves $S(a_0, \dots, e_0, g, \Psi, T)$, we may apply Theorem 3.1 (1). It follows:

$$|w|_\delta\leqq K_\Omega(\delta, M+mM+2M^2, m)(|\Psi|_2+|w|_0).$$

(Here it is essential that in Theorem 3.1 (1) the assumption $|c|_0\leqq K_1$ is sufficient. If we had to assume $|c|_\alpha\leqq K_1$ in that theorem, the preceding argument would not go through.)

Recalling the definition of $P\equiv P(\delta, m, M)$, and using

$$|\Psi|_2\leqq|\Psi|_{2,\delta}\leqq R, \qquad |w|_0\leqq R, \qquad \text{we obtain:} \quad |w|_\delta\leqq PR.$$

Next we observe

$$|c_0+d_0 g|_\delta,\ |2d_0 D_i g|_\delta,\ |d_0\Delta g+e_0|_\delta\leqq M+mM.$$

This, together with (2), implies by Theorem 3.1 (2):

(3) $|w|_{2+\delta}\leqq L_\Omega(\delta, M+mM, m)(|a_0 g w+b_0 w^2|_\delta+|\Psi|_{2,\delta}+|w|_0)$.

On the other hand, for $u, \tilde{u}\in C_\delta(\bar{Z}_T)$, we have the estimate

(4) $|u\tilde{u}|_\delta\leqq|u|_0|\tilde{u}|_0+|u|_0|\tilde{u}|_\delta+|u|_\delta|\tilde{u}|_0$.

Applying (4) on the term $a_0 g w+b_0 w^2$, and using $|g|_\delta, |w|_\delta\leqq PR$, $|g|_0, |w|_0\leqq R$, and $0\leqq a_0, b_0\leqq M^2/R$, we obtain:

(5) $|a_0 g w+b_0 w^2|_\delta\leqq 2RM^2(1+2P)$.

Putting (5) and the estimates $|\Psi|_{2,\delta}, |w|_0\leqq R$ into (3), it follows:

$$|w|_{2+\delta}\leqq L_\Omega(\delta, M+mM, m)(2RM^2(1+2P)+2R).$$

Hence from the definition of R:

$$|w|_{2+\delta}\leqq m.$$

Theorem 4.3. *Take* $\delta\in(0,1)$, $m, M\in\mathbb{R}$ *with* $0<m\leqq M$, $\tilde{a}_i\in[0, M^2/R]$, $\tilde{b}_i\in[mM/R, M^2/R]$, $\tilde{c}_i\in[m, M]$, $\tilde{d}_i\in[0, M/2]$, $\tilde{e}_i\in[0, mM/2]$, $\phi_i\in C_{2,\delta}(\bar{\Omega})$ *with* $\phi_i\geqq 0$, $\frac{\partial\phi_i}{\partial v}(x)=0$ *for* $x\in\partial\Omega$, $|\phi_i|_{2,\delta}\leqq R$ $(i=1, 2)$.

Let T belong to $[1,\infty)$.

Then there is a solution $(\tilde{u},\tilde{v})$ *to* $Q(\tilde{a},\dots,\tilde{e},\phi,T)$ *such that for* $w\in\{\tilde{u},\tilde{v}\}$ *the following statements are true:*

$$w\in C^{2,1}(\bar{Z}_T),\qquad 0\leqq w\leqq R,\qquad |w|_\delta\leqq PR,\qquad |w|_{2+\delta}\leqq m$$

(with $P\equiv P(\delta,m,M)$, $R\equiv R(\delta,m,M)$ *defined in Theorem 4.1).*

Proof. Define $V:=\{g\in C^{2,1}(\bar{Z}_T)\colon 0\leqq g\leqq R,\ |g|_\delta\leqq PR,\ |g|_{2+\delta}\leqq m\}$. V is a compact, convex subset of the Banach space $C_{2+\delta/2}(\bar{Z}_T)$.

The set $B:=(C_{2+\delta/2}(\bar{Z}_T))^2$, endowed with the norm

$$\|(u,v)\|:=|u|_{2+\delta/2}+|v|_{2+\delta/2},$$

is also a Banach space, and $V\times V$ is a convex, compact subset of B. Now, for $g\in V$, $i\in\{1,2\}$, we define $S_i g\in C^{2,1}(\bar{Z}_T)$ as the solution to the initial-boundary-value problem $S(\tilde{a}_i,\dots,\tilde{e}_i,g,\phi_i,T)$.

From Theorem 4.1 and 4.2 it follows that $S_i g$ is well defined and admits the estimates

$$0\leqq S_i g\leqq R,\qquad |S_i g|_\delta\leqq PR,\qquad |S_i g|_{2+\delta}\leqq m\quad (g\in V,\ i\in\{1,2\}).$$

This means that $S_i g\in V$ for $g\in V$, $i\in\{1,2\}$.

For $(u,v)\in V\times V$ we set $S(u,v):=(S_1 v, S_2 u)$. Then S is an operator from $V\times V$ into $V\times V$. We want to show that S is continuous in the norm $\|\ \|$ of the Banach space B. To this end, we define for $g, h\in V$, $1\leqq k\leqq N$:

$$\begin{aligned}
&A(g):=\tilde{c}_1+\tilde{d}_1 g,\\
&B_k(g):=2\tilde{d}_1 D_k g,\\
&C(g,h):=\tilde{d}_1\Delta g+\tilde{e}_1-\tilde{a}_1 g-\tilde{b}_1(S_1 g+S_1 h),\\
&F(g,h):=(-\tilde{d}_1\Delta(S_1 h)+\tilde{a}_1 S_1 h)(g-h)-2\tilde{d}_1\nabla(S_1 h)\nabla(g-h)-\tilde{d}_1 S_1 h\Delta(g-h).
\end{aligned}$$

Then we have for $g, h\in V$.

$$\begin{aligned}
(1)\quad & A(g)\Delta(S_1 g-S_1 h)+\sum_{k=1}^{N} B_k(g)D_k(S_1 g-S_1 h)\\
&\quad +C(g,h)(S_1 g-S_1 h)-D_{N+1}(S_1 g-S_1 h)=F(g,h),\\
&(S_1 g-S_1 h)(x,0)=0\quad (x\in\bar{\Omega}),\qquad \frac{\partial(S_1 g-S_1 h)}{\partial v}(x,s)=0\quad ((x,s)\in S_T).
\end{aligned}$$

Take $\rho>0$ such that $|u|_{\delta/2}\leqq\rho|u|_\delta$ for $u\in C_\delta(\bar{Z}_T)$.

Set $E:=\max\{\tilde{a}_i,\tilde{b}_i,\tilde{c}_i,\tilde{d}_i,\tilde{e}_i\colon 1\leqq i\leqq 2\}$. Observing that $|g|_{2+\delta}$, $|S_1 g|_{2+\delta}\leqq m$, $c_1+d_1 g\geqq m$ for $g\in V$, we find:

(2) $|A(g)|_{\delta/2}, \ |B_k(g)|_{\delta/2}, \ |C(g, h)|_{\delta/2} \leqq \rho(E+4m) \quad (1 \leqq k \leqq N, \ g, h \in V)$;

(3) $A(g)(x, s) \geqq m \quad ((x, s) \in \bar{Z}_T, \ g \in V)$.

From (1)–(3) and [6] Theorem IV.5.3 it follows that there exists a constant $\tilde{K}_1 > 0$ so that

(4) $|S_1 g - S_1 h|_{2+\delta/2} \leqq \tilde{K}_1 |F(g, h)|_{\delta/2} \quad$ for $g, h \in V$.

We may estimate $|F(g, h)|_{\delta/2}$ by

$$|F(g, h)|_{\delta/2} \leqq 5E|S_1 h|_{2+\delta/2} |g-h|_{2+\delta/2} \leqq 5E\rho m |g-h|_{2+\delta/2}.$$

Putting this into (4), we obtain

$$|S_1 g - S_1 h|_{2+\delta/2} \leqq \tilde{K}_2 |g-h|_{2+\delta/2} \quad \text{for } g, h \in V,$$

where $\tilde{K}_2 := \tilde{K}_1 \, 5E\rho m$.

The term $|S_2 g - S_2 h|_{2+\delta/2}$ may be estimated in the same way. Recalling how the norm $\| \ \|$ and the operator S were defined, we arrive at the estimate

$$\|S(g, g') - S(h, h')\| \leqq \tilde{K}_2 \|(g, g') - (h, h')\| \quad (g, g', h, h' \in V).$$

Now we may apply Schauder's fixed point theorem, in the form of [3] Theorem 10.1. This theorem yields the existence of functions $u, v \in V$ such that $S(\tilde{u}, \tilde{v}) = (\tilde{u}, \tilde{v})$; i.e.: $S_1 \tilde{v} = \tilde{u}$, $S_2 \tilde{u} = \tilde{v}$.

From the definition of V and S, it follows: $\tilde{u}, \tilde{v} \in C^{2,1}(\bar{Z}_T)$; $(\tilde{u}, \tilde{v})$ is a solution to $Q(\tilde{a}, \ldots, \tilde{e}, \phi, T)$; for $w \in \{\tilde{u}, \tilde{v}\}$, the estimates stated in Theorem 4.3 are valid.

Corollary 4.1. *Take* $\delta \in (0, 1)$, $m, M \in \mathbb{R}$ *with* $0 < m \leqq M$, $a_i \in [0, M]$, $b_i, c_i \in [m, M]$, $d_i \in [0, R/2]$, $e_i \in [0, mM/2]$, $W_i \in C_{2,\delta}(\bar{\Omega})$ *with* $W_i \geqq 0$, $\frac{\partial W_i}{\partial v}(x) = 0$ *for* $x \in \partial\Omega$, $|W_i|_{2,\delta} \leqq M$ $(i=1, 2)$.

Let T belong to $[1, \infty)$.

Then there is a solution (u, v) *to* $Q(a, \ldots, e, W, T)$ *such that for* $w \in \{u, v\}$

$$w \in C^{2,1}(\bar{Z}_T), \quad 0 \leqq w \leqq M, \quad |w|_{2+\delta} \leqq mM/R$$

(with $R \equiv R(\delta, m, M)$ *defined in Theorem 4.1).*

Remark. The assumption $d_i \in [0, R/2]$ is a smallness condition on d_i; see Remark to Theorem 4.1.

Proof of Corollary 4.1. Set $\tilde{a}_i := \frac{M}{R} a_i$, $\tilde{b}_i := \frac{M}{R} b_i$, $\tilde{c}_i := c_i$, $\tilde{d}_i := \frac{M}{R} d_i$, $\tilde{e}_i := e_i$, $\phi_i := \frac{R}{M} W_i$ $(i=1, 2)$.

Then for $\delta, m, M, \tilde{a}_i, \ldots, \tilde{e}_i, \phi_i, T$ the assumptions of Theorem 4.3 are satisfied. Hence there is a solution $(\tilde{u}, \tilde{v})$ of $Q(\tilde{a}, \ldots, \tilde{e}, \phi, T)$ with

$$\tilde{w} \in C^{2,1}(\bar{Z}_T), \quad 0 \leqq \tilde{w} \leqq R, \quad |\tilde{w}|_{2+\delta} \leqq m, \quad \text{for } \tilde{w} \in \{\tilde{u}, \tilde{v}\}.$$

Define $u := \frac{M}{R} \tilde{u}$, $v := \frac{M}{R} \tilde{v}$. Then the functions u, v have the desired properties.

Theorem 4.4. *Let $a_i, b_i, c_i, d_i, e_i \in [0, \infty)$ with $c_i > 0$, $W_i \in C^2(\bar{\Omega})$ with $W_i \geqq 0$ $(i=1, 2)$, $T \in (0, \infty]$, (u, v) a solution to $Q(a, \ldots, e, W, T)$ with $u, v \in C^{2,1}(\bar{Z}_T)$.*

Then $u, v \geqq 0$.

Proof. Assume there is an element $(x, s) \in \bar{Z}_T$ with $u(x, s) < 0$ or $v(x, s) < 0$. For $w \in \{u, v\}$, define $s_w := \sup\{r \in [0, T] \colon w(x, t) \geqq 0$ for $(x, t) \in \bar{Z}_r\}$.

Since $u(x, 0)$, $v(x, 0) \geqq 0$ for $x \in \bar{\Omega}$, s_w is well defined. Without loss of generality we may assume $s_u \leqq s_v$. Then it follows from the assumption at the beginning of the proof: $s_u < T$. Furthermore, we have $u(x, s_u)$, $v(x, s_u) \geqq 0$ $(x \in \bar{\Omega})$. Thus we may choose a number $\varepsilon > 0$ such that $\bar{\Omega} \times [s_u, s_u + \varepsilon] \subset \bar{Z}_T$, and $c_1 + d_1 v(x, s) \geqq c_1/2$ for $(x, s) \in \bar{\Omega} \times [s_u, s_u + \varepsilon]$. (Note that $c_1 > 0$ and $v \in C^0(\bar{Z}_T)$.) Finally, there is an element

(1) $(x, s) \in \bar{\Omega} \times [s_u, s_u + \varepsilon]$ with $u(x, s) < 0$.

Now we set for $1 \leqq i, j \leqq N$, $(x, t) \in \bar{Z}_\varepsilon$:

$$a_{ii}(x, t) := c_1 + d_1 v(x, t + s_u), \qquad a_{ij}(x, t) := 0 \quad \text{if } i \neq j,$$
$$b_i(x, t) := 2 d_1 D_i v(x, t + s_u),$$
$$c(x, t) := (d_1 \Delta v + e_1 - a_1 v - b_1 u)(x, t + s_u)$$
$$w(x, t) := u(x, t + s_u).$$

Then the assumptions of Corollary 3.2 are satisfied, with T, m, u replaced by ε, $c_1/2$, w.

Noting that $w(x, 0) \equiv u(x, s_u) \geqq 0$ for $x \in \bar{\Omega}$, it follows from Corollary 3.2: $w \geqq 0$. This means:

$$u(x, s) \geqq 0 \qquad \text{for } (x, s) \in \bar{\Omega} \times [s_u, s_u + \varepsilon],$$

a contradiction to (1).

Theorem 4.5. *For δ, m, M, a_i, b_i, c_i, d_i, e_i, W_i, T as in Corollary 4.1, there is one and only one solution (u, v) to $Q(a, \ldots, e, W, T)$ such that $u, v \in C^{2,1}(\bar{Z}_T)$.*

Proof. From Corollary 4.1 we know the existence of a solution (u, v) to $Q(a, \ldots, e, W, T)$ such that $w \in C^{2,1}(\bar{Z}_T)$, $0 \leqq w \leqq M$, $|w|_{2+\delta} \leqq mM/R$ for $w \in \{u, v\}$. Now let U, V be some functions from $C^{2,1}(\bar{Z}_T)$ such that (U, V) is a solution to $Q(a, \ldots, e, W, T)$.

Since (u, v) and (U, V) satisfy the first equation in $Q(a, \ldots, e, W, T)$, we get by subtraction:

(1) $(c_1 + d_1 V) \Delta(U - u) + 2 d_1 \nabla V \nabla(U - u) + C(U - u) - D_{N+1}(U - u) = F,$

where

$$C := d_1 \Delta V + e_1 - a_1 V - b_1 U - b_1 u,$$
$$F := (-d_1 \Delta u + a_1 u)(V - v) - 2 d_1 \nabla u \nabla(V - v) - d_1 u \Delta(V - v).$$

For $\gamma \in \mathbb{R}$, we set $C^{(\gamma)} := C - \gamma$,

$$z^{(\gamma)}(x, s) := (U - u)(x, s) e^{-\gamma s},$$
$$F^{(\gamma)}(x, s) := F(x, s) e^{-\gamma s} \qquad ((x, s) \in \bar{Z}_T).$$

Then, from (1), for $\gamma \in \mathbb{R}$:

(2) $(c_1 + d_1 v)\,\Delta z^{(\gamma)} + 2d_1\,\nabla V\,\nabla z^{(\gamma)} + C^{(\gamma)} z^{(\gamma)} - D_{N+1} z^{(\gamma)} = F^{(\gamma)}$.

We multiply both sides of (2) with $-z^{(\gamma)}$ and integrate over Z_T. Note that for $\gamma \in \mathbb{R}$,

$$\int_{Z_T} \Big((c_1 + d_1 V)\,\Delta z^{(\gamma)}(-z^{(\gamma)}) + 2d_1\,\nabla V\,\nabla z^{(\gamma)}(-z^{(\gamma)}) \Big)\, d(x,s)$$

$$= \int_{Z_T} \Big((c_1 + d_1 V)\,|\nabla z^{(\gamma)}|^2 + \frac{d_1}{2}\,\Delta V (z^{(\gamma)})^2 \Big)\, d(x,s)$$

$$\geqq \int_{Z_T} \Big(c_1 |\nabla z^{(\gamma)}|^2 + \frac{d_1}{2}\,\Delta V (z^{(\gamma)})^2 \Big)\, d(x,s).$$

In the last inequality, we used the estimate $V \geqq 0$, which follows from Theorem 4.4.

Moreover, since $z^{(\gamma)}(x,0) = 0$ $(x \in \bar{\Omega})$, we have

$$\int_{Z_T} (-D_{N+1} z^{(\gamma)})(-z^{(\gamma)})\, d(x,s) = \tfrac{1}{2} \int_{\Omega} (z^{(\gamma)}(x,T))^2\, dx \geqq 0 \qquad (\gamma \in \mathbb{R}).$$

Thus we obtain from (2):

(3) $$\int_{Z_T} \Big(c_1 |\nabla z^{(\gamma)}|^2 + \Big(\frac{d_1}{2}\,\Delta V - C^{(\gamma)}\Big)(z^{(\gamma)})^2 \Big)\, d(x,s) \leqq \int_{Z_T} F^{(\gamma)}(-z^{(\gamma)})\, d(x,s) \qquad (\gamma \in \mathbb{R}).$$

The right side in (3) may be estimated as follows:

(4) $$\Big| \int_{Z_T} F^{(\gamma)}(-z^{(\gamma)})\, d(x,s) \Big|$$

$$= \Big| \int_{Z_T} e^{-\gamma s} \{ -d_1 \nabla u (V - v) \nabla z^{(\gamma)} - a_1 u (V - v) z^{(\gamma)} - d_1 u \nabla (V - v) \nabla z^{(\gamma)} \}\, d(x,s) \Big|$$

$$\leqq \int_{Z_T} \Big\{ \Big(\frac{c_1}{2} + \frac{d_1}{2} |u|_0 \Big) |\nabla z^{(\gamma)}|^2 + \Big(\frac{d_1^2}{2c_1} |\nabla u|^2 + 1 \Big) (\tilde{z}^{(\gamma)})^2$$

$$+ \frac{1}{4} a_1^2 u^2 (z^{(\gamma)})^2 + \frac{d_1}{2} |u|_0 |\nabla \tilde{z}^{(\gamma)}|^2 \Big\}\, d(x,s) \qquad (\gamma \in \mathbb{R}),$$

with $\tilde{z}^{(\gamma)}(x,s) := (V - v)(x,s) e^{-\gamma s}$ $((x,s) \in \bar{Z}_T,\ \gamma \in \mathbb{R})$.

From the second equation in $Q(a, \ldots, e, W, T)$, it follows analogously, for $\gamma \in \mathbb{R}$:

(3′) $$\int_{Z_T} \Big(c_2 |\nabla \tilde{z}^{(\gamma)}|^2 + \Big(\frac{d^2}{2}\,\Delta U - \tilde{C}^{(\gamma)} \Big) (\tilde{z}^{(\gamma)})^2 \Big)\, d(x,s) \leqq \int_{Z_T} \tilde{F}^{(\gamma)}(-\tilde{z}^{(\gamma)})\, d(x,s),$$

(4′) $$\Big| \int_{Z_T} \tilde{F}^{(\gamma)}(-\tilde{z}^{(\gamma)})\, d(x,s) \Big| \leqq \int_{Z_T} \Big\{ \Big(\frac{c_2}{2} + \frac{d_2}{2} |v|_0 \Big) |\nabla \tilde{z}^{(\gamma)}|^2 + \Big(\frac{d_2^2}{2c_2} |\nabla v|^2 + 1 \Big) (z^{(\gamma)})^2$$

$$+ \frac{1}{4} a_2^2 v^2 (\tilde{z}^{(\gamma)})^2 + \frac{d_2}{2} |v|_0 |\nabla z^{(\gamma)}|^2 \Big\}\, d(x,s),$$

where

$$\tilde{C}^{(\gamma)} := d_2 \Delta U + e_2 - a_2 U - b_2 V - b_2 v - \gamma,$$

$$\tilde{F}^{(\gamma)}(x, s) := [(-d_2 \Delta v + a_2 v)(U - u) - 2 d_2 \nabla v \nabla (U - u) - d_2 v \Delta (U - u)](x, s) e^{-\gamma s}$$

$((x, s) \in \bar{Z}_T)$.

Next we add (3) and (3'). Then we estimate the right side of the resulting inequality by using (4) and (4'). After some rearranging, we arrive at the following result:

$$(5)\quad \int_{Z_T} \left\{ \left(\frac{c_1}{2} - \frac{d_1}{2} |u|_0 - \frac{d_2}{2} |v|_0 \right) |\nabla z^{(\gamma)}|^2 + \left(\frac{c_2}{2} - \frac{d_1}{2} |u|_0 - \frac{d_2}{2} |v|_0 \right) |\nabla \tilde{z}^{(\gamma)}|^2 + E^{(\gamma)} (z^{(\gamma)})^2 + \tilde{E}^{(\gamma)} (\tilde{z}^{(\gamma)})^2 \right\} d(x, s) \leqq 0 \quad (\gamma \in \mathbb{R}),$$

where

$$E^{(\gamma)} := -\frac{d_1}{2} \Delta V - e_1 + a_1 V + b_1 U + b_1 u - \frac{1}{4} a_1^2 u^2 - \frac{d_2^2}{2 c_2} |\nabla v|^2 - 1 + \gamma,$$

$$\tilde{E}^{(\gamma)} := -\frac{d_2}{2} \Delta U - e_2 + a_2 U + b_2 V + b_2 v - \frac{1}{4} a_2^2 v^2 - \frac{d_1^2}{2 c_1} |\nabla u|^2 - 1 + \gamma,$$

Using $|u|_0$, $|v|_0 \leqq M$, $c_i \geqq m$, $d_i \leqq \frac{R}{2} \leqq m/(2M)$ $(i = 1, 2)$ (see the definition of $R \equiv R(\delta, m, M)$ in Theorem 4.1), one finds:

$$(6)\quad \frac{c_i}{2} - \frac{d_1}{2} |u|_0 - \frac{d_2}{2} |v|_0 \geqq 0 \quad (i = 1, 2).$$

(Note that our uniqueness proof would fail here if we did not know the existence of a solution (u, v) to $Q(a, \ldots, e, W, T)$ with

$$c_i/2 - \frac{d_1}{2} |u|_0 - \frac{d_2}{2} |v|_0 \geqq 0 \quad (i = 1, 2),$$

as implied by Corollary 4.1. This is the reason why we assumed that $a_i, \ldots, e_i, W_i, T$ satisfy the conditions of Corollary 4.1.)

We obtain from (5) and (6):

$$(7)\quad \int_{Z_T} \{ E^{(\gamma)} (z^{(\gamma)})^2 + \tilde{E}^{(\gamma)} (\tilde{z}^{(\gamma)})^2 \} d(x, s) \leqq 0 \quad (\gamma \in \mathbb{R}).$$

Since $u, U, v, V \in C^{2,1}(\bar{Z}_T)$ and $T < \infty$, we may choose $\gamma > 0$ such that $E^{(\gamma)}, \tilde{E}^{(\gamma)} \geqq 1$. Thus it follows from (7): $z^{(\gamma)} = \tilde{z}^{(\gamma)} = 0$. This means: $u = U$, $v = V$.

Corollary 4.2. *Let $\delta, m, M, a_i, b_i, c_i, d_i, e_i, W_i$ $(i = 1, 2)$ be given as in Corollary 4.1. Then there are uniquely determined functions $u, v \in C^{2,1}(\bar{Z}_\infty)$ such that (u, v) is a solution to $Q(a, \ldots, e, W, \infty)$.*

For $w \in \{u, v\}$, the estimates $0 \leqq w \leqq M$, $|w|_{2+\delta} \leqq \frac{mM}{R}$ are valid.

Proof. We know from Corollary 4.1 that for $T\in[1,\infty)$, there are functions u_T, v_T such that (u_T, v_T) solves $Q(a, \ldots, e, W, T)$; furthermore, the inequalities $0\leqq w\leqq M$, $|w|_{2+\delta}\leqq mM/R$ are valid for $w\in\{u_T, v_T\}$. Take two reals $T_1, T_2\in[1,\infty)$ with $T_1<T_2$. Then it follows from Theorem 4.5:

$$u_{T_1}=(u_{T_2})|\bar{Z}_{T_1}, \qquad v_{T_1}=(v_{T_2})|\bar{Z}_{T_1}.$$

Hence we may define functions $u, v\colon \bar{Z}_\infty\to\mathbb{R}$ by setting $u|\bar{Z}_T:=u_T$, $v|\bar{Z}_T:=v_T$ for $T\in[1,\infty)$. Then $u, v\in C^{2,1}(\bar{Z}_\infty)$, and (u, v) is a solution to $Q(a, \ldots, e, W, \infty)$. Moreover, we have for $w\in\{u, v\}$: $0\leqq w\leqq R$, $|w|_{2+\delta}\leqq mM/R$.

If U, V are also functions in $C^{2,1}(\bar{Z}_\infty)$ such that (U, V) solves $Q(a, \ldots, e, W, \infty)$, then one may conclude from Theorem 4.5:

$$u|\bar{Z}_T=U|\bar{Z}_T, \qquad v|\bar{Z}_T=V|\bar{Z}_T \qquad \text{for } T\in[1,\infty).$$

Hence $u=U$, $v=V$.

Appendix

Theorem A. *Let* $\alpha\in(0,1)$, $0<K_2\leqq K_1$, $T\in(0,\infty)$. *Then there is a number* $Q:=Q(\alpha, K_1, K_2, \Omega, T)>0$ *with the following properties:*

If $a_{ij}, b_i, c, f\in C_\alpha(\bar{Z}_T)$, $\Psi\in C_{2,\alpha}(\bar{\Omega})$, $u\in C^{2,1}(\bar{Z}_T)$ *with* $a_{ij}=a_{ji}$, $|a_{ij}|_\alpha$, $|b_i|_0$, $|c|_0 \leqq K_1$ $(1\leqq i, j\leqq N)$,

$$\sum_{i,j=1}^{N} a_{ij}(x,s)\xi_i\xi_j\geqq K_2|\xi|^2 \qquad ((x,s)\in\bar{Z}_T,\ \xi\in\mathbb{R}^N),$$

$$\sum_{i,j=1}^{N} a_{ij}D_iD_ju+\sum_{i=1}^{N} b_iD_iu+cu-D_{N+1}u=f,$$

$$\sum_{i,j=1}^{N} a_{ij}(x,s)D_iu(x,s)\,n_j(x)=0 \qquad ((x,s)\in S_T),$$

$$u(x,0)=\Psi(x) \qquad (x\in\bar{\Omega}),$$

then $|u|_\alpha\leqq Q(|\Psi|_2+|f|_0)$.

Remark. Some assumptions in the preceding theorem are stronger than necessary. For example, one could weaken the conditions $b_i, c, f\in C_\alpha(\bar{Z}_T)$, $\Psi\in C_{2,\alpha}(\bar{\Omega})$, $u\in C^{2,1}(\bar{Z}_T)$. Furthermore, it is not necessary in Theorem A that Ω is C^3-bounded, as assumed throughout this paper (Ω $C^{1+\beta}$-bounded, for some $\beta\in(0,1)$, would be sufficient). But in the context of this paper, we do not need a more general version of Theorem A.

Since a detailed proof of this theorem would be very long, we shall restrict ourselves to the most important points.

Proof of Theorem A. We consider the case $N\geqq 2$. (The following proof carries over to the case $N=1$ after some modifications.)

In the following, the letter q denotes constants which only depend on α, K_1, K_2, Ω, T.

Let $a_{ij}, b_i, c, f, \Psi, u$ $(1 \leqq i, j \leqq N)$ be given as in Theorem A.

Without loss of generality, we may assume $\Psi = 0$. Then we may represent the function u as follows:

$$u(x,t) = W(x,t) + V(x,t),$$

with

$$W(x,t) := \int_0^t \int_\Omega \Gamma(x,t,\xi,\tau) f(\xi,\tau)\, d\xi\, d\tau,$$

$$V(x,t) := \int_0^t \int_{\partial\Omega} \Gamma(x,t,\xi,\tau)\, \varphi(\xi,\tau)\, dS(\xi)\, d\tau \qquad ((x,t) \in \bar{\Omega} \times [0,T]),$$

where φ is a certain function from $C^0(\partial\Omega \times [0,T])$, with

(1) $|\varphi|_0 \leqq q |f|_0$,

and Γ is given by

(2) $\Gamma(x,t,\xi,\tau) = Z(x,t,\xi,\tau) + Z_0(x,t,\xi,\tau) \qquad (x, \xi \in \bar{\Omega},\ t, \tau \in [T_0, T_1] \text{ with } t > \tau)$,

where Z, Z_0 are certain functions from $C^0(\{(x,t,\xi,\tau) \in \mathbb{R}^N \times \mathbb{R} \times \bar{\Omega} \times [0,T]: t > \tau\}$ and $C^0(\{(x,t,\xi,\tau) \in (\bar{\Omega} \times [0,T])^2 : t > \tau\}$, respectively. We do not consider these functions in detail; we only note some properties which will be needed later on:

$$Z(\cdot,\cdot,\xi,\tau) \in C^{2,1}(\mathbb{R}^N \times (\tau,\infty)) \quad \text{for } (\xi,\tau) \in \bar{\Omega} \times [0,T];$$

for $\mu \in [0, N/2]$, there is a number $A_\mu > 0$, depending on μ, α, K_1, K_2, Ω, T, such that

(3) $|D_t^l D_x^a Z(x,t,\xi,\tau)| \leqq A_\mu (t-\tau)^{-\mu} |x-\xi|^{-N-2l-a_1-\dots-a_N+2\mu}$

for $l \in \mathbb{N}_0$, $a \in \mathbb{N}_0^N$ with $2l + a_1 + \dots + a_N \leqq 2$, $x \in \mathbb{R}^N$, $\xi \in \bar{\Omega}$ with $x \neq \xi$, $t \in \mathbb{R}$, $\tau \in [0,T]$ with $t > \tau$;

(4) $|Z_0(x,t,\xi,\tau)| \leqq A_\mu (t-\tau)^{-\mu} |x-\xi|^{-N+2\mu}$

for $x, \xi \in \bar{\Omega}$ with $x \neq \xi$, $t, \tau \in [0,T]$ with $t > \tau$;

(5) $|Z_0(x,t,\xi,\tau) - Z_0(x',t,\xi,\tau)| \leqq A_\mu |x-x'|^\alpha (t-\tau)^{-\mu} |x-\xi|^{-N+2\mu}$

for $x, x', \xi \in \bar{\Omega}$ with $|x-\xi| > 2|x'-x|$, $t, \tau \in [0,T]$ with $t > \tau$;

(6) $|Z_0(x,t,\xi,\tau) - Z_0(x,t',\xi,\tau)| \leqq A_\mu (t-t')^{\alpha/2} (t'-\tau)^{-\mu} |x-\xi|^{-N+2\mu}$

for $x, \xi \in \bar{\Omega}$ with $x \neq \xi$, $t, t', \tau \in [0,T]$ with $t > t' > \tau$.

For these results, with the exception of (5) and (6), we refer to [2] Sects. I.1–I.5, V.1–V.3. Inequality (5) is an easy consequence of [2] Lemma 1.3 and of the definition of Z_0 (see [2] (5.1.3)). Inequality (6) also follows from [2] Lemma 1.3 and [2] (5.1.3), but its proof is more difficult; here it matters that we only only assumed $|b_i|_0$, $|c|_0 \leqq K_1$. Under the assumptions $|b_i|_\alpha$, $|c|_\alpha \leqq K_1$, one may find a better result (compare [7] (IV.13.5)).

As shown in [2], p.193/194, it follows from (1) and (3):

$$|W|_\alpha \leqq q|f|_0.$$

It remains to be proved that

(7) $|V|_\alpha \leqq q|f|_0$.

For this result Friedman [2], p. 149, refers to Pogorzelski [9]. But since inequality (7) is not easily understood from [9], we shall prove (7) by a straightforward computation, although we cannot give all the details here:

Take $x \in \bar{\Omega}$, $t, t' \in [0, T]$ with $t > t'$. Then

$$\begin{aligned} |V(x,t)-V(x,t')| \leqq & \int_{t'}^{t} \int_{\partial\Omega} |\Gamma(x,t,\xi,\tau)\,\varphi(\xi,\tau)|\,dS(\xi)\,d\tau \\ & + \int_0^{t'} \int_{\partial\Omega} |(Z_0(x,t,\xi,\tau)-Z_0(x,t',\xi,\tau))\,\varphi(\xi,\tau)|\,dS(\xi)\,d\tau \\ & + \int_0^{t'} \int_{\partial\Omega} |(Z(x,t,\xi,\tau)-Z(x,t',\xi,\tau))\,\varphi(\xi,\tau)|\,dS(\xi)\,d\tau. \end{aligned}$$

The first and the second of the preceding integrals may be estimated by $q|f|_0|t-t'|^{\alpha/2}$; this is shown by using (1)–(4) and (6), We denote by $\mathfrak{J}$ the third of the preceding integrals. Then

$$\mathfrak{J} = \int_0^{t'} \sum_{i=1}^{k} \int_{U_i} |(Z(x,t,\overset{i}{h}(r),\tau)-Z(x,t',\overset{i}{h}(r),\tau))\,\varphi(\overset{i}{h}(r),\tau)|\,J_i(r)\,dr\,d\tau.$$

Here we applied the definition of a surface integral over $\partial\Omega$. Thus, k is some integer, $U_1, \ldots, U_k$ are certain subsets of $\mathbb{R}^{N-1}$, which are open and bounded. Furthermore, for $i \in \{1, \ldots, k\}$, $\overset{i}{h} := (\overset{i}{h}_1, \ldots, \overset{i}{h}_N)$ is a certain function from U_i into $\partial\Omega$, and J_i is a certain function from U_i into $[0, \infty)$. Of course, k, U_i, $\overset{i}{h}$, J_i ($i \in \{1, \ldots, k\}$) depend only on Ω. We may assume that J_i is bounded, and $\overset{i}{h}$ is of the form

$$\overset{i}{h}(r) = (r_1, \ldots, r_{j_i-1}, \overset{i}{h}_{j_i}(r), r_{j_i}, \ldots, r_{N-1}),$$

for $r \in U_i$, with some $j_i \in \{1, \ldots, N\}$.

Take $i \in \{1, \ldots, k\}$. For $y \in \mathbb{R}^N$, we abbreviate

$$\tilde{y} := (y_1, \ldots\, y_{j_i-1}, y_{j_i+1}, \ldots, y_N).$$

Set $h := \overset{i}{h}$, $U := U_i$. Then we find:

$$\int_0^{t'} \int_U |(Z(x,t,h(r),\tau) - Z(x,t',h(r),\tau))|\, dr\, d\tau$$

$$\leqq q \int_0^{t'} \int_{\substack{U \\ |r-\tilde{x}|<(t-t')^{1/2}}} (|Z(x,t,h(r),\tau)| + |Z(x,t',h(r),\tau)|)\, dr\, d\tau$$

$$+ q \int_0^{t'} \int_{\substack{U \\ |r-\tilde{x}|\geqq(t-t')^{1/2}}} (t-t') \int_0^1 |D_{N+1} Z(x, t'+\vartheta(t-t'), h(r), \tau)|\, d\vartheta\, dr\, d\tau$$

$$\leqq q \int_0^{t'} \int_{|r-\tilde{x}|<(t-t')^{1/2}} (t'-\tau)^{-(1+\alpha)/2} |x-h(r)|^{-N+1+\alpha}\, dr\, d\tau$$

$$+ q(t-t') \int_0^1 \int_0^{t'} \int_{|\tilde{x}-r|\geqq(t-t')^{1/2}} (t'+\vartheta(t-t')-\tau)^{-(1+\alpha)/2} |x-h(r)|^{-N-1+\alpha}\, dr\, d\tau\, d\vartheta$$

(Here we used (3), with $\mu := (1+\alpha)/2$)

$$\leqq q \int_{|r-\tilde{x}|<(t-t')^{1/2}} |\tilde{x}-r|^{-N+1+\alpha}\, dr + q(t-t') \int_{|r-\tilde{x}|\geqq(t-t')^{1/2}} |\tilde{x}-r|^{-N-1+\alpha}\, dr$$

$$\leqq q(t-t')^{\alpha/2}.$$

Thus we have shown:

(8) $|V(x,t) - V(x,t')| \leqq q |f|_0 (t-t')^{\alpha/2}$.

Using (1)–(5), one may show in a similar way:

(9) $|V(x,t) - V(x',t)| \leqq q |f|_0 |x-x'|^\alpha \quad (x, x' \in \bar{\Omega},\ t \in [0, T])$,

(10) $|V|_0 \leqq q |f|_0$.

Inequality (7) then follows from (8)–(10).

Theorem B. *Let* $\delta \in (0,1)$, $d, m, B, T > 0$, $a, b_i, c \in C_\delta(\bar{Z}_T)$ *with* $a(x,s) \geqq m$ *for* $(x,s) \in \bar{Z}_T$.

Let $\Psi \in C_{2,\delta}(\bar{\Omega})$ *with* $\frac{\partial \Psi}{\partial v}(x) = 0$ *for* $x \in \partial\Omega$.

Then there is a function $u \in C_{2+\delta}(\bar{Z}_T)$ *with*

(1) $a\Delta u + \sum_{i=1}^{N} b_i D_i u + cu - D_{N+1} u = d \min\{B, u^2\}$,

$$\frac{\partial u}{\partial v}(x,s) = 0 \quad ((x,s) \in S_T), \quad u(x,0) = \Psi(x) \quad (x \in \bar{\Omega}).$$

Proof. For any $v \in C_\delta(\bar{Z}_T)$, [6] Theorem IV.5.3 yields existence of a uniquely determined function $w := Sv \in C_{2+\delta}(\bar{Z}_T)$ such that

$$a\Delta w + \sum_{i=1}^{N} b_i D_i w + cw - D_{N+1} w = d \min\{B, v^2\},$$

$$\frac{\partial w}{\partial v}(x,s) = 0 \quad ((x,s) \in S_T), \quad w(x,0) = \Psi(x) \quad (x \in \bar{\Omega}).$$

Furthermore, it follows from [6] Theorem IV.5.3. that there is a constant $M_1>0$ with

$$(2)\quad |Sv|_{2+\delta} \leqq M_1(|\Psi|_{2,\delta}+|d\min\{B,v^2\}|_\delta) \leqq M_1(|\Psi|_{2,\delta}+d|v|_\delta^2) \quad (v\in C_\delta(\bar{Z}_T)).$$

From Theorem A we know there is a number $M_2>0$ such that for $v\in C_\delta(\bar{Z}_T)$, $|Sv|_\delta \leqq M_2(|\Psi|_2+|d\min\{B,v^2\}|_0)$.

Thus, setting $M_3 := M_2(|\Psi|_2+dB)$, we have

$$(3)\quad |Sv|_\delta \leqq M_3 \quad \text{for } v\in C_\delta(\bar{Z}_T).$$

Now define

$$W := \{v\in C_{2+\delta}(\bar{Z}_T): |v|_\delta \leqq M_3,\ |v|_{2+\delta} \leqq M_1(|\Psi|_{2,\delta}+dM_3^2)\}.$$

The restriction $S|W$ of S to W maps W into itself, as follows from (2) and (3). W is a compact, convex subset of $C_\delta(\bar{Z}_T)$. Using Theorem A, one can show that $S|W$ is continuous with respect to the norm $|\ |_\delta$. Now, applying Schauder's fixed point theorem (see [3] Theorem 10.1), we may conclude that there is a function $u\in W$ with $Su=u$. This function belongs to $C_{2+\delta}(\bar{Z}_T)$, and it is a solution to (1).

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