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SPATIAL DECAY OF TIME-DEPENDENT OSEEN FLOWS*

PAUL DEURING†

Abstract. We consider an initial-boundary value problem for the time-dependent Oseen system in a three-dimensional exterior domain. We show that weak solutions of this problem may be represented by a sum of volume and single layer potentials. This representation is then used in order to study spatial decay of weak solutions.

Key words. exterior domain, time-dependent Oseen system, decay, wake, potential theory, single layer potential

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1. Introduction.

Consider a rigid body moving with prescribed steady velocity $u_0 \in \mathbb{R}^3 \backslash \{0\}$ and without rotation in a viscous incompressible fluid. Assume that the flow in a vicinity of the body is not influenced by some distant boundaries. Moreover, suppose that the flow in question is described with respect to a coordinate system attached to the body. Of course, although the rigid object moves with constant velocity, the flow near its boundary need not be stationary in general. A situation of this kind is usually modeled by the evolutionary Navier–Stokes system in an exterior domain $U := \mathbb{R}^3 \backslash \overline{\Omega}$, with a homogeneous Dirichlet boundary condition on $S_T := \partial\Omega \times (0,T)$, a boundary condition at infinity, and an initial condition,

$$\begin{aligned}
&\partial_t v - \nu \cdot \Delta_x v + (v \cdot \nabla_x) v + \varrho^{-1} \cdot \nabla_x p \;=\; \varrho^{-1} \cdot f, \quad \mathrm{div}_x v = 0 \quad \text{in } Z_T,\\
&v \,|\, S_T = 0, \quad v(x,t) \to u_0 \; (|x| \to \infty) \;\text{ for } t \in (0,T),\\
&v(x,0) = a(x) \;\text{ for } x \in U,
\end{aligned}$$

where the open bounded set $\Omega \subset \mathbb{R}^3$ represents the rigid object, and where $Z_T := U \times (0,T)$. Besides Ω, the given quantities of this problem are the viscosity $\nu \in (0,\infty)$, the density $\varrho \in (0,\infty)$, the far-field velocity u_0, the initial velocity a, and the volume force f, whereas $T \in (0,\infty]$, the velocity v, and the pressure p are unknown. Without loss of generality, we may assume that $u_0 = \beta \cdot (1,0,0)$ for some $\beta > 0$. Since a nonhomogeneous boundary condition at infinity is difficult to handle mathematically, we transform the velocity by a translation. In addition, we normalize the problem with respect to the size of the domain Ω and the magnitude of the velocity u_0. In this way, we arrive at the following initial-boundary value problem, where we denote the transformed quantities in the same way as the original ones, and where $\tau \in (0,\infty)$ is the Reynolds number:

$$\partial_t v - \Delta_x v + \tau \cdot \partial_{x_1} v + \tau \cdot (v \cdot \nabla_x) v + \nabla_x p \;=\; f, \quad \mathrm{div}_x v = 0 \quad \text{in } Z_T, \tag{1.1}$$

$$v \,|\, S_T = (-1,0,0), \quad v(x,t) \to 0 \; (|x| \to \infty) \;\text{ for } t \in (0,T), \tag{1.2}$$

$$v(x,0) = a(x) \;\text{ for } x \in U. \tag{1.3}$$

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†Univ Lille Nord de France, F-59000 Lille, France, and ULCO, LMPA, F-62228 Calais, France (Paul.Deuring@lmpa.univ-littoral.fr).

Note that the term $\tau \cdot \partial_1 v$ (the "Oseen term") arises due to the transformation of the velocity by a translation. Of course, this translation is the reason that the boundary condition on S_T is now inhomogeneous. We will follow the usual custom and call the differential equation in (1.1) a "Navier–Stokes system," although it is different from this system because of the Oseen term.

A solution of (1.1)–(1.3) should reflect the main features of the physical flow in question. In particular, such a solution should exhibit a "wake." This means that in a paraboloidal downstream region, the velocity converges slower than elsewhere to its constant boundary value at infinity. Such an asymptotic behavior is well established for solutions of the stationary version of (1.1)–(1.3). In this respect, we refer to [16, section IX.8], the references therein, [12], [13], [8], and [2].

Less is known about the asymptotic behavior of flows solving the nonstationary problem (1.1)–(1.3). For such flows, two types of asymptotics—spatial and temporal—are of interest, with the wake being a feature of the spatial ones, of course. As far as we know, these spatial asymptotics were studied by only two authors up to now, namely Knightly [18] and Mizumachi [23]. Both succeeded in detecting a wake, but only under restrictive assumptions. Knightly [18] required various smallness conditions, and he had to introduce an assumption on the decay of the trace of the stress tensor on S_T, since it is not known how to control this trace. Mizumachi [23] assumed that the exterior force f vanishes, the initial velocity a is close to a stationary solution to (1.1)–(1.3), and the velocity decays pointwise in time and in space. Neither of these authors considered the spatial decay of the pressure or of the gradient of the velocity. Concerning temporal asymptotics, they were investigated by Masuda [22], Shibata [26], and Enomoto and Shibata [11], in the following way: Some L^p- or Sobolev norm of a solution to (1.1)–(1.3) is taken with respect to the space variables; then the behavior of this norm is studied for large values of the time variable. The question of how to exhibit a wake is not considered. It should be further remarked that in the case $\Omega = \emptyset$—which does not interest us here—the Oseen term $\tau \cdot \partial_1 v$ may be eliminated by a suitable change of variables. In this way, some results on spatial asymptotics of flows verifying (1.1)–(1.3) in the case $\Omega = \emptyset$ (wherein, of course, the boundary condition $v|S_T = (-1, 0, 0)$ in (1.2) must be dropped) may be deduced from the theory of the Navier–Stokes system without the Oseen term, a theory which is much more developed than that of problem (1.1)–(1.3). We refer to Takahashi [33, section 2] for that approach.

As may be seen by the preceding remarks, the spatial asymptotics of flows satisfying (1.1)–(1.3) have not yet been investigated extensively up to now. What is more, for the Oseen linearization of (1.1)–(1.3), these asymptotics have not been studied at all, as far as we know. So it seems to be natural to undertake such a study. This is the aim of the work at hand.

So we consider the nonstationary Oseen system,

$$\partial_t u - \Delta_x u + \tau \cdot \partial_{x_1} u + \nabla_x \pi = f, \quad \operatorname{div}_x u = 0 \quad \text{in } Z_\infty, \tag{1.4}$$

with boundary and initial conditions,

$$u \,|\, S_\infty = 0, \quad u(x,t) \to 0 \ (|x| \to \infty) \ \text{ for } t \in (0,\infty), \tag{1.5}$$

$$u(x,0) = a(x) \ \text{ for } x \in U, \tag{1.6}$$

where the velocity is now denoted by u, and the pressure by π. Since we are interested in results which are uniform in $T \in (0,\infty)$, we consider the case $T = \infty$ in (1.4)–(1.6).

We further note that in (1.5), we prescribe vanishing boundary values on S_∞, instead of the constant boundary data $(-1,0,0)$. This is no restriction of generality because the boundary data $(-1,0,0)$ are smooth, independent of t, and have zero flux on $\partial\Omega$. Therefore they may be extended to U by a solenoidal smooth function with compact support (see [15, Corollary II.3.3]) and then subsumed in f. Even boundary data which are smooth and stationary, but do not have zero flux on $\partial\Omega$, can be handled because they may be extended to U by a smooth solenoidal function with fast decay; cf. [5, Theorem 3.5] for more details. Concerning boundary data which depend on t, we will be more general than in (1.5) and admit functions on S_∞ belonging to a Sobolev space which we will denote by H_∞, and whose significance will be explained below.

Regularity results for problem (1.4)–(1.6), in Sobolev as well as in Hölder spaces, were proved by Solonnikov in [30], where a system more general than (1.4) was considered. Asymptotic behavior of solutions to (1.4)–(1.6) was studied by Shibata and colleagues [10], [11], [19], [26], who considered time decay of spatial L^p-norms of solutions to (1.4)–(1.6). As indicated above, the focus of the work at hand lies on spatial asymptotics. More precisely, we establish pointwise estimates of the velocity, the pressure, their spatial derivatives of arbitrary order, and the first time derivative of the velocity, for x outside a ball containing $\overline{\Omega}$ (Corollary 5.6). (Under our assumptions on f, the time derivative of the pressure and higher time derivatives of the velocity need not exist.) Our estimates are uniform in t, and their right-hand sides do not depend on the solution of the problem. In particular, the trace of the stress tensor on S_∞ does not enter into the upper bounds we derive. Moreover, we exhibit the wake phenomenon, which actually appears more clearly than in the available theory of the stationary case; see the remark following Corollary 5.6 in this respect. By reducing the spatial decay rate by an epsilon, we obtain estimates whose right-hand sides contain a factor tending to zero for large t (Corollary 5.7). Whether this factor may be replaced by a term of the form $(1+t)^{-\mu}$ for some $\mu > 0$ remains an open question.

The preceding results will be derived under the assumption that the given functions f and a have compact support with respect to the space variable, uniform with respect to t in the case of f. Actually, it would be enough to require a sufficiently strong decay of the data for $|x| \to \infty$, similarly as in [8] in the stationary case. But this type of condition would give rise to considerable additional technical difficulties. Also, the link between the decay rate of the derivatives of the solution and the order of these derivatives would no longer be clearly visible.

Of course, neither of these assumptions—compact support or strong decay of the data—are well suited to an eventual extension of our theory to the nonlinear problem (1.1)–(1.3). However, there is an aspect of our theory which we think makes the present article a significant contribution to a future theory on (1.1)–(1.3). In fact, our decay estimates are deduced from an integral representation of the solution to the linear problem (1.4)–(1.6) (see (5.2) or Corollary 5.2). This "representation formula" might serve as a starting point of a nonlinear theory, so when deriving this formula, we were striving to keep the assumptions on the data f and a as weak as possible. The assumptions on f in particular are crucial in view of the nonlinear case. In fact, any attempt to apply our linear theory to the nonlinear problem (1.1)–(1.3) would probably require that, in our representation formula, the function f be replaced by the term $f - \tau \cdot (v \cdot \nabla)v$, where v denotes the velocity part of a flow satisfying (1.1)–(1.3). So it is perhaps interesting to note that the formula in question will be established for a function f belonging to $L^2(0,\infty,L^1(\mathbb{R}^3)^3)$ and to $L^2\big(\mathfrak{U} \times (0,\infty)\big)^3$, with $\mathfrak{U}$ denoting a bounded vicinity of $\partial\Omega$ (Corollaries 5.1, 5.2). These conditions should be compared with the regularity of $(v \cdot \nabla)v$. In the case of a weak solution to (1.1)–(1.3), v may be expected to belong to $L^\infty(0,T,L^2(U)^3)$, and its gradient to $L^2(Z_T)^3$, at least if $T<\infty$, so $(v\cdot\nabla)v \in L^2(0,T,L^1(U)^3)$. If the solution in question is even a strong one, it should further be expected that $(v\cdot\nabla)v \in L^2\big(A\times(0,T)\big)^3$, at least for bounded subsets A of U and for $T<\infty$; cf. [27, Theorem V.2.3.1]. Due to results by Heywood [17, section 5], the preceding relations are indeed satisfied for strong solutions of (1.1)–(1.3) if $T<\infty$ and if the lifespan of the strong solution in question is somewhat larger than T. What seems to be an open question, however, is under which conditions these relations are valid for $T=\infty$. Actually, existence and regularity of solutions to (1.1)–(1.3) have not been studied very extensively: Besides [17], we know only of [30] (strong solutions, global in time for small data, or local in time), and of the articles [26], [11] by Shibata and Enomoto and Shibata, who constructed solutions with velocity part in $C^0([0,\infty), L^3(U)^3)$, under the assumption that the data are small.

Let us briefly outline our theory. We will begin by introducing two volume potentials (Definition 2.10), which we will denote by $\mathfrak{R}^{(\tau)}(f)$ and $\mathfrak{I}^{(\tau)}(a)$, respectively. The first one—$\mathfrak{R}^{(\tau)}(f)$—is defined as the convolution of f and an Oseen fundamental solution, with the convolution performed with respect to the time and space variables. The second one—$\mathfrak{I}^{(\tau)}(a)$—consists of a convolution with respect to the space variables only, involving the initial data a and a fundamental solution of the scalar Oseen equation $\partial_t w - \Delta_x w + \tau\cdot\partial_{x_1} w = g$. The sum $\mathfrak{R}^{(\tau)}(f)+\mathfrak{I}^{(\tau)}(a)$ constitutes the velocity part of a flow verifying the Cauchy problem

$$(1.7)\qquad \partial_t w - \Delta_x w + \tau\cdot\partial_{x_1} w + \nabla_x \varrho = f,\quad \operatorname{div}_x w = 0 \text{ in } \mathbb{R}^3\times(0,\infty),$$

$$(1.8)\qquad w(x,t)\to 0\ (|x|\to\infty) \text{ for } t\in(0,\infty),\quad w(x,0)=a(x) \text{ for } x\in\mathbb{R}^3$$

(Lemmas 2.11 and 2.14 and Theorem 2.16). Thus, in order to solve the initial-boundary value problem (1.4)–(1.6), we have to add a boundary correction. It will take the form of a single layer potential, which we will denote by $\mathfrak{V}^{(\tau)}(\phi)$, and which is defined as the integral of a convolution product of a layer function $\phi\in L^2(S_\infty)^3$ and an Oseen fundamental solution, with the space variables being integrated on $\partial\Omega$ (Definition 2.20). For any $\phi\in L^2(S_\infty)^3$, the sum $u(\phi) := \big(\mathfrak{R}^{(\tau)}(f)+\mathfrak{I}^{(\tau)}(a)+\mathfrak{V}^{(\tau)}(\phi)\big)\,|\,Z_\infty$ solves the equations in (1.4)–(1.6) (see Lemma 2.21, (1.7), (1.8)), except the boundary condition on S_T, which need not hold in general. In order to satisfy this latter condition as well, we have to look for a function ϕ verifying the integral equation

$$(1.9)\qquad \mathfrak{V}^{(\tau)}(\phi)\,|\,S_\infty = -\big(\mathfrak{R}^{(\tau)}(f)+\mathfrak{I}^{(\tau)}(a)\big)\,|\,S_\infty.$$

This means that we want to solve (1.4)–(1.6) by the method of integral equations, which reduces the initial-boundary value problem (1.4)–(1.6) to the preceding integral equation on S_∞. (Actually, since we will consider the more general boundary condition $u|S_\infty = b$, the boundary data b will appear as an additional term on the right-hand side of (1.9); see (5.1).) By this approach, we are led to the study of the integral equation

$$(1.10)\qquad \mathfrak{V}^{(\tau)}(\phi)\,|\,S_\infty = c,$$

where $c: S_\infty \mapsto \mathbb{R}^3$ is given and ϕ is unknown. Starting from a result by Shen [25] on the nonstationary Stokes system, we showed in [7] that (1.10) admits a unique solution $\phi\in L^2_n(S_\infty)$ if c belongs to H_∞, where the space $L^2_n(S_\infty)$ consists of all square-integrable vector fields on S_∞ with vanishing flux through $\partial\Omega$ for a.e. $t\in(0,\infty)$, and where H_∞ is a Sobolev space on S_∞ involving a fractional derivative with respect to t. Precise definitions may be found in section 2. The result from [7] is stated as Theorem 2.24 below.

The definition of the norm $\| \ \|_{H_\infty}$ of the space H_∞ looks rather complicated. In particular, the term involving the time derivative of the normal component of the velocity may seem strange. But the norm $\| \ \|_{H_\infty}$ was chosen in [25] in such a way that the inequality $\|\phi\|_2 \leq \mathfrak{C} \cdot \|\mathfrak{V}^{(0)}(\phi)|S_\infty\|_{H_\infty}$ holds for functions $\phi \in L^2_n(S_\infty)$, with $\mathfrak{V}^{(0)}(\phi)$ denoting the single layer potential associated with the time-dependent Stokes system (see [25, pp. 365–367]). The preceding inequality implies L^2-estimates of Stokes flows against the norm $\| \ \|_{H_\infty}$ of the Dirichlet boundary data (see [25, Theorem 5.2.1]). On the other hand, in the half-space case, a Stokes flow may be expressed explicitly in terms of the data (see Solonnnikov [29, p. 36]). This representation leads to L^p-estimates involving the time derivative of the normal component of the velocity; see [29, equation (145)], and compare with a similar remark in [25, p. 297]. Thus it is clear that an analogous term must appear in the case of an exterior domain; this term manifests itself in the definition of $\| \ \|_{H_\infty}$.

The result from [7] stating that (1.10) admits a solution if $c \in H_\infty$ does not suffice to solve (1.9). In fact, we still have to show that the functions $\mathfrak{R}^{(\tau)}(f) \,|\, S_\infty$ and $\mathfrak{I}^{(\tau)}(a) \,|\, S_\infty$ belong to H_∞, under assumptions on f and a which are as weak as possible, for the reasons explained above. The proof of this property of $\mathfrak{R}^{(\tau)}(f)$ and $\mathfrak{I}^{(\tau)}(a)$ turned out to be the main difficulty of our theory, due to the complicated structure of the norm of H_∞. In fact, this proof makes up the greatest part of this article (sections 3 and 4). Once it has been carried through, we may conclude that there is $\phi \in L^2_n(S_\infty)$ with (1.9) (Corollary 5.1). Thus the sum $\left(\mathfrak{R}^{(\tau)}(f) + \mathfrak{I}^{(\tau)}(a) + \mathfrak{V}^{(\tau)}(\phi)\right) | \, Z_\infty$ is a solution of (1.4)–(1.6), and at the same time an integral representation of that solution (Corollary 5.1). In fact, this sum is the representation formula we mentioned above. As these remarks indicate, our approach also yields an existence result for solutions to (1.4)–(1.6) (Corollary 5.1), but this result is neither the motivation nor the main objective of this work.

Less effort is necessary to show that the function $u(\phi) \,|\, Z_{T'}$ belongs to the spaces $L^2(0,T',H^1(U)^3)$ and $H^1(0,T',V')$ for any $T' \in (0,\infty)$, with V denoting the closure of the solenoidal C_0^∞-vector fields on U with respect to the norm of $H^1(U)^3$. It was already shown in [6] that $\mathfrak{V}^{(\tau)}(\phi) \,|\, Z_{T'}$ belongs to $L^2(0,T',H^1(U)^3)$ and to $H^1(0,T',V')$ (Theorem 2.22), so it suffices to derive analogous relations for $\mathfrak{R}^{(\tau)}(f)$ (Corollary 2.17) and $\mathfrak{I}^{(\tau)}(a)$ (Corollary 2.18). Note that the space $L^2(0,T',H^1(U)^3) \cap H^1(0,T',V')$ is interesting because it is a uniqueness space for weak solutions to (1.4)–(1.6). We refer to [34, section 3.1] in this respect. The theory presented there, pertaining to the nonstationary Stokes system, also yields existence and uniqueness of weak solutions to the Oseen initial-boundary value problem (1.4)–(1.6). In view of the preceding remarks, it is obvious that our representation formula holds for these weak solutions (Corollary 5.2), provided, of course, that the functions f and a verify the assumptions of our theory. These assumptions are somewhat stronger than those required for the existence of a weak solution.

2. Notations, definitions. Some results on Oseen potentials. Recall the bounded Lipschitz domain Ω and the notations $U := \mathbb{R}^3 \backslash \overline{\Omega}$, $Z_T := U \times (0,T)$, $S_T := \partial\Omega \times (0,T)$, for $T \in (0,\infty]$, introduced in section 1. The set Ω was supposed to have a connected boundary, so Ω and U are connected. Let $n^{(\Omega)}$ denote the outward unit normal to Ω.

Following [21, pp. 269–270, 305–306], we choose $k(\Omega) \in \mathbb{N}$, $\alpha(\Omega) \in (0,\infty)$, orthonormal matrices $A_1^{(\Omega)}, \dots, A_{k(\Omega)}^{(\Omega)} \in \mathbb{R}^{3\times 3}$, vectors $C_1^{(\Omega)}, \dots, C_{k(\Omega)}^{(\Omega)} \in \mathbb{R}^3$, and Lipschitz continuous functions $a_1^{(\Omega)}, \dots, a_{k(\Omega)}^{(\Omega)} : [-\alpha(\Omega), \alpha(\Omega)]^2 \mapsto \mathbb{R}$ such that the following properties hold true: Defining the sets $\Delta^\gamma, \Lambda_i^\gamma, U_i^\gamma$ by

$$\Delta^\gamma := \big(-\gamma\cdot\alpha(\Omega),\ \gamma\cdot\alpha(\Omega)\big)^2, \quad \Lambda_i^\gamma := \{A_i^{(\Omega)}\cdot(\eta, a_i^{(\Omega)}(\eta)) \ + \ C_i^{(\Omega)} \ : \ \eta\in\Delta^\gamma\},$$
$$U_i^\gamma := \{A_i^{(\Omega)}\cdot(\eta, a_i^{(\Omega)}(\eta)+r) \ + \ C_i^{(\Omega)} \ : \ \eta\in\Delta^\gamma,\ r\in(-\gamma\cdot\alpha(\Omega),\, \gamma\cdot\alpha(\Omega))\}$$

for $i \in \{1,\dots,k(\Omega)\}$, $\gamma\in(0,1]$, and the function $H^{(i)} : \Delta^1\times(-\alpha(\Omega),\alpha(\Omega)) \mapsto U_i^1$ by

$$H^{(i)}(\eta, r) := A_i^{(\Omega)}\cdot(\eta, a_i^{(\Omega)}(\eta)+r) \ + \ C_i^{(\Omega)} \quad \text{for } \eta\in\Delta^1,\ r\in(-\alpha(\Omega),\alpha(\Omega)),$$

we have

$$U_i^1\cap U \ = \ H^{(i)}(\Delta^1\times(-\alpha(\Omega),0)), \quad U_i^1\cap\Omega \ = \ H^{(i)}\big(\Delta^1\times(0,\alpha(\Omega))\big),$$
$$U_i^1\cap\partial\Omega = \Lambda_i^1 \ \text{ for } i\in\{1,\dots,k(\Omega)\}, \quad \partial\Omega = \bigcup_{i=1}^{k(\Omega)}\Lambda_i^{1/4}.$$

These relations imply that

$$\int_{U_i^1} g(x)\,dx \ = \int_{-\alpha(\Omega)}^{\alpha(\Omega)}\int_{\Delta^1}(g\circ H^{(i)})(\eta,s)\,d\eta\,ds \qquad \text{for } g\in L^1(U_i^1), \tag{2.1}$$

and that there is a constant $\mathcal{D}_1 > 0$ with

$$\begin{aligned} &|\,H^{(i)}(\rho,\kappa) \ - \ H^{(i)}(\eta,\kappa')\,| \quad \ge \quad \mathcal{D}_1\cdot(|\rho-\eta|+|\kappa-\kappa'|)\\ &\text{for } \rho,\eta\in\Delta^1,\ \ \kappa,\kappa'\in(-\alpha(\Omega),\alpha(\Omega)),\ \ i\in\{1,\dots,k(\Omega)\}. \end{aligned} \tag{2.2}$$

We further introduce functions $h^{(i)} : \Delta^1 \mapsto \Lambda_i^1$, $J_i : \Delta^1\mapsto\mathbb{R}$ by setting

$$h^{(i)}(\eta) := A_i^{(\Omega)}\cdot(\eta, a_i^{(\Omega)}(\eta)) \ + \ C_i^{(\Omega)}, \qquad J^{(i)}(\eta) := \left(1+\sum_{r=1}^{2}|\partial_r h^{(i)}(\eta)|^2\right)^{1/2}$$

for $\eta\in\Delta^1$, $i\in\{1,\dots,k(\Omega)\}$. Then we have for any integrable function $g:\partial\Omega\mapsto\mathbb{R}$ and for $i\in\{1,\dots,k(\Omega)\}$ that

$$\int_{\Lambda_i^1} g\,d\Omega \ = \int_{\Delta^1}(g\circ h^{(i)})(\eta)\cdot J^{(i)}(\eta)\,d\eta. \tag{2.3}$$

Moreover, let $m^{(\Omega)}\in C_0^\infty(\mathbb{R}^3)^3$ be a nontangential vector field to Ω. This means that the equation $|m^{(\Omega)}(x)| = 1$ holds for x from a neighborhood of $\partial\Omega$ in $\mathbb{R}^3$, and that there are constants $\mathcal{D}_2, \mathcal{D}_3\in(0,\infty)$ such that

$$|x+\delta\cdot m^{(\Omega)}(x) - x' - \delta'\cdot m^{(\Omega)}(x')| \ge \mathcal{D}_2\cdot\big(|x-x'|+|\delta-\delta|'|\big) \tag{2.4}$$

for $x, x'\in\partial\Omega$, $\delta,\delta'\in[-\mathcal{D}_3,\mathcal{D}_3]$, and

$$x+\delta\cdot m^{(\Omega)}(x)\in U,\ \ x-\delta\cdot m^{(\Omega)}(x)\in\Omega \quad \text{for } x\in\partial\Omega,\ \delta\in(0,\mathcal{D}_3]. \tag{2.5}$$

Some indications on how to construct such a field are given in [24, p. 246]. Note that since Ω is only Lipschitz bounded, the relations in (2.4) and (2.5) do not hold in general when $m^{(\Omega)}$ is replaced by the outward unit normal to Ω. We further observe that

$$\operatorname{dist}(U_i^{1/4},\ \partial\Omega\backslash\Lambda_i^{1/2}) > 0 \ \text{ for } \ i \in \{1,\dots,k(\Omega)\}, \ \text{ and } \ \operatorname{dist}(\partial\Omega,\ \mathbb{R}^3\backslash \cup_{i=1}^{k(\Omega)} U_i^{1/4}) > 0.$$

Thus there is a constant $\mathcal{D}_4 > 0$ such that

$$(2.6) \qquad \begin{aligned} &|x-y| \geq \mathcal{D}_4 \ \text{ for } \ x \in \partial\Omega, \quad y \in \mathbb{R}^3\backslash \cup_{i=1}^{k(\Omega)} U_i^{1/4}, \\ &\text{and for } \ x \in \partial\Omega\backslash\Lambda_i^1, \quad y \in U_i^{1/2}, \quad i \in \{1,\dots,k(\Omega)\}. \end{aligned}$$

As explained in the proof of [6, Lemma 3.4], the relations in (2.4) and (2.5) imply that there is a constant $\mathfrak{D}_5 > 0$ depending only on Ω such that

$$(2.7) \qquad \begin{aligned} &|x - y - \kappa \cdot m^{(\Omega)}(y)| \geq \mathfrak{D}_5 \cdot (|x-y| + \kappa), \\ &|z - y + \kappa \cdot m^{(\Omega)}(y)| \geq \mathfrak{D}_5 \cdot (|z-y| + \kappa) \end{aligned}$$

for $\kappa \in (0, \mathfrak{D}_3]$, $y \in \partial\Omega$, $x, z \in \mathbb{R}^3$ with $\operatorname{dist}(x, \overline{\Omega}) < \mathfrak{D}_2 \cdot \kappa/2$ and $\operatorname{dist}(z, U) < \mathfrak{D}_2 \cdot \kappa/2$.

Put $B_r := \{y \in \mathbb{R}^3 \,:\, |y| < r\}$. We fix some $R_0 > 0$ with $\overline{\Omega} \subset B_{R_0/2}$. We further define $|\alpha| := \alpha_1 + \alpha_2 + \alpha_3$ (length of α) for multi-indices $\alpha \in \mathbb{N}_0^3$. Put $e_1 := (1,0,0)$, and

$$(2.8) \qquad s(x) := |x| - x_1 \qquad \text{for } \ x \in \mathbb{R}^3.$$

Let $A \subset \mathbb{R}^3$. Put $A^c := \mathbb{R}^3\backslash A$. The symbol χ_A stands for the characteristic function of A (equal to 1 on A, zero elsewhere).

If $\mathfrak{H}$ is a space consisting of functions from a set B into $\mathbb{R}$, we put $\mathfrak{H}^3 := \{F : B \mapsto \mathbb{R}^3 \,:\, F_1, F_2, F_3 \in \mathfrak{H}\}$. If $\|\ \|$ is a norm on $\mathfrak{H}$, we will use the same notation $\|\ \|$ for the norm $(\sum_{j=1}^3 \|F_i\|^2)^{1/2}$ of $\mathfrak{H}^3$.

Suppose that $A \subset \mathbb{R}^3$ is open. Let $p \in [1,\infty]$. The usual norm of the Lebesgue space $L^p(A)$ is denoted by $\|\ \|_p$. We define $L^p_\sigma(A)$ as the closure of the set $\{\varphi \in C_0^\infty(A)^3 \,:\, \operatorname{div}\varphi = 0\}$ with respect to the norm of $L^p(A)^3$. If $m \in \mathbb{N}$, we write $H^m(A)$ for the usual Sobolev space of order m on A, with exponent 2. The usual norm of this space is denoted by $\|\ \ \|_{m,2}$. For $s \in (0,1)$, let $H^s(A)$ be the Sobolev space defined by the intrinsic norm with exponent $p = 2$ defined in [1, section 7.51]. This norm is denoted by $\|\ \|_{s,2}$. For $\epsilon \in (0,1]$, let $H^\epsilon_\sigma(U)$ denote the closure of the set $\{v \in C_0^\infty(U)^3 \,:\, \operatorname{div} v = 0\}$ with respect to the norm of $H^\epsilon(U)$. Put $V := H^1_\sigma(U)$. We equip V with the norm $\|\ \|_{1,2}$ of $H^1(U)^3$. The symbol $\|\ \|_{V'}$ is used for the canonical norm of the dual space V' of V.

The L^2-norm of functions on $\partial\Omega$ will be denoted by $\|\ \|_2$. The Sobolev space $H^1(\partial\Omega)$ is to be defined in the standard way (see [21, section III.6], for example). Let $\|\ \|_{1,2}$ denote the usual norm of this space with respect to some local coordinates of $\partial\Omega$ (see [21, section III.6.7]). The notation $\|\ \|_{H^1(\partial\Omega)'}$ stands for the canonical norm of the dual space $H^1(\partial\Omega)'$ of $H^1(\partial\Omega)$. Put

$$L^2_n(\partial\Omega) := \left\{ v \in L^2(\partial\Omega)^3 \,:\, \int_{\partial\Omega} v \cdot n^{(\Omega)}\, d\Omega = 0 \right\}.$$

Let $T \in (0,\infty]$. (Note that the case $T = \infty$ is admitted.) Then we put

$$\begin{aligned} &L^2_n(S_T) := \{v \in L^2(S_T)^3 \,:\, v(\,\cdot\,,t) \in L^2_n(\partial\Omega) \ \text{ for almost every } \ t \in (0,T)\}, \\ &\widetilde{H}_T := \{v\,|\,S_T \,:\, v \in C_0^\infty(\mathbb{R}^4)^3,\ v\,|\,\mathbb{R}^3 \times (-\infty, 0] = 0\}. \end{aligned}$$

For $v \in C^1\big((-\infty, T)\big)$ with $v \,|\, (-\infty, 0] = 0$, and for $t \in (0, T)$, we put

$$\partial_t^{1/2} v(t) := \Gamma(1/2)^{-1} \cdot \partial_t \left(\int_0^t (t-r)^{-1/2} \cdot v(r)\, dr \right)$$

("fractional derivative of v"), where Γ denotes the usual gamma function. We further define

$$\partial_4^{1/2} v(x,t) := \partial_t^{1/2}\big(v(x, \,\cdot\,)\big)(t) \quad \text{for } (x,t) \in \mathbb{R}^3 \times (0,T),\ v \in \widetilde{H}_T.$$

For any $v \in L^2(S_T)$, we may define $F_v \in L^2\big(0, T, H^1(\partial\Omega)'\big)$ by setting

$$F_v(t)(\sigma) := \int_{\partial\Omega} v(x,t) \cdot \sigma(x)\, dx \quad \text{for } \sigma \in H^1(\partial\Omega) \text{ and for almost every } t \in (0,T).$$

We will write v instead of F_v. For $v \in \widetilde{H}_T$, set

$$\|v\|_{H_T} := \left(\int_0^T \left(\|v(\,\cdot\,, t) \,|\, \partial\Omega\|_{1,2}^2 + \int_{\partial\Omega} \big| \partial_4^{1/2} v(x,t) \big|^2 \, d\Omega(x) + \|\partial_t v(\,\cdot\,, t) \cdot n^{(\Omega)}\|_{H^1(\partial\Omega)'}^2 \right) dt \right)^{1/2}.$$

The mapping $\| \ \|_{H_T}$ is a norm on $\widetilde{H}_T$. Let the space H_T consist of all functions $v \in L_n^2(S_T)$ such that there exists a sequence (w_n) in $\widetilde{H}_T$ with the property that $\|v - w_n \,|\, S_T\|_2 \to 0$, and such that $(w_n \,|\, S_T)$ is a Cauchy sequence with respect to the norm $\| \ \|_{H_T}$. This means in particular that the sequence $\big(\|w_n \,|\, S_T\|_{H_T}\big)$ is convergent. Its limit value does not depend on the choice of the sequence (w_n) with the above properties. Thus, for $v \in H_T$, we may define the quantity $\|v\|_{H_T}$ in an obvious way. The mapping $\| \ \|_{H_T}$ is a norm on H_T, and the pair $\big(H_T, \| \ \|_{H_T}\big)$ is a Banach space.

Again let A be an open set in $\mathbb{R}^3$, and let $p \in [1, \infty]$. Suppose that $T_1, T_2 \in [0, \infty]$ with $T_1 < T_2$. Take $\sigma = 1$ or $\sigma = 3$. Let $\mathfrak{B}$ be a Banach space consisting of functions, respectively, from A into $\mathbb{R}^\sigma$ or from $\partial\Omega$ into $\mathbb{R}^\sigma$. Then, if a function $g : A \times (T_1, T_2) \mapsto \mathbb{R}^\sigma$ or $g : \partial\Omega \times (T_1, T_2) \mapsto \mathbb{R}^\sigma$ may be considered as a mapping from $L^p(T_1, T_2, \mathfrak{B})$, we write $\|g\|_{L^p(T_1, T_2, \mathfrak{B})}$ for the corresponding norm of g. An analogous notation is used for the norm of the space $H^1(T_1, T_2, \mathfrak{B})$. The L^2-norm of a function g as above will be denoted by $\|g\|_2$.

To recapitulate, the notations $\| \ \|_p$ (for $p \in [1, \infty]$) and $\| \ \|_{s,2}$ (for $s \in \mathbb{N}$ or $s \in (0,1)$) stand for norms of functions defined on an open set A in $\mathbb{R}^3$. The symbols $\| \ \|_2$ and $\| \ \|_{1,2}$ are additionally used for functions with domain $\partial\Omega$, and in the case of $\| \ \|_2$, for functions depending on space and time. Other norms which will be applied to the latter type of functions are $\| \ \|_{H_T}$, $\| \ \|_{L^p(T_1,T_2,\mathfrak{B})}$ and $\| \ \|_{H^1(T_1,T_2,\mathfrak{B})}$.

Let us present some auxiliary results.

LEMMA 2.1 (see [13, Lemma 4.3]). *Let $\beta \in (1, \infty)$. Then there is a constant $C = C(\beta)$ such that*

$$\int_{\partial B_r} \big(1 + s(x)\big)^{-\beta} \, do_x \le C \cdot r \quad \text{for } r \in (0, \infty),$$

with $s(x)$ defined in (2.8).

LEMMA 2.2 (see [8, Lemma 4.8]). *There is $C > 0$ such that for $x, y \in \mathbb{R}^3$, $\kappa \in (0, \infty)$,*

$$\big(1 + \kappa \cdot s(x-y)\big)^{-1} \le C \cdot \max\{1, \kappa\} \cdot (1 + |y|) \cdot \big(1 + \kappa \cdot s(x)\big)^{-1}.$$

LEMMA 2.3 (see [7, Lemma 3]). *Let $\mathfrak{A}$, $\mathfrak{B}$ be measurable spaces equipped with the measure μ and ν, respectively. Let F, F_1, $F_2 : \mathfrak{A} \times \mathfrak{B} \mapsto [0, \infty)$, $f : \mathfrak{B} \mapsto [0, \infty)$ be measurable functions. Suppose that $F = F_1 \cdot F_2$. Then*

$$\left(\int_{\mathfrak{A}} \left(\int_{\mathfrak{B}} F(x,y) \cdot f(y) \, d\nu(y) \right)^2 d\mu(x) \right)^{1/2} \leq C \cdot \|f\|_2, \tag{2.9}$$

with

$$C := \left(\sup \left\{ \int_{\mathfrak{B}} F_1(x,y)^2 \, d\nu(y) \; : \; x \in \mathfrak{A} \right\} \cdot \sup \left\{ \int_{\mathfrak{A}} F_2(x,y)^2 \, d\mu(x) \; : \; y \in \mathfrak{B} \right\} \right)^{1/2}.$$

Note that in the preceding lemma, the functions f, F_1, F_2, and F are nonnegative. Therefore, applying a standard convention in the theory of Lebesgue integration, the term $\|f\|_2$ and all the integrals appearing in the lemma make sense, but may take the value ∞. This allows us to state the lemma in a concise way, because we may avoid a cumbersome list of assumptions on F_1, F_2, and F, and also a cumbersome list of conclusions. For example, the lemma states implicitly that if f is L^2, and if the quantities appearing in the definition of the constant C are finite, the integral on the left-hand side of (2.9) is finite.

THEOREM 2.4. *For $n \in \mathbb{N}$, put $\Omega_n := \{x \in \mathbb{R}^3 \; : \; dist(x, \overline{\Omega}) < \mathfrak{D}_2 \cdot \mathfrak{D}_3/(2 \cdot n)\}$, with $\mathfrak{D}_2$, $\mathfrak{D}_3$ from (2.4) and (2.5). (Note that $\overline{\Omega} \subset \Omega_n$.)*

For $v \in H^1(\partial\Omega)$, $n \in \mathbb{N}$, there is a function $E^{(n)}(v) \in C^\infty(\Omega_n)$ such that $\|E^{(n)}(v) \,|\, \partial\Omega - v\|_2 \to 0$ $(n \to \infty)$ for $v \in H^1(\partial\Omega)$, and such that there are constants C_1, $C_2(p)$ for $p \in [2,3)$ with

$$\|E^{(n)}(v) \,|\, \partial\Omega\|_2 + \|\nabla E^{(n)}(v) \,|\, \partial\Omega\|_2 \leq C_1 \cdot \|v\|_{1,2},$$
$$\|\nabla E^{(n)}(v) \,|\, \Omega\|_p \leq C_2(p) \cdot \|v\|_{1,2}$$

for $n \in \mathbb{N}$, $v \in H^1(\partial\Omega)$, $p \in [2,3)$.

Proof. In the following, the letter $\mathfrak{C}$ stands for constants depending only on Ω. Let $p \in [2,3)$. We will write $\mathfrak{C}(p)$ for constants depending on Ω and p.

We extend each $v \in H^1(\partial\Omega)$ to a harmonic function on Ω which has the form of a single layer potential. In fact, according to [35, Theorem 3.3], for $v \in H^1(\partial\Omega)$, there is a unique function $T(v) \in L^2(\partial\Omega)$ with

$$v(x) = \int_{\partial\Omega} (4 \cdot \pi)^{-1} \cdot |x-y|^{-1} \cdot T(v)(y) \, d\Omega(y) \quad \text{for } x \in \partial\Omega, \tag{2.10}$$
$$\|T(v)\|_2 \leq \mathfrak{C} \cdot \|v\|_{1,2}.$$

Put $\epsilon_n := \mathfrak{D}_3/n$ for $n \in \mathbb{N}$. Then, for $\phi \in L^2(\partial\Omega)$, we define

$$\mathfrak{E}^{(n)}(\phi)(x) := \int_{\partial\Omega} (4 \cdot \pi)^{-1} \cdot |x - y - \epsilon_n \cdot m^{(\Omega)}(y)|^{-1} \cdot \phi(y) \, d\Omega(y) \quad (n \in \mathbb{N}, \; x \in \Omega_n),$$
$$\mathfrak{E}(\phi)(x) := \int_{\partial\Omega} (4 \cdot \pi)^{-1} \cdot |x-y|^{-1} \cdot \phi(y) \, d\Omega(y) \quad (x \in \Omega).$$

By (2.7), we have $\mathfrak{E}^{(n)}(\phi) \in C^\infty(\Omega_n)$ for $n \in \mathbb{N}$, $\phi \in L^2(\partial\Omega)$. Obviously $\mathfrak{E}(\phi) \in C^\infty(\Omega)$. A simple estimate involving (2.7) shows that

$$\Big| \big(x - y - \epsilon_n \cdot m^{(\Omega)}(y)\big)_l \cdot |x - y - \epsilon_n \cdot m^{(\Omega)}(y)|^{-3} - \big(x - y - \epsilon_n \cdot m^{(\Omega)}(x)\big)_l$$
$$\cdot |x - y - \epsilon_n \cdot m^{(\Omega)}(x)|^{-3} \Big|$$
$$\leq \mathfrak{C} \cdot \epsilon_n^{1/2} \cdot |x-y|^{-3/2} \quad (n \in \mathbb{N}, \; 1 \leq l \leq 3, \; x, y \in \partial\Omega, \; x \neq y).$$

Thus, referring to Lemma 2.3 with $\mathfrak{A} = \mathfrak{B} = \partial\Omega$, we find

$$\text{(2.11)} \qquad \Big(\int_{\partial\Omega} |\partial^\alpha \mathfrak{E}^{(n)}(\phi)(x) - \big(\partial^\alpha \mathfrak{E}(\phi)\big)\big(x - \epsilon_n \cdot m^{(\Omega)}(x)\big)|^2 \, d\Omega(x) \Big)^{1/2}$$
$$\leq \mathfrak{C} \cdot \epsilon_n^{1/2} \cdot \|\phi\|_2 \quad (n \in \mathbb{N},\ \phi \in L^2(\partial\Omega),\ \alpha \in \mathbb{N}^3 \text{ with } |\alpha| \leq 1).$$

On the other hand, by [35, Lemma 1.3],

$$\text{(2.12)} \qquad \Big(\int_{\partial\Omega} |\big(\partial^\alpha \mathfrak{E}(\phi)\big)\big(x - \epsilon_n \cdot m^{(\Omega)}(x)\big)|^2 \, d\Omega(x) \Big)^{1/2} \leq \mathfrak{C} \cdot \|\phi\|_2$$

for n, ϕ, α as in (2.11). We may conclude from (2.10)–(2.12) that

$$\text{(2.13)} \qquad \|\mathfrak{E}^{(n)}\big(T(v)\big) \,|\, \partial\Omega\|_2 + \|\nabla \mathfrak{E}^{(n)}\big(T(v)\big) \,|\, \partial\Omega\|_2 \leq \mathfrak{C} \cdot \|v\|_{1,2}$$

for $n \in \mathbb{N}$, $v \in H^1(\partial\Omega)$. Moreover, since by Lemma 2.3

$$\Big(\int_{\partial\Omega} \Big(\int_{\partial\Omega} |x-y|^{-1} \cdot |\phi(y)| \, d\Omega(y) \Big)^2 d\Omega(x) \Big)^{1/2} \leq \mathfrak{C} \cdot \|\phi\|_2 \quad \text{for } \phi \in L^2(\partial\Omega),$$

we obtain $\|\big(\mathfrak{E}^{(n)}(T(v)) - \mathfrak{E}(T(v))\big) \,|\, \partial\Omega\|_2 \to 0$ for $n \to \infty$ by (2.7) and Lebesgue's theorem on dominated convergence. Thus, with (2.10), $\|\mathfrak{E}^{(n)}\big(T(v)\big) \,|\, \partial\Omega - v\|_2 \to 0$ $(n \to \infty)$.

Since $\partial\Omega \subset \cup_{i=1}^{k(\Omega)} \Lambda_i^{1/2}$, and in view of (2.7), we get for $n \in \mathbb{N}$, $\phi \in L^2(\partial\Omega)$ that

$$\text{(2.14)} \qquad \|\nabla \mathfrak{E}^{(n)}(\phi) \,|\, \Omega\|_p \leq \mathfrak{C} \cdot \Big(\int_\Omega \Big(\int_{\partial\Omega} |x-y|^{-2} \cdot |\phi(y)| \, d\Omega(y) \Big)^p dx \Big)^{1/p}$$
$$\leq \mathfrak{C} \cdot \sum_{i=1}^{k(\Omega)} \big(A_i(n,\phi) + B_i(n,\phi) \big),$$

where

$$A_i(n,\phi) := \Big(\int_{\Omega \backslash U_i^1} \Big(\int_{\Lambda_i^{1/2}} |x-y|^{-2} \cdot |\phi(y)| \, d\Omega(y) \Big)^p dx \Big)^{1/p},$$

and with $B_i(n,\phi)$ defined in the same way as $A_i(n,\phi)$, except that the domain of integration $\Omega \backslash U_i^1$ is replaced by $\Omega \cap U_i^1$ $\big(1 \leq i \leq k(\Omega)\big)$. Since $\text{dist}(\Lambda_i^{1/2}, \Omega \backslash U_i^1) > 0$ for $1 \leq i \leq k(\Omega)$, it is obvious that

$$\text{(2.15)} \qquad A_i(n,\phi) \leq \mathfrak{C} \cdot \|\phi\|_2 \quad \text{for } 1 \leq i \leq k(\Omega),\ n \in \mathbb{N},\ \phi \in L^2(\Omega).$$

Referring to (2.1) and (2.3), we have

$$B_i(n,\phi)$$
$$\leq \mathfrak{C} \cdot \Big(\int_{-\alpha(\Omega)}^{\alpha(\Omega)} \int_{\Delta^1} \Big(\int_{\Delta^{1/2}} |H^{(i)}(\varrho, r) - h^{(i)}(\eta)|^{-2} \cdot |\phi\big(h^{(i)}(\eta)\big)| \, d\eta \Big)^p d\varrho \, dr \Big)^{1/p}$$

(i, n, ϕ as in (2.15)). Now put $\alpha := 1 - 2/p$ if $p > 2$, and $\alpha := 1/4$ in the case $p = 2$. Then

$$B_i(n,\phi) \leq \mathfrak{C} \cdot \Big(\int_{-\alpha(\Omega)}^{\alpha(\Omega)} r^{-\alpha \cdot p} \cdot \int_{\Delta^1} \Big(\int_{\Delta^{1/2}} |\varrho - \eta|^{-2+\alpha} \cdot |\phi\big(h^{(i)}(\eta)\big)| \, d\eta \Big)^p d\varrho \, dr \Big)^{1/p}.$$

But $p < 3$, hence $\alpha \cdot p < 1$, so we may integrate in r. Then we apply the Hardy–Littlewood–Sobolev inequality (see [32, Theorem V.1.1]) in the case $p > 2$ (note that $1/p = 1/2 - \alpha/2$ in that case), and Lemma 2.3 if $p = 2$. It follows that $B_i(n, \phi) \le \mathfrak{C}(p) \cdot \|\phi \circ h^{(i)}\|_2$. This estimate, combined with (2.14) and (2.15), yields that $\|\nabla \mathfrak{E}^{(n)}(\phi)\|_p \le \mathfrak{C}(p) \cdot \|\phi\|_2$ for $\phi \in L^2(\partial\Omega)$; hence with (2.10), $\|\nabla \mathfrak{E}^{(n)}\big((T(v)\big) \,|\, \Omega\|_p \le \mathfrak{C}(p) \cdot \|v\|_{1,2}$. Thus we see that if we set $E^{(n)}(v) := \mathfrak{E}^{(n)}\big(T(v)\big)$ for $v \in H^1(\partial\Omega)$, $n \in \mathbb{N}$, all the properties stated in the theorem are satisfied. □

Next we introduce the fundamental solutions we will consider in what follows. Let $\mathfrak{H}$ denote the usual fundamental solution of the heat equation in $\mathbb{R}^3$, that is,

$$\mathfrak{H}(z,t) := (4 \cdot \pi \cdot t)^{-3/2} \cdot e^{-|z|^2/(4 \cdot t)} \quad \text{for } (z,t) \in \mathbb{R}^3 \times (0, \infty),$$
$$\mathfrak{H}(z,t) := 0 \quad \text{for } (z,t) \in \big(\mathbb{R}^3 \times (-\infty, 0]\big) \backslash \{0\}.$$

We further introduce a fundamental solution of the time-dependent Stokes system by setting as in [25]

$$\Gamma_{jk}(z,t) := \delta_{jk} \cdot \mathfrak{H}(z,t) + \int_t^\infty \partial_j \partial_k \mathfrak{H}(z,s)\, ds, \qquad (2.16)$$
$$E_k(x) := (4 \cdot \pi)^{-1} \cdot x_k \cdot |x|^{-3}$$

for $(z,t) \in G := \big(\mathbb{R}^3 \times [0,\infty)\big) \backslash \{0\}$, $x \in \mathbb{R}^3 \backslash \{0\}$, $1 \le j,k \le 3$. Finally we define the velocity part of a fundamental solution of the time-dependent Oseen system (with Reynolds number κ) by putting

$$\Lambda_{jk}(z,t,\kappa) := \Gamma_{jk}(z - \kappa \cdot t \cdot e_1, t) \quad \text{for } (z,t) \in G,\ j,k \in \{1, 2, 3\}.$$

An associated pressure part is given by the functions E_k introduced in (2.16).

LEMMA 2.5. *We have $\mathfrak{H} \in C^\infty(\mathbb{R}^4 \backslash \{0\})$, $\Gamma_{jk} \in C^\infty(G)$, and $E_k \in C^\infty(\mathbb{R}^3 \backslash \{0\})$ for $1 \le j,k \le 3$. Moreover, for $l \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^3$, there is $C = C(l,\alpha) > 0$, $\widetilde{C} = \widetilde{C}(\alpha) > 0$ with*

$$\big|\,\partial_t^l \partial_z^\alpha \mathfrak{H}(z,t)\,\big| + \big|\,\partial_t^l \partial_z^\alpha \Gamma_{jk}(z,t)\,\big| \le C \cdot (|z|^2 + t)^{-3/2 - |\alpha|/2 - l},$$
$$|\partial_x^\alpha E_k(x)| \le \widetilde{C} \cdot |x|^{-2-|\alpha|}$$

for $z \in \mathbb{R}^3$, $t \in (0,\infty)$, $x \in \mathbb{R}^3 \backslash \{0\}$, $1 \le j,k \le 3$.

Proof. The estimate of $\mathfrak{H}$ stated in the lemma was established in [28]. Concerning the estimate of Γ, we refer to [25, Proposition 2.1.9]. □

We remark that Solonnikov [31, section 2.3] could show that Green's function for the time-dependent Stokes system in half-space verifies estimates analogous to those given in the preceding lemma for the Stokes fundamental solution, of course with the difference that in half-space, there is one spatial coordinate playing a special role. It should perhaps be further indicated that a different fundamental solution of the time-dependent Stokes system is used in [29]. However, this solution satisfies the same estimates as Γ (see [29, Theorem 1]).

For the Oseen fundamental solution, the following estimate holds.

LEMMA 2.6. *The function $\Lambda_{jk}(\cdot\,, \cdot\,, \kappa)$ belongs to the space $C^\infty(G)$ for $\kappa > 0$, $1 \le j,k \le 3$. For any $K > 0$, $\alpha \in \mathbb{N}_0^3$, $l \in \mathbb{N}_0$, there is some $C(K,\alpha,l) > 0$ with*

$$\big|\,\partial_t^l \partial_z^\alpha \mathfrak{H}(z - \kappa \cdot t \cdot e_1,\, t)\,\big| + \big|\,\partial_t^l \partial_z^\alpha \Lambda_{jk}(z,t,\kappa)\,\big|$$
$$\le C(K,\alpha,l) \cdot \max\{1,\kappa\}^{3/2+|\alpha|/2+l} \cdot \big(\gamma(z,t)^{-3/2-|\alpha|/2-l} + \gamma(z,t)^{-3/2-|\alpha|/2-l/2}\big)$$

for z, t, j, k *as in Lemma* 2.5, *and for* $\kappa \in (0,\infty)$, *where*

$$\gamma(z,t) := |z|^2 + t \ \text{ if } |z| \le K, \quad \gamma(z,t) := |z| \cdot \big(1 + \kappa \cdot s(z)\big) + t \ \text{ if } |z| > K,$$

with $s(z)$ *defined in* (2.8).

Proof. By [3, Lemma 2], we have

$$(|z - \kappa \cdot t \cdot e_1|^2 + t)^{-1} \le C(K) \cdot \max\{1, \kappa\} \cdot \gamma(z,t)^{-1}$$

for z, t, j, k as in Lemma 2.5 and for $\kappa \in (0,\infty)$, so the lemma follows from Lemma 2.5. □

Let us now fix a Reynolds number $\tau \in (0,\infty)$. In the following, the symbol $\mathfrak{C}$ will always denote constants depending only on Ω, R_0, and τ. We write $\mathfrak{C}(\gamma_1, \dots, \gamma_n)$ for constants depending additionally on other parameters $\gamma_1, \dots, \gamma_n \in (0,\infty)$ for some $n \in \mathbb{N}$.

The ensuing lemmas will allow us to define our volume potentials.

LEMMA 2.7. *Let* $p, q \in [1,\infty]$, $r \in (1,\infty]$, $s \in [1,\infty)$ *with* $q < p$, $s \le r$.

Then there is $C = C(\tau, p, q, r, s) > 0$ *such that for* $f \in L^s\big(0,\infty, L^q(\mathbb{R}^3)\big)$, $M \in (0,\infty)$, $j, k \in \{1, 2, 3\}$, $\alpha \in \mathbb{N}_0^3$ *with* $|\alpha| \le 1$, *we have the following for* $W = (0, M)$ *if* $1 - |\alpha|/2 + 3 \cdot (1/p - 1/q)/2 > 1/s - 1/r$, *and for* $W = (M, \infty)$ *if* $1 - |\alpha|/2 + 3 \cdot (1/p - 1/q)/2 < 1/s - 1/r$:

$$\left(\int_0^\infty \left[\int_{\mathbb{R}^3} \left(\int_0^\infty \int_{\mathbb{R}^3} \chi_W(t-\sigma) \cdot |\partial_x^\alpha \Lambda_{jk}(x-y, t-\sigma, \tau)| \cdot |f(y,\sigma)| \, dy \, d\sigma \right)^p dx \right]^{r/p} dt \right)^{1/r} \le C \cdot M^{3\cdot(1/p-1/q)/2+1-|\alpha|/2-1/s+1/r} \cdot \|f\|_{L^s\left(0,\infty,L^q(\mathbb{R}^3)\right)}. \tag{2.17}$$

(Of course, if $r = \infty$ *and/or* $p = \infty$, *the preceding inequality has to be modified in an obvious way.)*

Proof. We consider the case $p < \infty$. If $p = \infty$, a similar argument is valid. Take f, M, j, k, α as in the lemma. We find for $W \in \{\, (0,M), (M,\infty) \,\}$, $t \in (0,\infty)$, $\sigma \in (0,t)$,

$$\begin{aligned} &\left(\int_{\mathbb{R}^3} \left(\int_0^\infty \int_{\mathbb{R}^3} \chi_W(t-\sigma) \cdot |\partial_x^\alpha \Lambda_{jk}(x-y,t-\sigma,\tau)| \cdot |f(y,\sigma)| \, dy \, d\sigma \right)^p dx \right)^{1/p} \\ &\le \int_0^\infty \chi_W(t-\sigma) \cdot \left(\int_{\mathbb{R}^3} |\partial_z^\alpha \Lambda_{jk}(z,t-\sigma,\tau)|^{(1-1/q+1/p)^{-1}} dz \right)^{1-1/q+1/p} \\ &\quad \cdot \|f(\,\cdot\,,\sigma)\|_q \, d\sigma, \end{aligned} \tag{2.18}$$

where we used Minkowski's and Young's inequality for integrals. By the change of variable $y = z - \tau \cdot (t - \sigma) \cdot e_1$, the integral over $\mathbb{R}^3$ on the right-hand side of (2.18) may be transformed into the integral $\int_{\mathbb{R}^3} |\partial_y^\alpha \Gamma_{jk}(y, t-\sigma)|^{(1-1/q+1/p)^{-1}} dy$. But

$$\begin{aligned} &\int_{\mathbb{R}^3} |\partial_y^\alpha \Gamma_{jk}(y,t-\sigma)|^{(1-1/q+1/p)^{-1}} dy \\ &\le \mathfrak{C} \cdot \int_{\mathbb{R}^3} \big(|y| + (t-\sigma)^{1/2} \big)^{-(3+|\alpha|)\cdot(1-1/q+1/p)^{-1}} dy \\ &\le \mathfrak{C} \cdot (t-\sigma)^{-(3+|\alpha|)\cdot(1-1/q+1/p)^{-1}/2 + 3/2} \end{aligned}$$

for $\sigma \in (0,t)$; see Lemma 2.5. Now the inequality stated in the lemma follows by another application of Young's inequality. □

THEOREM 2.8. *Let $q \in (1,3)$. Then there is $C = C(q) > 0$ such that for $\phi \in L^q(\mathbb{R}^3)^3$, $1 \le k \le 3$,*

$$\left(\int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} |E_k(x-y)| \cdot |\phi(y)| \, dy \right)^{(1/q-1/3)^{-1}} dx \right)^{(1/q-1/3)} \le C \cdot \|\phi\|_q,$$

with E_k from (2.16).

Proof. The theorem is a direct consequence of the Hardy–Littlewood–Sobolev inequality (see [32, Theorem V.1.1]). □

LEMMA 2.9. *Let $q \in [1,\infty)$, $\alpha \in \mathbb{N}_0^3$, $l \in \mathbb{N}_0$. Then there is $C = C(q,\alpha,l,\tau) > 0$ such that*

$$\begin{aligned} \text{(2.19)} \qquad & \int_{\mathbb{R}^3} |\partial_t^l \partial_x^\alpha \big(\mathfrak{H}(x - \tau \cdot t \cdot e_1 - y, t) \big)| \cdot |a(y)| \, dy \\ & \le C \cdot \big(t^{(-3/q-|\alpha|-l)/2} + t^{(-3/q-|\alpha|-2\cdot l)/2} \big) \cdot \|a\|_q \end{aligned}$$

for $x \in \mathbb{R}^3$, $t \in (0,\infty)$, $a \in L^q(\mathbb{R}^3)$.

Proof. Lemma 2.9 follows from [9, equation (4.3)], where a kernel much more general than $\mathfrak{H}$ is considered. In the present situation, the lemma also is an immediate consequence of Lemma 2.6 and the Hölder inequality. □

Now we may define the volume potentials we will consider in the following.

DEFINITION 2.10. *Let $q, s \in [1,\infty)$, $\widetilde{q} \in (1,3)$, let $A \subset \mathbb{R}^3$ be measurable, and let $T \in (0,\infty]$. Suppose that $f \in L^s\big(0,T, L^q(A)^3\big)$. Then, for a.e. $x \in \mathbb{R}^3$, $t \in (0,\infty)$, and for $j \in \{1, 2, 3\}$, we set*

$$\text{(2.20)} \qquad \mathfrak{R}_j^{(\tau)}(f)(x,t) := \int_0^t \int_{\mathbb{R}^3} \sum_{k=1}^3 \Lambda_{jk}(x-y, t-\sigma, \tau) \cdot \widetilde{f}_k(y,\sigma) \, dy \, d\sigma,$$

where $\widetilde{f}$ denotes the zero extension of f to $\mathbb{R}^3 \times (0,\infty)$. Moreover, suppose that $f(\,\cdot\,,t) \in L^{\widetilde{q}}(A)^3$ for a.e. $t \in (0,T)$. Then, for a.e. $x \in \mathbb{R}^3$, $t \in (0,\infty)$, we put

$$\mathfrak{P}(f)(x,t) := \int_{\mathbb{R}^3} \sum_{k=1}^3 E_k(x-y) \cdot \widetilde{f}_k(y,t) \, dy.$$

Let $p \in [1,\infty)$, $a \in L^p(A)^3$. Then we define

$$\mathfrak{I}_j^{(\tau)}(a)(x,t) := \int_A \mathfrak{H}(x - \tau \cdot t \cdot e_1 - y, t) \cdot a_j(y) \, dy$$

for $x \in \mathbb{R}^3$, $t \in (0,\infty)$, and for $1 \le j \le 3$.

In order to see that the definition of $\mathfrak{R}^{(\tau)}(f)$ makes sense, consider Lemma 2.7 with $r = s$ and with $p \in (q,\infty)$ so close to q that $1 + 3 \cdot (1/p - 1/q)/2 > 0$. Then $1 + 3 \cdot (1/p - 1/q)/2 > 1/s - 1/r$, so that the integral in (2.17) with $W = (0,M)$ is finite for any $M \in (0,\infty)$. This means that the integral in (2.20) exists for almost every $x \in \mathbb{R}^3$ and $t \in (0,\infty)$; hence $\mathfrak{R}^{(\tau)}(f)$ is well defined. The well-posedness of $\mathfrak{P}(f)$ and $\mathfrak{I}^{(\tau)}(a)$ is an immediate consequence of Theorem 2.8 and Lemma 2.9, respectively. Similar arguments yield the following lemma.

LEMMA 2.11. *Let* $q, s \in [1, \infty)$, $f \in L^s\big(0, \infty, L^q(\mathbb{R}^3)^3\big)$. *Then the weak derivative* $\partial_l \mathbb{R}_j(f)$ *exists for* $1 \le j, l \le 3$. *Moreover, the integral*

$$\int_0^t \int_{\mathbb{R}^3} \sum_{k=1}^3 \partial_{x_l} \Lambda_{jk}(x-y, t-\sigma, \tau) \cdot f_k(y, \sigma)\, dy\, d\sigma$$

exists for such j, l *and for a.e.* $x \in \mathbb{R}^3$, $t \in (0, \infty)$, *and is equal to* $\partial_l \mathfrak{R}_j^{(\tau)}(f)(x, t)$. *The equation* $\operatorname{div}_x \mathfrak{R}^{(\tau)}(f) = 0$ *holds.*

Proof. Use Lemma 2.7 and the equation $\sum_{j=1}^3 \partial_{z_j} \Lambda_{jk}(z, t, \tau) = 0$, which holds for $z \in \mathbb{R}^3$, $t > 0$, $1 \le k \le 3$. □

Next we consider integrals of $\mathfrak{R}^{(\tau)}(f)(\,\cdot\,, t)$ on $\partial\Omega$.

LEMMA 2.12. *Let* $q, s \in [1, \infty)$, $f \in L^s(0, \infty, L^q(\mathbb{R}^3)^3)$. *Then*

(2.21)
$$\left(\int_0^T \left(\int_{\partial\Omega} \left(\int_0^t \int_{\mathbb{R}^3} |\Lambda_{jk}(x-y, t-\sigma, \tau) \cdot f_k(y, \sigma)|\, dy\, d\sigma \right)^q d\Omega(x) \right)^{s/q} dt \right)^{1/s}$$
$$\le \mathfrak{C}(T, q) \cdot \|f\|_{L^s(0,\infty,L^q(\mathbb{R}^3)^3)} \quad \text{for } T \in (0, \infty),\ 1 \le j, k \le 3.$$

In particular, the integral $\int_0^t \int_{\mathbb{R}^3} \sum_{k=1}^3 \Lambda_{jk}(x-y, t-\sigma, \tau) \cdot f_k(y, \sigma)\, dy\, d\sigma$ *exists for* $1 \le j \le 3$, *a.e.* $t \in (0, \infty)$, *and a.e.* $x \in \partial\Omega$.

Proof. We consider the case $q > 1$. If $q = 1$, a similar but somewhat simpler argument holds. Let $T \in (0, \infty)$, $j, k \in \{1, 2, 3\}$. Put $\delta := 1/(6 \cdot q)$. Let $\mathfrak{A}$ denote the left-hand side of (2.21). Then, by Minkowski's inequality for integrals applied to the function

$$(0, T) \times \partial\Omega \ni (\sigma, x) \mapsto \int_{\mathbb{R}^3} \chi_{(0,\infty)}(t-\sigma) \cdot |\Lambda_{jk}(x-y, t-\sigma, \tau)| \cdot |f_k(y, \sigma)|\, dy \in [0, \infty)$$

for fixed $t \in (0, T)$, and by Hölder's inequality,

(2.22)
$$\mathfrak{A} \le \left(\int_0^T \left[\int_0^T \left(\int_{\partial\Omega} \left(\int_{\mathbb{R}^3} \chi_{(0,\infty)}(t-\sigma) \cdot |\Lambda_{jk}(x-y, t-\sigma, \tau)| \cdot f_k(y, \sigma)|\, dy \right)^q d\Omega(x) \right)^{1/q} d\sigma \right]^s dt \right)^{1/s}$$
$$\le \left(\int_0^T \left[\int_0^T \chi_{(0,\infty)}(t-\sigma) \cdot \left(\int_{\partial\Omega} \left(\int_{\mathbb{R}^3} |\Lambda_{jk}(x-y, t-\sigma, \tau)|^{1+\delta \cdot q'}\, dy \right)^{q-1} \cdot \left(\int_{\mathbb{R}^3} |\Lambda_{jk}(x-y, t-\sigma, \tau)|^{1-\delta \cdot q} \cdot |f_k(y, \sigma)|^q\, dy \right) d\Omega(x) \right)^{1/q} d\sigma \right]^s dt \right)^{1/s}.$$

But with Lemma 2.5, for $x \in \partial\Omega$, $t \in (0, T)$, $\sigma \in (0, t)$,

$$\int_{\mathbb{R}^3} |\Lambda_{jk}(x-y, t-\sigma, \tau)|^{1+\delta \cdot q'}\, dy = \int_{\mathbb{R}^3} |\Gamma_{jk}(z, t-\sigma)|^{1+\delta \cdot q'}\, dz$$
$$\le \mathfrak{C} \cdot \int_{\mathbb{R}^3} \big(|z| + (t-\sigma)^{1/2} \big)^{-3 \cdot (1+\delta \cdot q')/2}\, dz \le \mathfrak{C}(q) \cdot (t-\sigma)^{-3 \cdot \delta \cdot q'/2},$$

and by Lemma 2.6, with $K = 2 \cdot R_0$, for $y \in B_{R_0}$, and t, σ as before,

$$\int_{\partial\Omega} |\Lambda_{jk}(x-y, t-\sigma, \tau)|^{1-\delta\cdot q}\, d\Omega(x) \le \mathfrak{C} \cdot \int_{\partial\Omega} \big(|x-y| + (t-\sigma)^{1/2} \big)^{-3\cdot(1-\delta\cdot q)}\, d\Omega(x)$$
$$\le \mathfrak{C}(q) \cdot (t-\sigma)^{(-1+3\cdot\delta\cdot q)/2}.$$

The last inequality holds because $-3\cdot(1-\delta\cdot q) < -2$. In the case $|y| > R_0$, it is obvious by Lemma 2.6 that the left-hand side of the previous estimate is bounded by $\mathfrak{C}(q) \cdot \min\{1, T^{(1-3\cdot\delta\cdot q)/2}\} \cdot (t-\sigma)^{(-1+3\cdot\delta\cdot q)/2}$. Now we may conclude from (2.22)

$$\mathfrak{A} \le \mathfrak{C}(T, q) \cdot \left(\int_0^T \left[\int_0^T \chi_{(0,\infty)}(t-\sigma) \cdot (t-\sigma)^{-1/(2\cdot q)} \cdot \|f(\,\cdot\,, \sigma)\|_q \, d\sigma \right]^s dt \right)^{1/s},$$

so estimate (2.21) follows by Young's inequality. From (2.21) we may deduce that the integral $\int_0^t \int_{\mathbb{R}^3} \sum_{k=1}^3 \Lambda_{jk}(x-y, t-\sigma, \tau) \cdot f_k(y, \sigma)\, dy\, d\sigma$ exists for a.e. $t \in (0,\infty)$ and a.e. $x \in \partial\Omega$. □

COROLLARY 2.13. *Let $q, s \in [1,\infty)$, $f \in L^s(0,\infty, L^q(\mathbb{R}^3)^3)$, $j \in \{1, 2, 3\}$. Then the trace of $\mathfrak{R}_j^{(\tau)}(f)(\,\cdot\,, t)$ on $\partial\Omega$ exists for a.e. $t \in (0,\infty)$ and equals the integral $\int_0^t \int_{\mathbb{R}^3} \sum_{k=1}^3 \Lambda_{jk}(x-y, t-\sigma, \tau) \cdot f_k(y, \sigma)\, dy\, d\sigma$ for a.e. $x \in \partial\Omega$.*

Proof. Let (f_n) be a sequence in $C_0^\infty\big(\mathbb{R}^3 \times (0,\infty)\big)^3$ converging to f with respect to the norm of $L^s(0,\infty, L^q(\mathbb{R}^3)^3)$. For any $n \in \mathbb{N}$, the function $\mathfrak{R}_j^{(\tau)}(f_n)$ is C^∞ in $\mathbb{R}^3 \times (0,\infty)$. Let $T \in (0,\infty)$, and take $p \in (q,\infty)$ so close to q that $1/2 + 3\cdot(1/p - 1/q)/2 > 0$. Then, by Lemma 2.7 with $M = T$ and $r = s$, we may conclude that

$$\|\partial^\alpha \big(\mathfrak{R}_j^{(\tau)}(f_n) - \mathfrak{R}_j^{(\tau)}(f) \big) | \mathbb{R}^3 \times (0,T) \|_{L^s(0,T,L^p(\mathbb{R}^3))} \to 0 \quad (n \to \infty)$$

for $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$. Take $R \in (0,\infty)$ with $\overline{\Omega} \subset B_R$. Since $p > q$, it follows that

$$\|\partial^\alpha \big(\mathfrak{R}_j^{(\tau)}(f_n) - \mathfrak{R}_j^{(\tau)}(f) \big) | B_R \times (0,T) \|_{L^s(0,T,L^q(B_R))} \to 0 \quad (n \to \infty)$$

for α as before. Thus there is a subsequence (g_n) of (f_n) such that

$$\|\partial^\alpha \big(\mathfrak{R}_j^{(\tau)}(g_n) - \mathfrak{R}_j^{(\tau)}(f) \big)(\,\cdot\,, t) | B_R \|_q \to 0 \quad (n \to \infty)$$

for a.e. $t \in (0,T)$, and for α as before. For a.e. $x \in \partial\Omega$, $t \in (0,T)$, let $\mathfrak{A}_j(x,t)$ denote the integral mentioned in the corollary. Then, by (2.21),

$$\|\mathfrak{R}_j^{(\tau)}(g_n) \,|\, \partial\Omega \times (0,T) - \mathfrak{A}_j \|_{L^s(0,T,L^q(\partial\Omega))} \to 0,$$

so there is a subsequence (h_n) of (g_n) with $\|\mathfrak{R}_j^{(\tau)}(h_n)(\,\cdot\,, t) \,|\, \partial\Omega - \mathfrak{A}_j(\,\cdot\,, t)\|_q \to 0$ for a.e. $t \in (0,\infty)$. Now the corollary follows. □

In the ensuing lemma and theorem, we indicate some properties of $\mathfrak{I}^{(\tau)}(a)$.

LEMMA 2.14. *Take p, A, a as in Definition 2.10. Then the function $\mathfrak{I}^{(\tau)}(a)$ belongs to $C^\infty\big(\mathbb{R}^3 \times (0,\infty)\big)^3$, and*

$$\partial_t^l \partial_x^\alpha \mathfrak{I}^{(\tau)}(a)(x,t) = \int_A \partial_t^l \partial_x^\alpha \big(\mathfrak{H}(x - y - \tau\cdot t\cdot e_1, t) \big) \cdot a(y)\, dy$$
$$(x \in \mathbb{R}^3,\ t \in (0,\infty),\ l \in \mathbb{N}_0,\ \alpha \in \mathbb{N}_0^3),$$
$$\partial_t \mathfrak{I}^{(\tau)}(a)(x,t) - \Delta_x \mathfrak{I}^{(\tau)}(a)(x,t) + \tau \cdot \partial_{x_1} \mathfrak{I}^{(\tau)}(a)(x,t) = 0 \quad (x \in \mathbb{R}^3,\ t > 0),$$
$$\|\mathfrak{I}^{(\tau)}(a)(\,\cdot\,, t)\|_p \le \|a\|_p \quad (t > 0).$$

Furthermore $\operatorname{div}_x \mathfrak{I}^{(\tau)}(a) = 0$ *if* $a \in L^p_\sigma(\mathbb{R}^3)^3$.

Proof. The L^p-estimate in the preceding lemma follows from Lemma 2.3 and the equation $\int_{\mathbb{R}^3} \mathfrak{H}(x,t)\,dx = 1$ for $t > 0$. Lemma 2.9, a density argument, and a partial integration yield the last statement of that lemma. □

Concerning the gradient of $\mathfrak{I}^{(\tau)}(a)$, we note the following.

THEOREM 2.15. *Let* $\epsilon \in (0,1]$, $a \in H^\epsilon(\mathbb{R}^3)^3$. *Then*

$$\|\nabla_x \mathfrak{I}^{(\tau)}(a) \,|\, \mathbb{R}^3 \times (0,T)\|_2 \le \mathfrak{C}(\epsilon) \cdot T^{\epsilon/2} \cdot \|a\|_{\epsilon,2} \quad \textit{for } T \in (0,\infty).$$

Proof. Theorem 2.15 may be deduced from [9, equation (4.3)]. It can also be derived directly from Lemmas 2.14 and 2.5, Young's inequality, and an interpolation argument. (Consider the cases $\epsilon = 0$ and $\epsilon = 1$.) □

For smooth f and a, we have the following theorem.

THEOREM 2.16. *Let* $f \in C_0^\infty\big(\mathbb{R}^3 \times (0,\infty)\big)^3$, $a \in C_0^\infty(\mathbb{R}^3)^3$. *Then the functions* $\mathfrak{P}(f)$ *and* $\mathfrak{I}_j^{(\tau)}(a)$ *belong to* $C^\infty\big(\mathbb{R}^3 \times [0,\infty)\big)$, *and* $\mathfrak{R}_j^{(\tau)}(f)$ *belongs to* $C_0^\infty\big(\mathbb{R}^3 \times (0,\infty)\big)^3$ *for* $1 \le j \le 3$. *Moreover, for* $x \in \mathbb{R}^3$, $t \in (0,\infty)$,

(2.23)
$$\partial_t \mathfrak{R}^{(\tau)}(f)(x,t) - \Delta_x \mathfrak{R}^{(\tau)}(f)(x,t) + \tau \cdot \partial_{x_1} \mathfrak{R}^{(\tau)}(f)(x,t) + \nabla_x \mathfrak{P}(f)(x,t) = f(x,t),$$
$$\mathfrak{I}^{(\tau)}(a)(x,0) = a(x).$$

If $\operatorname{supp}(a) \subset U$, *there is a vicinity* $\mathfrak{V} = \mathfrak{V}(a)$ *of* $\partial\Omega$ *such that* $\partial_4^l \partial^\alpha \mathfrak{I}^{(\tau)}(a)(x,0) = 0$ *for* $x \in \mathfrak{V}$, $l \in \mathbb{N}_0, \alpha \in \mathbb{N}_0^3$.

Proof. Let us consider $\mathfrak{I}^{(\tau)}(a)$. We know by Lemma 2.14 that $\mathfrak{I}^{(\tau)}(a) \in C^\infty\big(\mathbb{R}^3 \times (0,\infty)\big)^3$. We further observe that by Lemma 2.14,

$$\begin{aligned}
&\partial_4^l \partial^\alpha \mathfrak{I}^{(\tau)}(a)(x,\sigma) \\
&= \sum_{j=0}^{l} \binom{l}{j} \cdot \int_{\mathbb{R}^3} (\partial_4^j \partial_1^{l-j} \partial^\alpha \mathfrak{H})(x - y - \tau \cdot \sigma \cdot e_1, \sigma) \cdot (-\tau)^{l-j} \cdot a(y)\, dy \\
&= \sum_{j=0}^{l} \binom{l}{j} \cdot (-\tau)^{l-j} \int_{\mathbb{R}^3} \mathfrak{H}(x-y,\sigma) \cdot \partial^\alpha \Delta^j \partial_1^{l-j} a(y - \tau \cdot \sigma \cdot e_1)\, dy
\end{aligned}$$

($x \in \mathbb{R}^3$, $\sigma \in (0,\infty)$, $l \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^3$), where we integrated by parts in the last equation; note that $\operatorname{supp}(a)$ is compact. By taking the limit $\sigma \downarrow 0$, and recalling the usual arguments applied to the heat kernel (see [14, Theorem 1.2.1], for example), we obtain that $\mathfrak{I}^{(\tau)}(a) \in C^\infty\big(\mathbb{R}^3 \times [0,\infty)\big)$ and

$$\partial_4^l \partial^\alpha \mathfrak{I}^{(\tau)}(a)(x,0) = \sum_{j=0}^{l} \binom{l}{j} \cdot (-\tau)^{l-j} \cdot \partial^\alpha \Delta^j \partial_1^{l-j} a(x) \quad \text{for } x \in \mathbb{R}^3.$$

The last statement of the theorem follows from this formula. As for the claims on $\mathfrak{R}^{(\tau)}(f)$, Lebesgue's theorem on dominated convergence implies that derivatives of $\mathfrak{R}^{(\tau)}(f)$ can be made to act on f. Equation (2.23) then follows by partial integration and the properties of Λ_{jk}. □

We turn to the question of whether $\mathfrak{R}^{(\tau)}(f)$ and $\mathfrak{I}^{(\tau)}(a)$ belong to $L^2(0,T,\,H^1(\mathbb{R}^3)^3)$ and to $H^1(0,T,V')$.

COROLLARY 2.17. *Let* $s \in [1,2]$, $q \in [1,2)$ *with* $7/4 > 1/s + 3/(2 \cdot q)$, $T \in (0,\infty)$, $f \in L^s(0,T,\, L^q(\mathbb{R}^3)^3)$. *Then* $\mathfrak{R}^{(\tau)}(f) \in L^2(0,T,\, H^1(\mathbb{R}^3)^3)$ *and*

$$(2.24) \qquad \|\mathfrak{R}^{(\tau)}(f)\|_{L^2(0,T,\,H^1(\mathbb{R}^3)^3)} \le \mathfrak{C} \cdot T^{\epsilon} \cdot \|f\|_{L^s(0,T,\,L^q(\mathbb{R}^3)^3)},$$

with $\epsilon := 7/4 - 3/(2 \cdot q) - 1/s$ *if* $T < 1$, *and* $\epsilon := 9/4 - 3/(2 \cdot q) - 1/s$ *else. If in addition* $f \,|\, Z_T \in L^2(0,T,V')$, *with* $f(t) \in V'$ *for* $t \in (0,\infty)$ *being defined by*

$$f(t)(v) := \int_U f(x,t) \cdot v(x)\, dx \quad \textit{for } v \in C_0^\infty(U)^3 \textit{ with } \operatorname{div} v = 0,$$

then $\mathfrak{R}^{(\tau)}(f) \,|\, Z_T \in H^1(0,T,V')$ *and*

(2.25)

$$\|\mathfrak{R}^{(\tau)}(f) \,|\, Z_T\|_{H^1(0,T,V')} \le \mathfrak{C} \cdot \big(T^{\epsilon} \cdot \|f\|_{L^s(0,T,\,L^q(\mathbb{R}^3)^3)} + \|f \,|\, Z_T\|_{L^2(0,T,V')} \big).$$

Note that $f \in L^2(0,T,V')$ *if, for example,* $f \in L^2(0,T,\, L^{6/5}(\mathbb{R}^3)^3)$.

Proof. By Lemma 2.7 with $p = r = 2$, we see that $\mathfrak{R}^{(\tau)}(f) \in L^2(0,T,\, H^1(\mathbb{R}^3)^3)$, and that inequality (2.24) holds. Take $v \in C_0^\infty(U)^3$ with $\operatorname{div} v = 0$, and choose a sequence (f_n) in the space $C_0^\infty\big(\mathbb{R}^3 \times (0,T)\big)^3$ with $\|f_n - f\|_{L^s(0,T,\,L^q(\mathbb{R}^3)^3)} \to 0$. Theorem 2.16 implies for $n \in \mathbb{N}$, $t \in (0,T)$ that

$$\begin{aligned}
&\Big| \int_U \partial_t \mathfrak{R}^{(\tau)}(f_n)(x,t) \cdot v(x)\, dx \Big| \\
&= \Big| \int_U \Big[-\nabla_x \mathfrak{R}^{(\tau)}(f_n)(x,t) \cdot \nabla v(x) \;+\; \big(-\tau \cdot \partial_1 \mathfrak{R}^{(\tau)}(f_n)(x,t) + f_n(x,t) \big) \cdot v(x) \Big]\, dx \Big| \\
&\le \mathfrak{C} \cdot \big(\|\mathfrak{R}^{(\tau)}(f_n)(\,\cdot\,,t)\|_{1,2} + \|f_n(\,\cdot\,,t) \,|\, U\|_{V'} \big) \cdot \|v\|_{1,2};
\end{aligned}$$

hence by (2.24),

$$(2.26) \qquad \begin{aligned} &\Big(\int_0^T \sup_{v \in V} \Big\{ \Big| \int_U \partial_t \mathfrak{R}^{(\tau)}(f_n)(x,t) \cdot v(x)\, dx \Big| / \|v\|_{1,2} \Big\}^2 dt \Big)^{1/2} \\ &\le \mathfrak{C} \cdot \big(T^{\epsilon} \cdot \|f_n\|_{L^s(0,T,L^q(\mathbb{R}^3)^3)} + \|f_n \,|\, Z_T\|_{L^2(0,T,V')} \big). \end{aligned}$$

On the other hand, $\|\big(\mathfrak{R}^{(\tau)}(f_n) - \mathfrak{R}^{(\tau)}(f)\big) \,|\, Z_T\|_{L^2(0,T,V')} \to 0$, as follows from (2.24) and the assumption $\|f_n - f\|_{L^s(0,T,\,L^q(\mathbb{R}^3)^3)} \to 0$. Thus the relation $\mathfrak{R}^{(\tau)}(f) \,|\, Z_T \in H^1(0,T,V')$ and inequality (2.25) may be deduced from (2.24) and (2.26). □

Similar arguments, based on Lemma 2.14 and Theorem 2.15, yield the following.

COROLLARY 2.18. *Let* $\epsilon \in (0,1]$, $a \in H^{\epsilon}(U)^3$, $T \in (0,\infty)$. *Then*

$$\begin{aligned}
&\mathfrak{I}^{(\tau)}(a) \,|\, \mathbb{R}^3 \times (0,T) \in L^2(0,T,\, H^1(\mathbb{R}^3)^3), \quad \mathfrak{I}^{(\tau)}(a) \,|\, Z_T \in H^1(0,T,V'), \\
&\|\mathfrak{I}^{(\tau)}(a) \,|\, \mathbb{R}^3 \times (0,T)\|_{L^2(0,T,\,H^1(\mathbb{R}^3)^3)} + \|\mathfrak{I}^{(\tau)}(a) \,|\, Z_T\|_{H^1(0,T,V')} \\
&\le \mathfrak{C} \cdot \max\{T^{1/2}, T^{\epsilon/2}\} \cdot \|a\|_{\epsilon,2}.
\end{aligned}$$

We have no control on the values of either $\mathfrak{R}^{(\tau)}(f)$ or $\mathfrak{I}^{(\tau)}(a)$ on S_∞. Therefore we have to introduce a boundary correction so that the boundary condition on S_∞ stated in (1.5) may be satisfied. This boundary correction will take the form of a single layer potential. For its definition, we take into account the following observation, which is an immediate consequence of Lemmas 2.3 and 2.6.

LEMMA 2.19. *For $\kappa \in (0,\infty)$, $\phi \in L^2(S_\infty)^3$, $T \in (0,\infty)$, $1 \le j,k \le 3$, the inequality*

$$\left(\int_0^T \int_{\partial\Omega} \left(\int_0^t \int_{\partial\Omega} |\Lambda_{jk}(x-y,t-\sigma,\kappa) \cdot \phi_k(y,\sigma)| \, d\Omega(y) \, d\sigma \right)^2 d\Omega(x) \, dt \right)^{1/2} \le C(\Omega,\kappa,T) \cdot \|\phi\|_2$$

holds. In particular, the integral $\int_0^t \int_{\partial\Omega} \sum_{k=1}^3 \Lambda_{jk}(x-y,t-\sigma,\kappa) \cdot \phi_k(y,\sigma) \, d\Omega(y) \, d\sigma$ exists for a.e. $(x,t) \in \partial\Omega \times (0,\infty)$.

DEFINITION 2.20. *For $T \in (0,\infty]$, $\phi \in L^2(S_T)^3$, $\kappa \in (0,\infty)$, $x \in \mathbb{R}^3 \backslash \partial\Omega$, $t \in [0,\infty)$, and for a.e. $(x,t) \in \partial\Omega \times (0,\infty)$, we put*

$$\mathfrak{V}^{(0)}(\phi)(x,t) := \left(\int_0^t \int_{\partial\Omega} \sum_{k=1}^3 \Gamma_{jk}(x-y,t-\sigma) \cdot \widetilde{\phi}_k(y,\sigma) \, d\Omega(y) \, d\sigma \right)_{1\le j\le 3},$$

$$\mathfrak{V}^{(\kappa)}(\phi)(x,t) := \left(\int_0^t \int_{\partial\Omega} \sum_{k=1}^3 \Lambda_{jk}(x-y,t-\sigma,\kappa) \cdot \widetilde{\phi}_k(y,\sigma) \, d\Omega(y) \, d\sigma \right)_{1\le j\le 3},$$

where $\widetilde{\phi}$ denotes the zero extension of ϕ from S_T to S_∞. We further set

$$Q(\phi)(x,t) := \int_{\partial\Omega} \sum_{k=1}^3 E_k(x-y) \cdot \widetilde{\phi}_k(y,t) \, d\Omega(y)$$

for $T \in (0,\infty]$, $\phi \in L^2(S_T)^3$, $x \in \mathbb{R}^3 \backslash \partial\Omega$, $t \in (0,\infty)$. We call the pair of functions $(\mathfrak{V}^{(0)}(\phi), Q(\phi))$ and $(\mathfrak{V}^{(\kappa)}(\phi), Q(\phi))$ "the single layer potential related to the time-dependent Stokes and Oseen system," respectively.

The following lemma follows from Lebesgue's theorem on dominated convergence and the equation $\partial_t \Lambda_{jk} - \Delta_x \Lambda_{jk} + \tau \cdot \partial_{x_1} \Lambda_{jk} = 0$. The details are not completely trivial, but they are essentially the same as in the Stokes case, and cannot be elaborated here.

LEMMA 2.21. *Let $T \in (0,\infty]$, $\kappa \in [0,\infty)$, $\phi \in L^2(S_T)^3$. Abbreviate*

$$v := \mathfrak{V}^{(\kappa)}(\phi) \,|\, (\mathbb{R}^3 \backslash \partial\Omega) \times [0,\infty), \quad q := Q(\phi).$$

Then $v_j(\,\cdot\,,t)$, $q(\,\cdot\,,t) \in C^\infty(\mathbb{R}^3 \backslash \partial\Omega)$ for $1 \le j \le 3$, $t \in [0,\infty)$, with

$$\partial_x^\alpha v_j(x,t) = \int_0^t \int_{\partial\Omega} \partial_x^\alpha \Lambda_{jk}(x-y,t-\sigma,\kappa) \cdot \phi_k(y,\sigma) \, d\Omega(y) \, d\sigma \tag{2.27}$$

$(\alpha \in \mathbb{N}_0^3,\ x \in \mathbb{R}^3 \backslash \partial\Omega)$.

For $\alpha \in \mathbb{N}_0^3$, the partial derivative $\partial_x^\alpha v(x,t)$ as a function of $x \in \mathbb{R}^3 \backslash \partial\Omega$ and $t \in [0,\infty)$ is continuous. The partial derivative $\partial_t v(x,t)$ exists for $x \in \mathbb{R}^3 \backslash \partial\Omega$ and for a.e. $t \in (0,\infty)$. This derivative also is the weak derivative of v with respect to t on $(\mathbb{R}^3 \backslash \partial\Omega) \times (0,\infty)$, and the weak derivative of $v(x,\,\cdot\,)$ on $(0,\infty)$, if $x \in \mathbb{R}^3 \backslash \partial\Omega$. The equation

$$\partial_t v(x,t) - \Delta_x v(x,t) + \kappa \cdot \partial_{x_1} v(x,t) + \nabla_x q(x,t) = 0$$

holds for $x \in \mathbb{R}^3 \backslash \partial\Omega$ and for a.e. $t \in (0,\infty)$. Moreover, $\operatorname{div}_x v(x,t) = 0$ and $v(x,0) = 0$ for $x \in \mathbb{R}^3 \backslash \partial\Omega$, $t \in [0,\infty)$.

Concerning Sobolev regularity of $\mathfrak{V}^{(\tau)}(\phi)$, the following theorem was shown in [6].

THEOREM 2.22. *There is* $C = C(\Omega, \tau) > 0$ *such that*

$$\|\mathfrak{W}^{(\tau)}(\phi) \,|\, Z_\infty\|_{L^\infty(0,\infty,\, L^2(U)^3)} + \|\nabla \mathfrak{W}^{(\tau)}(\phi) \,|\, Z_\infty\|_2 + \|\partial_t\big(\mathfrak{W}^{(\tau)}(\phi) \,|\, Z_\infty\big)\|_{L^2(0,\infty,\,V')}$$
$$\leq C \cdot \|\phi\|_2 \quad \text{for } \phi \in L^2(S_\infty)^3,$$

where

$$\partial_t\big(\mathfrak{W}^{(\tau)}(\phi) \,|\, Z_\infty\big)(t)(v) := \int_U \partial_t \mathfrak{W}^{(\tau)}(\phi)(x,t) \cdot v(x)\, dx$$

for $t \in (0, \infty)$, $v \in C_0^\infty(U)^3$ *with* $\operatorname{div} v = 0$.

In particular, $\mathfrak{W}^{(\tau)}(\phi) \,|\, Z_T \in L^\infty(0,T,L^2(U)^3) \cap L^2(0,T,V) \cap H^1(0,T,V')$ *for* $T \in (0, \infty)$, $\phi \in L^2(S_\infty)^3$.

We note another result proved in [6], which shows that the restriction of $\mathfrak{W}^{(\kappa)}(\phi)$ to S_T may be considered as the boundary value of $\mathfrak{W}^{(\kappa)}(\phi) \,|\, Z_T$ on S_T.

LEMMA 2.23. *Let* $\phi \in L^2(S_\infty)^3$. *Then, for a.e.* $t \in (0,\infty)$, *the trace of* $\big(\mathfrak{W}^{(\tau)}(\phi)(\,\cdot\,,t)\big) \,|\, U$ *coincides with* $\mathfrak{W}^{(\tau)}(\phi)(\,\cdot\,,t) \,|\, \partial\Omega$.

Concerning the integral equation (1.10), the following theorem was shown in [7].

THEOREM 2.24. *For* $T \in (0, \infty]$, $c \in H_T$, *there is a unique function* $\phi \in L^2_n(S_T)$ *which solves the equation* $\mathfrak{W}^{(\tau)}(\phi) \,|\, S_T = c$. *There is* $C = C(\tau, \Omega) > 0$ *such that*

$$\|\phi\|_2 \leq C \cdot \|\mathfrak{W}^{(\tau)}(\phi) \,|\, S_T\|_{H_T} \quad \text{for } T \in (0, \infty],\ \phi \in L^2_n(S_T).$$

In order to apply Theorem 2.24 to (1.9), we have to show that $\big(\mathfrak{R}^{(\tau)}(f) + \mathfrak{J}^{(\tau)}(a)\big) \,|\, S_\infty$ belongs to H_∞. This will be done in the next two sections.

3. Study of the potential $\mathfrak{J}^{(\tau)}(a)$. In this section, we show that the restriction $\mathfrak{J}^{(\tau)}(a) \,|\, S_\infty$ belongs to H_∞ under suitable assumptions on a, and we estimate the H_∞-norm of this restriction against certain Sobolev norms of a. Our results in this respect are stated in the next theorem.

THEOREM 3.1. *For* $\epsilon \in (0,\, 1/2]$, $a \in H_\sigma^{1/2+\epsilon}(U)^3$, *the relation* $\mathfrak{J}^{(\tau)}(a)|S_\infty \in H_\infty$ *holds, and* $\|\mathfrak{J}^{(\tau)}(a)|S_\infty\|_{H_\infty} \leq \mathfrak{C} \cdot \epsilon^{-1/2} \cdot \|a\|_{1/2+\epsilon,2}$.

Proof. Let $\epsilon \in (0,\, 1/2]$ and $a \in C_0^\infty(U)^3$ with $\operatorname{div} a = 0$. By Theorem 2.16, we have $\mathfrak{J}^{(\tau)}(a) \in C^\infty\big(\mathbb{R}^3 \times [0,\infty)\big)^3$, and $\partial_4^l \partial^\alpha \mathfrak{J}^{(\tau)}(a)(x,0) = 0$ for x from a vicinity of $\partial\Omega$ and for $l \in \mathbb{N}_0$, $\alpha \in \mathbb{N}_0^3$. Therefore we may conclude that $\mathfrak{J}^{(\tau)}(a)|S_\infty \in \widetilde{H}_\infty$. Let $j \in \{1,\, 2,\, 3\}$. Then, for $b \in C_0^\infty(U)^3$,

$$\begin{aligned}
&|\partial_\sigma \mathfrak{J}_j^{(\tau)}(b)(x,\sigma)| = |(\Delta_x - \tau \cdot \partial_{x_1}) \mathfrak{J}_j^{(\tau)}(b)(x,\sigma)| \\
&= \Big| \int_{\mathbb{R}^3} \Big(\sum_{k=1}^3 \partial_k \mathfrak{H}(x - y - \tau \cdot \sigma \cdot e_1,\, \sigma) \cdot \partial_k b_j(y) \\
&\qquad + \tau \cdot \partial_1 \mathfrak{H}(x - y - \tau \cdot \sigma \cdot e_1,\, \sigma) \cdot b_j(y) \Big)\, dy \Big| \\
&\leq \mathfrak{C} \cdot \int_{\mathbb{R}^3} (|x - y - \tau \cdot \sigma \cdot e_1|^2 + \sigma)^{-2} \cdot (|\nabla b| + |b|)(y)\, dy \quad \big(x \in \mathbb{R}^3,\ \sigma \in (0,\infty)\big);
\end{aligned}$$

hence by Lemma 2.3 with $F_1 = F_2 = F^{1/2}$, $\mathfrak{A} = \partial\Omega$, $\mathfrak{B} = \mathbb{R}^3$,

$$\Big(\int_{\partial\Omega} |\partial_\sigma \mathfrak{J}_j^{(\tau)}(b)(x,\sigma)|^2\, d\Omega(x) \Big)^{1/2} \leq \mathfrak{C} \cdot \sigma^{-3/4} \cdot \|b\|_{1,2}. \tag{3.1}$$

Observing that $\mathfrak{I}^{(\tau)}(b)(x,0) = b(x) = 0$ for $x \in \partial\Omega,\ b \in C_0^\infty(U)^3$ (Theorem 2.16), and using Minkowski's inequality for integrals, we now get

$$\begin{aligned}
(3.2)\qquad &\Big(\int_{\partial\Omega}\Big|\partial_t\Big(\int_0^t (t-\sigma)^{-1/2}\cdot \mathfrak{I}_j^{(\tau)}(b)(x,\sigma)\,d\sigma\Big)\Big|^2\,d\Omega(x)\Big)^{1/2}\\
&= \Big(\int_{\partial\Omega}\Big|\int_0^t r^{-1/2}\cdot \partial_t\mathfrak{I}_j^{(\tau)}(b)(x,\,t-r)\,dr\Big|^2\,d\Omega(x)\Big)^{1/2}\\
&\le \int_0^t (t-\sigma)^{-1/2}\cdot\Big(\int_{\partial\Omega}|\partial_\sigma\mathfrak{I}_j^{(\tau)}(b)(x,\sigma)|^2\,d\Omega(x)\Big)^{1/2} d\sigma\\
&\le \mathfrak{C}\cdot\|b\|_{1,2}\cdot\int_0^t (t-\sigma)^{-1/2}\cdot\sigma^{-3/4}\,d\sigma \;\le\; \mathfrak{C}\cdot\|b\|_{1,2}\cdot t^{-1/4}.
\end{aligned}$$

On the other hand, for $t \in (0,\infty),\ b \in C_0^\infty(U)^3,\ x \in \partial\Omega$,

(3.3)

$$\begin{aligned}
&\partial_t\Big(\int_0^t (t-\sigma)^{-1/2}\cdot\mathfrak{I}_j^{(\tau)}(b)(x,\sigma)\,d\sigma\Big) = \int_0^t r^{-1/2}\cdot\partial_t\mathfrak{I}_j^{(\tau)}(b)(x,\,t-r)\,dr\\
&= (-1/2)\cdot\int_0^{t/2}(t-\sigma)^{-3/2}\cdot\mathfrak{I}_j^{(\tau)}(b)(x,\sigma)\,d\sigma + \int_{t/2}^t (t-\sigma)^{-1/2}\cdot\partial_\sigma\mathfrak{I}_j^{(\tau)}(b)(x,\sigma)\,d\sigma\\
&\quad + (t/2)^{-1/2}\cdot\mathfrak{I}_j^{(\tau)}(b)(x,\,t/2),
\end{aligned}$$

and by Lemma 2.5 and Lemma 2.3 with $\mathfrak{A} = \partial\Omega,\ \mathfrak{B} = \mathbb{R}^3,\ F_1 = F_2 = F^{1/2}$, and by the relation $\int_{\mathbb{R}^3}\mathfrak{H}(x,t)\,dx = 1$ for $t > 0$,

$$(3.4)\qquad \|\mathfrak{I}_j^{(\tau)}(b)(\,\cdot\,,t)\,|\,\partial\Omega\|_2 \le \mathfrak{C}\cdot t^{-1/4}\cdot\|b\|_2.$$

We further find

(3.5)

$$\begin{aligned}
&\|\partial_t\mathfrak{I}_j^{(\tau)}(b)(\,\cdot\,,t)\,|\,\partial\Omega\|_2\\
&= \Big(\int_{\partial\Omega}\Big|\int_{\mathbb{R}^3}\big(-\tau\cdot\partial_1\mathfrak{H}(x-y-\tau\cdot t\cdot e_1,\,t) + \partial_4\mathfrak{H}(x-y-\tau\cdot t\cdot e_1,\,t)\big)\\
&\qquad\cdot b_j(y)\,dy\Big|^2\,d\Omega(x)\Big)^{1/2}\\
&\le \mathfrak{C}\cdot\Big(\int_{\partial\Omega}\Big(\int_{\mathbb{R}^3}(|x-y-t\cdot\tau\cdot e_1|^2+t)^{-4}\,dy\Big)^{1/2}\cdot\|b\|_2\,d\Omega(x)\Big)^{1/2}\\
&\quad + \mathfrak{C}\cdot\Big(\int_{\partial\Omega}\Big(\int_{\mathbb{R}^3}(|x-y-t\cdot\tau\cdot e_1|^2+t)^{-5/2}\cdot|b(y)|\,dy\Big)^2\,d\Omega(x)\Big)^{1/2},
\end{aligned}$$

where we used Hölder's inequality in order to estimate the term with $\partial_1\mathfrak{H}$. The second integral over $\partial\Omega$ on the right-hand side of (3.5) is evaluated by means of Lemma 2.3 with $F_1 = F_2 = F^{1/2}$. It follows that

$$(3.6)\qquad \|\partial_t\mathfrak{I}_j^{(\tau)}(b)(\,\cdot\,,t)\,|\,\partial\Omega\|_2 \le \mathfrak{C}\cdot\|b\|_2\cdot t^{-5/4}.$$

Now, starting with (3.3) and applying Minkowski's inequality for integrals, and then referring to (3.4) and (3.6), we obtain for t, b as above that

(3.7)

$$\begin{aligned}&\Big(\int_{\partial\Omega}\Big|\partial_t\Big(\int_0^t (t-\sigma)^{-1/2}\cdot \mathfrak{I}_j^{(\tau)}(b)(x,\sigma)\,d\sigma\Big)\Big|^2\,d\Omega(x)\Big)^{1/2}\\ &\le \mathfrak{C}\cdot\Big(\int_0^{t/2}(t-\sigma)^{-3/2}\cdot\|\mathfrak{I}_j^{(\tau)}(b)(\,\cdot\,,\sigma)\,|\,\partial\Omega\|_2\,d\sigma\\ &\quad+\int_{t/2}^t (t-\sigma)^{-1/2}\cdot\|\partial_\sigma\mathfrak{I}_j^{(\tau)}(b)(\,\cdot\,,\sigma)\,|\,\partial\Omega\|_2\,d\sigma+\;t^{-1/2}\cdot\|\mathfrak{I}_j^{(\tau)}(b)(\,\cdot\,,t/2)\,|\,\partial\Omega\|_2\Big)\\ &\le \mathfrak{C}\cdot\Big(\int_0^{t/2}(t-\sigma)^{-3/2}\cdot\sigma^{-1/4}\,d\sigma+\int_{t/2}^t(t-\sigma)^{-1/2}\cdot\sigma^{-5/4}\,d\sigma+t^{-3/4}\Big)\cdot\|b\|_2\\ &\le\;\mathfrak{C}\cdot t^{-3/4}\cdot\|b\|_2.\end{aligned}$$

Since (3.7) and (3.2) hold for any $b\in C_0^\infty(U)^3$, interpolation yields for the function a introduced at the beginning of the proof that

$$\begin{aligned}&\Big(\int_{\partial\Omega}\Big|\partial_t\Big(\int_0^t (t-\sigma)^{-1/2}\cdot \mathfrak{I}_j^{(\tau)}(a)(x,\sigma)\,d\sigma\Big)\Big|^2\,d\Omega(x)\Big)^{1/2}\\ &\le \mathfrak{C}\cdot t^{-1/2+\epsilon/2}\cdot\|a\|_{1/2+\epsilon,\,2}\end{aligned}\tag{3.8}$$

for $t\in(0,\infty)$. As a consequence of (3.7) and (3.8),

(3.9)

$$\begin{aligned}&\Big(\int_0^\infty\int_{\partial\Omega}\Big|\partial_t\Big(\int_0^t (t-\sigma)^{-1/2}\cdot \mathfrak{I}_j^{(\tau)}(a)(x,\sigma)\,d\sigma\Big)\Big|^2\,d\Omega(x)\,dt\Big)^{1/2}\\ &\le \mathfrak{C}\cdot\Big(\int_0^1 t^{-1+\epsilon}\,dt\cdot\|a\|^2_{1/2+\epsilon,\,2}\;+\;\int_1^\infty t^{-3/2}\,dt\cdot\|a\|_2^2\Big)^{1/2}\le\mathfrak{C}\cdot\epsilon^{-1/2}\cdot\|a\|_{1/2+\epsilon,\,2}.\end{aligned}$$

Again using Lemma 2.5 and Lemma 2.3 with $\mathfrak{A}=\partial\Omega$, $\mathfrak{B}=\mathbb{R}^3$, $F_1=F_2=F^{1/2}$, we find

$$\|\partial_k\mathfrak{I}_j^{(\tau)}(b)(\,\cdot\,,t)\,|\,\partial\Omega\|_2\le\mathfrak{C}\cdot t^{-3/4}\cdot\|b\|_2\tag{3.10}$$

for $t\in(0,\infty)$, $b\in C_0^\infty(U)^3$, $1\le k\le 3$. Moreover, according to (3.4), for t, b, k as before,

$$\|\partial_k\mathfrak{I}_j^{(\tau)}(b)(\,\cdot\,,t)\,|\,\partial\Omega\|_2=\|\mathfrak{I}_j^{(\tau)}(\partial_k b)(\,\cdot\,,t)\,|\,\partial\Omega\|_2\le\mathfrak{C}\cdot t^{-1/4}\cdot\|b\|_{1,2}.$$

Therefore by interpolation we get

$$\|\nabla\mathfrak{I}_j^{(\tau)}(a)(\,\cdot\,,t)\,|\,\partial\Omega\|_2\le\mathfrak{C}\cdot t^{-1/2+\epsilon/2}\cdot\|a\|_{1/2+\epsilon,\,2}\quad\big(t\in(0,\infty)\big).\tag{3.11}$$

In addition, with Hölder's inequality and Lemma 2.5,

$$\begin{aligned}&\|\mathfrak{I}_j^{(\tau)}(a)(\,\cdot\,,t)\,|\,\partial\Omega\|_2\\ &\le\Big(\int_{\partial\Omega}\Big(\int_{\mathbb{R}^3}\mathfrak{H}(x-y-\tau\cdot t\cdot e_1,\,t)^2\,dy\Big)\cdot\|a\|_2^2\,d\Omega(y)\Big)^{1/2}\le\mathfrak{C}\cdot t^{-3/4}\cdot\|a\|_2\end{aligned}\tag{3.12}$$

for $t \in (0,\infty)$. Combining (3.10) and (3.12) yields

$$\|\mathfrak{I}_j^{(\tau)}(a)(\,\cdot\,,t)\,|\,\partial\Omega\|_{1,2} \le \mathfrak{C}\cdot t^{-3/4}\cdot\|a\|_2 \quad \big(t\in(0,\infty)\big),$$

and the estimates in (3.4) and (3.11) imply

$$\|\mathfrak{I}_j^{(\tau)}(a)(\,\cdot\,,t)\,|\,\partial\Omega\|_{1,2} \le \mathfrak{C}\cdot t^{-1/2+\epsilon/2}\cdot\|a\|_{1/2+\epsilon,2} \quad \big(t\in(0,1)\big).$$

It follows as in (3.9) that

$$\left(\int_0^\infty \|\mathfrak{I}_j^{(\tau)}(a)(\,\cdot\,,t)\|_{1,2}^2\,dt\right)^{1/2} \le \mathfrak{C}\cdot\epsilon^{-1/2}\cdot\|a\|_{1/2+\epsilon,2}. \tag{3.13}$$

Let $v\in H^1(\partial\Omega)$, $\sigma\in(0,\infty)$. We deduce from (3.6)

$$\begin{aligned}&\left|\int_{\partial\Omega}\big(\partial_\sigma\mathfrak{I}^{(\tau)}(a)(x,\sigma)\cdot n^{(\Omega)}(x)\big)\cdot v(x)\,d\Omega(x)\right| \\ &\le \mathfrak{C}\cdot\|\partial_\sigma\mathfrak{I}^{(\tau)}(a)(\,\cdot\,,\sigma)\,|\,\partial\Omega\|_2\cdot\|v\|_2 \le \mathfrak{C}\cdot\sigma^{-5/4}\cdot\|a\|_2\cdot\|v\|_2.\end{aligned} \tag{3.14}$$

For $p\in\mathbb{N}$, let $E^{(p)}(v)$ be defined as in Theorem 2.4. An estimate as in (3.14) yields

$$\int_{\partial\Omega}\big(\partial_\sigma\mathfrak{I}^{(\tau)}(a)(x,\sigma)\cdot n^{(\Omega)}(x)\big)\cdot\big(v(x)-E^{(p)}(v)(x)\big)\,d\Omega(x) \to 0 \tag{3.15}$$

for $p\to\infty$. Now take $p\in\mathbb{N}$. Recalling the results on $\mathfrak{I}^{(\tau)}(a)$ in Theorem 2.16, in particular the equation $\operatorname{div}_x\mathfrak{I}^{(\tau)}(a)(\,\cdot\,,\sigma)=0$, we get

$$\begin{aligned}&\int_{\partial\Omega}\big(\partial_\sigma\mathfrak{I}^{(\tau)}(a)(x,\sigma)\cdot n^{(\Omega)}(x)\big)\cdot E^{(p)}(v)(x)\,d\Omega(x) \\ &= \int_\Omega \partial_\sigma\mathfrak{I}^{(\tau)}(a)(x,\sigma)\cdot\nabla E^{(p)}(v)(x)\,dx \\ &= \int_\Omega (\Delta_x-\tau\cdot\partial_{x_1})\mathfrak{I}^{(\tau)}(a)(x,\sigma)\cdot\nabla E^{(p)}(v)(x)\,dx.\end{aligned} \tag{3.16}$$

But for $b\in C_0^\infty(U)^3$, we may conclude from Lemma 2.5 and Lemma 2.3 with $\mathfrak{A}=\Omega$, $\mathfrak{B}=\mathbb{R}^3$, and from Theorem 2.4 that

$$\begin{aligned}&\left|\int_\Omega \partial_1\mathfrak{I}^{(\tau)}(b)(x,\sigma)\cdot\nabla E^{(p)}(v)(x)\,dx\right| \le \mathfrak{C}\cdot\|\partial_1\mathfrak{I}^{(\tau)}(b)(\,\cdot\,,\sigma)\,|\,\Omega\|_2\cdot\|\nabla E^{(p)}(v)\,|\,\Omega\|_2 \\ &\le \mathfrak{C}\cdot\left(\int_\Omega\left(\int_{\mathbb{R}^3}(|x-y-\tau\cdot\sigma\cdot e_1|^2+\sigma)^{-2}\cdot|b(y)|\,dy\right)^2dx\right)^{1/2}\cdot\|\nabla E^{(p)}(v)\,|\,\Omega\|_2 \\ &\le \mathfrak{C}\cdot\sigma^{-1/2}\cdot\|b\|_2\cdot\|v\|_{1,2}.\end{aligned}$$

Similarly, with the relation $\int_{\mathbb{R}^3}\mathfrak{H}(z,\sigma)\,dz=1$,

$$\begin{aligned}&\left|\int_\Omega \partial_1\mathfrak{I}^{(\tau)}(b)(x,\sigma)\cdot\nabla E^{(p)}(v)(x)\,dx\right| = \left|\int_\Omega \mathfrak{I}^{(\tau)}(\partial_1 b)(x,\sigma)\cdot\nabla E^{(p)}(v)(x)\,dx\right| \\ &\le \mathfrak{C}\cdot\|\partial_1 b\|_2\cdot\|v\|_{1,2} \le \mathfrak{C}\cdot\|b\|_{1,2}\cdot\|v\|_{1,2}\end{aligned}$$

for b as before. Now it follows by interpolation that

(3.17)
$$\left|\int_\Omega \partial_1 \mathfrak{J}^{(\tau)}(a)(x,\sigma)\cdot\nabla E^{(p)}(v)(x)\,dx\right| \le \mathfrak{C}\cdot\sigma^{-1/4+\epsilon/2}\cdot\|a\|_{1/2+\epsilon,2}\cdot\|v\|_{1,2}.$$

Moreover,

$$\begin{aligned}(3.18)\qquad &\int_\Omega \Delta_x \mathfrak{J}^{(\tau)}(a)(x,\sigma)\cdot\nabla E^{(p)}(v)(x)\,dx\\ &= -\sum_{j,k=1}^{3}\int_\Omega \partial_j\mathfrak{J}_k^{(\tau)}(a)(x,\sigma)\cdot\partial_j\partial_k E^{(p)}(v)(x)\,dx\\ &\quad+\sum_{j,k=1}^{3}\int_{\partial\Omega}\partial_j\mathfrak{J}_k^{(\tau)}(a)(x,\sigma)\cdot\partial_k E^{(p)}(v)(x)\cdot n_j^{(\Omega)}(x)\,d\Omega(x)\\ &= \sum_{j,k=1}^{3}\int_{\partial\Omega}\partial_j\mathfrak{J}_k^{(\tau)}(a)(x,\sigma)\\ &\quad\cdot\big(\partial_k E^{(p)}(v)(x)\cdot n_j^{(\Omega)}(x)-\partial_j E^{(p)}(v)(x)\cdot n_k^{(\Omega)}(x)\big)\,d\Omega(x).\end{aligned}$$

In the last equation, we used again that $\operatorname{div}_x \mathfrak{J}^{(\tau)}(a) = 0$ (Theorem 2.16). On the other hand, we find for $j,k,l,m \in \{1, 2, 3\}$ that

(3.19)
$$\begin{aligned}&\left|\int_{\partial\Omega}\partial_l\mathfrak{J}_m^{(\tau)}(a)(x,\sigma)\cdot\partial_j E^{(p)}(v)(x)\cdot n_k^{(\Omega)}(x)\,d\Omega(x)\right|\\ &\le \mathfrak{C}\cdot\|\nabla_x\mathfrak{J}^{(\tau)}(a)(\,\cdot\,,\sigma)\,|\,\partial\Omega\|_2\cdot\|\nabla E^{(p)}(v)\,|\,\partial\Omega\|_2 \le \mathfrak{C}\cdot\sigma^{-1/2+\epsilon/2}\cdot\|a\|_{1/2+\epsilon,2}\cdot\|v\|_{1,2},\end{aligned}$$

where the last estimate is a consequence of (3.11) and Theorem 2.4. By combining (3.16)–(3.19), we get in the case $\sigma \le 1$ that

$$\begin{aligned}&\left|\int_{\partial\Omega}\big(\partial_\sigma\mathfrak{J}^{(\tau)}(a)(x,\sigma)\cdot n^{(\Omega)}(x)\big)\cdot E^{(p)}(v)(x)\,d\Omega(x)\right|\\ &\le \mathfrak{C}\cdot\sigma^{-1/2+\epsilon/2}\cdot\|a\|_{1/2+\epsilon,2}\cdot\|v\|_{1,2}.\end{aligned}$$

Since this inequality is true for any $p \in \mathbb{N}$, we now find with (3.15) and (3.14)

$$\begin{aligned}&\|\partial_\sigma\mathfrak{J}^{(\tau)}(a)(\,\cdot\,,\sigma)\cdot n^{(\Omega)}\|_{H^1(\partial\Omega)'}\\ &\le \mathfrak{C}\cdot\big(\chi_{(0,1]}(\sigma)\cdot\sigma^{-1/2+\epsilon/2}+\chi_{(1,\infty)}(\sigma)\cdot\sigma^{-5/4}\big)\cdot\|a\|_{1/2+\epsilon,2}.\end{aligned}$$

It follows that

$$(3.20)\qquad \left(\int_0^\infty \|\partial_\sigma\mathfrak{J}^{(\tau)}(a)(\,\cdot\,,\sigma)\cdot n^{(\Omega)}\|^2_{H^1(\partial\Omega)'}\,d\sigma\right)^{1/2} \le \mathfrak{C}\cdot\epsilon^{-1/2}\cdot\|a\|_{1/2+\epsilon,2}.$$

The estimates in (3.20), (3.13), and (3.9) imply that $\|\mathfrak{J}^{(\tau)}(a)|S_\infty\|_{H_\infty} \le \mathfrak{C}\cdot\epsilon^{-1/2}\cdot \|a\|_{1/2+\epsilon,2}$. Observing that inequalities (3.4) and (3.12) remain valid if the functions b and a are replaced by an arbitrary function $\gamma \in L^2(U)^3$, we see that the theorem now follows by a density argument. ◻

4. Study of the potential $\mathfrak{R}^{(\tau)}(f)$. We want to find criteria on f which guarantee that $\mathfrak{R}^{(\tau)}(f)\,|\,S_\infty \in H_\infty$. A result in this respect was already proved in [4]; it may be stated as follows.

THEOREM 4.1 (see [4, Theorem 7]). *Let $q \in (1,2)$, $\alpha \in (1, q'/2)$, $\beta \in (4/3, 2)$, and $f \in L^2(\mathbb{R}^3)^3$ with $f|B^c_{R_0} \times (0,\infty) \in L^\alpha(0,\infty, L^q(B^c_{R_0})^3) \cap L^\beta(0,\infty, L^q(B^c_{R_0})^3)$. Then $\mathfrak{R}^{(\tau)}(f)\,|\,S_\infty \in H_\infty$ and*

$$\|\mathfrak{R}^{(\tau)}(f)\,|\,S_\infty\|_{H_\infty} \le C\cdot \big(\|f\|_2 + \|f\,|\,B^c_{R_0}\times(0,\infty)\|_{L^\alpha(0,\infty,L^q(B^c_{R_0})^3)} + \|f\,|\,B^c_{R_0}\times(0,\infty)\|_{L^\beta(0,\infty,L^q(B^c_{R_0})^3)}\big),$$

where R_0 is as introduced at the beginning of section 2, and where the constant $C > 0$ depends on Ω, R_0, τ, q, α, and β.

As mentioned above, a proof of this theorem may be found in [4]. However, we remark that the terms $(|z|^2+t)^{-3/2-|\alpha|/2-l/2}$ and $\big(|z|\cdot(1+\tau\cdot s(z))+t\big)^{-3/2-|\alpha|/2-l/2}$, respectively, are lacking on the right-hand side of the inequalities stated in [4, Lemma 1], which corresponds to Lemma 2.6 here. As a consequence, an additional term $\mathfrak{C}\cdot \int_0^t\int_{\mathbb{R}^3}\big(\varrho(x,y)+t-\sigma\big)^{-3/2}\cdot|f(y,\sigma)|\,dy\,d\sigma$, with $(x,t)\in S_\infty$, should appear on the right-hand side of [4, equation (14)]. Since this right-hand side is estimated in the norm of $L^2(S_\infty)$, such an estimate must also be performed for this additional term. But this was already done in [4, equations (18)–(23)], so the proof in [4] is indeed complete. Note that the function $\widetilde{v}$ introduced in [4] following [4, equation (23)] belongs to $H^{3/2-\kappa}(\Omega)$ for $\kappa\in[0, 1/2)$, and not for $\kappa \in (1,2)$, as stated in [4, p. 257].

Theorem 4.1 does not seem to be very convenient for eventual applications to the nonlinear problem (1.1)–(1.3). As explained in section 1, the ensuing theorem should be more interesting in this respect.

THEOREM 4.2. *Put $\mathfrak{U} := \Omega \cup \cup_{i=1}^{k(\Omega)} U_i^{1/2}$. Let $q \in (3/2, 2]$, $T\in(0,\infty]$, $f \in L^2(0,T,L^1(\mathbb{R}^3)^3)$ with $f\,|\,\mathfrak{U}\times(0,T) \in L^2(0,T,L^q(\mathfrak{U})^3)$. Then $\mathfrak{R}^{(\tau)}(f)\,|\,S_T\in H_T$ and*

$$\|\mathfrak{R}^{(\tau)}(f)|S_T\|_{H_T} \le \mathfrak{C}(q)\cdot\big(\|f\|_{L^2(0,T,L^1(\mathbb{R}^3)^3)} + \|f|\mathfrak{U}\times(0,T)\|_{L^2(0,T,L^q(\mathfrak{U})^3)}\big). \tag{4.1}$$

Proof. We use the notations introduced at the beginning of section 2. The following abbreviations will prove to be convenient:

$$Z_i := \partial\Omega\backslash\Lambda_i^1,\quad V_i := U_i^{1/2}\ \text{ for } 1\le i\le k(\Omega),\quad Z_{k(\Omega)+1} := \partial\Omega,$$
$$V_{k(\Omega)+1} := \mathbb{R}^3\backslash \cup_{i=1}^{k(\Omega)}\overline{U_i^{1/4}}.$$

For $x\in\partial\Omega$, $y\in\mathbb{R}^3\backslash\cup_{i=1}^{k(\Omega)}\overline{U_i^{1/4}}$, the inequality $|x-y|\ge\mathfrak{D}_4$ holds by (2.6). In the case $i\in\{1,\dots,k(\Omega)\}$, $x\in\partial\Omega\backslash\Lambda_i^1$, $y\in U_i^{1/2}$, we also have $|x-y|\ge\mathfrak{D}_4$ by (2.6). Thus $|x-y|\ge\mathfrak{D}_4$ for $x\in Z_i$, $y\in V_i$, $i\in\{1,\dots,k(\Omega)+1\}$, and we may conclude with Lemma 2.6 that

$$\begin{aligned}|\partial_t^l\partial_x^\alpha\Lambda_{jk}(x-y,t,\tau)| &\le \mathfrak{C}\cdot\big((1+t)^{-3/2-|\alpha|/2-l}+(1+t)^{-3/2-|\alpha|/2-l/2}\big)\\ &\le \mathfrak{C}\cdot(1+t)^{-3/2-|\alpha|-l/2}\end{aligned} \tag{4.2}$$

for such x and y, $t\in(0,\infty)$, $l\in\mathbb{N}_0$, $\alpha\in\mathbb{N}_0^3$ with $l+|\alpha|\le 1$, $1\le j,k\le 3$.

Now let $T\in(0,\infty]$, $f\in C_0^\infty\big(\mathbb{R}^3\times(0,T)\big)^3$. Since $\mathfrak{R}^{(\tau)}(f)\in C_0^\infty\big(\mathbb{R}^3\times(0,\infty)\big)^3$ (Theorem 2.16), we have $\mathfrak{R}^{(\tau)}(f)\,|\,S_T\in\widetilde{H}_T$. Without loss of generality, we may assume that $T=\infty$. The sets $V_1,\dots,V_{k(\Omega)+1}$ constitute an open covering of $\mathbb{R}^3$.

L with $R^{(\tau)}(f)|\mathbb{R}^3\times(0,\varepsilon] = 0$ for some $\varepsilon>0$

Thus we may choose a partition of unity $g_1, \dots, g_{k(\Omega)+1} \in C_0^\infty(\mathbb{R}^3)$ subordinate to that covering. Put $F_i(x,t) := g_i(x) \cdot f(x,t)$ for $x \in \mathbb{R}^3$, $t \in (0,\infty)$, $1 \le i \le k(\Omega)+1$. Then $F_i \in C_0^\infty\big(\mathbb{R}^3 \times (0,\infty)\big)^3$, $\operatorname{supp}\big(F_i(\,\cdot\,,t)\big) \subset V_i$ for $t \in (0,\infty)$, $1 \le i \le k(\Omega)+1$, and $f = \sum_{i=1}^{k(\Omega)+1} F_i$. Now we get

(4.3)

$$\begin{aligned}
&\|\mathfrak{R}^{(\tau)}(f) \,|\, S_\infty\|_{H_\infty} \\
&\le \mathfrak{C} \cdot \sum_{j=1}^{3} \sum_{i=1}^{k(\Omega)+1} \Bigg[\sum_{\alpha \in \mathbb{N}_0^3,\, |\alpha| \le 1} \|\partial^\alpha \mathfrak{R}_j^{(\tau)}(F_i) \,|\, S_\infty\|_2 \\
&\qquad + \Bigg(\int_0^\infty \int_{\partial\Omega} \Big| \partial_t \Big(\int_0^t (t-r)^{-1/2} \cdot \mathfrak{R}_j^{(\tau)}(F_i)(x,r)\, dr \Big) \Big|^2 d\Omega(x)\, dt \Bigg)^{1/2} \Bigg] \\
&\quad + \Bigg(\int_0^\infty \|\partial_t \mathfrak{R}^{(\tau)}(f)(\,\cdot\,,t) \cdot n^{(\Omega)}\|^2_{H^1(\partial\Omega)'}\, dt \Bigg)^{1/2} \\
&\le \mathfrak{C} \cdot \sum_{j=1}^{3} \Bigg(\sum_{i=1}^{k(\Omega)} (A_{i,j} + B_{i,j} + C_{i,j} + D_{i,j}) \;+\; C_{k(\Omega)+1,j} \;+\; D_{k(\Omega)+1,j} \Bigg) \\
&\quad + \Bigg(\int_0^\infty \|\partial_t \mathfrak{R}^{(\tau)}(f)(\,\cdot\,,t) \cdot n^{(\Omega)}\|^2_{H^1(\partial\Omega)'}\, dt \Bigg)^{1/2}
\end{aligned}$$

where we used the abbreviations

$$A_{i,j} := \sum_{\alpha \in \mathbb{N}_0^3,\, |\alpha| \le 1} \Bigg(\int_0^\infty \int_{\Lambda_i^1} |\partial^\alpha \mathfrak{R}_j^{(\tau)}(F_i)(x,t)|^2\, d\Omega(x)\, dt \Bigg)^{1/2},$$

$$B_{i,j} := \Bigg(\int_0^\infty \int_{\Lambda_i^1} \Big| \partial_t \Big(\int_0^t (t-r)^{-1/2} \cdot \mathfrak{R}_j^{(\tau)}(F_i)(x,r)\, dr \Big) \Big|^2 d\Omega(x)\, dt \Bigg)^{1/2}$$

for $1 \le j \le 3$, $1 \le i \le k(\Omega)$. The terms $C_{i,j}$ and $D_{i,j}$ are defined as $A_{i,j}$ and $B_{i,j}$, respectively, but with the domain of integration Λ_i replaced by Z_i, and with the index i running from 1 to $k(\Omega)+1$ instead of $k(\Omega)$.

Let $v \in H^1(\partial\Omega)$, $t \in (0,\infty)$, and recall the function $E^{(n)}(v)$ for $n \in \mathbb{N}$, introduced in Theorem 2.4. By Theorem 2.16 (note in particular that $\operatorname{div} \mathfrak{R}^{(\tau)}(f) = 0$), we get

$$\begin{aligned}
\mathfrak{Z} &:= \int_{\partial\Omega} \partial_t \mathfrak{R}^{(\tau)}(f)(x,t) \cdot n^{(\Omega)}(x) \cdot v(x)\, d\Omega(x) \\
&= \lim_{n\to\infty} \int_{\partial\Omega} \partial_t \mathfrak{R}^{(\tau)}(f)(x,t) \cdot n^{(\Omega)}(x) \cdot E^{(n)}(v)(x)\, d\Omega(x) \\
&= \lim_{n\to\infty} \int_{\Omega} \partial_t \mathfrak{R}^{(\tau)}(f)(x,t) \cdot \nabla E^{(n)}(v)(x)\, dx \\
&= \lim_{n\to\infty} \int_{\Omega} \big(\Delta_x \mathfrak{R}^{(\tau)}(f) - \tau \cdot \partial_{x_1} \mathfrak{R}^{(\tau)}(f) - \nabla \mathfrak{P}(f) + f \big)(x,t) \cdot \nabla E^{(n)}(v)(x)\, dx \\
&= \lim_{n\to\infty} \int_{\partial\Omega} \sum_{j=1}^{3} \Bigg[\sum_{k=1}^{3} \partial_k \mathfrak{R}_j^{(\tau)}(f)(x,t) \big(\partial_j E^{(n)}(v)(x) \cdot n_k^{(\Omega)}(x) - \partial_k E^{(n)}(v)(x) \cdot n_j^{(\Omega)}(x) \big) \\
&\qquad - \tau \cdot \partial_{x_1} \mathfrak{R}_j^{(\tau)}(f)(x,t) \cdot E^{(n)}(v)(x) \cdot n_j^{(\Omega)}(x) \Bigg] d\Omega(x) \\
&\quad + \lim_{n\to\infty} \int_{\Omega} \big(-\nabla \mathfrak{P}(f) + f \big)(x,t) \cdot \nabla E^{(n)}(v)(x)\, dx,
\end{aligned}$$

where the last equation follows by two partial integrations involving $\Delta_x \mathfrak{R}^{(\tau)}(f)$, and by a single partial integration of the product $\tau \cdot \partial_{x_1} \mathfrak{R}^{(\tau)}(f)(x,t) \cdot \nabla E^{(n)}(v)(x)$. Since $q \in (3/2, 2]$, we have $q' \in [2, 3)$, and thus we get, by referring to Theorem 2.4,

(4.4)

$$|\mathfrak{Z}| \le \mathfrak{C} \cdot \Big(\|\nabla \mathfrak{R}^{(\tau)}(f)(\,\cdot\,,t) \,|\, \partial\Omega\|_2 \cdot \sup_{n\in\mathbb{N}} \big(\|\nabla E^{(n)}(v) \,|\, \partial\Omega\|_2 + \|E^{(n)}(v) \,|\, \partial\Omega\|_2 \big)$$
$$+ \big(\|\nabla \mathfrak{P}(f)(\,\cdot\,,t) \,|\, \Omega\|_q + \|f(\,\cdot\,,t) \,|\, \Omega\|_q \big) \cdot \sup_{n\in\mathbb{N}} \|\nabla E^{(n)}(v) \,|\, \Omega\|_{q'} \Big)$$
$$\le \mathfrak{C} \cdot \big(\|\nabla \mathfrak{R}^{(\tau)}(f)(\,\cdot\,,t) \,|\, \partial\Omega\|_2 + \|\nabla \mathfrak{P}(f)(\,\cdot\,,t) \,|\, \Omega\|_q + \|f(\,\cdot\,,t) \,|\, \Omega\|_q \big) \cdot \|v\|_{1,2}.$$

By applying a well-known result based on the Calderon–Zygmund inequality, we get

$$\|\nabla \mathfrak{P}(F_i)(\,\cdot\,,t)\|_q \;\le\; \mathfrak{C}(q) \cdot \|F_i(\,\cdot\,,t)\|_q \;\le\; \mathfrak{C}(q) \cdot \|f(\,\cdot\,,t) \,|\, U_i^{1/2}\|_q$$

for $i \in \{1, \dots, k(\Omega)\}$. The same argument yields

$$\|\nabla \mathfrak{P}\big(F_{k(\Omega)+1} | \mathfrak{U} \times (0,\infty)\big)(\,\cdot\,,t)\|_q \le \mathfrak{C}(q) \cdot \|f(\,\cdot\,,t) | \mathfrak{U}\|_q.$$

Since $\partial\Omega \subset \cup_{i=1}^{k(\Omega)} U_i^{1/4}$, there is $\delta > 0$ such that $|x - y| \ge \delta$ for $x \in \Omega$, $y \in \mathbb{R}^3 \backslash \mathfrak{U}$. Therefore it is obvious that for $x \in \Omega$,

$$|\nabla \mathfrak{P}\big(F_{k(\Omega)+1} \,|\, (\mathbb{R}^3 \backslash \mathfrak{U}) \times (0,\infty)\big)(x,t)| \le \mathfrak{C} \cdot \|F_{k(\Omega)+1}(\,\cdot\,,t)\|_1 \le \mathfrak{C} \cdot \|f(\,\cdot\,,t)\|_1,$$

hence

$$\|\nabla \mathfrak{P}\big(F_{k(\Omega)+1} \,|\, (\mathbb{R}^3 \backslash \mathfrak{U}) \times (0,\infty)\big)(\,\cdot\,,t) \,|\, \Omega\|_q \;\le\; \mathfrak{C} \cdot \|f(\,\cdot\,,t)\|_1.$$

Since $f = \sum_{i=1}^{k(\Omega)+1} F_i$, we have thus found that

$$\|\nabla \mathfrak{P}(f)(\,\cdot\,,t) \,|\, \Omega\|_q \;\le\; \mathfrak{C}(q) \cdot \big(\|f(\,\cdot\,,t) \,|\, \mathfrak{U}\|_q + \|f(\,\cdot\,,t)\|_1 \big).$$

Since v was arbitrarily chosen in $H^1(\partial\Omega)$, and t in $(0,\infty)$, we may thus deduce from (4.4),

(4.5)

$$\Big(\int_0^\infty \|\partial_t \mathfrak{R}^{(\tau)}(f)(\,\cdot\,,t) \cdot n^{(\Omega)}\|^2_{H^1(\partial\Omega)'}\, dt \Big)^{1/2}$$
$$\le \mathfrak{C} \cdot \big(\|\nabla \mathfrak{R}^{(\tau)}(f) \,|\, S_\infty\|_2 + \|f | \mathfrak{U} \times (0,\infty)\|_{L^2(0,\infty,L^q(\mathfrak{U})^3)} + \|f\|_{L^2(0,\infty,L^1(\mathbb{R}^3)^3)} \big).$$

But by again referring to the equation $f = \sum_{i=1}^{k(\Omega)+1} F_i$, we see that

$$\|\nabla \mathfrak{R}^{(\tau)}(f) | S_\infty\|_2 \;\le\; \sum_{j=1}^{3} \Big(\sum_{i=1}^{k(\Omega)} (A_{i,j} + C_{i,j}) \;+\; C_{k(\Omega)+1,\,j} \Big).$$

This relation, with (4.5) and (4.4), implies

$$(4.6) \quad \|\mathfrak{R}^{(\tau)}(f) \,|\, S_\infty\|_{H_\infty}$$
$$\le \mathfrak{C} \cdot \sum_{j=1}^{3} \Big(\sum_{i=1}^{k(\Omega)} (A_{i,j} + B_{i,j} + C_{i,j} + D_{i,j}) \;+\; C_{k(\Omega)+1,\,j} \;+\; D_{k(\Omega)+1,\,j} \Big)$$
$$+ \mathfrak{C} \cdot \big(\|f\|_{L^2(0,\infty,L^1(\mathbb{R}^3)^3)} + \|f | \mathfrak{U} \times (0,\infty)\|_{L^2(0,\infty,L^q(\mathfrak{U})^3)} \big).$$

Take $j \in \{1, 2, 3\}$, $i \in \{1, \dots, k(\Omega)+1\}$, and let us estimate $C_{i,j}$ and $D_{i,j}$. By Lemma 2.11 and (4.2), we have

(4.7)

$$\begin{aligned}
&C_{i,j}^2 \\
&\leq \mathfrak{C} \cdot \sum_{\alpha \in \mathbb{N}_0^3,\, |\alpha| \leq 1} \int_0^\infty \int_{Z_i} \Big(\int_0^t \int_{V_i} \sum_{k=1}^3 |\partial^\alpha \Lambda_{jk}(x-y, t-\sigma, \tau)| \cdot |f(y,\sigma)| \, dy \, d\sigma \Big)^2 d\Omega(x) \, dt \\
&\leq \mathfrak{C} \cdot \int_0^\infty \int_{\partial\Omega} \Big(\int_0^t \int_{\mathbb{R}^3} (1+t-\sigma)^{-3/2-|\alpha|/2} \cdot |f(y,\sigma)| \, dy \, d\sigma \Big)^2 d\Omega(x) \, dt \\
&\leq \mathfrak{C} \cdot \int_0^\infty \Big(\int_0^\infty \chi_{(0,\infty)}(t-\sigma) \cdot (1+t-\sigma)^{-3/2-|\alpha|/2} \cdot \|f(\,\cdot\,,\sigma)\|_1 \, d\sigma \Big)^2 dt \\
&\leq \mathfrak{C} \cdot \|f\|^2_{L^2(0,\infty,L^1(\mathbb{R}^3)^3)},
\end{aligned}$$

where the last estimate follows by Young's inequality. In order to estimate $D_{i,j}$, we observe that

$$\begin{aligned}
(4.8) \qquad &\partial_t \Big(\int_0^t (t-r)^{-1/2} \cdot \mathfrak{R}_j^{(\tau)}(F_i)(x,r) \, dr \Big) \\
&= \sum_{k=1}^3 \int_0^t \int_{V_i} H_{jk}(x-y, t, \sigma) \cdot F_{i,k}(y,\sigma) \, dy \, d\sigma
\end{aligned}$$

for $x \in \partial\Omega$, $t \in (0,\infty)$, with

$$\begin{aligned}
&H_{jk}(z,t,\sigma) \\
&:= 2^{1/2} \cdot (t-\sigma)^{-1/2} \cdot \Lambda_{jk}(z, (t-\sigma)/2, \tau) + \int_{(t+\sigma)/2}^t (t-r)^{-1/2} \cdot \partial_r \Lambda_{jk}(z, r-\sigma, \tau) \, dr \\
&\quad - (1/2) \cdot \int_\sigma^{(t+\sigma)/2} (t-r)^{-3/2} \cdot \Lambda_{jk}(z, r-\sigma, \tau) \, dr \quad \text{for } z \in \mathbb{R}^3 \backslash \{0\},\ 1 \leq k \leq 3.
\end{aligned}$$

Some details of the computations leading to (4.8) may be found in [4, equation (13)]. On the other hand, for $x \in Z_i$, $y \in V_i$, $t \in (0,\infty)$, $\sigma \in (0,t)$, $1 \leq k \leq 3$, we get by referring to (4.2) that

(4.9)

$$\begin{aligned}
&|H_{jk}(x-y,t,\sigma)| \\
&\leq \mathfrak{C} \cdot \Big((t-\sigma)^{-1/2} \cdot (1+t-\sigma)^{-3/2} + \int_{(t+\sigma)/2}^t (t-r)^{-1/2} \cdot (1+r-\sigma)^{-2} \, dr \\
&\qquad + (t-\sigma)^{-3/2} \cdot \int_\sigma^{(t+\sigma)/2} (1+r-\sigma)^{-3/2} \, dr \Big) \\
&\leq \mathfrak{C} \cdot \Big((t-\sigma)^{-1/2} \cdot (1+t-\sigma)^{-3/2} + (1+t-\sigma)^{-2} \cdot \int_{(t+\sigma)/2}^t (t-r)^{-1/2} \, dr \\
&\qquad + (t-\sigma)^{-3/2} \cdot \big(\chi_{(0,1)}(t-\sigma) \cdot \int_\sigma^{(t+\sigma)/2} dr \\
&\qquad + \chi_{(1,\infty)}(t-\sigma) \cdot \int_\sigma^{(t+\sigma)/2} (1+r-\sigma)^{-15/16} \, dr \big) \Big)
\end{aligned}$$

$$\leq \mathfrak{C} \cdot \Big((t-\sigma)^{-1/2} \cdot (1+t-\sigma)^{-3/2} \;+\; (1+t-\sigma)^{-3/2} \;+\; \chi_{(0,1)}(t-\sigma) \cdot (t-\sigma)^{-1/2} \\ \quad + \; \chi_{(1,\infty)}(t-\sigma) \cdot (t-\sigma)^{-23/16} \Big)$$

$$\leq \mathfrak{C} \cdot \Big(\chi_{(0,1)}(t-\sigma) \cdot (t-\sigma)^{-1/2} + \chi_{(1,\infty)}(t-\sigma) \cdot (t-\sigma)^{-23/16} \Big).$$

Using (4.8) and (4.9), we get

(4.10)

$$|D_{i,j}| \\ \leq \mathfrak{C} \cdot \Big(\int_0^\infty \int_{Z_i} \Big| \sum_{k=1}^3 \int_0^t \int_{V_i} H(x-y,t,\sigma) \cdot F_{i,k}(y,\sigma)\, dy\, d\sigma \Big|^2 d\Omega(x)\, dt \Big)^{1/2} \\ \leq \mathfrak{C} \cdot \Big(\int_0^\infty \int_{Z_i} \Big| \int_0^t \int_{V_i} \Big(\chi_{(0,1)}(t-\sigma) \cdot (t-\sigma)^{-1/2} + \chi_{(1,\infty)}(t-\sigma) \cdot (t-\sigma)^{-23/16} \Big) \\ \qquad \cdot |f(y,\sigma)|\, dy\, d\sigma \Big|^2 d\Omega(x)\, dt \Big)^{1/2} \\ \leq \mathfrak{C} \cdot \int_0^\infty \Big(\chi_{(0,1)}(r) \cdot r^{-1/2} + \chi_{(1,\infty)}(r) \cdot r^{-23/16} \Big)\, dr \cdot \|f|V_i \times (0,\infty)\|_{L^2(0,\infty,L^1(V_i)^3)} \\ \leq \mathfrak{C} \cdot \|f\|_{L^2(0,\infty,L^1(\mathbb{R}^3)^3)},$$

where the penultimate estimate follows by Young's inequality.

Now take $j \in \{1, 2, 3\}$, $i \in \{1, \ldots, k(\Omega)\}$, and consider $A_{i,j}$ and $B_{i,j}$. Since $q > 3/2$, we have $1 - 3/(2 \cdot q) > 0$, so we may choose $\delta \in \big(0, 1 - 3/(2 \cdot q)\big)$ with $\delta < 1/2$. Let $\epsilon \in (0, 1/2)$.

For $x \in \Lambda_i^1$, $t \in (0,\infty)$, $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq 1$, by Lemmas 2.6 and 2.11 and the choice of F_i and V_i we obtain

(4.11)

$$|\partial^\alpha \mathfrak{R}_j^{(\tau)}(F_i)(x,t)| \\ \leq \mathfrak{C} \cdot \int_0^t \int_{U_i^{1/2}} (|x-y|^2 + t - \sigma)^{-3/2-|\alpha|/2} \cdot |f(y,\sigma)|\, dy\, d\sigma \\ \leq \mathfrak{C} \cdot \int_0^t \int_{U_i^{1/2}} \Big(\chi_{(0,1)}(t-\sigma) \cdot (t-\sigma)^{-1+\delta} \cdot |x-y|^{-1-|\alpha|-2\cdot\delta} \\ \qquad + \chi_{(1,\infty)}(t-\sigma) \cdot (t-\sigma)^{-1-\epsilon} \cdot |x-y|^{-1-|\alpha|+2\cdot\epsilon} \Big) \cdot |f(y,\sigma)|\, dy\, d\sigma \\ \leq \mathfrak{C} \cdot \int_0^t \int_{U_i^{1/2}} \Big(\chi_{(0,1)}(t-\sigma) \cdot (t-\sigma)^{-1+\delta} \cdot |x-y|^{-2-2\cdot\delta} \\ \qquad + \chi_{(1,\infty)}(t-\sigma) \cdot (t-\sigma)^{-1-\epsilon} \cdot |x-y|^{-2+2\cdot\epsilon} \Big) \cdot |f(y,\sigma)|\, dy\, d\sigma.$$

Moreover, with Lemma 2.6, for $t \in (0,\infty)$, $\sigma \in (0,t)$, $x \in \Lambda_i^1$, $y \in U_i^{1/2}$, $1 \leq k \leq 3$,

$$|H_{jk}(x-y,t,\sigma)| \\ \leq \mathfrak{C} \cdot \Big((t-\sigma)^{-1/2} \cdot (|x-y|^2 + t - \sigma)^{-3/2} \\ \qquad + \int_{(t+\sigma)/2}^t (t-r)^{-1/2} \cdot \sum_{\nu \in \{2, 5/2\}} (|x-y|^2 + r - \sigma)^{-\nu}\, dr$$

$$
\begin{aligned}
&\qquad + \int_\sigma^{(t+\sigma)/2} (t-r)^{-3/2} \cdot (|x-y|^2 + r - \sigma)^{-3/2}\, dr\Big) \\
&\le \mathfrak{C} \cdot \Big((t-\sigma)^{-1/2} \cdot (|x-y|^2 + t - \sigma)^{-3/2} \\
&\qquad + \sum_{\nu \in \{2, 5/2\}} (|x-y|^2 + t - \sigma)^{-\nu} \cdot \int_{(t+\sigma)/2}^{t} (t-r)^{-1/2}\, dr \\
&\qquad + (t-\sigma)^{-3/2} \cdot \int_\sigma^{(t+\sigma)/2} (|x-y|^2 + r - \sigma)^{-3/2}\, dr \Big) \\
&\le \mathfrak{C} \cdot \Big((t-\sigma)^{-1/2} \cdot (|x-y|^2 + t - \sigma)^{-3/2} \\
&\qquad + \Big[\sum_{\nu \in \{2, 5/2\}} (|x-y|^2 + t - \sigma)^{-\nu} \Big] \cdot (t-\sigma)^{1/2} \\
&\qquad + (t-\sigma)^{-3/2} \cdot \Big[\chi_{(0,1)}(t-\sigma) \cdot |x-y|^{-2-2\cdot\delta} \cdot \int_\sigma^{(t+\sigma)/2} (r-\sigma)^{-1/2+\delta}\, dr \\
&\qquad\qquad + \chi_{(1,\infty)}(t-\sigma) \cdot |x-y|^{-2+2\cdot\epsilon} \cdot \int_\sigma^{(t+\sigma)/2)} (r-\sigma)^{-1/2-\epsilon}\, dr \Big] \Big) \\
&\le \mathfrak{C} \cdot \Big((t-\sigma)^{-1/2} \cdot (|x-y|^2 + t - \sigma)^{-3/2} + \sum_{\nu \in \{2, 5/2\}} (|x-y|^2 + t - \sigma)^{-\nu+1/2} \\
&\qquad + \chi_{(0,1)}(t-\sigma) \cdot (t-\sigma)^{-1+\delta} \cdot |x-y|^{-2-2\cdot\delta} \\
&\qquad + \chi_{(1,\infty)}(t-\sigma) \cdot (t-\sigma)^{-1-\epsilon} \cdot |x-y|^{-2+2\cdot\epsilon} \Big) \\
&\le \mathfrak{C} \cdot \Big(\chi_{(0,1)}(t-\sigma) \cdot (t-\sigma)^{-1+\delta} \cdot |x-y|^{-2-2\cdot\delta} \\
&\qquad + \chi_{(1,\infty)}(t-\sigma) \cdot (t-\sigma)^{-1-\epsilon} \cdot |x-y|^{-2+2\cdot\epsilon} \\
&\qquad + \chi_{(0,1)}(t-\sigma) \cdot (|x-y|^2 + t - \sigma)^{-2} \\
&\qquad + \chi_{(1,\infty)}(t-\sigma) \cdot (|x-y|^2 + t - \sigma)^{-3/2} \Big) \\
&\le \mathfrak{C} \cdot \Big(\chi_{(0,1)}(t-\sigma) \cdot (t-\sigma)^{-1+\delta} \cdot |x-y|^{-2-2\cdot\delta} \\
&\qquad + \chi_{(1,\infty)}(t-\sigma) \cdot (t-\sigma)^{-1-\epsilon} \cdot |x-y|^{-2+2\cdot\epsilon} \Big).
\end{aligned}
$$

In the next to last inequality, we used that

$$\max_{\nu \in \{3/2, 2\}} (|x-y|^2 + t - \sigma)^{-\nu} \le \mathfrak{C} \cdot (|x-y|^2 + t - \sigma)^{-2} \quad \text{in the case } t - \sigma \le 1$$

and

$$\max_{\nu \in \{3/2, 2\}} (|x-y|^2 + t - \sigma)^{-\nu} \le \mathfrak{C} \cdot (|x-y|^2 + t - \sigma)^{-3/2} \quad \text{if } t - \sigma \ge 1.$$

Now we find with (4.8) that

(4.12)

$$\left| \partial_t \Big(\int_0^t (t-r)^{-1/2} \cdot \mathfrak{R}_j^{(\tau)}(F_i)(x,r)\, dr \Big) \right|$$

$$\leq \mathfrak{C} \cdot \int_0^t \int_{U_i^{1/2}} \Big(\chi_{(0,1)}(t-\sigma) \cdot (t-\sigma)^{-1+\delta} \cdot |x-y|^{-2-2\cdot\delta} + \chi_{(1,\infty)}(t-\sigma) \cdot (t-\sigma)^{-1-\epsilon} \cdot |x-y|^{-2+2\cdot\epsilon} \Big) \cdot |f(y,\sigma)| \, dy \, d\sigma$$

for $t \in (0,\infty)$, $x \in \Lambda_i^1$. Combining (4.11) and (4.12) yields

(4.13)

$$|A_{i,j}| + |B_{i,j}| \leq \mathfrak{C} \cdot \Big(\int_0^\infty \int_{\Lambda_i^1} \Big[\int_0^t \int_{U_i^{1/2}} \Big(\chi_{(0,1)}(t-\sigma) \cdot (t-\sigma)^{-1+\delta} \cdot |x-y|^{-2-2\cdot\delta} + \chi_{(1,\infty)}(t-\sigma) \cdot (t-\sigma)^{-1-\epsilon} \cdot |x-y|^{-2+2\cdot\epsilon} \Big) \cdot |f(y,\sigma)| \, dy \, d\sigma \Big]^2 d\Omega(x) \, dt \Big)^{1/2}.$$

The functions $H^{(i)}$ and $h^{(i)}$ appearing below were introduced in section 2. For brevity, we will write α instead of $\alpha(\Omega)$. Put

$$\widetilde{f}_i(\eta, r, \sigma) := f\big(H^{(i)}(\eta, r), \, \sigma \big) \quad \text{for } \eta \in \Delta^{1/2}, \ r \in (-\alpha, \alpha), \ \sigma \in (0,\infty),$$

and $\widetilde{f}_i(\eta, r, \sigma) := 0$ for any other $(\eta, r, \sigma) \in \mathbb{R}^3$. Note that

$$\|\widetilde{f}_i\|_{L^2(0,\infty,L^b(\mathbb{R}^3)^3)} \leq \mathfrak{C} \cdot \|f|U_i^{1/2} \times (0,\infty)\|_{L^2(0,\infty,L^b(U_i^{1/2})^3)} \quad \text{for } b \in [1,\infty),$$

as follows from (2.1). By the same reference and by (2.3) and (2.2), we get

(4.14)

$$\begin{aligned}
\mathfrak{K}_1 :=& \Big(\int_0^\infty \int_{\Lambda_i^1} \Big(\int_0^t \int_{U_i^{1/2}} \chi_{(0,1)}(t-\sigma) \cdot (t-\sigma)^{-1+\delta} \cdot |x-y|^{-2-2\cdot\delta} \cdot |f(y,\sigma)| \, dy \, d\sigma \Big)^2 d\Omega(x) \, dt \Big)^{1/2} \\
\leq& \mathfrak{C} \cdot \Big(\int_0^\infty \int_{\Delta^1} \Big(\int_0^t \int_{\Delta^{1/2}} \int_{-\alpha/2}^{\alpha/2} \chi_{(0,1)}(t-\sigma) \cdot (t-\sigma)^{-1+\delta} \cdot |h^{(i)}(\varrho) - H^{(i)}(\eta, r)|^{-2-2\cdot\delta} \cdot |\widetilde{f}_i(\eta, r, \sigma)| \, dr \, d\eta \, d\sigma \Big)^2 d\varrho \, dt \Big)^{1/2} \\
\leq& \mathfrak{C} \cdot \Big(\int_0^\infty \int_{\Delta^1} \Big(\int_0^t \int_{\Delta^{1/2}} \chi_{(0,1)}(t-\sigma) \cdot (t-\sigma)^{-1+\delta} \cdot \int_{-\alpha/2}^{\alpha/2} (|\varrho - \eta| + |r|)^{-2-2\cdot\delta} \cdot |\widetilde{f}_i(\eta, r, \sigma)| \, dr \, d\eta \, d\sigma \Big)^2 d\varrho \, dt \Big)^{1/2} \\
\leq& \mathfrak{C}(q) \cdot \Big(\int_0^\infty \int_{\Delta^1} \Big(\int_0^t \int_{\Delta^{1/2}} \chi_{(0,1)}(t-\sigma) \cdot (t-\sigma)^{-1+\delta} \cdot |\varrho - \eta|^{-1-2\cdot\delta-1/q} \cdot \|\widetilde{f}_i(\eta, \, \cdot \, , \sigma)\|_q \, d\eta \, d\sigma \Big)^2 d\varrho \, dt \Big)^{1/2},
\end{aligned}$$

where the last equation follows by Hölder's inequality. Next we apply Minkowski's and Young's inequality for integrals to obtain

$$\mathfrak{K}_1 \leq \mathfrak{C}(q) \cdot \Big(\int_0^\infty \Big(\int_0^\infty \chi_{(0,1)}(t-\sigma) \cdot (t-\sigma)^{-1+\delta} \cdot \Big[\int_{\mathbb{R}^2} \Big[\int_{\mathbb{R}^2} \chi_{(0,3\cdot\alpha)}(|\varrho-\eta|) \cdot |\varrho-\eta|^{-1-2\cdot\delta-1/q} \cdot \|\widetilde{f}_i(\eta, \cdot, \sigma)\|_q \, d\eta \Big]^2 d\varrho \Big]^{1/2} d\sigma \Big)^2 dt \Big)^{1/2}$$

$$\leq \mathfrak{C}(q) \cdot \Big(\int_0^\infty \Big(\int_0^\infty \chi_{(0,1)}(t-\sigma) \cdot (t-\sigma)^{-1+\delta} \cdot \Big[\int_{\mathbb{R}^2} \chi_{(0,3\cdot\alpha)}(|\zeta|) \cdot |\zeta|^{(-1-2\cdot\delta-1/q)\cdot(3/2-1/q)^{-1}} \, d\zeta \Big]^{3/2-1/q} \cdot \|\widetilde{f}_i(\cdot, \cdot, \sigma)\|_q \, d\sigma \Big)^2 dt \Big)^{1/2}$$

$$\leq \mathfrak{C}(q) \cdot \Big(\int_0^\infty \Big(\int_0^\infty \chi_{(0,1)}(t-\sigma) \cdot (t-\sigma)^{-1+\delta} \cdot \|\widetilde{f}_i(\cdot, \cdot, \sigma)\|_q \, d\sigma \Big)^2 dt \Big)^{1/2}.$$

Note that $(-1-2\cdot\delta-1/q)\cdot(3/2-1/q)^{-1} > -2$ by the choice of δ. Now we apply Young's inequality again, which yields

$$\mathfrak{K}_1 \leq \mathfrak{C}(q) \cdot \Big(\int_0^\infty \chi_{(0,1)}(s) \cdot s^{-1+\delta} \, ds \Big) \cdot \|\widetilde{f}_i\|_{L^2(0,\infty,L^q(\mathbb{R}^2)^3)} \leq \mathfrak{C}(q) \cdot \|f|U_i^{1/2} \times (0,\infty)\|_{L^2(0,\infty,L^q(U_i^{1/2})^3)}. \tag{4.15}$$

Similar arguments may be used in order to estimate

$$\mathfrak{K}_2 := \Big(\int_0^\infty \int_{\Lambda_i^1} \Big(\int_0^t \int_{U_i^{1/2}} \chi_{(1,\infty)}(t-\sigma) \cdot (t-\sigma)^{-1-\epsilon} \cdot |x-y|^{-2+2\cdot\epsilon} \cdot |f(y,\sigma)| \, dy \, d\sigma \Big)^2 d\Omega(x) \, dt \Big)^{1/2}.$$

First, in analogy to (4.14), we get

$$\mathfrak{K}_2 \leq \mathfrak{C}(q) \cdot \Big(\int_0^\infty \int_{\Delta^1} \Big(\int_0^t \int_{\Delta^{1/2}} \chi_{(1,\infty)}(t-\sigma) \cdot (t-\sigma)^{-1-\epsilon} \cdot |\varrho-\eta|^{-1+2\cdot\epsilon-1/q} \cdot \|\widetilde{f}_i(\eta, \cdot, \sigma)\|_q \, d\eta \, d\sigma \Big)^2 d\varrho \, dt \Big)^{1/2};$$

then, by Minkowksi's and Young's inequality,

$$\mathfrak{K}_2 \leq \mathfrak{C}(q) \cdot \Big(\int_0^\infty \Big(\int_0^\infty \chi_{(1,\infty)}(t-\sigma) \cdot (t-\sigma)^{-1-\epsilon} \cdot \|\widetilde{f}_i(\cdot, \cdot, \sigma)\|_q \, d\sigma \Big)^2 dt \Big)^{1/2},$$

where we used that $(-1+2\cdot\epsilon-1/q)\cdot(3/2-1/q)^{-1} > -2$. Finally, by another application of Young's inequality,

$$\mathfrak{K}_2 \leq \mathfrak{C}(q) \cdot \|\widetilde{f}_i\|_{L^2(0,\infty,L^q(\mathbb{R}^2)^3)} \leq \mathfrak{C}(q) \cdot \|f|U_i^{1/2} \times (0,\infty)\|_{L^2(0,\infty,L^q(U_i^{1/2})^3)}. \tag{4.16}$$

Combining (4.13), (4.15), and (4.16), we may conclude that

$$|A_{i,j}| + |B_{i,j}| \leq \mathfrak{C}(q) \cdot \|f|U_i^{1/2} \times (0,\infty)\|_{L^2(0,\infty,L^q(U_i^{1/2})^3)}. \tag{4.17}$$

A synthesis of (4.6), (4.7), (4.10), and (4.17) yields (4.1). Recall that we assumed $f \in C_0^\infty\big(\mathbb{R}^3 \times (0,\infty)\big)^3$. Inequalities (4.7) and (4.11) with $\alpha = 0$ remain valid if f belongs only to the space $L^2(0,\infty, L^1(\mathbb{R}^3)^3)$ with $f|\mathfrak{U} \times (0,\infty) \in L^2(0,\infty, L^q(\mathfrak{U})^3)$. Hence we may conclude with (4.14)–(4.17) that for such f,

$$\|\mathfrak{R}^{(\tau)}(f)\,|\,S_\infty\|_2 \le \mathfrak{C}(q) \cdot \big(\|f\|_{L^2(0,\infty,L^1(\mathbb{R}^3)^3)} + \|f|U_i^{1/2} \times (0,\infty)\|_{L^2(0,\infty,L^q(U_i^{1/2})^3)} \big).$$

Now the theorem follows by a density argument. □

5. Applications: Existence, a representation formula, and spatial decay. Although they are not the main point of the work at hand, we first state some existence results. They should indicate that we do, in fact, obtain solutions of problems (1.4)–(1.6), and that under some additional but not very stringent assumptions on f, these solutions belong to a uniqueness class, so that they can be identified with solutions obtained in other ways (as in [30], for example). But of course, interesting questions related to existence remain unanswered and cannot be addressed here. In particular, the precise regularity of the single layer potential near $\partial\Omega$ is an open problem in our situation, where Ω is assumed only to be Lipschitz bounded, and where $\phi \in L^2(S_\infty)^3$. (The result in Theorem 2.22 is not optimal.) Even in the Stokes case, this problem seems to be as yet unsolved. Away from $\partial\Omega$ and from the initial time $t = 0$, the regularity of our solution is determined by the volume potential $\mathfrak{R}^{(\tau)}(f)$, which may be studied via Fourier transform and multiplier theorems. This is also a point which we do not take up here. Concerning the potential $\mathfrak{I}^{(\tau)}(a)$, whose regularity is determined by its behavior at $t = 0$, the results in Lemma 2.14 and Theorem 2.15 should be the best possible in relation to our assumptions on a.

COROLLARY 5.1. *Let $b \in H_\infty$, $\epsilon \in (0, 1/2]$, $a \in H_\sigma^{1/2+\epsilon}(U)^3$, $q \in (3/2, 2]$, $f \in L^2(0,\infty, L^1(\mathbb{R}^3)^3)$ with $f\,|\,\mathfrak{U} \times (0,\infty) \in L^2(0,\infty, L^q(\mathfrak{U})^3)$, where $\mathfrak{U}$ was defined in Theorem 4.2. Then there is a unique function $\phi \in L_n^2(S_\infty)$ such that*

$$\mathfrak{V}^{(\tau)}(\phi)\,|\,S_\infty = b - \big(\mathfrak{R}^{(\tau)}(f) + \mathfrak{I}^{(\tau)}(a)\big)\,|\,S_\infty. \tag{5.1}$$

Put

$$u := \big(\mathfrak{R}^{(\tau)}(f) + \mathfrak{I}^{(\tau)}(a) + \mathfrak{V}^{(\tau)}(\phi)\big)\,|\,Z_\infty, \quad \pi := \big(\mathfrak{P}(f) + \dot{Q}(\phi)\big)\,|\,Z_\infty. \tag{5.2}$$

Then the weak derivatives $\partial_l u$ exist for $1 \le l \le 3$, and the equations $u|S_\infty = b$ and $\mathrm{div}_x u = 0$ *hold. The relation $u(x,t) \to 0$ $(|x| \to \infty)$ holds for a.e. $t \in (0,\infty)$ in the sense that $u(\,\cdot\,,t) \in L^2(U)^3$ for such t.*

Let $s \in [1,2]$, $q \in [1,2)$ with $7/4 > 1/s + 3/(2 \cdot q)$, and assume in addition that

$$f\,|\,\mathbb{R}^3 \times (0,T) \in L^s(0,T, L^q(\mathbb{R}^3)^3), \quad f\,|\,Z_T \in L^2(0,T,V') \quad \text{for } T \in (0,\infty), \tag{5.3}$$

where $f(t) \in V'$ $(t \in (0,\infty))$ is defined as in Corollary 2.17. Then u is the only function verifying the relations

$$u\,|\,Z_T \in L^2(0,T, H^1(U)^3) \cap H^1(0,T,V') \quad \text{for } T \in (0,\infty), \tag{5.4}$$

$$u\,|\,S_\infty = b, \quad \mathrm{div}_x u = 0, \tag{5.5}$$

$$\int_0^T \int_U \Big(u(x,t) \cdot v(x) \cdot \varphi'(t) + \big(\nabla_x u(x,t) \cdot \nabla v(x) + \tau \cdot \partial_1 u(x,t) \cdot v(x) - f(x,t) \cdot v(x)\big) \cdot \varphi(t)\Big)\, dx\, dt = \int_U a(x) \cdot v(x)\, dx \cdot \varphi(0) \tag{5.6}$$

for $v \in V$, $T \in (0,\infty)$, $\varphi \in C^1([0,T])$ *with* $\varphi(T) = 0$. *In other words, u is the velocity part of a weak solution to* (1.4)–(1.6), *with boundary condition* $u|S_\infty = b$.

If $f \in C_0^\infty\big(\mathbb{R}^3 \times (0,\infty)\big)^3$, $a \in C_0^\infty(U)^3$ *with* $\operatorname{div} a = 0$, *then* $u_j(\,\cdot\,,t)$, $\pi(\,\cdot\,,t) \in C^\infty(U)$ *for* $t \in (0,\infty)$, $1 \le j \le 3$, *the derivative* $\partial_t u(x,t)$ *exists for* $x \in U$ *and for a.e.* $t \in (0,\infty)$, *and the pair* $(u,\,\pi)$ *verifies* (1.4)–(1.6) *with boundary condition* $u|S_\infty = b$ *(classical solution).*

Proof. Lemmas 2.12 and 2.23 and Theorems 2.24, 3.1, and 4.2 yield that $u|S_\infty = b$. The equation $\operatorname{div}_x u = 0$ holds by Lemmas 2.11, 2.14, and 2.21. According to Lemma 2.7 with $q = 1$, $p = r = s = 2$, we have $\mathfrak{R}^{(\tau)}(f)\,|\,\mathbb{R}^3 \times (0,M) \in L^2(\mathbb{R}^3 \times (0,M))^3$ for $M \in (0,\infty)$, so $\mathfrak{R}^{(\tau)}(f)(\,\cdot\,,t) \in L^2(\mathbb{R}^3)^3$ for a.e. $t \in (0,\infty)$. Lemma 2.14 and Theorem 2.22 imply an analogous result for $\mathfrak{I}^{(\tau)}(a)$ and $\mathfrak{V}^{(\tau)}(\phi)$, respectively. Thus $u(\,\cdot\,,t) \in L^2(U)^3$ for a.e. $t \in (0,\infty)$.

Now suppose that f verifies the additional assumptions in (5.3). Then the relation in (5.4) follows from Corollaries 2.17 and 2.18, and Theorem 2.22. The equations in (5.5) were proved above. In order to establish (5.6), take $T \in (0,\infty)$ and a sequence (f_n) in $C_0^\infty(\mathbb{R}^3 \times (0,\infty))^3$ with $\|(f - f_n)\,|\,\mathbb{R}^3 \times (0,T)\|_{L^\beta(0,T,\,L^\gamma(\mathbb{R}^3)^3)} \to 0$ for some $\beta, \gamma \in [1,\infty)$. In view of our assumptions on f, we may, for example, choose $\beta = 2$, $\gamma = 1$, or $\beta = s$, $\gamma = q$. Moreover, let (a_n) be a sequence in $C_0^\infty(U)^3$ with $\operatorname{div} a_n = 0$ $(n \in \mathbb{N})$ and $\|a_n - a\|_{\epsilon,2} \to 0$. Put $u_n := \big(\mathfrak{R}^{(\tau)}(f_n) + \mathfrak{I}^{(\tau)}(a_n) + \mathfrak{V}^{(\tau)}(\phi)\big)\,|\,Z_\infty$ for $n \in \mathbb{N}$. Due to Lemma 2.14, Theorem 2.16, and Lemma 2.21, we see that u_n verifies (5.6) with f, a replaced by f_n, a_n, respectively, and for $v \in C_0^\infty(U)^3$ with $\operatorname{div} v = 0$.

On the other hand, we may conclude from Lemmas 2.7 and 2.18 that

$$\|\partial_x^\alpha\big(\mathfrak{R}^{(\tau)}(f_n) - \mathfrak{R}^{(\tau)}(f)\big)\,|\,Z_T\|_{L^\beta(0,T,\,L^p(U)^3)} \to 0 \quad (\alpha \in \mathbb{N}_0^3,\ |\alpha| \le 1),$$

$$\|\big(\mathfrak{I}^{(\tau)}(a_n) - \mathfrak{I}^{(\tau)}(a)\big)\,|\,Z_T\|_{L^2(0,T,\,H^1(U)^3)} \to 0,$$

with $p \in (\gamma,\infty)$ so close to γ that $1/2 + 3\cdot(1/p - 1/\gamma)/2 > 0$. Thus it follows that (5.6) is valid for $v \in C_0^\infty(U)^3$ with $\operatorname{div} v = 0$. Finally, a density argument based on Corollaries 2.17 and 2.18 and Theorem 2.22 yields that (5.6) holds for $v \in V$. As concerns the uniqueness statement of the corollary, it follows with the argument in [34, p. 176].

In the case that f and a are C^∞-regular, the last part of the theorem follows from Lemma 2.14, Theorem 2.16, and Lemma 2.21. □

As an immediate consequence of Corollary 5.1, we note the following.

COROLLARY 5.2. *Let* f, a, b *satisfy the conditions required in Corollary* 5.1 *for* (5.4)–(5.6) *to hold. Then there exists* $\phi \in L_n^2(S_T)$ *with* (5.1).

Suppose that $u : U \mapsto \mathbb{R}^3$ *verifies the relations in* (5.4)–(5.6). *Then the equation* $u = \big(\mathfrak{R}^{(\tau)}(f) + \mathfrak{I}^{(\tau)}(a) + \mathfrak{V}^{(\tau)}(\phi)\big)\,|\,U$ *holds.*

The assumptions on f in (5.6) simplify strongly if it is assumed that there is $R > 0$ such that $\operatorname{supp}(f(\,\cdot\,,t)) \subset B_R$ for all $t \in (0,\infty)$; see Corollary 5.6 below.

Next we are going to use the preceding results in order to obtain spatial decay rates of solutions to the time-dependent Oseen system. We will assume that the data f and a admit a compact support with respect to the space variable; see our comments in section 2. In the ensuing three lemmas, we present estimates of the single layer potential $\mathfrak{V}^{(\tau)}(\phi)$ and of the volume potentials $\mathfrak{I}^{(\tau)}(a)$ and $\mathfrak{R}^{(\tau)}(f)$.

LEMMA 5.3. *The inequalities*

$$(5.7) \qquad |\partial_x^\alpha \mathfrak{V}^{(\tau)}(\phi)(x,t)| \le \mathfrak{C}(\alpha)\cdot\Big(\big(|x|\cdot(1+\tau\cdot s(x)) + t\big)^{-1-|\alpha|/2}\cdot\|\phi|S_{t/2}\|_2 + \big(|x|\cdot(1+\tau\cdot s(x))\big)^{-1-|\alpha|/2}\cdot\|\phi\,|\,S_\infty\backslash S_{t/2}\|_2\Big),$$

$$(5.8) \qquad |\partial_x^\alpha Q(\phi)(x,t)| \;\leq\; \mathfrak{C}(\alpha)\cdot|x|^{-2-|\alpha|}\cdot\|\phi(\,\cdot\,,t)\|_2$$

hold for $\phi \in L^2(S_\infty)^3$, $x \in B^c_{R_0}$, $t \in (0,\infty)$, $\alpha \in \mathbb{N}_0^3$, *where the constant* R_0 *was introduced in section* 2, *and the term* $s(x)$ *in* (2.8).

Proof. Since $\overline{\Omega} \subset B_{R_0/2}$, we have

$$(5.9) \qquad |x-y| \geq |x|/2 \geq R_0/2 \quad \text{for } x \in B^c_{R_0},\ y \in \partial\Omega.$$

Thus, (2.27), Lemma 2.6 with $K = R_0/2$, and Lemma 2.2 yield that

$$|\partial_x^\alpha \mathfrak{V}^{(\tau)}(\phi)(x,t)| \leq \mathfrak{C}(\alpha)\cdot\int_0^t \big(|x|\cdot(1+\tau\cdot s(x)\big)+t-\sigma\big)^{-3/2-|\alpha|/2}\cdot\int_{\partial\Omega}|\phi(y,\sigma)|\,d\Omega(y)\,d\sigma$$

for ϕ, x, t, α as in the lemma. By Hölder's inequality, it follows that

$$\begin{aligned}&|\partial_x^\alpha \mathfrak{V}^{(\tau)}(\phi)(x,t)|\\ &\leq \mathfrak{C}(\alpha)\cdot\int_{\partial\Omega}\Bigg[\bigg(\int_0^{t/2}\big(|x|\cdot(1+\tau\cdot s(x)\big)+t-\sigma\big)^{-3-|\alpha|}\,d\sigma\bigg)^{1/2}\\ &\quad\cdot\bigg(\int_0^{t/2}|\phi(y,\sigma)|^2\,d\sigma\bigg)^{1/2}\\ &+\bigg(\int_{t/2}^t\big(|x|\cdot(1+\tau\cdot s(x)\big)+t-\sigma\big)^{-3-|\alpha|}\,d\sigma\bigg)^{1/2}\cdot\bigg(\int_{t/2}^t|\phi(y,\sigma)|^2\,d\sigma\bigg)^{1/2}\Bigg]d\Omega(x).\end{aligned}$$

Now inequality (5.7) may be obtained by integration with respect to σ and by Hölder's inequality applied to the integral on $\partial\Omega$. Inequality (5.8) is an obvious consequence of (5.9). □

It was already indicated in the proof of [3, Lemma 14] that estimate (5.7) follows from (5.9) and Lemma 2.2. (We remark that the estimate of a time derivative stated in [3, Lemma 14] seems to be incorrect.)

LEMMA 5.4. *Let* $R > 0$, $f \in L^2(0,\infty,L^1(B_{R/2})^3)$, $t \in [0,\infty)$. *Then the functions* $\mathfrak{R}_j^{(\tau)}(f)(\,\cdot\,,t)\,|\,\overline{B_R}^c$ *and* $\mathfrak{P}(f)(\,\cdot\,,t)\,|\,\overline{B_R}^c$ *belong to* $C^\infty(\overline{B_R}^c)$ *for* $1 \leq j \leq 3$, $t \in (0,\infty)$. *The derivative* $\partial_t\partial_x^\alpha\mathfrak{R}^{(\tau)}(f)(x,t)$ *exists for* $\alpha \in \mathbb{N}_0^3$, *a.e.* $t \in (0,\infty)$, $x \in \overline{B_R}^c$, *and the pair* $\big(\mathfrak{R}^{(\tau)}(f),\,\mathfrak{P}(f)\big)$ *satisfies* (1.4) *pointwise on* $\overline{B_R}^c \times (0,\infty)$, *with vanishing right-hand side. The inequalities*

$$\begin{aligned}&|\partial_x^\alpha\mathfrak{R}^{(\tau)}(f)(x,t)|\\ &\leq \mathfrak{C}(\alpha,R)\cdot\Big(\big(|x|\cdot(1+\tau\cdot s(x))+t\big)^{-1-|\alpha|/2}\cdot\|f\,|\,B_{R/2}\times(0,\,t/2)\|_{L^2(0,t/2,\,L^1(B_{R/2})^3)}\\ &\qquad+\big(|x|\cdot(1+\tau\cdot s(x))\big)^{-1-|\alpha|/2}\cdot\|f\,|\,B_{R/2}\times(t/2,\,\infty)\|_{L^2(t/2,\infty,\,L^1(B_{R/2})^3)}\Big),\end{aligned}$$

$$|\partial_x^\alpha\mathfrak{P}(f)(x,t)| \leq \mathfrak{C}(\alpha,R)\cdot|x|^{-2-|\alpha|}\cdot\|f(\,\cdot\,,t)\|_1$$

hold for $\alpha \in \mathbb{N}_0^3$, $x \in \overline{B_R}^c$, $t \in (0,\infty)$.

Proof. The differentiability properties of $\mathfrak{R}^{(\tau)}(f)$ and $\mathfrak{P}(f)$ stated in the lemma, as well as the observation with respect to (1.4), may be shown in an analogous way as Lemma 2.21 (whose proof we did not elaborate). In fact, here and in Lemma 2.21, the domain of integration of the space variables in the definition of the potentials involved, and the domain where differentiations with respect to the space variables are performed, have empty intersection.

The estimates stated in Lemma 5.4 follow by the same arguments as those in Lemma 5.3, but with the role of R_0 taken by R, and the integral of $\phi(\,\cdot\,,t)$ over $\partial\Omega$ replaced by that of $f(\,\cdot\,,t)$ on $B_{R/2}$. □

LEMMA 5.5. *Let $R>0$, $a\in L^1(B_{R/2})^3$, $x\in B_R^c$, $t\in[0,\infty)$, $\alpha\in\mathbb{N}_0^3$. Then*

$$|\partial_x^\alpha \mathfrak{J}^{(\tau)}(a)(x,t)| \le \mathfrak{C}(\alpha,R)\cdot\big(|x|\cdot(1+\tau\cdot s(x))+t\big)^{-3/2-|\alpha|/2}\cdot\|a\|_1.$$

Proof. The lemma is an obvious consequence of Lemma 2.14 and Lemma 2.6 with $K=R/2$, and Lemma 2.2. □

Now we obtain decay results for solutions of the nonstationary Oseen equation.

COROLLARY 5.6. *Let $R\in(0,\infty)$, $\epsilon\in(0,1/2]$, $a\in H_\sigma^{1/2+\epsilon}(U)^3$ with* supp$(a)\subset$ *$B_{R/2}$. Moreover, let $q\in(3/2,2]$, $f\in L^2(0,\infty,L^q(B_{R/2})^3)$, and $b\in H_\infty$.*

Then f, a, and b verify the conditions required in Corollary 5.1 for (5.4)–(5.6) to hold.

There is a solution $\phi\in L_n^2(S_T)$ of (5.1). Put

$$u := \big(\mathfrak{R}^{(\tau)}(f)+\mathfrak{J}^{(\tau)}(a)+\mathfrak{V}^{(\tau)}(\phi)\big)\,|\,Z_\infty,$$

and let $\mathfrak{B}$ denote the complement in $\mathbb{R}^3$ of the closure of the ball $B_{\max\{R,R_0\}}$. Then the functions $u_j(\,\cdot\,,t)|\mathfrak{B}$ and $\big(\mathfrak{P}(f)+Q(\phi)\big)(\,\cdot\,,t)|\mathfrak{B}$ belong to $C^\infty(\mathfrak{B})$ for $t\in(0,\infty)$, $1\le j\le 3$, the derivative $\partial_t\partial_x^\alpha u(x,t)$ exists for $\alpha\in\mathbb{N}_0^3$, $t\in(0,\infty)$, $x\in\mathfrak{B}$, and

(5.10)

$$\begin{aligned}
|\partial_x^\alpha u(x,t)| &\le \mathfrak{C}(R,\alpha,q,\epsilon)\cdot\big(|x|\cdot(1+\tau\cdot s(x))\big)^{-1-|\alpha|/2}\\
&\quad\cdot\big(\|f\|_{L^2(0,\infty,L^q(B_{R/2})^3)}+\|a\|_{1/2+\epsilon,2}+\|b\|_{H_\infty}\big),\\
|\partial_x^\alpha\big(\mathfrak{P}(f)(x,t)+Q(\phi)(x,t)\big)| &\le \mathfrak{C}(R,\alpha)\cdot|x|^{-2-|\alpha|}\cdot\big(\|f(\,\cdot\,,t)\|_1+\|\phi(\,\cdot\,,t)\|_2\big),\\
|\partial_t\partial_x^\alpha u(x,t)| &\le \mathfrak{C}(R,\alpha,q)\cdot\big(|x|\cdot(1+\tau\cdot s(x))\big)^{-3/2-|\alpha|/2}\\
&\quad\cdot\big(\|f\|_{L^2(0,\infty,L^1(B_{R/2})^3)}+\|a\|_1+\|\phi\|_2+\|\phi(\,\cdot\,,t)\|_2\big)
\end{aligned}$$

for $x\in\mathfrak{B}$, $\alpha\in\mathbb{N}_0^3$, $t\in(0,\infty)$ (a.e. $t\in(0,\infty)$ in the case of the last inequality). The term $s(x)$ was defined in (2.8).

Proof. The statements pertaining to the differentiability of u and $\mathfrak{P}(f)+Q(\phi)$ follow from Lemmas 5.4, 2.14, and 2.21. The first two estimates in the lemma may be deduced from Lemmas 5.3–5.5 and from the inequality

$$\|\phi\|_2\le\mathfrak{C}(R,q)\cdot\big(\|f\|_{L^2(0,\infty,L^q(B_{R/2})^3)}+\|a\|_{1/2+\epsilon,2}+\|b\|_{H_\infty}\big)$$

(Theorems 2.24, 3.1, and 4.2). From Lemmas 5.4, 2.14, and 2.21, we see that the pair (u,π), with $\pi:=\big(\mathfrak{P}(f)+Q(\phi)\big)\,|\,Z_\infty$, solves (1.4) pointwise on $\mathfrak{B}\times(0,\infty)$ with vanishing right-hand side. Therefore the last estimate in Corollary 5.6 follows from the first two and from the inequality $|x|\ge\mathfrak{C}(R)\cdot\big(1+\tau\cdot s(x)\big)$ valid for $x\in B^c_{\max\{R,R_0\}}$. □

We remark that the factor $1+\tau\cdot s(x)$ on the right-hand side of (5.10) is the mathematical manifestation of the wake phenomenon. Inequality (5.10) exhibits the wake phenomenon more clearly than the decay estimates available for the stationary case. In fact, if $u(x)$ denotes the velocity part of a stationary Oseen flow associated with an exterior force $f(x)$ with $\text{supp}(f) \subset B_{R/2}$ for some $R > 0$, then the estimate

$$|\partial_x^\alpha u(x)| \le \mathfrak{C}(R) \cdot \big(|x|^{-1-|\alpha|} + \big(|x| \cdot (1 + \tau \cdot s(x)) \big)^{-1-|\alpha|/2} \big)$$

holds for $|x| \ge R$, $\alpha \in \mathbb{N}_0^3$ with $1 \le |\alpha| \le 2$, according to [15, Theorem VII.6.2], [20, equation (1.15)]. Thus there is an additional term $|x|^{-1-|\alpha|}$ which does not arise in the nonstationary case.

In the next corollary, which is also an immediate consequence of Lemmas 5.3–5.5, we present an estimate of the spatial decay of the velocity, with the special feature that on the right-hand side of this estimate, there is a factor which tends to zero for t tending to infinity.

COROLLARY 5.7. *Consider the situation of Corollary* 5.6. *Let* $\delta \in (0, 1]$. *Then, for* x, t, α *as in Corollary* 5.6,

$$\begin{aligned}|\partial_x^\alpha u(x,t)| &\le \mathfrak{C}(R, \alpha, q) \cdot \big(|x| \cdot (1 + \tau \cdot s(x)) \big)^{-1-|\alpha|/2+\delta} \\ &\cdot \Big((1+t)^{-\delta} \cdot (\|f\|_{L^2(0,\infty,L^1(B_{R/2})^3)} + \|a\|_1 + \|\phi\|_2) \\ &+ \| f \,|\, B_{R/2} \times (t/2, \infty)\|_{L^2(t/2,\infty,L^1(B_{R/2})^3)} + \|\phi | S_\infty \backslash S_{t/2}\|_2 \Big).\end{aligned}$$

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