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THE STOKES RESOLVENT IN 3D DOMAINS WITH CONICAL BOUNDARY POINTS: NONREGULARITY IN L^P-SPACES

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Abstract. It is shown that solutions of the 3D Stokes resolvent problem in domains with conical boundary points, under homogeneous Dirichlet boundary conditions, do not satisfy the usual resolvent estimate in L^p-spaces if p is close to infinity or close to 1.

1. Introduction. In recent years a great deal of research work was devoted to studying the nonstationary Navier-Stokes system in an L^p-framework with p different from 2, under homogeneous boundary conditions. Among the many articles dealing with this subject, we mention [21] and [24], where further references may be found. A more extensive presentation is given in the monographs [35] and [36]. All these studies deal with the Navier-Stokes systems in domains with smooth boundary, of class C^2 or better. If bounded Lipschitz domains are admitted, weak and strong solutions are only known to exist in L^2-spaces; see [23] and [12]. Thus it seems to be an open problem whether the nonstationary Navier-Stokes system over bounded Lipschitz domains, under homogeneous Dirichlet boundary conditions, may be solved in L^p-spaces with p different from 2.

In order to deal with this problem, one might think of using the functional analytic method invented by Fujita and Kato [18]. When bounded domains with smooth boundary are considered, this method–in an L^p-framework with $p \neq 2$–depends on the following result pertaining to the Stokes resolvent problem (1.1):

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Theorem 1.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^2-boundary $\partial\Omega$. Then, for any $p \in (1,\infty)$, $\lambda \in \mathbb{C}\backslash(-\infty,0]$ and $f \in L^p(\Omega)^3$, there exists a uniquely determined pair of functions $(u,\pi) = (u(\Omega,f,\lambda),\pi(\Omega,f,\lambda))$ such that $u \in W^{2,p}(\Omega)^3, \pi \in W^{1,p}(\Omega), \pi$ has vanishing mean value, and*

$$-\Delta u + \lambda u + \nabla\pi = f, \quad div\, u = 0, \tag{1.1}$$

$$u \,|\, \partial\Omega = 0. \tag{1.2}$$

For any $p \in (1,\infty)$, $\vartheta \in [0, \pi)$, there is a constant $C = C(\Omega,p,\vartheta)$ with

$$\big\| u(\Omega,f,\lambda) \big\|_p \le C|\lambda|^{-1}\big\| f \big\|_p \qquad \text{for } f \in L^p(\Omega)^3,\ \lambda \in \mathbb{C}\backslash\{0\} \tag{1.3}$$

with $|\arg\lambda| \le \vartheta$.

Proofs of this theorem may be found in [32] and [20]. Similar results related to the Stokes resolvent system in unbounded domains were established in [27], [1], [5], [6], [7], [8], [16].

Inequality (1.3) is the critical point of Theorem 1.1. In view of the question of whether the Fujita-Kato method may be applied to the nonstationary Navier-Stokes system in bounded Lipschitz domains, a proof of estimate (1.3) should be attempted under the assumption that Ω is only Lipschitz bounded. We show that such an attempt cannot succeed: if Ω is a domain with a narrow reentrant corner, inequality (1.3) is false for some values $p \in (1,\infty)$.

In order to explain this result in more detail, let us introduce some notations. If $N \in \mathbb{N}$, we write $\mathbb{B}_N(x,r)$ for an open ball in $\mathbb{R}^N$ with center $x \in \mathbb{R}^N$ and radius $r > 0$. For $\varphi \in (0, \pi/2]$, we put $\mathbb{K}(\varphi) := \big\{\big(\eta, |\eta|\cot\varphi + r\big) : \eta \in \mathbb{R}^2, r \in (0,\infty)\big\}$. This means $\mathbb{K}(\varphi)$ is an open circular infinite cone, with vertex in the origin, semiaperture φ and axis pointing in the positive x_3-direction. If $\varphi \in (0,\pi/2]$, let $\Omega_{\pi-\varphi}$ be a bounded domain with connected boundary supposed to be smooth everywhere except at a single point x_0. Without loss of generality, we may assume $x_0 = 0$. In a neighbourhood of this point, the set $\Omega_{\pi-\varphi}$ is to coincide with $\mathbb{R}^3\backslash\overline{\mathbb{K}(\varphi)}$. More concretely, we assume

$$\Omega_{\pi-\varphi} \cap \mathbb{B}_3(0,2) = \big(\mathbb{R}^3\backslash\overline{\mathbb{K}(\varphi)}\big) \cap \mathbb{B}_3(0,2). \tag{1.4}$$

Moreover, if A is an arbitrary set, $\gamma \in \mathbb{N}$, and $f : A \mapsto \mathbb{C}^\gamma$ a function, we shall frequently use the notation $|f|_0$ defined by $|f|_0 := \sup\big\{\,|\,f(x)\,|\ :\ x \in A\big\}$.

According to [26] or [11, Corollary 2.2, Theorem 2.6], boundary value problem (1.1), (1.2) in $\Omega_{\pi-\varphi}$ may be solved in the following sense:

Theorem 1.2. *Let* $\varphi \in (0, \pi/2]$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $p \in [2, \infty)$. *Then, for any* $f \in L^p(\Omega_{\pi-\varphi})^3$, *there is a uniquely determined pair of functions*

$$(u, \pi) = \big(u(f, \lambda, \varphi),\, \pi(f, \lambda, \varphi)\big) \in W^{2,\,p}_{loc}(\Omega_{\pi-\varphi})^3 \times W^{1,\,p}_{loc}(\Omega_{\pi-\varphi})$$

which satisfies (1.1) *and fulfills the relations*

$$u \in W^{1,\,2}_0(\Omega_{\pi-\varphi})^3 \cap L^p(\Omega_{\pi-\varphi})^3, \quad \pi \in L^2(\Omega_{\pi-\varphi}), \quad \int_{\Omega_{\pi-\varphi}} \pi\, dx = 0. \quad (1.5)$$

In particular, boundary condition (1.2) *is satisfied in the trace sense. Moreover, there is a constant* $C = C(p, \lambda, \Omega_{\pi-\varphi})$ *with*

$$\big\| u(f, \lambda, \varphi)\big\|_p \;\le\; C\big\| f\big\|_p \qquad \textit{for } f \in L^p(\Omega_{\pi-\varphi})^3. \quad (1.6)$$

It is proved in [3] that the relation $u \in W^{1,\,2}_0(\Omega_{\pi-\varphi})^3$ in (1.5) may be strengthened to $u \in W^{3/2,\,2}(\Omega_{\pi-\varphi})^3 \cap W^{1,\,q}_0(\Omega_{\pi-\varphi})^3$, for some $q \in (3, \infty)$. However, this fact will not be needed in the present context.

We show that for some exponents $p \in (1, \infty)$ with $p \neq 2$, the solutions of (1.1), (1.2) introduced in Theorem 1.2 do not satisfy inequality (1.3). More precisely, concerning exponents $p > 2$, the following result will be established:

Theorem 1.3. *Let* $\vartheta \in [0, \pi)$. *Then there are numbers* $\varphi \in (0, \pi/2)$ *and* $S \in (3, \infty)$ *with the following property:*

If $r \in [S, \infty)$ *and* $C_1 > 0$, $C_2 \ge 0$, *there exists a number* $M \in (C_2, \infty)$ *and a function* $f \in C_0^\infty(\Omega_{\pi-\varphi})^3$ *such that*

$$\big\| u(f,\, Me^{i\vartheta},\, \varphi)\big\|_r \;\ge\; C_1 M^{-1}\big\| f\big\|_r,$$

where $u(f,\, Me^{i\vartheta},\, \varphi)$ *was introduced in Theorem* 1.2.

As the proof of Theorem 1.3 will indicate, the exceptional values of φ mentioned in that theorem tend to be close to zero, so the corresponding domains $\Omega_{\pi-\varphi}$ exhibit a narrow reentrant corner. The question remains open as to what is the smallest number $S \in (2, \infty)$ with the properties stated in Theorem 1.3. The number S which will arise in our proof is larger than 3, but it is not clear whether the exponent 3 is significant in this context.

We deal with the case $p < 2$ by combining Theorem 1.3 with a duality argument, to obtain

Corollary 1.1. *Let* $\vartheta \in [0, \pi)$. *Then there are numbers* $\varphi \in (0, \pi/2)$, $\widetilde{S} \in (1, 3/2)$ *such that for any* $r \in (1, \widetilde{S}]$, $C_1 > 0$, $C_2 \geq 0$, *there exists* $M \in (C_2, \infty)$, $f \in C_0^\infty(\Omega_{\pi-\varphi})^3$ *with* $\|u(f, Me^{i\vartheta}, \varphi)\|_r \geq C_1 M^{-1} \|f\|_r$.

We do not treat the problem of whether in the case $p < 2$, boundary value problem (1.1), (1.2) in $\Omega_{\pi-\varphi}$ can be solved for an arbitrary function $f \in L^p(\Omega_{\pi-\varphi})^3$. When the Poisson equation is considered under analogous assumptions, solutions may be obtained by making use of the boundary potential method ([34, Theorem 5.1]). It is not clear, however, whether the same approach works when applied to the Stokes resolvent system (1.1). Alternatively, one may turn to Kondratiev's theory, which provides another access to the Dirichlet problem (1.1), (1.2) in $\Omega_{\pi-\varphi}$. This theory admits all exponents $p \in (1, \infty)$ and angles $\varphi \in (0, \pi/2]$, apart from some exceptional values. We refer to [29] for details.

Theorem 1.3 and Corollary 1.1 should be compared to a result by Shen ([30]) pertaining to strongly elliptic resolvent systems. Shen proved that for any $p \in (1, \infty)$, solutions to such systems in general Lipschitz domains satisfy L^p-estimates analogous to inequality (1.3). Thus our results point out a major difference between strongly elliptic systems and the Stokes equations. We remark that another such difference is related to resolvent estimates in Hölder norms ([9]).

Let us give some indications of the proof of Theorem 1.3. To this end put $\widetilde{g}_1(r) := e^{-r} + r^{-2}(re^{-r} + e^{-r} - 1)$, $\widetilde{g}_2(r) := e^{-r} + 3r^{-2}(re^{-r} + e^{-r} - 1)$ for $r \in \mathbb{C}\backslash\{0\}$, and $\widetilde{E}^\lambda_{jk}(z) := (4\pi|z|)^{-1}(\delta_{jk}\widetilde{g}_1(\sqrt{\lambda}|z|) - z_j z_k |z|^{-2} \widetilde{g}_2(\sqrt{\lambda}|z|))$, $E_{4k}(z) := (4\pi|z|^3)^{-1} z_k$ for $z \in \mathbb{R}^3\backslash\{0\}, j, k \in \{1, 2, 3\}, \lambda \in \mathbb{C}\backslash(-\infty, 0]$. Then a fundamental solution of the Stokes resolvent system (1.1) is given by the matrix-valued function $(\widetilde{E}^\lambda_{1k}, \widetilde{E}^\lambda_{2k}, \widetilde{E}^\lambda_{3k}, E_{4k})_{1\leq k\leq 3}$. We further introduce the stress tensor $\widetilde{\mathcal{D}}^\lambda : \mathbb{R}^3\backslash\{0\} \mapsto \mathbb{C}^{3\times3\times3}$ related to this fundamental solution by setting

$$\widetilde{\mathcal{D}}^\lambda_{jkl} := D_j \widetilde{E}^\lambda_{kl} + D_k \widetilde{E}^\lambda_{jl} - \delta_{jk} E_{4l} \text{ for } \ j,\, k,\, l \in \{1,\, 2,\, 3\},\ \lambda \in \mathbb{C}\backslash(-\infty, 0],$$

where the symbols D_j, D_k denote the partial derivative with respect to the j-th and k-th variable, respectively. For $\epsilon > 0$, $\varphi \in (0, \pi/2]$, we define the domain $\mathbb{L}(\varphi, \epsilon)$ by rounding off the vertex of $\mathbb{K}(\varphi)$. More precisely, we choose a monotone increasing function $\widetilde{\Psi} \in C^\infty([0, \infty))$ with $\widetilde{\Psi}(r) = r$ for $r \in [1, \infty)$, $r \leq \widetilde{\Psi}(r) \leq 1$ for $r \in [0, 1]$, $\Psi(0) > 0$; see [10, pages 30–31]. Then for $\epsilon > 0$, $\varphi \in (0, \pi/2]$ we put $\beta^{(\varphi,\epsilon)}(\eta) := \epsilon \cot\varphi \widetilde{\Psi}(|\eta|/\epsilon)$ for $\eta \in \mathbb{R}^2$, $\mathbb{L}(\varphi, \epsilon) := \{(\eta, \beta^{(\varphi,\epsilon)}(\eta) + r) : \eta \in \mathbb{R}^2, r \in (0, \infty)\}$; compare [10, pages 30–31].

Obviously, $\overline{\mathbb{L}(\varphi,\epsilon)} \subset \overline{\mathbb{K}(\varphi)}\backslash\{0\}$ and

$$\mathbb{L}(\varphi,\epsilon)\backslash\mathbb{B}_3(0,\epsilon/\sin\varphi) = \mathbb{K}(\varphi)\backslash\mathbb{B}_3(0,\epsilon/\sin\varphi) \quad (\epsilon>0, \varphi\in(0,\pi/2]). \quad (1.7)$$

This means the domain $\mathbb{L}(\varphi,\epsilon)$ and the cone $\mathbb{K}(\varphi)$ coincide everywhere except in a neighbourhood of the vertex of $\mathbb{K}(\varphi)$, with $\mathbb{L}(\varphi,\epsilon)$ being smoothly bounded everywhere. Take $p \in (1,\infty)$, $\lambda \in \mathbb{C}\backslash(-\infty,0]$, $\tau \in \{-1,1\}$, $\epsilon \in (0,\infty)$, $\varphi \in (0,\pi/2]$, and let $B \in \big\{\Omega_{\pi-\varphi}, \mathbb{K}(\varphi), \mathbb{L}(\varphi,\epsilon)\big\}$. Introduce the operators $\Pi\big(\tau,\, p,\, B\big) : L^p(\partial B) \mapsto L^p(\partial B)$ and $\Gamma\big(\tau,\, p,\, \lambda,\, B\big) : L^p(\partial B)^3 \mapsto L^p(\partial B)^3$ by setting

$$\begin{aligned} &\Pi\big(\tau,p,B\big)(\Phi)(x) && (1.8)\\ &:= (\tau/2)\Phi(x) - \int_{\partial B} (4\pi)^{-1}\big((x-y)n^{(B)}(y)\big)|x-y|^{-3}\Phi(y)\,dB(y), \\ &\Gamma\big(\tau,p,\lambda,B\big)(\Psi)(x) && (1.9)\\ &:= (\tau/2)\Psi(x) + \Bigg(\int_{\partial B}\sum_{j,k=1}^{3} \widetilde{\mathcal{D}}^{\lambda}_{jkl}(x-y)n_k^{(B)}(y)\Psi_j(y)\,dB(y)\Bigg)_{1\le l\le 3}, \end{aligned}$$

for $\Phi \in L^p(\partial B)$, $\Psi \in L^p(\partial B)^3$, $x \in \partial B$. The symbol $n^{(B)}$ denotes the outward unit normal to B.

These mappings are well defined, and they are bounded with respect to the norm of $L^p(\partial B)$. In the case $B \in \big\{\mathbb{K}(\varphi),\ \mathbb{L}(\varphi,\epsilon)\big\}$, we refer to [10, Lemmas 6.2, 6.5] for this fact. If $B = \Omega_{\pi-\varphi}$, this result follows by combining the preceding reference with the well-known theory of singular integrals on smooth manifolds, as presented in [13, Sections 4, 5], for example. It should be remarked that even if B is a general bounded Lipschitz domain, the operators introduced in (1.8) and (1.9) are bounded with respect to the norm of $L^p(\partial B)$; see [4]. However, this deep-lying result will not be needed here.

Let us indicate the argument of this paper. Suppose some $\varphi \in (0,\pi/2)$ is given. Then, as is known from [14, pages 101–102] (see also [10, Theorem 8.1]), there is at most a countable number of exponents $q \in (1,2)$ such that $\Pi(1,q,\mathbb{K}(\varphi))$ is not bijective. On the other hand, [11, Theorem 2.8] states there is at least one such exceptional exponent. Now choose $\varphi = \varphi_0$ so close to zero that the first eigenvalue $\gamma_1^{(\varphi_0)}$ of the Laplace-Beltrami operator on $\mathbb{K}(\varphi_0) \cap \partial\mathbb{B}_3(0,1)$ with homogeneous Dirichlet boundary values is larger than or equal to 2. Assume that if problem (1.1), (1.2) is considered on the domain $\Omega_{\pi-\varphi_0}$, the critical resolvent estimate (1.3) is valid for any $p \in (2,\infty)$.

The work at hand describes how one may deduce from this assumption that $\Pi(1, q, \mathbb{K}(\varphi_0))$ is bijective for all $q \in (1, 2]$, without any exceptional value. This conclusion contradicts [11, Theorem 2.8], and thus implies Theorem 1.3.

In order to set up this contradiction, we show in Sections 3–5 that if the parameters $\varphi \in (0, \pi/2]$ and $p \in [2, \infty)$ satisfy certain conditions, and if inequality (1.3) holds on $\Omega_{\pi-\varphi}$, then $\Gamma(-1, p, \lambda, \mathbb{K}(\varphi))$ is one-to-one (Theorem 5.2). As a first step to prove this assertion, we estimate certain potential functions. Then these estimates will be used to derive two Green's formulas on the unbounded domain $\mathbb{L}(\varphi, \epsilon)$; see Theorem 3.2 and 4.4. Once these Green's formulas are available, we establish the desired uniqueness result about $\Gamma(-1, p, \lambda, \mathbb{K}(\varphi))$; see Section 5. In the last part of our theory, presented in Section 7, we will make use of functional analytic arguments, mainly involving the theory of semi-Fredholm operators. These arguments and our result on $\Gamma(-1, p, \lambda, \mathbb{K}(\varphi))$ (Theorem 5.2) imply that for φ_0 chosen as above, the operator $\Pi(1, p', \mathbb{K}(\varphi))$ is bijective for any $p \in (2, \infty)$–a statement which gives rise to a contradiction, as mentioned before.

We remark that the properties of the angle φ_0 described above are essential because at some point of our theory–in the proof of Theorem 4.1 – we shall need that for $\varphi \in (0, \varphi_0]$, $g \in C_0^\infty(\mathbb{K}(\varphi))$, the Dirichlet problem $\Delta v = g \text{ in } \mathbb{K}(\varphi), v|\partial\mathbb{K}(\varphi) = 0$, may be solved in certain Kondratiev spaces.

We further remark that the present article heavily draws on results from reference [10], which is mainly devoted to studying the operator $\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi))$. In particular, it is shown in [10] that for any $\lambda \in \mathbb{C}\backslash(-\infty, 0], \tau \in \{-1, 1\}$, there are values of p and φ such that $\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi))$ is not Fredholm ([10, Corollary 12.9]). We conjectured in [10, Section 1] that such a behaviour of the operator $\Gamma(\tau, p, \lambda, \mathbb{K}(\varphi))$ might indicate that estimate (1.3) does not hold for all $p \in (1, \infty)$. This suspicion is confirmed by Theorem 1.3 and Corollary 1.1 in the work at hand.

When problem (1.1), (1.2) is studied under the assumption that Ω is a bounded domain in $\mathbb{R}^n$ ([20]), or an exterior domain in $\mathbb{R}^3$ ([5], [6], [7]), or a half-space in $\mathbb{R}^n$ ([27]), the proof of estimate (1.3) may be reduced to L^p-estimates for the Laplacian with Neumann boundary conditions ([20, Lemmas 2.3, 2.5], [5, (5.11)], [27, Theorems 6.2, 6.5]).

For example, Giga [20, Proof of Lemma 2.5] uses the fact that if Ω is a bounded smooth domain in $\mathbb{R}^n$ with $n \in \mathbb{N}$, $n \geq 2$, and if $p \in (1, \infty)$, $g \in W^{-1/p,p}(\partial\Omega)$, then the solution $v \in W^{1,p}(\Omega)$ of the boundary value problem $\Delta v = 0$ in Ω, $\partial v/\partial n = g$ on $\partial\Omega$, admits the estimate $\|v\|_{1,p} \leq C(\Omega, p)\|g\|_{-1/p,p}$. However, if Ω is only supposed to be a Lipschitz domain, this estimate is, in general, only valid for exponents $p \in (3/2-\epsilon(\Omega), 3+\epsilon(\Omega))$, with some value $\epsilon(\Omega) > 0$ depending on Ω; see [15, Corollaries 9.3, 12.3].

Of course this observation does not prove that inequality (1.3) is false for certain exponents $p \in (1,\infty)$ if Ω is a Lipschitz domain. However, they indicate it should not be surprising that a negation of (1.3) may be derived from the Fredholm properties of the operator $\Pi(1,p,\mathbb{K}(\varphi))$. In fact, the adjoint of this operator is linked to the Laplace equation in $\mathbb{R}^3\backslash\overline{\mathbb{K}(\varphi)}$ under Neumann boundary conditions. The Laplace equation arises due to the pressure part π of solutions (u,π) to (1.1), (1.2): if the right-hand side f in (1.1) is smooth and solenoidal, the pressure π satisfies the equation $\Delta\pi = 0$. Thus the fact that the operator $\Pi(1,q,\mathbb{K}(\varphi))$ plays a critical role in our proof indicates that it is the influence of the pressure, or equivalently, of the divergence condition $\operatorname{div} u = 0$, which causes inequality (1.3) to fail for certain values of p and φ. This observation explains why no such exceptional values arise in the case of strongly elliptic systems ([30]).

2. Some definitions and auxiliary results. As indicated before, we will frequently refer to the theory in [10] which is basic to this article. Moreover, we will make use of several results from [11]. Theorem 1.2, for example, was proved in the latter reference. In the ensuing two theorems, we state some further essential tools, which pertain to the Stokes system and to the divergence equation.

Theorem 2.1. ([19, p. 232]) *Let Ω be a bounded domain with C^2-boundary. Denote by $n^{(\Omega)}$ the outward unit normal to Ω. Let $p \in (1,\infty)$. For any $F \in (W_0^{1,p'}(\Omega)^3)'$, $b \in W^{1-1/p,p}(\partial\Omega)^3$ with $\int_{\partial\Omega} n^{(\Omega)} b\, d\Omega = 0$, there is a uniquely determined pair $(u,\pi) \in W^{1,p}(\Omega)^3 \times L^p(\Omega)$ with $\operatorname{div} u = 0, u|\partial\Omega = b$, and*

$$\int_\Omega \sum_{j,k=1}^{3} (D_k u_j - \delta_{jk}\cdot\pi) D_k v_j\, dx = F(v) \text{ for } v \in C_0^\infty(\Omega)^3,\ \int_\Omega \pi\, dx = 0. \quad (2.1)$$

There is a constant $C > 0$ such that

$$\|u\|_{1,p} + \|\pi/\mathbb{R}\|_p \le C(\|F\|_{-1,p} + \|b\|_{1-1/p,p}) \quad (2.2)$$

for F, b, u, π as before, with $\|F\|_{-1,p} := \sup\{\|F(w)\|/\|w\|_{1,p'} : w \in W_0^{1,p'}(\Omega)^3, w \neq 0\}$.

Theorem 2.2. ([19, Theorem 3.2], [2]) *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain, and let $p \in (1,\infty)$. Then there is a constant $C = C(\Omega,p)$ and an* *operator*

$$\mathcal{D}(\Omega,p):\Big\{f\in W_0^{1,p}(\Omega):\int_\Omega f\,dx=0\Big\}\longmapsto W_0^{2,p}(\Omega)^3$$

such that the following holds for $f\in W_0^{1,p}(\Omega)$ *with* $\int_\Omega f\,dx=0$:

$$div\,\mathcal{D}(\Omega,p)(f)=f,\quad \|\mathcal{D}(\Omega,p)(f)\|_{1,p}\le C\|f\|_p,\quad \|\mathcal{D}(\Omega,p)(f)\|_{2,p}\le C\|f\|_{1,p}.$$

Next we consider the Laplace-Beltrami operator in three dimensions. Let $\varphi\in(0,\pi/2]$ and put $C_\varphi:=\mathbb{K}(\varphi)\cap\partial\mathbb{B}_3(0,1)$. Let Δ'_φ denote the Laplace-Beltrami operator with domain $W_0^{1,2}(C_\varphi)\cap W^{2,2}(C_\varphi)$. This means Δ'_φ is the uniquely determined operator from $W_0^{1,2}(C_\varphi)\cap W^{2,2}(C_\varphi)$ into $L^2(C_\varphi)$ such that

$$\begin{aligned}&\big(\Delta'_\varphi u\big)\big(g_\varphi(\theta,\vartheta)\big)\\&=\sin^{-1}(\vartheta)\partial/\partial\vartheta\big(\sin\vartheta\,\partial/\partial\vartheta(u\circ g_\varphi)(\theta,\vartheta)\big)+\sin^{-2}(\vartheta)\partial^2/\partial\theta^2(u\circ g_\varphi)(\theta,\vartheta)\end{aligned}$$

for $u\in W_0^{1,2}(C_\varphi)\cap W^{2,2}(C_\varphi)$, $\theta\in(0,2\pi)$, $\vartheta\in(0,\varphi)$, where $g_\varphi:(0,2\pi)\times(0,\varphi)\mapsto C_\varphi$ is defined by

$$g_\varphi(\theta,\vartheta):=\big(\sin\vartheta\cos\theta,\sin\vartheta\sin\theta,\cos\vartheta\big)\quad\text{for }\theta\in(0,2\pi),\ \vartheta\in(0,\varphi).$$

The operator $-\Delta'_\varphi$ is self-adjoint and positive in $L^2(C_\varphi)$; see [22, pp. 74/75], [31]. Thus, by the theory of self-adjoint operators, there is a sequence $\big(\gamma_n^{(\varphi)}\big)_{n\in\mathbb{N}}$ in $(0,\infty)$ such that the set of eigenvalues of $-\Delta'_\varphi$ coincides with $\{\gamma_n^{(\varphi)}:n\in\mathbb{N}\}$, and it holds that $\gamma_n^{(\varphi)}<\gamma_{n+1}^{(\varphi)}$ for $n\in\mathbb{N}$, $\gamma_n^{(\varphi)}\to\infty$ $(n\to\infty)$. Moreover, applying Courant's theory of eigenvalues, we get

$$\gamma_1^{(\varphi)}\le\gamma_1^{(\tau)}\qquad\text{for}\quad\varphi,\tau\in(0,\pi/2)\quad\text{with}\quad\varphi\ge\tau.\tag{2.3}$$

We further have

$$\gamma_1^{(\varphi)}\uparrow\infty\quad(\varphi\downarrow 0).\tag{2.4}$$

A proof of (2.4) may be based on the observation that

$$f(\theta,\vartheta)=f(\theta,\vartheta)-f(\theta,\varphi)=\int_\varphi^\vartheta D_2f(\theta,t)\,dt$$

for $\vartheta, \varphi \in [0,\pi],\ \theta \in [0,2\pi]$, and for smooth functions $f : [0,2\pi] \times [0,\pi] \mapsto \mathbb{R}$ with $f(\cdot,\varphi) = 0$. It follows for $\varphi \in (0,\pi/2),\ h \in W_0^{1,2}(C_\varphi),\ \theta \in (0,2\pi),\ \vartheta \in (0,\varphi)$ that

$$\big|(h \circ g_\varphi)(\theta,\vartheta)\big| \sin^{1/2}(\vartheta) \le (\varphi-\vartheta)^{1/2}\Big(\int_\vartheta^\varphi \big|D_2(h\circ g_\varphi)(\theta,t)\big|^2 \sin t dt\Big)^{1/2},$$

hence

$$\begin{aligned}
&\int_0^{2\pi}\int_0^\varphi \big|(h\circ g_\varphi)(\theta,\vartheta)\big|^2 \sin\vartheta\, d\vartheta\, d\theta \\
&\le K\varphi^2 \int_0^{2\pi}\int_0^\varphi \Big(\sin^{-1}(\vartheta)\big|D_1(h\circ g_\varphi)(\theta,\vartheta)\big|^2 \\
&\qquad + \sin\vartheta\big|D_2(h\circ g_\varphi)(\theta,\vartheta)\big|^2\Big) d\vartheta\, d\theta \\
&= K\varphi^2 \int_0^{2\pi}\int_0^\varphi \big((-\Delta'_\varphi h)\circ g_\varphi\big)(\theta,\vartheta)(h\circ g_\varphi)(\theta,\vartheta)\sin\vartheta\, d\vartheta\, d\theta,
\end{aligned}$$

with a numerical constant K. Thus we get

$$\|h\|_2^2 \ \le\ K\varphi^2 \int_{C_\varphi} -\Delta'_\varphi h h\, dC_\varphi \quad \text{for} \quad h \in W_0^{1,2}(C_\varphi).$$

The last inequality implies (2.4).

Let us introduce Kondratiev's spaces on $\mathbb{K}(\varphi)$. For $\varphi \in (0,\pi/2]$, $l \in \mathbb{N}_0$, $\beta \in \mathbb{R}$, $p \in (1,\infty)$, $u \in W^{l,1}_{loc}(\mathbb{K}(\varphi))$, put

$$\|u\|^{(l)}_{p,\beta} := \Big(\int_{\mathbb{K}(\varphi)} \sum_{\alpha\in\mathbb{N}_0^3, |\alpha|_*\le l} |x|^{p(\beta-l+|\alpha|_*)}\big|D^\alpha u(x)\big|^p\ dx\Big)^{1/p},$$

where we used the abbreviation $|\alpha|_* := \alpha_1+\alpha_2+\alpha_3$ for $\alpha = (\alpha_1,\alpha_2,\alpha_3) \in \mathbb{N}_0^3$. For φ, l, β, p as before, we further set

$$V^l_{p,\beta}\big(\mathbb{K}(\varphi)\big) := \big\{u \in W^{l,1}_{loc}(\mathbb{K}(\varphi))\ :\ \|u\|^{(l)}_{p,\beta} < \infty\big\}.$$

We need the following results pertaining to the spaces $V^l_{p,\beta}\big(\mathbb{K}(\varphi)\big)$:

Theorem 2.3. ([29, pages 79, 82, 92; Theorem 6.6, 6.10]). *Let* $p \in (1,\infty)$, $\beta \in \mathbb{R}$, $\varphi \in (0,\pi/2]$. *Assume* $\beta-2+3/p \notin \{(1/2)(1\pm(4\gamma_n^{(\varphi)}+1)^{1/2}) : n \in \mathbb{N}\}$. *Then the operator*

$$A^\varphi_{p,\beta} : \big\{u \in V^2_{p,\beta}\big(\mathbb{K}(\varphi)\big) : u|\partial\mathbb{K}(\varphi) = 0\big\} \longmapsto V^0_{p,\beta}\big(\mathbb{K}(\varphi)\big),\quad A^\varphi_{p,\beta}(u) := -\Delta u,$$

is an isomorphism. Let $\delta \in \mathbb{R}$ *with* $\beta < \delta$. *Assume*

$$[\beta - 2 + 3/p,\ \delta - 2 + 3/p] \cap \big\{(1/2)\big(1 \pm (4\gamma_n^{(\varphi)} + 1)^{1/2}\big) : n \in \mathbb{N}\big\} = \emptyset.$$

Let $f \in V^0_{p,\beta}(\mathbb{K}(\varphi)) \cap V^0_{p,\delta}(\mathbb{K}(\varphi))$. *Then* $(A^{\varphi}_{p,\beta})^{-1}(f) \in V^2_{p,\delta}(\mathbb{K}(\varphi))$.

Of course, the assumption $\varphi \le \pi/2$ in the definition of Δ'_φ and in Theorem 2.3 may be relaxed to $\varphi < \pi$. However, values $\varphi \in (\pi/2, \pi)$ will not be considered in the following, and we want to maintain the convention, already used in [10] and [11], that $\mathbb{K}(\varphi)$ denotes a cone with semiaperture $\varphi \le \pi/2$.

Next we introduce local coordinates for $\partial\mathbb{K}(\varphi)$ and $\partial\mathbb{L}(\varphi, \epsilon)$. Let $\epsilon \in (0, \infty)$, $\varphi \in (0, \pi/2]$. Put

$$g^{(\varphi)}(\eta) := (\eta, |\eta| \cot\varphi), \quad \gamma^{(\varphi,\epsilon)}(\eta) := \big(\eta, \beta^{(\varphi,\epsilon)}(\eta)\big) \text{ for } \ \eta \in \mathbb{R}^2,$$

where $\beta^{(\varphi,\epsilon)}$ was introduced in Section 1. The functions $g^{(\varphi)}, \gamma^{(\varphi,\epsilon)}$ are parametric representations of $\partial\mathbb{K}(\varphi)$ and $\partial\mathbb{L}(\varphi, \epsilon)$, respectively, with

$$\mathbb{K}(\varphi) = g^{(\varphi)}(\mathbb{R}^2), \quad \mathbb{L}(\varphi, \epsilon) = \gamma^{(\varphi,\epsilon)}(\mathbb{R}^2);$$

see [10, pages 29–32]. We shall write $J^{(\varphi,\epsilon)}$ for the area element related to $\gamma^{(\varphi,\epsilon)}$. Note that the area element induced by $g^{(\varphi)}$ is constant and equals $\sin^{-1}(\varphi)$. The outward unit normal to $\mathbb{K}(\varphi)$ and $\mathbb{L}(\varphi, \epsilon)$ will be denoted by, respectively, $n^{(\varphi)}$ and $n^{(\varphi,\epsilon)}$. Sometimes it will be convenient to use the notation

$$\mathbb{L}(\varphi, 0) := \mathbb{K}(\varphi), \quad \gamma^{(\varphi,0)} := g^{(\varphi)}, \quad n^{(\varphi,0)} := n^{(\varphi)}, \quad J^{(\varphi,0)} := \sin^{-1}(\varphi).$$

We make frequent use of the ensuing relations:

$$\begin{aligned} &\big|\gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta) + (0,0,r)\big| \\ &\ge (1/4)\sin\varphi\big(\big|\gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta)\big| + |r|\big) \ge (1/4)\sin\varphi(|\xi - \eta| + |r|) \end{aligned} \tag{2.5}$$

for $\xi, \eta \in \mathbb{R}^2,\ r \in \mathbb{R},\ \epsilon \ge 0,\ \varphi \in (0, \pi/2]$;

$$\int_{\mathbb{R}^3} f\, dx = \int_{\mathbb{R}} \int_{\mathbb{R}^2} f\big(\gamma^{(\varphi,\epsilon)}(\xi) + (0,0,r)\big)\, d\xi\, dr \tag{2.6}$$

for $f \in L^1(\mathbb{R}^3)$, $\epsilon \ge 0$, $\varphi \in (0, \pi/2]$. We refer to [10, Lemma 3.4] for inequality (2.5), and to [10, Lemma 3.5] for equation (2.6).

Next we state an estimate of the fundamental solution of (1.1) introduced in Section 1: for any $\vartheta \in [0, \pi)$, there is a constant $C(\vartheta) > 0$ with

$$\left| D^a \widetilde{E}^\lambda_{jk}(z) \right| \le C(\vartheta) |\lambda|^{-\gamma} |z|^{-1-2\gamma-|a|_*} \tag{2.7}$$

for $\lambda \in \mathbb{C}\backslash\{0\}$ with $|\arg \lambda| \le \vartheta,\ z \in \mathbb{R}^3\backslash\{0\},\ 1 \le j, k \le 3,\ \gamma \in [0,1],\ a \in \mathbb{N}_0^3$ with $|a|_* \le 2$. A proof of (2.7) is indicated in [10, page 65].

For $\lambda \in \mathbb{C}\backslash(-\infty, 0],\ h \in C_0^\infty(\mathbb{R}^3)^3$, we define the volume potentials $\widetilde{R}^\lambda(h) : \mathbb{R}^3 \mapsto \mathbb{R}^3,\ S(h) : \mathbb{R}^3 \mapsto \mathbb{R}$ by setting

$$\widetilde{R}^\lambda(h)(x) := \Big(\int_{\mathbb{R}^3} \sum_{k=1}^3 \widetilde{E}^\lambda_{jk}(x-y) h_k(y)\, dy \Big)_{1 \le j \le 3},$$

$$S(h)(x) := \int_{\mathbb{R}^3} \sum_{k=1}^3 E_{4k}(x-y) h_k(y)\, dy \quad \text{for} \quad x \in \mathbb{R}^3.$$

Some properties of these potential functions are collected in the ensuing theorem.

Theorem 2.4. *For $\lambda \in \mathbb{C}\backslash(-\infty, 0],\ h \in C_0^\infty(\mathbb{R}^3)^3$, it holds that $\widetilde{R}^\lambda(h) \in C^\infty(\mathbb{R}^3)^3 \cap W^{2,s}(\mathbb{R}^3)^3$ for $s \in (1, \infty)$, $S(h) \in C^\infty(\mathbb{R}^3) \cap L^s(\mathbb{R}^3)$ for $s \in (3/2, \infty)$, $\nabla S(h) \in L^s(\mathbb{R}^3)^3$ for $s \in (1, \infty)$, $-\Delta \widetilde{R}^\lambda(h) + \lambda \widetilde{R}^\lambda(h) + \nabla S(h) = h$, $div \widetilde{R}^\lambda(h) = 0$. It further holds for $j, k \in \{1, 2, 3\},\ \epsilon \ge 0,\ \varphi \in (0, \pi/2],\ s \in (1, \infty)$ that*

$$\widetilde{R}^\lambda_k(h)|\partial\mathbb{L}(\varphi, \epsilon),\ \ D_j \widetilde{R}^\lambda_k(h)|\partial\mathbb{L}(\varphi, \epsilon),\ \ S(h)|\partial\mathbb{L}(\varphi, \epsilon) \ \in\ L^s\big(\partial\mathbb{L}(\varphi, \epsilon)\big). \tag{2.8}$$

The assertions of this theorem except those in (2.8) are proved in [27] and [13, Section 1]. The relations in (2.8) may easily be obtained from (2.7) and the assumption $h \in C_0^\infty(\mathbb{R}^3)^3$.

Next we introduce some boundary potentials. Take $\varphi \in (0, \pi/2]$, $\epsilon \ge 0$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $\tau \in \{-1, 1\}$, $p \in (1, \infty)$. Let the operator

$$\Gamma^*\big(\tau, p, \lambda, \mathbb{L}(\varphi, \epsilon)\big) : L^p\big(\partial\mathbb{L}(\varphi, \epsilon)\big)^3 \mapsto L^p\big(\partial\mathbb{L}(\varphi, \epsilon)\big)^3$$

be given by

$$\Gamma^*\big(\tau, p, \lambda, \mathbb{L}(\varphi, \epsilon)\big)(\Phi)(x) :=$$
$$(\tau/2)\Phi(x) - \Big(\int_{\partial\mathbb{L}(\varphi,\epsilon)} \sum_{k,l=1}^3 \widetilde{\mathcal{D}}^\lambda_{jkl}(x-y) n_k^{(\varphi,\epsilon)}(x) \Phi_l(y) d\mathbb{L}(\varphi, \epsilon)(y) \Big)_{1 \le j \le 3}$$

for $x \in \partial\mathbb{L}(\varphi,\epsilon)$, $\Phi \in L^p\big(\partial\mathbb{L}(\varphi,\epsilon)\big)^3$. This operator is well defined ([10, Lemma 6.5]) and adjoint to the mapping $\Gamma\big(\tau, p', \lambda, \mathbb{L}(\varphi,\epsilon)\big)$ introduced in Section 1. Here and in the following, we use the notation $p' := (1-1/p)^{-1}$. An essential property of these potentials is stated in

Theorem 2.5. ([10, Corollary 12.2, 13.3]). *Let* $\varphi \in (0,\pi/2]$, $\tau \in \{-1,1\}$, $\lambda \in \mathbb{C}\backslash(-\infty,0]$, $\epsilon \in [0,\infty)$. *Then the operators* $\Gamma(\tau,2,\lambda,\mathbb{L}(\varphi,\epsilon))$ *and* $\Gamma^*(\tau,2,\lambda,\mathbb{L}(\varphi,\epsilon))$ *are bijective.* (*Note that the case* $\epsilon = 0$ *is admitted.*)

Let $A \in \{\mathbb{L}(\varphi,\epsilon),\ \Omega_{\pi-\varphi}\}$, $\Phi \in L^p(\partial A)^3$. We introduce the single layer potentials $\widetilde{V}^\lambda\big(\partial A\big)(\Phi) : \mathbb{R}^3 \mapsto \mathbb{C}^3,\ Q\big(\partial A\big)(\Phi) : \mathbb{R}^3\backslash\partial A \mapsto \mathbb{C}$ by setting

$$\widetilde{V}^\lambda\big(\partial A\big)(\Phi)(x) := \Big(\int_{\partial A}\sum_{k=1}^{3}\widetilde{E}^\lambda_{jk}(x-y)\Phi_k(y)\,dA(y)\Big)_{1\le j\le 3} \ \text{ for }\ x \in \mathbb{R}^3,$$

$$Q\big(\partial A\big)(\Phi)(x) := \int_{\partial A}\sum_{k=1}^{3} E_{4k}(x-y)\Phi_k(y)\,dA(y) \text{ for }\ x \in \mathbb{R}^3\backslash\partial A;$$

compare [10, Definition 9.2]. Let $B \in \{\mathbb{K}(\varphi), \mathbb{R}^3\backslash\overline{\mathbb{K}(\varphi)}, \Omega_{\pi-\varphi}\}$, $\Phi \in L^p(\partial B)^3$. Define the double-layer potentials $W(\lambda,B)(\Phi) : \overline{B} \mapsto \mathbb{C}^3$, $P(\lambda,B)(\Phi) : B \mapsto \mathbb{C}$ by

$$W(\lambda,B)(\Phi)(x) := \Big(\int_{\partial B}\sum_{j,k=1}^{3}\widetilde{\mathcal{D}}^\lambda_{jkl}(x-y)n^{(B)}_k(y)\Phi_j(y)\,dB(y)\Big)_{1\le l\le 3}$$
$$\text{for } x \in B,$$
$$W(\lambda,B)(\Phi)(x) := \Gamma\big(1,p,\lambda,B\big)(\Phi)(x) \ \text{ for } x \in \partial B \text{ if } B \in \{\mathbb{K}(\varphi), \Omega_{\pi-\varphi}\},$$
$$W(\lambda,B)(\Phi)(x) := -\Gamma\big(-1,p,\lambda,B\big)(\Phi)(x) \ \text{ for } x \in \partial B,\ \text{if } B = \mathbb{R}^3\backslash\overline{\mathbb{K}(\varphi)},$$
$$P(\lambda,B)(\Phi)(x) := \int_{\partial B}\sum_{j,k=1}^{3}\big(2D_jE_{4k}(x-y) - \lambda(4\pi)^{-1}$$
$$\delta_{jk}\big(|x-y|^{-1} - \sigma_B(y)\big)\big)n^{(B)}_k(y)\Phi_j(y)\,dB(y)$$

for $x \in B$, where the symbol $n^{(B)}$ denotes the outward unit normal to B, and the function $\sigma_B : \partial B \mapsto \mathbb{R}$ is given by

$$\sigma_B(y) := 0 \ \text{ for }\ y \in \partial B \ \text{ if }\ B = \Omega_{\pi-\varphi},$$
$$\sigma_B(y) := \big|(0,0,1) - y\big|^{-1} \ \text{ for }\ y \in \partial B \ \text{ if }\ B \in \big\{\mathbb{K}(\varphi), \mathbb{R}^3\backslash\overline{\mathbb{K}(\varphi)}\big\}.$$

Note that in the case $B \in \{\mathbb{K}(\varphi), \mathbb{R}^3\backslash\overline{\mathbb{K}(\varphi)}\}$, the integral appearing in the definition of the function $P(\lambda, B)(\Phi)$ would not converge in general if the term $\sigma_B(y)$ were dropped. We further indicate that for $V \in \{\mathbb{K}(\varphi), \mathbb{R}^3\backslash\overline{\mathbb{K}(\varphi)}\}$, $x \in V$, we have

$$P(\lambda, V)(\Phi)(x) = \tau\big(\mathrm{div}Z^{(\varphi)}(\Phi)(x) + \lambda S^{(\varphi)}(\Phi)(x)\big), \tag{2.9}$$

where $\tau := 1$ if $V = \mathbb{K}(\varphi)$, $\tau := -1$ else;

$$Z^{(\varphi)}(\Phi)(x) := \tag{2.10}$$
$$\Big((2\pi)^{-1}\int_{\partial\mathbb{K}(\varphi)}\sum_{k=1}^{3}(x-y)_k|x-y|^{-3}n_k^{(\varphi)}(y)\Phi_j(y)\,d\mathbb{K}(\varphi)(y)\Big)_{1\le j\le 3},$$
$$S^{(\varphi)}(\Phi)(z) := \tag{2.11}$$
$$-(4\pi)^{-1}\int_{\partial\mathbb{K}(\varphi)}\big(|z-y|^{-1} - |(0,0,1)-y|^{-1}\big)\big(n^{(\varphi)}(y)\Phi(y)\big)\,d\mathbb{K}(\varphi)(y)$$

for $x \in \mathbb{R}^3\backslash\partial\mathbb{K}(\varphi)$, $z \in \mathbb{R}^3$. In the ensuing two lemmas, we state some obvious properties of the preceding single- and double-layer potentials.

Lemma 2.1. (see [10, pages 154–156]) *Let* $\varphi \in (0, \pi/2]$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $p \in (1, \infty)$, $B \in \{\mathbb{K}(\varphi), \mathbb{R}^3\backslash\overline{\mathbb{K}(\varphi)}, \Omega_{\pi-\varphi}\}$, $\Phi \in L^p(\partial B)^3$, $\epsilon \ge 0$, $\psi \in L^p(\partial\mathbb{L}(\varphi, \epsilon))^3$. *Let* (u,π) *be given by* $(u,\pi) := (\widetilde{V}^\lambda(\partial\mathbb{L}(\varphi,\epsilon))(\psi)|\mathbb{R}^3\backslash\partial\mathbb{L}(\varphi,\epsilon), Q(\partial\mathbb{L}(\varphi,\epsilon))(\psi))$, *or* $(u, \pi) := \big(W(\lambda, B)(\Phi)|B, P(\lambda, B)(\Phi)\big)$. *Then* u *and* π *are* C^∞*-functions with* $-\Delta u + \lambda u + \nabla\pi = 0$, $div\, u = 0$.

Lemma 2.2. *Let* $p \in (1, \infty)$, $\varphi \in (0, \pi/2]$, $\Phi \in L^p\big(\partial\mathbb{K}(\varphi)\big)^3$. *Then the functions* $Z^{(\varphi)}(\Phi)$ *and* $S^{(\varphi)}(\Phi)|\mathbb{R}^3\backslash\partial\mathbb{K}(\varphi)$ *are of class* C^∞. *Suppose that* $p > 2$. *Then* $S^{(\varphi)}(\Phi) \in C^0(\mathbb{R}^3)$.

Proof. In order to prove continuity of $S^{(\varphi)}(\Phi)$ in the case $p > 2$, take $x, x' \in \mathbb{R}^3$, and decompose the domain of integration $\partial\mathbb{K}(\varphi)$ appearing in the definition of $S^{(\varphi)}(\Phi)$ into the parts $\partial\mathbb{K}(\varphi) \cap \mathbb{B}_3(x, 2|x-x'|)$ and $\partial\mathbb{K}(\varphi)\backslash\mathbb{B}_3(x, 2|x-x'|)$. Estimating the two integrals arising in this way, we obtain $|S^{(\varphi)}(\Phi)(x) - S^{(\varphi)}(\Phi)(x')| \le \mathcal{C}|x-x'|^{-1+2/p'}$, where the constant $\mathcal{C}$ is independent of x and x'. The other statements of Lemma 2.2 are obvious. □

A result much more deep-lying than these lemmas concerns the behaviour of $W(\lambda, B)(\Phi)$ near ∂B. In fact, the function $W(\lambda, B)(\Phi)$ is defined on ∂B in such a way that $W(\lambda, B)(\Phi)|\partial B$ is the boundary value of $W(\lambda, B)(\Phi)|B$, in a sense which depends on the smoothness of Φ. We shall exploit the following version of this result:

Theorem 2.6. *Take* $p \in (1,\infty)$, $\varphi \in (0,\pi/2]$, $\lambda \in \mathbb{C}\backslash(-\infty,0]$, $B \in \{\mathbb{K}(\varphi), \mathbb{R}^3\backslash\overline{\mathbb{K}(\varphi)}\}$. *Let* $\Phi \in L^p(\partial\mathbb{K}(\varphi))^3, \tau \in \{-1,1\}$ *with* $\Gamma\big(\tau,p,\lambda,\mathbb{K}(\varphi)\big)(\Phi) = 0$. *Then it holds*

$$\big(\Phi \circ g^{(\varphi)}\big)|\mathbb{B}_2(0,b)\backslash\overline{\mathbb{B}_2(0,a)} \in C^{1,\alpha}\big(\mathbb{B}_2(0,b)\backslash\overline{\mathbb{B}_2(0,a)}\big)^3 \tag{2.12}$$

for $\alpha \in (0,1)$, $a,b \in (0,\infty)$ *with* $a<b$. (*For an open set* $U \subset \mathbb{R}^2$ *and for* $\alpha \in (0,1)$, *we denote by* $C^{1,\alpha}(U)$ *the space of all* C^1*-functions* $f : U \mapsto \mathbb{C}$ *such that* $D_j f$ *is Hölder continuous in* U *with Hölder exponent* α, *for* $1 \le j \le 2$.) *If* $\Phi \in L^p(\partial\mathbb{K}(\varphi))^3$ *satisfies* (2.12), *then the function* $W(\lambda,B)(\Phi)$ *belongs to* $C^1(\overline{B}\backslash\{0\})^3$, *the function* $P(\lambda,B)(\Phi)$ *may be continuously extended to* $\overline{B}\backslash\{0\}$, *and*

$$\begin{aligned} &W(\lambda,B)(\Phi)|B\cap U_{a,b} \in W^{2,s}(B\cap U_{a,b})^3, \\ &P(\lambda,B)(\Phi)|B\cap U_{a,b} \in W^{1,s}(B\cap U_{a,b}) \end{aligned} \tag{2.13}$$

for $s \in (1,\infty)$, $a,b \in (0,\infty)$ *with* $a<b$, *with the abbreviation* $U_{a,b} := \mathbb{B}_3(0,b)\backslash\overline{\mathbb{B}_3(0,a)}$.

Theorem 2.6 is mute as concerns the behaviour of $W(\lambda,B)(\Phi)$ and $P(\lambda,B)(\Phi)$ near the vertex of $\mathbb{K}(\varphi)$ or $\mathbb{R}^3\backslash\overline{\mathbb{K}(\varphi)}$. Therefore, this theorem may be reduced to the regularity theory of the Stokes resolvent problem (1.1) on smoothly bounded domains. In fact, the claim in (2.12) may be proved by cutting off the function Φ near the origin and near infinity, and then using potential theoretic arguments as in the case of bounded domains with smooth boundary; see [13, Sections 4 and 7], for example.

Once the relations in (2.12) are available, it may be shown by arguments as in [13, Section 4] that $W(\lambda,B)(\Phi)|B\cap U_{a,b}$ may be continuously extended to $\overline{B}\backslash U_{a,b}$, with this extension coinciding on $\partial B\cap U_{a,b}$ with $W(\lambda,B)(\Phi)|\partial B\cap U_{a,b}$. Next the relations in (2.13) may be established by a cut-off argument and by the regularity theory for the Stokes system on bounded, smooth domains ([19, Theorem IV.6.1]). Finally, by referring to Sobolev's lemma, we see that $W(\lambda,B)(\Phi)|\overline{B}\cap\overline{U_{a,b}}$ is a C^1-function, and $P(\lambda,B)(\Phi)|B\cap U_{a,b}$ may be continuously extended to $\overline{B}\cap\overline{U_{a,b}}$. Of course, these indications amount to hardly more than a rough sketch of a proof of Theorem 2.6. We do not want, however, to enter into details because this proof is essentially well known.

Another tool we need in the following is a result on the stress tensor of the double-layer potential related to the Stokes resolvent system (1.1). Here we mean by "stress tensor" the operator T defined by $T(u,\pi) =$

$(T_{jk}(u,\pi))_{1\le j,k\le 3} := (D_j u_k + D_k u_j - \delta_{jk}\pi)_{1\le j,k\le 3}$, for vector-valued functions u and scalar functions π which are sufficiently smooth. The result in question may be stated as follows:

Theorem 2.7. ([17], [33, page 351, 3.31]). *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^3-boundary. Let $\alpha \in (0,1)$, $\Phi \in C^{1,\alpha}(\partial\Omega)^3$. Put*

$$u(x) := \Big(\int_{\partial\Omega} \sum_{j,k=1}^{3} \widetilde{\mathcal{D}}^{\lambda}_{jkl}(x-y) n^{(\Omega)}_k(y) \Phi_j(y)\, d\Omega(y) \Big)_{1\le l\le 3},$$

$$\pi(x) := \int_{\partial\Omega} \sum_{j,k=1}^{3} \big(2D_j E_{4k}(x-y) - \lambda(4\pi)^{-1}\delta_{jk}|x-y|^{-1} \big) n^{(\Omega)}_k(y) \Phi_j(y)\, d\Omega(y)$$

for $x \in \mathbb{R}^3 \backslash \partial\Omega$, where $n^{(\Omega)}$ denotes the outward unit normal to Ω. Then the following holds for $j \in \{1,2,3\}$, $x \in \partial\Omega$:

$$\sum_{k=1}^{3} T_{jk}(u,\pi)\big(x + \kappa n^{(\Omega)}(x)\big) n^{(\Omega)}_k(x) - \sum_{k=1}^{3} T_{jk}(u,\pi)\big(x - \kappa n^{(\Omega)}(x)\big) n^{(\Omega)}_k(x) \longrightarrow 0 \quad (\kappa \downarrow 0).$$

3. Estimates of potential functions. A first Green's formula. As indicated in Section 1, a main difficulty of our theory consists in proving certain Green's formulas. Two such formulas will be considered, both of them related to the resolvent problem (1.1) on the infinite domain $\mathbb{L}(\varphi,\epsilon)$, for $\epsilon > 0$. The first of these formulas involves the single-layer potential $\widetilde{V}^{\lambda}\big(\partial\mathbb{L}(\varphi,\epsilon)\big)(\Phi)$ (see Theorem 3.2), the second one the double-layer potential $W\big(\lambda,\mathbb{K}(\varphi)\big)(\Phi)$ (see Theorem 4.4), with the function Φ satisfying certain integral equations. The double-layer potential will turn out to be more difficult to handle, due to the slow decay of $W\big(\lambda,\mathbb{K}(\varphi)\big)(\Phi)(x)$ and $P\big(\lambda,\mathbb{K}(\varphi)\big)(\Phi)(x)$ for $|x| \to \infty$. Our approach, however, will be the same in both cases: first we smooth out our potential functions by slightly modifying their kernels, then we apply the divergence theorem on the domain $\mathbb{B}_3(0,2n) \cap \mathbb{L}(\varphi,\epsilon)$, for $n \in \mathbb{N}$. Finally we let n tend to infinity. Of course, it is in this last step that difficulties arise.

In this section we present some preparatory results which will be needed in order to carry out this program. In addition, we shall prove the Green's formula related to $\widetilde{V}^{\lambda}\big(\partial\mathbb{L}(\varphi,\epsilon)\big)(\Phi)$ (Theorem 3.2).

For the rest of this section, we fix $\varphi \in (0, \pi/2]$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $\epsilon \geq 0$, $\tau \in \{-1, 1\}$. Put $V := \mathbb{L}(\varphi, \epsilon)$, $J := (0, \infty)$ if $\tau = 1$, $V := \mathbb{R}^3\backslash\overline{\mathbb{L}(\varphi, \epsilon)}$, $J := (-\infty, 0)$ if $\tau = -1$. Note that

$$\big|s + \tau r\big| \geq |s| \quad \text{for} \quad s \in J,\, r \in (0, \infty). \tag{3.1}$$

Inequalities (2.5), (2.7) and (3.1) imply there is a constant $\mathcal{C} > 0$ such that the following holds for $\xi, \eta \in \mathbb{R}^2$ with $\xi \neq \eta$, $s \in J$, $r \in (0, \infty)$, $\sigma \in [0, 2]$, $j, k, l \in \{1, 2, 3\}$, $a \in \mathbb{N}_0^3$ with $|a|_* \leq 1$:

$$\begin{aligned} &\big|D^a \widetilde{E}^{\lambda}_{jk}\big(\gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta) + (0, 0, s + \tau r)\big)\big| \\ &\qquad \leq \mathcal{C}(\max\{|\xi - \eta|, |s|\})^{-3-|a|_*+\sigma}, \end{aligned} \tag{3.2}$$

$$|\widetilde{\mathcal{D}}^{\lambda}_{jkl}(\gamma^{(\varphi,\epsilon)}(\xi) - \gamma^{(\varphi,\epsilon)}(\eta) + (0, 0, s + \tau r))| \leq \mathcal{C}(\max\{|\xi - \eta|, |s|\})^{-2}. \tag{3.3}$$

For $r \in (0, \infty)$, put $V_r := (0, 0, -\tau r) + V$. Then V_r is an open set with $\overline{V} \subset V_r$ $(r \in (0, \infty))$. Moreover, if $r \in (0, \infty)$, $\Phi \in L^q\big(\partial\mathbb{L}(\varphi, \epsilon)\big)^3$ for some $q \in (1, \infty)$, then we define the functions $A^{(r)}(\Phi) : V_r \mapsto \mathbb{C}^3$, $B^{(r)}(\Phi) : V_r \mapsto \mathbb{C}$ by setting for $x \in V_r$

$$A^{(r)}(\Phi)(x) := \Big(\int_{\partial\mathbb{L}(\varphi,\epsilon)} \sum_{l=1}^{3} \widetilde{E}^{\lambda}_{jl}\big(x - y + (0, 0, \tau r)\big)\Phi_l(y)\, d\mathbb{L}(\varphi, \epsilon)(y)\Big)_{1\leq j\leq 3},$$

$$B^{(r)}(\Phi)(x) := \int_{\partial\mathbb{L}(\varphi,\epsilon)} \sum_{l=1}^{3} E_{4l}\big(x - y + (0, 0, \tau r)\big)\Phi_l(y)\, d\mathbb{L}(\varphi, \epsilon)(y).$$

As mentioned in [10, pages 156–157, Corollary 9.1], it holds that

$$A^{(r)}(\Phi) \in C^{\infty}(V_r)^3, \quad B^{(r)}(\Phi) \in C^{\infty}(V_r), \tag{3.4}$$

$$-\Delta A^{(r)}(\Phi) + \lambda A^{(r)}(\Phi) + \nabla B^{(r)}(\Phi) = 0, \quad \operatorname{div} A^{(r)}(\Phi) = 0. \tag{3.5}$$

Later it will be necessary to control the potentials $A^{(r)}(\Phi)$ and $B^{(r)}(\Phi)$ with respect to r. We consider this point in Lemmas 3.1 and 3.6 and Corollaries 3.1 and 3.2 below.

Lemma 3.1. *Let $p, q \in (1, 2]$ with either $p = 2$ or $q = 2$. Take $\Phi \in L^p\big(\partial\mathbb{L}(\varphi, \epsilon)\big)^3$, $\Psi \in L^q\big(\partial\mathbb{L}(\varphi, \epsilon)\big)^3$ and $j, k \in \{1, 2, 3\}$. Abbreviate*

$$\mathcal{A}(r, h) := \int_V \big|A^{(r)}_j(\Phi) T_{jk}\big(A^{(r)}(\Psi), B^{(r)}(\Psi)\big)h\big|\, dx$$

for $r \in (0,\infty)$ *and* $h \in C^0(\mathbb{R}^3)$ *with supp*(h) *compact. Then there are constants* $C > 0$, $\kappa \in (0,1)$ *such that* $\mathcal{A}(r,h) \le C|h|_0 M^\kappa$ *for* $r, M \in (0,\infty)$, $h \in C^0(\mathbb{R}^3)$ *with supp*$(h) \subset \mathbb{B}_3(0,M)$.

Proof. We slightly modify the reasoning given in [10, pages 163–165]. Take r, M, h as in the lemma. Let $\mathcal{C}$ denote constants which do not depend on these quantities. Due to (2.6), (3.1), (3.2) and (3.3), we get

$$\mathcal{A}(r,h) \le \mathcal{C}|h|_0 \int_J \int_{\mathbb{B}_2(0,M)} A(\xi,s)B(\xi,s)d\xi\, ds, \tag{3.6}$$

with

$$A(\xi,s) := \int_{\mathbb{R}^2} \min\Big\{\big(|\xi-\eta|+|s|\big)^{-1}, \big(|\xi-\eta|+|s|\big)^{-3}\Big\}\big|\Phi\circ\gamma^{(\varphi,\epsilon)}(\eta)\big|d\eta,$$
$$B(\xi,s) := \int_{\mathbb{R}^2} \big(|\xi-\eta|+|s|\big)^{-2}\big|\Psi\circ\gamma^{(\varphi,\epsilon)}(\eta)\big|d\eta$$

for $\xi \in \mathbb{B}_2(0,M)$, $s \in J$. We put $\epsilon_1 := 2(1/p - 1/q')$, $\delta := p/q'$, $\gamma := 0$ in the case $q' > p$, and $\epsilon_1 := 1/8$, $\delta := 7/8$, $\gamma := 1/8$ if $q' \le p$, that is, if $p = q = 2$. Then it follows by the Hardy-Littlewood-Sobolev inequality, for $s \in J$ that

$$\begin{aligned}&\Big(\int_{\mathbb{B}_2(0,M)} |A(\xi,s)|^{q'}\, d\xi\Big)^{1/q'} \qquad (3.7)\\ &\le \mathcal{C}M^\gamma \min\{1,|s|^{-1-\epsilon_1}\}\Big(\int_{\mathbb{B}_2(0,M)}\Big(\int_{\mathbb{R}^2}|\xi-\eta|^{-2+\epsilon_1}\big|\Phi\circ\gamma^{(\varphi,\epsilon)}(\eta)\big|\,d\eta\Big)^{p/\delta}d\xi\Big)^{\delta/p}\\ &\le \mathcal{C}M^\gamma \min\big\{1,|s|^{-1-\epsilon_1}\big\}\|\Phi\|_p,\end{aligned}$$

and, with $\epsilon_2 := (1-\gamma)/2$, $t := (1/q - \epsilon_2/2)^{-1}$,

$$\begin{aligned}&\Big(\int_{\mathbb{B}_2(0,M)} |B(\xi,s)|^{q}\, d\xi\Big)^{1/q} \qquad (3.8)\\ &\le \mathcal{C}M^{2(1/q-1/t)}s^{-\epsilon_2}\Big(\int_{\mathbb{B}_2(0,M)}\Big(\int_{\mathbb{R}^2}|\xi-\eta|^{-2+\epsilon_2}\big|\Psi\circ\gamma^{(\varphi,\epsilon)}(\eta)\big|d\eta\Big)^t d\xi\Big)^{1/t}\\ &\le \mathcal{C}M^{2(1/q-1/t)}s^{-\epsilon_2}\|\Psi\|_q.\end{aligned}$$

Thus, starting from (3.6), applying (3.7) and (3.8), and integrating with respect to s, we obtain the inequality stated in the lemma, with $\kappa := \gamma + 2(1/q - 1/t) = \gamma + \epsilon_2 = 1/2 + \gamma/2 < 1$. □

Lemma 3.2. *Let* $p \in (1,\infty)$. *Then there is a constant* $C > 0$ *with*

$$\mathcal{A}(M,\Phi) := \Big(\int_{\mathbb{B}_3(0,M)} \Big(\int_{\partial\mathbb{L}(\varphi,\epsilon)} |x-y|^{-2} |\Phi(y)| \, d\mathbb{L}(\varphi,\epsilon)(y) \Big)^p \, dx \Big)^{1/p}$$
$$\leq C M^{1/p} \|\Phi\|_p \qquad \text{for} \;\; \Phi \in L^p\big(\partial\mathbb{L}(\varphi,\epsilon)\big)^3,\; M \in (\epsilon,\infty).$$

Proof. By Hölder's inequality, $\mathcal{A}(M,\Phi) \leq \mathcal{A}_1(M,\Phi) + \mathcal{A}_2(M,\Phi)$, where the expressions $\mathcal{A}_1(M,\Phi)$ and $\mathcal{A}_2(M,\Phi)$ are defined by

$$\mathcal{A}_j(M,\Phi) := \Big(\int_{\mathbb{B}_3(0,M)} \Big(\int_{U_j} \big|x - \gamma^{(\varphi,\epsilon)}(\eta)\big|^{-2+\alpha_j p'} \cdot J^{(\varphi,\epsilon)}(\eta) \, d\eta \Big)^{p-1} \tag{3.9}$$
$$\times \Big(\int_{U_j} |x - \gamma^{(\varphi,\epsilon)}(\eta)\big|^{-2-\alpha_j p} \big|\Phi\big(\gamma^{(\varphi,\epsilon)}(\eta)\big)\big|^p J^{(\varphi,\epsilon)}(\eta) \, d\eta \Big) \, dx \Big)^{1/p},$$

for $j \in \{1,2\}$, with $\alpha_1 := 1/(2p)$, $\alpha_2 := -2/p$, $U_1 := \mathbb{B}_2(0,2M)$, $U_2 := \mathbb{R}^2 \backslash \mathbb{B}_2(0,2M)$. The first integral in η on the right-hand side of (3.9) may be estimated against $\mathcal{C}_1 M^{\alpha_j p'}$, for $j \in \{1,2\}$, with $\mathcal{C}_1 > 0$ independent of M, j and x. Since $M \geq \epsilon$, it holds that

$$\big|\gamma^{(\varphi,\epsilon)}(\eta)\big| \leq |\eta| + \cot\varphi \max\{\epsilon, |\eta|\} \leq \mathcal{C}_2 M \quad \text{for} \;\; \eta \in \mathbb{B}_2(0,2M),$$

where $\mathcal{C}_2 > 0$ does not depend on η or M. It follows that

$$\mathbb{B}_3(0,M) \subset \mathbb{B}_3\big(\gamma^{(\varphi,\epsilon)}(\eta), (\mathcal{C}_2+1)M\big) \qquad \text{for} \;\; \eta \in \mathbb{B}_2(0,2M). \tag{3.10}$$

Thus, having estimated the first integral in η on the right-hand side of (3.9), we now exchange the order of integration of the remaining integrals, make use of (3.10) in the case $j = 1$, and integrate with respect to x. Then the lemma follows by some easy computations. □

Lemma 3.3. *Let* $p \in (2,\infty)$, $\Phi \in L^p\big(\partial\mathbb{K}(\varphi)\big)^3$. *Then there is a constant* $\mathcal{C} > 0$ *such that*

$$\|S^{(\varphi)}(\Phi)|\mathbb{B}_3(0,M)\|_p \;\leq\; \mathcal{C} M^{1+1/p} \quad \text{for} \;\; M \in [1,\infty),$$

where $S^{(\varphi)}(\Phi)$ *was introduced in* (2.11).

Proof. Let $M \in [1,\infty)$. Split the domain of integration appearing in (2.11) into the set $\partial\mathbb{K}(\varphi) \cap \mathbb{B}_3(0, 4M/\sin\varphi)$ and $\partial\mathbb{K}(\varphi) \backslash \mathbb{B}_3(0, 4M/\sin\varphi)$. Then the absolute value of both integrals arising in this way is bounded by $\mathcal{C} M^{-1+2/p'}$ for $x \in \mathbb{B}_3(0,M)$, with $\mathcal{C}$ independent of M and x. This result, which is a rather straightforward consequence of (2.5), implies the lemma. □

Let us draw some consequences of Lemma 3.2.

Corollary 3.1. *Let* $p \in (1,\infty),\ \Phi \in L^p\big(\partial\mathbb{L}(\varphi,\epsilon)\big)^3,\ j,k \in \{1,2,3\},\ a \in \mathbb{N}_0^3$ *with* $|a|_* \le 1,\ h \in C^0(\mathbb{R}^3)$ *with supp*(h) *compact. Then*

$$\int_V h(x)\Big(D^a A_j^{(r)}(\Phi)(x) - \partial^a/\partial x^a \widetilde{V}^\lambda\big(\partial\mathbb{L}(\varphi,\epsilon)\big)(\Phi)(x)\Big)dx \to 0\ (r\downarrow 0),$$

$$\int_V h(x)\Big(B^{(r)}(\Phi)(x) - Q\big(\partial\mathbb{L}(\varphi,\epsilon)\big)(\Phi)(x)\Big)dx \to 0\ (r\downarrow 0).$$

Proof. The following holds for $f \in L^p(\mathbb{R}^3) : \big\|f\big(\cdot + (0,0,r)\big) - f\big\|_p \to 0$ $(r \to 0)$. Thus Corollary 3.1 follows from (2.7) and Lemma 3.2. $\square$

Corollary 3.2. *Let* $p \in (1,\infty),\ \Phi \in L^p\big(\partial\mathbb{L}(\varphi,\epsilon)\big)^3,\ f \in L^{p'}(\mathbb{R}^3)^3$ *or* $f \in L^s(\mathbb{R}^3)^3$ *for all* $s \in (3/2,\infty)$*. Then there are constants* $C > 0,\ \delta \in (0,1)$ *such that the following holds for* $r \in (0,\infty),\ M \in [\epsilon,\infty),\ h \in C^0(\mathbb{R}^3)$ *with supp*$(h) \subset \mathbb{B}_3(0,M),\ j,k \in \{1,2,3\},\ a \in \mathbb{N}_0^3$ *with* $|a|_* \le 1:$

$$\int_V \big(\big|fD^aA^{(r)}h\big| + |fB^{(r)}h|\big)\ dx\ \le\ C|h|_0 M^\delta.$$

Proof. The relations in (2.7), (2.5), (3.1), Lemma 3.2 together with Hölder's inequality imply that the left-hand side of the estimate appearing in this corollary is bounded by $\mathcal{C}|h|_0\|f|\mathbb{B}_3(0,M)\|_{p'}\cdot M^{1/p}$, with $\mathcal{C} > 0$ independent of h, M and f. This estimate proves the corollary if $f \in L^{p'}(\mathbb{R}^3)^3$. In the case $p' < 3/2,\ f \in L^s(\mathbb{R}^3)^3$ for $s \in (3/2,\infty)$, we note that $M^{1/p}\|f|\mathbb{B}_3(0,M)\|_{p'} \le \mathcal{C}M^{1-1/(2p)}\|f\|_{(2/3-1/(2p))^{-1}}$. $\square$

We further state a jump relation:

Theorem 3.1. *Let* $p \in (1,\infty),\ \Phi \in L^p(\partial\mathbb{L}(\varphi,\epsilon))^3,\ j \in \{1,2,3\}$*. Then*

$$\int_{\partial\mathbb{L}(\varphi,\epsilon)}\Big|\Gamma_j^*\big(-\tau,p,\lambda,\mathbb{L}(\varphi,\epsilon)\big)(\Phi)(x) + \sum_{k=1}^{3} T_{jk}\big(A^{(r)}(\Phi),B^{(r)}(\Phi)\big)(x)n_k^{(\varphi,\epsilon)}(x)\Big|^p d\mathbb{L}(\varphi,\epsilon)(x) \quad\longrightarrow\quad 0 \quad (r\downarrow 0).$$

(*Recall that the quantities* $\varphi,\lambda,\tau,\epsilon$ *were fixed at the beginning of this section.*)

For a proof of this theorem we refer to [10, pages 150–153].

We further mention a result which may be reduced, via (3.1), to the Hardy-Littlewood-Sobolev inequality. Details of such an argument may be found in [10, page 154, 158/159].

Lemma 3.4. *Let $j \in \{1,2,3\}$, $q \in (1,\infty)$, $t \in (1,\ (1/q-1/2)^{-1}]$ in the case $q<2$, $t \in (1,\infty)$ if $q \geq 2$. Then there is some $\mathcal{C}>0$ with*

$$\Big(\int_{\partial\mathbb{L}(\varphi,\epsilon)} \big|\widetilde{V}^{\lambda}\big(\partial\mathbb{L}(\varphi,\epsilon)\big)(\Phi)\big(x+(0,0,\tau r)\big)\big|^{t} d\mathbb{L}(\varphi,\epsilon)(x)\Big)^{1/t} \leq \mathcal{C}\|\Phi\|_q$$

for $\Phi \in L^q\big(\partial\mathbb{L}(\varphi,\epsilon)\big)^3$, $r \in [0,\infty)$. Moreover, the following holds for Φ as before, and for $g \in C^0\big(\partial\mathbb{L}(\varphi,\epsilon)\big)^3$ with supp(g) bounded:

$$\int_{\partial\mathbb{L}(\varphi,\epsilon)} \big|\widetilde{V}^{\lambda}\big(\partial\mathbb{L}(\varphi,\epsilon)\big)(\Phi)(x) - A^{(r)}(\Phi)(x)\big|^{t}|g(x)|d\mathbb{L}(\varphi,\epsilon)(x) \longrightarrow 0\ (r \downarrow 0).$$

The next lemma will be the starting point for proving the Green's formulas mentioned at the beginning of this section.

Lemma 3.5. *Let $v \in C^2(\overline{V})^3$, $\varrho \in C^1(\overline{V})$, $u \in C^1(\overline{V})^3$, $\pi \in C^0(\overline{V})$, $p \in (1,\infty)$ with $u|V\cap\mathbb{B}_3(0,b) \in W^{2,p}\big(V\cap\mathbb{B}_3(0,b)\big)^3$, $\pi|V\cap\mathbb{B}_3(0,b) \in W^{1,p}(V\cap\mathbb{B}_3(0,b))$ for $b \in (0,\infty)$. Further, assume div$\,v =$ div$\,u = 0$. Put $f := -\Delta u + \lambda u + \nabla\pi$, $g := -\Delta v + \lambda v + \nabla\varrho$. Let $\widetilde{\varphi} \in C^1(\mathbb{R}^3)$ with supp$(\widetilde{\varphi})$ compact. Then*

$$\int_V (fv-ug)\widetilde{\varphi}\,dx = \int_V \sum_{j,k=1}^{3}\Big(-u_jT_{jk}(v,\varrho)+T_{jk}(u,\pi)v_j\Big)D_k\widetilde{\varphi}\,dx$$
$$+\,\tau\int_{\partial\mathbb{L}(\varphi,\epsilon)}\sum_{j,k=1}^{3}\Big(u_jT_{jk}(v,\varrho)-T_{jk}(u,\pi)v_j\Big)n_k^{(\varphi,\epsilon)}\widetilde{\varphi}\,d\mathbb{L}(\varphi,\epsilon).$$

This lemma readily follows from the divergence theorem. Note that the outward unit normal to V is given by $\tau n^{(\varphi,\epsilon)}$. The assumptions on u,π,v,ϱ required in this lemma might, of course, be relaxed. However, we have chosen such a level of generality as will be needed in the following.

Now we are in a position to establish our first Green's formula. We recall that the quantities ϵ,τ,λ,V were fixed at the beginning of this section.

Theorem 3.2. (first Green's formula) *Let $q \in (1,2]$. Assume that $\Phi \in L^2\big(\partial\mathbb{L}(\varphi,\epsilon)\big)^3$ with $\Gamma^*\big(-\tau,2,\lambda,\mathbb{L}(\varphi,\epsilon)\big)(\Phi)\Big|\partial\mathbb{L}(\varphi,\epsilon)\backslash\mathcal{M} = 0$, where $\mathcal{M}$ is some bounded, measurable subset of $\partial\mathbb{L}(\varphi,\epsilon)$. Let $h \in C_0^\infty(\mathbb{R}^3)^3$, $\psi \in L^q(\partial\mathbb{L}(\varphi,\epsilon))^3$ with $\Gamma^*\big(-\tau,q,\lambda,\mathbb{L}(\varphi,\epsilon)\big)(\psi) = T\big(\widetilde{R}^{\lambda}(h),S(h)\big)n^{(\varphi,\epsilon)}$. Then it holds that*

$$\int_V h\widetilde{V}^{\lambda}\big(\partial\mathbb{L}(\varphi,\epsilon)\big)(\Phi)\,dx \tag{3.11}$$
$$=(-\tau)\int_{\partial\mathbb{L}(\varphi,\epsilon)\cap\mathcal{M}}(\widetilde{R}^{\lambda}(h)+\widetilde{V}^{\lambda}(\partial\mathbb{L}(\varphi,\epsilon))(\psi))\Gamma^*(-\tau,2,\lambda,\mathbb{L}(\varphi,\epsilon))(\Phi)\,d\mathbb{L}(\varphi,\epsilon).$$

Proof. Choose a sequence $(\widetilde{\varphi}_n)$ in $C_0^\infty(\mathbb{R}^3)$ with

$$supp(\widetilde{\varphi_n}) \subset \mathbb{B}_3(0,2n),\ 0 \leq \widetilde{\varphi}_n \leq 1, \quad \widetilde{\varphi}_n|\mathbb{B}_3(0,n) = 1 \ \text{ for } n \in \mathbb{N}, \\ |\nabla\widetilde{\varphi_n}| \leq \mathcal{C}n^{-1} \text{ for } n \in \mathbb{N}, \quad (3.12)$$

with a constant $\mathcal{C} > 0$ independent of n. Let $n_0 \in \mathbb{N}$ be so large that $supp(h) \cup \mathcal{M} \subset \mathbb{B}_3(0,n_0)$. By Corollary 3.1, we have for $n \geq n_0$

$$\int_V h\widetilde{V}^\lambda\big(\partial\mathbb{L}(\varphi,\epsilon)\big)(\Phi)\,dx = \lim_{r\downarrow 0}\int_V hA^{(r)}(\Phi)\varphi_n\,dx. \tag{3.13}$$

Applying Lemma 3.5, Theorem 2.4, (3.4) and (3.5), we get for $r \in (0,\infty)$, $n \in \mathbb{N}$ with $n \geq n_0$

$$\int_V hA^{(r)}(\Phi)\widetilde{\varphi}_n\,dx = \sum_{\upsilon=1}^{4} M_\upsilon^{(r,n)} + \sum_{\upsilon=1}^{4} N_\upsilon^{(r,n)}, \tag{3.14}$$

where the expressions $M_1^{(r,n)},\dots,M_4^{(r,n)}$, $N_1^{(r,n)},\dots,N_4^{(r,n)}$ are given by

$$M_\upsilon^{(r,n)} := \int_{V\cap\mathbb{B}_3(0,2n)} \sum_{j,k=1}^{3} a_jT_{jk}(b,c)D_k\widetilde{\varphi}_n\,dx,$$

$$N_\upsilon^{(r,n)} := \int_{\partial\mathbb{L}(\varphi,\epsilon)\cap\mathbb{B}_3(0,2n)} (-\tau)\sum_{j,k=1}^{3} a_jT_{jk}(b,c)n_k^{(\varphi,\epsilon)}\widetilde{\varphi}_n\,d\mathbb{L}(\varphi,\epsilon),$$

with

$$\begin{aligned} a = \widetilde{R}^\lambda(h), \quad b = -A^{(r)}(\Phi), \quad c = -B^{(r)}(\Phi) \quad &\text{if } \upsilon = 1,\\ a = A^{(r)}(\psi), \quad b = -A^{(r)}(\Phi), \quad c = -B^{(r)}(\Phi) \quad &\text{if } \upsilon = 2,\\ a = A^{(r)}(\Phi), \quad b = \widetilde{R}^\lambda(h), \quad c = S(h) \quad &\text{if } \upsilon = 3,\\ a = A^{(r)}(\Phi), \quad b = A^{(r)}(\psi), \quad c = B^{(r)}(\psi) \quad &\text{if } \upsilon = 4. \end{aligned}$$

Let $n \in \mathbb{N}$ with $n \geq n_0$ be fixed. Referring to (3.13), Theorem 3.1 and Lemma 3.4, we get by letting r tend to zero in (3.14)

$$\int_V h\widetilde{V}^\lambda\big(\partial\mathbb{L}(\varphi,\epsilon)\big)(\Phi)\,dx = \lim_{r\downarrow 0}\sum_{\upsilon=1}^{4} M_\upsilon^{(r,n)} + \int_{\partial\mathbb{L}(\varphi,\epsilon)}(-\tau)\sum_{\upsilon=1}^{2} P_\upsilon\widetilde{\varphi}_n\,d\mathbb{L}(\varphi,\epsilon), \tag{3.15}$$

with

$$P_1 := \big(\widetilde{R}^\lambda(h) + \widetilde{V}^\lambda\big(\partial\mathbb{L}(\varphi,\epsilon)\big)(\psi)\big)\Gamma^*\big(-\tau, 2, \lambda, \mathbb{L}(\varphi,\epsilon)\big)(\Phi),$$
$$P_2 := \big(T\big(\widetilde{R}^\lambda(h), S(h)\big)n^{(\varphi,\epsilon)} - \Gamma^*\big(-\tau, q, \lambda, \mathbb{L}(\varphi,\epsilon)\big)(\psi)\big)\widetilde{V}^\lambda\big(\partial\mathbb{L}(\varphi,\epsilon)\big)(\Phi).$$

Due to the choice of ψ, it holds that $P_2 = 0$. Moreover, recalling that $n \geq n_0$ and $\mathcal{M} \subset \mathbb{B}_3(0, n_0)$, we get $P_1|\mathbb{L}(\varphi,\epsilon)\backslash\mathcal{M} = 0$, $\widetilde{\varphi}_n|\partial\mathbb{L}(\varphi,\epsilon) \cap \mathcal{M} = 1$. Thus, the second summand on the right-hand side of (3.15) equals the right-hand side in (3.11). Referring to (3.12), Theorem 2.4, Lemma 3.1 and Corollary 3.2, we further obtain

$$\sup\{|M_v^{(r,n)}| : r \in (0,\infty)\} \longrightarrow 0 \ (n \to \infty) \ \text{ for } \ v \in \{1,2,3,4\}.$$

Therefore equation (3.11) follows by letting n tend to infinity in (3.15). □

In the rest of this section, we will establish some further estimates of potential functions.

Lemma 3.6. *Let $p \in (1,\infty)$, $\Phi \in L^p\big(\partial\mathbb{L}(\varphi,\epsilon)\big)^3$. Then there is a constant $C \in (0,\infty)$ such that*

$$\int_V \big|\partial^a/\partial x^a \widetilde{V}_j^\lambda\big(\partial\mathbb{L}(\varphi,\epsilon)\big)(\Phi)\big(x + (0,0,\tau r)\big)\big|^p \, dx \leq C \qquad (3.16)$$

for $r \in [0,\infty)$, $1 \leq j \leq 3$, $a \in \mathbb{N}_0^3$ with $|a|_* \leq 1$.

Proof. Take r, j, a as in the lemma. We shall prove inequality (3.16) if $|a|_* = 1$. The case $a = 0$ may be handled in a similar way. Using (2.6) and (3.2), we see that the left-hand side in (3.16) is bounded by

$$\mathcal{C}\int_0^{|\lambda|^{-1/2}} \mathcal{W}(s)\, ds + \mathcal{C}\int_{|\lambda|^{-1/2}}^\infty \mathcal{W}(s)\, ds,$$

where the letter $\mathcal{C}$ denotes constants independent of r, and where $\mathcal{W}(s)$ is defined by

$$\mathcal{W}(s) := \int_{\mathbb{R}^2}\Big(\int_{\mathbb{R}^2}\Big[\chi_{(0,|\lambda|^{-1/2})}(|\xi-\eta|)(|\xi-\eta|+s)^{-2} + \chi_{(|\lambda|^{-1/2},\infty)}(|\xi-\eta|)(|\xi-\eta|+s)^{-4}\Big]\Big|\Phi\big(\gamma^{(\varphi,\epsilon)}(\eta)\big)\Big|\, d\eta\Big)^p d\xi,$$

for $s \in (0,\infty)$. In order to estimate the integral of $\mathcal{W}(s)$ over $(0, |\lambda|^{-1/2})$, we use the inequality

$$(|\xi-\eta|+s)^{-\gamma} \geq |\xi-\eta|^{-\gamma+1/(2p)} s^{-1/(2p)}$$

for $\xi, \eta \in \mathbb{R}^2$, $\xi \neq \eta,, s \in (0, \infty), \gamma \in \{2, 4\}$. Concerning the integral of $\mathcal{W}(s)$ over $(|\lambda|^{-1/2}, \infty)$, we refer to the estimate

$$(|\xi - \eta| + s)^{-\gamma} \geq |\xi - \eta|^{-\gamma + 3/(2p)} s^{-3/(2p)}, \quad \text{for } \xi, \eta, s, \gamma \text{ as before.}$$

Due to these relations, we may apply a trivial version of Young's inequality ([10, Lemma 4.9]), which yields (3.16). □

Lemma 3.7. *Assume $\epsilon = 0$; that is, $\mathbb{L}(\varphi, \epsilon) = \mathbb{K}(\varphi)$. Let $r, p \in (1, \infty)$ with $r > 3p/2$, $\Phi \in L^p\big(\partial\mathbb{K}(\varphi)\big)^3$ with $\Phi \circ g^{(\varphi)}|\mathbb{R}^2\backslash\mathbb{B}_2(0, \delta) \in L^t\big(\mathbb{R}^2\backslash\mathbb{B}_2(0, \delta)\big)^3$ for $\delta > 0$, $t \in [p, \infty)$. Put*

$$\mathcal{W}(x) := \int_{\partial\mathbb{K}(\varphi)} |x - y|^{-2} |\Phi(y)| \, d\mathbb{K}(\varphi)(y) \quad \textit{for } x \in V.$$

Then $\mathcal{W}|V\backslash\mathbb{B}_3(0, 1)$ is an L^r-function.

Proof. In the following, the letter $\mathcal{C}$ denotes positive real numbers. If $\xi \in \mathbb{R}^2$, $s \in \mathbb{R}$ with $1 < \big|g^{(\varphi)}(\xi) + (0, 0, s)\big|$, it follows that $\sin\varphi \leq |\xi| + |s|$; hence, by (2.6) and (2.5) $\big\|\mathcal{W}|V\backslash\mathbb{B}_3(0, 1)\big\|_r^r \leq \mathcal{C}\mathcal{A}$, with

$$\mathcal{A} := \int_{\mathbb{R}^2} \int_0^\infty \chi_{(\sin\varphi, \infty)}(|\xi| + s) \Big(\int_{\mathbb{R}^2} (|\xi - \eta| + s)^{-2} \big|\Phi\big(g^{(\varphi)}(\eta)\big)\big| \, d\eta \Big)^r ds \, d\xi. \tag{3.17}$$

Let $\mathcal{A}_1, \mathcal{A}_2$ be defined by the right-hand side of (3.17), with the factor $\chi_{(0, (\sin\varphi)/2)}(|\eta|)$ and $\chi_{((\sin\varphi)/2, \infty)}(|\eta|)$, respectively, inserted into the innermost integral. It follows that $\mathcal{A} \leq 2^{r-1}(\mathcal{A}_1 + \mathcal{A}_2)$. Thus the lemma is proved if we can show that $\mathcal{A}_j < \infty$ for $j = 1, 2$. But $|\xi - \eta| + s \geq (|\xi| + s)/2$ for $\xi, \eta \in \mathbb{R}^2$, $s > 0$ with $|\xi| + s \geq \sin\varphi$, $|\eta| < (\sin\varphi)/2$. By this observation, and because $r > 3p/2 > 3/2$, we may conclude that $\mathcal{A}_1 < \infty$. Turning to the estimate of $\mathcal{A}_2$, we set $\delta := 2r/p - 3$. Due to the assumption $r > 3p/2$, it holds that $\delta > 0$. We observe that $\mathcal{A}_2 \leq \mathcal{C}\big(\big\|B|(0, 1)\big\|_1 + \big\|B|(1, \infty)\big\|_1\big)$, with

$$B(s) := \int_{\mathbb{R}^2} \Big(\int_{\mathbb{R}^2} (|\xi - \eta| + s)^{-2} \chi_{((\sin\varphi)/2, \infty)}(|\eta|) \big|\Phi\big(g^{(\varphi)}(\eta)\big)\big| \, d\eta \Big)^r d\xi$$

for $s > 0$. Both terms $\big\|B|(0, 1)\big\|_1$ and $\big\|B|(1, \infty)\big\|_1$ are finite. In fact, as concerns $\|B|(0, 1)\|_1$, this may be seen by applying the Hardy-Littlewood-Sobolev inequality, our assumptions on Φ, as well as the estimate

$$|\xi - \eta| + s \geq |\xi - \eta|^{1 - 1/(4r)} s^{1/(4r)} \quad \big(\xi, \eta \in \mathbb{R}^2, \ s > 0\big).$$

The term $\|B|(1,\infty)\|_1$ may be estimated in almost the same way, the only difference being that the preceding inequality is replaced by

$$|\xi-\eta|+s \geq |\xi-\eta|^{1-(1+\delta)/(2r)} s^{(1+\delta)/(2r)} \quad (\xi,\eta\in\mathbb{R}^2,\ s>0).$$

The ensuing result was proved in [10].

Theorem 3.3. ([10, Lemmas 6.2, 6.5]). *Let $p\in(1,\infty)$, $\Phi\in L^p\big(\partial\mathbb{K}(\varphi)\big)^3$. Then there is a constant $C\in(0,\infty)$ such that it holds for $r\in(0,\infty)$ that*

$$\int_{\mathbb{R}^2}(|W(\lambda,\mathbb{K}(\varphi))(\Phi)(g^{(\varphi)}(\xi)+(0,0,r))|^p+|Z^{(\varphi)}(\Phi)(g^{(\varphi)}(\xi)+(0,0,r))|^p)\,d\xi \\ \leq C,$$

where $Z^{(\varphi)}(\Phi)$ was introduced in (2.10) and λ was fixed at the beginning of this section.

We shall make use of this theorem in the proof of

Theorem 3.4. *Let $p\in(2,\infty)$, $q\in(1,2]$ with $p'\leq q<3p'/2$. Recall the quantities $\lambda,\varphi,\epsilon,\tau,V$ fixed at the beginning of this section. Suppose that $\tau=1$, $V=\mathbb{K}(\varphi)$. Let $\Phi\in L^p(\partial\mathbb{K}(\varphi))^3$ with $\Gamma(\sigma,p,\lambda,\mathbb{K}(\varphi))(\Phi)=0$ for some $\sigma\in\{-1,1\}$, $\Psi\in L^q(\partial\mathbb{L}(\varphi,\epsilon))^3$, $h\in C_0^\infty(\mathbb{R}^3)^3$, $j,k\in\{1,2,3\}$, $(\widetilde{\varphi}_n)$ a sequence in $C_0^\infty(\mathbb{R}^3)$ with*

$$\widetilde{\varphi}_n|\mathbb{B}_3(0,n)=1, \quad supp(\widetilde{\varphi}_n)\subset\mathbb{B}_3(0,2n) \ \ for \ \ n\in\mathbb{N}. \tag{3.18}$$

Assume there is a constant $\mathcal{C}_1>0$ with

$$|\nabla\widetilde{\varphi}_n|_0\leq\mathcal{C}_1 n^{-1},\ |D^2\widetilde{\varphi}_n|_0\leq\mathcal{C}_1 n^{-2} \ \ for \ \ n\in\mathbb{N}. \tag{3.19}$$

Then

$$\int_{\mathbb{L}(\varphi,\epsilon)}\big(\widetilde{R}_j^\lambda(h)+\widetilde{V}^\lambda\big(\partial\mathbb{L}(\varphi,\epsilon)\big)(\Psi)\big)D_k\widetilde{\varphi}_n \\ \times T_{jk}\big(W\big(\lambda,\mathbb{K}(\varphi)\big)(\Phi),\ P\big(\lambda,\mathbb{K}(\varphi)\big)(\Phi)-\lambda S^{(\varphi)}(\Phi)\big)\,dx \longrightarrow 0 \quad (n\to\infty).$$

In order to establish this theorem, we shall not refer to the second inequality in (3.19). Instead we shall only use that $|D^2\widetilde{\varphi_n}|_0\leq\mathcal{C}_1 n^{-1}$ ($n\in\mathbb{N}$). Later on, however, we will need sequences $(\widetilde{\varphi}_n)$ which satisfy (3.19) in full; so we introduced this inequality already here.

Proof. Abbreviate $H:=(\widetilde{R}_j^\lambda(h)+\widetilde{V}^\lambda(\partial\mathbb{L}(\varphi,\epsilon))(\Psi))|\mathbb{L}(\varphi,\epsilon)$, $W:=W(\lambda,\mathbb{K}(\varphi))(\Phi)$, $P:=P(\lambda,\mathbb{K}(\varphi))(\Phi)$, $S:=S^{(\varphi)}(\Phi)$, $Z:=Z^{(\varphi)}(\Phi)$, with $Z^{(\varphi)}(\Phi)$,

$S^{(\varphi)}(\Phi)$ introduced in (2.10) and (2.11), respectively. Due to our assumptions on Φ, Theorem 2.6 implies that $W \in C^1(\overline{\mathbb{K}(\varphi)}\backslash\{0\})^3$, and that there is $\varrho \in C^0(\overline{\mathbb{K}(\varphi)}\backslash\{0\})$ with $\varrho|\mathbb{K}(\varphi) = P$. On the other hand, $\underline{S^{(\varphi)}}(\Phi) \in C^0(\mathbb{R}^3)$ according to Lemma 2.2, so by (2.9) there is $\gamma \in C^0(\overline{\mathbb{K}(\varphi)}\backslash\{0\})$ with $\gamma|\mathbb{K}(\varphi) = \text{div}Z$. By Theorem 2.4 and Lemma 3.6 we know that $H \in W^{1,q}(\mathbb{L}(\varphi,\epsilon))^3$.

For $r \in (0,\infty)$, put $L_r := (0,0,r) + \mathbb{L}(\varphi,\epsilon)$. Note that $\overline{L_r} \subset \mathbb{L}(\varphi,\epsilon) \subset \mathbb{K}(\varphi)$ for $r > 0$, $\chi_{L_r}(x) \to 1 (r \downarrow 0)$ for $x \in \mathbb{L}(\varphi,\epsilon)$. Thus, recalling (2.9) and the fact that $supp(D_k\widetilde{\varphi}_n)$ is compact, we may conclude that

$$\int_{\mathbb{L}(\varphi,\epsilon)} HD_k\widetilde{\varphi}_n T_{jk}(W, P-\lambda S)\,dx = \int_{\mathbb{L}(\varphi,\epsilon)} HD_k\widetilde{\varphi}_n T_{jk}(W,\text{div}Z)\,dx$$
$$= \lim_{r\downarrow 0}\int_{L_r} HD_k\widetilde{\varphi}_n T_{jk}(W,\text{div}Z)\,dx = \lim_{r\downarrow 0}\big(C^{(1)}_{n,r} + C^{(2)}_{n,r} + C^{(3)}_{n,r}\big) \tag{3.20}$$

for $n \in \mathbb{N}$, where

$$C^{(1)}_{n,r} := \int_{\partial L_r} HD_k\widetilde{\varphi}_n(W_j m^{(r)}_k + W_k m^{(r)}_j - \delta_{jk}(m^{(r)}Z))\,dL_r,$$
$$C^{(2)}_{n,r} := -\int_{L_r}((D_jHD_k\widetilde{\varphi}_n + HD_jD_k\widetilde{\varphi}_n)W_k + (D_kHD_k\widetilde{\varphi}_n + HD^2_k\widetilde{\varphi}_n)W_j)\,dx,$$
$$C^{(3)}_{n,r} := \int_{L_r}\delta_{jk}\sum_{l=1}^{3}(D_lHD_k\widetilde{\varphi}_n + HD_lD_k\widetilde{\varphi}_n)Z_l\,dx,$$

with $m^{(r)}$ denoting the outward unit normal to $L_r(n \in \mathbb{N},\ r \in (0,\infty))$. Using (3.18) and (3.19), and recalling (1.7), we get for $n \in \mathbb{N}$, $n \geq \epsilon/\sin\varphi$, $r \in (0,\infty)$

$$|C^{(2)}_{n,r}| + |C^{(3)}_{n,r}| \leq \mathcal{C}n^{-1}\|H\|_{1,q}\sum_{l=1}^{3}\big(\|W_l|\mathcal{U}_n\|_{q'} + \|Z_l|\mathcal{U}_n\|_{q'}\big)$$
$$\leq \mathcal{C}n^{-1+3/q'-3/p}\|H\|_{1,q}\sum_{l=1}^{3}\big(\|W_l|\mathcal{U}_n\|_p + \|Z_l|\mathcal{U}_n\|_p\big),$$

with $\mathcal{U}_n := \mathbb{K}(\varphi)\cap\big(\mathbb{B}_3(0,2n)\backslash\mathbb{B}_3(0,n)\big)$. Here and in the following, the letter $\mathcal{C}$ denotes constants which do not depend on n or r. Recalling that $\|H\|_{1,q} < \infty$, we get by Lemma 3.2 and inequality (2.7)

$$|C^{(2)}_{n,r}| + |C^{(3)}_{n,r}| \leq \mathcal{C}n^{-1+3/q'-2/p} \quad \text{for} \quad n \in \mathbb{N}, n \geq \epsilon/\sin\varphi, r \in (0,\infty). \tag{3.21}$$

In order to estimate the term $C^{(1)}_{n,r}$, we note that the mapping $T_r : \mathbb{R}^2 \ni \eta \mapsto \gamma^{(\varphi,\epsilon)}(\eta) + (0,0,r) \in \partial L_r$ is a parametric representation of ∂L_r, with $T_r(\mathbb{R}^2) = \partial L_r$ $(r \in (0,\infty))$. The area element related to this representation is the function $J^{(\varphi,\epsilon)}$ mentioned in Section 2. Since

$$\mathrm{supp}(D_k\widetilde{\varphi_n}) \subset \mathbb{B}_3(0,2n)\backslash\mathbb{B}_3(0,n) \quad \text{for} \quad n \in \mathbb{N},$$

we get for $n \in \mathbb{N}$, $r \in (0,\infty)$

$$|C^{(1)}_{n,r}| \le |J^{(\varphi,\epsilon)}|_0 \int_{\{\xi\in\mathbb{R}^2 : n \le |T_r(\xi)| \le 2n\}} (|H||D_k\widetilde{\varphi}_n|(2|W| + |Q|)) \circ T_r \, d\xi. \tag{3.22}$$

Referring to the definition of $\gamma^{(\varphi,\epsilon)}$ in Section 2, we see that for $n \in \mathbb{N}$ with $n \ge (1+\cot\varphi)\epsilon + 1$, $r \in (0,1]$, $\xi \in \mathbb{R}^2$ with $|T_r(\xi)| > n$, it holds that $|\xi| \ge \epsilon$, so that $T_r(\xi) = \widetilde{T}_r(\xi)$, with the mapping $\widetilde{T}_r$ being defined by $\widetilde{T}_r : \mathbb{R}^2 \ni \eta \mapsto (\eta, |\eta|\cot\varphi + r) \in \partial\mathbb{K}(\varphi)$.

For n, r as before, and for $\xi \in \mathbb{R}^2$ with $|T_r(\xi)| \in [n, 2n]$, we may then conclude: $(n-1)\sin\varphi \le |\xi| \le 2n$. It follows with (3.22) and (3.19), for $n \in \mathbb{N}$ with $n \ge (1+\cot\varphi)\epsilon + 1, r \in (0,1]$

$$\begin{aligned} |C^{(1)}_{n,r}| &\le \mathcal{C}n^{-1}\|H \circ T_r|\mathcal{V}_n\|_q\big(\|W \circ \widetilde{T}_r|\mathcal{V}_n\|_{q'} + \|Z \circ \widetilde{T}_r|\mathcal{V}_n\|_{q'}\big) \qquad (3.23) \\ &\le \mathcal{C}n^{-1+2/q'-2/p}\|H \circ T_r|\mathcal{V}_n\|_q\big(\|W \circ \widetilde{T}_r|\mathcal{V}_n\|_p + \|Z \circ \widetilde{T}_r|\mathcal{V}_n\|_p\big), \end{aligned}$$

where we used the abbreviation $\mathcal{V}_n := \mathbb{B}_2(0,2n)\backslash\mathbb{B}_2\big(0,(n-1)\sin\varphi\big)$. But we know by Theorem 3.3 that

$$\|W \circ \widetilde{T}_r\|_p + \|Z \circ \widetilde{T}_r\|_p \ \le\ \mathcal{C} \quad \text{for} \quad r \in (0,\infty).$$

Moreover, according to Lemma 3.4, it holds that

$$\|\widetilde{V}^{\lambda}_j\big(\partial\mathbb{L}(\varphi,\epsilon)\big)(\Psi) \circ T_r\|_q \le \mathcal{C} \quad \text{for} \quad r \in (0,\infty).$$

Taking $n_0 \in \mathbb{N}$ so large that *supp*$(h) \subset \mathbb{B}_3\big(0,(n_0-1)(\sin\varphi)/2\big)$ and $(n_0 - 1)\sin\varphi \ge \epsilon/\sin\varphi$, $n_0 \ge (1+\cot\varphi)\epsilon + 1$, we deduce from inequality (2.7) that $\|\widetilde{R}^{\lambda}(h) \circ T_r|\mathcal{V}_n\|_q \le \mathcal{C}$ for $n \in \mathbb{N}$, $n \ge n_0, r \in (0,\infty)$. Thus inequality (3.23) implies

$$|C^{(1)}_{n,r}| \le \mathcal{C}n^{-1+2/q'-2/p} \qquad \text{for} \quad n \in \mathbb{N},\ n \ge n_0,\ r \in (0,1]. \tag{3.24}$$

Combining (3.20), (3.21) and (3.24) yields

$$\Big|\int_{\mathbb{L}(\varphi,\epsilon)} HD_k\widetilde{\varphi}_n T_{jk}(W, P - \lambda S)\, dx\Big| \le n^{-1+3/q'-2/p}$$

for $n \in \mathbb{N}$, $n \ge n_0$, $r \in (0,1]$. But due to our assumption $q < 3p'/2$, we have $-1 + 3/q' - 2/p < 0$, so Theorem 3.4 follows. □

4. A second Green's formula. In this section, we shall prove our second Green's formula (Theorem 4.4). To this end, we fix $\epsilon \in (0, \infty)$ and $\lambda \in \mathbb{C}\backslash(-\infty, 0]$. Moreover, we take $\varphi \in (0, \pi/2)$, $q \in (1, 2]$ with

$$(1/2)\big(1 - (4\gamma_1^{(\varphi)} + 1)^{1/2}\big) \; < \; -2 + 3/q'. \tag{4.1}$$

The number $\gamma_1^{(\varphi)}$ was introduced in Section 2. As may be seen by (2.4), the condition in (4.1) is satisfied if φ is close enough to zero. We shall return to this point in Section 7.

The ensuing theorem states that certain potential functions are bounded in a negative norm.

Theorem 4.1. *Let $h \in C_0^\infty(\mathbb{R}^3)$, $\Psi \in L^q\big(\partial\mathbb{L}(\varphi, \epsilon)\big)^3$ with*

$$\Gamma^*\big(-1, q, \lambda, \mathbb{L}(\varphi, \epsilon)\big)(\Psi) = T\big(\widetilde{R}^\lambda(h), S(h)\big)n^{(\varphi,\epsilon)}. \tag{4.2}$$

Then there is a constant $C > 0$ with

$$\Big|\int_{\mathbb{L}(\varphi,\epsilon)} \big(\widetilde{R}^\lambda(h) + \widetilde{V}^\lambda\big(\partial\mathbb{L}(\varphi, \epsilon)\big)(\Psi)\big) f \, dx\Big| \leq C\big\|\nabla f\big\|_{q'}$$

for $f \in C_0^\infty(\mathbb{R}^3)^3$ with $supp(h) \cap supp(f) = \emptyset$.

Proof. Choose $\sigma \in C^\infty(\mathbb{R}^3)$ with

$$\sigma|\mathbb{R}^3\backslash\mathbb{B}_3(0, 2\epsilon/\sin\varphi) = 1, \quad \sigma|\mathbb{B}_3(0, \epsilon/\sin\varphi) = 0, \quad 0 \leq \sigma \leq 1. \tag{4.3}$$

Let $R \in (\epsilon/\sin\varphi, \infty)$ be so large that $supp(h) \subset \mathbb{B}_3(0, R)$. Take $f \in C_0^\infty(\mathbb{R}^3)^3$. In this proof, the letter $\mathcal{C}$ will denote constants which do not depend on f. Due to (4.1), the operator $A^\varphi_{q',0}$ introduced in Theorem 2.3 is invertible. Put $v(f) := (A^\varphi_{q',0})^{-1}\big(-\text{div} f|\mathbb{K}(\varphi)\big)$. By standard elliptic regularity theory, it holds that $v(f) \in C^\infty(\overline{\mathbb{K}(\varphi)}\backslash\{0\})$. Put $\overline{v}(f) := -\Delta\sigma v(f) - \nabla\sigma\nabla v(f)$, $w(f) := \sigma v(f)$. Then we have, $w(f)|\partial\mathbb{K}(\varphi) = 0$,

$$w(f) \in C^\infty(\overline{\mathbb{K}(\varphi)}) \cap V^2_{q',0}(\mathbb{K}(\varphi)), \quad A^\varphi_{q',0}(w(f)) = -\Delta w(f) = \overline{v}(f) - \text{div} f. \tag{4.4}$$

Obviously the function $\overline{v}(f) - \text{div} f$ belongs to $V^0_{q',1}\big(\mathbb{K}(\varphi)\big) \cap V^0_{q',0}\big(\mathbb{K}(\varphi)\big)$. We assumed $q \leq 2$, hence $-1 + 3/q' \leq 1/2$, so assumption (4.1) implies

$$[-2 + 3/q', -1 + 3/q'] \cap \big\{(1/2)\big(1 \pm \; (4\gamma_n^{(\varphi)} + 1)^{1/2}\big) : n \in \mathbb{N}\big\} = \emptyset.$$

Thus we may conclude by Theorem 2.3 and (4.4) that $w(f) \in V^2_{q',1}\big(\mathbb{K}(\varphi)\big)$. This fact, the boundedness of $(A^{\varphi}_{q',0})^{-1}$ (Theorem 2.3), and the relations in (1.7), (4.3) and (4.4) imply

$$\big\|\overline{v}(f)\big\|_{q'} \le \mathcal{C}\big\|\mathrm{div}f\big\|_{q'}, \tag{4.5}$$

$$\big\|D^a w(f)|\mathbb{L}(\varphi,\epsilon)\big\|_{q'} \le \mathcal{C}\big\|\mathrm{div}f\big\|_{q'} \qquad \text{for } \ a \in \mathbb{N}_0^3 \text{ with } |a|_* = 2, \tag{4.6}$$

$$\big\|\nabla w(f)|\mathbb{L}(\varphi,\epsilon) \cap \mathbb{B}_3(0,R)\big\|_{q'} \le \mathcal{C}\big\|\mathrm{div}f\big\|_{q'}, \tag{4.7}$$

$$\big\|D^a w(f)|\mathbb{L}(\varphi,\epsilon) \cap \big(\mathbb{B}_3(0,2M)\backslash\mathbb{B}_3(0,M)\big)\big\|_{q'} \le \mathcal{D}M^{1-|a|_*} \tag{4.8}$$

for $a \in \mathbb{N}_0^3$ with $|a|_* \le 1$, $M \in (R,\infty)$, where the constant $\mathcal{D}$ is independent of M, but depends on f. Note that since the support of $\overline{v}(f)$ is independent of f, the constants in inequalities (4.5)–(4.7) do not depend on f.

Now we choose a sequence $(\widetilde{\varphi}_n)$ in $C_0^\infty(\mathbb{R}^3)$ with properties as listed in the proof of Theorem 3.2. For $r \in (0,\infty)$, we define $A^{(r)}(\Psi), B^{(r)}(\Psi)$ in the same way as at the beginning of Section 3, but with the restriction $\tau = 1$, $V = \mathbb{L}(\varphi,\epsilon)$. Due to (1.7), the definition of $v(f)$ and the relation $\sigma|\mathbb{B}_3(0,\epsilon/\sin\varphi) = 0$, it holds that $w(f)|\partial\mathbb{L}(\varphi,\epsilon) = 0$. Therefore we obtain for $n \in \mathbb{N}$, $r \in (0,\infty)$, making use of Theorem 2.4, (3.4) and (3.5),

$$\int_{\mathbb{L}(\varphi,\epsilon)} (\widetilde{R}^\lambda(h) + A^{(r)}(\Psi))\widetilde{\varphi}_n \nabla w(f)\,dx = -\int_{\mathbb{L}(\varphi,\epsilon)} (\widetilde{R}^\lambda(h) + A^{(r)}(\Psi))\nabla\widetilde{\varphi}_n w(f)\,dx.$$

Letting r tend to zero, and referring to Corollary 3.1, we see the preceding equation remains valid when the term $A^{(r)}(\Psi)$ is replaced by $\widetilde{V}^\lambda(\partial\mathbb{L}(\varphi,\epsilon))(\Psi)$. On the other hand, it follows by (3.12), (4.8), Lemma 3.6, Theorem 2.4 and Lebesque's dominant convergence theorem that

$$\int_{\mathbb{L}(\varphi,\epsilon)} \big(\widetilde{R}^\lambda(h) + \widetilde{V}^\lambda\big(\partial\mathbb{L}(\varphi,\epsilon)\big)(\Psi)\big)\nabla\widetilde{\varphi}_n w(f)\;dx \;\longrightarrow\; 0 \quad (n\to\infty).$$

Thus we have

$$\int_{\mathbb{L}(\varphi,\epsilon)} \big(\widetilde{R}^\lambda(h) \;+\; \widetilde{V}^\lambda\big(\partial\mathbb{L}(\varphi,\epsilon)\big)(\Psi)\big) f\;dx = \lim_{n\to\infty} A(n), \tag{4.9}$$

with $A(n) := \int_{\mathbb{L}(\varphi,\epsilon)}\big(\widetilde{R}^\lambda(h) + \widetilde{V}^\lambda\big(\partial\mathbb{L}(\varphi,\epsilon)\big)(\Psi)\big)\widetilde{\varphi}_n\big(f - \nabla w(f)\big)\;dx$. But for $n \in \mathbb{N}$, we find by Theorem 2.4, Corollary 3.1, (3.4), and (3.5) that

$$A(n) = \lambda^{-1}\lim_{r\downarrow 0}\sum_{\upsilon=1}^{5}\mathcal{B}_\upsilon(n,r), \tag{4.10}$$

where we used the abbreviations

$$\mathcal{B}_1(n,r) := \int_{\partial\mathbb{L}(\varphi,\epsilon)\cap\mathbb{B}_3(0,2n)} \sum_{j,k=1}^{3} T_{jk}\big(\widetilde{R}^{\lambda}(h) + A^{(r)}(\Psi), S(h) + B^{(r)}(\Psi)\big)$$
$$\times\, n_k^{(\varphi,\epsilon)} \widetilde{\varphi}_n \big(f - \nabla w(f)\big)_j \, d\mathbb{L}(\varphi,\epsilon),$$
$$\mathcal{B}_2(n,r) := \int_{\mathbb{L}(\varphi,\epsilon)} - \sum_{j,k=1}^{3} T_{jk}\big(\widetilde{R}^{\lambda}(h) + A^{(r)}(\Psi), S(h) + B^{(r)}(\Psi)\big)$$
$$\times\, D_k \widetilde{\varphi}_n \big(f - \nabla w(f)\big)_j \, dx,$$
$$\mathcal{B}_3(n,r) := \int_{\mathbb{L}(\varphi,\epsilon)} - \sum_{j,k=1}^{3} \big(D_j \widetilde{R}_k^{\lambda}(h) + D_k \widetilde{R}_j^{\lambda}(h) + D_j A_k^{(r)}(\Psi) + D_k A_j^{(r)}(\Psi)\big)$$
$$\times\, \widetilde{\varphi}_n D_k \big(f - \nabla w(f)\big)_j \, dx,$$
$$\mathcal{B}_4(n,r) := \int_{\mathbb{L}(\varphi,\epsilon)\cap\mathbb{B}_3(0,2n)} \big(S(h) + B^{(r)}(\Psi)\big) \widetilde{\varphi}_n \mathrm{div}\big(f - \nabla w(f)\big) \, dx,$$
$$\mathcal{B}_5(n,r) := \int_{\mathbb{L}(\varphi,\epsilon)\cap\mathbb{B}_3(0,2n)} h \widetilde{\varphi}_n \big(f - \nabla w(f)\big) \, dx,$$

for $n \in \mathbb{N}$, $r \in (0,\infty)$. As a consequence of Theorem 3.1 and the choice of Ψ, we get $\mathcal{B}_1(r,n) \to 0$ $(r \downarrow 0)$, for $n \in \mathbb{N}$. Thus, again referring to Corollary 3.1, we obtain from (4.10) that

$$A(n) = \lambda^{-1} \sum_{\upsilon=2}^{5} \mathcal{B}_\upsilon(n,0) \qquad \text{for} \quad n \in \mathbb{N}, \tag{4.11}$$

where $\mathcal{B}_\upsilon(n,0)$ is defined in the same way as $\mathcal{B}_\upsilon(n,r)$, but with the expressions $A^{(r)}(\Psi), B^{(r)}(\Psi)$ replaced by $\widetilde{V}^{\lambda}\big(\partial\mathbb{L}(\varphi,\epsilon)\big)(\Psi)$ and $Q\big(\partial\mathbb{L}(\varphi,\epsilon)\big)(\Psi)$, respectively.

By (4.8), (3.12), Lemmas 3.6, 3.2, and Theorem 2.4, we see there are numbers $\sigma, \mathcal{K} \in (0,\infty)$ such that $\big|\mathcal{B}_2(n,0)\big| \le \mathcal{K} n^{-\sigma}$ for $n \in \mathbb{N}$. This implies that $\mathcal{B}_2(n,0) \to 0$ $(n \to \infty)$. It is clear by (4.6), Lemma 3.6 and Theorem 2.4 that

$$\big|\mathcal{B}_3(n,0)\big| \le \mathcal{C} \big\|\nabla f\big\|_{q'} \qquad (n \in \mathbb{N}). \tag{4.12}$$

Using (4.4), the equation $\mathrm{div}\big(f - \nabla w(h)\big) = \overline{v}(f)$, and the fact that $supp(h) \cup$

$supp\big(\overline{v}(f)\big)$ is compact, we find that

$$\mathcal{B}_4(n,0)+\mathcal{B}_5(n,0) \longrightarrow \int_{\mathbb{L}(\varphi,\epsilon)} \big(S(h)+Q\big(\partial\mathbb{L}(\varphi,\epsilon)\big)(\Psi)\big)\overline{v}(f)\,dx + \int_{\mathbb{L}(\varphi,\epsilon)} h\big(f-\nabla w(f)\big)\,dx \quad (n\to\infty). \tag{4.13}$$

Further observe that $supp\big(\overline{v}(f)\big) \subset \mathbb{B}_3(0,2\epsilon/\sin\varphi)$ and $supp(h)\subset \mathbb{B}_3(0,R)$. Thus, if we assume that $supp(h)\cap supp(f)=\emptyset$, and if we denote the limit in (4.13) by $\mathcal{L}$, we get by (4.5), (4.7), Theorem 2.4 and Lemma 3.2 that

$$\mathcal{L} \le \mathcal{C}\big\|\mathrm{div} f\big\|_{q'}. \tag{4.14}$$

Collecting the results from (4.9), (4.11), (4.12), (4.13) and (4.14), we obtain Theorem 4.1. □

Corollary 4.1. *Let* h,Ψ *be given as in Theorem* 4.1. *Let* R *be so large that* $supp(h)\subset\mathbb{B}_3(0,R)$. *Put for* $v\in C_0^\infty(\mathbb{R}^3)^3$

$$\begin{aligned} A(v) := \int_{\mathbb{L}(\varphi,\epsilon)} \sum_{j,k=1}^{3} \big(&\partial/\partial x_k \widetilde{R}_j^\lambda(h)(x) + \partial/\partial x_k \widetilde{V}_j^\lambda\big(\partial\mathbb{L}(\varphi,\epsilon)\big)(\Psi)(x) \\ &- \delta_{jk}S(h)(x) - \delta_{jk}Q\big(\partial\mathbb{L}(\varphi,\epsilon)\big)(\Psi)(x)\big)D_k v_j(x)\,dx, \\ B(v) := \int_{\mathbb{L}(\varphi,\epsilon)} \sum_{j,k=1}^{3} T_{jk}\big(&\widetilde{R}^\lambda(h) \\ &+ \widetilde{V}^\lambda\big(\partial\mathbb{L}(\varphi,\epsilon)\big)(\Psi), S(h)+Q\big(\partial\mathbb{L}(\varphi,\epsilon)\big)(\Psi)\big)D_k v_j\,dx. \end{aligned}$$

Then there is a constant $C>0$ *with*

$$\big|A(v)\big| \le C\big\|\nabla v\big\|_{q'} \qquad \textit{for}\ \ v\in C_0^\infty(\mathbb{L}(\varphi,\epsilon)\backslash\overline{\mathbb{B}_3(0,R)})^3,$$
$$\big|B(v)\big| \le C\big\|\nabla v\big\|_{q'} \qquad \textit{for}\ \ v\in C_0^\infty(\mathbb{R}^3\backslash\overline{\mathbb{B}_3(0,R)})^3.$$

Note these estimates do not follow by Hölder's inequality because in general, we may at best hope to get

$$\big(S(h)+Q\big(\partial\mathbb{L}(\varphi,\epsilon)\big)(\Psi)\big)\Big|\mathbb{L}(\varphi,\epsilon) \in L^{3q/2}\big(\mathbb{L}(\varphi,\epsilon)\big);$$

see Theorem 2.4 and the proof of Lemma 3.7.

Proof. For $r \in (0,\infty)$, define $A^{(r)}(\Psi), B^{(r)}(\Psi)$ in the same way as at the beginning of Section 3, but with the restriction $\tau = 1$, $V = \mathbb{L}(\varphi,\epsilon)$. Let $v \in C_0^\infty(\mathbb{L}(\varphi,\epsilon)\backslash\overline{\mathbb{B}_3(0,R)})^3$. It holds by Theorem 2.4, (3.4), (3.5), for $r \in (0,\infty)$ that

$$\int_{\mathbb{L}(\varphi,\epsilon)} \sum_{j,k=1}^{3} \big(D_k\widetilde{R}_j^\lambda(h) + D_kA_j^{(r)}(\Psi) - \delta_{jk}S(h) - \delta_{jk}B^{(r)}(\Psi)\big)D_kv_j\,dx$$
$$= -\lambda\int_{\mathbb{L}(\varphi,\epsilon)} \big(\widetilde{R}^\lambda(h) + A^{(r)}(\Psi)\big)v\,dx; \tag{4.15}$$

note that $supp(h) \cap supp(v) = \emptyset$. Now Corollary 3.1 yields

$$A(v) = -\lambda\int_{\mathbb{L}(\varphi,\epsilon)} \big(\widetilde{R}^\lambda(h) + \widetilde{V}^\lambda\big(\partial\mathbb{L}(\varphi,\epsilon)\big)(\Psi)\big)v\,dx;$$

hence, by Theorem 4.1, $|A(v)| \le \mathcal{C}\|\nabla v\|_{q'}$, with $\mathcal{C} > 0$ independent of v. The estimate for $|B(v)|$ claimed in Corollary 4.1 may be established in a similar way. This time, however, we have to consider functions $v \in C_0^\infty(\mathbb{R}^3\backslash\overline{\mathbb{B}_3(0,R)})^3$. Thus, when integrating by parts as in (4.15), we additionally obtain a surface integral over $\partial\mathbb{L}(\varphi,\epsilon)$. But this integral vanishes for $r \downarrow 0$, as follows by (4.2) and Theorem 3.1. $\square$

Corollary 4.1 will be used in order to estimate the pressure function $S(h) + Q\big(\partial\mathbb{L}(\varphi,\epsilon)\big)(\Psi)$, for suitable functions h and Ψ. Another tool needed for this purpose is

Theorem 4.2. *Let $U_\varphi \subset \mathbb{R}^3$ be a bounded domain with C^∞-boundary such that $U_\varphi \cap (\mathbb{B}_3(0,2)\backslash\mathbb{B}_3(0,1)) = \mathbb{K}(\varphi)\cap(\mathbb{B}_3(0,2)\backslash\mathbb{B}_3(0,1))$, $U_\varphi \subset \mathbb{K}(\varphi)$. Denote by $n^{(U_\varphi)}$ the outward unit normal to U_φ. Choose a function $\varrho_\varphi \in C_0^\infty(\mathbb{R}^3)^3$ with $\varrho_\varphi|\partial U_\varphi = n^{(U_\varphi)}$. Then there is a constant $C > 0$ such that for $u \in W^{1,q}(U_\varphi)^3$, $\pi \in L^q(U_\varphi)$ with $div\,u = 0$, it holds that*

$$\|\pi\|_q \le C\big(\|u\|_{1,q} + A(u,\pi) + B(u,\pi)\big), \tag{4.16}$$

where $A(u,\pi) := \sup\big\{\big|\int_{U_\varphi}\sum_{j,k=1}^3(D_ku_j - \delta_{jk}\pi)D_kv_j\ dx\big|/\|\nabla v\|_{q'} : v \in C_0^\infty(U_\varphi)^3,\ v \ne 0\big\}$, $B(u,\pi) := \big|\int_{U_\varphi}\sum_{j,k=1}^3 T_{jk}(u,\pi)D_k\varrho_{\varphi,j}\,dx\big|/\|\nabla\varrho_\varphi|U_\varphi\|_{q'}$.

We recall that the exponent q was fixed at the beginning of this section.

Proof. Put $\mathcal{W}_1 := W_0^{1,q'}(U_\varphi)^3$, $\mathcal{W}_2 := span\{\varrho_\varphi|U_\varphi\}$. For $(v,w) \in \mathcal{W}_1\times\mathcal{W}_2$, set $|||(v,w)||| := \|v\|_{1,q'} + \|\nabla w\|_{q'}$. Since

$$\int_{U_\varphi} \mathrm{div}\varrho_\varphi\,dx > 0, \tag{4.17}$$

the mapping $|||\cdot|||$ is a norm. Obviously the space $\mathcal{W}_1 \times \mathcal{W}_2$ equipped with this norm is a Banach space. Denote by $|||\cdot|||'$ the usual norm of the dual space $(\mathcal{W}_1 \times \mathcal{W}_2)'$.

Let $b \in W^{1-1/q,q}(\partial U_\varphi)^3,\ F \in (\mathcal{W}_1 \times \mathcal{W}_2)'$. Then there is a uniquely determined function $u \in W^{1,q}(U_\varphi)^3,\ \pi \in L^q(U_\varphi)$ with $\operatorname{div} u = 0,\ u|\partial U_\varphi = b$,

$$\int_{U_\varphi} \sum_{j,k=1}^{3} (D_k u_j - \delta_{jk}\pi) D_k v_j \, dx = F(v,0) \qquad \text{for} \ \ v \in W_0^{1,q'}(U_\varphi)^3,$$

$$\int_{U_\varphi} \sum_{j,k=1}^{3} T_{jk}(u,\pi) D_k \varrho_{\varphi,j} \, dx = F(0, \varrho_\varphi|U_\varphi).$$

In fact, the first of these two equations may be solved by referring to Theorem 2.1. In view of (4.17), a suitable constant may be added to the pressure part of this solution so that the second equation is valid as well.

Put $\mathcal{F}(b,F) := (u,\pi)$. Then we have defined a mapping

$$\mathcal{F} : W^{1-1/q,q}(\partial U_\varphi)^3 \times (\mathcal{W}_1 \times \mathcal{W}_2)' \longmapsto \big\{ u \in W^{1,q}(U_\varphi)^3 : \operatorname{div} u = 0 \big\} \times L^q(U_\varphi).$$

Obviously this mapping is one-to-one, linear and onto. Moreover its inverse is bounded if its domain is equipped with the norm $|(b,F)|_D := \|b\|_{1-1/q,q} + |||F|||'$, and its range with the norm $|(u,\pi)|_R := \|u\|_{1,q} + \|\pi\|_q$. Now Theorem 4.2 follows by the open mapping theorem and a standard trace theorem. □

Note that Theorem 4.2 is valid without assumption (4.1) on q and φ. The next theorem, however, will be deduced from Corollary 4.1 and Theorem 4.2, so condition (4.1) will be necessary once more.

Theorem 4.3. *Let h and Ψ be given as in Theorem* 4.1. *Put* $u := \big(\widetilde{R}^\lambda(h) + \widetilde{V}^\lambda\big(\partial\mathbb{L}(\varphi,\epsilon)\big)(\Psi)\big)\Big|\mathbb{L}(\varphi,\epsilon),\ \pi := \big(S(h) + Q\big(\partial\mathbb{L}(\varphi,\epsilon)\big)(\Psi)\big)\Big|\mathbb{L}(\varphi,\epsilon)$. *Then it holds that*

$$u \in W^{1,q}\big(\mathbb{L}(\varphi,\epsilon)\big)^3 \cap C^\infty\big(\mathbb{L}(\varphi,\epsilon)\big)^3, \quad \pi \in C^\infty\big(\mathbb{L}(\varphi,\epsilon)\big), \tag{4.18}$$

$$-\Delta u + \lambda u + \nabla \pi = h, \quad \mathit{div}\, u = 0. \tag{4.19}$$

Moreover, there are numbers $C_0 \in (0,\infty), M_0 \in [\epsilon/\sin\varphi, \infty)$ with

$$\big\|\pi|\mathcal{B}\big(M,\mathbb{K}(\varphi)\big)\big\|_q \le C_0 \qquad \mathit{for} \ \ M \in (M_0,\infty), \tag{4.20}$$

where the set $\mathcal{B}\big(M,\mathbb{K}(\varphi)\big)$ is defined in the following way, for $M > \epsilon/\sin\varphi$:

$$\begin{aligned}\mathcal{B}(M,\mathbb{K}(\varphi)) &:= \mathbb{K}(\varphi) \cap (\mathbb{B}_3(0,2M)\backslash \mathbb{B}_3(0,M)) \\ &= \mathbb{L}(\varphi,\epsilon) \cap (\mathbb{B}_3(0,2M)\backslash \mathbb{B}_3(0,M)).\end{aligned}$$

Inequality (4.20) is remarkable because on the basis of Lemma 3.2, one would expect an additional factor $M^{1/q}$ to arise on the right-hand side of (4.20). However, due to the assumptions (4.1) and (4.2) on φ, q and Ψ, such a factor does not show up.

Proof. Concerning the claims in (4.18) and (4.19), we refer to Theorem 2.4 and Lemmas 2.1 and 3.6. This leaves us to establish (4.20). To this end, take $R \in (0,\infty)$ so large that $supp(h) \subset \mathbb{B}_3(0,R)$. Furthermore, choose $U_\varphi, n^{(U_\varphi)}, \varrho_\varphi$ as in Theorem 4.2, take $a \in (0,1)$ such that $U_\varphi \subset \mathbb{K}(\varphi)\backslash\mathbb{B}_3(0,a)$, and put $M_0 := \max\{\epsilon/(a\sin\varphi), R/a\}$.

Let $M \in [M_0,\infty)$. Combining (1.7), the relation $U_\varphi \subset \mathbb{K}(\varphi)\backslash\mathbb{B}_3(0,a)$ and the inequality $M \geq M_0 \geq \epsilon/(a\sin\varphi)$, we obtain $MU_\varphi \subset \mathbb{K}(\varphi)\backslash\mathbb{B}_3(0,\epsilon/\sin\varphi)$ $\subset \mathbb{L}(\varphi,\epsilon)$. Therefore we may define functions $u_M, h_M : U_\varphi \mapsto \mathbb{C}^3,\ \pi : U_\varphi \mapsto \mathbb{C}$ by setting

$$u_M(x) := u(Mx), \quad h_M(x) := M^2 h(Mx),\ \pi_M(x) := M\pi(Mx) \ \text{ for } \ x \in U_\varphi.$$

By Theorem 2.4, Lemma 3.2 and 3.6, it holds that $u_M \in W^{1,q}(U_\varphi)^3,\ \pi_M \in L^q(U_\varphi)$. Referring to Theorem 4.2, we see there is a constant C independent of M such that inequality (4.16) holds with (u,π) replaced by (u_M,π_M). On the other hand, since $M > \epsilon/\sin\varphi$ and $\big(\mathbb{B}_3(0,2)\backslash\mathbb{B}_3(0,1)\big)\cap\mathbb{K}(\varphi) \subset U_\varphi$, we conclude that $\mathcal{B}\big(M,\mathbb{K}(\varphi)\big) \subset MU_\varphi$. Thus, combining (4.16) and a scaling argument, we arrive at the estimate

$$\|\pi|\mathcal{B}(M,\mathbb{K}(\varphi))\|_q \leq CM^{3/q-1}(\|u_M\|_{1,q} + A(u_M,\pi_M) + B(u_M,\pi_M)), \tag{4.21}$$

with C from (4.16), and with $A(u_M,\pi_M), B(u_M,\pi_M)$ defined as in Theorem 4.2. But $M \geq M_0 \geq R/a$ and $U_\varphi \subset \mathbb{R}^3\backslash\mathbb{B}_3(0,a)$; hence, $MU\varphi \subset \mathbb{L}(\varphi,\epsilon)\backslash\mathbb{B}_3(0,R)$. This means that if $v \in C_0^\infty(U_\varphi)^3$, the function $v^{(M)}$ defined by $v^{(M)} := v(M^{-1}x)$ for $x \in \mathbb{R}^3$ belongs to $C_0^\infty\big(\mathbb{L}(\varphi,\epsilon)\backslash\mathbb{B}_3(0,R)\big)^3$. Combining this observation with (4.21), Lemma 3.6, Theorem 2.4 and Corollary 4.1, and using a scaling argument once more, we see there is a constant C_0 satisfying (4.20). □

Let us note another consequence of Theorem 4.1.

Corollary 4.2. *Let $p \in (2,\infty)$ with $p' \leq q < 3p'/2$, $\Phi \in L^p(\partial\mathbb{K}(\varphi))^3$, $\Psi \in L^q(\partial\mathbb{K}(\varphi))^3$, $h \in C_0^\infty(\mathbb{R}^3)^3$, with h, Ψ satisfying* (4.2). (*The quantities ϵ, λ, φ, q were fixed at the beginning of this section.*) *Choose a sequence $(\widetilde{\varphi}_n)$ and a constant $\mathcal{C}_1$ as in Theorem* 3.4. *Let $\alpha \in \mathbb{C}$, and put*

$$A_n := \int_{\mathbb{L}(\varphi,\epsilon)} \Big(\widetilde{R}^\lambda(h) + \widetilde{V}^\lambda\big(\partial\mathbb{L}(\varphi,\epsilon)\big)(\Psi)\Big)\nabla\widetilde{\varphi}_n\big(S^{(\varphi)}(\Phi)+\alpha\big)\,dx \ \ \textit{for} \ \ n \in \mathbb{N}.$$

Then $A_n \to 0\,(n \to \infty)$.

Proof. Take $n \in \mathbb{N}$ with $supp(h) \subset \mathbb{B}_3(0,n)$. Let $\mathcal{C}$ denote constants which do not depend on n. According to Lemma 2.2 and 3.2, it holds that

$$S^{(\varphi)}(\Phi)|\mathbb{B}_3(0,4n) \in W^{1,p}\big(\mathbb{B}_3(0,4n)\big).$$

But $q' \le p$, so the function $S^{(\varphi)}(\Phi)|\mathbb{B}_3(0,4n)$ belongs to $W^{1,q'}\big(\mathbb{B}_3(0,4n)\big)$. Since the support of $\nabla\widetilde{\varphi}_n$ is contained in $\mathbb{B}_3(0,2n)\backslash\mathbb{B}_3(0,n)$, we have

$$f_n := \nabla\widetilde{\varphi}_n\big(S^{(\varphi)}(\Phi)+\alpha\big)|\mathbb{B}_3(0,2n)\backslash\overline{\mathbb{B}_3(0,n)} \in W_0^{1,q'}\big(\mathbb{B}_3(0,2n)\backslash\overline{\mathbb{B}_3(0,n)}\big)^3,$$

and $supp(h) \cap supp(f_n) = \emptyset$. We further note that by Theorem 2.4 and Lemma 3.6, $\big(\widetilde{R}^\lambda(h) + \widetilde{V}^\lambda\big(\partial\mathbb{L}(\varphi,\epsilon)\big)(\Psi)\big)|\mathbb{L}(\varphi,\epsilon) \in W^{1,q}\big(\mathbb{L}(\varphi,\epsilon)\big)^3$. Now it follows from Theorem 4.1 and a density argument that $|A_n| \le \mathcal{C}\|\nabla f_n\|_{q'}$; hence,

$$|A_n| \;\le\; \mathcal{C}n^{3/q'-3/p}\|\nabla f_n\|_p,$$

with the letter $\mathcal{C}$ denoting constants which do not depend on n. But Lemmas 3.2, 3.3 and inequality (3.19) yield $\|\nabla f_n\|_p \le \mathcal{C}n^{-1+1/p}$; hence, $|A_n| \le \mathcal{C}n^{-1+3/q'-2/p}$. Since $q < 3p'/2$, we have $-1+3/q'-2/p < 0$, so Corollary 4.2 follows. □

Now we are in a position to prove the main result of this section.

Theorem 4.4. (second Green's formula) *Let* $\epsilon, \lambda, \varphi, q$ *be given as at the beginning of this section. Let* $p \in (2,\infty)$ *with* $p' \le q < 3p'/2$, $\Phi \in L^p(\partial\mathbb{K}(\varphi))^3$ *with* $\Gamma(-1,p,\lambda,\mathbb{K}(\varphi))(\Phi) = 0$. *Abbreviate* $v := W\big(\lambda,\mathbb{K}(\varphi)\big)(\Phi)$. *According to Theorem* 2.6, *it holds that* $v \in C^1(\overline{\mathbb{K}(\varphi)}\backslash\{0\})^3$, *and there is* $\varrho \in C^0(\overline{\mathbb{K}(\varphi)}\backslash\{0\})$ *with* $\varrho|\mathbb{K}(\varphi) = P(\lambda,\mathbb{K}(\varphi))(\Phi)$. *Further suppose there is some number* $\alpha \in \mathbb{C}$ *and a bounded measurable set* $\mathcal{M} \subset \partial\mathbb{L}(\varphi,\epsilon)$ *with* $T(v,\varrho+\alpha)n^{(\varphi,\epsilon)}|\partial\mathbb{L}(\varphi,\epsilon)\backslash\mathcal{M} = 0$. *Let* $h \in C_0^\infty(\mathbb{R}^3)^3$, $\Psi \in L^q(\partial\mathbb{L}(\varphi,\epsilon))^3$ *with* $\Gamma^*(-1,\,q,\lambda,\mathbb{L}(\varphi,\epsilon))(\Psi) = T(\widetilde{R}^\lambda(h),S(h))n^{(\varphi,\epsilon)}$. *Then*

$$\int_{\mathbb{L}(\varphi,\epsilon)} hv\,dx = \int_{\partial\mathbb{L}(\varphi,\epsilon)\cap\mathcal{M}} \big(\widetilde{R}^\lambda(h) + \widetilde{V}^\lambda\big(\partial\mathbb{L}(\varphi,\epsilon)\big)(\Psi)\big)\big(T(v,\varrho+\alpha)n^{(\varphi,\epsilon)}\big)d\mathbb{L}(\varphi,\epsilon). \tag{4.22}$$

Proof. Let a sequence $(\widetilde{\varphi}_n)$ in $C_0^\infty(\mathbb{R}^3)$ be chosen as in Theorem 3.4. For $r \in (0,\infty)$, let $A^{(r)}(\Psi)$, $B^{(r)}(\Psi)$ be defined as at the beginning of Section 3, with $\tau = 1$, $V = \mathbb{L}(\varphi,\epsilon)$, and with Φ replaced by Ψ. Then, if $r \in (0,\infty)$, $n \in$

$\mathbb{N}$ with $supp(h) \subset \mathbb{B}_3(0,n)$, it holds by (3.4), (3.5), Theorems 2.4, 2.6 and Lemma 3.5 that

$$\int_{\mathbb{L}(\varphi,\epsilon)} hv\,dx = \lim_{n\to\infty} \int_{\mathbb{L}(\varphi,\epsilon)} hv\widetilde{\varphi}_n\,dx = \lim_{n\to\infty} \sum_{\upsilon=1}^{4} T_\upsilon(n,r),$$

with

$$\mathcal{T}_1(n,r) := \int_{\mathbb{L}(\varphi,\epsilon)} - \sum_{j,k=1}^{3} \big(\widetilde{R}^\lambda(h) + A^{(r)}(\Psi)\big)_j T_{jk}(v,\varrho+\alpha) D_k\widetilde{\varphi}_n\,dx,$$

$$\mathcal{T}_2(n,r) := \int_{\mathbb{L}(\varphi,\epsilon)} \sum_{j,k=1}^{3} T_{jk}\big(\widetilde{R}^\lambda(h) + A^{(r)}(\Psi), S(h) + B^{(r)}(\Psi)\big) v_j D_k\widetilde{\varphi}_n\,dx,$$

$$\mathcal{T}_3(n,r) := \int_{\partial\mathbb{L}(\varphi,\epsilon)} \sum_{j=1}^{3} \Big[\big(\widetilde{R}^\lambda(h) + A^{(r)}(\Psi)\big)_j \Big(\sum_{k=1}^{3} T_{jk}(v,\varrho+\alpha) n_k^{(\varphi,\epsilon)}\Big)\Big] \times \widetilde{\varphi}_n\,d\mathbb{L}(\varphi,\epsilon),$$

$$\mathcal{T}_4(n,r) := \int_{\partial\mathbb{L}(\varphi,\epsilon)} - \sum_{j,k=1}^{3} T_{jk}\big(\widetilde{R}^\lambda(h) + A^{(r)}(\Psi), S(h) + B^{(r)}(\Psi)\big) \times n_k^{(\varphi,\epsilon)} v_j \widetilde{\varphi}_n\,d\mathbb{L}(\varphi,\epsilon).$$

Letting r tend to zero, we get by Corollary 3.1, Theorem 3.1, Lemma 3.4 and the choice of Ψ

$$\int_{\mathbb{L}(\varphi,\epsilon)} hv\,dx = \sum_{\upsilon=1}^{2} \mathcal{T}_\upsilon(n) + \mathcal{T}_3 \qquad \text{for } n \in \mathbb{N} \text{ with } \mathcal{M} \subset \mathbb{B}_3(0,n),$$

where the expression $\mathcal{T}_\upsilon(n)$ for $\upsilon \in \{1,2\}$ is defined in the same way as $\mathcal{T}_\upsilon(n,r)$, but with $A^{(r)}(\Psi)$ and $B^{(r)}(\Psi)$ replaced by $\widetilde{V}^\lambda\big(\partial\mathbb{L}(\varphi,\epsilon)\big)(\Psi)$ and $Q\big(\partial\mathbb{L}(\varphi,\epsilon)\big)(\Psi)$, respectively. The expression $\mathcal{T}_3$ is an abbreviation for the right-hand side of (4.22). It is clear by (3.19), Theorem 3.4 and Corollary 4.2 that $\mathcal{T}_1(n) \to 0$ $(n \to \infty)$. We may conclude from (1.7), (2.7), (3.19), Lemmas 3.2 and 3.6 and Theorems 2.4 and 4.3 that

$$\begin{aligned} |\mathcal{T}_2(n)| &\le \mathcal{C} n^{-1} \|v|\mathbb{K}(\varphi) \cap \big(\mathbb{B}_3(0,2n)\backslash\mathbb{B}_3(0,n)\big)\|_{q'} \\ &\le \mathcal{C} n^{-1+3/q'-3/p} \|v|\mathbb{K}(\varphi) \cap \big(\mathbb{B}_3(0,2n)\backslash\mathbb{B}_3(0,n)\big)\|_p \le \mathcal{C} n^{-1+3/q'-2/p} \end{aligned}$$

for $n \in \mathbb{N},\ n \ge \epsilon/\sin\varphi$. Since $q < 3p'/2$, $-1 + 3/q' - 2/p < 0$; it follows that $\mathcal{T}_2(n) \to 0$ $(n \to \infty)$. Combining these results yields Theorem 4.4. □

5. Estimate (1.3) and the operator $\Gamma(-1, p, \lambda, \mathbb{K}(\varphi))$**.** In this section, we shall derive a relationship between estimate (1.3) and the properties of the operator $\Gamma(-1, p, \lambda, \mathbb{K}(\varphi))$. We begin by noting a consequence of Theorem 2.1:

Lemma 5.1. *Let* $\varphi \in (0, \pi/2]$*, and abbreviate* $V := \mathbb{R}^3 \backslash \overline{\mathbb{K}(\varphi)}, \mathcal{U} := V \cap (\mathbb{B}_3(0,4) \backslash \mathbb{B}_3(0,1/4))$*. Let* Ω *be a bounded* C^∞*-domain in* $\mathbb{R}^3$ *with* $\Omega \subset V$*, and* $\Omega \cap (\mathbb{B}_3(0,8) \backslash \mathbb{B}_3(0,1/8)) = V \cap (\mathbb{B}_3(0,8) \backslash \mathbb{B}_3(0,1/8))$*,* $\zeta \in C_0^\infty(\mathbb{R}^3)$ *with* $\zeta | \mathbb{B}_3(0,2) \backslash \mathbb{B}_3(0,1/2) = 1$*,* $supp(\zeta) \subset \mathbb{B}_3(0,4) \backslash \mathbb{B}_3(0,1/4)$*. Take* $r \in (1, \infty)$*. Then there is a constant* $C > 0$ *with the following properties:*

For $\lambda \in \mathbb{C} \backslash (-\infty, 0]$*,* $u \in C^1(\overline{V} \backslash \{0\})^3 \cap W^{2,1}_{loc}(V)^3$*,* $\pi \in C^0(\overline{V} \backslash \{0\}) \cap W^{1,1}_{loc}(V)$ *with*

$$-\Delta u + \lambda u + \nabla \pi = 0, \qquad div\, u = 0, \qquad u | \partial \mathbb{K}(\varphi) = 0, \tag{5.1}$$

there exists some $c \in \mathbb{C}$ *with*

$$\begin{aligned} &\|\nabla u | V \cap \big(\mathbb{B}_3(0,2) \backslash \mathbb{B}_3(0,1/2)\big)\|_r + \|(\pi + c) | \big(\mathbb{B}_3(0,2) \backslash \mathbb{B}_3(0,1/2)\big)\|_r \\ &\le C\big((1 + |\lambda|)\|u|\mathcal{U}\|_r + \|\nabla \zeta \pi | \Omega\|_{-1,r}\big). \end{aligned}$$

Proof. Let λ, u, π be given as in the lemma, and let $\mathcal{D} := \mathcal{D}(\Omega, r)$ be the operator introduced in Theorem 2.2. Put

$$\begin{aligned} &g := \operatorname{div}(\zeta u)|\Omega, \quad v := \big(\zeta u - \mathcal{D}(g)\big)|\Omega, \quad \varrho := (\zeta \pi)|\Omega, \\ &F(w) := \int_\Omega \Big(-\Delta \zeta u - \sum_{k=1}^3 D_k \zeta D_k u + \Delta \mathcal{D}(g) - \lambda \zeta u + \nabla \zeta \pi\Big) w \, dx \\ &= \int_\Omega \Big((-\Delta \zeta - \lambda \zeta) w u + \sum_{j,k=1}^3 (D_k^2 \zeta w_j + D_k \zeta D_k w_j) u_j \\ &- \sum_{j,k=1}^3 D_k \mathcal{D}(g)_j D_k w_j + \nabla \zeta \pi w\Big) \, dx \end{aligned}$$

for $w \in W_0^{1,r'}(\Omega)^3$. Due to our assumptions on Ω, ζ and u, we have $v \in W_0^{1,r}(\Omega)^3$, $\operatorname{div} v = 0$. Taking into account (5.1), we may apply Theorem 2.1 to obtain $c \in \mathbb{C}$ such that $\|v\|_{1,r} + \|\pi + c\|_r \le \mathcal{C}\|F\|_{-1,r}$, with $\mathcal{C}$ independent of v, π and F. On the other hand, we get for $w \in W_0^{1,r'}(\Omega)^3$

$$|F(w)| \le \mathcal{C}\big((1 + |\lambda|)\|u|\mathcal{U}\|_r + \|\nabla \mathcal{D}(g)\|_r + \|\nabla \zeta \pi | \Omega\|_{-1,r}\big)\|w\|_{1,r'},$$

where the letter $\mathcal{C}$ denotes constants which are independent of w, u, π and λ. But by Theorem 2.2, we have

$$\|\nabla \mathcal{D}(g)\|_r \le \mathcal{C}\|g\|_r = \mathcal{C}\|\nabla\zeta u|\Omega\|_r \le \mathcal{C}\|u|\mathcal{U}\|_r,$$

with the same meaning of $\mathcal{C}$ as before. Lemma 5.1 follows by collecting the preceding estimates. □

Lemma 5.2. *Let* φ, $V, \mathcal{U}$, Ω, ζ, r *be given as in Lemma* 5.1. *Take* $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $\Phi \in L^r(\partial\mathbb{K}(\varphi))^3$ *with* $\Gamma\big(-1, r, \lambda, \mathbb{K}(\varphi)\big)(\Phi) = 0$. *Then there is* $C > 0$ *and for each* $n \in \mathbb{N}$ *a number* $c_n \in \mathbb{C}$ *with*

$$\|\nabla\zeta\big(P(\lambda, V)(\Phi)(n\,\cdot) + c_n\big)|\Omega\|_{-1,r} \le Cn\big(\|Z^{(\varphi)}(\Phi)(n\,\cdot)|\mathcal{U}\|_r + \|\big(\nabla S^{(\varphi)}(\Phi)\big)(n\,\cdot)|\mathcal{U}\|_r\big) \quad \text{for } n \in \mathbb{N}, \tag{5.2}$$

where the functions $Z^{(\varphi)}(\Phi)$, $S^{(\varphi)}(\Phi)$ *were introduced in* (2.10) *and* (2.11), *respectively.*

Proof. Let $n \in \mathbb{N}$. By Theorem 2.6 and our assumptions on Φ, there is $\varrho \in C^0(\overline{V}\backslash\{0\})$ with $\varrho|V = P(\lambda, V)(\Phi)(n\,\cdot)$. In particular, $\|P(\lambda, V)(\Phi)(n\,\cdot)|\mathcal{U}\|_r < \infty$, so the left-hand side in (5.2) is well defined. We find for $w \in C_0^\infty(\Omega)^3$, $c \in \mathbb{C}$, using (2.9), that

$$\begin{aligned}
&\Big|\int_\Omega \nabla\zeta\big(P(\lambda, V)(\Phi)(n\,\cdot) + c\big)w\,dx\Big| \\
&= \Big|\int_\Omega\Big(\sum_{j,k=1}^{3} n^{-1}(D_jD_k\zeta w_k + D_k\zeta D_j w_k)Z_j^{(\varphi)}(\Phi)(n\,\cdot) \\
&\quad + \nabla\zeta w\big(S^{(\varphi)}(\Phi)(n\,\cdot) + c\big)\Big)dx\Big| \\
&\le \mathcal{C}\Big(n^{-1}\|w\|_{1,r'}\|Z^{(\varphi)}(\Phi)(n\,\cdot)|\mathcal{U}\|_r + \|w\|_{r'}\|\big(S^{(\varphi)}(\Phi)(n\,\cdot) + c\big)|\mathcal{U}\|_r\Big).
\end{aligned} \tag{5.3}$$

Here and in the following, the letter $\mathcal{C}$ denotes constants which do not depend on n. On the other hand, using Poincaré's inequality in the version of [19, Theorem 4.3], we see there is $c_n \in \mathbb{C}$ with

$$\begin{aligned}
\|\big(S^{(\varphi)}(\Phi)(n\,\cdot) + c_n\big)|\mathcal{U}\|_r &\le \mathcal{C}\|\nabla\big(S^{(\varphi)}(\Phi)(n\,\cdot)\big)|\mathcal{U}\big)\|_r \\
&= \mathcal{C}n\|\big(\nabla S^{(\varphi)}(\Phi)\big)(n\,\cdot)|\mathcal{U}\|_r.
\end{aligned} \tag{5.4}$$

Combining (5.3) and (5.4) yields Lemma 5.2. □

Corollary 5.1. *Take* $\varphi, V, \mathcal{U}, r$ *as in Lemma* 5.1. *Let* $\lambda \in \mathbb{C}\backslash(-\infty,0]$, $\Phi \in L^r\big(\partial\mathbb{K}(\varphi)\big)^3$ *such that* $\Gamma\big(-1,r,\lambda,\mathbb{K}(\varphi)\big)(\Phi) = 0$. *Then there is a constant* $C > 0$ *and for* $n \in \mathbb{N}$ *a number* $\gamma_n \in \mathbb{C}$ *such that for* $n \in \mathbb{N}$,

$$\begin{aligned} &\Big\|\nabla\Big(W(\lambda,V)(\Phi)(n\,\cdot)\big)|V \cap \big(\mathbb{B}_3(0,2)\backslash\mathbb{B}_3(0,1/2)\big)\Big\|_r \qquad (5.5)\\ &+ \Big\|n\Big(P(\lambda,V)(\Phi)(n\,\cdot) + \gamma_n\big)|V \cap \big(\mathbb{B}_3(0,2)\backslash\mathbb{B}_3(0,1/2)\big)\Big\|_r \le Cn^2 \\ &\times \Big(\|W(\lambda,V)(\Phi)(n\,\cdot)|\mathcal{U}\|_r + \|Z^{(\varphi)}(\Phi)(n\,\cdot)|\mathcal{U}\|_r + \|\big(\nabla S^{(\varphi)}(\Phi)\big)(n\,\cdot)|\mathcal{U}\|_r\Big). \end{aligned}$$

Proof. Let $n \in \mathbb{N}$, and put $u := W(\lambda,V)(\Phi)(n\,\cdot)$. By Theorem 2.6 and our assumption on Φ, we have $u \in C^1(\overline{V}\backslash\{0\})^3 \cap C^\infty(V)^3$, $u|\partial\mathbb{K}(\varphi) = 0$, and there is $\pi \in C^0(\overline{V}\backslash\{0\})$ such that $\pi|V = nP(\lambda,V)(\Phi)(n\,\cdot)$, with $P(\lambda,V)(\Phi)(n\,\cdot)$ belonging to $C^\infty(V)$. Recalling Lemma 2.1, we thus see that the pair (u,π) satisfies the assumptions of Lemma 5.1, with λ replaced by λn^2. Now Corollary 5.1 follows from Lemmas 5.1 and 5.2. □

The next result will later imply that if Theorem 1.3 is not valid, then $\Gamma\big(-1,p,\lambda,\mathbb{K}(\varphi)\big)$ must be one-to-one provided that $p > 2$ and φ is close to zero; see the proof of Theorem 7.2.

Theorem 5.1. *Let* $\varphi \in (0,\pi/2]$, $\lambda \in \mathbb{C}\backslash(-\infty,0]$, $q \in (1,2]$, $p \in (2,\infty)$ *with* $p' \le q < (3/2)p'$. *Assume the quantities* q, φ *satisfy assumption* (4.1). *Further suppose the operator* $\Gamma^*(-1,q,\lambda,\overline{\mathbb{L}(\varphi,1)})$ *is onto. Let* $\Phi \in L^p(\partial\mathbb{K}(\varphi))^3$ *with* $W(\lambda,\mathbb{R}^3\backslash\overline{\mathbb{K}(\varphi)})(\Phi) = 0$. (*This means in particular that* $\Gamma(-1,p,\lambda,\mathbb{K}(\varphi))(\Phi) = 0$.) *Then it follows that* $\Phi = 0$.

Proof. Abbreviate $v := W(\lambda,\mathbb{K}(\varphi))(\Phi)$, $\widetilde{v} := W(\lambda,\mathbb{R}^3\backslash\overline{\mathbb{K}(\varphi)})(\Phi)$. According to Theorem 2.6, the function $\Phi \circ g^{(\varphi)}|\mathbb{B}_2(0,1/\epsilon)\backslash\mathbb{B}_2(0,\epsilon)$ is of class $C^{1,\alpha}$, for $\alpha \in (0,1)$, $\epsilon > 0$, and it holds that

$$v \in C^1\big(\overline{\mathbb{K}(\varphi)}\backslash\{0\}\big)^3, \quad \widetilde{v} \in C^1((\mathbb{R}^3\backslash\mathbb{K}(\varphi))\backslash\{0\})^3.$$

Theorem 2.6 further implies there are functions $\varrho \in C^0\big(\overline{\mathbb{K}(\varphi)}\backslash\{0\}\big)$, $\widetilde{\varrho} \in C^0\Big(\big(\mathbb{R}^3\backslash\mathbb{K}(\varphi)\big)\backslash\{0\}\Big)$ such that

$$\varrho|\mathbb{K}(\varphi) = P\big(\lambda,\mathbb{K}(\varphi)\big)(\Phi), \quad \widetilde{\varrho}|\mathbb{R}^3\backslash\overline{\mathbb{K}(\varphi)} = P\big(\lambda,\mathbb{R}^3\backslash\overline{\mathbb{K}(\varphi)}\big)(\Phi).$$

It follows by Theorem 2.7 and a cut-off argument that

$$T(v,\varrho)(x)n^{(\varphi)}(x) = -T(\widetilde{v},\widetilde{\varrho})(x)n^{(\varphi)}(x) \qquad \text{for } x \in \partial\mathbb{K}(\varphi)\backslash\{0\}.$$

Thus, recalling our assumption $\widetilde{v} = 0$, and referring to Lemma 2.1, we may conclude there is a number $\alpha \in \mathbb{C}$ with

$$T(v, \varrho + \alpha)(x) n^{(\varphi)}(x) = 0 \qquad \text{for } \; x \in \partial\mathbb{K}(\varphi)\backslash\{0\}. \tag{5.6}$$

Put $\mathcal{M} := \gamma^{(\varphi,1)}\big(\mathbb{B}_2(0,1)\big)$, with $\gamma^{(\varphi,1)}$ the parametric representation of $\mathbb{L}(\varphi, 1)$ introduced in Section 2. It follows from (5.6) that

$$T(v, \varrho + \alpha) n^{(\varphi,1)} | \partial\mathbb{L}(\varphi, 1)\backslash\mathcal{M} = 0. \tag{5.7}$$

But $\mathcal{M} \subset \overline{\mathbb{K}(\varphi)}\backslash\{0\}$ and $v \in C^1\big(\overline{\mathbb{K}(\varphi)}\backslash\{0\}\big)^3$, $\varrho \in C^0\big(\overline{\mathbb{K}(\varphi)}\backslash\{0\}\big)$; hence, $T(v, \varrho + \alpha) n^{(\varphi,1)} \in L^2\big(\partial\mathbb{L}(\varphi, 1)\big)^3$. On the other hand, we know by Theorem 2.5 that the operator $\Gamma^*\big(-1, 2, \lambda, \mathbb{L}(\varphi, 1)\big)$ is onto. Therefore we may choose $\vartheta \in L^2\big(\partial\mathbb{L}(\varphi, 1)\big)^3$ with $\Gamma^*\big(-1, 2, \lambda, \mathbb{L}(\varphi, 1)\big)(\vartheta) = -T(v, \varrho + \alpha) n^{(\varphi,1)}$. The results discussed up to this point will now be exploited in order to establish the equation

$$v|\mathbb{L}(\varphi, 1) = \widetilde{V}^\lambda\big(\partial\mathbb{L}(\varphi, 1)\big)(\vartheta)|\mathbb{L}(\varphi, 1). \tag{5.8}$$

Take $h \in C_0^\infty(\mathbb{L}(\varphi, 1))^3$. By (2.8) and assumptions on $\Gamma^*(-1, q, \lambda, \mathbb{L}(\varphi, 1))$, we may choose $\Psi \in L^q(\partial\mathbb{L}(\varphi, 1))^3$ with

$$\Gamma^*\big(-1, q, \lambda, \mathbb{L}(\varphi, 1)\big)(\Psi) = T\big(\widetilde{R}^\lambda(h), S(h)\big) n^{(\varphi,1)}. \tag{5.9}$$

Taking account of (5.7), we get by our first Green's formula (Theorem 3.2) that

$$\begin{aligned} &\int_{\mathbb{L}(\varphi,1)} h \widetilde{V}^\lambda\big(\partial\mathbb{L}(\varphi, 1)\big)(\vartheta)\, dx \qquad (5.10)\\ &= -\int_{\partial\mathbb{L}(\varphi,1)\cap\mathcal{M}} \big(\widetilde{R}^\lambda(h) + \widetilde{V}^\lambda\big(\partial\mathbb{L}(\varphi, 1)\big)(\Psi)\big)\Gamma^*\big(-1, 2, \lambda, \mathbb{L}(\varphi, 1)\big)(\vartheta)\, d\mathbb{L}(\varphi, 1). \end{aligned}$$

Our second Green's formula (Theorem 4.4) implies

$$\begin{aligned} &\int_{\mathbb{L}(\varphi,1)} hv\, dx \qquad (5.11)\\ &= \int_{\partial\mathbb{L}(\varphi,1)\cap\mathcal{M}} \big(\widetilde{R}^\lambda(h) + \widetilde{V}^\lambda\big(\partial\mathbb{L}(\varphi, 1)\big)(\Psi)\big)\big(T(v, \varrho + \alpha) n^{(\varphi,1)}\big)\, d\mathbb{L}(\varphi, 1). \end{aligned}$$

It follows from (5.10), (5.11) and our choice of ϑ that

$$\int_{\mathbb{L}(\varphi,1)} h\big(\widetilde{V}^\lambda\big(\partial\mathbb{L}(\varphi, 1)\big)(\vartheta) - v\big)\, dx = 0.$$

Since h was arbitrarily chosen in $C_0^\infty(\mathbb{L}(\varphi,1))^3$, we have established (5.8). This result combined with (1.7), Lemma 3.4 and the definition of $W(\lambda,\mathbb{K}(\varphi))(\Phi)$ implies

$$\Gamma\big(1,p,\lambda,\mathbb{K}(\varphi)\big)(\Phi)\Big|\partial\mathbb{K}(\varphi)\backslash\mathbb{B}_3(0,1/\sin\varphi)$$
$$= \widetilde{V}^\lambda\big(\partial\mathbb{L}(\varphi,1)\big)(\vartheta)\Big|\partial\mathbb{K}(\varphi)\backslash\mathbb{B}_3(0,1/\sin\varphi) \in L^2\big(\partial\mathbb{K}(\varphi)\backslash\mathbb{B}_3(0,1/\sin\varphi)\big)^3.$$

We may now conclude $\Gamma\big(1,p,\lambda,\mathbb{K}(\varphi)\big)(\Phi)\in L^2\big(\partial\mathbb{K}(\varphi)\big)^3$. On the other hand, one of our assumptions states that $\Gamma\big(-1,p,\lambda,\mathbb{K}(\varphi)\big)(\Phi)$ vanishes, so we have shown $\Phi\in L^2\big(\partial\mathbb{K}(\varphi)\big)^2$. But the operator $\Gamma\big(-1,2,\lambda,\mathbb{K}(\varphi)\big)$ is one-to-one (Theorem 2.5); hence, it follows that $\Phi=0$. □

In the rest of this section, we consider the situation arising when Theorem 1.3 is assumed to be false. A preliminary result in this respect is

Theorem 5.2. *Let* $\varphi\in(0,\pi/2)$, $\vartheta\in[0,\pi)$, $q\in(1,2]$, $p\in(2,\infty)$ *with* $p'\le q<3p'/2$. *Assume the parameters* q,φ *satisfy* (4.1) *and the operator* $\Gamma^*\big(-1,q,e^{i\vartheta},\mathbb{L}(\varphi,1)\big)$ *is onto. Further suppose there are numbers* $r\in(3p/2,\infty)$, $C_1>0$, $C_2\ge 0$ *such that*

$$\|u(f,Me^{i\vartheta},\varphi)\|_r\le C_1M^{-1}\|f\|_r \text{ for } f\in L^r(\Omega_{\pi-\varphi})^3,\ M\in(C_2,\infty), \tag{5.12}$$

where the set $\Omega_{\pi-\varphi}$ *and the function* $u(f,Me^{i\vartheta},\varphi)$ *were introduced in Section* 1 (*see* (1.4) *and Theorem* 1.2). *Then the operator* $\Gamma(-1,p,e^{i\vartheta},\mathbb{K}(\varphi))$ *is one-to-one.*

Proof. Abbreviate $\lambda:=e^{i\vartheta}$, $V:=\mathbb{R}^3\backslash\overline{\mathbb{K}(\varphi)}$. Let $\Phi\in L^p\big(\partial\mathbb{K}(\varphi)\big)^3$ with $\Gamma\big(-1,p,\lambda,\mathbb{K}(\varphi)\big)(\Phi)=0$. In view of Theorem 5.1, we intend to prove that $W(\lambda,V)(\Phi)=0$. To this end, let $n\in\mathbb{N}$, and put $\Phi^{(n)}(x):=\Phi(nx)$ for $x\in\partial\mathbb{K}(\varphi)$. We claim that

$$\begin{aligned}&W(n^2\lambda,V)(\Phi^{(n)})|\Omega_{\pi-\varphi}\in W^{1,2}(\Omega_{\pi-\varphi})^3,\\&P(n^2\lambda,V)(\Phi^{(n)})|\Omega_{\pi-\varphi}\in L^2(\Omega_{\pi-\varphi}).\end{aligned}\tag{5.13}$$

For a proof of this assertion, choose a function $\widetilde{\alpha}\in C_0^\infty(\mathbb{R}^3)$ with $0\le\widetilde{\alpha}\le 1$, $\widetilde{\alpha}|\mathbb{B}_3(0,1)=1$, $\widetilde{\alpha}|\mathbb{R}^3\backslash\mathbb{B}_3(0,3/2)=0$. Define $\widetilde{\Phi}:\partial\Omega_{\pi-\varphi}\mapsto\mathbb{C}^3$ by $\widetilde{\Phi}(x):=\widetilde{\alpha}(x)\Phi^{(n)}(x)$ for $x\in\partial\Omega_{\pi-\varphi}\cap\mathbb{B}_3(0,2)$, $\widetilde{\Phi}(x):=0$ for $x\in\partial\Omega_{\pi-\varphi}\backslash\mathbb{B}_3(0,2)$. Our assumption on Φ implies $\Gamma\big(-1,p,n^2\lambda,\mathbb{K}(\varphi)\big)(\Phi^{(n)})=0$. It follows that

$$\Gamma\big(1,p,n^2\lambda,\Omega_{\pi-\varphi}\big)(\widetilde{\Phi})\Big|\partial\Omega_{\pi-\varphi}\cap\mathbb{B}_3(0,1/2)\in W^{1,2}\big(\partial\Omega_{\pi-\varphi}\cap\mathbb{B}_3(0,1/2)\big)^3. \tag{5.14}$$

On the other hand, we know by Theorem 2.6 that $W(n^2\lambda, V)(\widetilde{\alpha}\Phi^{(n)}) \in C^1(\overline{V}\backslash\{0\})^3$. This result combined with (5.14) yields $\Gamma(1,p,n^2\lambda,\Omega_{\pi-\varphi})(\widetilde{\Phi}) \in W^{1,2}(\partial\Omega_{\pi-\varphi})^3$. Now we may apply [11, Theorem 2.6] to obtain

$$W(n^2\lambda,\Omega_{\pi-\varphi})(\widetilde{\Phi}) \in W^{1,2}(\Omega_{\pi-\varphi})^3, \quad P(n^2\lambda,\Omega_{\pi-\varphi})(\widetilde{\Phi}) \in L^2(\Omega_{\pi-\varphi}). \quad (5.15)$$

As a consequence of Theorem 2.6, and because the function $(1-\widetilde{\alpha})\cdot\Phi^{(n)}$ vanishes near the origin, we get

$$W(n^2\lambda, V)((1-\widetilde{\alpha})\Phi^{(n)}) \in C^1(\overline{V})^3, \quad P(n^2\lambda, V)((1-\widetilde{\alpha})\Phi^{(n)}) \in C^0(\overline{V}). \quad (5.16)$$

Combining the relations in (5.15) and (5.16) yields (5.13). We further point out that by [11, Corollary 4.2], it holds that $\Phi \in L^r\big(\partial\mathbb{K}(\varphi)\big)^3$. Thus it follows from [11, Corollary 2.2] that $W(n^2\lambda,\Omega_{\pi-\varphi})(\widetilde{\Phi}) \in L^r(\Omega_{\pi-\varphi})^3$; hence, $W(n^2\lambda, V)(\Phi^{(n)})|\Omega_{\pi-\varphi} \in L^r(\Omega_{\pi-\varphi})^3$. We abbreviate $W_n := W(\lambda, V)(\Phi)(n\cdot)$, $P_n := nP(\lambda, V)(\Phi)(n\,\cdot)$. It holds that

$$W_n = W(n^2\lambda, V)(\Phi^{(n)}), \qquad P_n = P(n^2\lambda, V)(\Phi^{(n)}) + \beta_n,$$

with some constant $\beta_n \in \mathbb{C}$, as may be seen by some easy computations. As a consequence, we have $W_n|\Omega_{\pi-\varphi} \in W^{1,2}(\Omega_{\varphi-\pi})^3 \cap L^r(\Omega_{\pi-\varphi})^3 \cap W^{2,r}_{loc}(\Omega_{\pi-\varphi})^3$, $P_n|\Omega_{\pi-\varphi} \in L^2(\Omega_{\varphi-\pi}) \cap W^{1,r}_{loc}(\Omega_{\pi-\varphi})$. We further note that by Theorem 2.6 and the choice of Φ, we have $W_n \in C^1(\overline{V}\backslash\{0\})^3$, and there is $\pi \in C^0(\overline{V}\backslash\{0\})$ with $\pi|V = P_n$. Put $g_n := \operatorname{div}(\widetilde{\alpha}W_n|\overline{\Omega_{\pi-\varphi}}) = \nabla\widetilde{\alpha}\cdot W_n|\overline{\Omega_{\pi-\varphi}}$, where we used the fact that $\operatorname{div} W(\lambda, V)(\Phi) = 0$. Since $\nabla\widetilde{\alpha}(x) = 0$ for $x \in \mathbb{B}_3(0,1)$, the preceding results imply $g_n \in C^1(\overline{\Omega_{\pi-\varphi}})$. By the choice of $\widetilde{\alpha}$ and Φ, we have $g_n|\partial\Omega_{\pi-\varphi} = 0$; hence, $g_n|\Omega_{\pi-\varphi} \in W^{1,r}_0(\Omega_{\pi-\varphi})$. Moreover the function g_n has mean value zero over $\Omega_{\pi-\varphi}$. Thus we may apply to g_n the operator $\mathcal{D} := \mathcal{D}(\Omega_{\pi-\varphi}, r)$ introduced in Theorem 2.2. This is true for any $n \in \mathbb{N}$.

Since $\Phi \in L^r\big(\partial\mathbb{K}(\varphi)\big)^3$, as remarked above, Corollary 5.1 shows there is a sequence (γ_n) in $\mathbb{C}$ and a constant $C > 0$ such that inequality (5.5) holds for all $n \in \mathbb{N}$. Put $v_n := (\widetilde{\alpha}W_n)|\Omega_{\pi-\varphi} - \mathcal{D}(g_n)$, $\varrho_n := \widetilde{\alpha}(P_n + n\gamma_n)|\Omega_{\pi-\varphi}$, $f_n := (-\Delta\widetilde{\alpha}W_n - \sum_{k=1}^3 D_k\widetilde{\alpha}D_kW_n + \Delta\mathcal{D}(g_n) - n^2\lambda\mathcal{D}(g_n) + \nabla\widetilde{\alpha}(P_n + n\cdot\gamma_n))|\Omega_{\pi-\varphi}$ for $n \in \mathbb{N}$. Then $f_n \in L^r(\Omega_{\pi-\varphi})^3$, $v_n \in W^{1,2}_0(\Omega_{\pi-\varphi})^3 \cap L^r(\Omega_{\pi-\varphi})^3 \cap W^{2,r}_{loc}(\Omega_{\pi-\varphi})^3$, $\varrho_n \in L^2(\Omega_{\pi-\varphi}) \cap W^{1,r}_{loc}(\Omega_{\pi-\varphi})$, $-\Delta v_n + n^2\lambda v_n + \nabla\varrho_n = f_n$, $\operatorname{div} v_n = 0$ for $n \in \mathbb{N}$. At this point we may apply our assumption (5.12), which yields

$$\|v_n\|_r \le C_1 n^{-2}\|f_n\|_r \qquad \text{for } n \in \mathbb{N},\ n > (C_2)^{1/2}.$$

Thus we find for such n that

$$\begin{aligned}&\|W(\lambda,V)(\Phi)|V\cap\mathbb{B}_3(0,n)\|_r = n^{3/r}\|W_n|V\cap\mathbb{B}_3(0,1)\|_r \le n^{3/r}\|W_n|\Omega_{\pi-\varphi}\|_r\\ &\le n^{3/r}\big(\|v_n\|_r+\|\mathcal{D}(g_n)\|_r\big)\le \mathcal{C}n^{3/r}\big(n^{-2}\|f_n\|_r+\|\mathcal{D}(g_n)\|_r\big)\\ &\le \mathcal{C}n^{3/r}\big(n^{-2}\big(\|W_n|\mathcal{V}\|_{1,r}+\|(P_n+n\gamma_n)|\mathcal{V}\|_r+\|\Delta\mathcal{D}(g_n)\|_r+n^2\|\mathcal{D}(g_n)\|_r\big)\\ &+\|\mathcal{D}(g_n)\|_r\big),\ \text{with}\ \ \mathcal{V}:=V\cap\big(\mathbb{B}_3(0,2)\backslash\overline{\mathbb{B}_3(0,1/2)}\big).\end{aligned}$$

Here and in the following, the letter $\mathcal{C}$ denotes constants which do not depend on n. Referring to Theorem 2.2, we may conclude for $n\in\mathbb{N},\ n>(C_2)^{1/2}$ that

$$\begin{aligned}&\|W(\lambda,V)(\Phi)|V\cap\mathbb{B}_3(0,n)\|_r \qquad (5.17)\\ &\le \mathcal{C}n^{3/r}\big(n^{-2}\big(\|W_n|\mathcal{V}\|_{1,r}+\|(P_n+n\gamma_n)|\mathcal{V}\|_r+n^2\|W_n|\mathcal{V}\|_r\big)+\|W_n|\mathcal{V}\|_r\big)\\ &\le \mathcal{C}n^{3/r}\big(\|W_n|\mathcal{V}\|_r+n^{-2}\|\nabla W_n|\mathcal{V}\|_r+n^{-2}\|(P_n+n\gamma_n)|\mathcal{V}\|_r\big).\end{aligned}$$

Put $\mathcal{U}:=V\cap\big(\mathbb{B}_3(0,4)\backslash\mathbb{B}_3(0,1/4)\big),\mathcal{B}_n:=V\cap\big(\mathbb{B}_3(0,4n)\backslash\overline{\mathbb{B}_3(0,n/4)}\big)$ for $n\in\mathbb{N}$. Then we get by (5.17), Corollary 5.1 and (2.7), for $n\in\mathbb{N},\ n>(C_2)^{1/2}$ that

$$\begin{aligned}&\|W(\lambda,V)(\Phi)|V\cap\mathbb{B}_3(0,n)\|_r \qquad (5.18)\\ &\le \mathcal{C}n^{3/r}\Big(\|W_n|\mathcal{U}\|_r+\|Z^{(\varphi)}(\Phi)(n\,\cdot)|\mathcal{U}\|_r+\|\big(\nabla S^{(\varphi)}(\Phi)\big)(n\,\cdot)|\mathcal{U}\|_r\big)\\ &=\mathcal{C}\big(\|W(\lambda,V)(\Phi)|\mathcal{B}_n\|_r+\|Z^{(\varphi)}(\Phi)|\mathcal{B}_n\|_r+\|\nabla S^{(\varphi)}(\Phi)|\mathcal{B}_n\|_r\big)\\ &\le \mathcal{C}\Big(\int_{\mathcal{B}_n}\Big(\int_{\partial\mathbb{K}(\varphi)}|x-y|^{-2}|\Phi(y)|\,d\mathbb{K}(\varphi)(y)\Big)^r\,dx\Big)^{1/r}.\end{aligned}$$

By [11, Corollary 4.2], we have $\Phi\in L^s\big(\partial\mathbb{K}(\varphi)\big)^3$ for $s\in[p,\infty)$. Since $r>3p/2$, it follows from Lemma 3.7 and Lebesgue's theorem on dominated convergence that the right-hand side in (5.18) tends to zero for $n\to\infty$. This implies Theorem 5.2 □

6. On potential functions on $\partial\mathbb{K}(\varphi)$. In this section we shall repeat or slightly modify some results from [10] pertaining to certain potential functions on $\partial\mathbb{K}(\varphi)$. These potential functions, except those already introduced in Section 1 or 2, are presented in the ensuing definition.

Definition 6.1. Let $\varphi\in(0,\pi/2]$. For $\gamma\in\{1,3\},\ A\subset\mathbb{R}^2,\ B\subset\partial\mathbb{K}(\varphi),\ f: A\mapsto\mathbb{C}^\gamma,\ \Phi: B\mapsto\mathbb{C}^\gamma$, we denote by $\widetilde{f}$ the zero extension of f to $\mathbb{R}^2$, and by $\widetilde{\Phi}$ the zero extension of Φ to $\partial\mathbb{K}(\varphi)$. Let $R \in (0,\infty),\ S \in [1,\infty),\ p \in (1,\infty),\ \lambda \in \mathbb{C}\backslash(-\infty,0],\ \tau \in \{-1,1\}$. Then we define the operators

$$\Pi^*\big(\tau,p,\mathbb{K}(\varphi)\big) : L^p\big(\partial\mathbb{K}(\varphi)\big) \mapsto L^p\big(\partial\mathbb{K}(\varphi)\big),$$
$$H^* := H^*(\tau,p,\varphi,R,S) \ :\ L^p\big(\mathbb{R}^2\backslash\mathbb{B}_2(0,RS)\big)^3 \mapsto L^p\big(\mathbb{R}^2\backslash\mathbb{B}_2(0,R)\big)^3$$

by

$$\begin{aligned}&\Pi^*\big(\tau,p,\mathbb{K}(\varphi)\big)(\Phi)(x)\\ &:= \tau\Phi(x)/2 + \int_{\partial K(\varphi)} (4\pi)^{-1}\big(n^{(\varphi)}(x)(x-y)\big)|x-y|^{-3}\Phi(y)\ d\mathbb{K}(\varphi)(y),\\ &H^*(h)(\xi) \ :=\ \tau\widetilde{h}(\xi)/2 - \Big(\int_{\mathbb{R}^2\backslash\mathbb{B}_2(0,RS)} (4\pi)^{-1}\big(g^{(\varphi)}(\xi)\ -\ g^{(\varphi)}(\eta)\big)_l\\ &\times \big|g^{(\varphi)}(\xi)\ -\ g^{(\varphi)}(\eta)\big|^{-3}\big((n^{(\varphi)}\circ g^{(\varphi)})(\eta)h(\eta)\big)\sin^{-1}(\varphi)d\eta\Big)_{1\le l\le 3}\end{aligned}$$

for $\Phi \in L^p\big(\partial\mathbb{K}(\varphi)\big),\ x \in \partial\mathbb{K}(\varphi),\ h \in L^p\big(\mathbb{R}^2\backslash\mathbb{B}_2(0,RS)\big)^3,\ \xi \in \mathbb{R}^2\backslash\mathbb{B}_2(0,R)$; compare [10, Definition 6.1, (6.10)]. We further define

$$\begin{aligned}&\Gamma^{(inf)} := \Gamma^{(inf)}(\tau,p,\lambda,\varphi,R) : L^p(g^{(\varphi)}(\mathbb{R}^2\backslash\mathbb{B}_2(0,R)))^3\\ &\qquad \mapsto L^p\big(g^{(\varphi)}(\mathbb{R}^2\backslash\mathbb{B}_2(0,R))\big)^3,\\ &\Gamma^{(ver)} := \Gamma^{(ver)}(\tau,p,\lambda,\varphi,R) : L^p\big(g^{(\varphi)}\big(\mathbb{B}_2(0,R)\big)\big)^3 \mapsto L^p\Big(g^{(\varphi)}\big(\mathbb{B}_2(0,R)\big)\Big)^3,\\ &J := J(\tau,p,\lambda,\varphi,R,S) : L^p\big(\mathbb{R}^2\backslash\mathbb{B}_2(0,RS)\big)^3 \mapsto L^p\big(\mathbb{R}^2\backslash\mathbb{B}_2(0,R)\big)^3,\\ &F^* := F^*(\tau,p,\varphi,R,S) : L^p\big(\mathbb{R}^2\backslash\mathbb{B}_2(0,RS)\big) \mapsto L^p\big(\mathbb{R}^2\backslash\mathbb{B}_2(0,R)\big)\end{aligned}$$

by setting

$$\begin{aligned}&\Gamma^{(inf)}(\Phi) := \Gamma\big(\tau,p,\lambda,\mathbb{K}(\varphi)\big)(\widetilde{\Phi})|g^{(\varphi)}\big(\mathbb{R}^2\backslash\mathbb{B}_2(0,R)\big),\\ &\Gamma^{(ver)}(\Psi) := \Gamma\big(\tau,p,\lambda,\mathbb{K}(\varphi)\big)(\widetilde{\Psi})|g^{(\varphi)}\big(\mathbb{B}_2(0,R)\big),\\ &J(h) := \big[\Gamma\big(\tau,p,\lambda,\mathbb{K}(\varphi)\big)\big(\widetilde{h}\circ(g^{(\varphi)})^{-1}\big)\big]\circ g^{(\varphi)}|\mathbb{R}^2\backslash\mathbb{B}_2(0,R),\\ &F^*(f) := -\big[\Pi^*\big(-\tau,p,\mathbb{K}(\varphi)\big)\big(\widetilde{f}\circ(g^{(\varphi)})^{-1}\big)\big]\circ g^{(\varphi)}|\mathbb{R}^2\backslash\mathbb{B}_2(0,R)\end{aligned}$$

for $\Phi \in L^p(g^{(\varphi)}(\mathbb{R}^2\backslash\mathbb{B}_2(0,R)))^3,\ \Psi \in L^p(g^{(\varphi)}(\mathbb{B}_2(0,R)))^3,\ h \in L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0, RS))^3,\ f \in L^p(\mathbb{R}^2\backslash\mathbb{B}_2(0,RS))$.

For explicit representations of $\Gamma^{(inf)}, \Gamma^{(ver)}, J$ and F^* by integrals see [10, Definition 6.1, (6.8), (6.11)]. The integral appearing in the definition of H^* is to be understood as a principal value in $L^p\big(\mathbb{R}^2\backslash\mathbb{B}_2(0,R)\big)$ (see [10, Corollary 4.2]), whereas the integral defining $\Pi^*\big(\tau,p,\mathbb{K}(\varphi)\big)$ exists as a standard Lebesgue integral ([10, Lemma 6.2]).

If X and Y are Banach spaces and $\mathcal{T} : X \mapsto Y$ is a linear bounded operator with closed range and finite-dimensional kernel, we say that $\mathcal{T}$ is an F_+-operator, or more shortly, that $\mathcal{T}$ is F_+.

We note some obvious properties of the operators introduced above:

Lemma 6.1. *Let $\varphi, R, S, p, \lambda, \tau$ be given as in the preceding definition. Then the operator $\Pi^*(\tau,p,\mathbb{K}(\varphi))$ is adjoint to $\Pi(\tau,p',\mathbb{K}(\varphi))$. The operator $J(\tau,p,\lambda,\varphi,R,1)$ is Fredholm (F_+) if and only if $\Gamma^{(inf)}(\tau,p,\lambda,\varphi,R)$ is Fredholm (F_+). If these two operators are F_+, they have the same index.*

Lemma 6.2. *Let φ, R, S, p, λ, τ be given as in Definition* 6.1. *Denote by $\mathcal{I}_{RS/\mu}, \mathcal{I}_R$ the identical mapping of the sets $\mathbb{R}^2\backslash\mathbb{B}_2(0,RS/\mu)$ and $\mathbb{R}^2\backslash\mathbb{B}_2(0,R)$, respectively ($\mu \in (0,\infty)$). Then it holds for $f \in L^2\big(\mathbb{R}^2\backslash\mathbb{B}_2(0,RS)\big)^3$, $\mu \in (0,\infty)$ that*

$$J(\tau,p,\lambda,\varphi,R,S)(f) = \big(J(\tau,p,\mu^2\lambda,\varphi,R/\mu,S)\big(f\circ(\mu\mathcal{I}_{RS/\mu})\big)\big)\circ\big((1/\mu)\mathcal{I}_R\big).$$

We further state some results which may not be completely obvious but still are rather easy to prove.

Lemma 6.3. *For φ, R, S, p, τ as in Definition* 6.1, *and for $h \in L^p\big(\mathbb{R}^2\backslash\mathbb{B}_2(0,RS)\big)^3$, it holds that*

$$(n^{(\varphi)}\circ g^{(\varphi)})H^*(\tau,p,\varphi,R,S)(h) = F^*(\tau,p,\varphi,R,S)\big((n^{(\varphi)}\circ g^{(\varphi)})h\big).$$

If $F^*(\tau,p,\varphi,R,1)$ is an F_+-operator, the mapping $H^*(\tau,p,\varphi,R,1)$ admits the same property.

The first statement of this lemma corresponds to [10, Lemma 6.8]. The second one follows by a variant of the arguments in [10, page 225].

Lemma 6.4. *Take $\varphi, R, S, p, \lambda, \tau$ as in Definition* 6.1. *Then the operator $\Gamma\big(\tau,p,\lambda,\Omega_{\pi-\varphi}\big)$ is F_+ if and only if the operator $\Gamma^{(ver)}(-\tau,p,\lambda,\varphi,R)$ has the same property. In case these operators are F_+, they have the same index.*

This lemma is a consequence of the choice of $\Omega_{\pi-\varphi}$. Its proof is almost identical to that of [10, Lemma 13.7].

Lemma 6.5. *Let $\varphi, R, S, p, \lambda, \tau$ be given as in Definition* 6.1. *Then the operators $\Gamma^{(inf)} := \Gamma^{(inf)}(\tau, p, \lambda, \varphi, R)$ and $\Gamma^{(ver)} := \Gamma^{(ver)}(\tau, p, \lambda, \varphi, R)$ are Fredholm (F_+) if and only if $\Gamma := \Gamma\big(\tau, p, \lambda, \mathbb{K}(\varphi)\big)$ is Fredholm (F_+). In case these operators are F_+, it holds that*

$$index(\Gamma) = index(\Gamma^{(inf)}) + index(\Gamma^{(ver)}).$$

If $\Pi^*\big(\tau, p, \mathbb{K}(\varphi)\big)$ is Fredholm, then $F^*(-\tau, p, \varphi, R, S)$ is an F_+-operator.

The operator $J(\tau, p, \lambda, \varphi, R, S)$ is F_+ if and only if $J(\tau, p, \lambda, \varphi, RS, 1)$ has the same property.

If $R_1, R_2 \in (0, \infty)$, the operators $J(\tau, p, \lambda, \varphi, R_1, S)$ and $J(\tau, p, \lambda, \varphi, R_2, S)$ are linked in the same way.

This lemma is proved by choosing suitable compact operators which are added to or subtracted from the preceding operators. For details we refer to [10, Lemmas 6.9, 6.10, Corollary 6.1–6.3].

Lemma 6.6. *Take $\varphi, R, S, p, \lambda, \tau$ as in Definition* 6.1. *Let $\epsilon \in (0, \infty)$. If $J(\tau, p, \lambda, \varphi, R, 1)$ is a Fredholm operator with index zero, then $\Gamma(\tau, p, \lambda, \mathbb{L}(\varphi, \epsilon))$ has the same property.*

If $F^*(\tau, p, \varphi, R, S)$ is F_+, then $\Pi^*\big(-\tau, p, \mathbb{L}(\varphi, \epsilon)\big)$ is also F_+.

This lemma is valid because the boundary of $\mathbb{L}(\varphi, \epsilon)$ is smooth, so the operators $J(\tau, p, \lambda, \varphi, R, S)$ and $\Gamma\big(\tau, p, \lambda, \mathbb{L}(\varphi, \epsilon)\big)$ differ only by compact operators, modulo a transformation into local coordinates. The same remark holds with respect to $F^*(\tau, p, \varphi, R, S)$ and $\Pi^*\big(-\tau, p, \mathbb{L}(\varphi, \epsilon)\big)$; see [10, Corollaries 6.1, 6.2, 6.4].

Lemma 6.7. *Let $p \in (1, \infty)$, $\tau \in \{-1, 1\}$, $\varphi \in (0, \pi/2]$, $\epsilon > 0$. Suppose there is $\mathcal{C} > 0$ with*

$$\|\Phi\|_p \le \mathcal{C}\|\Pi\big(\tau, p, \mathbb{L}(\varphi, 1)\big)(\Phi)\|_p \qquad \textit{for } \Phi \in L^p\big(\partial\mathbb{L}(\varphi, 1)\big).$$

Then there is $\widetilde{\mathcal{C}} > 0$ with

$$\|\Phi\|_p \le \widetilde{\mathcal{C}}\|\Pi\big(\tau, p, \mathbb{K}(\varphi)\big)(\Phi)\|_p \qquad \textit{for } \Phi \in L^p\big(\partial\mathbb{K}(\varphi)\big).$$

This lemma is shown by a scaling argument; see [10, Lemmas 12.4, 12.5].

Lemma 6.8. *Let $p \in (1, \infty)$, $\tau \in \{-1, 1\}$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $\varphi_1, \varphi_2 \in (0, \pi/2]$ with $\varphi_1 < \varphi_2$. Assume that $\Pi^*\big(\tau, p, \mathbb{K}(\varphi)\big)$ is F_+ for $\varphi \in [\varphi_1, \varphi_2]$, and $index\big(\Pi^*\big(\tau, p, \mathbb{K}(\varphi_1)\big)\big) = 0$. Then $index\big(\Pi^*\big(\tau, p, \mathbb{K}(\varphi_2)\big)\big) = 0$.*

Lemma 6.8 follows by a homotopy argument; see [10, Lemma 6.17].

We further state some more deep-lying results pertaining to the operators introduced in Definition 6.1 or in Section 1 or 2. (One such result–Theorem 2.5–was already presented in Section 2 and exploited in Section 5; it will be needed again later on.)

Theorem 6.1. ([10, Theorem 11.1]) *Let* $p \in (1,\infty)$, $\vartheta \in [0,\pi)$, $\varphi \in (0,\pi/2]$. *Then there exists* $\mathcal{C} > 0$ *such that for* $\lambda \in \mathbb{C}\backslash\{0\}$ *with* $|\arg\lambda| \leq \vartheta$, $|\lambda| \geq 1$, $N \in \mathbb{N}$, $N \geq 24, R, S \in (2,\infty)$, $\tau \in \{-1,1\}$, $f \in L^p\big(\mathbb{R}^2\backslash\mathbb{B}_2(0,RS)\big)^3$, *the following inequality holds true:*

$$\begin{aligned}&\|H^*(\tau,p,\varphi,R,S)(f)\|_p\\&\leq \mathcal{C}(N\|J(\tau,p,\lambda,\varphi,R,S)(f)\|_p + (N^{-\frac{1}{2}} + NS^{-(1\wedge(2/p))} + N^6|\lambda|^{-\frac{1}{2}})\|f\|_p).\end{aligned}$$

Theorem 6.2. ([10, Corollaries 9.3, 9.4]) *Let* $\varphi \in (0,\pi/2]$, $\tau \in \{-1,1\}$, $\lambda \in \mathbb{C}\backslash(-\infty,0]$, $\epsilon \in (0,\infty)$. *Then the operator* $\Pi\big(\tau,p,\mathbb{L}(\varphi,\epsilon)\big)$ *is one-to-one for all* $p \in (1,\infty)$, *and* $\Gamma^*\big(\tau,p,\lambda,\mathbb{L}(\varphi,\epsilon)\big)$ *is one-to-one for* $p \in (1,2]$.

Theorem 6.3. *Let* φ, R, p, λ, τ *be given as in Definition* 6.1. *Assume* $\Gamma^{(inf)}(\tau,p,\lambda,\varphi,R)$ *is a Fredholm operator with index zero. Then the operator* $F^*(\tau,p,\varphi,1,1)$ *is* F_+.

Proof. According to Lemma 6.1, the assumptions in Theorem 6.3 imply that $J(\tau,p,\lambda,\varphi,R,1)$ is Fredholm with index zero. In this situation, it was shown in [10, pages 232–234] that the operator $F^*(\tau,p,\varphi,1,1)$ is F_+. In fact, if we set $\vartheta := \arg\lambda$, $\mu := |\lambda|$ in [10, pages 232–234], the arguments given there as part of the proof of [10, Theorem 12.2] may be used here without any modification. □

Theorem 6.4. ([14, page 102], [10, Theorem 8.1], [11, Theorem 2.8]) *Let* $\varphi \in (0,\pi/2]$, $\tau \in \{-1,1\}$. *Then the set*

$$\{p \in (1,\infty) : \Pi^*(\tau,p,\mathbb{K}(\varphi))\textit{is not bijective}\}$$

is countable. For any $\varphi \in (0,\pi/2)$, *there is at least one exponent* $q \in (1,2)$ *such that* $\Pi\big(1,q,\mathbb{K}(\varphi)\big)$ *is not bijective.*

Theorem 6.5. ([28, Lemma I.2.1]) *Let* X, Y *be Banach spaces with norm* $\|\ \|_X, \|\ \|_Y$, *respectively, and let* $\mathcal{F} : X \mapsto Y$ *be a linear, bounded operator. Then* $\mathcal{F}$ *is* F_+ *if and only if there exists a constant* $\mathcal{C} > 0$ *and a linear compact operator* $T : X \mapsto Y$ *such that*

$$\|x\|_X \leq \mathcal{C}\big(\|\mathcal{F}(x)\|_Y + \|T(x)\|_Y\big) \qquad \textit{for } \ x \in X.$$

Making use of these results, we establish

Theorem 6.6. *Let* $p \in (1,\infty)$, $\varphi \in (0,\pi/2]$, $\tau \in \{-1,1\}$ *such that* $\Pi^*(-\tau,p,\mathbb{K}(\varphi))$ *is Fredholm. Take* $\lambda \in \mathbb{C}\backslash(-\infty,0]$, $R \in (0,\infty)$. *Then the operator* $\Gamma^{(inf)}(\tau,p,\lambda,\varphi,R)$ *is* F_+.

Proof. We modify the proof of [10, Theorem 12.1]. Since $\Pi^*\big(-\tau,p,\mathbb{K}(\varphi)\big)$ is Fredholm, the operator $F^*(R,S) := F^*(\tau,p,\varphi,R,S)$ is F_+, for $S \in [1,\infty)$ (Lemma 6.5). It follows by Lemma 6.3 that $H^*(R,S) := H^*(\tau,p,\varphi,R,S)$ is also an F_+-operator if $S = 1$. Thus, according to Theorem 6.5, there is a constant $\mathcal{C}_1 > 0$ and a linear compact operator $T : L^p\big(\mathbb{R}^2\backslash\mathbb{B}_2(0,R)\big)^3 \mapsto L^p\big(\mathbb{R}^2\backslash\mathbb{B}_2(0,R)\big)^3$ such that

$$\|g\|_p \le \mathcal{C}_1\big(\|H^*(R,1)(g)\|_p + \|T(g)\|_p\big) \quad \text{for} \quad g \in L^p\big(\mathbb{R}^2\backslash\mathbb{B}_2(0,R)\big)^3. \tag{6.1}$$

By Theorem 6.1, there exists $\mathcal{C}_2 > 0$ with

$$\begin{aligned} &\|H^*(R,S)(f)\|_p \\ &\le \mathcal{C}_2 N\|J(\mu,S)(f)\|_p + \mathcal{C}_2\big(N^{-1/2} + NS^{-\left(1\wedge(2/p)\right)} + N^6\mu^{-1/2}\big)\|f\|_p \end{aligned} \tag{6.2}$$

for $S \in (2,\infty)$, $\mu \in (1,\infty)$, $N \in \mathbb{N}$ with $N \ge 24$, $f \in L^p\big(\mathbb{R}^2\backslash\mathbb{B}_2(0,RS)\big)^3$, where the mapping $J(\mu,S)$ is defined by $J(\mu,S) := J(\tau,p,\mu|\lambda|^{-1}\lambda,\varphi,R,S)$.

If we choose N, S and μ large enough, we obtain from (6.1) and (6.2) that

$$\|f\|_p \le \mathcal{C}\big(\|J(\mu,S)(f)\|_p + \|T(\widetilde{f})\|_p\big) + (1/2)\|f\|_p \tag{6.3}$$

for $f \in L^p\big(\mathbb{R}^2\backslash\mathbb{B}_2(0,RS)\big)^3$, with a constant $\mathcal{C} > 0$ independent of f, and with $\widetilde{f}$ denoting the zero extension of f to $\mathbb{R}^2\backslash\mathbb{B}_2(0,R)$. Due to (6.3), we may again apply Theorem 6.5, to obtain that $J(\mu,S)$ is F_+. Theorem 6.6 now follows by Lemmas 6.1, 6.2 and 6.5. □

7. Proof of Theorem 1.3. Starting from Theorem 5.2, we use functional analytic arguments in order to derive Theorem 1.3. In the first step, we consider a situation as in Theorem 5.2. This time, though, we require in addition that $\Pi^*\big(1,p,\mathbb{K}(\varphi)\big)$ is bijective for some $p \in (2,\infty)$. Then the operator $\Gamma\big(-1,s,\lambda,\mathbb{K}(\varphi)\big)$ will turn out to be one-to-one not only if $s = p$, but for any $s \in [2,p]$.

Theorem 7.1. *Let the assumptions of Theorem* 5.2 *be valid. Suppose in addition the operator* $\Pi^*(1,p,\mathbb{K}(\varphi))$ *is bijective. Put* $\lambda := e^{i\vartheta}$. *Then the operator* $\Gamma(-1,s,\lambda,\mathbb{K}(\varphi))$ *is bijective for* $s \in [2,p]$.

Proof. Set

$$\Gamma^{(ver)} := \Gamma^{(ver)}\big(-1, p, \lambda, \varphi, 1\big), \quad \Gamma^{(inf)} := \Gamma^{(inf)}\big(-1, p, \lambda, \varphi, 1\big).$$

By [11, Theorem 2.5], the operator $\Gamma\big(1, p, \lambda, \Omega_{\pi-\varphi}\big)$ is Fredholm. This observation and Lemma 6.4 imply $\Gamma^{(ver)}$ is Fredholm too. On the other hand, by Theorem 6.6 and our assumption on $\Pi^*\big(1, p, \mathbb{K}(\varphi)\big)$, the operator $\Gamma^{(inf)}$ is F_+. Now it follows by Lemma 6.5 that $\Gamma\big(-1, p, \lambda, \mathbb{K}(\varphi)\big)$ has closed range. But Theorem 5.2 states the latter operator is one-to-one, and by [11, Corollary 4.4], its adjoint is one-to-one as well. Thus the operator $\Gamma\big(-1, p, \lambda, \mathbb{K}(\varphi)\big)$ must be bijective. According to Theorem 2.5, the operator $\Gamma\big(-1, 2, \lambda, \mathbb{K}(\varphi)\big)$ exhibits the same feature.

Now take $\Psi \in L^p\big(\partial\mathbb{K}(\varphi)\big)^3 \cap L^2\big(\partial\mathbb{K}(\varphi)\big)^3$. By the results established in the first part of this proof, we may choose $\Phi^{(2)} \in L^2\big(\partial\mathbb{K}(\varphi)\big)^3$, $\Phi^{(p)} \in L^p\big(\partial\mathbb{K}(\varphi)\big)^3$ with

$$\Gamma\big(-1, 2, \lambda, \mathbb{K}(\varphi)\big)(\Phi^{(2)}) = \Psi = \Gamma\big(-1, p, \lambda, \mathbb{K}(\varphi)\big)(\Phi^{(p)}).$$

Referring to [11, Corollary 4.2], we conclude that $\Phi^{(2)} \in L^p\big(\partial\mathbb{K}(\varphi)\big)^3$. But as mentioned above, Theorem 5.2 states the operator $\Gamma\big(-1, p, \lambda, \mathbb{K}(\varphi)\big)$ is one-to-one, so we obtain $\Phi^{(2)} = \Phi^{(p)}$. Now we may apply the Riesz-Thorin interpolation theorem to the inverse of the operators $\Gamma\big(-1, 2, \lambda, \mathbb{K}(\varphi)\big)$ and $\Gamma\big(-1, p, \lambda, \mathbb{K}(\varphi)\big)$. Then Theorem 7.1 follows by a standard argument. □

Next we remove some of the assumptions required in Theorem 7.1.

Theorem 7.2. *Let $\varphi \in (0, \pi/2)$ with $\gamma_1^{(\varphi)} \geq 2$ (see Section 2). Take $\vartheta \in [0, \pi)$, $K \in (2, \infty)$. Assume there are numbers $r \in [3K/2, \infty)$, $C_1 > 0$, $C_2 \geq 0$ such that inequality (5.12) is valid. Then the operator $\Gamma\big(-1, p, e^{i\vartheta}, \mathbb{K}(\varphi)\big)$ is bijective for any $p \in [2, K)$.*

Proof. Let $\epsilon \in \big(0, \min\{2, K-2\}\big)$. By Theorem 6.4, we may choose $p \in (1, \infty)$ such that $\min\{4, K\} - \epsilon < p < \min\{4, K\}$ and $\Pi^*\big(1, p, \mathbb{K}(\varphi)\big)$ is bijective. On the other hand, according to Theorem 2.5, the operator $\Gamma^*\big(-1, 2, e^{i\vartheta}, \mathbb{L}(\varphi, 1)\big)$ is onto. We further mention that due to the choice of φ, condition (4.1) is satisfied with $q = 2$. It further holds that $p' < 2 < 3p'/2$, $r > 3p/2$. Thus we are in a position to apply Theorem 7.1 with $q = 2$, which yields that the operator $\Gamma\big(-1, s, e^{i\vartheta}, \mathbb{K}(\varphi)\big)$ is bijective for $s \in [2, p]$. Recalling the choice of p, we conclude the latter operator is bijective for $s \in [2, \min\{4, K\})$. In particular, Theorem 7.2 is proved if $K \leq 4$.

Consider the case $K > 4$. Take $q \in \big(3/2, (3/2)(2r/3)'\big)$. This means in particular $q' \in (2,3)$; hence, the operator $\Gamma\big(-1, q', e^{i\vartheta}, \mathbb{K}(\varphi)\big)$ is bijective according to the first part of this proof. It follows with Lemma 6.5 that the operators

$$\Gamma^{(inf)} := \Gamma^{(inf)}(-1, q', e^{i\vartheta}, \varphi, 1) \quad \text{and} \quad \Gamma^{(ver)} := \Gamma^{(ver)}(-1, q', e^{i\vartheta}, \varphi, 1)$$

are Fredholm, and

$$0 = \text{index}(\Gamma^{(inf)}) + \text{index}(\Gamma^{(ver)}). \tag{7.1}$$

On the other hand, the operator $\Gamma(1, q', e^{i\vartheta}, \Omega_{\pi-\varphi})$ is Fredholm with index zero ([11, Theorem 2.5]); hence, we obtain by Lemma 6.4 that index $(\Gamma^{(ver)}) = 0$. Now equation (7.1) implies that the index of $\Gamma^{(inf)}$ vanishes as well. Referring to Lemma 6.1 and 6.6, we conclude the operator $\Gamma\big(-1, q', e^{i\vartheta}, \mathbb{L}(\varphi, 1)\big)$ is Fredholm with index zero. The adjoint $\Gamma^*\big(-1, q, e^{i\vartheta}, \mathbb{L}(\varphi, 1)\big)$ of this operator is one-to-one (Theorem 6.2); hence, it must be onto.

It holds that $2r/3 \geq K > 4$. Let $\epsilon \in (0, 2r/3 - 4)$. According to Theorem 6.4, there is some $p \in (2r/3 - \epsilon, 2r/3)$ such that $\Pi^*\big(1, p, \mathbb{K}(\varphi)\big)$ is bijective. Then $p' < 4/3 < q < 3p'/2,\ r > 3p/2$, and the parameters φ, q satisfy (4.1). We thus have shown that all assumptions of Theorem 7.1 are satisfied. This theorem, in turn, implies Theorem 7.2. □

Now we are in a position to conduct the

Proof of Theorem 1.3. Choose $\varphi_0 \in (0, \pi/2)$ with $\gamma_1^{(\varphi_0)} \geq 2$; see (2.4). Assume Theorem 1.3 is false. Then, for any $\varphi \in (0, \varphi_0],\ S \in (3, \infty)$, there are numbers $r \in [S, \infty),\ C_1 > 0,\ C_2 \geq 0$ such that inequality (5.12) is valid for $f \in C_0^\infty(\Omega_{\pi-\varphi})^3,\ M \in (C_2, \infty)$. By (1.6) and a density argument, it follows that (5.12) is valid for $f \in L^r(\Omega_{\pi-\varphi})^3,\ M \in (C_2, \infty)$.

Let $\varphi \in (0, \varphi_0]$. We deduce from (2.3) and the choice of φ_0 that $\gamma_1^{(\varphi)} \geq \gamma_1^{(\varphi_0)} \geq 2$. Thus we may apply Theorem 7.2 with $K = 2S/3$, for any $S \in (3, \infty)$. This theorem implies the operator $\Gamma\big(-1, p, e^{i\vartheta}, \mathbb{K}(\varphi)\big)$ is bijective for $p \in [2, \infty)$. Let $p \in (2, \infty)$. According to [11, Theorem 2.5], the operator $\Gamma\big(1, p, e^{i\vartheta}, \Omega_{\pi-\varphi}\big)$ is Fredholm with index zero, so we may conclude with Lemma 6.4 that $\Gamma^{(ver)} := \Gamma^{(ver)}(-1, p, e^{i\vartheta}, \varphi, 1)$ is also Fredholm with index zero. It follows with Lemma 6.5 that $\Gamma^{(inf)} := \Gamma^{(inf)}(-1, p, e^{i\vartheta}, \varphi, 1)$ has the same property. Now Theorem 6.3 yields that $F^*(-1, p, \varphi, 1, 1)$ is F_+; hence, we may conclude by Lemma 6.6 that $\Pi^*\big(1, p, \mathbb{L}(\varphi, 1)\big)$ is an operator of the same type. But its adjoint $\Pi\big(1, p', \mathbb{L}(\varphi, 1)\big)$ is one-to-one (Theorem

6.2). Therefore the open mapping theorem yields a constant $\mathcal{C} > 0$ with

$$\|\Phi\|_{p'} \leq \mathcal{C}\big\|\Pi\big(1, p', \mathbb{L}(\varphi, 1)\big)(\Phi)\big\|_{p'} \qquad \text{for } \Phi \in L^{p'}\big(\partial\mathbb{L}(\varphi, 1)\big).$$

By referring to Lemma 6.7, we may deduce from the preceding estimate there is $\overline{\mathcal{C}} > 0$ with

$$\|\Phi\|_{p'} \leq \overline{\mathcal{C}}\big\|\Pi\big(1, p', \mathbb{K}(\varphi)\big)(\Phi)\big\|_{p'} \qquad \text{for } \Phi \in L^{p'}\big(\partial\mathbb{K}(\varphi)\big).$$

This implies the operator $\Pi\big(1, p', \mathbb{K}(\varphi)\big)$ is one-to-one and has closed range. In particular it is F_+. Recall that φ was arbitrarily chosen in $(0, \varphi_0]$.

By [11, Theorem 2.9], there is some $\overline{\varphi} \in (0, \varphi_0)$ such that $\Pi\big(1, p', \mathbb{K}(\overline{\varphi})\big)$ is bijective. Since we have shown that for any $\varphi \in (0, \varphi_0]$, the operator $\Pi\big(1, p', \mathbb{K}(\varphi)\big)$ is F_+, we may use Lemma 6.8 to obtain

$$\operatorname{index}\big(\Pi\big(1, p', \mathbb{K}(\varphi)\big)\big) = 0 \qquad \text{for } \varphi \in [\overline{\varphi}, \varphi_0].$$

Thus the operator $\Pi\big(1, p', \mathbb{K}(\varphi_0)\big)$ is bijective. But the exponent p was arbitrarily chosen in $(2, \infty)$, so we have arrived at a contradiction to the second statement of Theorem 6.4, that is, to [11, Theorem 2.8]. □

8. The case $p < 2$. In this section, we prove Corollary 1.1 by reducing it to Theorem 1.3 via an interpolation argument.

Let $p \in [2, \infty)$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $\varphi \in (0, \pi/2]$. We define the operator $A_{p,\lambda,\varphi} : L^p(\Omega_{\pi-\varphi})^3 \mapsto L^p(\Omega_{\pi-\varphi})^3$ by setting $A_{p,\lambda,\varphi}(f) := u(f, \lambda, \varphi)$ for $f \in L^p(\Omega_{\pi-\varphi})^3$, where the function $u(f, \lambda, \varphi)$ was introduced in Theorem 1.2. Due to the uniqueness result in Theorem 1.2, this operator must be linear, and by (1.6), it is bounded. Furthermore it is in a certain sense symmetric, as is indicated by the ensuing lemma.

Lemma 8.1. *Let $p \in [2, \infty)$, $\lambda \in \mathbb{C}\backslash(-\infty, 0]$, $f \in C_0^\infty(\Omega_{\pi-\varphi})^3$. Denote the adjoint of $A_{p,\lambda,\varphi}$ by $(A_{p,\lambda,\varphi})'$. Then $(A_{p,\lambda,\varphi})'(f) = u(f, \lambda, \varphi)$, with $u(f, \lambda, \varphi)$ introduced in Theorem* 1.2.

Proof. Let $g \in C^\infty(\Omega_{\pi-\varphi})^3$. By [11, Theorem 5.1], we may choose functions $\Phi, \Psi \in L^2(\Omega_{\pi-\varphi})^3$ with

$$u(f, \lambda, \varphi) = \big(\widetilde{R}^\lambda(f) + \widetilde{V}^\lambda\big(\partial\Omega_{\pi-\varphi}\big)(\Phi)\big)\Big|\Omega_{\pi-\varphi},$$

$$u(g, \lambda, \varphi) = \big(\widetilde{R}^\lambda(g) + \widetilde{V}^\lambda\big(\partial\Omega_{\pi-\varphi}\big)(\Psi)\big)\Big|\Omega_{\pi-\varphi}.$$

In analogy to Lemma 3.5, we have

$$\int_\Omega f\big(\widetilde{R}^\lambda(g)+V(\Psi)\big)dx$$

$$= \int_{\partial\Omega}\sum_{j=1}^{3}\Big[\big(\widetilde{R}^\lambda(f)+V(\Phi)\big)_j\Big(\sum_{k=1}^{3}T_{jk}\big(\widetilde{R}^\lambda(g),S(g)\big)n_k-\Gamma^*\big(-1,2,\lambda,\Omega\big)(\Psi)\Big)$$

$$-\ \Big(\sum_{k=1}^{3}T_{jk}\big(\widetilde{R}^\lambda(f),S(f)\big)n_k-\Gamma^*\big(-1,2,\lambda,\Omega\big)(\Phi)\Big)\big(\widetilde{R}^\lambda(g)+V(\Psi)\big)_j\Big]d\Omega$$

$$+\int_\Omega\big(\widetilde{R}^\lambda(f)+V(\Phi)\big)g\ dx,$$

where $\Omega := \Omega_{\pi-\varphi}$, the letter n denotes the outward unit normal to $\Omega_{\pi-\varphi}$, and $V(\Phi) := \widetilde{V}^\lambda\big(\partial\Omega_{\pi-\varphi}\big)(\Phi)$, $V(\Psi) := \widetilde{V}^\lambda\big(\partial\Omega_{\pi-\varphi}\big)(\Psi)$. This result, which is well known, may be proved by a jump relation similar to the one in Theorem 3.1; compare the proof of [11, Lemma 5.1]. (In the case $f = g = 0$, the preceding formula coincides with [11, (5.2)], and in the case $g = 0,\ \Phi = 0$, with [11, (5.3)].) Since $u(g,\lambda,\varphi)|\partial\Omega_{\pi-\varphi} = 0 = u(f,\lambda,\varphi)|\partial\Omega_{\pi-\varphi}$, we conclude that

$$\int_{\Omega_{\pi-\varphi}} fu(g,\lambda,\varphi)\ dx = \int_{\Omega_{\pi-\varphi}} u(f,\lambda,\varphi)g\ dx.$$

Therefore Lemma 8.1 follows by the definition of $A_{p,\lambda,\varphi}$. □

Now we are in a position to give the

Proof of Corollary 1.1. Take $\varphi \in (0,\pi/2)$, $S \in (3,\infty)$, and assume that Corollary 1.1 is false. Since $S' \in (1,3/2)$, it follows that there are numbers $r \in (1,S']$, $C_1 > 0$, $C_2 \geq 0$ such that

$$\big\|u(f,Me^{i\vartheta},\varphi)\big\|_r \leq C_1M^{-1}\big\|f\big\|_r \tag{8.1}$$

for $f \in C_0^\infty(\Omega_{\pi-\varphi})^3, M \in (C_2,\infty)$. Due to Lemma 8.1, we may conclude that

$$\big\|(A_{r',Me^{i\vartheta},\varphi})'(f)\big\|_r \leq C_1M^{-1}\big\|f\big\|_r \tag{8.2}$$

for f, M as in (8.1). But the operator $(A_{r',Me^{i\vartheta},\varphi})'$ is bounded, so inequality (8.2) holds for any $f \in L^r(\Omega_{\pi-\varphi})^3$. It follows that

$$\big\|A_{r',Me^{i\vartheta},\varphi}(f)\big\|_{r'} \leq C_1M^{-1}\big\|f\big\|_{r'}$$

for $f \in L^{r'}(\Omega_{\pi-\varphi})^3, M \in (C_2,\infty)$. Since $r' \geq S$, and because S was arbitrarily chosen in $(3,\infty)$, we have arrived at a contradiction to Theorem 1.3.

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