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Computation of 3D exterior stationary incompressible Navier-Stokes flows with non-zero velocity at infinity.

We aim to find numerical approximations of stationary incompressible Navier-Stokes flows in 3D exterior domains. To this end, we consider the Navier-Stokes system with local artificial boundary conditions on a truncated exterior domain. This boundary value problem is solved approximately by a finite element method. The corresponding error is estimated with respect to the discretisation depth and the size of the truncated exterior domain.

1. Introduction

We consider the steady motion of a rigid body in a viscous incompressible fluid. The body is represented by a bounded open polyhedron Ω in $\mathbb{R}^3$. If we suppose that the reference frame is attached to the body, such a motion is modeled by the stationary incompressible Navier-Stokes equations, with a boundary condition on $\partial\Omega$ and at infinity:

$$-\Delta v \;+\; \mathcal{R}\cdot(v\cdot\nabla)v \;+\; \nabla p \;=\; g, \quad \operatorname{div} v \;=\; 0 \quad \text{in } \mathbb{R}^3\backslash\overline{\Omega}, \quad v\,|\,\partial\Omega = 0, \quad v(x)\to\kappa\cdot(1,0,0)\;\;(|x|\to\infty),$$

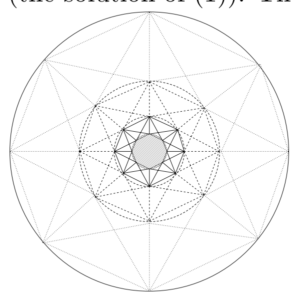
where $\mathcal{R},\kappa\in(0,\infty)$ et $g:\mathbb{R}^3\backslash\overline{\Omega}\mapsto\mathbb{R}^3$ are given. For simplicity, we assumed that the body moves without rotation; otherwise an inhomogeneous boundary condition on $\partial\Omega$ has to be imposed. The preceding boundary value problem may be transformed into a problem of the following type:

$$\begin{cases} -\Delta u \;+\; \tau\cdot D_1 u \;+\; \tilde{\tau}\cdot(u\cdot\nabla)u \;+\; \nabla\pi \;=\; f, \quad \operatorname{div} u \;=\; 0 \quad \text{in } \overline{\Omega}^c, \\ u\,|\,\partial\Omega \;=\; (1,0,0), \quad u(x)\to 0\;\;(|x|\to\infty), \end{cases} \tag{1}$$

where $\tau\in(0,\infty)$ and $\tilde{\tau}\in[0,\tau]$. A solution (u,π) of this problem exists under appropriate assumption on f; see [3, Theorem IX.4.1].

2. Choice of truncated exterior domains and finite element methods.

Choose $S\in(0,\infty)$ in such a way that $\overline{\Omega}$ is contained in $B(0,S)$, and suppose that $\Omega_S := B(0,S)\setminus\overline{\Omega}$ is the neighbourhood of Ω where we actually want to know the details of the exterior flow (u,π) (the solution of (1)). Then for $h\in(0,S/2)$, $R\in(2S,\infty)$, we consider a truncated exterior domain $P_{h,R}$ which is polyhedral, verifies the relation $B(0,R(1-h^2/S^2)^{1/2})\setminus\overline{\Omega}\subset P_{h,R}\subset B(0,R)\setminus\overline{\Omega}$, and satisfies some additional technical assumptions with respect to the exterior ("artificial") part $\partial P_{h,R}\setminus\partial\Omega$ of its boundary. Simultaneously with the choice of $P_{h,R}$, we introduce a decomposition $T_{h,R}=(K_l)_{1\le l\le k}=(K_l^{h,R})_{1\le h\le k_{h,R}}$ of $P_{h,R}$ into tetrahedrons K_l. Denoting $U_0:=\Omega_S$, $U_j:=B(0,2^j\cdot S)\setminus B(0,2^{j-1}\cdot S)$ for $j\in\mathbb{N}$, we suppose that the diameter of each tetrahedron K_l with $K_l\cap U_j\neq\emptyset$ for some $j\in\mathbb{N}$ is of order $2^j\cdot h$. This means that when R increases, the number of vertices of the meshes $T_{h,R}$ only grows as $\log(R)$. Meshes of this type were proposed by Goldstein [3]. On the artificial boundary $\partial P_{h,R}\setminus\partial\Omega$, we prescribe the boundary condition



Projection in 2d of our mesh.

$$\sum_{j=1}^{3}(D_j v_k-\delta_{jk}\cdot\pi-\frac{\tilde{\tau}}{2}\cdot v_j\cdot v_k)(x)\cdot n_j^{h,R}(x) \;+\; (R^{-1}+\frac{\tau}{2}\cdot(1-n_k^{h,R}(x))\cdot v(x) \;=\; 0$$

for $1\le k\le 3,\; x\in\partial P_{h,R}\backslash\partial\Omega$, where $n^{h,R}$ denotes the outward unit normal to $P_{h,R}$; compare [2].

We choose P1-P1 finite elements and a stabilization procedure called "term by term stabilization" and due to Rebollo [5]. With this procedure, seperate terms are introduced for stabilizing the influence of the incompressibility constraint on the one hand, and the effects of the convection term on the other. To describe our method more precisely, we define

$$V_{h,R}=\{v\in C^0(\overline{P_{h,R}}):v|K_l\in P_1(K_l)^3\text{ for }1\le l\le k_{h,R}\},\; M_{h,R}=\{\rho\in C^0(\overline{P_{h,R}}):\rho|K_l\in P_1(K_l)\text{ for }1\le l\le k_{h,R}\},$$

$$a(v,w) := \int_{P_{h,R}} (\nabla v \nabla w + \tau D_1 v w)\, dx + \int_{\partial P_{h,R}\setminus\partial\Omega} (\frac{1}{R} + \frac{\tau}{2}(1 - n_1^{(h,R)}))uv\, dS \quad \text{for } v, w \in V_{h,R},$$

$$b(z,v,w) := \widetilde{\tau} \int_{P_{h,R}} (\,(z\nabla)vw + \frac{1}{2}(\operatorname{div} z)\, vw\,)\, dx - \frac{\widetilde{\tau}}{2} \int_{\partial P_{h,R}\setminus\partial\Omega} (zn^{(h,R)})vw\, dS \quad \text{for } z, v, w \in V_{h,R}.$$

Then we consider the following discrete variational problem on $P_{h,R}$:

$$\begin{cases} \text{Find } v = v_{h,R} \in V_{h,R},\ \varrho = \varrho_{h,R} \in M_{h,R} \text{ such that } v|\partial\Omega = (1,0,0), \\ a(v,w) + b(v,v,w) - \int_{P_{h,R}} \varrho \operatorname{div} w\, dx + \sum_{K\in T_{h,R}} \tau_{1K} \int_K (\overline{v}_K \nabla v)(\overline{v}_K \nabla w)\, dx = \int_{P_{h,R}} fw\, dx \quad \text{for } w \in V_{h,R}, \\ \int_{P_{h,R}} \operatorname{div} v\, q\, dx + \sum_{K\in T_{h,R}} \tau_{2K} \int_K \nabla\pi\nabla q\, dx = 0 \quad \text{for } q \in M_{h,R}, \end{cases} \tag{2}$$

where $\tau_{2K} = 840^{-2}\text{meas}(K)|\hat{b}_K|_1^{-2}$, with $\hat{b}_K$ denoting the usual bubble function on K ($K \in T_{h,R}$); see [5, p. 304-305] for this choice of τ_{2K}. The numbers τ_{1K} and vectors $\overline{v}_K$ are also chosen as in [5, p. 304-305] and correspond to a variant of the SUPG (Streamline Upwind Petrov Galerkin) method. Thus the sum involving the coefficients τ_{1K} is meant to stabilize the effects of the convective term, whereas the other sum in (2), with coefficients τ_{2K}, corresponds to pressure stabilization. The following error estimates are valid.

Theorem . *There is $\tau_0 \in (0,\infty)$ such that for $\tau \in (0,\infty)$, $\widetilde{\tau} \in [0,\tau]$ with $\widetilde{\tau} \le \tau_0$, $h \in (0, S/2]$, $R \in [2S,\infty)$, $f \in L^2(\mathbb{R}^3)^3$, there exists a solution $(v_{h,R}, \varrho_{h,R}) \in V_{h,R} \times M_{h,R}$ of (2).*

Suppose in addition there is $\sigma > 4$, $\gamma > 0$ with $|f(x)| \le \gamma|x|^{-\sigma}$ for $x \in \mathbb{R}^3\setminus B(0,S)$. Then there exists $h_0 > 0$ such that for $h \in (0,h_0]$, $R = 1/h$, for $\tau, \widetilde{\tau}$, $v_{h,R}, \varrho_{h,R}$ as above, and for the exterior flow (u,π) from (1), the ensuing estimate holds true:

$$||\,(u - u_{h,R})|\Omega_S\,||_2 \ \le Ch(1 + \ln(R/S))^{1/2}, \qquad ||\,(\pi - \pi_{h,R})|\Omega_S\,||_2 \ \le Ch^{1/2}(1 + \ln(R/S))^{1/2}$$

with a constant $C > 0$ independent of h (and hence of $R = 1/h$).

3. Numerical results

We present some preliminary results pertaining to the case $\Omega = B_1(0)$, $S = 2.1$, $\widetilde{\tau} = 0$, $\tau_{1K} = 0$ for $K \in T_{h,R}$ (linear problem), $\tau = 1$, $u(x) = 10\,\text{rot}(g(x), g(x), g(x))$, $\pi(x) = 10\,(|x|-1)^2|x|^{-5}$, where $g(x) = (|x|-1)^4|x|^{-4}$ ($x \in \mathbb{R}^3\setminus B_1$). (Then $u|\partial\Omega = 0$, which is not the boundary condition considered in the theorem above, but this discrepancy does not change the error estimates given there.) The resulting algebraic system was solved by an exact Uzawa method, with BiCGSTAB in the inner and outer loop. The latter solver was preconditioned by the diagonal of the discrete operator $-\Delta + \tau \cdot D_1$ (inner loop) and an approximate diagonal of the Schur complement (outer loop), respectively. Starting from a hexahedral grid, we performed N refinement steps ($N = 0, 1, 2, 3$), and then decomposed each hexahedron in 24 tetrahedrons. The latter step corresponds to an additional refinement; see [1] for more details. The values we obtained for the L^2-error $\|(u - v_{h,R})|\Omega_S\|_2$, with $R = 2^J$, $J = 2, 3, 4, 5$, are indicated in the table on the right.

	J=2	J=3	J=4	J=5	J=6
N=0	4.88	1.89	1.49	1.27	1.16
N=1	5.85	2.16	1.22	0.73	0.55
N=2	6.14	2.15	1.01	0.44	0.21
N=3	6.14	2.14	1.02	0.36	0.13

4. References

1 Deuring, P. Calculating Stokes flows around a sphere: comparison of artificial boundary conditions. Ann. Univ. Ferrara - Sez. VII- Sc. Mat. Vol. XLVI, 1-9 (2000)

2 Deuring, P., Kračmar, S. Exterior stationary Navier-Stokes flows in 3D with non-zero velocity at infinity: approximation by flows in bounded domains. Submitted.

3 Galdi, G.P An introduction to the mathematical theory of the Navier-Stokes equations. Volume II. Nonlinear steady problems, Springer, New York,(1994)

4 Goldstein, C. I. The finite element method with nonuniform mesh sizes for unbounded domains. Math. Comp. **36** (1981), 387-404.

5 Rebollo, T. C. A term by term stabilization algorithm for finite element solution of incompressible fluid flow problems. Numer. Math. **79** (1998), 283-319.

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