On Volume Potentials Related to the Time-dependent Oseen System

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Abstract: The article treats volume potentials associated with the time-dependent Oseen system. It is shown that under suitable assumptions on the weight function, these potentials belong to a certain Sobolev space on the boundary of a domain. This result is an element in the construction of a representation formula for exterior evolutionary Oseen flows.

Key Words: Exterior domain, Navier-Stokes system, time-dependent Oseen system.

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1 Introduction

For a bounded, open set $\Omega \subset \mathbb{R}^3$, we consider the following initial-boundary value problem for the time-dependent Oseen system,

$$\begin{aligned}
&\partial_t u(x,t) - \Delta_x u(x,t) + \tau \cdot \partial_{x_1} u(x,t) \qquad (1)\\
&\quad +\nabla_x \pi(x,t) = f(x,t), \;\; \mathrm{div}_x u(x,t) = 0\\
&\quad \text{for } x \in \mathbb{R}^3 \backslash \overline{\Omega}, \;\; t \in (0,\infty),\\
&u(x,t) = 0 \;\text{ for }\; x \in \partial\Omega, \;\; t \in (0,\infty), \qquad (2)\\
&u(x,t) \to 0 \;(|x| \to \infty) \;\text{ for }\; t \in (0,\infty), \qquad (3)\\
&u(x,0) = a(x) \;\text{ for }\; x \in \mathbb{R}^3 \backslash \overline{\Omega}, \qquad (4)
\end{aligned}$$

where the velocity $u : (\mathbb{R}^3 \backslash \overline{\Omega}) \times (0,\infty) \mapsto \mathbb{R}^3$ and the pressure $\pi : (\mathbb{R}^3 \backslash \overline{\Omega}) \times (0,\infty) \mapsto \mathbb{R}$ are the unknowns. In [1], we explained why this problem is of interest, and we mentioned previous studies dealing with it. Also in [1], we indicated how to construct a solution to (1) - (4) by means of the method of integral equations. This method automatically yields an integral representation of u and π. Such a representation, in turn, may be used to obtain estimates of the spatial decay of the solutions of (1) - (4).

For the convenience of the reader, in Section 2 and 3 below we will indicate the main steps of the approach from [1]. Also some auxiliary results are stated in those sections. In Section 4, it will be shown that a volume potential related to (1) belongs to a Sobolev space which is the range of a certain integral operator on $\partial\Omega$ (Theorem 7). Once this result is established, the volume integral under consideration may be included into our representation formula (Corollary 9).

2 Notations. Volume potentials.

We fix $\tau \in (0,\infty)$, an open bounded set $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary, as well as functions $f \in C_0^\infty\big(\mathbb{R}^3 \times (0,\infty)\big)^3$, $a \in C_0^\infty(\mathbb{R}^3 \backslash \overline{\Omega})^3$ with $\mathrm{div}\, a = 0$. Of course, the smoothness assumptions on f and a may be reduced, but this is a point we do not want to pursue here. Let N denote the outward unit normal to Ω.

Put $B_r := \{y \in \mathbb{R}^3 : |y| < r\}$, $B_r^c := \mathbb{R}^3 \backslash B_r$ for $r \in (0,\infty)$. We fix some $R_0 > 0$ with $\overline{\Omega} \subset B_{R_0}/2$. The letter C will denote constants which only depend on Ω, R_0 or τ.

We further define $S_T := \partial\Omega \times (0,T)$ for $T \in (0,\infty]$, $|\alpha|_1 := \alpha_1 + \alpha_2 + \alpha_3$ (length of α) for multi-indices $\alpha \in \mathbb{N}_0^3$, $e_1 := (1,0,0)$, $s(x) := |x| - x_1$ for $x \in \mathbb{R}^3$. We write $< \, , \, >$ for the usual scalar product in $\mathbb{R}^3$. Let H denote the usual fundamental solution of the heat equation in $\mathbb{R}^3$, that is,

$$H(z,t) := (4 \cdot \pi \cdot t)^{-3/2} \cdot e^{-|z|^2/(4 \cdot t)}$$

for $t \in (0,\infty)$, $z \in \mathbb{R}^3$. We further introduce a fundamental solution of the time-dependent Stokes system by setting as in [4]:

$$\Gamma_{jk}(z,t) := \delta_{jk} \cdot H(z,t) + \int_t^\infty \partial_j \partial_k H(z,s)\, ds,$$

$$E_k(x) := (4 \cdot \pi)^{-1} \cdot x_k \cdot |x|^{-3}$$

for $z \in \mathbb{R}^3$, $t \in (0,\infty)$, $x \in \mathbb{R}^3 \backslash \{0\}$, $1 \le j,k \le 3$. Finally we define a fundamental solution of the time-dependent Oseen system by putting

$$\Lambda_{jk}(z,t,\kappa) := \Gamma_{jk}(z - \kappa \cdot t \cdot e_1, t)$$

for $z \in \mathbb{R}^3,\ t,\kappa \in (0,\infty),\ j,k \in \{1,2,3\}$. The ensuing properties of Λ_{jk} will be important in the following.

Lemma 1 *The function $\Lambda_{jk}(\,\cdot\,,\,\cdot\,,\kappa)$ belongs to the space $C^\infty(\mathbb{R}^3 \times (0,\infty))$, for $\kappa > 0$, $1 \le j,k \le 3$.*

For any $K > 0$, there exists a constant $C(K) > 0$ such that

$$|\,\partial_t^l \partial_z^\alpha \Lambda_{jk}(z,t,\kappa)\,| \le C(K)\cdot \max\{1,\kappa\}^{3/2+|\alpha|_1/2+l} \cdot(|z|^2+t)^{-3/2-|\alpha|_1/2-l}, \quad \text{if } z \in B_K,$$

$$|\,\partial_t^l \partial_z^\alpha \Lambda_{jk}(z,t,\kappa)\,| \le C(K)\cdot \max\{1,\kappa\}^{3/2+|\alpha|_1/2+l} \cdot\big(\,|z|\cdot(1+\tau\cdot s(z))+t\,\big)^{-3/2-|\alpha|_1/2-l}, \quad \text{if } z \in B_K^c,$$

for $z \in \mathbb{R}^3,\ t,\kappa \in (0,\infty),\ j,\ k \in \{1,2,3\},\ \alpha \in \mathbb{N}_0^3,\ l \in \mathbb{N}_0$ with $|\alpha|_1 + 2\cdot l \le 2$.

Proof: See [1, Section 2].

Next we introduce some volume potentials related to the time-dependent Oseen system. We put

$$\mathcal{R}_j(f)(x,t) := \int_0^t \int_{\mathbb{R}^3} \sum_{k=1}^{3} \Lambda_{jk}(x-y,t-\sigma,\tau)\cdot f_k(y,\sigma)\,dy\,d\sigma,$$

$$\mathcal{P}(f)(x,t) := \int_{\mathbb{R}^3} \sum_{k=1}^{3} E_k(x-y)\cdot f_k(y,t)\,dy,$$

$$\mathcal{I}_j(a)(x,r) := \int_{\mathbb{R}^3} H(x-\tau\cdot r\cdot e_1 - y, r)\cdot a_j(y)\,dy,$$

$$\mathcal{I}(a)(x,0) = a(x)$$

for $x \in \mathbb{R}^3,\ t \in [0,\infty),\ r \in (0,\infty),\ 1 \le j \le 3$. Then we have

Theorem 2 *The functions $\mathcal{R}_j(f)$, $\mathcal{P}(f)$ and $\mathcal{I}_j(a)$ belong to $C^\infty(\mathbb{R}^3 \times [0,\infty))$, for $1 \le j \le 3$. Moreover, for $x \in \mathbb{R}^3,\ t \in (0,\infty)$,*

$$\partial_t\mathcal{R}(f)(x,t) - \Delta_x\mathcal{R}(f)(x,t) + \tau\cdot\partial_{x_1}\mathcal{R}(f)(x,t) + \nabla_x\mathcal{P}(f)(x,t) = f(x,t),$$

$$\mathrm{div}_x\mathcal{R}(f)(x,t) = 0, \quad \mathcal{R}(f)(x,0) = 0,$$

$$\partial_t\mathcal{I}(a)(x,t) - \Delta_x\mathcal{I}(a)(x,t) + \tau\cdot\partial_{x_1}\mathcal{I}(a)(x,t) = 0,$$

$$\mathrm{div}_x\mathcal{I}(a)(x,t) = 0, \quad \mathcal{I}(a)(x,0) = a(x).$$

3 The single-layer potential related to the evolutionary Oseen system.

We want to apply a result from the proof of [4, Theorem 5.2.1]. To this end, we introduce some function spaces used in that reference. Put

$$L_n^2(\partial\Omega) := \{v \in L^2(\partial\Omega)^3 : \int_{\partial\Omega} < v, N > \,d\Omega = 0\}.$$

Let $T \in (0,\infty]$. (Note that the case $T = \infty$ is admitted.) Then we put

$$L_n^2(S_T) := \{v \in L^2(S_T)^3 : v(\,\cdot\,,t) \in L_n^2(\partial\Omega) \text{ for almost every } t \in (0,T)\},$$

$$\mathcal{H}_T := \{v\,|\,S_T : v \in C_0^\infty(\mathbb{R}^4)^3,\ v\,|\,\mathbb{R}^3\times(-\infty,0] = 0\}.$$

For $v \in C^1((-\infty,T))$ with $v\,|\,(-\infty,0] = 0$, and for $t \in (0,T)$, we put

$$\partial_t^{1/2}v(t) := \Gamma(1/2)^{-1}\cdot\partial/\partial_t\Big(\int_0^t (t-s)^{-1/2}\cdot v(s)\,ds\Big)$$

("fractional derivative of v"), where Γ is the usual gamma function. We further define for $v \in \mathcal{H}_T$:

$$\|v\|_{H^{1,1/2}(S_T)} := \Big(\int_0^T \big(\,\|v(\,\cdot\,,t)\|_{1,2}^2 + \int_{\partial\Omega} |\,\partial_t^{1/2}v(x,t)\,|^2\,d\Omega(x)\,\big)\,dt\Big)^{1/2},$$

where $\|\ \|_{1,2}$ denotes the usual norm of $H^1(\partial\Omega)$. The mapping $\|\ \|_{H^{1,1/2}(S_T)}$ is a norm on $\mathcal{H}_T$. For any $v \in L^2(S_T)$, we may define $F_v \in L^2(0,T,H^1(\partial\Omega)')$ by setting for $\sigma \in H^1(\partial\Omega)$ and for almost every $t \in (0,T)$:

$$F_v(t)(\sigma) := \int_{\partial\Omega} v(x,t)\cdot\sigma(x)\,dx.$$

We will write v instead of F_v. For $v \in \mathcal{H}_T$, set

$$< \partial_t v, N > (x,t) := < \partial_t v(x,t), N(x) > \quad \text{for } (x,t) \in S_T,$$

$$\|v\|_{H_T} := \|v\|_{H^{1,1/2}(S_T)} + \|< \partial_t v, N >\|_{L^2(0,T,H^1(\partial\Omega)')}.$$

The mapping $\|\ \|_{H_T}$ is a norm on $\mathcal{H}_T$. Let the space H_T consist of all functions $v \in L_n^2(S_T)$ such that there exists a sequence (w_n) in $C_0^\infty(\mathbb{R}^4)^3$ with the properties that

$$w_n\,|\,S_T \in \mathcal{H}_T \text{ for } n \in \mathbb{N}, \quad \|v - w_n\,|\,S_T\|_2 \to 0,$$

and $(w_n\,|\,S_T)$ is a Cauchy sequence with respect to the norm $\|\ \|_{H_T}$. This means in particular that the sequences $\big(\| < \partial_t w_n, N > \|_{L^2(0,T,H^1(\partial\Omega)')} \big)$ and $\big(\|w_n \mid S_T\|_{H^{1,1/2}(S_T)} \big)$ are convergent. Their limit value does not depend on the choice of the sequence (w_n) with the above properties, for a given $v \in H_T$. Thus, for $v \in H_T$, we may define the quantity $\|v\|_{H_T}$ in an obvious way. The mapping $\| \;\; \|_{H_T}$ is a norm on H_T, and the pair $\big(H_T, \| \;\; \|_{H_T} \big)$ is a Banach space.

Next we introduce our single-layer potentials. For $\phi \in L^2(S_T)^3$, $\kappa \in (0,\infty)$, $x \in \mathbb{R}^3$, $t \in [0,T] \cap \mathbb{R}$, we put

$$\mathcal{V}_T^{(0)}(\phi)(x,t) := \Big(\int_0^t \sum_{k=1}^3 \Gamma_{jk}(x-y, t-\sigma) \cdot\phi_k(y,\sigma)\, d\Omega(y)\, d\sigma \Big)_{1\le j\le 3},$$

$$\mathcal{V}_T^{(\kappa)}(\phi)(x,t) := \Big(\int_0^t \sum_{k=1}^3 \Lambda_{jk}(x-y, t-\sigma, \kappa) \cdot\phi_k(y,\sigma)\, d\Omega(y)\, d\sigma \Big)_{1\le j\le 3}.$$

We call $\mathcal{V}^{(0)}(\phi)$ and $\mathcal{V}^{(\kappa)}(\phi)$ the "single-layer potential related to the time-dependent Stokes- and Oseen-system", respectively. We further set

$$Q_T(\phi)(x,t) := \int_{\partial\Omega} \sum_{k=1}^3 E_k(x-y) \cdot \phi(y,t)\, d\Omega(y)$$

for $\phi \in L^2(S_T)^3$, $x \in \mathbb{R}^3\backslash\partial\Omega$, $t \in [0,T] \cap \mathbb{R}$. The following lemma holds:

Lemma 3 *Let $T \in (0,\infty]$, $\kappa \in [0,\infty)$, $\phi \in L^2(S_T)^3$. Then the function*

$$v := \mathcal{V}_T^{(\kappa)}(\phi) \mid (\mathbb{R}^3\backslash\partial\Omega) \times ([0,T] \cap \mathbb{R})$$

belongs to $C^\infty\big((\mathbb{R}^3\backslash\partial\Omega) \times ([0,T] \cap \mathbb{R}) \big)^3$.

Set $q := Q_T(\phi)$. Then $q(\,\cdot\,,t) \in C^\infty(\mathbb{R}^3\backslash\partial\Omega)$ and

$$\begin{aligned} &\partial_t v(x,t) - \Delta_x v(x,t) \\ &\quad + \tau \cdot \partial_{x_1} v(x,t) + \nabla_x q(x,t) = 0, \\ &\quad \operatorname{div}_x v(x,t) = 0, \quad v(x,0) = 0 \\ &\quad \textit{for } x \in \mathbb{R}^3\backslash\partial\Omega,\ t \in (0,T). \end{aligned}$$

The restriction $\mathcal{V}_T^{(\kappa)}(\phi) \mid S_T$ may be considered as the boundary value of $\mathcal{V}_T^{(\kappa)}(\phi) \mid (\mathbb{R}^3\backslash\overline{\Omega}) \times (0,T)$ on S_T in the following sense:

Lemma 4 *Take T, κ, ϕ as in Lemma 3. Then*

$$\mathcal{V}_T^{(\kappa)}(\phi) \mid S_T \in L^2(S_T)^3,$$

$$\int_0^T \int_{\partial\Omega} \Big| \mathcal{V}_T^{(\kappa)}(\phi)(x + \epsilon \cdot m^{(\Omega)}(x), t) - \mathcal{V}_T^{(\kappa)}(\phi)(x,t) \Big|^2 d\Omega(x)\, dt \longrightarrow 0 \ (\epsilon \downarrow 0).$$

Stronger convergence results are probably valid; for example, see [4, Theorem 2.1.7] in the case $T < \infty$ and $\kappa = 0$. But this is also a point we do not want to pursue here.

Now we may see how the problem of finding a solution to (1) - (4) reduces to an integral equation on S_∞:

Lemma 5 *Let $\phi \in L^2(S_\infty)^3$with*

$$\mathcal{V}_\infty^{(\tau)}(\phi) \mid S_\infty = \big(-\mathcal{R}(f) - \mathcal{I}(a) \big) \mid S_\infty. \tag{5}$$

Put

$$\begin{aligned} u := \big(\mathcal{R}(f) + \mathcal{I}(a) \qquad\qquad & (6) \\ + \mathcal{V}_\infty^{(\tau)}(\phi) \big) \mid (\mathbb{R}^3\backslash\overline{\Omega}) \times [0,\infty), & \end{aligned}$$

$$\pi := \big(\mathcal{P}(f) + Q(\phi) \big) \mid (\mathbb{R}^3\backslash\overline{\Omega}) \times (0,\infty). \tag{7}$$

Then $u \in C^\infty\big((\mathbb{R}^3\backslash\overline{\Omega}) \times [0,\infty) \big)^3$, and $\pi(\,\cdot\,,t)$ is a C^∞-function on $\mathbb{R}^3\backslash\overline{\Omega}$ for almost every $t \in (0,T)$. Moreover, the pair (u,π) is a solution of (1) - (4), with the boundary condition on S_∞ verified in the sense that

$$\int_0^\infty \int_{\partial\Omega} \big| u(x + \epsilon \cdot m^{(\Omega)}(x), t) \big|^2 d\Omega(x)\, dt \longrightarrow 0 \ \textit{ for } \epsilon \downarrow 0. \tag{8}$$

Note that the equation in (5) is in fact an integral equation on S_∞, with ϕ as unknown. In order to solve (5), we introduce operators $\mathcal{F}_T^{(\kappa)} : L_n^2(S_T) \mapsto H_T$, for $\kappa \in [0,\infty)$, $T \in (0,\infty]$, by setting

$$\mathcal{F}_T^{(\kappa)}(\phi) := \mathcal{V}_T^{(\kappa)}(\phi) \mid S_T \quad \text{for } \phi \in L_n^2(S_T).$$

According to the following theorem, equation (5) is solvable if the right-hand side of (5) is in H_T.

Theorem 6 *The mapping $\mathcal{F}_\infty^{(\tau)} : L_n^2(S_\infty) \mapsto H_\infty$ is bijective.*

Proof: See [1, Section 4], where this theorem is reduced to a result from [4] pertaining to the Stokes system.

The next two theorems state conditions on f and a, respectively, ensuring that the right-hand side of (5) belongs to H_T.

Theorem 7 *The function $\mathcal{R}(f)\,|\,S_\infty$ belongs to H_∞. More precisely, let $q \in (1,2)$, $\alpha \in (1, q/2)$, $\beta \in (4/3, 2)$. Then*

$$\begin{aligned}\|\mathcal{R}(f)\,|\,S_\infty\|_{H_T} \;\leq\; & C(q,\alpha,\beta)\cdot\big(\,\|f\|_2 \\ & + \|f\,|\,(\mathbb{R}^3\backslash B_{R_0})\times(0,\infty)\|_{L^\alpha(0,\infty,L^q(B^c_{R_0})^3)} \\ & + \|f\,|\,(\mathbb{R}^3\backslash B_{R_0})\times(0,\infty)\|_{L^\beta(0,\infty,L^q(B^c_{R_0})^3)}\big),\end{aligned}$$

where R_0 was introduced at the beginning of Section 2, and where the constant $C(q,\alpha,\beta) > 0$ depends on Ω, R_0, τ, q, α and β.

It is the aim of the work at hand to prove this theorem. This proof is given in Section 4 below.

Theorem 8 *We have $\mathcal{I}(a)\,|\,S_\infty \in H_\infty$. More precisely, let $q \in (1,2)$. Then*

$$\|\mathcal{I}(a)\,|\,S_\infty\|_{H_\infty} \;\leq\; C(q)\cdot(\|a\|_2 + \|a\|_q),$$

with $C(q) > 0$ depending on Ω, R_0, τ and q.

As a consequence of Theorem 6, 7 and 8, we arrive at the following result:

Corollary 9 *There is a solution (u,π) of (1) - (4) such that $u \in C^\infty\big((\mathbb{R}^3\backslash\overline{\Omega})\times[0,\infty)\big)^3$, $\pi(\,\cdot\,,t) \in C^\infty(\mathbb{R}^3\backslash\overline{\Omega})$ for $t \in (0,\infty)$, and such that the boundary condition on S_∞ is verified as in (8). This solution is given by (6), (7), with $\phi \in L^2_n(S_\infty)$ fulfilling (5).*

4 Proof of Theorem 7.

Since $\overline{\Omega} \subset B_{R_0/2}$, we have $|x-y| \geq |y|/2 \geq R_0/2$ for $x \in \overline{\Omega}$, $y \in B^c_{R_0}$. On the other hand, by [2, Lemma 4.8], the inequality

$$\begin{aligned}&\big(1+\tau\cdot s(x-y)\big)^{-1} \\ &\quad\leq\; \mathcal{C}\cdot\big(1+s(x)\big)\cdot\big(1+\tau\cdot s(y)\big)^{-1}\end{aligned}$$

holds for $x,y \in \mathbb{R}^3$. Therefore, for $x \in \overline{\Omega}$, $y \in B^c_{R_0}$:

$$\big(1+\tau\cdot s(x-y)\big)^{-1} \leq \mathcal{C}\cdot\big(1+\tau\cdot s(y)\big)^{-1}.$$

For $x,y \in \mathbb{R}^3$, put

$$\begin{aligned}&\varrho(x,y) := |x-y|^2, \;\; \text{if } y \in B_{R_0}, \\ &\quad\varrho(x,y) := \big(1+\tau\cdot s(y)\big)\cdot|y|, \;\; \text{if } y \in B^c_{R_0}.\end{aligned}$$

Since we have $|x-y| \geq R_0/2$ for $x \in \overline{\Omega}$, $y \in B^c_{R_0}$ (see above), and $|x-y| \leq |x|+|y| \leq 2\cdot R_0$ for $x \in \overline{\Omega}$, $y \in B_{R_0}$, we may now deduce from Lemma 1 with $K = 2\cdot R_0$ and $K = R_0/2$, respectively:

$$\begin{aligned}&\big|\,\partial^l_t\partial^\alpha_x\Lambda_{jk}(x-y,t-\sigma,\tau)\,\big| \qquad (9)\\ &\quad\leq\; \mathcal{C}\cdot\big(\varrho(x,y)+t-\sigma\big)^{-3/2-|\alpha|_1/2-l}\end{aligned}$$

for $x \in \overline{\Omega}$, $y \in \mathbb{R}^3$, $j,k \in \{1,2,3\}$, $\alpha \in \mathbb{N}^3_0$, $l \in \mathbb{N}_0$ with $|\alpha|_1 + 2\cdot l \leq 2$, $t \in (0,\infty)$, $\sigma \in (0,t)$. According to [3], there is a constant $C(\tau,R_0,\epsilon) > 0$ for any $\epsilon \in (0,\infty)$ such that

$$\int_{\partial B_r}\big(1+\tau\cdot s(y)\big)^{-1-\epsilon}\,do_y \;\leq\; C(\tau,R_0,\epsilon)\cdot r$$

for $r \in [R_0,\infty)$. It follows for $\kappa \in (2,\infty)$:

$$\begin{aligned}&\int_{B^c_{R_0}}\big[\big(1+\tau\cdot s(y)\big)\cdot|y|\big]^{-\kappa}\,dy \qquad (10)\\ &\quad\leq\; \int_{R_0}^\infty r^{-\kappa}\cdot\int_{\partial B_r}\big(1+\tau\cdot s(y)\big)^{-\kappa}\,do_y\,dr \\ &\quad\leq\; \overline{C}(\tau,R_0,\kappa),\end{aligned}$$

with a constant $\overline{C}(\tau,R_0,\kappa) > 0$ only depending on τ, R_0 and κ.

Since $q < 2$, hence $1-2/q' > 0$, we may choose $\epsilon \in (0,\,1-2/q')$. Moreover, since $\beta > 4/3$, we have $-1/2+1/\beta < 1/4$, so we may choose $\delta \in (0,1/4)$ with $\delta > -1/2+1/\beta$. Let $\widetilde{\delta} \in (0,\,1/2-2\cdot\delta)$. We then have the following relations:

$$\begin{aligned}&(-1+\epsilon)\cdot q' < -2, \qquad (11)\\ &(-1+\delta)\cdot(3/2-1/\beta)^{-1} > -1, \\ &-2-4\cdot\delta-\widetilde{\delta} > -3, \\ &(3/2-1/\beta)^{-1} > 1.\end{aligned}$$

The first one of these inequalities holds because $\epsilon < 1-2/q'$, the second one is a consequence of the relation $\delta > -1/2+1/\beta$, the third one is due to the inequality $\delta < 1/4$, and the fourth one follows from the relation $\beta < 2$. For $r \in \mathbb{R}$, put

$$F(r) := \chi_{(0,1)}(r)\cdot r^{-1+\delta} + \chi_{(1,\infty)}(r)\cdot r^{-1-\epsilon}.$$

Let $j \in \{1,2,3\}$. Then, due to (9), we have for $l \in \{1,2,3\}$, $x \in \mathbb{R}^3$, $t \in (0,\infty)$,

$$\begin{aligned}&\big|\,\partial_{x_l}\mathcal{R}_j(f)(x,t)\,\big| \;=\; \Big|\int_0^t\int_{\mathbb{R}^3}\sum_{k=1}^3 \qquad (12)\\ &\qquad\partial_{x_l}\Lambda_{jk}(x-y,t-\sigma,\tau)\cdot f_k(y,\sigma)\,dy\,d\sigma\,\Big| \\ &\quad\leq \mathcal{C}\cdot\int_0^t\int_{\mathbb{R}^3}\big(\varrho(x,y)+t-\sigma\big)^{-2}\cdot|f(y,\sigma)|\,dy\,d\sigma \\ &\quad\leq \mathcal{C}\cdot\int_0^t\int_{\mathbb{R}^3}\Big(\chi_{(0,1)}(t-\sigma)\cdot(t-\sigma)^{-1+\delta} \\ &\qquad\cdot\varrho(x,y)^{-1-\delta} + \chi_{(1,\infty)}(t-\sigma)\cdot(t-\sigma)^{-1-\epsilon} \\ &\qquad\cdot\varrho(x,y)^{-1+\epsilon}\Big)\cdot|f(y,\sigma)|\,dy\,d\sigma \\ &\quad\leq \mathcal{C}\cdot\int_0^t\int_{\mathbb{R}^3}F(t-\sigma)\cdot\max\{\varrho(x,y)^{-1-\delta}, \\ &\qquad\qquad\varrho(x,y)^{-1+\epsilon}\}\cdot|f(y,\sigma)|\,dy\,d\sigma.\end{aligned}$$

We further find for $t \in (0,\infty),\ \sigma \in (0,t),\ x \in \mathbb{R}^3$:

$$\begin{aligned}
&\partial_t\Big(\int_0^t (t-r)^{-1/2}\cdot \mathcal{R}_j(f)(x,r)\,dr\Big) \qquad (13)\\
&= \partial_t\Big(\sum_{k=1}^3 \int_{\mathbb{R}^3}\int_0^t f_k(y,\sigma)\cdot \int_\sigma^t (t-r)^{-1/2}\\
&\quad \cdot\Lambda_{jk}(x-y,r-\sigma,\tau)\,dr\,d\sigma\,dy\Big)\\
&= \sum_{k=1}^3 \int_{\mathbb{R}^3} \partial_t\Big[\int_0^t\Big(2\cdot\int_{(t+\sigma)/2}^t (t-r)^{1/2}\\
&\quad \cdot\partial_r\Lambda_{jk}(x-y,r-\sigma,\tau)\,dr\\
&\quad + 2^{1/2}\cdot(t-\sigma)^{1/2}\cdot\Lambda_{jk}(x-y,(t-\sigma)/2,\ \tau)\\
&\quad + \int_\sigma^{(t+\sigma)/2}(t-r)^{-1/2}\\
&\quad \cdot\Lambda_{jk}(x-y,r-\sigma,\tau)\,dr\Big)\cdot f_k(y,\sigma)\,d\sigma\Big]\,dy\\
&= \sum_{k=1}^3\int_0^t\int_{\mathbb{R}^3} H_{jk}(x-y,t,\sigma)\cdot f_k(y,\sigma)\,dy\,d\sigma,
\end{aligned}$$

with

$$\begin{aligned}
&H_{jk}(z,t,\sigma) :=\\
&\quad 2^{1/2}\cdot(t-\sigma)^{-1/2}\cdot\Lambda_{jk}(z,(t-\sigma)/2,\ \tau)\\
&+ \int_{(t+\sigma)/2}^t (t-r)^{-1/2}\cdot\partial_r\Lambda_{jk}(z,r-\sigma,\tau)\,dr\\
&-(1/2)\cdot\int_\sigma^{(t+\sigma)/2}(t-r)^{-3/2}\cdot\Lambda_{jk}(z,r-\sigma,\tau)\,dr
\end{aligned}$$

for $z \in \mathbb{R}^3\backslash\{0\},\ t \in (0,\infty),\ \sigma \in (0,t),\ j,k \in \{1,2,3\}$. But for such $t,\ \sigma,\ j,\ k$, and for $x \in \overline{\Omega},\ y \in \mathbb{R}^3$, we find

$$\begin{aligned}
&|\, H_{jk}(x-y,t-\sigma)|\\
&\le C\cdot\Big((t-\sigma)^{-1/2}\cdot\big(\varrho(x,y)+t-\sigma\big)^{-3/2}\\
&+\int_{(t+\sigma)/2}^t (t-r)^{-1/2}\cdot\big(\varrho(x,y)+r-\sigma\big)^{-5/2}\,dr\\
&+(t-\sigma)^{-3/2}\cdot\int_\sigma^{(t+\sigma)/2}\big(\varrho(x,y)+r-\sigma\big)^{-3/2}\,dr\Big)\\
&\le C\cdot\Big((t-\sigma)^{-1/2}\cdot\big(\varrho(x,y)+t-\sigma\big)^{-3/2}\\
&\quad+\big(\varrho(x,y)+t-\sigma\big)^{-5/2}\cdot(t-\sigma)^{1/2}\\
&\quad+(t-\sigma)^{-3/2}\cdot\Big[\chi_{(0,1)}(t-\sigma)\cdot\varrho(x,y)^{-1-\delta}\\
&\qquad\cdot(t-\sigma)^{1/2+\delta}\\
&\quad+\chi_{(1,\infty)}(t-\sigma)\cdot\varrho(x,y)^{-1+\epsilon}\cdot(t-\sigma)^{1/2-\epsilon}\Big]\Big)\\
&\le C\cdot F(t-\sigma)\cdot\max\{\varrho(x,y)^{-1+\epsilon},\ \varrho(x,y)^{-1-\delta}\}.
\end{aligned}$$

Thus we may conclude from (13), for $x \in \partial\Omega,\ t \in (0,\infty)$:

$$\begin{aligned}
&\Big|\,\partial_t\Big(\int_0^t (t-r)^{-1/2}\cdot\mathcal{R}_j(f)(x,r)\,dr\,\Big| \qquad (14)\\
&\le C\cdot\int_0^t\int_{\mathbb{R}^3} F(t-\sigma)\\
&\quad\cdot\max\{\varrho(x,y)^{-1+\epsilon},\varrho(x,y)^{-1-\delta}\}\cdot|f(y,\sigma)|\,dy\,d\sigma.
\end{aligned}$$

On the other hand, we find by the definition of $\varrho(x,y)$, (9), (11) and Young's inequality:

$$\begin{aligned}
&\Big(\int_0^\infty\int_{\partial\Omega}\Big(\int_0^t\int_{B_{R_0}} F(t-\sigma) \qquad (15)\\
&\quad\cdot\max\{\varrho(x,y)^{-1+\epsilon},\ \varrho(x,y)^{-1-\delta}\}\\
&\quad\cdot|f(y,\sigma)|\,dy\,d\sigma\Big)^2\,d\Omega(x)\,dt\Big)^{1/2}\\
&\le C\cdot\Big(\int_{\partial\Omega}\int_0^\infty\Big(\int_0^t F(t-\sigma)\\
&\quad\cdot\int_{B_{R_0}}|x-y|^{-2-2\cdot\delta}\cdot|f(y,\sigma)|\,dy\,d\sigma\Big)^2\\
&\qquad dt\,d\Omega(x)\Big)^{1/2}\\
&\le C\cdot\Big(\int_{\partial\Omega}\Big(\int_{\mathbb{R}}F(r)\,dr\Big)^2\cdot\int_0^\infty\Big(\int_{B_{R_0}}\\
&\quad|x-y|^{-2-2\cdot\delta}\cdot|f(y,\sigma)|\,dy\Big)^2\,d\sigma\,d\Omega(x)\Big)^{1/2}\\
&\le C\cdot\Big(\int_0^\infty\int_{\partial\Omega}\Big(\int_{B_{R_0}}|x-y|^{-2-2\cdot\delta}\\
&\quad\cdot|f(y,\sigma)|\,dy\Big)^2\,d\Omega(x)\,d\sigma\Big)^{1/2}\\
&\le C\cdot\Big(\int_0^\infty\int_{\partial\Omega}\Big(\int_{B_{R_0}}|x-y|^{-2-4\cdot\delta-\tilde{\delta}}\,dy\Big)\\
&\quad\cdot\Big(\int_{B_{R_0}}|x-y|^{-2+\tilde{\delta}}\cdot|f(y,\sigma)|^2\,dy\Big)\,d\Omega(x)\,d\sigma\Big)^{1/2}\\
&\le C\cdot\Big(\int_0^\infty\int_{B_{R_0}}\int_{\partial\Omega}|x-y|^{-2+\tilde{\delta}}\,d\Omega(x)\\
&\quad\cdot|f(y,\sigma)|^2\,dy\,d\sigma\Big)^{1/2}\\
&\le C\cdot\|f\|_2,
\end{aligned}$$

where we used Fubini's theorem in the last but one inequality. We further find with (10), (11),

$$\begin{aligned}
&\Big(\int_0^\infty\int_{\partial\Omega}\Big(\int_0^t\int_{B^c_{R_0}} F(t-\sigma) \qquad (16)\\
&\quad\cdot\max\{\varrho(x,y)^{-1+\epsilon},\ \varrho(x,y)^{-1-\delta}\}\\
&\quad\cdot|f(y,\sigma)|\,dy\,d\sigma\Big)^2\,d\Omega(x)\,dt\Big)^{1/2}
\end{aligned}$$

$$\begin{aligned}
&\le C \cdot \Big(\int_0^\infty \int_{\partial\Omega} \Big(\int_0^t \int_{B^c_{R_0}} F(t-\sigma) \\
&\quad \cdot \big[\, (1 + \tau \cdot s(y)) \cdot |y| \,\big]^{-1+\epsilon} \\
&\quad \cdot |f(y,\sigma)| \, dy \, d\sigma \Big)^2 \, d\Omega(x) \, dt \Big)^{1/2} \\
&\le C \cdot \Big(\int_0^\infty \Big(\int_0^t F(t-\sigma) \\
&\quad \cdot \Big(\int_{B^c_{R_0}} \big[\, (1 + \tau \cdot s(y)) \cdot |y| \,\big]^{(-1+\epsilon)\cdot q'} dy \Big)^{1/q'} \\
&\quad \cdot \| f(\,\cdot\,,\sigma) \,|\, B^c_{R_0} \|_q \, d\sigma \Big)^2 \, dt \Big)^{1/2} \\
&\le C \cdot \Big(\int_0^\infty \Big(\int_0^\infty F(t-\sigma) \\
&\qquad \cdot \| f(\,\cdot\,,\sigma) \,|\, B^c_{R_0} \|_q \, d\sigma \Big)^2 \, dt \Big)^{1/2} \\
&\le C \cdot \| f \,|\, B^c_{R_0} \times (0,\infty) \|_{L^\beta(0,\infty,\, L^q(B^c_{R_0})^3)},
\end{aligned}$$

where the last estimate follows by Young's inequality; note that $\int_0^\infty F(r)^{(3/2-1/\beta)^{-1}} < \infty$ by (11). We may now conclude from (12), (14), (15) and (16), for $l \in \{1, 2, 3\}$,

$$\begin{aligned}
&\| \partial_l \mathcal{R}_j(f) \,|\, S_\infty \|_2 + \| \partial_t^{1/2} \mathcal{R}_j(f) \,|\, S_\infty \|_2 \qquad (17) \\
&\quad \le C \cdot \big(\|f\|_2 \\
&\qquad + \| f \,|\, B^c_{R_0} \times (0,\infty) \|_{L^\beta(0,\infty,\, L^q(B^c_{R_0})^3)} \big).
\end{aligned}$$

Moreover, by (9),

$$\begin{aligned}
&\| \mathcal{R}_j(f) \,|\, S_\infty \|_2 \qquad (18) \\
&\quad \le \Big(\int_0^\infty \int_{\partial\Omega} \Big(\int_0^t \int_{\mathbb{R}^3} \big(\varrho(x,y) + t - \sigma \big)^{-3/2} \\
&\qquad \cdot |f(y,\sigma)| \, dy \, d\sigma \Big)^2 \, d\Omega(x) \, dt \Big)^{1/2}
\end{aligned}$$

But for $x \in \partial\Omega$, $y \in B_{R_0}$, $t \in (0,\infty)$, $\sigma \in (0,t)$, we have

$$\begin{aligned}
&\big(\varrho(x,y) + t - \sigma \big)^{-3/2} \\
&\quad \le C \cdot \Big(\chi_{(0,1)}(t-\sigma) \cdot (t-\sigma)^{-15/16} \\
&\quad + \chi_{(1,\infty)}(t-\sigma) \cdot (t-\sigma)^{-17/16} \Big) \cdot |x-y|^{-9/8}.
\end{aligned}$$

Thus it follows by Young's inequality,

$$\begin{aligned}
&\Big(\int_0^\infty \int_{\partial\Omega} \Big(\int_0^t \int_{B_{R_0}} \big(\varrho(x,y) + t - \sigma \big)^{-3/2} \qquad (19) \\
&\qquad \cdot |f(y,\sigma)| \, dy \, d\sigma \Big)^2 \, d\Omega(x) \, dt \Big)^{1/2} \\
&\le C \cdot \|f\|_2.
\end{aligned}$$

Since $q < 2$, in particular $q < 4$, we obtain the inequality $-2/q' + 3/2 > 0$. Moreover, since $\alpha < q'/2$, we get $-2/q' + 3/2 > 3/2 - 1/\alpha$. Thus we may choose

$$\tilde{\epsilon} \in (3/2 - 1/\alpha,\ -2/q' + 3/2) \quad \text{with } \tilde{\epsilon} > 0.$$

It follows that

$$\begin{aligned}
-\tilde{\epsilon} \cdot (3/2 - 1/\alpha)^{-1} &< -1, \qquad (20) \\
(-3/2 + \tilde{\epsilon}) \cdot q' &< -2.
\end{aligned}$$

Using the second of these two inequalities as well as (10), we find for $t \in (0,\infty)$, $\sigma \in (0,t)$, $x \in \partial\Omega$,

$$\begin{aligned}
&\int_{B^c_{R_0}} \big(\varrho(x,y) + t - \sigma \big)^{-3/2} \cdot |f(y,\sigma)| \, dy \qquad (21) \\
&= \int_{B^c_{R_0}} \big[\, (1 + \tau \cdot s(y)) \cdot |y| + t - \sigma \,\big]^{-3/2} \\
&\qquad \cdot |f(y,\sigma)| \, dy \\
&\le C \cdot (R_0 + t - \sigma)^{-\tilde{\epsilon}} \\
&\quad \cdot \Big(\int_{B^c_{R_0}} \big[\, (1 + \tau \cdot s(y)) \cdot |y| \,\big]^{(-3/2+\tilde{\epsilon})\cdot q'} dy \Big)^{1/q'} \\
&\quad \cdot \| f(\,\cdot\,,\sigma) \,|\, B^c_{R_0} \|_q \\
&\le C \cdot (R_0 + t - \sigma)^{-\tilde{\epsilon}} \cdot \| f(\,\cdot\,,\sigma) \,|\, B^c_{R_0} \|_q.
\end{aligned}$$

But due to (20) and Young's inequality, we have

$$\begin{aligned}
&\Big(\int_0^\infty \Big(\int_0^t (R_0 + t - \sigma)^{-\tilde{\epsilon}} \qquad (22) \\
&\qquad \cdot \| f(\,\cdot\,,\sigma) \,|\, B^c_{R_0} \|_q \, d\sigma \Big)^2 \, dt \Big)^{1/2} \\
&\le C \cdot \| f \,|\, B^c_{R_0} \times (0,\infty) \|_{L^\alpha(0,\infty,\, L^q(B^c_{R_0})^3)}.
\end{aligned}$$

Combining (18), (19), (21) and (22) yields

$$\begin{aligned}
&\| \mathcal{R}_j(f) \,|\, S_\infty \|_2 \le C \cdot \big(\|f\|_2 \qquad (23) \\
&\qquad + \| f \,|\, B^c_{R_0} \times (0,\infty) \|_{L^\alpha(0,\infty,\, L^q(B^c_{R_0})^3)} \big).
\end{aligned}$$

According to [5], there is some $C > 0$ and for any $v \in H^1(\partial\Omega)$ a function $\tilde{v} \in \cap_{\kappa \in (1,2)} H^{3/2-\kappa}(\Omega) \cap H^2_{loc}(\Omega)$ with

$$\tilde{v} \,|\, \partial\Omega = v, \quad \Delta\tilde{v} = 0, \quad \|\tilde{v}\|_{5/4,2} \le C \cdot \|v\|_{1,2}.$$

(The relation $\Delta\tilde{v} = 0$ will not be needed in the following.) Let $v \in H^1(\partial\Omega)$, $t \in (0,\infty)$. Then we get by Theorem 2,

$$\begin{aligned}
&\int_\Omega v(x) \cdot < \partial_t \mathcal{R}(f)(x,t),\, N(x) > \, dx \qquad (24) \\
&\quad = \sum_{i=1}^5 A_i(t),
\end{aligned}$$

with

$$A_1(t) := -\tau \cdot \int_\Omega \nabla\tilde{v}(x) \cdot \partial_{x_1}\mathcal{R}(f)(x,t)\, dx,$$
$$A_2(t) := -\int_\Omega \nabla\tilde{v}(x) \cdot \nabla_x\mathcal{S}(f)(x,t)\, dx,$$
$$A_3(t) := \int_\Omega \nabla\tilde{v}(x) \cdot f(x,t)\, dx,$$
$$A_4(t) := \int_\Omega \nabla\tilde{v}(x) \cdot \Delta_x\mathcal{R}(f\,|\,B^c_{R_0})(x,t)\, dx,$$
$$A_5(t) := \int_\Omega \nabla\tilde{v}(x) \cdot \Delta_x\mathcal{R}(f\,|\,B_{R_0})(x,t)\, dx.$$

The integral in the definition of $A_1(t)$ is retransformed into an integral over $\partial\Omega$. We then get

$$|A_1(t)| \le C \cdot \|v\|_2 \cdot \|\partial_1\mathcal{R}(f)(\,\cdot\,,t)\,|\,\partial\Omega\|_2. \quad (25)$$

In order to evaluate $A_2(t)$, we use the well-known estimate $\|\partial_i\mathcal{S}(f)(\,\cdot\,,t)\|_2 \le C \cdot \|f(\,\cdot\,,t)\|_2$ for any $i \in \{1, 2, 3\}$. We thus get

$$|A_2(t)| \le C \cdot \|\nabla\tilde{v}\|_2 \cdot \|f(\,\cdot\,,t)\|_2. \quad (26)$$

Obviously

$$|A_3(t)| \le C \cdot \|\nabla\tilde{v}\|_2 \cdot \|f(\,\cdot\,,t)\|_2. \quad (27)$$

Moreover, by (9),

$$|A_4(t)| \le C \cdot \int_\Omega |\nabla\tilde{v}(x)| \cdot \int_0^t \int_{B^c_{R_0}} \big[\big(1+\tau\cdot s(y)\big)\cdot|y| + t - \sigma\big]^{-5/2} \cdot |f(y,\sigma)|\, dy\, d\sigma\, dx.$$

Observing that

$$\big[\big(1+\tau\cdot s(y)\big)\cdot|y| + t-\sigma\big]^{-5/2} \le C \cdot \Big(\chi_{(0,1)}(t-\sigma) + \chi_{(1,\infty)}(t-\sigma)\cdot(t-\sigma)^{-5/4}\Big) \cdot \big[\big(1+\tau\cdot s(y)\big)\cdot|y|\big]^{-5/4}$$

for $y \in B^c_{R_0}$, $t \in (0,\infty)$, $\sigma \in (0,t)$, it easily follows with (10) and the Hölder inequality that

$$|A_4(t)| \le C \cdot \|\nabla\tilde{v}\|_2 \cdot \int_0^\infty \big[\chi_{(0,1)}(t-\sigma) + \chi_{(1,\infty)}(t-\sigma)\cdot(t-\sigma)^{-5/4}\big] \cdot \|f(\,\cdot\,,\sigma)\|_2\, d\sigma. \quad (28)$$

For $w \in H^2(\Omega)$, $\sigma \in (0,t)$, put

$$B(w,t,\sigma) := \sum_{j=1}^3 \int_\Omega \partial_{x_j}w(x) \cdot \int_{B_{R_0}} \sum_{k=1}^3 \Delta_x\Lambda_{jk}(x-y,t-\sigma,\tau)\cdot f_k(y,\sigma)\, dy\, dx.$$

A straightforward application of (9), of Hölder's and Young's inequality yields for $w \in H^2(\Omega)$, $\sigma \in (0,t)$ with $t-\sigma < 1$:

$$|B(w,t,\sigma)| \le C\cdot(t-\sigma)^{-1}\cdot\|\nabla w\|_2\cdot\|f(\,\cdot\,,\sigma)\|_2. \quad (29)$$

By first performing a partial integration with respect to x, and only then applying (9), we get for w, σ as before,

$$\begin{aligned}|B(w,t,\sigma)| &\le C\cdot\sum_{j,k,l=1}^3 \Big|\int_{\partial\Omega}\partial_{x_i}w(x)\cdot N_l(x)\cdot\int_{B_{R_0}}\partial_{x_l}\Lambda_{jk}(x-y,t-\sigma,\tau)\cdot f_k(y,\sigma)\,dy\,dx \\ &\quad - \int_\Omega \partial_{x_i}\partial_{x_k}w(x)\cdot\int_{B_{R_0}}\partial_{x_l}\Lambda_{jk}(x-y,t-\sigma,\tau)\cdot f_k(y,\sigma)\,dy\,dx\Big| \\ &\le C\cdot\sum_{j,k,l=1}^3\Big[\|\nabla w\,|\,\partial\Omega\|_2\cdot\Big(\int_{\partial\Omega}\Big|\int_{B_{R_0}}\partial_{x_l}\Lambda_{jk}(x-y,t-\sigma,\tau)\cdot f_k(y,\sigma)\,dy\Big|^2 d\Omega(x)\Big)^{1/2} \\ &\quad + \|\nabla w\|_{1,2}\cdot\Big(\int_\Omega\Big|\int_{B_{R_0}}\partial_{x_l}\Lambda_{jk}(x-y,t-\sigma,\tau)\cdot f_k(y,\sigma)\,dy\Big|^2 dx\Big)^{1/2} \\ &\le C\cdot(t-\sigma)^{-3/4}\cdot\|\nabla w\|_{1,2}\cdot\|f(\,\cdot\,,\sigma)\|_2;\end{aligned} \quad (30)$$

note that $\|\nabla w\,|\,\partial\Omega\|_2 \le C\cdot\|\nabla w\|_{1,2}$ by a standard trace estimate.

Inequalities (29) and (30) yield by interpolation:

$$|B(w,t,\sigma)| \le C\cdot(t-\sigma)^{-15/16}\cdot\|\nabla w\|_{1/4,2}\cdot\|f(\,\cdot\,,\sigma)\|_2,$$

for w, σ as before. For $\sigma \in (0,t)$ with $t-\sigma \ge 1$, it is easy to get

$$|B(\tilde{v},t,\sigma)| \le C\cdot\|\nabla\tilde{v}\|_2\cdot(t-\sigma)^{-5/2}\cdot\|f(\,\cdot\,,\sigma)\|_2.$$

Now we may conclude that

$$|A_5(t)| \le C\cdot\|\nabla\tilde{v}\|_{1/4,2} \cdot \int_0^\infty\big[\chi_{(0,1)}(t-\sigma)\cdot(t-\sigma)^{-15/16} + \chi_{(1,\infty)}(t-\sigma)\cdot(t-\sigma)^{-5/2}\big]\cdot\|f(\,\cdot\,,\sigma)\|_2\,d\sigma. \quad (31)$$

Now we combine (24) - (28) and (31). Recalling the inequality $\|\widetilde{v}\|_{5/4,\,2} \le C \cdot \|v\|_{1,\,2}$, we obtain

$$\begin{aligned} &\Big| \int_{\partial\Omega} < \partial_t \mathcal{R}(f)(x,t),\, N(x) > \cdot v(x)\; d\Omega(x) \Big| \\ &\le\; C \cdot \|v\|_{1,2} \cdot \Big(\|\partial_1 \mathcal{R}(f)(\,\cdot\,,t)\,|\,\partial\Omega\|_2 \\ &\qquad + \|f(\,\cdot\,,t)\|_2 \;+\; Z(t) \Big), \end{aligned}$$

where we used the abbreviation

$$\begin{aligned} Z(t) := \int_0^\infty &\big[\chi_{(0,1)}(t-\sigma) \cdot (t-\sigma)^{-15/16} \\ &+ \chi_{(1,\infty)}(t-\sigma) \cdot (t-\sigma)^{-5/4} \big] \cdot \|f(\,\cdot\,,\sigma)\|_2 \; d\sigma. \end{aligned}$$

This means that

$$\begin{aligned} &\| < \partial_t \mathcal{R}(f)(\,\cdot\,,t),\, N > \|_{H^1(\partial\Omega)'} \\ &\le\; C \cdot \big(\|\partial_1 \mathcal{R}(f)(\,\cdot\,,t)\,|\,\partial\Omega\|_2 \;+\; \|f(\,\cdot\,,t)\|_2 \\ &\qquad + Z(t) \big). \end{aligned}$$

Now inequality (17) and Young's inequality yield

$$\begin{aligned} &\| < \partial_t \mathcal{R}(f),\, N > \|_{L^2(0,\infty,\,H^1(\partial\Omega)')} \qquad (32) \\ &\le\; C \cdot \big(\|f\|_2 \\ &\qquad + \|f\,|\,B^c_{R_0} \times (0,\infty)\|_{L^\beta(0,\infty,\,L^q(B^c_{R_0})^3)} \big). \end{aligned}$$

Combining (17), (23) and (32) yields the theorem. □

5 Conclusion

It was shown that the volume potential $\mathcal{R}(f)$ related to the time-dependent Oseen system belongs to the Sobolev space H_T on $\partial\Omega$. Moreover the restriction $\mathcal{R}(f)\,|\,S_\infty$ was estimated in the norm of that space. These results are elements of a new access introduced in [1] to obtain pointwise decay estimates with repect to space variables for exterior evolutionary Oseen flows.

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