Analysis 20, 65 – 90 (2000)

Analysis

ARTIFICIAL BOUNDARY CONDITIONS FOR THE OSEEN SYSTEM IN 3D EXTERIOR DOMAINS.

Paul Deuring Stanislav Kračmar *

Received : April 26, 1999; revised: November 3, 1999

Abstract

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. We consider a solution (u, π) of the exterior Oseen problem

$$-\Delta u + \lambda \cdot D_1 u + \nabla \pi = f, \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3 \backslash \overline{\Omega},$$

satisfying the boundary conditions

$$u \,|\, \partial\Omega = b, \qquad u(x) \to 0 \ \ (|x| \to \infty).$$

This exterior Oseen flow is approximated by solutions of the Oseen problem in a truncated exterior domain $\Omega_R := \{\, y \in \mathbb{R}^3 \backslash \overline{\Omega} \,:\, |y| < R \,\}$, for $R > 0$, with a local artificial boundary condition prescribed on the truncating sphere $|y| = R$. The corresponding truncation error is estimated in L^2-norms.

AMS Subject Classification: **Primary** 35J99, 35Q72. **Secondary** 73K12, 73K15.

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain, and let λ be a positive real. It is well known that the Oseen system

(1.1) $-\Delta u + \lambda \cdot D_1 u + \nabla \pi = f, \quad \operatorname{div} u = 0$

in $\mathbb{R}^3 \setminus \overline{\Omega}$, with a Dirichlet boundary condition on $\partial\Omega$

(1.2) $u \,|\, \partial\Omega = b,$

and with a decay condition near infinity

(1.3) $u(x) \to 0$ for $|x| \to \infty,$

has a unique solution (u, π) ("exterior Oseen flow") in appropriate Sobolev spaces; see [14, pp. 355 ff.] and Theorem 3.8 below.

*Supported by grant No. 201/99/0267 from the Grant Agency of the Czech Republic, and partly supported by grant No. A2120801/98 from the Academy of Sciences of the Czech Republic.

In this paper, we shall study the question of how to approximate the exterior Oseen flow (u, π) by a suitable solution of the Oseen system on a truncated exterior domain $\Omega_R := B_R \setminus \overline{\Omega}$, where B_R denotes an open ball in $\mathbb{R}^3$ centered in the origin, with radius $R > 0$. More precisely, we shall consider the Oseen system

(1.4) $-\Delta v + \lambda \cdot D_1 v + \nabla \rho = f|\Omega_R, \quad \operatorname{div} v = 0 \quad \text{in } \Omega_R,$

under a Dirichlet boundary condition on $\partial\Omega$,

(1.5) $v \,|\, \partial\Omega = b,$

and under a so called "artificial boundary condition" on the sphere ∂B_R,

(1.6) $\mathcal{L}_R(v, \rho)(x) = 0 \quad \text{for } x \in \partial B_R,$

with the boundary operator $\mathcal{L}_R$ defined by

(1.7) $$\mathcal{L}_R(v, \rho)(x) := \left[\sum_{k=1}^{3} (D_k v_j(x) - \delta_{jk} \cdot \rho(x)) \cdot \frac{x_k}{R} + \frac{\lambda}{2} \cdot \left(1 - \frac{x_1}{R}\right) \cdot v_j(x) \right]_{1 \le j \le 3}$$

for $x \in \partial B_R$. It will turn out that problem (1.4) - (1.6) has a unique weak solution (v, ρ) in $W^{1,2}(\Omega_R)^3 \times L^2(\Omega_R)$. The difference between this solution and the exterior Oseen flow (u, π) ("truncation error") will be shown to decay with the following rates (see Theorem 4.5 and Corollary 4.4 below):

(1.8) $\|\nabla u|\Omega_R - \nabla v\|_2 = \mathcal{O}(R^{-1}), \quad \|u|\Omega_S - v|\Omega_S\|_2 = \mathcal{O}(R^{-1}), \quad \|\pi|\Omega_R - \rho\|_2 = \mathcal{O}(1)$

for $R \to \infty$. Here S is some fixed positive real with $\overline{\Omega} \subset B_S$. The relations in (1.8) hold under the assumptions $f \in L^2(\mathbb{R}^3)^3$, $f(x) = \mathcal{O}(|x|^{-4-\varepsilon})$ for some $\varepsilon > 0$. Our approach additionally yields pointwise decay rates for the exterior Oseen flow (u, π) (Theorem 3.9). We have for example

$$|\,u(x)\,| \cdot (1 - x_1/|x|) \;\le\; C \cdot |x|^{-2},$$

under assumptions on f and b as stated above. Our results may be extended to the following, more general variant of the Oseen system,

$$-\Delta \tilde{u} + \tilde{u}_0 \cdot \nabla \tilde{u} + \nabla \tilde{\pi} = \tilde{f}, \quad \operatorname{div} \tilde{u} = 0$$

in an exterior domain $\mathbb{R}^3 \setminus \overline{U}$, where $\tilde{u}_0$ is a vector in $\mathbb{R}^3$, and $U \subset \mathbb{R}^3$ is a bounded Lipschitz domain. Moreover, boundary condition (1.3) near infinity may be replaced by the more general condition $\tilde{u}(x) \to \tilde{u}_\infty$ for $|x| \to \infty$, for a fixed vector $\tilde{u}_\infty \in \mathbb{R}^3$. The reduction to (1.3) may, of course, be achieved by the translation $u = \tilde{u} - \tilde{u}_\infty$.

Problem (1.4) - (1.6) is of interest because it may yield a discretization of the exterior Oseen flow (u, π). In fact, since the domain Ω_R is bounded, standard numerical methods should yield an approximate solution of (1.4) - (1.6), which may then be considered an approximation of the exterior Oseen flow. Of course, this point of view must be justified by suitable error estimates. Previously a similar approach was applied to the Poisson equation ([17], [18], [1]) and to the Stokes system ([21], [6], [7], [8]). In the work at hand, we propose a crucial first step in order to adapt this method to the Oseen system: as indicated above, we present a well posed boundary value problem for the Oseen system in the domain Ω_R, and then estimate the truncation error arising between the solution of this problem on one hand and the exterior Oseen flow on the other.

Note that our artificial boundary condition (1.6) is local, or in other words, a pointwise condition. Thus, contrary to the situation arising with nonlocal conditions, we admit functions f which do not have compact support. The choice of a local boundary condition further implies that a finite element method should suffice to discretize problem (1.4) - (1.6); there is no need to couple finite element with boundary element methods. On the negative side, a solution of (1.4) - (1.6) will not, in general, coincide with the restriction of the exterior Oseen flow to Ω_R; in other words, a truncation error arises. However, this is not a serious drawback as long as the truncation error may be kept smaller than an eventual discretisation error. We refer to [17], [18] for a discussion of this point, which is only of marginal interest here because we shall restrict ourselves to treating the continuous problem.

To cite some results related to ours, we remark that local artificial boundary conditions for the Poisson equation and related estimates for the truncation error were derived in [17], [18], [31]. Similar results for the Stokes system may be found in [6], [21], [20], [32].

There are many articles dealing with pointwise boundary conditions for various partial differential equations, but without giving estimates of the truncation error. We only mention the papers [24], [2], [26], [28], [29] treating the instationary Navier-Stokes system. Otherwise we refer to the survey paper [37] by Tsynkov and the monograph [16] by Givoli. Concerning nonlocal artificial boundary conditions, we mention the paper [3] pertaining to the Poisson equation, the references [19], [23], [35], [36] treating the Stokes system, and the article [22], where the instationary Oseen system in halfspace is considered, with a hyperplane as artificial boundary. Further references may be found in [37].

Let us briefly explain our choice of the operator $\mathcal{L}_R$. To this end suppose that $R \in (0,\infty)$ with $\overline{\Omega} \subset B_R$, and that $(v_R, \varrho_R) \in H^2(\Omega_R)^3 \times H^1(\Omega_R)$ is a solution of system (1.4) under boundary condition (1.5) on $\partial\Omega$. Further suppose the exterior Oseen flow (u,π) satisfies the condition $u\,|\,\Omega_R \in H^2(\Omega_R)^3$, $\pi\,|\,\Omega_R \in H^1(\Omega_R)$. In order to estimate the difference between $u\,|\,\Omega_R$ and v_R in the gradient norm, we observe

$$
\begin{aligned}
(1.9)\quad & \|\nabla(u\,|\,\Omega_R - v_R)\|_2^2 \\
&= \int_{\Omega_R} \Big[\sum_{j,k=1}^{3} (D_j u_k - D_j v_{R,k})^2 - \operatorname{div}(u - v_R)\cdot(\pi - \varrho_R) \Big]\, dx \\
&\quad + \int_{\partial B_R} (\lambda/2)\cdot(u - v_R)^2(x)\cdot(x_1/R)\, do_x \\
&\quad - \int_{\partial B_R} (\lambda/2)\cdot(u - v_R)^2(x)\cdot(x_1/R)\, do_x \\
&= \int_{\partial B_R} [\widetilde{\mathcal{L}}_R(u,\pi) - \widetilde{\mathcal{L}}_R(v_R,\varrho_R)]\cdot(u - v_R)\, do_x,
\end{aligned}
$$

where

$$
\widetilde{\mathcal{L}}_R(w,\sigma)(x) := \Big[\sum_{k=1}^{3} (D_k w_j(x) - \delta_{jk}\cdot\sigma(x))\cdot(x_k/R) - (\lambda/2)\cdot w_j(x)\cdot(x_1/R) \Big]_{1\le j\le 3}
$$

for $x \in \partial B_R$, $(w,\sigma) \in \{(u,\pi), (v_R,\varrho_R)\}$. Equation (1.9) is valid because (u,π) is a solution of (1.1), and v_R solves (1.4). Now assume that $\widetilde{\mathcal{L}}_R(v_R,\varrho_R) = 0$. Then we have by (1.9)

$$
(1.10)\quad \|\nabla(u\,|\,\Omega_R - v_R)\|_2^2 = \int_{\partial B_R} \widetilde{\mathcal{L}}_R(u,\pi)\cdot(u - v_R)\, do_x
$$

$$\leq \ (4 \cdot \epsilon)^{-1} \cdot \| \tilde{\mathcal{L}}_R(u, \pi) \|_2^2 \ + \ \epsilon \cdot \| (u - v_R) \,|\, \partial B_R \|_2^2 \qquad \text{for } \epsilon > 0.$$

Thus, if the term $\epsilon \cdot \| (u - v_R) \,|\, \partial B_R \|_2^2$ were dominated by the left-hand side of (1.10), for ϵ small but independent of R, we would have reduced an estimate of the truncation error $u \,|\, \Omega_R - v_R$ to an estimate of the term $\| \tilde{\mathcal{L}}_R(u, \pi) \|_2$. This latter expression only depends on the exterior flow (u, π). At this point, three questions arise. First, does problem (1.4), (1.5) with the artificial boundary condition $\tilde{\mathcal{L}}_R(v_R, \varrho_R) = 0$ on ∂B_R admit a solution? Second, if such a solution exists, may the term $\epsilon \cdot \| \tilde{\mathcal{L}}_R(v_R, \varrho_R) \|_2^2$ be dominated by the left-hand side in (1.10), and third, what is the asymptotic behaviour of $\| \tilde{\mathcal{L}}_R(u, \pi) \|_2$ when R tends to infinity?

We were only able to answer the last question, and this answer was disappointing. In fact, by Corollary 3.8 and the method used for its proof, we obtain

$$(1.11) \ \ \| \nabla u \,|\, \partial B_R \|_2 = O(R^{-1}), \quad \| \pi \,|\, \partial B_R \|_2 = O(R^{-1}), \quad \| u(x) \cdot x_1/R \,|\, \partial B_R \|_2 = O(1)$$

for $R \to \infty$. This means the components of the term $\tilde{\mathcal{L}}_R(v_R, \varrho_R)$ are not well balanced as regards their asymptotic behaviour for $R \to \infty$.

The situation changes when the term $w(x) \cdot x_1/R$ in the definition of the boundary operator $\tilde{\mathcal{L}}_R$ is replaced by $w(x) \cdot (1 - x_1/R)$, that is, when the operator $\tilde{\mathcal{L}}_R$ is replaced by $\mathcal{L}_R$. Then problem (1.4), (1.5) with boundary condition (1.6) on ∂B_R may be written as a mixed variational problem admitting a unique solution (v_R, ϱ_R), and the truncation error $\| \nabla(u \,|\, \Omega_R - v_R) \|_2$ may be estimated against $\| \mathcal{L}_R(u, \pi) \|_2$, due to a variant of inequality (1.10); see Section 4. Moreover we have

$$\| u(x) \cdot (1 - x_1/R) \,|\, \partial B_R \|_2 \ = \ O(R^{-1}) \quad (R \to \infty)$$

(Corollary 3.8), so that by (1.11)

$$\| \mathcal{L}_R(u, \pi) \|_2 \ = \ O(R^{-1}) \quad (R \to \infty).$$

It remains an open question whether there is a different boundary operator leading to a higher rate of decay of the truncation error.

We remark that our decay estimates of the term $\| \mathcal{L}_R(u, \pi) \|_2$ are first reduced to the case of boundary data b orthogonal to $n^{(\Omega)}$ with respect to the L^2-scalar product on $\partial\Omega$, where $n^{(\Omega)}$ denotes the outward unit normal to Ω. In that case the exterior Oseen flow may be represented by the sum of a single layer and a volume potential. This integral representation, derived in Section 3, is more specific than the one given in [14, Theorem VII.6.2], and allows us to quantify the influence of the data on the decay of the exterior Oseen flow.

2 Some notations

For a multiindex $b \in \mathbb{N}_0^3$, the length $|b|_*$ of b is defined by $|b|_* := b_1 + b_2 + b_3$. If $A \subset \mathbb{R}^3$, then ∂A denotes the boundary, and $\chi_A : \mathbb{R}^3 \mapsto \{0, 1\}$ the characteristic function of A. Assume that A is Lebesgue measurable. For $p \in [1, \infty)$, we denote by $L^p(A)$ the set of all p-integrable functions from A into $\mathbb{R}$. The symbol $\| \ \|_p$ stands for the usual norm of this space.

Let $G \subset \mathbb{R}^3$ be open. If $N \in \mathbb{N}$ and f is a function from G into $\mathbb{R}^N$ with suitable smoothness, the symbols $D_k f, D_k D_l f, D^b f$, for $k, l \in \{1, 2, 3\}$, $b \in \mathbb{N}_0^3$, are used in an obvious way in order to denote partial derivatives. We further introduce the abbreviations ∇f, Δf, div$f$ for the gradient of f, the Laplacian applied to f, and the divergence of f, respectively.

For $p \in (1, \infty)$, $k \in (0, \infty)$, and under appropriate assumptions on G, we denote by $W^{k,p}(G)$, $W^{k,p}(\partial G)$ the usual Sobolev spaces; see [13, pp. 255-256, 330, 332]. In the case $p = 2$, we shall write $H^k(G)$, $H^k(\partial G)$ instead of $W^{k,2}(G)$, $W^{k,2}(\partial G)$, respectively.

The parameter $\lambda \in (0, \infty)$ will be kept fixed throughout. We assume that $\Omega \subset \mathbb{R}^3$ is a bounded domain in $\mathbb{R}^3$ with Lipschitz boundary. This domain will be kept fixed as well. The case $\Omega = \emptyset$ is admitted, although we shall not explicitly deal with this case. In fact, if $\Omega = \emptyset$, our arguments may be somewhat simplified. The outward unit normal to Ω will be denoted by $n^{(\Omega)}$.

As mentioned in the preceding section, we write B_R for the open ball in $\mathbb{R}^3$ with radius $R > 0$ and center in the origin. The truncated exterior domain Ω_R is defined by $\Omega_R := B_R \setminus \overline{\Omega}$, for $R \in (0, \infty)$ with $\overline{\Omega} \subset B_R$. We fix a number $S > 0$ with the property that $\overline{\Omega} \subset B_S$.

Finally, we define functions s, η^α_β by

$$s(x) := |x| - x_1, \quad \eta^\alpha_\beta(x) := (1 + |x|)^\alpha \cdot (1 + s(x))^\beta \quad \text{for } x \in \mathbb{R}^3, \quad \alpha, \beta \in \mathbb{R},$$

where $|x| := (x_1^2 + x_2^2 + x_3^2)^{1/2}$.

3 Solutions of the Oseen system in exterior domains

In this section we consider the Oseen problem (1.1) – (1.3). We will present some results on existence, regularity and decay of solutions to this problem. To begin with, we introduce some notations.

Let $E := (E_{jk})_{1 \le j \le 4,\, 1 \le k \le 3}$ denote the fundamental solution of the 3D Oseen system (1.1) presented in [14, Section VII.3], that is,

$$E_{jk}(z) := E_{jk}(z; \lambda) := (\delta_{jk} \cdot \Delta - D_j D_k) \Phi(z; \lambda), \qquad E_{4k}(z) := (4 \cdot \pi)^{-1} \cdot z_k \cdot |z|^{-3},$$

for $z \in \mathbb{R}^3 \setminus \{0\}$, $1 \le j, k \le 3$, where

$$\Phi(z; \lambda) := (4 \cdot \pi \cdot \lambda)^{-1} \cdot \psi(\lambda \cdot s(z)/2), \quad \psi(z) := \int_0^z (1 - e^{-t}) \cdot t^{-1}\, dt, \quad s(z) := |z| - z_1.$$

Moreover, put

$$U_{jk}(z) := (8 \cdot \pi)^{-1} \cdot \left(|z|^{-1} \cdot \delta_{jk} + z_j \cdot z_k \cdot |z|^{-3} \right), \qquad U_{4k}(z) := (4 \cdot \pi)^{-1} \cdot z_k \cdot |z|^{-3},$$

for $z \in \mathbb{R}^3 \setminus \{0\}$, $1 \le j, k \le 3$. Thus the matrix-valued function $(U_{jk})_{1 \le j \le 4,\, 1 \le k \le 3}$ is a fundamental solution of the Stokes system $-\Delta u + \nabla \pi = f$, $\operatorname{div} u = 0$, and it holds $E_{4k} = U_{4k}$ for $1 \le k \le 3$. We shall need the following properties of E:

THEOREM 3.1 *Let $j, k \in \{1, 2, 3\}$. Then $E_{jk}, E_{4k} \in C^\infty(\mathbb{R}^3 \setminus \{0\})$, and*

$$-\Delta E_{jk} + \lambda \cdot D_1 E_{jk} + D_j E_{4k} = 0, \qquad \sum_{i=1}^{3} D_i E_{ik} = 0.$$

If $a = (a_1, a_2, a_3) \in \mathbb{N}_0^3$, $|a|_* \le 1$, then there is a constant $C = C(\lambda) > 0$ such that

(3.1) $\quad |D^a E_{jk}(z)| \le C \cdot |z|^{-1-|a|_*}, \quad z \in B_1 \setminus \{0\},$

(3.2) $|D^\alpha E_{4k}(z)| \le C \cdot |z|^{-2-|\alpha|_\bullet}, \quad z \in \mathbb{R}^3 \setminus \{0\},$

(3.3) $|D^\alpha E_{jk}(z)| \le C \cdot \eta^{-(2+|\alpha|_\bullet)/2}_{-(2+|\alpha|_\bullet)/2}(z), \quad z \in \mathbb{R}^3 \setminus B_1$

(3.4) $|D^\alpha E_{jk}(z)| \le C \cdot \eta^{-(3+|\alpha|_\bullet)/2}_{-(1+|\alpha|_\bullet)/2}(z), \quad z \in \mathbb{R}^3 \setminus B_1, \ a_1 \neq 0,$

(3.5) $|D^\alpha E_{jk}(z) - D^\alpha U_{jk}(z)| \le C \cdot |z|^{-|\alpha|_\bullet}, \quad z \in B_1 \setminus \{0\}.$

Proof: For a proof of estimates (3.1) - (3.4), we refer to [14, Section VII.3]. Inequality (3.5) is proved in [27, (1.12)]. □

LEMMA 3.1 *For $\alpha, \beta \in \mathbb{R}$ and $S > 0$, there exists $C = C(S, \alpha, \beta) > 0$ such that*

$$(3.6)\quad \int_{\partial B_R} \eta^{-\alpha}_{-\beta}(x)\, do_x \quad \le \quad C \cdot R^{2-\alpha-\min\{1,\beta\}} \cdot \left\{ \begin{array}{ll} \ln(1+R), & \beta = 1 \\ 1 & \beta \neq 1 \end{array} \right\} \quad \textit{for } R \ge S.$$

For a proof we refer to [12, Lemma 2.3].

Let $U \subset \mathbb{R}^3$ be a bounded Lipschitz domain. We introduce single-layer potentials $V(\partial U)(\Phi) : \mathbb{R}^3 \mapsto \mathbb{R}^3$ and $Q(\partial U)(\Phi) : \mathbb{R}^3 \setminus \partial U \mapsto \mathbb{R}$ by

$$V(\partial U)(\Phi)(x) \;:=\; \left(\int_{\partial U} \sum_{k=1}^{3} E_{jk}(x-y) \cdot \Phi_k(y)\, dU_y \right)_{1 \le j \le 3},$$

$$Q(\partial U)(\Phi)(x') \;:=\; \int_{\partial U} \sum_{k=1}^{3} E_{4k}(x'-y) \cdot \Phi_k(y)\, dU_y$$

for $\Phi \in L^p(\partial U)^3$, $p \in (1,\infty)$, $x \in \mathbb{R}^3$, $x' \in \mathbb{R}^3 \setminus \partial U$. In order to establish some fundamental properties of these potientials, we note

LEMMA 3.2 *Let $p \in (1,\infty)$, $k \in (-\infty, 2)$. Then there is some $C = C(k, \Omega, p) > 0$ such that it holds for $f \in L^p(\partial\Omega)$:*

$$\left(\int_{\partial\Omega} \left(\int_{\partial\Omega} |x-y|^{-k} \cdot |f(y)|\, d\Omega_y \right)^p d\Omega_x \right)^{1/p} \quad \le \quad C \cdot \|f\|_p.$$

The lemma follows by Hölder's inequality; see [5, Lemma 4.9]. As a first application, we observe

LEMMA 3.3 *Let $p \in (1,\infty)$, $\Phi \in L^p(\partial\Omega)^3$. Then the functions $V(\partial\Omega)(\Phi) : \mathbb{R}^3 \mapsto \mathbb{R}^3$, $Q(\partial\Omega)(\Phi) : \mathbb{R}^3 \setminus \partial\Omega \mapsto \mathbb{R}$ are well defined. It holds $V(\partial\Omega)(\Phi)|\mathbb{R}^3 \setminus \partial\Omega \in C^\infty(\mathbb{R}^3 \setminus \partial\Omega)^3$, $Q(\partial\Omega)(\Phi) \in C^\infty(\mathbb{R}^3 \setminus \partial\Omega)$ and*

$$D^\alpha \left[V(\partial\Omega)(\Phi)|\mathbb{R}^3 \setminus \partial\Omega \right](x) \;=\; \left(\int_{\partial\Omega} \sum_{k=1}^{3} D^\alpha E_{jk}(x-y) \cdot \Phi_k(y)\, d\Omega_y \right)_{1 \le j \le 3},$$

$$D^\alpha Q(\partial\Omega)(\Phi)(x) \;=\; \int_{\partial\Omega} \sum_{k=1}^{3} D^\alpha E_{4k}(x-y) \cdot \Phi_k(y)\, d\Omega_y$$

for $a \in \mathbb{N}_0^3$, $x \in \mathbb{R}^3 \setminus \partial\Omega$. Moreover

$$-\Delta \left[V(\partial\Omega)(\Phi)|\mathbb{R}^3 \setminus \partial\Omega \right] + \lambda \cdot D_1 \left[V(\partial\Omega)(\Phi)|\mathbb{R}^3 \setminus \partial\Omega \right] + \nabla Q(\partial\Omega)(\Phi) \;=\; 0,$$

$$\operatorname{div}\left[V(\partial\Omega)(\Phi)|\mathbb{R}^3\setminus\partial\Omega\right] = 0.$$

Proof: We know by Lemma 3.2, (3.1) and (3.3) that

$$\int_{\partial\Omega}\left(\int_{\partial\Omega}\left|\sum_{k=1}^{3}E_{jk}(x-y)\cdot\Phi_k(y)\right|\,\mathrm{d}\Omega_y\right)^p\mathrm{d}\Omega_x < \infty.$$

This means the integral $\int_{\partial\Omega}\left|\sum_{k=1}^{3}E_{jk}(x-y)\cdot\Phi_k(y)\right|\mathrm{d}\Omega_y$ exists for $1\le j\le 3$ and for almost every $x\in\partial\Omega$. Obviously this integral exists as well for any $x\in\mathbb{R}^3\setminus\partial\Omega$. Thus the function $V(\partial\Omega)(\Phi)$ is well defined. The other statements of the lemma are obvious. □

LEMMA 3.4 *Let $p\in(1,\infty)$. Recall that $S\in(0,\infty)$ with $\overline{\Omega}\subset B_S$. There is a constant $C=C(\lambda,\Omega,S,p)>0$ such that*

(3.7) $\quad |D^a V(\partial\Omega)(\Phi)(x)| \le C\cdot\|\Phi\|_p\cdot|x|^{-(2+|a|_*)/2}\cdot(1+s(x))^{-(2+|a|_*)/2},$

(3.8) $\quad |Q(\partial\Omega)(\Phi)(x)| \le C\cdot\|\Phi\|_p\cdot|x|^{-2},$

for $\Phi\in L^p(\partial\Omega)^3$, $x\in\mathbb{R}^3\setminus B_{2S}$, $a\in\mathbb{N}_0^3$ with $|a|_*\le 1$. Moreover

$$V(\partial\Omega)(\Phi)\in L^6(\mathbb{R}^3)^3,\qquad \nabla\left[V(\partial\Omega)(\Phi)|\mathbb{R}^3\setminus\partial\Omega\right]\in L^2(\mathbb{R}^3\setminus\partial\Omega)^9,$$

$$Q(\partial\Omega)(\Phi)\in L^2(\mathbb{R}^3\setminus\partial\Omega)\qquad \textit{for } \Phi\in L^2(\partial\Omega)^3.$$

Proof: The estimates in (3.7) and (3.8) are obvious consequences of (3.2) and (3.3). Let $\Phi\in L^2(\partial\Omega)^3$. By [9, Lemma 5.7] it holds

$$V(\partial\Omega)(\Phi)|B_{2S}\setminus\overline{\Omega}\in W^{1,2}(B_{2S}\setminus\overline{\Omega})^3,\qquad Q(\partial\Omega)(\Phi)|B_{2S}\setminus\overline{\Omega}\in L^2(B_{2S}\setminus\overline{\Omega}),$$

$$V(\partial\Omega)(\Phi)|\Omega\in W^{1,2}(\Omega)^3,\qquad Q(\partial\Omega)(\Phi)|\Omega\in L^2(\Omega).$$

It follows $V(\partial\Omega)|B_{2S}\in L^6(B_{2S})^3$. ~~It is clear by (3.7), (3.8) and~~ We conclude from Lemma 3.1 that

$$V(\partial\Omega)(\Phi)|\mathbb{R}^3\setminus B_{2S}\in L^6(\mathbb{R}^3\setminus B_{2S})^3,\qquad \nabla\left[V(\partial\Omega)(\Phi)|\mathbb{R}^3\setminus B_{2S}\right]\in L^2(\mathbb{R}^3\setminus B_{2S})^9,$$

$$Q(\partial\Omega)(\Phi)|\mathbb{R}^3\setminus B_{2S}\in L^2(\mathbb{R}^3\setminus B_{2S}).$$

Combining the preceding relations yields the lemma. □

We introduce some notations related to the boundary of Ω. Let $\varepsilon,\delta\in(0,\infty)$, $y,z\in\mathbb{R}^3$ with $|z|=1$. Then we define the cone $K(y,z,\delta,\varepsilon)$ with vertex y and with axis pointing in the direction z by setting

$$K(y,z,\delta,\varepsilon) := \left\{y+t\cdot b\,:\,t\in(0,\delta),\ b\in\mathbb{R}^3,\ |b|=1,\ |b-z|<\varepsilon\right\}.$$

We choose a non-tangential vector field $m^{(\Omega)}:\partial\Omega\mapsto\mathbb{R}^3$. This means it holds $|m^{(\Omega)}(x)|=1$ for $x\in\partial\Omega$, there is a function $\widetilde{m}\in C^\infty(\mathbb{R}^3)^3$ with $\widetilde{m}|\partial\Omega=m^{(\Omega)}$, and there are constants $\mathcal{D}_1=\mathcal{D}_1(\Omega)>0,\cdots,\mathcal{D}_4=\mathcal{D}_4(\Omega)>0$ with

$$K(x,m^{(\Omega)}(x),\mathcal{D}_3,\mathcal{D}_4)\subset\mathbb{R}^3\setminus\overline{\Omega},\quad K(x,-m^{(\Omega)}(x),\mathcal{D}_3,\mathcal{D}_4)\subset\Omega\quad \text{for } x\in\partial\Omega,$$

(3.9) $\quad \left|x+\kappa\cdot m^{(\Omega)}(x)-x'-\kappa'\cdot m^{(\Omega)}(x')\right| \ge \mathcal{D}_2\cdot(|x-x'|+|\kappa-\kappa'|)$

for $x,x'\in\partial\Omega$, $\kappa,\kappa'\in(-\mathcal{D}_1,\mathcal{D}_1)$. Some indications on the construction of such a function

$m^{(\Omega)}$ are given in [33, p. 246] and [9, p. 119]. Note that we may require in addition that there is some $\delta > 0$ with

(3.10) $|n^{(\Omega)}(x) \cdot m^{(\Omega)}(x)| \geq \delta \quad$ for $x \in \partial\Omega$,

where $n^{(\Omega)}$ denotes the outward unit normal to Ω. For $\kappa \in (0, \mathcal{D}_1)$, we define

$$\Omega^{(-1,\kappa)} := \left\{x \in \mathbb{R}^3 : \operatorname{dist}(x, \mathbb{R}^3 \setminus \Omega) < \mathcal{D}_2 \cdot \kappa\right\},$$

$$\Omega^{(1,\kappa)} := \left\{x \in \mathbb{R}^3 : \operatorname{dist}(x, \overline{\Omega}) < \mathcal{D}_2 \cdot \kappa\right\}.$$

For $\tau \in \{-1, 1\}$, $\kappa \in (0, \mathcal{D}_1)$, the set $\Omega^{(\tau,\kappa)}$ is open, and $\overline{\Omega} \subset \Omega^{(1,\kappa)}$, $\mathbb{R}^3 \setminus \Omega \subset \Omega^{(-1,\kappa)}$.

Let $\kappa \in (0, \mathcal{D}_1)$, $x \in \Omega^{(1,\kappa)}$, $y \in \partial\Omega$, and distinguish two cases. First, assume $x \in \Omega^{(1,\kappa)} \cap \Omega^{(-1,\kappa)}$. Then there exists $z \in \partial\Omega$ with $|x - z| < \mathcal{D}_2 \cdot \kappa$, and it follows by (3.9)

$$\left|x - y - \kappa \cdot m^{(\Omega)}(y)\right| \geq \left|z - y - \kappa \cdot m^{(\Omega)}(y)\right| - |x - z| \geq \mathcal{D}_2 \cdot \kappa - |x - z| > 0.$$

Second, we assume $x \in \Omega^{(1,\kappa)} \setminus \Omega^{(-1,\kappa)}$. Then we may conclude $\operatorname{dist}(x, \mathbb{R}^3 \setminus \Omega) \geq \mathcal{D}_2 \cdot \kappa$. Since $y + \kappa \cdot m^{(\Omega)}(y) \in \mathbb{R}^3 \setminus \Omega$, we obtain $|x - y - \kappa \cdot m^{(\Omega)}(y)| \geq \mathcal{D}_2 \cdot \kappa > 0$.

A similar argument holds if $x \in \Omega^{(-1,\kappa)}$. Thus we have shown

(3.11) $\left|x - y - \tau \cdot \kappa \cdot m^{(\Omega)}(y)\right| > 0 \quad$ for $\kappa \in (0, \mathcal{D}_1)$, $\tau \in \{-1, 1\}$, $y \in \partial\Omega$, $x \in \Omega^{(\tau,\kappa)}$.

This observation yields an easy way to approximate the single layer potentials $V(\partial\Omega)(\Phi)$ and $Q(\partial\Omega)(\Phi)$ by smooth functions. In fact, let $\kappa \in (0, \mathcal{D}_1)$, $\tau \in \{-1, 1\}$, $p \in (1, \infty)$, $\Phi \in L^p(\partial\Omega)^3$. Then we define functions $V^{(\tau,\kappa)} := V^{(\tau,\kappa)}(\Omega)(\Phi) : \Omega^{(\tau,\kappa)} \mapsto \mathbb{R}^3$ and $Q^{(\tau,\kappa)} := Q^{(\tau,\kappa)}(\Omega)(\Phi) : \Omega^{(\tau,\kappa)} \mapsto \mathbb{R}$ by

$$V^{(\tau,\kappa)}(x) := \left(\int_{\partial\Omega} \sum_{k=1}^{3} E_{jk}(x - y - \tau \cdot \kappa \cdot m^{(\Omega)}(y)) \cdot \Phi_k(y)\, d\Omega_y\right)_{1 \leq j \leq 3},$$

$$Q^{(\tau,\kappa)}(x) := \int_{\partial\Omega} \sum_{k=1}^{3} E_{4k}(x - y - \tau \cdot \kappa \cdot m^{(\Omega)}(y)) \cdot \Phi_k(y)\, d\Omega_y \qquad \text{for } x \in \Omega^{(\tau,\kappa)}.$$

LEMMA 3.5 *Let* $p \in (1, \infty)$, $\Phi \in L^p(\partial\Omega)^3$, $\kappa \in (0, \mathcal{D}_1)$, $\tau \in \{-1, 1\}$. *Then* $V^{(\tau,\kappa)} \in C^\infty(\Omega^{(\tau,\kappa)})^3$, $Q^{(\tau,\kappa)} \in C^\infty(\Omega^{(\tau,\kappa)})$,

$$D^a V_j^{(\tau,\kappa)}(x) = \int_{\partial\Omega} \sum_{k=1}^{3} D^a E_{jk}(x - y - \tau \cdot \kappa \cdot m^{(\Omega)}(y)) \cdot \Phi_k(y)\, d\Omega_y,$$

$$D^a Q^{(\tau,\kappa)}(x) = \int_{\partial\Omega} \sum_{k=1}^{3} D^a E_{4k}(x - y - \tau \cdot \kappa \cdot m^{(\Omega)}(y)) \cdot \Phi_k(y)\, d\Omega_y$$

for $x \in \Omega^{(\tau,\kappa)}$, $a \in \mathbb{N}_0^3$, $1 \leq j \leq 3$. *Moreover,*

$$-\Delta V^{(\tau,\kappa)} + \lambda \cdot D_1 V^{(\tau,\kappa)} + \nabla Q^{(\tau,\kappa)} = 0, \qquad \operatorname{div} V^{(\tau,\kappa)} = 0.$$

Proof: Obvious by (3.11) and Lebesgue's theorem on dominated convergence. □

Let us now make precise our claim that the functions $V^{(\tau,\kappa)}$ and $Q^{(\tau,\kappa)}$ approximate the layer potential $V(\partial\Omega)(\Phi)$ and $Q(\partial\Omega)(\Phi)$, respectively.

LEMMA 3.6 *Let* $p \in (1,\infty)$, $\Phi \in L^p(\partial\Omega)^3$, $\tau \in \{-1,1\}$. *Then*

$$\int_{\partial\Omega} \left|V^{(\tau,\kappa)} - V(\partial\Omega)(\Phi)\right|^p d\Omega \longrightarrow 0 \quad \text{if} \quad \kappa \downarrow 0.$$

Proof: Use (3.9), Lemma 3.2 and Lebesgue's theorem on dominated convergence. □

LEMMA 3.7 *Let* $p \in (1,\infty)$, $\Phi \in L^p(\partial\Omega)^3$, $R \in (0,\infty)$ *with* $\overline{\Omega} \subset B_{R/2}$, $a \in \mathbb{N}_0^3$ *with* $|a|_* \leq 1$. *Then*

$$\int_{\partial B_R} \left|\partial^a/\partial x^a V^{(-1,\kappa)}(x) - \partial^a/\partial x^a V(\partial\Omega)(\Phi)(x)\right|^p d\Omega(x) \longrightarrow 0,$$

$$\int_{\partial B_R} \left|Q^{(-1,\kappa)} - Q(\partial\Omega)(\Phi)\right|^p d\Omega \longrightarrow 0 \quad \text{for } \kappa \downarrow 0.$$

Proof: Obvious by Lebesgue's theorem on dominated convergence. □

LEMMA 3.8 *Let* $p \in (4/3,\infty)$, $\Phi \in L^p(\partial\Omega)^3$, $j \in \{1,2,3\}$, $a \in \mathbb{N}_0^3$ *with* $|a|_* \leq 1$. $R \in (0,\infty)$ *with* $\overline{\Omega} \subset B_R$. *Then*

$$\int_{B_R\setminus\overline{\Omega}} \left|D^a V_j^{(-1,\kappa)}(x) \;-\; \partial^a/x^a\left[V_j(\partial\Omega)(\Phi)(x)\right]\right|^2 dx \;\longrightarrow\; 0,$$

$$\int_{\Omega} \left|D^a V_j^{(1,\kappa)}(x) \;-\; \partial^a/x^a\left[V_j(\partial\Omega)(\Phi)(x)\right]\right|^2 dx \;\longrightarrow\; 0 \quad \text{for} \quad \kappa \downarrow 0.$$

Proof: Lemma 3.8 follows from (3.1), (3.3), by the same arguments as used in the proof of [5, p. 249, Lemma 13.1]. □

Let us note a consequence of Lemma 3.6 and 3.8:

COROLLARY 3.1 *Let* $p \in (4/3,\infty)$, $\Phi \in L^p(\partial\Omega)^3$. *Then*

$$V(\partial\Omega)(\Phi)|\partial\Omega \;=\; \text{trace}\left[V(\partial\Omega)(\Phi)|\mathbb{R}^3 \setminus \overline{\Omega}\right] \;=\; \text{trace}\left[V(\partial\Omega)(\Phi)|\Omega\right].$$

Next we consider double-layer potentials with the kernel $D_jE_{kl} - D_jU_{kl} + D_kE_{jl} - D_kU_{jl}$.

LEMMA 3.9 *Let* $p \in (1,\infty)$, $\Phi \in L^p(\partial\Omega)^3$. *Then the function* $\mathcal{K} : \mathbb{R}^3 \to \mathbb{R}^3$, *given by*

$$\mathcal{K}_j(x) := \int_{\partial\Omega} \sum_{k,l=1}^{3} (D_jE_{kl} - D_jU_{kl} + D_kE_{jl} - D_kU_{jl})(x-y)\cdot n_k^{(\Omega)}(x)\cdot \Phi_l(y)\, d\Omega_y$$

for $x \in \mathbb{R}^3$, $j \in \{1,2,3\}$, *is well defined, and it holds*

$$\int_{\partial\Omega} \left|\mathcal{K}(x + \kappa\cdot m^{(\Omega)}(x)) - \mathcal{K}(x)\right|^p d\Omega_x \longrightarrow 0 \quad \text{for} \quad \kappa \to 0.$$

Moreover the operator $\mathcal{K}^*(p,\Omega) : L^p(\partial\Omega)^3 \to L^p(\partial\Omega)^3$, *introduced by*

$$\text{(3.12)} \quad \mathcal{K}_j^*(p,\Omega)(\Psi)(x) := \int_{\partial\Omega} \sum_{k,l=1}^{3} (D_jE_{kl} - D_jU_{kl} + D_kE_{jl} - D_kU_{jl})(x-y)\cdot n_k^{(\Omega)}(x)$$
$$\cdot\Psi_l(y)\, d\Omega_y \qquad \text{for } x \in \partial\Omega,\; j \in \{1,2,3\},\; \Psi \in L^p(\partial\Omega)^3,$$

is well defined, linear and bounded.

Proof: By (3.5) and Lemma 3.2, we have

(3.13) $$\int_{\partial\Omega} |(D_l E_{jk} - D_l U_{jk})(x-y) \cdot \Phi_k(y)| \, d\Omega_y \;<\; +\infty$$

for $1 \le j,k,l \le 3$ and for almost every $x \in \partial\Omega$. Obviously the relation in (3.13) also holds if $x \in \mathbb{R}^3 \setminus \partial\Omega$. Thus the function $\mathcal{K}$ is well defined. Referring to (3.5), (3.9), Lemma 3.2 and Lebesgue's theorem on dominated convergence, we obtain (3.12).

It further follows from (3.5) and Lemma 3.2 that $\mathcal{K}^*(p,\Omega)(\Psi) \in L^p(\partial\Omega)^3$ for $\Psi \in L^p(\partial\Omega)^3$, and that $\mathcal{K}^*(p,\Omega)$ is bounded. □

Let $\tau \in \{-1,1\}$, $p \in (1,\infty)$, $\Phi \in L^p(\partial\Omega)^3$. Define $\Gamma^* = \Gamma^*(\tau,p,\lambda,\Omega)(\Phi) : \partial\Omega \to \mathbb{R}^3$ by

$$\Gamma^* := \frac{\tau}{2}\Phi \;-\; L^p(\partial\Omega)^3\text{-}\lim_{\varepsilon\downarrow 0}\Big(\int_{\partial\Omega} \chi_{(\varepsilon,\infty)}(|I-y|) \cdot \sum_{k,l=1}^{3} (D_j E_{kl} + D_k E_{jl} - \delta_{jk}\cdot E_{4l})(I-y)\cdot n_k^{(\Omega)}(I)\cdot \Phi_l(y)\, d\Omega_y \Big)_{1\le j\le 3}$$

where I denotes the identical mapping of the set $\partial\Omega$ onto itself, and the limit is to be understood in the sense of convergence in the space $L^p(\partial\Omega)^3$.

THEOREM 3.2 *Let $\tau \in \{-1,1\}$, $p \in (1,\infty)$ and $\Phi \in L^p(\partial\Omega)^3$. Then the function $\Gamma^* = \Gamma^*(\tau,p,\lambda,\Omega)(\Phi)$ is well defined, and it holds for $1 \le j \le 3$:*

(3.14) $$\int_{\partial\Omega} \Big| \Gamma_j^*(\tau,p,\lambda,\Omega)(\Phi) + \sum_{k=1}^{3} (D_j V_k + D_k V_j - \delta_{jk}\cdot Q)\left(x + \tau\cdot\kappa\cdot m^{(\Omega)}(x)\right)\cdot n_k^{(\Omega)}(x)\Big|^p do_x \longrightarrow 0,$$

for $\kappa \downarrow 0$, where we used the abbreviations $V := V(\partial\Omega)(\Phi)\backslash\partial\Omega$, $Q := Q(\partial\Omega)(\Phi)$.

Proof: It is known from [4] that the family $\left(\Lambda^{(\varepsilon)}\right)_{\varepsilon>0}$ of functions, with

$$\Lambda_j^{(\varepsilon)}(x) := \int_{\partial\Omega} \chi_{(\varepsilon,\infty)}(|x-y|)\cdot \sum_{k,l=1}^{3} (D_j U_{kl} + D_k U_{jl} - \delta_{jk}\cdot E_{4l})(x-y)\cdot n_k^{(\Omega)}(x) \cdot\Phi_l(y)\, d\Omega_y \qquad \text{for } x \in \partial\Omega,\ j \in \{1,2,3\},$$

converges in $L^p(\partial\Omega)^3$ for $\varepsilon \downarrow 0$; see [11, pp. 773/774], [38, pp. 578-582], [30, pp. 266-270]. Abbreviate $\Lambda^* := \frac{\tau}{2}\Phi - L^p(\partial\Omega)^3\text{-}\lim_{\varepsilon\downarrow 0}\Lambda^{(\varepsilon)}$. Since by (3.5) and Lemma 3.2

$$\int_{\partial\Omega}\Big(\int_{\partial\Omega}\big|(D_j E_{kl} - D_j U_{kl} + D_k E_{jl} - D_k U_{jl})(x-y)\cdot n_k^{(\Omega)}(x)\cdot\Phi_l(y)\big|\, d\Omega_y\Big)^p d\Omega_x < \infty$$

for $1 \le j,k,l \le 3$, we see the function $\Gamma^* = \Gamma^*(\tau,p,\lambda,\Omega)(\Phi)$ is well defined, with

(3.15) $$\Gamma^* = \Lambda^* + \mathcal{K}^*(p,\Omega)(\Phi),$$

where $\mathcal{K}^*(p,\Omega)(\Phi)$ was introduced in Lemma 3.9. We have by [11, p. 774, (0.9), (0.10)]:

(3.16) $$\int_{\partial\Omega}\Big|\Lambda_j^*(x) \;+\; \int_{\partial\Omega}\sum_{k,l=1}^{3} (D_j U_{kl} + D_k U_{jl} - \delta_{jk}\cdot E_{4l})(x + \tau\cdot\kappa\cdot m^{(\Omega)}(x) - y) \cdot n_k^{(\Omega)}(x)\cdot\Phi_l(y)\, d\Omega_y\Big|^p d\Omega_x \to 0 \qquad \text{for } \kappa\downarrow 0,\ 1\le j\le 3:$$

see also [30, pp. 266-270]. Now, combining (3.12), (3.15) and (3.17), we obtain (3.14). □

Put

$$L^2_n(\partial\Omega) := \left\{ \Phi \in L^2(\partial\Omega)^3 \; : \; \int_{\partial\Omega} \Phi \cdot n^{(\Omega)} \, d\Omega \; = \; 0 \right\}.$$

Define the operator $\mathcal{E}(\partial\Omega) : L^2_n(\partial\Omega) \to H^1(\partial\Omega)^3 \cap L^2_n(\partial\Omega)$ by

(3.17) $\mathcal{E}(\partial\Omega)(\Phi) := V(\partial\Omega)(\Phi)|\partial\Omega$ for $\Phi \in L^2_n(\partial\Omega)$.

Our aim is to show that $\mathcal{E}(\partial\Omega)$ is well defined and bijective. As a preliminary result in this direction, we note

LEMMA 3.10 *Let $\Phi \in L^2(\partial\Omega)^3$. Then $V(\partial\Omega)(\Phi)|\partial\Omega \in L^2_n(\partial\Omega)$.*

Proof: We know by Lemma 3.2 that $V(\partial\Omega)(\Phi)|\partial\Omega \in L^2(\partial\Omega)^3$.

Setting $V^{(1,\kappa)} := V^{(1,\kappa)}(\Omega)(\Phi)$ for $\kappa \in (0, \mathcal{D}_1)$. and using Lemma 3.5 and 3.6, we get

$$\int_{\partial\Omega} V(\partial\Omega)(\Phi) \cdot n^{(\Omega)} \, d\Omega \; = \; \lim_{\kappa \downarrow 0} \int_{\partial\Omega} V^{(1,\kappa)} \cdot n^{(\Omega)} \, d\Omega \; = \; \lim_{\kappa \downarrow 0} \int_{\Omega} \operatorname{div} V^{(1,\kappa)} \, dx \; = \; 0.$$

□

The next theorem is needed in order to show that $\mathcal{E}(\partial\Omega)$ is one-to-one:

THEOREM 3.3 *Let $\Phi \in L^2_n(\partial\Omega)$ with $V(\partial\Omega)(\Phi)|\partial\Omega = 0$. Then $\Phi = 0$.*

Proof: Put $V^{(\tau,\kappa)} := V^{(\tau,\kappa)}(\Omega)(\Phi)$, $Q^{(\tau,\kappa)} := Q^{(\tau,\kappa)}(\Omega)(\Phi)$, for $\tau \in \{-1, 1\}$, $\kappa \in (0, \mathcal{D}_1)$, with $\mathcal{D}_1$ from (3.9). For $R \in (0, \infty)$ with $\overline{\Omega} \subset B_R$, $\varepsilon, \kappa \in (0, \mathcal{D}_1)$, we get by partial integration and by Lemma 3.5:

$$\frac{1}{2} \cdot \int_{B_R \setminus \overline{\Omega}} \sum_{j,k=1}^{3} \left(D_j V_k^{(-1,\kappa)} + D_k V_j^{(-1,\kappa)} \right) \cdot \left(D_j V_k^{(-1,\varepsilon)} + D_k V_j^{(-1,\varepsilon)} \right) dx$$
$$+ \int_{B_R \setminus \overline{\Omega}} \lambda \cdot D_1 V^{(-1,\kappa)} \cdot V^{(-1,\varepsilon)} \, dx$$
$$= \int_{\partial\Omega \cup \partial B_R} \sum_{j,k=1}^{3} \left(-\delta_{jk} \cdot Q^{(-1,\kappa)} + D_j V_k^{(-1,\kappa)} + D_k V_j^{(-1,\kappa)} \right) \cdot V_j^{(-1,\varepsilon)} \cdot n_k \, d\Omega,$$

with $n(y) := -n^{(\Omega)}(y)$ for $y \in \partial\Omega$, $n(y) := R^{-1} \cdot y$ for $y \in \partial B_R$. Letting ε tend to zero, and applying Lemma 3.6, 3.7 and 3.8, as well as the assumption $V(\partial\Omega)(\Phi)|\partial\Omega = 0$, we get for $\kappa \in (0, \mathcal{D}_1)$

$$\frac{1}{2} \cdot \int_{B_R \setminus \overline{\Omega}} \sum_{j,k=1}^{3} \left(D_j V_k^{(-1,\kappa)} + D_k V_j^{(-1,\kappa)} \right) \cdot (D_j u_k + D_k u_j) \, dx + \int_{B_R \setminus \overline{\Omega}} \lambda \cdot D_1 V^{(-1,\kappa)} \cdot u \, dx$$
$$= \int_{\partial B_R} \sum_{j,k=1}^{3} \left(-\delta_{jk} \cdot Q^{(-1,\kappa)} + D_j V_k^{(-1,\kappa)} + D_k V_j^{(-1,\kappa)} \right)(x) \cdot u_j(x) \cdot (x_k/R) \, do_x,$$

where $u := V(\partial\Omega)(\Phi)$, $\pi := Q(\partial\Omega)(\Phi)$. But according to Lemma 3.4, we have

(3.18) $u|B_R \setminus \overline{\Omega} \in W^{1,2}(B_R \setminus \overline{\Omega})^3, \qquad \pi|B_R \setminus \overline{\Omega} \in L^2(B_R \setminus \overline{\Omega})$.

Thus we conclude from Lemma 3.7 and 3.8, by letting κ tend to zero:

$$\frac{1}{2}\cdot\int_{B_R\setminus\overline{\Omega}}\sum_{j,k=1}^{3}(D_ju_k+D_ku_j)^2\,dx \;+\; \int_{B_R\setminus\overline{\Omega}}\lambda\cdot D_1u\cdot u\,dx$$

$$= \int_{\partial B_R}\sum_{j,k=1}^{3}(-\delta_{jk}\cdot\pi+D_ju_k+D_ku_j)(x)\cdot u_j(x)\cdot(x_k/R)\,do_x.$$

Since $V(\partial\Omega)(\Phi)|\partial\Omega=0$, it follows with (3.18) and Corollary 3.1

$$\int_{B_R\setminus\overline{\Omega}}D_1u\cdot u\,dx \;=\; (1/2)\cdot\int_{B_R\setminus\overline{\Omega}}D_1(u\cdot u)\,dx \;=\; (1/2)\cdot\int_{\partial B_R}(u\cdot u)(x)\cdot(x_1/R)\,do_x,$$

so that

$$\text{(3.19)}\quad \frac{1}{2}\cdot\int_{B_R\setminus\overline{\Omega}}\sum_{j,k=1}^{3}(D_ju_k+D_ku_j)^2\,dx \;=\; -(\lambda/2)\cdot\int_{\partial B_R}|u(x)|^2\cdot(x_1/R)\,do_x$$

$$+\int_{\partial B_R}\sum_{j,k=1}^{3}(-\delta_{jk}\cdot\pi+D_ju_k+D_ku_j)(x)\cdot u_j(x)\cdot(x_k/R)\,do_x.$$

It is obvious by (3.7), (3.8) and Lemma 3.1 that

$$\int_{\partial B_R}\sum_{j,k=1}^{3}(-\delta_{jk}\cdot\pi+D_ju_k+D_ku_j)(x)\cdot u_j(x)\cdot(x_k/R)\,do_x\longrightarrow 0,$$

$$\int_{\partial B_R}|u(x)|^2\cdot(x_1/R)\,do_x\longrightarrow 0\quad\text{for } R\to+\infty.$$

Thus we may conclude from (3.19)

$$\int_{\mathbb{R}^3\setminus\overline{\Omega}}\sum_{j,k=1}^{3}(D_ju_k+D_ku_j)^2\,dx \;=\; 0,\qquad\text{i.e.}\qquad (D_ju_k+D_ku_j)|\,\mathbb{R}^3\setminus\overline{\Omega} \;=\; 0$$

for $1\le j,k\le 3$. But for any domain $B\subset\mathbb{R}^3$, we have

$$\text{(3.20)}\quad \left\{f\in C^1(B)^3: D_jf_k+D_kf_j=0 \text{ for } 1\le j,k\le 3\right\} \;=\; \left\{a+b\times \mathrm{id}(B): a,b\in\mathbb{R}^3\right\},$$

with $\mathrm{id}(B)(x):=x$ for $x\in B$; see [10, p. 173] for example. Furthermore, it holds $u(x)=O(|x|^{-1})$ for $|x|\to\infty$; see (3.7). Thus we may conclude $u|\mathbb{R}^3\setminus\overline{\Omega}=0$. This, in turn, means by Lemma 3.3 there is a constant $\gamma\in\mathbb{R}$ with $\pi(x)=\gamma$ for $x\in\mathbb{R}^3\setminus\overline{\Omega}$. But the function π also decays for large values of $|x|$ (see (3.8)), hence $\pi|\mathbb{R}^3\setminus\overline{\Omega}=0$. Now we may conclude from (3.14): $\Gamma^*(1,2,\lambda,\Omega)(\Phi)=0$.

Analogously, using Lemma 3.3, 3.4, 3.5, 3.6, 3.8, Corollary 3.1, (3.20) and (3.14), we may deduce $\Gamma^*(-1,2,\lambda,\Omega)(\Phi)=-\gamma\cdot n^{(\Omega)}$.

Combining the two preceding equations yields $\Phi=\gamma\cdot n^{(\Omega)}$, hence $\int_{\partial\Omega}\Phi\cdot n^{(\Omega)}\,d\Omega=\gamma\cdot\int_{\partial\Omega}d\Omega$. But $\Phi\in L^2_n(\partial\Omega)$, so it follows $\gamma=0$, hence $\Phi=0$. □

Next we intend to show that the operator $\mathcal{E}(\partial\Omega)$ is well defined and bijective. We begin with some auxiliary results.

<u>LEMMA</u> 3.11 *Let $A,B\subset\mathbb{R}^2$ be bounded, measurable sets, and $K: A\times B\to\mathbb{R}^{3\times 3}$ a* *measurable function. Assume there is* $\mathcal{C} > 0$, $\kappa \in (-\infty, 2)$ *with*

$$|K(\xi,\eta)| \le \mathcal{C} \cdot |\xi - \eta|^{-\kappa} \quad \text{for} \quad \xi, \eta \in \mathbb{R}^2 \quad \text{such that} \quad \xi \ne \eta.$$

Let $p \in (1,\infty)$. *Then the operator* $T : L^p(B)^3 \to L^p(A)^3$,

$$T(f)(\xi) := \int_B K(\xi,\eta) \cdot f(\eta)\, d\eta \quad \text{for} \quad \xi \in A,\ f \in L^p(B)^3,$$

is well defined, linear and compact.

Proof: See [5, p. 83, Lemma 6.3], where the case $\kappa = 1$ is considered. The proof given there carries through for any $\kappa \in (-\infty, 2)$. It reduces compactness of T to compactness of Hille-Tamarkin operators ([25, p. 275, Theorem 11.6]). □

LEMMA 3.12 *Take* $U, V \subset \mathbb{R}^2$ *be open and bounded, with* $V \subset U$. *Let* $\rho : U \to \mathbb{R}^3$ *be a Lipschitz continuous, regular, injective function. Let* $Z \in C^1(\mathbb{R}^3 \setminus \{0\})^{3\times 3}$ *and assume there is a constant* $\mathcal{C} > 0$ *with*

$$|Z(x)| \le \mathcal{C}, \quad |D_l\, Z(x)| \le \mathcal{C} \cdot |x|^{-1} \quad \text{for} \quad x \in \mathbb{R}^3 \setminus \{0\},\ 1 \le l \le 3.$$

Let $p \in (1,\infty)$. *Introduce the operator* $T := T(p, Z, U, V, \rho) : L^p(\rho(U))^3 \to W^{1,p}(V)^3$ *by*

$$T(\Phi)(\xi) := \int_U Z(\rho(\xi) - \rho(\eta)) \cdot \Phi(\rho(\eta))\, d\eta \quad \text{for} \quad \Phi \in L^p(\rho(U))^3,\ \xi \in V.$$

Then T *is well defined, linear and compact.*

Proof: By Hölder's inequality, applied as in the proof [5, Lemma 4.9], we see that for any $\kappa \in (-\infty, 2)$, there is some $\mathcal{C}_1 > 0$ such that

$$\Big(\int_V \Big(\int_U |\xi - \eta|^{-\kappa} \cdot |f(\eta)|\, \mathrm{d}\eta\Big)^p\, \mathrm{d}\xi\Big)^{1/p} \quad \le \quad \mathcal{C}_1 \cdot \|f\|_p \quad \text{for} \quad f \in L^p(U);$$

compare Lemma 3.2. This means that the integral

$$\int_U D^a Z(\rho(\xi) - \rho(\eta)) \cdot \Phi(\rho(\eta))\, \mathrm{d}\eta$$

exists for $a \in \mathbb{N}_0^3$ with $|a|_* \le 1$, $\Phi \in L^p(\rho(U))^3$, and for almost any $\xi \in V$. It further follows there is a constant $\mathcal{C}_2 > 0$ with

$$(3.21)\quad \Big(\int_V \Big|\int_U D^a Z(\rho(\xi) - \rho(\eta)) \cdot \Phi(\rho(\eta))\, \mathrm{d}\eta\Big|^p\, \mathrm{d}\xi\Big)^{1/p} \quad \le \quad \mathcal{C}_2 \cdot \|\Phi\|_p$$

for $a \in \mathbb{N}_0^3$ with $|a|_* \le 1$, $\Phi \in L^p(\rho(U))^3$. For $j \in \{1,2\}$, $\Phi \in L^p(\rho(U))^3$, define

$$\mathcal{K}_j(\Phi)(\xi) := \sum_{k=1}^{3} D_j \rho_k(\xi) \cdot \int_U D_k Z(\rho(\xi) - \rho(\eta)) \cdot \Phi(\rho(\eta))\, \mathrm{d}\eta$$

for $\Phi \in L^p(\rho(U))^3$, $\xi \in V$. We see by (3.21) there is $\mathcal{C}_3 > 0$ with

$$(3.22)\quad \|\mathcal{K}_j(\Phi)\|_p \quad \le \quad \mathcal{C}_3 \cdot \|\Phi\|_p \quad \text{for} \quad j \in \{1,2\},\ \Phi \in L^p(\rho(U))^3.$$

Let $\Phi \in L^p(\rho(U))^3$, $j \in \{1,2\}$, $\mathcal{S} \in C_0^\infty(V)^3$. Then

$$\int_V D_j \mathcal{S} \cdot T(\Phi)\, \mathrm{d}\xi \quad = \quad \int_U \int_V D_j \mathcal{S}(\xi) \cdot Z(\rho(\xi) - \rho(\eta)) \cdot \Phi(\rho(\eta))\, \mathrm{d}\xi\, \mathrm{d}\eta$$

$$= \quad -\int_U \int_V \mathcal{S}(\xi) \cdot \frac{\partial}{\partial \xi_j}[Z(\rho(\xi) - \rho(\eta))] \cdot \Phi(\rho(\eta))\, \mathrm{d}\xi\, \mathrm{d}\eta,$$

where the last equation follows from a well-known theorem about the derivative of a Lipschitz continuous function; see [34, p. 148, Theorem 7.20]. Thus we get

$$\int_V D_j\mathcal{S}\cdot T(\Phi)\,d\xi = -\int_U\int_V \mathcal{S}(\xi)\cdot\sum_{k=1}^{3} D_j\rho_k(\xi)\cdot D_kZ(\rho(\xi)-\rho(\eta))\cdot\Phi(\rho(\eta))\,d\xi\,d\eta$$

$$= -\int_V \mathcal{S}\cdot\mathcal{K}_j(\Phi)\,d\xi.$$

This equation implies the function $T(\Phi)$ has weak derivatives of order 1, and

(3.23) $D_jT(\Phi) = \mathcal{K}_j(\Phi)$ for $j\in\{1,2\}$.

We may conclude from (3.22) and (3.23) that $T(\Phi)\in W^{1,p}(V)^3$ for $\Phi\in L^p(\rho(U))$. Thus the operator T is well defined. In order to show that it is compact, define $\mathcal{L}_j : L^p(U)^3 \to L^p(V)^3$ for $j\in\{0,1,2\}$ by

$$\mathcal{L}_0(f)(\xi) := \int_U Z(\rho(\xi)-\rho(\eta))\cdot f(\eta)\,d\eta$$

$$\mathcal{L}_j(f)(\xi) := \int_U \sum_{k=1}^{3} D_j\rho_k(\xi)\cdot D_kZ(\rho(\xi)-\rho(\eta))\cdot f(\eta)\,d\eta$$

for $f\in L^p(U)^3$, $1\le j\le 2$, $\xi\in V$. By Lemma 3.11 and the assumptions on Z, the operators $\mathcal{L}_0,\mathcal{L}_1,\mathcal{L}_2$ are compact. Let $(\Phi^{(n)})$ be a bounded sequence in $L^p(\rho(U))^3$. For $n\in\mathbb{N}$, put $f^{(n)} := \Phi^{(n)}\circ\rho$. Then $(f^{(n)})$ is a bounded sequence in $L^p(U)^3$, so there is a strictly increasing mapping $\sigma:\mathbb{N}\to\mathbb{N}$ such that the sequence $(\mathcal{L}_j(f^{(\sigma(n))}))_{n\in\mathbb{N}}$ converges in $L^p(V)^3$, for $j\in\{0,1,2\}$. This means the sequences $(T(\Phi^{(\sigma(n))}))_{n\in\mathbb{N}}$ and $(\mathcal{K}_j(\Phi^{(\sigma(n))}))_{n\in\mathbb{N}}$ converge in $L^p(V)^3$, for $j\in\{1,2\}$. It follows by (3.23) that $(T(\Phi^{(\sigma(n))}))$ converges in $W^{1,p}(V)^3$. Thus T is compact. □

<u>**COROLLARY**</u> **3.2** *Let* $Z\in C^1(\mathbb{R}^3\setminus\{0\})^3$ *with* $|Z(x)|\le\mathcal{C}$, $|D_l Z(x)|\le\mathcal{C}\cdot|x|^{-1}$ *for* $x\in\mathbb{R}^3\setminus\{0\}$, $1\le l\le 3$, *with some constant* $\mathcal{C}>0$. *Let* $p\in(1,\infty)$. *Introduce the operator* $\mathcal{T} := \mathcal{T}(Z,\Omega,p) : L^p(\partial\Omega)^3\to W^{1,p}(\partial\Omega)^3$ *by*

$$\mathcal{T}(\Phi)(x) := \int_{\partial\Omega} Z(x-y)\cdot\Phi(y)\,d\Omega_y \quad \text{for} \quad x\in\partial\Omega,\ \Phi\in L^p(\partial\Omega)^3.$$

Then the operator $\mathcal{T}$ *is well defined, linear and compact.*

In fact, we arrive at this corollary by transforming the preceding integral into local coordinates and applying Lemma 3.12.

<u>**COROLLARY**</u> **3.3** *Let* $p\in(1,\infty)$. *Define the operator* $\tilde{\mathcal{E}}_p(\partial\Omega) : L^p(\partial\Omega)^3\to W^{1,p}(\partial\Omega)^3$ *by*

$$\tilde{\mathcal{E}}_p(\partial\Omega)(\Phi)(x) := \left(\int_{\partial\Omega}\sum_{k=1}^{3}(E_{jk}-U_{jk})(x-y)\cdot\Phi_k(y)\,d\Omega_y\right)_{1\le j\le 3}$$

for $\Phi\in L^p(\partial\Omega)^3$, $x\in\partial\Omega$. *Then* $\tilde{\mathcal{E}}_p(\partial\Omega)$ *is well defined, linear and compact.*

Proof: Combine (3.5) and Corollary 3.2. □

We further need a deep-lying result on single-layer potentials on Lipschitz boundaries:

THEOREM 3.4 *The operator* $\overline{\mathcal{E}}(\partial\Omega) : L^2_n(\partial\Omega) \to H^1(\partial\Omega)^3 \cap L^2_n(\partial\Omega)$,

$$\overline{\mathcal{E}}(\partial\Omega)(\Phi)(x) := \left(\int_{\partial\Omega} \sum_{k=1}^{3} U_{jk}(x-y) \cdot \Phi_k(y)\, d\Omega_y \right)_{1 \le j \le 3}$$

for $\Phi \in L^2_n(\partial\Omega)$, $x \in \partial\Omega$, *is well defined, linear, bounded and bijective. In particular, it is Fredholm with index* 0.

For this theorem we refer to [11, p. 792].

COROLLARY 3.4 *The operator* $\mathcal{E}(\partial\Omega) : L^2_n(\partial\Omega) \to H^1(\partial\Omega)^3 \cap L^2_n(\partial\Omega)$ *(see* (3.17)*) is linear, bounded and bijective.*

Proof: Since $\mathcal{E}(\partial\Omega) = \overline{\mathcal{E}}(\partial\Omega) + \tilde{\mathcal{E}}_2(\partial\Omega)|L^2_n(\partial\Omega)$, it follows from Theorem 3.4 and Corollary 3.3 that $\mathcal{E}(\partial\Omega)$ is Fredholm with index zero. But Theorem 3.3 states that $\mathcal{E}(\partial\Omega)$ is one-to-one, hence this operator must be bijective. □

COROLLARY 3.5 *There is a constant* $C > 0$ *such that it holds for* $b \in H^1(\partial\Omega)^3 \cap L^2_n(\partial\Omega)$, $\Phi := \mathcal{E}(\partial\Omega)^{-1}(b)$

$$\sup\Big\{ \Big| \int_{\partial\Omega} \Phi \cdot g \, d\Omega \Big| \Big/ \| g \|_{1/2,2} \;:\; g \in H^1(\partial\Omega)^3 \cap L^2_n(\partial\Omega),\; g \neq 0 \Big\} \;\le\; C \cdot \| b \|_{1/2,2}.$$

Proof: Corollary 3.5 follows from Corollary 3.4 by duality and interpolation. □

Next we introduce some volume potentials. If $t \in (1, 3/2)$, $s \in (1,3)$, $f \in L^t(\mathbb{R}^3)^3$ and $g \in L^s(\mathbb{R}^3)^3$, we set for $x \in \mathbb{R}^3$

$$\mathcal{R}(f)(x) := \left(\int_{\mathbb{R}^3} \sum_{k=1}^{3} E_{jk}(x-y) \cdot f_k(y)\, \mathrm{d}y \right)_{1 \le j \le 3},$$

$$\mathcal{S}(g)(x) := \int_{\mathbb{R}^3} \sum_{k=1}^{3} E_{4k}(x-y) \cdot g_k(y)\, \mathrm{d}y.$$

THEOREM 3.5 *Let* $t \in (1, 3/2)$, $s \in (1,3)$. *For* $f \in L^t(\mathbb{R}^3)^3$, $g \in L^s(\mathbb{R}^3)^3$, *the functions* $\mathcal{R}(f)$, $\mathcal{S}(g)$ *are well defined. There is a constant* $C = C(s,t) > 0$ *such that for* f, g *as before*

$$\|\mathcal{R}(f)\|_{(1/t-2/3)^{-1}} \le C \cdot \|f\|_t, \quad \|\mathcal{S}(g)\|_{(1/s-1/3)^{-1}} \le C \cdot \|g\|_s,$$

$$\left(\int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} \left| \sum_{k=1}^{3} D_l E_{jk}(x-y) \cdot g_k(y) \right| dy \right)^{(1/s-1/3)^{-1}} dx \right)^{1/s-1/3} \le C \cdot \|g\|_s$$

for $j, l \in \{1, 2, 3\}$.

Proof: By (3.1) and (3.3), there is a constant $C_1 > 0$ with

$$| E_{jk}(z) | \le C_1 \cdot |z|^{-1} \quad \text{for } z \in \mathbb{R}^3 \backslash \{0\},\ 1 \le j,k \le 3.$$

Let f, g be given as in the theorem. By C, we denote constants which do not depend on f or g. The preceding estimate, inequality (3.2) and the Hardy-Littlewood-Sobolev inequality imply that the functions $\mathcal{R}(f)$ and $\mathcal{S}(g)$ are well defined, with

$$\| \mathcal{R}(f) \|_{1/t-2/3} \le C \cdot \| f \|_t, \quad \| \mathcal{S}(g) \|_{1/s-1/3} \le C \cdot \| g \|_s.$$

Similarly, we may conclude from (3.1) and the Hardy-Littlewood Sobolev inequality:

$$(3.24)\quad \Big(\int_{\mathbb{R}^3}\Big(\int_{\mathbb{R}^3}\Big|\sum_{k=1}^{3} D_l E_{jk}(x-y)\cdot \chi_{(0,1)}(|x-y|)\cdot g_k(y)\Big|\,dy\Big)^{(1/s-1/3)^{-1}}dx\Big)^{1/s-1/3}$$
$$\leq\ \mathcal{C}\cdot\|\,g\,\|_s \qquad \text{for } j,l\in\{1,\,2,\,3\}.$$

Using (3.3), we find with Lemma 3.1

$$\int_{\mathbb{R}^3\backslash B_1}|\,D_l E_{jk}(z)\,|^{3/2}dz \quad\leq\quad \mathcal{D}\cdot\int_{\mathbb{R}^3\backslash B_1}(\,\eta_{-3/2}^{-3/2}\,)^{3/2}(z)\,dz$$
$$=\ \mathcal{D}\cdot\int_1^\infty\int_{\partial B_r}\eta_{-9/4}^{-9/4}(z)\,do_z\,dr \quad\leq\quad \mathcal{D}\cdot\int_1^\infty r^{-5/4}\,dr\ <\ \infty$$

for $j,k,l\in\{1,\,2,\,3\}$, with the letter $\mathcal{D}$ denoting numerical constants. Thus, by Young's inequality, the estimate in (3.24) remains valid if the factor $\chi_{(0,1)}(|x-y|)$ is replaced by $\chi_{(1,\infty)}(|x-y|)$. This observation completes the proof of Theorem 3.5. □

<u>THEOREM</u> 3.6 *Let $f\in L^{6/5}(\mathbb{R}^3)^3\cap L^2(\mathbb{R}^3)^3$. Then $\mathcal{R}(f)\in H^2_{loc}(\mathbb{R}^3)^3$, $\mathcal{S}(f)\in H^1_{loc}(\mathbb{R}^3)$,*

$$-\Delta\mathcal{R}(f)+\lambda\cdot D_1\mathcal{R}(f)+\nabla\mathcal{S}(f)\ =\ f,\quad \operatorname{div}\mathcal{R}(f)\ =\ 0,$$

$$(3.25)\quad D_l\mathcal{R}(f)(x)\ =\ \Big(\int_{\mathbb{R}^3}\sum_{k=1}^{3} D_l E_{jk}(x-y)\cdot f_k(y)\,dy\Big)_{1\leq j\leq 3} \qquad \textit{for } x\in\mathbb{R}^3,\ l\in\{1,2,3\}.$$

In addition, there is a constant $C>0$ such that

$$\|\mathcal{R}(g)\|_6\ \leq\ C\cdot\|g\|_{6/5},\quad \|\,\nabla\mathcal{R}(g)\,\|_2\ +\ \|\mathcal{S}(g)\|_2\ \leq\ C\cdot\|g\|_{6/5}\quad \textit{for } g\in L^{6/5}(\mathbb{R}^3)^3.$$

Proof: For the first part of the theorem, up to but not including (3.25), we refer to [14, Theorem VII.4.1]. As for the second part, it is an immediate consequence of Theorem 3.5. □

<u>COROLLARY</u> 3.6 *There is a constant $C>0$ such that*

$$\|\mathcal{R}(f)|\partial\Omega\|_{1/2,2}\ \leq\ C\cdot\|f\|_{6/5}\quad \textit{for}\quad f\in L^{6/5}(\mathbb{R}^3)^3.$$

Proof: Use a standard trace theorem and the fact that there is a constant $\mathcal{C}>0$ with $\|\,\mathcal{R}(f)\,|\,\Omega\,\|_{1,2}\leq\mathcal{C}\cdot\|\,f\,\|_{6/5}$ for f as above; see Theorem 3.6. □

We finally state a consequence of some results from [14]:

<u>THEOREM</u> 3.7 ([14, Theorem VII.1.1, VII.2.1, VII.6.2]) *There is a uniquely determined pair of functions $(W,P)=(\,W(\Omega),\,P(\Omega)\,)$ with*

$$W\in C^\infty(\mathbb{R}^3\backslash\overline{\Omega})^3\cap L^6(\mathbb{R}^3\backslash\overline{\Omega})^3,\quad \nabla W\in L^2(\mathbb{R}^3\backslash\overline{\Omega})^9,\quad P\in C^\infty(\mathbb{R}^3\backslash\overline{\Omega})\cap L^2(\mathbb{R}^3\backslash\overline{\Omega})$$

satisfying (1.1) - (1.3) with $f=0$, $b=m^{(\Omega)}$.
Moreover there is a function $\Psi=\Psi(\Omega,S)\in L^2(\partial B_S)^3$ with

$$(3.26)\quad W_j(x)=V_j(\partial B_S)(\Psi)(x)\ +\ \int_{\partial B_S}\sum_{m,k=1}^{3} T_{jkm}(x-y)\cdot W_k(y)\cdot(y_m/S)\,do_y,$$

$$P(x) = Q(\partial B_S)(\Psi)(x) + \int_{\partial B_S} \sum_{m,k=1}^{3} 2 \cdot D_m E_{4k}(x-y) \cdot W_k(y) \cdot (y_m/S)\, do_y$$

for $x \in \mathbb{R}^3 \backslash \overline{B_S}$, $1 \le j \le 3$, *where* $T_{jkm} := D_m E_{jk} + D_k E_{jm} - \delta_{km} \cdot E_{4j}$ *for* $1 \le j, k, m \le 3$.

Now we are in a position to prove an existence and representation theorem for solutions of the Oseen problem (1.1) - (1.3).

THEOREM 3.8 *Let* $f \in L^2(\mathbb{R}^3)^3 \cap L^{6/5}(\mathbb{R}^3)^3$, $b \in H^1(\partial\Omega)^3$. *Then there exists a uniquely determined pair of functions*

$$(u, \pi) := (u(b,f), \pi(b,f)) \in H^2_{loc}(\mathbb{R}^3 \setminus \overline{\Omega})^3 \times H^1_{loc}(\mathbb{R}^3 \setminus \overline{\Omega})$$

such that $u \in L^6(\mathbb{R}^3 \setminus \overline{\Omega})^3$, $\nabla u \in L^2(\mathbb{R}^3 \setminus \overline{\Omega})^9$, $\pi \in L^2(\mathbb{R}^3 \setminus \overline{\Omega})$, *and such that the pair* (u, π) *satisfies* (1.1), (1.2). *(Relation (1.3) is valid in a weak form because* $u \in L^6(\mathbb{R}^3 \setminus \overline{\Omega})^3$; *see [14, Lemma II.5.2, Theorem II.5.1].) Put*

$$\mu := \mu(b) := \int_{\partial\Omega} b \cdot n^{(\Omega)}\, d\Omega \,/ \Big(\int_{\partial\Omega} n^{(\Omega)} \cdot m^{(\Omega)}\, d\Omega \Big), \quad \tilde{b} := b - \mu \cdot m^{(\Omega)} \,|\, \partial\Omega,$$

$$\Phi := \Phi(f, \tilde{b}) := \mathcal{E}(\partial\Omega)^{-1}(\tilde{b} - \mathcal{R}(f) \,|\, \partial\Omega).$$

(Recall that $n^{(\Omega)}$ *denotes the outward unit normal to* Ω. *According to (3.10), the definition of* μ *makes sense. For the definition of the operator* $\mathcal{E}(\partial\Omega)$, *see (3.17).) Then* $\Phi \in L^2_n(\partial\Omega)$. *and*

$$(3.27)\quad u(x) = \mathcal{R}(f)(x) + V(\partial\Omega)(\Phi)(x) + \mu \cdot W(x),$$

$$(3.28)\quad \pi(x) = \mathcal{S}(f)(x) + Q(\partial\Omega)(\Phi)(x) + \mu \cdot P(x) \quad \text{for} \quad x \in \mathbb{R}^3 \setminus \overline{\Omega},$$

with $W = W(\Omega)$, $P = P(\Omega)$ *introduced in Theorem 3.7*

In view of our assumptions on the data f and b, our solution (u, π) should exhibit a somewhat higher regularity than indicated in the preceding theorem. However, the result given there is sufficient for our purpose.

Proof: The uniqueness result stated in Theorem 3.8 holds according to [14, Theorem VII.2.1]. Concerning existence, we note that $\tilde{b}$ is orthogonal to $n^{(\Omega)}$ with respect to the scalar product of $L^2(\partial\Omega)^3$, hence $\tilde{b} \in L^2_n(\partial\Omega)$. This means $\Phi = \Phi(f, b)$ is well defined. Thus, if we introduce u by the right-hand side in (3.27), and π by the right-hand side in (3.28), it follows by Lemma 3.3, 3.4, Corollary 3.1, Theorem 3.6 and 3.7 that the pair of functions (u, π) has all the properties listed in Theorem 3.8. □

COROLLARY 3.7 *There is a constant* $C > 0$ *such that*

$$\sup\Big\{ \Big| \int_{\partial\Omega} \Phi(f,b) \cdot g\, d\Omega \Big| \Big/ \| g \|_{1/2,2} \; : \; g \in H^1(\partial\Omega)^3 \cap L^2_n(\partial\Omega),\ g \neq 0 \Big\}$$
$$\le C \cdot \Big(\|f\|_{6/5} + \|b\|_{1/2,2} \Big)$$

for $f \in L^2(\mathbb{R}^3)^3 \cap L^{6/5}(\mathbb{R}^3)^3$, $b \in H^1(\partial\Omega)^3 \cap L^2_n(\partial\Omega)$, *with* $\Phi(b, f)$ *introduced in Theorem 3.8.*

Proof: Combine Corollary 3.5 and 3.6. □

Next we prove a decay result for the functions u, π from Theorem 3.8.

THEOREM 3.9 *Let $\sigma \in (5/2, \infty)$. Then there exists a constant $C = C(\Omega, S, \sigma, \lambda) > 0$ with the ensuing properties:*

If $f \in L^2(\mathbb{R}^3)^3$, $\gamma \in (0,\infty)$ with $|f(y)| \le \gamma \cdot |y|^{-\sigma}$ for $y \in \mathbb{R}^3 \backslash B_S$ (in particular $f \in L^{6/5}(\mathbb{R}^3)^3$), $b \in H^1(\partial\Omega)^3$, $j, l \in \{1, 2, 3\}$, $x \in \mathbb{R}^3 \backslash B_{2 \cdot S}$, then

$$\begin{aligned} &|u(b,f)_j(x)| \cdot s(x)/|x| \; + \; |D_l u(b,f)_j(x)| + |\pi(b,f)(x)| \\ &\le \; C \cdot (\gamma + \| f \,|\, B_S \|_2 + \| b \|_{1/2,2}) \cdot |x|^{-2} \cdot \mathcal{M}(|x|, \sigma), \end{aligned}$$

where $\mathcal{M}(r,\sigma) := 1$ if $\sigma > 4$, $\mathcal{M}(r,\sigma) := \max\{1, \ln r\}$ if $\sigma = 4$, $\mathcal{M}(r,\sigma) := r^{-\sigma+4}$ if $\sigma < 4$, for $r \in (0,\infty)$.

Proof: Let f, γ, b, j, l, x be given as in the theorem. By the letter C, we shall denote constants which only depend on Ω, S, σ or λ. We shall show

$$(3.29) \quad |u(b,f)(x)| \cdot s(x)/|x| \; \le \; C \cdot (\gamma + \| f \,|\, B_S \|_2 + \| b \|_{1/2,2}) \cdot |x|^{-2} \cdot \mathcal{M}(|x|, \sigma).$$

Analogous estimates for $|D_l u(b,f)_j(x)|$ and $|\pi(b,f)(x)|$ may be established by a similar, but somewhat less complicated reasoning. In order to prove (3.29), we put

$$R := |x|, \quad \mathcal{B}_1 := B_S, \quad \mathcal{B}_2 := B_{R/2} \backslash B_S, \quad \mathcal{B}_3 := B_{4 \cdot R} \backslash B_{R/2}, \quad \mathcal{B}_4 := \mathbb{R}^3 \backslash B_{4 \cdot R}.$$

By (3.27) we have

$$\begin{aligned} (3.30) \quad u(b,f)_j(x) \; &= \; \int_{\partial\Omega} \sum_{k=1}^{3} E_{jk}(x-y) \cdot \Phi_k(y) \, d\Omega(y) \; + \; \mu \cdot W_j(x) \\ &+ \sum_{\nu=1}^{4} \int_{\mathcal{B}_\nu} \sum_{k=1}^{3} E_{jk}(x-y) \cdot f_k(y) \, dy, \end{aligned}$$

where $\Phi := \Phi(b,f)$ and $\mu := \mu(b)$ were introduced in Theorem 3.8, and $W := W(\Omega)$ in Theorem 3.7.

Let us consider the integral over $\mathcal{B}_2$ appearing on the right-hand side of (3.30). It is the estimate of this integral which requires the strongest decay properties of f. In fact, we have by (3.3)

$$\begin{aligned} &\int_{\mathcal{B}_2} \sum_{k=1}^{3} | E_{jk}(x-y) | \cdot s(x)/|x| \cdot | f_k(y) | \, dy \\ &\le \; C \cdot \gamma \cdot |x|^{-1} \cdot \int_{\mathcal{B}_2} \eta_{-1}^{-1}(x-y) \cdot (|x| - |y| - (x-y)_1 + |y| - y_1) \cdot |y|^{-\sigma} \, dy \\ &\le \; C \cdot \gamma \cdot |x|^{-1} \cdot \Big(\int_{\mathcal{B}_2} \eta_{-1}^{-1}(x-y) \cdot (|x-y| - (x-y)_1) \cdot |y|^{-\sigma} \, dy \\ &\quad + \int_{\mathcal{B}_2} \eta_{-1}^{-1}(x-y) \cdot |y|^{-\sigma+1} \, dy \Big) \\ &\le \; C \cdot \gamma \cdot |x|^{-1} \cdot \int_{\mathcal{B}_2} |x-y|^{-1} \cdot (|y|^{-\sigma} + |y|^{-\sigma+1}) \, dy \\ &\le \; C \cdot \gamma \cdot |x|^{-2} \cdot \int_{\mathcal{B}_2} |y|^{-\sigma+1} \, dy \quad \le \quad C \cdot \gamma \cdot |x|^{-2} \cdot \mathcal{M}(|x|, \sigma). \end{aligned}$$

Those expressions on the right-hand side of (3.30) which correspond to the indeces $\nu = 1, 3, 4$ may be dealt with in a similar way, the main difference being that some estimates may be based on Lemma 3.1. We leave the details to the reader, and instead consider the first term on the right-hand side of (3.30). Put

$$g_k(y) := E_{jk}(x-y) \quad \text{for } y \in \overline{\Omega},\ 1 \le k \le 3.$$

Then $g \in C^\infty(\overline{\Omega})^3$, and g is orthogonal to $n^{(\Omega)}$ with respect to the scalar product of $L^2(\partial\Omega)^3$. (Note that $x \in \mathbb{R}^3 \backslash \overline{\Omega}$.) It follows by Corollary 3.7, (3.3) and a standard trace theorem

$$\begin{aligned}
&\Big| \int_{\partial\Omega} \sum_{k=1}^{3} E_{jk}(x-y) \cdot s(x)/|x| \cdot \Phi(y)\, d\Omega(y) \Big| \\
&\le\ C \cdot (\, \|f\|_{6/5} + \|b\|_{1/2,2}\,) \cdot \|\, g \,|\, \partial\Omega \,\|_{1/2,2} \cdot s(x)/|x| \\
&\le\ C \cdot (\, \|f\|_{6/5} + \|b\|_{1/2,2}\,) \cdot \sup\{\, \eta_{-1}^{-1}(x-y)\ :\ y \in \overline{\Omega}\,\} \cdot s(x)/|x|.
\end{aligned}$$

On the other hand, taking into account that $x \in B_2 \subset \mathbb{R}^3 \backslash B_{2\cdot S}$, we find $y \in \Omega \subset B_S$

$$\begin{aligned}
\eta_{-1}^{-1}(x-y) \cdot s(x)/|x| \ &=\ |x|^{-1} \cdot \eta_{-1}^{-1}(x-y) \cdot (\, |x| - |y| - (x-y)_1 + |y| - y_1\,) \\
&\le\ |x|^{-1} \cdot \eta_{-1}^{-1}(x-y) \cdot (\, |x-y| - (x-y)_1 + 2 \cdot S\,) \\
&\le\ (1 + 2 \cdot S) \cdot |x|^{-1} \cdot |x-y|^{-1}\ \le\ 2 \cdot (1 + 2 \cdot S) \cdot |x|^{-2}.
\end{aligned}$$

A similar argument may be used in order to estimate $\mu \cdot W_j(x)$. □

The preceding theorem implies immediately

COROLLARY 3.8 *Let $\sigma \in (5/2, \infty)$. Then there is a constant $C = C(\Omega, S, \sigma, \gamma)$ such that for f, γ, b as in Theorem 3.9, and for $R \in [2 \cdot S, \infty)$*

$$\begin{aligned}
&\Big(\int_{\partial B_R} \big(|\, u(b,f)(x) \,|^2 \cdot s(x)^2/|x|^2\, do_x \big)^{1/2} \ +\ \|\, \nabla u(b,f) \,|\, \partial B_R \,\|_2 \ +\ \|\, \pi(b,f) \,|\, \partial B_R \,\|_2 \\
&\le\ C \cdot (\, \gamma + \|\, f \,|\, B_S \,\|_2 + \|b\|_{1/2,2}\,) \cdot R^{-1} \cdot \mathcal{M}(R, \sigma).
\end{aligned}$$

4 The Oseen system in a truncated exterior domain

In this section, we solve the Oseen system (1.4), (1.5) under the artificial boundary condition (1.6) on ∂B_R, and we compare this solution with the exterior Oseen flow introduced in Theorem 3.8. To this end, we introduce the subspace W_R of $H^1(\Omega_R)^3$ by setting

$$W_R := \big\{ v \in H^1(\Omega_R)^3 : v|\partial\Omega = 0 \big\}.$$

We recall three results which were proved in [6]:

LEMMA 4.1 ([6, Lemma 4.1]) *There is a constant $C > 0$ such that*

$$\|u\|_2 \ \le\ C \cdot \big(R \cdot \|\nabla u\|_2 + R^{1/2} \cdot \|u|\partial B_R\|_2 \big)$$

for $R \in (0, \infty)$ with $\overline{\Omega} \subset B_R$, $u \in W_R$.

LEMMA 4.2 ([6, Lemma 5.1]) *There is a constant $C = C(S) > 0$ with*

$$\|u|\Omega_S\|_2 \ \le\ C \cdot \big(\|\nabla u\|_2 + R^{-(1/2)} \cdot \|u|\partial B_R\|_2 \big) \qquad \textit{for } R,\ u \textit{ as in Lemma 4.1.}$$

THEOREM 4.1 **([6, Theorem 4.1])** *There is a constant $C = C(\Omega) > 0$ and for any $R \in (0,\infty)$ with $\overline{\Omega} \subset B_R$ a linear operator $\mathcal{D}_R : L^2(\Omega_R) \mapsto W_R$ such that for $\pi \in L^2(\Omega_R)$*

$$\operatorname{div} \mathcal{D}_R(\pi) = \pi, \qquad \|\nabla \mathcal{D}_R(\pi)\|_2 + R^{-(1/2)} \cdot \|\mathcal{D}_R(\pi)|\partial B_R\|_2 \leq C \cdot \|\pi\|_2.$$

The ensuing two Theorems are also well known.

THEOREM 4.2 *There is a bounded operator $\mathcal{F} : H^{1/2}(\partial\Omega)^3 \mapsto H^1(\Omega_{2S})^3$ such that $\mathcal{F}(b)|\partial\Omega = b$, $\mathcal{F}(b)|\partial B_{2S} = 0$ for $b \in H^{1/2}(\partial\Omega)^3$.*

THEOREM 4.3 **([14, Theorem III.3.2])** *There is a bounded linear operator*

$$\mathcal{O} : \left\{\rho \in L^2(\Omega_{2S}) : \int_{\Omega_{2S}} \rho \, dx = 0\right\} \longmapsto H_0^1(\Omega_{2S})^3$$

such that

$$\operatorname{div} \mathcal{O}(\rho) = \rho \quad \text{for} \quad \rho \in L^2(\Omega_{2S}) \quad \text{with} \quad \int_{\Omega_{2S}} \rho \, dx = 0.$$

Let us draw some conclusions from these theorems.

COROLLARY 4.1 *There is a linear bounded operator*

$$\mathcal{A} : \left\{b \in H^{1/2}(\partial\Omega)^3 : \int_{\partial\Omega} b \cdot n^{(\Omega)} d\Omega = 0\right\} \longmapsto H^1(\Omega_{2S})^3$$

such that $\mathcal{A}(b)|\partial\Omega = b$, $\mathcal{A}(b)|\partial B_{2S} = 0$,

$$\operatorname{div} \mathcal{A}(b) = 0 \quad \text{for } b \in H^{1/2}(\partial\Omega)^3 \text{ with } \int_{\partial\Omega} b \cdot n^{(\Omega)} d\Omega = 0.$$

Proof: Let $b \in H^{1/2}(\partial\Omega)^3$ with $\int_{\partial\Omega} b \cdot n^{(\Omega)} d\Omega = 0$. Then we have $\operatorname{div} \mathcal{F}(b) \in L^2(\Omega_{2S})$,

$$\int_{\Omega_{2S}} \operatorname{div} \mathcal{F}(b) \, dx = \int_{\partial\Omega} \mathcal{F}(b) \cdot n^{(\Omega)} d\Omega = \int_{\partial\Omega} b \cdot n^{(\Omega)} d\Omega = 0.$$

Therefore we may apply the operator $\mathcal{O}$ from Theorem 4.3 to $\operatorname{div} \mathcal{F}(b)$, to obtain

$$\mathcal{O}(\operatorname{div} \mathcal{F}(b)) \in H^1(\Omega_{2S})^3, \quad \operatorname{div} \mathcal{O}(\operatorname{div} \mathcal{F}(b)) = \operatorname{div}(\mathcal{F}(b)), \quad \mathcal{O}(\operatorname{div} \mathcal{F}(b))|\partial\Omega_{2S} = 0.$$

For $b \in H^{1/2}(\partial\Omega)^3$ with $\int_{\partial\Omega} b \cdot n^{(\Omega)} d\Omega = 0$, put $\mathcal{A}(b) := \mathcal{F}(b) - \mathcal{O}(\operatorname{div} \mathcal{F}(b))$. This operator $\mathcal{A}$ exhibits the desired properties. □

COROLLARY 4.2 *There is a constant $C = C(\Omega, S) > 0$ and for any $R \in [2 \cdot S, \infty)$ a linear operator*

$$\mathcal{A}_R : \left\{b \in H^{1/2}(\partial\Omega)^3 : \int_{\partial\Omega} b \cdot n^{(\Omega)} d\Omega = 0\right\} \longmapsto H^1(\Omega_R)^3$$

such that $\operatorname{div} \mathcal{A}_R(b) = 0$, $supp(\mathcal{A}_R(b)) \subset B_{2S}$, $\mathcal{A}_R(b)|\partial\Omega = b$,

$$\|\mathcal{A}_R(b)\|_{1,2} \leq C \cdot \|b\|_{1/2,2} \quad \text{for } b \in H^{1/2}(\partial\Omega)^3 \text{ with } \int_{\partial\Omega} b \cdot n^{(\Omega)} d\Omega = 0.$$

Proof: Recall the operator $\mathcal{A}$ introduced in Corollary 4.1. Take $R \in [2 \cdot S, \infty)$. For $b \in H^{1/2}(\partial\Omega)^3$ with $\int_{\partial\Omega} b \cdot n^{(\Omega)} d\Omega = 0$, define $\mathcal{A}_R(b)$ as the zero extension of $\mathcal{A}(b)$ to Ω_R. Since $\mathcal{A}(b)|\partial B_{2S} = 0$, Corollary 4.2 follows immediately. □

We introduce an inner product $(\cdot,\cdot)^{(R)}$ on W_R by defining

$$(v,w)^{(R)} := \int_{\Omega_R} \sum_{j,k=1}^{3} D_j v_k \cdot D_j w_k \, dx \; + \; \int_{\partial B_R} (\lambda/2) \cdot v \cdot w \, do_x \quad \text{for } v,w \in W_R.$$

The space W_R equipped with this inner product is a Hilbert space. The norm induced by $(\cdot,\cdot)^{(R)}$ is denoted by $|\cdot|^{(R)}$, that is,

$$|v|^{(R)} := \left(\|\nabla v\|_2^2 \; + \; (\lambda/2) \cdot \|v|\partial B_R\|_2^2\right)^{1/2} \quad \text{for } v \in W_R.$$

Define the bilinear forms $\alpha_R : H^1(\Omega_R)^3 \times H^1(\Omega_R)^3 \to \mathbb{R}$, $\beta_R : H^1(\Omega_R)^3 \times L^2(\Omega_R) \to \mathbb{R}$ by

$$\alpha_R(v,w) := \int_{\Omega_R} \left[\sum_{j,k=1}^{3} D_j v_k \cdot D_j w_k + \lambda \cdot D_1 v \cdot w \right] dx$$

$$+ \frac{\lambda}{2} \int_{\partial B_R} v(x) \cdot w(x) \cdot \left(1 - \frac{x_1}{R}\right) do_x,$$

$$\beta_R(w,\sigma) := -\int_{\Omega_R} \operatorname{div}(w) \cdot \sigma \, dx \qquad \text{for } v,w \in H^1(\Omega_R)^3,\ \sigma \in L^2(\Omega_R),\ R \in [S,\infty).$$

The ensuing lemma follows from Lemma 4.1:

LEMMA 4.3 *There is a constant $C = C(\Omega, S) > 0$ such that*

$$\begin{aligned} \alpha_R(v,w) &\le 9 \cdot \|\nabla v\|_2 \cdot \|\nabla w\|_2 + \lambda \cdot \|\nabla v\|_2 \cdot \|w\|_2 + (\lambda/2) \cdot \|v|\partial B_R\|_2 \cdot \|w|\partial B_R\|_2 \\ &\le C \cdot |v|^{(R)} \cdot |w|^{(R)} \qquad \textit{for } v,w \in H^1(\Omega_R)^3,\ R \in [S,\infty). \end{aligned}$$

Lemma 4.3 states that α_R is bounded with respect to the norm $|\cdot|^{(R)}$. In addition, the bilinear form α_R is positive definite on W_R with respect to this norm:

LEMMA 4.4 *Let $R \in [S,\infty)$, $w \in W_R$. Then $(|w|^{(R)})^2 = \alpha_R(w,w)$.*

Proof: Obvious by the definition of W_R, $|\ |^{(R)}$ and α_R. □

As a consequence of Theorem 4.1 we obtain that the bilinear form β_R is stable (compare [6, pp. 256/257]):

COROLLARY 4.3 *Let $R \in [S,\infty)$. Then there is a constant $C = C(R,\Omega)$ such that*

$$\inf_{\rho \in L^2(\Omega_R),\ \rho \neq 0} \ \sup_{v \in W_R,\ v \neq 0} \frac{\beta_R(v,\rho)}{|v|^{(R)} \cdot \|\rho\|_2} \ \ge \ C.$$

Note that the constant C in Corollary 4.3 depends on R. However, this will not matter in the following.

Proof: Let $\rho \in L^2(\Omega_R)$. Recall the operator $\mathcal{D}_R$ from Theorem 4.1, and put $v := \mathcal{D}_R(-\rho)$. Then $v \in W_R$ and

$$\begin{aligned} \beta_R(v,\rho) &= \|\rho\|_2^2 \ \ge \ C \cdot \left(\|\nabla v\|_2 + R^{-1/2} \cdot \|v|\partial B_R\|_2\right) \cdot \|\rho\|_2 \\ &\ge C \cdot \min\{\,1,\ \sqrt{2} \cdot (R \cdot \lambda)^{-1/2}\} \cdot (\,\|\nabla v\|_2 + (\lambda/2)^{1/2} \cdot \|v|\partial B_R\|_2\,) \cdot \|\rho\|_2 \end{aligned}$$

$$\geq C \cdot \min\{ 1, \sqrt{2} \cdot (R \cdot \lambda)^{-1/2} \} \cdot |v|^{(R)} \cdot \|\rho\|_2,$$

where C denotes constants independent of v, ϱ, R and λ. □

By the standard theory of mixed variational problems (see [15, pp. 57-61]), we may conclude

THEOREM 4.4 *Let $R \in [2 \cdot S, \infty)$, $f \in L^2(\Omega_R)^3$, $b \in H^{1/2}(\partial\Omega)^3$. Then there is a uniquely determined pair of functions $(\tilde{v}, \tilde{\rho}) = (\tilde{v}(R,f), \tilde{\rho}(R,f)) \in W_R \times L^2(\Omega_R)$ such that*

(4.1) $\alpha_R(\tilde{v}, w) + \beta_R(w, \tilde{\rho})$

$$= \int_{\Omega_R} f \cdot w \, dx \; - \; \alpha_R((\mathcal{A}_R(b) + \mu \cdot W) \,|\, \Omega_R, \, w) \; - \; \beta_R(w, \, \mu \cdot P \,|\, \Omega_R) \quad \textit{for } w \in W_R,$$

(4.2) $\beta_R(\tilde{v}, \sigma) = 0 \quad$ *for* $\sigma \in L^2(\Omega_R)$,

where the operator $\mathcal{A}_R$ was introduced in Corollary 4.2, the number $\mu := \mu(b)$ in Theorem 3.8, and the functions $W := W(\Omega)$, $P := P(\Omega)$ in Theorem 3.7.

Let us interpret variational problem (4.1), (4.2) as a boundary value problem:

LEMMA 4.5 *Take $R \in [2 \cdot S, \infty)$, $f \in L^2(\Omega_R)^3$. Assume that Ω is C^2-bounded, let $b \in H^{3/2}(\partial\Omega)^3$, choose $\mu = \mu(b)$ as in Theorem 3.8, $W = W(\Omega)$, $P = W(\Omega)$ as in Theorem 3.7, and $\tilde{v} = \tilde{v}(R,f)$, $\tilde{\varrho} = \tilde{\varrho}(R,f)$ as in the preceding theorem. Assume that $\mathcal{A}_R(b)$, $W \,|\, \Omega_R$, $\tilde{v} \in H^2(\Omega_R)^3$ and $\tilde{\rho}$, $P \,|\, \Omega_R \in H^1(\Omega_R)$.*

Then the pair of functions $(v_R, \varrho_R) := (\tilde{v} + \mathcal{A}_R(b) + \mu \cdot W, \, \tilde{\varrho} + \mu \cdot P)$ solves (1.4) - (1.6).

The proof of this lemma is obvious. Lemma 4.5 implies that in the situation of Theorem 4.4, the pair of functions (v_R, ϱ_R) introduced in Lemma 4.5 may be considered a weak solution of problem (1.4) - (1.6).

The solution of (4.1), (4.2) will now be compared to the exterior Oseen flow introduced in Theorem 3.8. We shall estimate the truncation error between the solution of (4.1), (4.2) on one hand and the exterior Oseen flow on the other.

THEOREM 4.5 *Let $\sigma \in (5/2, \infty)$. Then there is a constant $C = C(\Omega, S, \sigma, \lambda) > 0$ with the following properties:*

Let $f \in L^2(\mathbb{R}^3)^3$, $\gamma > 0$ with $|f(x)| \leq \gamma \cdot |x|^{-\sigma}$ for $x \in \mathbb{R}^3 \setminus B_S$. Take $b \in H^{1/2}(\partial\Omega)^3$, $R \in [2 \cdot S, \infty)$. Then

$$|\, u \,|\, \Omega_R \, - \, v \,|^{(R)} \;\leq\; C \cdot (\gamma + \|f|B_S\|_2 + \|b\|_{1/2,2}) \cdot R^{-1} \cdot \mathcal{M}(R, \sigma),$$

$$\|\, \pi \,|\, \Omega_R - \rho \,\|_2 \;\leq\; C \cdot (\gamma + \|f|B_S\|_2 + \|b\|_{1/2,2}) \cdot \mathcal{M}(R, \sigma),$$

where

$$u := u(b,f), \quad \pi := \pi(b,f), \quad v := \tilde{v}(R, \, f \,|\, \Omega_R) \; + \; \mathcal{A}_R(b) \; + \; \mu(b) \cdot W(\Omega),$$

$$\varrho := \tilde{\varrho}(R, \, f \,|\, \Omega_R) \; + \; \mu(b) \cdot P(\Omega),$$

with the exterior Oseen flow $(u(b,f), \pi(b,f))$ introduced in Theorem 3.8. The expression $\mathcal{M}(R, \sigma)$ was defined in Theorem 3.9, the number $\mu(b)$ in Theorem 3.8, and the functions $W(\Omega)$, $P(\Omega)$ in Theorem 3.7.

Proof: We suppose $b \in H^1(\partial\Omega)^3$. The case $b \in H^{1/2}(\partial\Omega)^3$ may be dealt with by a density argument. For brevity, we write $\tilde{v}$, $\tilde{\varrho}$, μ, W, P instead of $\tilde{v}(R, f\,|\,\Omega_R)$, $\tilde{\varrho}(R, f\,|\,\Omega_R)$, $\mu(b)$, $W(\Omega)$, $P(\Omega)$, respectively. Put $w := u\,|\,\Omega_R - v$, $\kappa := \pi\,|\,\Omega_R - \rho$. Then we observe that

$$\beta_R(w,\kappa) = -\int_{\Omega_R} \operatorname{div} u \cdot \kappa \, dx \; - \; \beta_R(\tilde{v},\kappa) \; + \; \int_{\Omega_R} (\operatorname{div}\mathcal{A}_R(b) \; + \; \mu \cdot \operatorname{div} W) \cdot \kappa \, dx \; = \; 0.$$

Since $w \in W_R$, it follows with Lemma 4.4

(4.3) $\quad (|w|^{(R)})^2 = \alpha_R(w,w) = \alpha_R(w,w) + \beta_R(w,\kappa).$

On the other hand, we find for $g \in W_R$:

$$\begin{aligned}
\alpha_R(w,g) + \beta_R(g,\kappa) \;&=\; \alpha_R(u\,|\,\Omega_R,\, g) \;+\; \beta_R(g,\, \pi\,|\,\Omega_R) \;-\; \alpha_R(\tilde{v},g) \;-\; \beta_R(g,\tilde{\rho}) \\
&\quad - \alpha_R(\,(\mathcal{A}_R(b) \;+\; \mu \cdot W)\,|\,\Omega_R,\, g\,) \;-\; \beta_R(g,\, \mu \cdot P\,|\,\Omega_R) \\
&= \int_{\Omega_R} \left(\sum_{j,k=1}^{3} D_j u_k \cdot D_j g_k \;+\; \lambda \cdot D_1 u \cdot g \;-\; \operatorname{div} g \cdot \pi \right) dx \\
&\quad + \frac{\lambda}{2} \cdot \int_{\partial B_R} u(x) \cdot g(x) \cdot \left(1 - \frac{x_1}{R}\right) do_x \;-\; \int_{\Omega_R} f \cdot g \, dx \\
&= \int_{\partial B_R} \mathcal{L}_R(u,\pi)(x) \cdot g(x) \, do_x \;\leq\; \Big[9 \cdot \|\nabla u|\partial B_R\|_2 \;+\; \|\pi|\partial B_R\|_2 \\
&\qquad + (\lambda/2) \cdot \Big(\int_{\partial B_R} |u(x)|^2 \cdot (1 - x_1/R)^2 \, do_x \Big)^{1/2} \Big] \cdot \|g|\partial B_R\|_2,
\end{aligned}$$

where the boundary operator $\mathcal{L}_R$ was defined in (1.7). Let $\mathcal{C}$ denote constants which only depend on Ω, S, σ or λ. Then the preceding inequality and Corollary 3.8 imply for $g \in W_R$

(4.4) $\quad \alpha_R(w,g) \;+\; \beta_R(g,\kappa) \;\leq\; \mathcal{C} \cdot (\gamma + \|f|B_S\|_2 + \|b\|_{1/2,2}) \cdot R^{-1} \cdot \mathcal{M}(R,\sigma) \cdot |g|^{(R)}.$

Referring to (4.3), and setting $g = w$, we may conclude

(4.5) $\quad |w|^{(R)} \;\leq\; \mathcal{C} \cdot (\gamma + \|f|B_S\|_2 + \|b\|_{1/2,2}) \cdot R^{-1} \cdot \mathcal{M}(R,\sigma).$

In order to estimate $\|\kappa\|_2$, put $\tilde{g} := \mathcal{D}_R(-\kappa)$, with the operator $\mathcal{D}_R$ introduced in Theorem 4.1. Then we get by using Lemma 4.1 and 4.3:

$$\begin{aligned}
\alpha_R(w,\tilde{g}) \;&\leq\; 9 \cdot \|\nabla w\|_2 \cdot \|\nabla \tilde{g}\|_2 + \lambda \cdot \|\nabla w\|_2 \cdot \|\tilde{g}\|_2 + (\lambda/2) \cdot \|w|\partial B_R\|_2 \cdot \|\tilde{g}|\partial B_R\| \\
&\leq\; 9 \cdot |w|^{(R)} \cdot \|\nabla \tilde{g}\|_2 \;+\; \mathcal{C} \cdot |w|^{(R)} \cdot R \cdot (\|\nabla \tilde{g}\|_2 + R^{-1/2} \cdot \|\tilde{g}|\partial B_R\|_2) \\
&\quad + (\lambda/2) \cdot |w|^{(R)} \cdot \|\tilde{g}|\partial B_R\|_2 \\
&\leq\; (9 + \mathcal{C} \cdot R + (\lambda/2) \cdot R^{1/2}) \cdot |w|^{(R)} \cdot (\, \|\nabla \tilde{g}\|_2 + R^{-1/2} \cdot \|\tilde{g}|\partial B_R\|_2\,) \\
&\leq\; \mathcal{C} \cdot R \cdot |w|^{(R)} \cdot (\, \|\nabla \tilde{g}\|_2 + R^{-1/2} \cdot \|\tilde{g}|\partial B_R\|_2\,) \;\leq\; \mathcal{C} \cdot R \cdot |w|^{(R)} \cdot \|\kappa\|_2.
\end{aligned}$$

Combining this estimate with (4.4), (4.5) and Theorem 4.1 yields

$$\begin{aligned}
\|\kappa\|_2^2 \;&=\; \beta_R(\tilde{g},\kappa) \;=\; \alpha_R(w,\tilde{g}) + \beta_R(\tilde{g},\kappa) - \alpha_R(w,\tilde{g}) \\
&\leq\; \mathcal{C} \cdot \Big((\gamma + \|f|B_S\|_2 + \|b\|_{1,2}) \cdot R^{-1} \cdot \mathcal{M}(R,\sigma) + R \cdot |w|^{(R)} \Big) \cdot \|\kappa\|_2 \\
&\leq\; \mathcal{C} \cdot (\gamma + \|f|B_S\|_2 + \|b\|_{1,2}) \cdot \mathcal{M}(R,\sigma) \cdot \|\kappa\|_2,
\end{aligned}$$

hence

$$\|\kappa\|_2 \leq C \cdot (\gamma + \|f|B_S\|_2 + \|b\|_{1,2}) \cdot \mathcal{M}(R, \sigma).$$

□

The previous theorem may be used in order to obtain a local L^2-estimate of the truncation error, that is, an L^2-estimate of the truncation error in a neighbourhood of Ω :

COROLLARY 4.4 *Let $\sigma \in (5/2, \infty)$. Then there is a constant $C = C(\Omega, S, \sigma, \lambda) > 0$ such that*

$$\|u|\Omega_S - v|\Omega_S\|_2 \leq C \cdot (\gamma + \|f|B_S\|_2 + \|b\|_{1,2}) \cdot R^{-1} \cdot \mathcal{M}(R, \sigma)$$

for u, v, f, γ, b, R as in Theorem 4.5.

Proof: Combine Lemma 4.2 and Theorem 4.5. □

References

[1] Bänsch, E., Dörfler, W.: Adaptive finite elements for exterior domain problems. Numer. Math. **80**, 497-523 (1998).

[2] Bruneau, C.-H., Fabrie, P.: New efficient boundary conditions for incompressible Navier-Stokes equations: a well-posedness result. RAIRO Modél. Math. Anal. Numér. **30**, 815-840 (1996).

[3] Carstensen, C., Funken, S. A.: Coupling of nonconforming finite element elements. I: a priori estimates. II: a posteriori-estimates and adaptive mesh-refinement. Computing **62**, 229-241, 243-259 (1999).

[4] Coifman, R., McIntosh, A., Meyer, Y.: L'intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes Lipschitziennes. Ann. of Math. **116**, 361-388 (1982).

[5] Deuring, P.: The Stokes System in an Infinite Cone. Berlin: Akademie Verlag 1994.

[6] Deuring, P.: Finite element methods for the Stokes system in three-dimensional exterior domains. Math. Methods Appl. Sci. **20**, 245-269 (1997).

[7] Deuring, P.: A mixed finite element method on truncated exterior domains. RAIRO Modél. Math. Anal. Numér. **32**, 283-305 (1998).

[8] Deuring, P.: Approximating exterior flows by mixed finite element methods: error estimates and implementation. Preprint, University of Halle. 1997.

[9] Deuring, P., Wahl, W. von: Strong solutions of the Navier-Stokes system in Lipschitz bounded domains. Math. Nachr. **171**, 111-148 (1995).

[10] Deuring, P., Wahl, W. von, Weidemaier, P.: Das lineare Stokes System in $\mathbb{R}^3$. I. Vorlesung über das Innenraumproblem. Bayreuth. Math. Schr. **27**, 1-252 (1988).

[11] Fabes, E. B., Kenig, C. E., Verchota, C. G.: The Dirichlet problem for the Stokes system on Lipschitz domains. *Duke Math. J.* **57**, 769-793 (1988).

[12] Farwig, R.: The stationary exterior 3D-problem of Oseen and Navier-Stokes equations in anisotropically weighted Sobolev spaces. Math. Z. **211**, 409-447 (1992).

[13] Fučík, S.. John, O., Kufner, A.: Function Spaces. Leyden: Noordhoff 1977.

[14] Galdi, G. P.: An Introduction to the Mathematical Theory of the Navier-Stokes Equations. Vol. I. Linearized Steady Problems (Rev. Ed.). New York e.a.: Springer 1998.

[15] Girault, V.. Raviart, P.-A.: Finite Element Methods for the Navier-Stokes Equations. Berlin e.a.: Springer 1986.

[16] Givoli, D.: Numerical methods for problems in infinite domains. Amsterdam: Elsevier 1992.

[17] Goldstein. C. I.: The finite element method with nonuniform mesh sizes for unbounded domains. Math. Comp. **36**, 387-404 (1981).

[18] Goldstein. C. I.: Multigrid methods for elliptic problems in unbounded domains. SIAM J. Numer. Anal. **30**, 159-183 (1993).

[19] Guirguis. G. H.: On the coupling of boundary integral and finite element methods for the exterior Stokes problem in 3D. SIAM J. Numer. Anal. **24**, 310-322 (1987).

[20] Guirguis. G. H.: A third-order boundary condition for the exterior Stokes problem in three dimensions. Math. Comp. **49**, 379-389 (1987).

[21] Guirguis. G. H., Gunzberger, M. D.: On the approximation of the exterior Stokes problem in three dimensions. RAIRO Modél. Math. Anal. Numér. **21**, 445-464 (1987).

[22] Halpern. L.. Schatzman, M.: Artificial boundary conditions for incompressible viscous flows. SIAM J. Math. Anal. **20**, 308-353 (1989).

[23] Hebeker, F. K.: Efficient boundary element methods for three-dimensional exterior viscous flows. Numer. Meth. Partial Diff. Equ. **2**, 273-297 (1986).

[24] Heywood. J. G., Rannacher, R., Turek, S.: Artificial boundaries and flux and pressure conditions for the incompressible Navier-Stokes equations. Internat. J. Numer. Methods Fluids **22**. 325-352 (1996).

[25] Jörgens, K.: Linear Integral Operators. Boston e.a.: Pitman 1982.

[26] Kračmar. S.. Neustupa, J.: Global existence of weak solutions of a nonsteady variational inequality of the Navier-Stokes type with mixed boundary conditions. In: Proceedings of the International Symposium on Numerical Analysis, 156-177. Prague: Charles University 1993.

[27] Kračmar. S.. Novotný, A., Pokorný, M.: Estimates of three dimensional Oseen kernels in L^p spaces. In: Applied Nonlinear Analysis, 281-316. London-New York: Plenum Press 1999.

[28] Kučera, P.: Solutions to the Navier-Stokes equations with mixed boundary conditions in a bounded domain. In: Analysis, Numerics and Applications of Differential and Integral Equations. Pitman Res. Notes Math. Ser., Vol. 379. London: Addison Wesley 1998.

[29] Kučera, P.: A structure of the set of critical points to the Navier-Stokes equations with mixed boundary conditions. In: Navier-Stokes Equations: Theory and Numerical Methods. Pitman Res. Notes Math. Ser., Vol. 380. London: Addison Wesley 1998.

[30] Meyer, Y., Coifman, R.: Wavelets, Calderon-Zygmund Operators and Multilinear Operators. Cambridge: Cambridge University Press 1997.

[31] Nazarov, S. A., Specovius-Neugebauer, M.: Approximation of exterior problems. Optimal conditions for the Laplacian. Analysis **16**, 305-324 (1996).

[32] Nazarov, S. A., Specovius-Neugebauer, M.: Approximation of exterior boundary value problems for the Stokes system. Asymptotic Anal. **14**, 233-255 (1997).

[33] Nečas, J.: Les Méthodes Directes en Théorie des Équations Elliptiques. Paris: Masson 1967.

[34] Rudin, W.: Real and Complex Analysis (3rd ed.). New York e.a.: McGraw-Hill 1987.

[35] Sequeira, A.: The coupling of boundary integral and finite element methods for the bidimensional exterior steady Stokes problem. Math. Methods Appl. Sci. **5**, 356-375 (1983).

[36] Sequeira, A.: On the computer implementation of a coupled boundary and finite element method for the bidimensional exterior steady Stokes problem. Math. Methods Appl. Sci. **8**, 117-133 (1986).

[37] Tsynkov, S. V.: Numerical solution of problems on unbounded domains. A review. Applied Numer. Math. **27**, 465-532 (1998).

[38] Verchota, C. G.: Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains. J. Funct. Anal. **59**, 572-611 (1984).

P. Deuring
Centre Universitaire de la Mi-Voix
Université du Littoral
Laboratoire de Mathematiques Pures et Appliquées
B.P. 699
F-62228 Calais Cedex
France

S. Kračmar
Department of Technical Mathematics
Faculty of Mechanical Engineering
Czech Technical University
Karlovo nám. 13
Praha 2 CZ 125 53
Czech Republic