A Potential-Theoretic Approach to the Time-Dependent Oseen System

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Abstract We consider an initial-boundary value problem for the time-dependent Oseen system in a 3D exterior domain. This problem is reduced to an integral equation for the single layer potential related to the the Oseen system. The resolution of this integral equation, in turn, is reduced to a result by Shen, American Journal of Mathematics, 113, 293–373, 1991 on the nonstationary Stokes system.

Keywords Time dependent Oseen system · Single layer potential · Integral equation

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain with connected boundary, and let U denote the exterior domain $\mathbb{R}^3 \backslash \overline{\Omega}$. Take $T \in (0, \infty]$. Then we consider the time-dependent Oseen system on $Z_T := U \times (0, T)$,

$$\partial_t u - \Delta_x u + \tau \cdot \partial_{x_1} u + \nabla_x p = f, \quad \mathrm{div}_x u = 0 \quad \text{in } Z_T, \tag{1}$$

with a Dirichlet boundary condition on $\partial\Omega$ and zero velocity at infinity,

$$u \mid S_T = b, \quad u(x,t) \to 0 \ \ (|x| \to \infty) \ \text{ for } \ t \in (0,T), \tag{2}$$

where $S_T := \partial\Omega \times (0, T)$. In addition, we impose an initial condition,

$$u(x, 0) = a(x) \ \text{ for } \ x \in U. \tag{3}$$

Our aim is to solve this initial-boundary value problem by means of potential theory. As usual with this access, the given problem will be split into two subproblems. The first is the Cauchy problem

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$$\partial_t v - \Delta_x v + \tau \cdot \partial_{x_1} v + \nabla_x \varrho = f, \quad \operatorname{div}_x v = 0 \quad \text{in } \mathbb{R}^3 \times (0,T), \tag{4}$$

$$v(x,t) \to 0 \;\; (|x| \to \infty) \;\text{ for }\; t \in (0,T), \tag{5}$$

$$v(x,0) = a(x) \;\text{ for }\; x \in \mathbb{R}^3, \tag{6}$$

and the second the ensuing initial-boundary value problem:

$$\partial_t w - \Delta_x w + \tau \cdot \partial_{x_1} w + \nabla_x \pi = 0, \quad \operatorname{div}_x w = 0 \quad \text{in } Z_T, \tag{7}$$

$$w \mid S_T = b - v|S_T, \quad w(x,t) \to 0 \;\; (|x| \to \infty) \;\text{ for }\; t \in (0,T), \tag{8}$$

$$w(x,0) = 0 \;\text{ for }\; x \in \Omega. \tag{9}$$

The function v in (8) is the velocity part of the solution to (4), (5) and (6). Under suitable assumptions on the functions a and f, a solution of the Cauchy problem (4), (5) and (6) is given by the convolution of an Oseen fundamental solution with f and a, respectively. The convolution with f is performed with respect to the space and the time variables and will be denoted by $\mathfrak{R}^{(\tau)}(f)$. The convolution with a only involves the space variables; we will denote it by $\mathfrak{I}^{(\tau)}(a)$. (Precise definitions will be given in Sect. 2.) The function $\mathfrak{R}^{(\tau)}(f)$ solves (4), (5) and (6) with $a = 0$, and $\mathfrak{I}^{(\tau)}(a)$ is a solution of (4), (5) and (6) with $f = 0$, so the sum $\mathfrak{R}^{(\tau)}(f) + \mathfrak{I}^{(\tau)}(a)$ satisfies (4), (5) and (6) without f or a necessarily vanishing. (Here and in the rest of this introduction, when we discuss a solution of the Oseen system, we actually consider only the velocity part of such a solution. Regarding the pressure part, we refer to the main body of the paper.)

As concerns the second subproblem, that is, (7), (8) and (9), we want to solve it by a single layer potential $\mathfrak{V}^{(\tau)}(\varphi)$ with a suitable layer function $\varphi \in L^2(S_T)^3$. The potential $\mathfrak{V}^{(\tau)}(\varphi)$ is defined as a Volterra integral in time and as a surface integral on $\partial\Omega$ in space (see (19)), and satisfies (7) and (9) for any $\varphi \in L^2(S_T)^3$. That latter function is determined by the remaining condition (8), which takes the form

$$\mathfrak{V}^{(\tau)}(\varphi) \mid S_T = b - \big(\mathfrak{R}^{(\tau)}(f) + \mathfrak{I}^{(\tau)}(a)\big) \mid S_T. \tag{10}$$

This is an integral equation on S_T with φ as unknown. Thus we are led to study the integral equation

$$\mathfrak{V}^{(\tau)}(\varphi) \mid S_T = c, \tag{11}$$

where c is given and φ is looked for. It is the main purpose of the work at hand to solve this problem.

To this end, we start from a result by Shen [19] involving the single layer potential $\mathfrak{V}^{(0)}(\varphi)$ associated to the time-dependent Stokes system (Eq. (1) with $\tau = 0$). Shen shows that the equation $\mathfrak{V}^{(0)}(\varphi) \mid S_T = c$ admits a unique solution in the space $L^2_n(S_T)$ of L^2-vector fields on S_T having zero flux on $\partial\Omega$ for a. e. $t \in (0,T)$, provided the data c belongs to a certain Sobolev space H_T involving a fractional derivative with respect to the time variable. It is not straightforward to generalize this result to the Oseen system because if $T = \infty$, the mapping $\varphi \mapsto \big(\mathfrak{V}^{(\tau)}(\varphi) - \mathfrak{V}^{(0)}(\varphi)\big) \mid S_T$ does not seem to be compact as an operator from $L^2_n(S_T)$ into H_T.

Instead of compactness, our argument uses the theory of Fredholm operators. With this theory in mind, we show that the term $\big(\mathfrak{V}^{(\kappa)}(\varphi)-\mathfrak{V}^{(0)}(\varphi)\big) \,|\, S_T$, considered as a function of the Reynolds number $\kappa \in [0, \tau]$, is continuous with respect to the norm of operators from $L^2(S_T)$ into H_T (Theorem 4). The proof of this result is the main difficulty we have to handle in order to solve (11).

Still there is another point which requires some effort, namely the proof of uniqueness of the operator $\mathfrak{V}^{(\tau)}(\varphi) \,|\, S_T$ if φ is taken from $L^2_n(S_T)$ (Theorem 6). Although the general approach for obtaining this result is well known, in the present context we have to deal with the problem of giving a sense to certain surface integrals.

Once these two crucial points are settled, it may be shown by Fredholm theory that $\mathfrak{V}^{(\tau)}(\varphi) \,|\, S_T$ as an operator from $L^2_n(S_T)$ into H_T is bijective. Thus, for any $c \in H_T$, there is a unique solution $\varphi \in L^2_n(S_T)$ to equation (11); see Corollary 3. This is the main result of the present article.

However, this result is not sufficient to solve equation (10). In fact, in order to apply our theory for (11) to (10), we first have to make sure that $\big(\mathfrak{R}^{(\tau)}(f) + \mathfrak{I}^{(\tau)}(a)\big) \,|\, S_T$ belongs to H_T. If we do not want to restrict ourselves to smooth functions, this relation is by no means obvious, in particular if $T = \infty$, and will be discussed in another paper [6]. Actually already in [4] we showed that $\mathfrak{R}^{(\tau)}(f) \,|\, S_T \in H_T$ if the function f belongs to certain L^p-spaces. But the criterion in that reference is not well adapted to an eventual application to the nonlinear case (Navier-Stokes system with Oseen term) and should be considered as preliminary. The result we prove in [6] should be more useful in this respect, although we did not yet study the nonlinear case. Also in [6], we show that $\mathfrak{I}^{(\tau)}(a) \,|\, S_T \in H_T$ if $a \in H^\sigma(U)^3$ for some $\sigma \in (1/2, 1]$. Thus, in reference [6], we are in a position to solve the integral equation (10) by applying Corollary 3 below to (11) with $c = b - \big(\mathfrak{R}^{(\tau)}(f) + \mathfrak{I}^{(\tau)}(a)\big) \,|\, S_T$. This means the function $\big(\mathfrak{R}^{(\tau)}(f)+\mathfrak{I}^{(\tau)}(a)+\mathfrak{V}^{(\tau)}(\varphi)\big) \,|\, Z_T$ is a solution of (1), (2) and (3). In [6], we further show this solution to belong to a function space in which the usual weak solution of (1), (2) and (3) is looked for. Since this space is a uniqueness space for (1), (2) and (3), we may conclude that weak solutions may be represented by the sum $\big(\mathfrak{R}^{(\tau)}(f) + \mathfrak{I}^{(\tau)}(a) + \mathfrak{V}^{(\tau)}(\varphi)\big) \,|\, Z_T$ ([6, Corollary 5.1]), at least if the data f, a and b verify the assumptions we require in our theory. This representation formula, in turn, is exploited in [6] in order to derive estimates of the spatial decay of solutions to (1), (2) and (3) ([6, Corollary 5.2]).

The preceding indications, some of which will be made precise in Sect. 2 (Theorem 1, Lemma 6, Theorem 2, Lemma 7, Corollary 1), should serve to convince the reader that the resolution of Eq. (11) – the subject of the work at hand – is an important element in a wider theory, and thus is worthwhile to be studied. We remark that in [3], we already sketched the approach we present in the following in order to solve (11). But reference [3] does not give a proof of the two crucial result mentioned above (continuous dependence from the Reynolds number and uniqueness of the single layer potential). What we did prove in [3], however, was a useful estimate of the Oseen fundamental solution, although a term was forgotten when this estimate was stated in [3, Lemma 3]. A correct formulation may be found below in Lemma 5. We finally remark that estimates of $\mathfrak{R}^{(\tau)}(f)$ were derived in [12, 13] in a different context. For other studies, based on different methods, of the time-dependent Oseen system or of the resolvent problem associated to the Oseen system, we refer to [8, 9, 15, 16, 20].

2 Notations, Definitions, Auxiliary Results

Recall the bounded Lipschitz domain Ω and the notations $U := \mathbb{R}^3 \backslash \overline{\Omega}$, $Z_T := U \times (0,T)$, $S_T := \partial\Omega \times (0,T)$ $(T \in (0,\infty])$ introduced in Sect. 1. The set Ω was supposed to have a connected boundary, so Ω and U are connected. Let $n^{(\Omega)}$ denote the outward unit normal to Ω. We choose a non-tangential vector field $m^{(\Omega)} \in C_0^\infty(\mathbb{R}^3)^3$ to Ω. This means that $m^{(\Omega)}(x) = 1$ for x from a neighbourhood of $\partial\Omega$, and there are constants $\mathfrak{D}_1,\ \mathfrak{D}_2 \in (0,\infty)$ with

$$|x + \delta \cdot m^{(\Omega)}(x) - x' - \delta' \cdot m^{(\Omega)}(x')| \geq \mathfrak{D}_1 \cdot \big(|x - x'| + |\delta - \delta'| \big) \tag{12}$$

for $x, x' \in \partial\Omega,\ \delta, \delta' \in [-\mathfrak{D}_2,\ \mathfrak{D}_2]$, and

$$x + \delta \cdot m^{(\Omega)}(x) \in U,\quad x - \delta \cdot m^{(\Omega)}(x) \in \Omega \quad \text{for } x \in \partial\Omega,\ \delta \in (0, \mathfrak{D}_2]. \tag{13}$$

Some indications on how to construct such a field are given in [18, p. 246]. Since Ω is only Lipschitz bounded, the relations in (12) and (13) do not hold in general when $m^{(\Omega)}$ is replaced by the outward unit normal to Ω.

As explained in the proof of [5, Lemma 3.4], the relations in (12) and (13) imply there is a constant $\mathfrak{D}_3 > 0$ only depending on Ω such that

$$\begin{aligned} |x - y - \kappa \cdot m^{(\Omega)}(y)| &\geq \mathfrak{D}_3 \cdot (|x - y| + \kappa), \\ |z - y + \kappa \cdot m^{(\Omega)}(y)| &\geq \mathfrak{D}_3 \cdot (|z - y| + \kappa) \end{aligned} \tag{14}$$

for $\kappa \in (0, \mathfrak{D}_2],\ y \in \partial\Omega,\ x, z \in \mathbb{R}^3$ with $\text{dist}(x, \Omega) < \mathfrak{D}_1 \cdot \kappa/2$ and $\text{dist}(z, U) < \mathfrak{D}_1 \cdot \kappa/2$.

Put $B_r := \{y \in \mathbb{R}^3 : |y| < r\}$ for $r \in (0,\infty)$. We fix some $R_0 > 0$ with $\overline{\Omega} \subset B_{R_0/2}$. We further define $|\alpha| := \alpha_1 + \alpha_2 + \alpha_3$ (length of α) for multi-indices $\alpha \in \mathbb{N}_0^3$. Put $e_1 := (1,0,0)$ and $s(x) := |x| - x_1$ for $x \in \mathbb{R}^3$.

Let $A \subset \mathbb{R}^3$ be open. Then we write $H^1(A)$ for the usual Sobolev space of order 1 and exponent 2. This space is to be equipped with its standard norm. Denote by V the closure of the set $\big\{ v \in C_0^\infty(U)^3 \ :\ \text{div}\, v = 0 \big\}$ with respect to that norm (with $A = U$). The Sobolev spaces $H^{1/2}(\partial\Omega)$ and $H^1(\partial\Omega)$ are to be defined in the standard way; see [11, Chapter III.6], for example. Let $\|\ \ \|_{1/2,2}$ and $\|\ \ \|_{1,2}$, respectively, denote their usual norm with respect to some local coordinates of $\partial\Omega$ ([11, Section III.6.7]). Put

$$L_n^2(\partial\Omega) := \{v \in L^2(\partial\Omega)^3 \ :\ \int_{\partial\Omega} v \cdot n^{(\Omega)}\, d\Omega = 0\}.$$

Let $T \in (0, \infty]$. (Note that the case $T = \infty$ is admitted.) Then we put

$$L^2_n(S_T) := \{v \in L^2(S_T)^3 \ : \ v(\,\cdot\,, t) \in L^2_n(\partial\Omega) \ \text{ for almost every } \ t \in (0, T)\},$$
$$\widetilde{H}_T := \{v \,|\, S_T \ : \ v \in C_0^\infty(\mathbb{R}^4)^3, \ v \,|\, \mathbb{R}^3 \times (-\infty, 0] = 0\}.$$

For $v \in C^1\big((-\infty, T)\big)$ with $v\,|\,(-\infty, 0] = 0$, and for $t \in (0, T)$, we put

$$\partial_t^{1/2} v(t) := \Gamma(1/2)^{-1} \cdot \partial_t\Big(\int_0^t (t-r)^{-1/2} \cdot v(r)\, dr\Big)$$

("fractional derivative of v"), where Γ denotes the usual gamma function. We further define for $v \in \widetilde{H}_T$:

$$\partial_4^{1/2} v(x,t) := \partial_t^{1/2}\big(\,v(x,\ \cdot\,)\big)(t) \quad \text{for } \ (x,t) \in \partial\Omega \times (0,T).$$

For any $v \in L^2(S_T)$, we may define $F_v \in L^2\big(0, T, H^1(\partial\Omega)'\big)$ by setting

$$F_v(t)(\sigma) := \int_{\partial\Omega} v(x,t)\cdot\sigma(x)\,dx \quad \text{for } \ \sigma \in H^1(\partial\Omega) \ \text{ and for a. e. } \ t \in (0,T).$$

We will write v instead of F_v. For $v \in \widetilde{H}_T$, set

$$\|v\|_{H_T} := \Big(\int_0^T \Big(\|v(\,\cdot\,,t)\,|\,\partial\Omega\|^2_{1,2} + \int_{\partial\Omega} \big|\,\partial_4^{1/2} v(x,t)\,\big|^2\, d\Omega(x)$$
$$+ \|\partial_t v(\,\cdot\,,t)\cdot n^{(\Omega)}\|^2_{H^1(\partial\Omega)'}\Big)\,dt\Big)^{1/2}.$$

The mapping $\|\ \|_{H_T}$ is a norm on $\widetilde{H}_T$. Let the space H_T consist of all functions $v \in L^2_n(S_T)$ such that there exists a sequence (w_n) in $\widetilde{H}_T$ with the property that $\|v - w_n\,|\,S_T\|_2 \to 0$, and such that $(w_n\,|\,S_T)$ is a Cauchy sequence with respect to the norm $\|\ \|_{H_T}$. This means in particular that the sequence $\big(\|w_n\,|\,S_T\|_{H_T}\big)$ is convergent. Its limit value does not depend on the choice of the sequence (w_n) with the above properties. Thus, for $v \in H_T$, we may define the quantity $\|v\|_{H_T}$ in an obvious way. The mapping $\|\ \ \|_{H_T}$ is a norm on H_T, and the pair $\big(H_T,\ \|\ \ \|_{H_T}\big)$ is a Banach space.

Let us present some auxiliary results.

Lemma 1 *([10, Lemma 4.3]) Let $\beta \in (1, \infty)$. Then there is a constant $C = C(\beta)$ such that $\int_{\partial B_r}\big(1 + s(x)\big)^{-\beta}\, do_x \le C \cdot r \quad for \ \ r \in (0, \infty)$.*

Lemma 2 *([7, Lemma 4.8]) There is $C > 0$ such that for $x, y \in \mathbb{R}^3$, $\kappa \in (0, \infty)$, the following inequality holds:*

$$\big(1 + \kappa \cdot s(x-y)\big)^{-1} \ \le \ C \cdot \max\{1, \kappa\} \cdot (1 + |y|) \cdot \big(1 + \kappa \cdot s(x)\big)^{-1}.$$

Lemma 3 *(Compare [2, Lemma 4.9]) Let $\mathfrak{A}$, $\mathfrak{B}$ be measurable spaces equipped with the measure μ and ν, respectively. Let $F,\ F_1,\ F_2 : \mathfrak{A} \times \mathfrak{B} \mapsto [0, \infty),\ \ f : \mathfrak{B} \mapsto [0, \infty)$ be measurable functions. Suppose that $F = F_1 \cdot F_2$. Then*

$$\Big(\int_{\mathfrak{A}} \Big(\int_{\mathfrak{B}} F(x, y) \cdot f(y)\, d\nu(y)\Big)^2 d\mu(x)\Big)^{1/2} \;\le\; C \cdot \|f\|_2, \tag{15}$$

with

$$C := \Big(\sup_{x \in \mathfrak{A}} \Big\{ \int_{\mathfrak{B}} F_1(x, y)^2\, d\nu(y) \Big\} \cdot \sup_{y \in \mathfrak{B}} \Big\{ \int_{\mathfrak{A}} F_2(x, y)^2\, d\mu(x) \Big\}\Big)^{1/2}.$$

Proof Hölder's inequality yields that the left-hand side of (15) is bounded by

$$\Big(\int_{\mathfrak{A}} \Big(\int_{\mathfrak{B}} F_1(x, y)^2\, d\nu(y)\Big) \cdot \Big(\int_{\mathfrak{B}} F_2(x, y)^2 \cdot f(y)^2\, d\nu(y)\Big)\, d\mu(x)\Big)^{1/2}.$$

Thus the lemma follows after an application of Fubini's theorem. □

Next we introduce the fundamental solutions we will consider in what follows. Let $\mathfrak{H}$ denote the usual fundamental solution of the heat equation in $\mathbb{R}^3$, that is,

$$\mathfrak{H}(z, t) := (4 \cdot \pi \cdot t)^{-3/2} \cdot e^{-|z|^2/(4 \cdot t)} \quad \text{for } (z, t) \in \mathbb{R}^3 \times (0, \infty),$$
$$\mathfrak{H}(z, t) := 0 \;\text{ for }\; (z, t) \in \big(\mathbb{R}^3 \times (-\infty, 0]\big)\backslash\{0\}.$$

We further introduce a fundamental solution of the time-dependent Stokes system by setting as in [19]

$$\begin{gathered} \Gamma_{jk}(z, t) := \delta_{jk} \cdot \mathfrak{H}(z, t) + \int_t^\infty \partial_j \partial_k \mathfrak{H}(z, s)\, ds, \\ E_k(x) := (4 \cdot \pi)^{-1} \cdot x_k \cdot |x|^{-3} \end{gathered} \tag{16}$$

for $(z, t) \in G := \big(\mathbb{R}^3 \times [0, \infty)\big)\backslash\{0\},\ x \in \mathbb{R}^3\backslash\{0\},\ 1 \le j, k \le 3$. Finally we define the velocity part of a fundamental solution of the time-dependent Oseen system (with Reynolds number κ) by putting

$$\Lambda_{jk}(z, t, \kappa) := \Gamma_{jk}(z - \kappa \cdot t \cdot e_1, t) \quad \text{for } (z, t) \in G,\ j, k \in \{1, 2, 3\}.$$

An associated pressure part is given by the functions E_k introduced in (16).

Lemma 4 *The relations $\mathfrak{H} \in C^\infty(\mathbb{R}^4\backslash\{0\})$, $\Gamma_{jk} \in C^\infty(G)$ and $E_k \in C^\infty(\mathbb{R}^3\backslash\{0\})$ hold for $1 \le j, k \le 3$. Moreover, for $l \in \mathbb{N}_0$, $\alpha \in \mathbb{N}^3$, there is $C = C(l, \alpha) > 0$, $\widetilde{C} = \widetilde{C}(\alpha) > 0$ with*

$$\big|\, \partial_t^l \partial_z^\alpha \mathfrak{H}(z, t) \,\big| + \big|\, \partial_t^l \partial_z^\alpha \Gamma_{jk}(z, t) \,\big| \le C \cdot (|z|^2 + t)^{-3/2 - |\alpha|/2 - l},$$

$$|\partial_x^\alpha E_k(x)| \le \widetilde{C} \cdot |x|^{-2-|\alpha|}$$

for $z \in \mathbb{R}^3,\ t \in (0,\infty),\ x \in \mathbb{R}^3 \backslash \{0\},\ 1 \le j,k \le 3.$

For the Oseen fundamental solution, the following estimate holds:

Lemma 5 *The function* $\Lambda_{jk}(\,\cdot\,,\,\cdot\,,\kappa)$ *belongs to the space* $C^\infty(G)$, *for* $\kappa > 0,\ 1 \le j,k \le 3$. *For any* $K > 0,\ \alpha \in \mathbb{N}_0^3,\ l \in \mathbb{N}_0$, *there is some* $C(K,\alpha,l) > 0$ *with*

$$\begin{aligned}&\left|\,\partial_t^l \partial_z^\alpha \Lambda_{jk}(z,t,\kappa)\,\right| \\ &\le C(K,\alpha,l) \cdot \max\{1,\kappa\}^{\frac{3}{2}+\frac{|\alpha|}{2}+l} \cdot \left(\gamma(z,t)^{-\frac{3}{2}-\frac{|\alpha|}{2}-l} + \gamma(z,t)^{-\frac{3}{2}-\frac{|\alpha|}{2}-\frac{l}{2}}\right),\end{aligned}$$

for $z,\ t,\ j,\ k$ *as in Lemma 4, and for* $\kappa \in (0,\infty)$, *where*

$$\gamma(z,t) := |z|^2 + t \ \text{ if } \ |z| \ \le \ K, \quad \gamma(z,t) := |z| \cdot \big(1 + \kappa \cdot s(z)\big) + t \ \text{ if } \ |z| > K.$$

Proof Reference [3, Lemma 2], Lemma 4. □

Let us now fix a Reynolds number $\tau \in (0,\infty)$. In the following, the symbol $\mathfrak{C}$ will always denote constants only depending on $\Omega,\ R_0$ and τ. We write $\mathfrak{C}(\gamma_1,\ \ldots,\ \gamma_n)$ for constants depending additionally on other parameters $\gamma_1,\ \ldots,\ \gamma_n \in (0,\infty)$, for some $n \in \mathbb{N}$.

Let us introduce the volume potentials mentioned in Sect. 1. Take $f \in C_0^\infty\big(\mathbb{R}^3 \times (0,\infty)\big)^3$. Then, for $x \in \mathbb{R}^3,\ t \in [0,\infty),\ j \in \{1,\ 2,\ 3\}$, we put

$$\mathfrak{R}_j^{(\tau)}(f)(x,t) := \int_0^t \int_{\mathbb{R}^3} \sum_{k=1}^3 \Lambda_{jk}(x-y, t-\sigma, \tau) \cdot f_k(y,\sigma)\, dy\, d\sigma, \qquad (17)$$

$$\mathfrak{P}(f)(x,t) := \int_{\mathbb{R}^3} \sum_{k=1}^3 E_k(x-y) \cdot f_k(y,t)\, dy.$$

Let $a \in C_0^\infty(U)^3$. Then we define

$$\mathfrak{I}_j^{(\tau)}(a)(x,t) := \int_U \mathfrak{H}(x - \tau \cdot t \cdot e_1 - y, t) \cdot a_j(y)\, dy, \qquad (18)$$

for $x,\ t$ and j as above.

These potentials solve the Cauchy problem (4), (5) and (6):

Theorem 1 *Take* f *and* a *as in (17), (18). Then the functions* $\mathfrak{R}_j^{(\tau)}(f)$, $\mathfrak{P}(f)$ *and* $\mathfrak{I}_j^{(\tau)}(a)$ *belong to* $C^\infty\big(\mathbb{R}^3 \times [0,\infty)\big)$, *for* $1 \le j \le 3$. *Moreover,* $\mathfrak{R}_j^{(\tau)}(f)$ *and* $\mathfrak{I}_j^{(\tau)}(a)$ *verify (5), and the following relations hold for* $x \in \mathbb{R}^3,\ t \in (0,\infty)$*:*

$$\partial_t \mathfrak{R}^{(\tau)}(f)(x,t) - \Delta_x \mathfrak{R}^{(\tau)}(f)(x,t) + \tau \cdot \partial_{x_1} \mathfrak{R}^{(\tau)}(f)(x,t)$$

$$+ \nabla_x \mathfrak{P}(f)(x,t) = f(x,t),$$
$$div_x \mathfrak{R}^{(\tau)}(f)(x,t) = 0, \quad \mathfrak{R}^{(\tau)}(f)(x,0) = 0, \quad \mathfrak{I}^{(\tau)}(a)(x,0) = a(x),$$
$$\partial_t \mathfrak{I}^{(\tau)}(a)(x,t) - \Delta_x \mathfrak{I}^{(\tau)}(a)(x,t) + \tau \cdot \partial_{x_1} \mathfrak{I}^{(\tau)}(a)(x,t) = 0.$$

If $div\, a = 0$, *then* $div_x \mathfrak{I}^{(\tau)}(a)(x,t) = 0$ *for* $x \in \mathbb{R}^3,\ t \in (0,\infty)$.

This theorem follows from Lebesgue's theorem on dominated convergence and from some standard properties of $\mathfrak{H}$ (like the equation $\int_{\mathbb{R}^3} \mathfrak{H}(x,t)\, dx = 1$ for $t \in (0,\infty)$). The details of a proof are tedious but simple.

Next we define the single layer potentials mentioned in Sect. 1. For $T \in (0,\infty],\ \varphi \in L^2(S_T)^3,\ \kappa \in (0,\infty),\ x \in \mathbb{R}^3,\ t \in [0,\infty)$, we put

$$\mathfrak{V}^{(0)}(\varphi)(x,t) := \Big(\int_0^t \int_{\partial\Omega} \sum_{k=1}^3 \Gamma_{jk}(x-y, t-\sigma) \cdot \widetilde{\varphi}_k(y,\sigma)\, d\Omega(y)\, d\sigma \Big)_{1 \le j \le 3},$$
$$\mathfrak{V}^{(\kappa)}(\varphi)(x,t) := \Big(\int_0^t \int_{\partial\Omega} \sum_{k=1}^3 \Lambda_{jk}(x-y, t-\sigma, \kappa) \cdot \widetilde{\varphi}_k(y,\sigma)\, d\Omega(y)\, d\sigma \Big)_{1 \le j \le 3},$$

where $\widetilde{\varphi}$ denotes the zero extension of φ from S_T to S_∞ if $T < \infty$, and $\widetilde{\varphi} := \varphi$ else. We further set

$$Q(\varphi)(x,t) := \int_{\partial\Omega} \sum_{k=1}^3 E_k(x-y) \cdot \widetilde{\varphi}_k(y,t)\, d\Omega(y)$$

for $\varphi \in L^2(S_T)^3,\ x \in \mathbb{R}^3 \backslash \partial\Omega,\ t \in (0,\infty)$. We call the function pairs $\big(\mathfrak{V}^{(0)}(\varphi), Q(\varphi)\big)$ and $\big(\mathfrak{V}^{(\kappa)}(\varphi),\ Q(\varphi)\big)$ the "single layer potential related to the time-dependent Stokes and Oseen system", respectively.

The following lemma holds:

Lemma 6 *Let* $T \in (0,\infty],\ \kappa \in [0,\infty),\ \varphi \in L^2(S_T)^3$. *Abbreviate*

$$w := \mathfrak{V}^{(\kappa)}(\varphi) \,|\, (\mathbb{R}^3 \backslash \partial\Omega) \times [0,\infty), \quad \pi := Q(\varphi).$$

Then $w_j(\,\cdot\,,t),\ \pi(\,\cdot\,,t) \in C^\infty(\mathbb{R}^3 \backslash \partial\Omega)$ *for* $1 \le j \le 3,\ t \in [0,\infty)$. *For* $\alpha \in \mathbb{N}_0^3$, *the partial derivative* $\partial_x^\alpha w(x,t)$ *as a function of* $x \in \mathbb{R}^3 \backslash \partial\Omega$ *and* $t \in [0,\infty)$ *is continuous. The partial derivative* $\partial_t w(x,t)$ *exists for* $x \in \mathbb{R}^3 \backslash \partial\Omega$ *and for a.e.* $t \in (0,\infty)$. *This derivative also is the weak derivative of* w *with respect to* t *on* $(\mathbb{R}^3 \backslash \partial\Omega) \times (0,\infty)$, *and the weak derivative of* $w(x,\ \cdot\,)$ *on* $(0,\infty)$, *for* $x \in \mathbb{R}^3 \backslash \partial\Omega$. *The equations (7) (with* τ *replaced by* κ*) and (9) are satisfied.*

We mention a regularity result on $\mathfrak{V}^{(\tau)}(\varphi)$ established in [5].

Theorem 2 $\mathfrak{V}^{(\tau)}(\varphi) \,|\, Z_T \in L^\infty(0,T,L^2(U)^3) \cap L^2(0,T,V) \cap H^1(0,T,V')$ *for* $T \in (0,\infty),\ \varphi \in L^2(S_\infty)^3$.

We note another result proved in [5], and which shows that the restriction of $\mathfrak{V}^{(\kappa)}(\varphi)$ to S_T may be considered as the boundary value of $\mathfrak{V}^{(\kappa)}(\varphi) \,|\, Z_T$ on S_T:

Lemma 7 *Let $\varphi \in L^2(S_\infty)^3$. Then the trace of $\big(\mathfrak{V}^{(\tau)}(\varphi)(\,\cdot\,,t)\big) \,|\, U$ coincides with $\mathfrak{V}^{(\tau)}(\varphi)(\,\cdot\,,t) \,|\, \partial\Omega$, for a. e. $t \in (0,\infty)$.*

Now we can state how the problem of finding a solution to (1), (2) and (3) reduces to an integral equation on S_∞, as indicated in Sect. 1. Since this result is only mentioned in order to motivate the study of Eq. (11), we did not make an effort to weaken the assumptions on f and a.

Corollary 1 *Let f, a be given as in (17), (18), and suppose that* $\operatorname{div} a = 0$. *Let $b \in L^2(S_\infty)^3$, and suppose there is $\varphi \in L^2(S_\infty)^3$ with*

$$\mathfrak{V}^{(\tau)}(\varphi) \,|\, S_\infty = b - \big(\mathfrak{R}^{(\tau)}(f) + \mathfrak{I}^{(\tau)}(a)\big) \,|\, S_\infty. \tag{19}$$

Put

$$u := \big(\mathfrak{R}^{(\tau)}(f) + \mathfrak{I}^{(\tau)}(a) + \mathfrak{V}^{(\tau)}(\varphi)\big) \,|\, Z_\infty, \quad p := \big(\mathfrak{P}(f) + Q(\varphi)\big) \,|\, Z_\infty. \tag{20}$$

Then $u_j(\,\cdot\,,t)$, $p(\,\cdot\,,t) \in C^\infty(U)$ for $t \in (0,\infty)$, $1 \le j \le 3$, the derivative $\partial_t u(x,t)$ exists for $x \in U$ and for a. e. $t \in (0,\infty)$, and the pair (u,p) verifies (1), (2) and (3).

3 Solving Integral Equation (11)

We start by stating a result from [19].

Theorem 3 *For $T \in (0,\infty]$, the operator $\mathfrak{F}_T^{(0)} : L_n^2(S_T) \ni \varphi \mapsto \mathfrak{V}^{(0)}(\varphi) \,|\, S_T \in H_T$ is well defined, linear, bounded and bijective. In particular, $\mathfrak{F}_T^{(0)}$ is Fredholm with index 0.*

Proof See [19, pp. 365–367]. Note that the arguments given there also hold for $T = \infty$, although only the case $T < \infty$ is considered. □

The ensuing theorem will allow to reduce the invertibility of the Oseen single layer potential to Theorem 3.

Theorem 4 *Let $T \in (0,\infty]$. Then $\big(\mathfrak{V}^{(\kappa_2)}(\varphi) - \mathfrak{V}^{(\kappa_1)}(\varphi)\big) \,|\, S_T \in H_T$ and*

$$\big\| \big(\mathfrak{V}^{(\kappa_2)}(\varphi) - \mathfrak{V}^{(\kappa_1)}(\varphi)\big) \,|\, S_T \big\|_{H_T} \;\le\; C(\kappa_2) \cdot (\kappa_2 - \kappa_1)^{1/2} \cdot \|\varphi\|_2 \tag{21}$$

for $\varphi \in L^2(S_T)^3$, $\kappa_1, \kappa_2 \in [0,\infty)$ with $\kappa_1 < \kappa_2$, where the constant $C(\kappa)$, for $\kappa \in (0,\infty)$, depends on Ω and κ, and is an increasing function of κ. In view of Theorem 3, this means in particular that the operator

$$\mathfrak{F}_T^{(\kappa)} : L_n^2(S_T) \ni \varphi \mapsto \mathfrak{V}^{(\kappa)}(\varphi) \,|\, S_T \in H_T$$

is well defined, linear and bounded. Abbreviate $\varepsilon_p := min\,\{1,\ \mathfrak{D}_2\}/p \quad for \quad p \in \mathbb{N}$,

$$v := v(\kappa_1, \kappa_2, \varphi) := \big(\mathfrak{V}^{(\kappa_2)}(\varphi) - \mathfrak{V}^{(\kappa_1)}(\varphi)\big)\,|\,(\mathbb{R}^3 \backslash \partial\Omega) \times (0, T),$$

where $\kappa_1, \kappa_2 \in [0, \infty),\ \varphi \in L^2(S_T)^3$. *Then the relation*

$$\partial_l v\big(\,x + \varepsilon_p \cdot m^{(\Omega)}(x),\ t\,\big) - \partial_l v\big(\,x - \varepsilon_p \cdot m^{(\Omega)}(x),\ t\,\big) \to 0 \,\text{for}\ \ p \to \infty \tag{22}$$

holds for $l \in \{1,\ 2,\ 3\},\ \ x \in \partial\Omega$ *and* $t \in (0, T)$.

Finally, if $T < \infty$*, the operator* $\mathfrak{F}_T^{(\kappa_2)} - \mathfrak{F}_T^{(\kappa_1)} \,:\, L^2_n(S_T) \mapsto H_T$ *is compact for* $\kappa_1, \kappa_2 \in [0, \infty)$.

Proof For $\varrho_1, \varrho_2 \in [0, \infty)$ with $\varrho_1 < \varrho_2,\ z \in \mathbb{R}^3,\ s \in (0, \infty),\ j, k \in \{1,\ 2,\ 3\}$, put

$$K_{jk}^{(\varrho_1, \varrho_2)}(z, s) := \Lambda_{jk}(z, s, \varrho_2) - \Lambda_{jk}(z, s, \varrho_1) \quad \text{if}\ \varrho_1 > 0.$$

In the case $\varrho_1 = 0$, the term $\Lambda_{jk}(z, t, \varrho_1)$ is replaced by $\Gamma_{jk}(z, t)$.

Let $\kappa_1,\ \kappa_2 \in [0, \infty)$ with $\kappa_1 < \kappa_2$. We will write K_{jk} instead of $K_{jk}^{(\kappa_1, \kappa_2)}$ in what follows. All the constants appearing in the estimates in this proof and depending on κ_2 are increasing functions of κ_2.

Lemma 4 and 5 with $K = 2 \cdot R_0$ yield

$$|\partial_t^l \partial_z^\alpha K_{jk}(z, t)| \ \leq\ \mathfrak{C}(\kappa_2) \cdot \Big((|z|^2 + t)^{-\frac{3}{2} - \frac{|\alpha|}{2} - \frac{l}{2}} + (|z|^2 + t)^{-\frac{3}{2} - \frac{|\alpha|}{2} - l} \Big) \tag{23}$$

for $l \in \{0, 1\},\ \ \alpha \in \mathbb{N}_0^3$ with $|\alpha| + l \leq 2,\ \ z \in B_{2 \cdot R_0},\ \ t \in (0, \infty),\ \ j, k \in \{1,\ 2,\ 3\}$. Moreover

$$\begin{aligned}
&|\partial_t^l \partial_z^\alpha K_{jk}(z, t)| \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (24)\\
&= \Big| \int_0^1 \Big[\partial_z^\alpha \partial_1^{l+1} \Lambda_{jk}\big(\,z, t, \kappa_1 + \vartheta \cdot (\kappa_2 - \kappa_1)\big) \cdot \big(-\kappa_1 - \vartheta \cdot (\kappa_2 - \kappa_1) \big)^l \cdot t \\
&\qquad + \delta_{l1} \cdot \partial_z^\alpha \partial_1 \partial_4 \Lambda_{jk}\big(\,z, t, \kappa_1 + \vartheta \cdot (\kappa_2 - \kappa_1)\big) \cdot t \\
&\qquad + \delta_{l1} \cdot \partial_z^\alpha \partial_1 \Lambda_{jk}\big(\,z, t, \kappa_1 + \vartheta \cdot (\kappa_2 - \kappa_1)\big) \Big]\, d\vartheta\ \Big| \cdot (\kappa_2 - \kappa_1) \\
&\leq\ \mathfrak{C}(\kappa_2) \cdot (\kappa_2 - \kappa_1) \cdot \big((|z|^2 + t)^{-1 - |\alpha|/2 - l/2} + (|z|^2 + t)^{-1 - |\alpha|/2 - l} \big).
\end{aligned}$$

By taking the average of the right-hand side of (23) and (24), we get for $l,\ \alpha,\ z,\ t,\ j,\ k$ as before,

$$\begin{aligned}
&|\partial_t^l \partial_z^\alpha K_{jk}(z, t)| \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (25)\\
&\leq \mathfrak{C}(\kappa_2) \cdot (\kappa_2 - \kappa_1)^{1/2} \cdot \big((|z|^2 + t)^{-5/4 - |\alpha|/2 - l/2} + (|z|^2 + t)^{-5/4 - |\alpha|/2 - l} \big).
\end{aligned}$$

We further remark that for $z \in \mathbb{R}^3 \backslash \{0\},\ \ j, k \in \{1,\ 2,\ 3\},\ \ \alpha \in \mathbb{N}_0^3,\ \ l \in \mathbb{N}_0$, we have

$$\partial_t^l \partial_z^\alpha K_{jk}(z,t) \to 0 \quad \text{for} \ \ t \downarrow 0. \tag{26}$$

For $\delta \in (0, \mathfrak{D}_2]$, let $\varphi^{(\delta)} \in C_0^\infty(\mathbb{R}^3)$ be such that $\varphi^{(\delta)}(x) = 1$ for $x \in \mathbb{R}^3$ with $\text{dist}(x, \overline{\Omega}) \le \mathfrak{D}_1 \cdot \delta/4$, and $\varphi^{(\delta)}(x) = 0$ if $dist(x, \overline{\Omega}) \ge 3 \cdot \mathfrak{D}_1 \cdot \delta/8$. Next choose $\zeta \in C^\infty(\mathbb{R})$ with $\zeta \,|\, (-\infty, 1] = 1$ and $\zeta \,|\, [2, \infty) = 0$. Put $\zeta^{(\delta)}(t) := \zeta(\delta \cdot t)$ for $t \in \mathbb{R},\ \delta \in (0,\ \min\{1, \mathfrak{D}_2\}\,]$. Then $supp(\zeta^{(\delta)}) \subset (-\infty,\ 2/\delta]$ and $\zeta^{(\delta)} \,|\, (-\infty,\ 1/\delta] = 1$, hence

$$|\zeta^{(\delta)\prime}(t)| \ \le\ \mathfrak{C} \cdot \chi_{(1/\delta,\,2/\delta)}(t) \cdot t^{-1} \ \le\ \mathfrak{C} \cdot \chi_{(1,\infty)}(t) \cdot t^{-1} \ \le\ \mathfrak{C} \tag{27}$$

for $t \in (0, \infty)$, $\delta \in (0, \mathfrak{D}_2]$ with $\delta \le 1$. Using the number ε_p introduced in Theorem 4, we define

$$\begin{aligned} w_{p,\,j}^{(\varrho_1,\varrho_2)}(\varphi)(x,t) &:= \varphi^{(\varepsilon_p)}(x) \cdot \zeta^{(\varepsilon_p)}(t) \\ &\quad \cdot \int_0^t \int_{\partial\Omega} \sum_{k=1}^3 K_{jk}^{(\varrho_1,\varrho_2)}(x - y - \varepsilon_p \cdot m^{(\Omega)}(y),\ t - \sigma) \cdot \varphi_k(y,\sigma)\, d\Omega(y)\, d\sigma \end{aligned}$$

for $p \in \mathbb{N},\ 1 \le j \le 3,\ \varphi \in L^2(S_T)^3,\ x \in \mathbb{R}^3,\ t \in [0,T] \cap \mathbb{R},\ \varrho_1, \varrho_2 \in [0,\infty)$ with $\varrho_1 < \varrho_2$. In the following, we will write $w_p(\varphi)$ instead of $w_p^{(\kappa_1,\kappa_2)}(\varphi)$. Due to (13), (14), (26) and the choice of $\varphi^{(\varepsilon_p)}$ and $\zeta^{(\varepsilon_p)}$, we have $w_p \,|\, S_T \in \widetilde{H}_T \cap L_n^2(S_T)$.

It was shown in the proof of [3, Lemma 9] that $(\mathfrak{F}_T^{(\varrho)} - \mathfrak{F}_T^{(0)})(\varphi) \in H_T$ and

$$\|w_p^{(\varrho,0)}(\varphi) - (\mathfrak{F}_T^{(\varrho)} - \mathfrak{F}_T^{(0)})(\varphi)\|_{H_T} \to 0 \ \ (p \to \infty) \tag{28}$$

for $\varphi \in L^2(S_T)^3,\ \varrho \in (0,\infty)$. We note that the term $\mathfrak{C} \cdot \gamma(x,t)^{-3/2-|\alpha|/2-l/2}$ is lacking on the right-hand side of the estimate in [3, Lemma 3], which corresponds to Lemma 5 here. As a consequence of this oversight, inequality [3, (18)] in the proof of [3, Lemma 9] is incomplete; the term $\mathfrak{C} \cdot (|z|^2 + t)^{-3/2-|\alpha|/2-l/2}$ has to be added on the right-hand side. This modification, in turn, means that the function H_2 in [3, p. 123] has to be changed: the integral over $\big(0,\ (t-\sigma)/2\big)$ in the definition of H_2 should apply to the function

$$r^{-\frac{1}{2}} \cdot \big[\, \chi_{(1,\infty)}(t) \cdot t^{-1} \cdot (|x-y|^2 + t - \sigma - r)^{-\frac{3}{2}} + (|x-y|^2 + t - \sigma - r)^{-2} \,\big].$$

(Distinguish the cases $|x-y|^2 + t - \sigma - r \le 1$ and $|x-y|^2 + t - \sigma - r \ge 1$.) The rest of the proof of [3, Lemma 9] remains valid with a modification of the upper bound of $H_2(x,y,t,\sigma)$, which may be chosen as

$$\begin{aligned} &\mathfrak{C} \cdot \chi_{(0,1)}(t-\sigma) \cdot |x-y|^{-\frac{15}{8}} \cdot \big(\chi_{(1,\infty)}(t) \cdot t^{-1} \cdot (t-\sigma)^{-\frac{1}{16}} + (t-\sigma)^{-\frac{9}{16}} \big) \\ &\quad + \mathfrak{C} \cdot \chi_{(1,\infty)}(t-\sigma) \cdot |x-y|^{-\frac{1}{8}} \cdot \big(\chi_{(1,\infty)}(t) \cdot t^{-1} \cdot (t-\sigma)^{-\frac{15}{16}} + (t-\sigma)^{-\frac{23}{16}} \big). \end{aligned}$$

Since $w_p(\varphi) = w_p^{(\kappa_2,0)}(\varphi) - w_p^{(\kappa_1,0)}(\varphi)$ for $p \in \mathbb{N},\ \varphi \in L^2(S_T)^3$, we may conclude from (28) that

$$\| w_p(\varphi) - (\mathfrak{F}_T^{(\kappa_2)} - \mathfrak{F}_T^{(\kappa_1)})(\varphi) \|_{H_T} \to 0 \ \ (p \to \infty), \quad \text{for } \ \varphi \in L^2(S_T)^3. \tag{29}$$

Let $\varphi \in L^2(S_T)^3,\ \ p \in \mathbb{N}$. We are going to estimate $\|w_p(\varphi)\|_{H_T}$. To this end, using the abbreviation

$$\beta_p(x, y) := x - y - \varepsilon_p \cdot m^{(\Omega)}(y) \quad \text{for } \ x \in \overline{\Omega},\ \ y \in \partial\Omega, \tag{30}$$

we define for $j, k, m \in \{1,\ 2,\ 3\},\ x \in \overline{\Omega},\ y \in \partial\Omega,\ t \in (0, \infty),\ \sigma \in (0, t)$,

$$\begin{aligned}
&\mathfrak{K}_{p,j,k,0}(x, y, t, \sigma) := \zeta^{(\varepsilon_p)}(t) \cdot K_{jk}\big(\beta_p(x, y),\ t - \sigma \big),\\
&\mathfrak{K}_{p,j,k,m}(x, y, t, \sigma) := \zeta^{(\varepsilon_p)}(t) \cdot \partial_{x_m} K_{jk}\big(\beta_p(x, y),\ t - \sigma \big),\\
&\mathfrak{L}_{p,j,k}(x, y, t, \sigma) := \int_0^{(t-\sigma)/2} r^{-1/2} \cdot \big[\zeta^{(\varepsilon_p)\prime}(t - r) \cdot K_{jk}\big(\beta_p(x, y),\ t - \sigma - r \big)\\
&\qquad + \zeta^{(\varepsilon_p)}(t - r) \cdot \partial_t K_{jk}\big(\beta_p(x, y),\ t - \sigma - r \big) \big]\, dr\\
&\qquad + 2^{1/2} \cdot (t - \sigma)^{-1/2} \cdot \zeta^{(\varepsilon_p)}\big((t + \sigma)/2 \big) \cdot K_{jk}\big(\beta_p(x, y),\ (t - \sigma)/2 \big)\\
&\qquad - (1/2) \cdot \int_{(t-\sigma)/2}^{t-\sigma} r^{-3/2} \cdot \zeta^{(\varepsilon_p)}(t - r) \cdot K_{jk}\big(\beta_p(x, y),\ t - \sigma - r \big)\, dr,\\
&\mathfrak{M}_{p,j,k}(x, y, t, \sigma) := \zeta^{(\varepsilon_p)\prime}(t) \cdot K_{jk}\big(\beta_p(x, y),\ t - \sigma \big)\\
&\qquad\qquad\qquad\qquad + \zeta^{(\varepsilon_p)}(t) \cdot \partial_t K_{jk}\big(\beta_p(x, y),\ t - \sigma \big).
\end{aligned}$$

Inequalities (23), (27) and (14) imply

$$|\, \partial_t^l \partial_x^\alpha \big[\zeta^{(\varepsilon_p)}(t) \cdot K_{jk}\big(\beta_p(x, y),\ t - \sigma \big) \big] | \le \mathfrak{C}(\kappa_2) \cdot \varepsilon_p^{-3/2 - |\alpha|/2 - l} \tag{31}$$

for $x \in \overline{\Omega},\ \ y \in \partial\Omega,\ \ t \in (0, T),\ \ \sigma \in (0, t)$, and for $\alpha \in \mathbb{N}_0^3,\ l \in \mathbb{N}_0$ with $|\alpha| + l \le 1$. Thus we see that the functions $\mathfrak{K}_{p,j,k,m}$ and $\mathfrak{M}_{p,j,k}$ are bounded, and

$$|\mathfrak{L}_{p,j,k}(x, y, t, \sigma)| \le \mathfrak{C}(\varepsilon_p, \kappa_2) \cdot \big((t - \sigma)^{1/2} + (t - \sigma)^{-1/2} \big) \tag{32}$$

for $x, y \in \partial\Omega,\ \ t \in (0, \infty),\ \ \sigma \in (0, t),\ \ 1 \le j, k \le 3$. As a consequence, we may define

$$\mathfrak{A}_{p,j,k,m}(\psi)(x, t) := \int_0^t \int_{\partial\Omega} \mathfrak{K}_{p,j,k,m}(x, y, t, \sigma) \cdot \psi(y, \sigma)\, d\Omega(y)\, d\sigma$$

for $j, k \in \{1,\ 2,\ 3\},\ \ m \in \{0,\ 1,\ 2,\ 3\},\ \ x \in \partial\Omega,\ \ t \in (0, T),\ \ \psi \in L^2(S_T)$. In addition, we may introduce $\mathfrak{B}_{p,j,k}(\psi)$ and $\mathfrak{C}_{p,j,k}(\psi)$ in the same way as $\mathfrak{A}_{p,j,k,m}(\psi)$, but with the kernel $\mathfrak{K}_{p,j,k,m}$ replaced by $\mathfrak{L}_{p,j,k}$ and $\mathfrak{M}_{p,j,k}$, respectively, and with

$x \in \Omega$ instead of $x \in \partial\Omega$ in the case of $\mathfrak{C}_{p,j,k}(\psi)$. By referring to (26), it may be shown that

$$\begin{aligned}
&\partial_t\Big(\int_0^t (t-r)^{-1/2} \cdot w_{p,j}(\varphi)(x,r)\,dr\Big) \qquad (33)\\
&= \sum_{k=1}^{3} \int_{\partial\Omega} \partial_t\Big(\int_0^t \varphi_k(y,\sigma) \cdot \int_\sigma^t (t-r)^{-1/2} \cdot \zeta^{(\varepsilon_p)}(r)\\
&\qquad\qquad \cdot K_{jk}\big(\,\beta_p(x,y),\; r-\sigma\,\big)\,dr\,d\sigma\Big)\,d\Omega(x)\\
&= \sum_{k=1}^{3} \int_0^t \int_{\partial\Omega} \mathfrak{L}_{p,j,k}(x,y,t,\sigma) \cdot \varphi_k(y,\sigma)\,d\Omega(y)\,d\sigma = \sum_{k=1}^{3} \mathfrak{B}_{p,j,k}(\varphi_k)(x,t)
\end{aligned}$$

for $j \in \{1, 2, 3\},\; x \in \partial\Omega,\; t \in (0,T)$. Let $E : H^{1/2}(\partial\Omega) \mapsto H^1(\Omega)$ be a linear bounded extension operator. Using the equation $\mathrm{div}_x\, w_p(\varphi)(x,t) = 0\ \big(x \in \Omega,\; t \in (0,T)\big)$, we get for $t \in (0,T),\; v \in H^1(\partial\Omega)$,

$$\begin{aligned}
&\int_{\partial\Omega} \big(\,\partial_t w_p(\varphi)(x,t) \cdot n^{(\Omega)}(x)\,\big) \cdot v(x)\,d\Omega(x) \qquad (34)\\
&= \sum_{j=1}^{3} \int_\Omega \partial_t w_{p,j}(\varphi)(x,t) \cdot \partial_j E(v)(x)\,dx\\
&= \sum_{j,k=1}^{3} \int_\Omega \int_0^t \int_{\partial\Omega} \mathfrak{M}_{p,j,k}(x,y,t,\sigma) \cdot \varphi_k(y,\sigma)\,d\Omega(y)\,d\sigma \cdot \partial_j E(v)(x)\,dx\\
&= \sum_{j,k=1}^{3} \int_\Omega \mathfrak{C}_{p,j,k}(\varphi_k)(x,t) \cdot \partial_j E(v)(x)\,dx\\
&\le \sum_{j,k=1}^{3} \|\mathfrak{C}_{p,j,k}(\varphi_k)(\,\cdot\,,t)\|_2 \cdot \|v\|_{1/2,\,2}.
\end{aligned}$$

Since $\|v\|_{1/2,2} \le \mathfrak{C} \cdot \|v\|_{1,2}$ for $v \in H^1(\partial\Omega)$, and in view of (33) and (34), we may conclude

$$\begin{aligned}
&\|w_p(\varphi)\,|\,S_T\,\|_{H_T} \qquad (35)\\
&\le \mathfrak{C} \cdot \sum_{j,k=1}^{3} \Big(\sum_{m=0}^{3} \|\mathfrak{A}_{p,j,k,m}(\varphi_k)\|_2 + \|\mathfrak{B}_{p,j,k}(\varphi_k)\|_2 + \|\mathfrak{C}_{p,j,k}(\varphi_k)\|_2\Big).
\end{aligned}$$

Next we indicate an observation which follows from (25) and (14), and which holds for $x, y \in \partial\Omega,\; t \in (0,T),\; \sigma \in (0,t),\; j,k \in \{1, 2, 3\}$ and $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$:

$$|\partial_x^\alpha K_{jk}\big(\beta_p(x,y),\ t-\sigma)| \le \mathfrak{C}(\kappa_2)\cdot(\kappa_2-\kappa_1)^{1/2}\cdot H_1(x-y,t-\sigma), \tag{36}$$

with

$$H_1(z,r) := \chi_{(0,1)}(r)\cdot r^{-15/16}\cdot |z|^{-13/8} + \chi_{(1,\infty)}(r)\cdot r^{-5/4}$$

for $r \in (0,\infty)$, $z \in \mathbb{R}^3\backslash\{0\}$. By referring to (25), (27), (14), we obtain for x, y, t, σ, j, k as in (36),

$$\begin{aligned}
&|\mathfrak{L}_{jk}(x,y,t,\sigma)| \le \mathfrak{C}(\kappa_2)\cdot(\kappa_2-\kappa_1)^{1/2}\\
&\quad\cdot\Big(\int_0^{(t-\sigma)/2} r^{-1/2}\cdot\big[\,\chi_{(1,\infty)}(t-r)\cdot(t-r)^{-1}\cdot(|x-y|^2+t-\sigma-r)^{-5/4}\\
&\quad+(|x-y|^2+t-\sigma-r)^{-7/4}+(|x-y|^2+t-\sigma-r)^{-9/4}\,\big]\,dr\\
&\quad+(t-\sigma)^{-1/2}\cdot(|x-y|^2+t-\sigma)^{-5/4}\\
&\quad+\int_{(t-\sigma)/2}^{t-\sigma} r^{-3/2}\cdot(|x-y|^2+t-\sigma-r)^{-5/4}\,dr\Big)\\
&\le \mathfrak{C}(\kappa_2)\cdot(\kappa_2-\kappa_1)^{1/2}\cdot\Big(\big[\,(t-\sigma)^{-1}\cdot(|x-y|^2+t-\sigma)^{-5/4}\\
&\quad+(|x-y|^2+t-\sigma)^{-7/4}+(|x-y|^2+t-\sigma)^{-9/4}\,\big]\cdot\int_0^{(t-\sigma)/2} r^{-1/2}\,dr\\
&\quad+(t-\sigma)^{-1/2}\cdot(|x-y|^2+t-\sigma)^{-5/4}\\
&\quad+(t-\sigma)^{-3/2}\cdot\int_{(t-\sigma)/2}^{t-\sigma}(|x-y|^2+t-\sigma-r)^{-5/4}\,dr\Big)\\
&\le \mathfrak{C}(\kappa_2)\cdot(\kappa_2-\kappa_1)^{1/2}\cdot\Big(\big(1+(t-\sigma)^{-1/2}\big)\cdot(|x-y|^2+t-\sigma)^{-5/4}\\
&\quad+\chi_{(1,\infty)}(t-\sigma)\cdot(t-\sigma)^{-3/2}\cdot|x-y|^{-1/2}\\
&\quad+\chi_{(0,1)}(t-\sigma)\cdot(t-\sigma)^{-3/2}\cdot|x-y|^{-13/8}\cdot\int_{(t-\sigma)/2}^{t-\sigma}(t-\sigma-r)^{-7/16}\,dr\Big)\\
&\le \mathfrak{C}(\kappa_2)\cdot(\kappa_2-\kappa_1)^{1/2}\cdot H_2(x-y,t-\sigma),
\end{aligned} \tag{37}$$

where we used the following notation, for $z \in \mathbb{R}^3\backslash\{0\}$, $r \in (0,\infty)$,

$$H_2(z,r) := \chi_{(1,\infty)}(r)\cdot r^{-5/4}\cdot|z|^{-1/2} + \chi_{(0,1)}(r)\cdot r^{-15/16}\cdot|z|^{-13/8}.$$

Moreover, due to (24), (25), (27) and (14), we get for $x \in \Omega$ and for y, t, σ, j, k as in (36),

$$\begin{aligned}
&|\mathfrak{M}_{p,j,k}(x,y,t,\sigma)|\\
&\le \mathfrak{C}\cdot\big(\,|\zeta^{(\varepsilon_p)\prime}(t)|\cdot|K_{jk}\big(\beta_p(x,y),\ t-\sigma)|+|\partial_t K_{jk}\big(\beta_p(x,y),\ t-\sigma)|\,\big)
\end{aligned} \tag{38}$$

$$\le \mathfrak{C}(\kappa_2) \cdot (\kappa_2 - \kappa_1)^{1/2} \cdot \big[(|x-y|^2 + t - \sigma)^{-5/4} + (|x-y|^2 + t - \sigma)^{-3/2} + (|x-y|^2 + t - \sigma)^{-2} \big]$$
$$\le \mathfrak{C}(\kappa_2) \cdot (\kappa_2 - \kappa_1)^{1/2} \cdot H_3(x-y, t-\sigma),$$

with

$$H_3(z, r) := \chi_{(1,\infty)}(r) \cdot r^{-5/4} + \chi_{(0,1)}(r) \cdot r^{-15/16} \cdot |z|^{-17/8}$$

for $z \in \mathbb{R}^3 \backslash \{0\}$, $r \in (0, \infty)$. Applying Lemma 3 with $\mathfrak{A} = \mathfrak{B} = S_T$, $F_1 = F_2 = F^{1/2}$, we get in the case $i \in \{1, 2\}$ that

$$\Big(\int_0^T \int_{\partial\Omega} \Big(\int_0^T \int_{\partial\Omega} H_i(x-y, t-\sigma) \cdot |\varphi(y,\sigma)| \, d\Omega(y) \, d\sigma \Big)^2 d\Omega(x) \, dt \Big)^{1/2} \le \mathfrak{C} \cdot \|\varphi\|_2. \tag{39}$$

Also by Lemma 3, this time with $\mathfrak{A} = \Omega \times (0, T)$, $\mathfrak{B} = S_T$, $F_1 = F_2 = F^{1/2}$,

$$\Big(\int_0^T \int_{\Omega} \Big(\int_0^T \int_{\partial\Omega} \chi_{(1,\infty)}(t-\sigma) \cdot (t-\sigma)^{-5/4} \cdot |\varphi(y,\sigma)| \, d\Omega(y) \, d\sigma \Big)^2 dx \, dt \Big)^{1/2} \le \mathfrak{C} \cdot \|\varphi\|_2. \tag{40}$$

When we apply Lemma 3 with $\mathfrak{A} = \Omega \times (0, T)$, $\mathfrak{B} = S_T$,

$$F_1(x, t, y, \sigma) := \chi_{(0,1)}(t-\sigma) \cdot (t-\sigma)^{-15/32} \cdot |x-y|^{-7/8}$$

for $x \in \Omega$, $y \in \partial\Omega$, $t, \sigma \in (0, \infty)$, and with F_2 defined in the same way as F_1, except that the exponent $-7/8$ is replaced by $-5/4$, we arrive at the estimate

$$\Big(\int_0^T \int_{\Omega} \Big(\int_0^T \int_{\partial\Omega} \chi_{(0,1)}(t-\sigma) \cdot (t-\sigma)^{-15/16} \cdot |x-y|^{-17/8} \cdot |\varphi(y,\sigma)| \, d\Omega(y) \, d\sigma \Big)^2 dx \, dt \Big)^{1/2} \le \mathfrak{C} \cdot \|\varphi\|_2. \tag{41}$$

Inequalities (40) and (41) imply

$$\Big(\int_0^T \int_{\Omega} \Big(\int_0^T \int_{\partial\Omega} H_3(x-y, t-\sigma) \cdot |\varphi(y,\sigma)| \, d\Omega(y) \, d\sigma \Big)^2 dx \, dt \Big)^{1/2} \le \mathfrak{C} \cdot \|\varphi\|_2. \tag{42}$$

Combining (35), (36), (37), (38), (39) and (42) yields $\| w_p(\varphi) \,|\, S_T \|_{H_T} \le \mathfrak{C}(\kappa_2) \cdot (\kappa_2 - \kappa_1)^{1/2} \cdot \|\varphi\|_2$. This is true for any $p \in \mathbb{N}$. Thus inequality (21) follows with (29). An obvious application of (24), (14) and the mean value theorem yields

$$\Big|\sum_{k=1}^{3}\Big(\partial_i K_{jk}(x-y+\varepsilon\cdot m^{(\Omega)}(x),t-\sigma)-\partial_i K_{jk}(x-y-\varepsilon\cdot m^{(\Omega)}(x),t-\sigma)\Big)\Big|$$
$$\leq \mathfrak{C}(\kappa_2)\cdot \min\big\{(|x-y|^2+t-\sigma)^{-3/2},\ \varepsilon\cdot(|x-y|^2+t-\sigma)^{-2}\big\}$$
$$\leq \mathfrak{C}(\kappa_2)\cdot\varepsilon^{1/4}\cdot(|x-y|^2+t-\sigma)^{-13/8}$$

for $x, y\in\partial\Omega,\ t\in(0,T),\ \sigma\in(0,t),\ 1\leq i, j\leq 3,\ \varepsilon\in(0,\mathfrak{D}_2]$. Now the relation in (22) follows after an integration over S_t, for $t\in(0,T)$.

Let us finally suppose that $T<\infty$. Recalling the abbreviation introduced in (30), we may conclude from (23), (24), (14) that for $x\in\overline{\Omega},\ y\in\partial\Omega,\ t\in(0,T),\ p,q\in\mathbb{N},\ j,k\in\{1,2,3\},\ l\in\mathbb{N},\ \alpha\in\mathbb{N}_0^3$ with $|\alpha|+l\leq 1$,

$$\begin{aligned}
&|\partial_t^l\partial_x^\alpha K_{jk}\big(\beta_p(x,y),\ t\big)-\partial_t^l\partial_x^\alpha K_{jk}\big(\beta_q(x,y),\ t\big)|\\
&\leq \sum_{\nu\in\{p,q\}}|\partial_t^l\partial_x^\alpha K_{jk}\big(\beta_\nu(x,y),\ t\big)|^{7/8}\\
&\quad\cdot|\partial_t^l\partial_x^\alpha K_{jk}\big(\beta_p(x,y),\ t\big)-\partial_t^l\partial_x^\alpha K_{jk}\big(\beta_q(x,y),\ t\big)|^{1/8}\\
&\leq \mathfrak{C}(\kappa_2,T)\cdot(\kappa_2-\kappa_1)^{7/8}\cdot(|x-y|^2+t)^{7\cdot(-1-|\alpha|/2-l)/8}\\
&\quad\cdot\Big|\int_0^1\sum_{i=1}^{3}\partial_t^l\partial_x^\alpha\partial_i K_{jk}\big(x-y-(\varepsilon_q+\vartheta\cdot(\varepsilon_p-\varepsilon_q))\cdot m^{(\Omega)}(y),\ t\big)\\
&\quad\cdot(\varepsilon_p-\varepsilon_q)\cdot m_i^{(\Omega)}(y)\,d\vartheta\Big|^{1/8}\\
&\leq \mathfrak{C}(T,\kappa_2)\cdot(|x-y|^2+t)^{\frac{7}{8}(-1-\frac{|\alpha|}{2}-l)}\cdot\\
&\quad|\varepsilon_p-\varepsilon_q|^{\frac{1}{8}}\cdot(|x-y|^2+t)^{(-2-\frac{|\alpha|}{2}-\frac{l}{8})}\\
&\leq \mathfrak{C}(T,\kappa_2)\cdot|\varepsilon_p-\varepsilon_q|^{1/8}\cdot(|x-y|^2+t)^{-9/8-|\alpha|/2-l}.
\end{aligned}\tag{43}$$

Now it is easy to show with (24) and (27) that in the situation of (43), with $\sigma\in(0,t)$,

$$|\partial_t^l\partial_x^\alpha\big[\zeta^{(\varepsilon_p)}(t)K_{jk}\big(\beta_p(x,y),\ t-\sigma\big)\big]-\partial_t^l\partial_x^\alpha\big[\zeta^{(\varepsilon_q)}(t)K_{jk}\big(\beta_q(x,y),\ t-\sigma\big)\big]|$$
$$\leq \mathfrak{C}(T,\kappa_2)|\varepsilon_p-\varepsilon_q|^{1/8}(|x-y|^2+t-\sigma)^{-9/8-|\alpha|/2-l}.$$

Due to this inequality, one may use similar arguments as in the proof of (21), but with simplifications because of the assumption $T<\infty$, to obtain

$$\|\big(w_p(\varphi)-w_q(\varphi)\big)\mid S_T\|_{H_T}\leq\mathfrak{C}(T,\kappa_2)\cdot|\varepsilon_p-\varepsilon_q|^{1/8}\cdot\|\varphi\|_2\quad\text{for}\quad p,q\in\mathbb{N}.$$

It follows with (29) that

$$\sup\big\{\,\|w_p(\varphi)-(\mathfrak{F}_T^{(\kappa_2)}-\mathfrak{F}_T^{(\kappa_1)})(\varphi)\|_{H_T}/\|\varphi\|_2\ :\ \varphi\in L_n^2(S_T)\backslash\{0\}\big\}\longrightarrow 0\tag{44}$$

for $n \to \infty$. On the other hand, take $p \in \mathbb{N}$. Since $T < \infty$ and the functions $\mathfrak{K}_{p,j,k,m}$ and $\mathfrak{M}_{p,j,k}$ are bounded (see (31)), and because of (32), we may conclude that $\mathfrak{A}_{p,j,k,m}$, $\mathfrak{B}_{p,j,k}$ and $\mathfrak{C}_{p,j,k}$ are compact as operators from $L^2(S_T)$ into $L^2(S_T)$. Thus, due to (35), it follows that the operator $W_p : L^2_n(S_T) \ni \psi \mapsto w_p(\psi) \,|\, S_T \in H_T$ is compact. In view of (44), this means that $\mathfrak{F}_T^{(\kappa_2)} - \mathfrak{F}_T^{(\kappa_1)} : L^2_n(S_T) \mapsto H_T$ is compact. □

Theorem 5 (jump relations) *Let* $T \in (0, \infty]$, $\kappa \in [0, \infty)$, $\varphi \in L^2(S_T)^3$, $j \in \{1, 2, 3\}$, *and put* $V := \mathfrak{V}^{(\kappa)}(\varphi) \,|\, (\mathbb{R}^3 \backslash \partial\Omega) \times (0, T)$. *Then*

$$Q(\varphi)\big(x + \varepsilon \cdot m^{(\Omega)}(x),\, t\big) - Q(\varphi)\big(x - \varepsilon \cdot m^{(\Omega)}(x),\, t\big) \longrightarrow n^{(\Omega)}(x) \cdot \varphi(x, t),$$

$$\sum_{k=1}^{3}\Big(\big(\partial_k V_j - \delta_{jk} \cdot Q(\varphi)\big)\big(x + \varepsilon \cdot m^{(\Omega)}(x),\, t\big)\Big) \cdot n_k^{(\Omega)}(x)$$
$$- \sum_{k=1}^{3}\Big(\big(\partial_k V_j - \delta_{jk} \cdot Q(\varphi)\big)\big(x - \varepsilon \cdot m^{(\Omega)}(x),\, t\big)\Big) \cdot n_k^{(\Omega)}(x) \longrightarrow -\varphi_j(x, t)$$

for $\varepsilon \downarrow 0$, *for a. e.* $x \in \partial\Omega$, $t \in (0, T)$.

Proof See (13) and [19, Lemma 2.3.4] in the case $\kappa = 0$. The statement for $\kappa > 0$ follows with (22). □

Now we may establish a result which serves to prove uniqueness of a solution to the integral equation (11). The general approach for proving this result is standard, but in our particular situation, the problem consists in interpreting all the integrals involved in a rigorous way.

Theorem 6 *Let* $T \in (0, \infty]$, $\kappa \in [0, \infty)$ $\varphi \in L^2_n(S_T)$ *with* $\mathfrak{F}_T^{(\kappa)}(\varphi) = 0$. *Then* $\varphi = 0$.

Proof Let $\widetilde{T} \in (0, T)$. For $p \in \mathbb{N}$, we put $\varepsilon_p := \min\{1, \mathfrak{D}_2\}/p$, and we set for $j \in \{1, 2, 3\}$, $x \in \mathbb{R}^3$ with $\operatorname{dist}(x, U) < \mathfrak{D}_1 \cdot \varepsilon_p/2$, $t \in [0, \widetilde{T}]$,

$$V_j^{(p)}(x, t) := \int_0^t \int_{\partial\Omega} \sum_{k=1}^{3} \Lambda_{jk}\big(x - y + \varepsilon_p \cdot m^{(\Omega)}(y),\, t - \sigma,\, \kappa\big) \cdot \varphi_k(y, \sigma)\, d\Omega(y)\, d\sigma, \tag{45}$$

$$Q^{(p)}(x, t) := \int_{\partial\Omega} \sum_{k=1}^{3} E_k\big(x - y + \varepsilon_p \cdot m^{(\Omega)}(y)\big) \cdot \varphi_k(y, t)\, d\Omega(y). \tag{46}$$

By (13), (14) and Lemma 5, we have for $1 \le j, k \le 3$, $p \in \mathbb{N}$, $t \in [0, \widetilde{T}]$ that

$$V_j^{(p)}(\,\cdot\,, t),\ Q^{(p)}(\,\cdot\,, t) \in C^\infty(U_p), \quad \partial_k V^{(p)},\ V^{(p)} \in C^0\big(U_p \times [0, \widetilde{T}]\big)^3$$

with $U_p := \{x \in \mathbb{R}^3 : \operatorname{dist}(x, U) < \mathfrak{D}_1 \cdot \varepsilon_p/2\}$. It further follows that the derivative $\partial_t V^{(p)}(x, t)$ exists for $x \in U_p$ and for a. e. $t \in (0, \widetilde{T})$, with

$$\partial_t V_j^{(p)}(x,t) = \int_0^t \int_{\partial\Omega} \sum_{k=1}^3 \partial_t \Lambda_{jk}\big(x - y + \varepsilon_p \cdot m^{(\Omega)}(y),\ t-\sigma, \kappa\big) \qquad (47)$$
$$\cdot \varphi_k(y,\sigma)\, d\Omega(y)\, d\sigma \ -\ \partial_{x_j} Q^{(p)}(x,t),$$
$$\partial_t V_j^{(p)}(x,t) - \Delta_x V_j^{(p)}(x,t) + \tau \cdot \partial_{x_1} V_j^{(p)}(x,t) + \partial_{x_j} Q^{(p)}(x,t) = 0. \qquad (48)$$

We may conclude from (47) and (14) that

$$|\partial_t V_j^{(p)}(x,t)| \le \mathfrak{C}(\widetilde{T}, p) \cdot \|\varphi(\,\cdot\,,t)\|_2 \quad \text{for } x \in U_p,\ t \in (0,\widetilde{T}). \qquad (49)$$

This means in particular the function $\partial_t V^{(p)}(x,\ \cdot\ )$ is square integrable on $(0,\widetilde{T})$. On the other hand, this function is not only the pointwise, but also the weak derivative of $V^{(p)}(x,\ \cdot\ )$ on $(0,\widetilde{T})$. Since $V^{(p)}(x,\ \cdot\ )$ is continuous on $[0,\widetilde{T}]$, we thus have $V^{(p)}(x,\ \cdot\ ) \in H^1\big((0,\widetilde{T})\big)^3$. Moreover, for $p, q \in \mathbb{N},\ R \in [R_0,\infty)$, we get from (48) that

$$0 = \int_0^{\widetilde{T}} \int_{B_R\setminus\overline{\Omega}} \big(\partial_t V^{(p)} - \Delta_x V^{(p)} + \tau \cdot \partial_{x_1} V^{(p)} + \nabla_x Q^{(p)}\big) \cdot V^{(q)}\, dx\, dt,$$

hence by partial integration and the equation $\operatorname{div}_x V^{(p)} = 0$,

$$\begin{aligned} 0 = &\int_0^{\widetilde{T}}\int_{\partial B_R\cup\partial\Omega} \sum_{j,k=1}^3 \Big((-\partial_k V_j^{(p)} + \delta_{jk} Q^{(p)})V_j^{(q)}\Big)(x,t) N_k^{(R)}(x)\, do_x\, dt \\ &+ \int_0^{\widetilde{T}}\int_{\Omega_R} \Big((\nabla_x V^{(p)} \nabla_x V^{(q)}) + \tau(\partial_{x_1} V^{(p)} V^{(q)}) + (\partial_t V^{(p)} V^{(q)})\Big)(x,t)\, dx\, dt, \end{aligned} \qquad (50)$$

where $N^{(R)} : \partial\Omega_R \mapsto \mathbb{R}^3$ denotes the outward unit normal to $\Omega_R := B_R\setminus\overline{\Omega}$, that is, $N^{(R)}(x) := R^{-1} \cdot x$ for $x \in \partial B_R,\ N^{(R)} := -n^{(\Omega)}(x)$ for $x \in \partial\Omega$. Now abbreviate $\mathfrak{V} := \mathfrak{V}^{(\tau)}(\varphi)$. As explained in the proof of [5, Theorem 2.4], we have

$$\begin{aligned} \|\big(V^{(p)} - \mathfrak{V}\big) \,|\, \Omega_R \times (0,\widetilde{T})\|_{1,2} \to 0, \quad (p\to\infty) \\ \|(V^{(p)} - \mathfrak{V}) \,|\, S_{\widetilde{T}}\|_2 \to 0, \quad (p\to\infty). \end{aligned} \qquad (51)$$

Recalling that $\mathfrak{V}|S_T = \mathfrak{F}_T^{(\kappa)}(\varphi) = 0$ by the choice of φ, and noting that the function $\mathfrak{V} \,|\, \Omega_R \times (0,T)$ belongs to $L^2(0,T,\ H^1(\Omega_R)^3)$ by Theorem 2, we may thus deduce from (50) and (49), by first letting q, and then p, tend to zero,

$$0 = \int_0^{\widetilde{T}}\int_{\partial B_R} \Big(\sum_{j,k=1}^3 \Big((-\partial_{x_k}\mathfrak{V}_j(x,t) + \delta_{jk} Q(\varphi)(x,t))\mathfrak{V}_j(x,t)\Big) x_k/R\, do_x\, dt$$
$$+ \int_0^{\widetilde{T}} \int_{\Omega_R} \big(|\nabla_x \mathfrak{V}(x,t)|^2 + \tau\partial_{x_1}\mathfrak{V}(x,t)\mathfrak{V}(x,t)\big)\, dx\, dt$$

$$+\lim_{p\to\infty}\int_0^{\widetilde{T}}\int_{\Omega_R}\partial_t V^{(p)}(x,t)\mathfrak{V}(x,t)\,dx\,dt. \tag{52}$$

On the other hand, referring to (51) and the relation $\mathfrak{V}|S_T=\mathfrak{F}_T^{(\kappa)}(\varphi)=0$, we get

$$\begin{aligned}
&\int_0^{\widetilde{T}}\int_{\Omega_R}\partial_{x_1}\mathfrak{V}(x,t)\cdot\mathfrak{V}(x,t)\;dx\;dt \qquad (53)\\
&=\lim_{p\to\infty}\int_0^{\widetilde{T}}\int_{\Omega_R}\partial_{x_1}V^{(p)}(x,t)\cdot V^{(p)}(x,t)\;dx\;dt\\
&=\lim_{p\to\infty}\int_0^{\widetilde{T}}\int_{\partial B_R\cup\partial\Omega}(1/2)\cdot|V^{(p)}(x,t)|^2\cdot N^{(R)}(x)\;do_x\;dt\\
&=\int_0^{\widetilde{T}}\int_{\partial B_R}(1/2)\cdot|\mathfrak{V}(x,t)|^2\cdot x_1/R\;do_x\;dt.
\end{aligned}$$

Let us determine the remaining limit in (52). To this end, we first observe that for $j\in\{1,\,2,\,3\},\;\;p\in\mathbb{N},\;\;x\in\Omega_R,\;\;t\in(0,\widetilde{T})$, the following relations are valid:

$$\begin{aligned}
&|V^{(p)}(x,t)-\mathfrak{V}(x,t)|\\
&=\Big|\int_0^t\int_{\partial\Omega}\sum_{k,l=1}^{3}\int_0^1\partial_l\Lambda_{jk}\big(x-y+\vartheta\cdot\varepsilon_p\cdot m^{(\Omega)}(y),\;t-\sigma,\;\kappa\big)\;d\vartheta\cdot\varepsilon_p\\
&\qquad\qquad\cdot m_l^{(\Omega)}(y)\cdot\varphi_k(y,\sigma)\;d\Omega(y)\;d\sigma\Big|\\
&\le\mathfrak{C}(\kappa,R)\cdot\varepsilon_p\cdot\int_0^t\int_{\partial\Omega}(|x-y|^2+t-\sigma)^{-2}\cdot|\varphi(y,\sigma)|\;d\Omega(y)\;d\sigma,
\end{aligned}$$

where we used Lemma 5 with $K=2\cdot R$ and (14) in the last estimate. By Lemma 3 with $\mathfrak{A}=\Omega_R\times(0,\widetilde{T}),\;\;\mathfrak{B}=S_{\widetilde{T}}$,

$$\begin{aligned}
F_1(x,t,y,\sigma)&:=\chi_{(0,\infty)}(t-\sigma)\cdot(|x-y|^2+t-\sigma)^{-7/8},\\
F_2(x,t,y,\sigma)&:=\chi_{(0,\infty)}(t-\sigma)\cdot(|x-y|^2+t-\sigma)^{-9/8}
\end{aligned}$$

for $x\in\Omega_R,\;\;y\in\partial\Omega,\;\;t,\sigma\in(0,\widetilde{T})$, we get

$$\|(V^{(p)}-\mathfrak{V})\,|\,\Omega_R\times(0,\widetilde{T})\|_2\le\mathfrak{C}(\kappa,\widetilde{T},R)\cdot\varepsilon_p\cdot\|\varphi\|_2\quad\text{for}\;\;p\in\mathbb{N}. \tag{54}$$

On the other hand, we refer to (14), (47), Lemma 5 with $K=2\cdot R$, and to Lemma 3 with $\mathfrak{A}$, $\mathfrak{B}$, $F_2,\;F_1$ as above, except that the exponent $-7/8$ in the definition of F_1 is replaced by $-15/16$. It follows for $p\in\mathbb{N},\;\;1\le j\le 3$,

$$\Big(\int_0^{\widetilde{T}}\int_{\Omega_R}|\partial_t V_j^{(p)}(x,t)+\partial_{x_j}Q^{(p)}(x,t)|^2\;dx\;dt\Big)^{1/2}$$

$$\leq \Big(\int_0^{\widetilde{T}} \int_{\Omega_R} \Big| \int_0^t \int_{\partial\Omega} \sum_{k=1}^3 \partial_t \Lambda_{jk}\big(x - y + \varepsilon_p m^{(\Omega)}(y),\ t - \sigma, \kappa \big)$$

$$\cdot\ \varphi_k(y,\sigma)\, d\Omega(y)\, d\sigma \Big|^2 dx\, dt \Big)^{1/2}$$

$$\leq \mathfrak{C}(R) \cdot$$

$$\cdot \Big(\int_0^{\widetilde{T}} \int_{\Omega_R} \Big(\int_0^t \int_{\partial\Omega} \leq \big((|x-y| + \varepsilon_p)^2 + t - \sigma \big)^{-5/2} |\varphi(y,\sigma)|\, d\Omega(y)\, d\sigma \Big)^2 dx\, dt \Big)^{1/2}$$

$$\leq \mathfrak{C}(R) \varepsilon_p^{-7/8} \cdot$$

$$\cdot \Big(\int_0^{\widetilde{T}} \int_{\Omega_R} \Big(\int_0^t \int_{\partial\Omega} (|x-y|^2 + t - \sigma)^{-33/16} \cdot |\varphi(y,\sigma)|\, d\Omega(y)\, d\sigma \Big)^2 dx\, dt \Big)^{1/2} \tag{55}$$

$$\leq \mathfrak{C}(\kappa, \widetilde{T}, R) \cdot \varepsilon_p^{-7/8} \cdot \|\varphi\|_2.$$

A similar argument yields

$$\begin{aligned} &\Big(\int_0^{\widetilde{T}} \int_{\Omega_R} |\partial_{x_j} Q^{(p)}(x,t)|^2\, dx\, dt \Big)^{1/2} \\ &\leq \mathfrak{C} \cdot \varepsilon_p^{-7/8} \cdot \Big(\int_0^{\widetilde{T}} \int_{\Omega_R} \Big(\int_{\partial\Omega} |x-y|^{-17/8} \cdot |\varphi(y,t)|\, d\Omega(y) \Big)^2 dx\, dt \Big)^{1/2} \\ &\leq \mathfrak{C}(\widetilde{T}, R) \cdot \varepsilon^{-7/8} \cdot \|\varphi\|_2 \end{aligned}$$

for $p \in \mathbb{N},\ 1 \leq j \leq 3$. Due to (54), (55), (47) and the preceding estimate, we may conclude for $p \in \mathbb{N}$ that

$$\Big| \int_0^{\widetilde{T}} \int_{\Omega_R} \partial_t V^{(p)} \cdot (\mathfrak{V} - V^{(p)})\, dx\, dt \Big| \leq \mathfrak{C}(\kappa, \widetilde{T}, R) \cdot \varepsilon_p^{1/8} \cdot \|\varphi\|_2^2.$$

Thus we get

$$\begin{aligned} &\lim_{p\to\infty} \int_0^{\widetilde{T}} \int_{\Omega_R} \partial_t V^{(p)} \cdot \mathfrak{V}\, dx\, dt = \lim_{p\to\infty} \int_0^{\widetilde{T}} \int_{\Omega_R} \partial_t V^{(p)} \cdot V^{(p)}\, dx\, dt \\ &= \lim_{p\to\infty} \int_{\Omega_R} \big(|V^{(p)}(x,\widetilde{T})|^2 - |V^{(p)}(x,0)|^2 \big)/2\, dx = \int_{\Omega_R} |\mathfrak{V}(x,\widetilde{T})|^2/2\, dx. \end{aligned} \tag{56}$$

We note that the partial integration in (56) is justified because the function $V^{(p)}(x,\ \cdot\ )$ belongs to $C^0([0,\widetilde{T}])^3$ and to $H^1\big((0,\widetilde{T})\big)^3$, for $x \in U_p,\ p \in \mathbb{N}$, as observed above. The last equation in (56) follows from (14) and Lebesgue's theorem on dominated convergence; compare the proof of [5, Theorem 2.4], in particular [5, (33)]. Also note that $V^{(p)}(x, 0) = 0$ for $p \in \mathbb{N}$, $x \in U_p$. Equation (52), (53) and (56) yield

$$0 = \int_0^{\widetilde{T}} \int_{\partial B_R} \Big(\sum_{j,k=1}^{3} -\partial_{x_k} \mathfrak{V}_j(x,t) \mathfrak{V}_j(x,t) x_k / R \tag{57}$$

$$+ (\tau/2) |\mathfrak{V}(x,t)|^2 x_1 / R \; + \; Q(\varphi)(x,t) R^{-1} \big(x \mathfrak{V}(x,t) \big) \Big) \, do_x \, dt$$

$$+ \int_0^t \int_{\Omega_R} |\nabla_x \mathfrak{V}(x,t)|^2 \, dx \, dt \; + \; (1/2) \int_{\Omega_R} |\mathfrak{V}(x, \widetilde{T})|^2 \, dx.$$

We have $\int_0^{\widetilde{T}} \int_U |\nabla_x \mathfrak{V}(x,t)|^2 \, dx \, dt \, < \infty$ and $\int_U |\mathfrak{V}(x, \widetilde{T})|^2 \, dx \, < \infty$, according to [5, Corollary 3.1]. Moreover, by Lemma 5 with $K = R_0/2$,

$$|\partial^\alpha \mathfrak{V}(x,t)| \le \mathfrak{C}(\kappa, \widetilde{T}, R_0) \cdot \|\varphi\|_2 \cdot |x|^{-3/2 - |\alpha|/2},$$
$$|Q(\varphi)(x,t)| \le \mathfrak{C}(\kappa, \widetilde{T}) \|\varphi\|_2 \cdot |x|^{-2}$$

for $x \in B_{R_0}^c, \;\; t \in (0, \widetilde{T}), \;\; \alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 1$. Thus, letting R tend to infinity in (57), we obtain

$$0 = \int_0^{\widetilde{T}} \int_U |\nabla_x \mathfrak{V}(x,t)|^2 \, dx \, dt + (1/2) \int_U |\mathfrak{V}(x, \widetilde{T})|^2 \, dx. \tag{58}$$

Since $\widetilde{T}$ is an arbitrary number in $(0, T)$, and because $\mathfrak{V}|Z_T$ is continuous, we may conclude that $\mathfrak{V}|Z_T = 0$. Therefore Eq. (7) (verified according to Lemma 6), the assumption that U is connected, and the decay of $Q(\varphi)(x,t)$ for $|x| \to \infty$ imply that $Q(\varphi)(x,t) = 0$ for $x \in U$ and for a. e. $t \in (0, T)$.

Next we define $V^{(-p)}(x,t)$ and $Q^{(-p)}(x,t)$ in the same way as respectively $V^{(p)}(x,t)$ and $Q^{(p)}(x,t)$ in (45) and (46), but for $x \in \mathbb{R}^3$ with $\mathrm{dist}(x, \overline{\Omega}) < \mathfrak{D}_1 \cdot \varepsilon_p / 2$ instead of $x \in U_p$, and with the term $-y + \varepsilon_p \cdot m^{(\Omega)}(y)$ replaced by $-y - \varepsilon_p \cdot m^{(\Omega)}(y)$. By a reasoning analogous to (but somewhat simpler than) the one which led to (58), we may show that $\mathfrak{V} \mid \Omega \times (0, T) = 0$ and that for a. e. $t \in (0, T)$, there is $c_{in}(t) \in \mathbb{R}$ with $Q(\varphi)(x,t) = c_{in}(t)$ for $x \in \Omega$. It follows that

$$Q(\varphi)(x + \varepsilon_p \cdot m^{(\Omega)}(x), \; t) - Q(\varphi)(x - \varepsilon_p \cdot m^{(\Omega)}(x), \; t) \;\to\; -c_{in}(t) \quad (p \to \infty)$$

for $x \in \partial\Omega$ and for a. e. $t \in (0, T)$. Thus we may conclude from Theorem 5 that $-c_{in}(t) = n^{(\Omega)}(x) \cdot \varphi(x,t)$ for a. e. $x \in \partial\Omega, \;\; t \in (0, T)$, hence

$$-c_{in}(t) \cdot \int_{\partial\Omega} d\Omega(x) = \int_{\partial\Omega} n^{(\Omega)}(x) \cdot \varphi(x,t) \, d\Omega(x) \quad \text{for a. e. } \; t \in (0, T).$$

Since $\varphi \in L^2_n(\partial\Omega)$, we arrive at the equation $c_{\text{in}}(t) = 0$ for a. e. $t \in (0, T)$. But $\mathfrak{V} \,|\, (\mathbb{R}^3\backslash\partial\Omega) \times (0, T) = 0$, as proved above, so we have

$$\sum_{k=1}^{3}\Big(\partial_k(\mathfrak{V}_j|\mathfrak{O}) - \delta_{jk} \cdot Q(\varphi)\Big)\big(x + \varepsilon_p \cdot m^{(\Omega)}(x),\ t \big) \cdot n_k^{(\Omega)}(x) -$$
$$\sum_{k=1}^{3}\Big(\partial_k(\mathfrak{V}_j|\mathfrak{O}) - \delta_{jk} \cdot Q(\varphi)\Big)\big(x - \varepsilon_p \cdot m^{(\Omega)}(x),\ t \big) \cdot n_k^{(\Omega)}(x) \ \to 0 \quad (p \to \infty)$$

for $1 \le j \le 3$, a. e. $x \in \partial\Omega,\ t \in (0, T)$, where we used the abbreviation $\mathfrak{O} := (\mathbb{R}^3\backslash\partial\Omega) \times (0, T)$. Now Theorem 5 yields $\varphi = 0$. □

At this point we may prove the invertibility of the operator $\mathfrak{F}_T^{(\tau)}$.

Corollary 2 *Let $T \in (0, \infty]$. Then the operator $\mathcal{F}_T^{(\tau)} : L^2_n(S_T) \mapsto H_T$ is linear, bounded and bijective.*

Proof Let $\kappa \in (0, \infty)$. For any $T' \in (0, \infty)$, the operator $\mathfrak{F}_{T'}^{(\kappa)} - \mathfrak{F}_{T'}^{(0)}$ is linear, bounded and compact; see Theorem 4. Theorem 3 now implies that $\mathfrak{F}_{T'}^{(\kappa)}$ is linear, bounded and closed range for $T' \in (0, \infty)$. Since $\mathfrak{F}_{T'}^{(\kappa)}(\varphi|S_{T'}) = \mathfrak{F}_T^{(\kappa)}(\varphi) \,|\, S_{T'}$ for $T' \in (0, T),\ \varphi \in L^2_n(S_T)$, it is not difficult to conclude that $\mathfrak{F}_T^{(\kappa)}$ is also closed-range even if $T = \infty$. A detailed argument may be found in the proof of [3, Lemma 12]. On the other hand, we know by Theorem 6 that the operator $\mathfrak{F}_T^{(\kappa)}$ is one-to-one, and by Theorem 3 and 4 that it is bounded. Thus this operator is linear, bounded and semi-Fredholm. This is true for any $\kappa \in (0, \infty)$. Due to inequality (21), the index of this operator is stable with respect to κ. In this respect, we refer to [17, Theorem I.3.11], where Fredholm operators are considered. But the result stated there also holds for semi-Fredholm operators. To see this, it suffices to replace the reference to [17, Theorem I.3.5] in [17] by a reference to [14, p. 235, Theorem 5.17] (stability of the index of semi-Fredholm operators with respect to small perturbations). But by Theorem 3, the index of $\mathfrak{F}_T^{(0)}$ equals zero. Thus we may conclude that the operator $\mathfrak{F}_T^{(\tau)}$ also has index zero. Recalling that the latter operator is one-to-one (Theorem 6), we thus obtain that it is even bijective. □

The preceding corollary means that we may solve the integral equation in (11) if its right-hand side is in H_T:

Corollary 3 *Let $T \in (0, \infty],\ c \in H_T$. Then there is a unique function $\varphi \in L^2_n(S_T)$ which solves (11). There is $C = C(\tau, \Omega) > 0$ such that $\|\varphi\|_2 \le C \cdot \|\mathfrak{F}_T^{(\tau)}(\varphi)\|_{H_T}$ for $\varphi \in L^2_n(S_T)$.*

Proof We still have to show that the constant C does not depend on $T \in (0, \infty]$. To this end, suppose that $T \in (0, \infty)$. For $b \in H_T,\ x \in \partial\Omega,\ t \in (0, \infty)$, define

$$G_T(b)(x, t) := b(x, t) \ \text{ if } t \le T, \quad G_T(b)(x, t) := b(x, 2T - t) \ \text{ if } \ t \in (T,\ 2T],$$
$$G_T(b)(x, t) := 0 \ \text{ if } \ t > 2 \cdot T.$$

Then $G_T(b) \in H_T$ and

$$\|G_T(b)\|_{H_\infty} \leq \mathfrak{C} \cdot \|b\|_{H_T} \quad \text{for } b \in H_T. \tag{59}$$

The proof of this result is not trivial; we refer to [1] for details. To indicate a possible way to proceed, we mention that for $x \in \mathbb{R}^3$, $t \in [0, T]$, $v \in \widetilde{H}_T$,

$$v(x,t) = B(1/2, 1/2)^{-1/2} \cdot \int_0^t (t-s)^{-1/2} \cdot \partial_4^{1/2} v(x,s)\, ds, \tag{60}$$

where B denotes the usual beta function. By applying (60) to $v(x,t)$ for $t \leq T$ and to $v(x, 2 \cdot T - t)$ for $t \in (T, 2 \cdot T)$, it is possible to prove that

$$\|\partial_4^{1/2} G_T(v|S_T)\|_2 \leq \mathfrak{C} \cdot \left(\|v|S_T\|_2 + \|\partial_4^{1/2} v|S_T\|_2 \right) \quad \text{for } v \in \widetilde{H}_T.$$

Then inequality (59) may be established by some obvious additional estimates. Once an extension operator G_T from H_T to H_∞ with (59) is available, the corollary follows from Corollary 2 with $T = \infty$. □

References

1. Brown, R. M.: Layer Potentials and Boundary Value Problems for the Heat Equation in Lipschitz Domains. Thesis, University of Minnesota (1987)
2. Deuring, P.: *The Stokes System in an Infinite Cone.* Akademie Verlag, Berlin (1994)
3. Deuring, P.: The single-layer potential associated with the time-dependent Oseen system. Proceedings of the 2006 IASME/WSEAS International Conference on Continuum Mechanics. Chalkida, Greeece, May 11–13, 117–125 (2006)
4. Deuring, P.: On volume potentials related to the time-dependent Oseen system. WSEAS Trans. Math. **5**, 252–259 (2006)
5. Deuring, P.: On boundary-driven time-dependent Oseen flows. Banach Center Publ. **81**, 119–132 (2008)
6. Deuring, P.: Spatial decay of time-dependent Oseen flows. SIAM J. Math. Anal. **41**, 886–922 (2009)
7. Deuring, P., Kračmar, S.: Exterior stationary Navier-Stokes flows in 3D with non-zero velocity at infinity: approximation by flows in bounded domains. Math. Nachr. **269–270**, 86–115 (2004)
8. Enomoto, Y., Shibata, Y.: Local energy decay of solutions to the Oseen equation in the exterior domain. Indiana Univ. Math. J. **53**, 1291–1330 (2004)
9. Enomoto, Y., Shibata, Y.: On the rate of decay of the Oseen semigroup in exterior domains and its application to Navier-Stokes equation. J. Math. Fluid Mech. **7**, 339–367 (2005)
10. Farwig, R.: The stationary exterior 3D-problem of Oseen and Navier-Stokes equations in anisotropically weighted Sobolev spaces. Math. Z. **211**, 409–447 (1992)
11. Fučik, S., John, O., Kufner, A.: *Function Spaces.* Noordhoff, Leyden (1977)
12. Galdi, G. P., Silvestre, A. S.: Strong solutions to the Navier-Stokes equations around a rotating obstacle. Arch. Rat. Mech. Appl. **176**, 331–350 (2005)
13. Galdi, G. P., Silvestre, S. A.: The steady motion of a Navier-Stokes liquid around a rigid body. Arch. Rat. Mech. Appl. **184**, 371–400 (2007)
14. Kato, T.: *Perturbation Theory of Linear Operators.* Springer, Berlin e.a. (1996)

15. Knightly, G. H.: Some decay properties of solutions of the Navier-Stokes equations. In: R. Rautmann (ed.), *Approximation Methods for Navier-Stokes Problems.* Lecture Notes in Math. **771**, 287–298, Springer (1979)
16. Kobayashi, T., Shibata, Y.: On the Oseen equation in three dimensional exterior domains. Math. Ann. **310**, 1–45 (1998)
17. Mikhlin, S. G., Prössdorf, S.: *Singular Integral Operators.* Springer, Berlin e.a. (1986)
18. Nečas, J.: *Les Méthodes Directes en Théorie des Équations Elliptiques.* Masson, Paris (1967)
19. Shen, Z.: Boundary value problems for parabolic Lamé systems and a nonstationary linearized system of Navier-Stokes equations in Lipschitz cylinders. Am. J. Math. **113**, 293–373 (1991)
20. Shibata, Y.: On an exterior initial boundary value problem for Navier-Stokes equations. Quarterly Appl. Math. **57**, 117–155 (1999)